Name: Shivani Patel **GSI:** Madeline Brandt **DISC #:** 103

Math 55, Homework 5.

Prob 1. Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor$.

$$\sum_{k=0}^{m} \lfloor k^{1/3} \rfloor = \sum_{i=0}^{m^{1/3}} i \left| \{ 0 \le k \le m : \lfloor k^{1/3} \rfloor = i \} \right|$$

$$= \sum_{i=0}^{\lfloor m^{1/3} \rfloor} i \left| \{ 0 \le k \le m : i^3 \le k < (i+1)^3 \} \right|$$

$$= \sum_{i=0}^{\lfloor m^{1/3} \rfloor - 1} i((i+1)^3 - i^3) + \lfloor m^{1/3} \rfloor (m - \lfloor m^{1/3} \rfloor^3 + 1)$$

$$= \sum_{i=0}^{\lfloor m^{1/3} \rfloor - 1} (3i^3 + 3i^2 + i) + \lfloor m^{1/3} \rfloor m - \lfloor m^{1/3} \rfloor^3 + 1)$$

$$= \frac{1}{4}(\lfloor m^{1/3} \rfloor - 1 \lfloor m^{1/3} \rfloor^2 (3 \lfloor m^{1/3} \rfloor + 1) + \lfloor m^{1/3} \rfloor (m - \lfloor m^{1/3} \rfloor^3 + 1)$$

Prob 2. (a) Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate r if a payment P is made on the loan each month.

$B(k) = B(k-1)\left(1+\frac{r}{12}\right) - P$

(b) Determine what the monthly payment P should be so that the loan is paid off after T months.

$B(k) = B(0)\left(1+\frac{r}{12}\right)^k - P - P\left(1+\frac{r}{12}\right) - P\left(1+\frac{r}{12}\right)^2 - \ldots - P\left(1+\frac{r}{12}\right)^{k-1}$

$P\frac{\left(1+\frac{r}{12}\right)^k - 1}{\left(1+\frac{r}{12}-1\right)} = \frac{12P}{r}\left(\left(1+\frac{r}{12}\right)^k - 1\right)$, so $B(k) = \left(B(0) - \frac{12P}{r}\right)\left(1+\frac{r}{12}\right)^k + \frac{12P}{r}$

to pay loan after T months we calculate $B(T)=0$

$P = \frac{r\,B(0)}{12} \cdot \frac{\left(1+\frac{r}{12}\right)^T}{\left(1+\frac{r}{12}\right)^T - 1}$

(c) Suppose you take out a fixed-rate mortgage for $\$1M$ at the current (historically low) rate 3% and want to pay it off in 20 years. What monthly payment should you make?

$r = 0.03$, $T = 20 \cdot 12 = 240$ & $B(0) = 1{,}000{,}000$

$P = 2500 \cdot \frac{1.0025^{240}}{1.0025^{240} - 1} \approx 5545.97$

this shows that the fixed monthly payment should be $\$5{,}545.97$ ($\approx \$5{,}546$)

(d) Now suppose the same mortgage of $\$1M$ but you have qualified only for the rate 5% and the maximum monthly payment you can afford is \$5K. How many years will it take you to pay off that mortgage?

$\left(\frac{12P}{r} - B(0)\right)\left(1+\frac{r}{12}\right)^T = \frac{12P}{r} \Longrightarrow \left(1+\frac{r}{12}\right)^T = \frac{12P}{12P - r\,B(0)} \Longrightarrow T = \frac{\ln(12P) - \ln(12P - r\,B(0))}{\ln\left(1+\frac{r}{12}\right)}$

Now for the values: $P = 5{,}000$, $r = 0.05$ & $B(0) = 1{,}000{,}000$

$T = \frac{\ln 6}{\ln 1.00416667} \approx 430.918 \approx \boxed{35.9 \text{ years}}$

Prob 3. Write down the full addition and multiplication tables for $\mathbb{Z}_9$ (where addition means $+_9$ and multiplication means $\cdot_9$).

$+_9$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

$\cdot_9$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1	3	5	7
3	0	3	6	0	3	6	0	3	6
4	0	4	8	3	7	2	6	8	5
5	0	5	1	6	2	7	3	8	4
6	0	6	3	0	6	3	0	6	3
7	0	7	5	3	8	8	6	4	2
8	0	8	7	6	5	4	3	2	1

Prob 4. (a) Prove that, if p is a prime, then all positive integers less than p except for 1 and $p-1$ can be split into $(p-3)/2$ pairs such that each pair consists of integers that are inverses of each other modulo p.

the proof of $p=2$ is vacuous.

the product of all numbers in the set S is congruent to 1 mod $p \longrightarrow$ product of $(p-3)/2$ pairs which are inverses of each other

(b) Conclude from part (a) that $(p-1)! \equiv -1 \pmod{p}$ whenever p is prime.

$$(p-1)! = 1\left(\prod_{j\in S} j\right)(p-1) \equiv 1(-1) \equiv -1 \pmod{p}$$

(c) What can we conclude if n is a positive integer such that $(n-1)! \not\equiv -1 \pmod{n}$?

If $(n-1)! \not\equiv -1 \pmod{n}$, then n is composite

If n was prime then we would conclude part b

Prob 5. Prove or disprove that there are infinitely many primes of the form $6k + 5$, $k \in \mathbb{Z}_+$.

If there are only finite many primes of $6k+5$ (represented by $p_1, p_2, p_3, \cdots, p_n$ for $n \in \mathbb{N}$) For example if we think about $6(p_1 \cdots p_n)-1$ and then plug into the equation showing $6(p_1, \cdots, p_n - 1)+5$. We see that none of the primes $p_1, p_2, \cdots p_n$ divide N because $N \equiv -1 \pmod{p_j}$ $\forall j = 1, \ldots, n$.

case 1:

If N is prime then N is not in $(p_j)_{j=1}^{n}$ bc of p_j values not dividing into N

case 2:

If N is composite then we have to look at the prime divisors. N cannot divide by 2 or 3 because $N \equiv -1 \pmod 6$

therefore all prime divisors of N are either -1 or 1 (mod 6) from cases shown above. If they were all equal to 1 (mod 6) then N would also have to equal 1 (mod 6) but we saw that $N \equiv -1$ therefore this gives evidence that at least one prime divisors of N is equal to $-1 \equiv 5 \pmod 6$

through this proof we can see there are infinitely many primes of the form $6k+5$