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Math 55, Homework 8.

Prob 1. Prove the **Multinomial Theorem**: If $m, n \in \mathbb{N}$, then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} \frac{n!}{n_1!n_2!\cdots n_m!} x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}.$$

we expand the left hand side, then we get m^n terms of the form $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ with $n_1+n_2+\ldots+n_m=n$, many of which will be repeated. when collecting similar terms, we will see the same monomial $x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}$ whenever we pick x_1 from n_1 factors $(x_1+x_2+\ldots+x_m)$, x_2 from n_2 factors $(x_1+x_2+\cdots+x_m)$ etc. x_m from n_m factors $(x_1+x_2+\ldots+x_m)$. we now recall that the multinomial coefficient $\frac{n!}{n_1!n_2!\cdots n_m!}$ counts the number of permutations of n elements where each element j is repeated n_j times, $j=1,\cdots m$. Therfore, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ is exactly the multinomial coefficient $\frac{n!}{n_1!n_2!\cdots n_m!}$. this is true for each monomial in the right-hand side, so this completes the proof

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Prob 2. Solve the problem posed by Chevalier de Méré to Blaise Pascal and Pierre de Fermat:

(a) Find the probability that a double six comes up at least once when a pair of fair dice is rolled 24 times. Is this probability greater than 1/2?

The probability of getting a double six is $1/36$, so the probability of not getting a double six once is $35/36$ so when we toss a pair of dice 24 times, the probability of not getting a double six even once is $(35/36)^{24}$, so the probability of getting a double six at least once is

$$1-\left(\frac{35}{36}\right)^{24} \approx 0.4914 < \frac{1}{2}$$

(b) Is it more likely that a six comes up at least once when a fair die is rolled four times or that a double six comes up at least once when a pair of fair dice is rolled 24 times?

the prob that a six never comes up when a fair die is rolled four times is $(5/6)^4$. so the prob that it comes up at least once is

$$1-\left(\frac{5}{6}\right)^{4} \approx 0.5177 > \frac{1}{2} > 0.4914$$

Prob 3. A player in the Mega Millions lottery picks five different integers between 1 and 56, inclusive, and a sixth integer between 1 and 46, which may duplicate one of the earlier five integers. The player wins the jackpot of the first five numbers match the first five numbers drawn (irrespective of the order) and the sixth number matches the sixth number drawn.

(a) What is the probability a player wins the jackpot?

$= \binom{56}{5}$ + # of ways to chose the 6th integer

$$= \frac{1}{\left(\binom{56}{5} * 46\right)} = \boxed{\frac{1}{175,711,536}}$$

(b) What is the probability a player wins \$250,000, the prize for matching only the first five numbers?

$$\frac{46}{\binom{56}{5} * 46} = \frac{46}{175,711,536} = \boxed{\frac{1}{3,819,816}}$$

Prob 4. Suppose that p and q are primes and that $n = pq$. What is the probability that a randomly chosen positive integer less than n is not divisible by p or q?

Let p and q be primes

$n = pq$

since $n = pq$ and since p is prime

$p, 2p, 3p, \ldots, (q-1)p$ are all divisible by p

since $n = pq$ & since q is prime

$q, 2q, 3q, \ldots, (p-1)q$ are all divisible by q

thus there are $p-1$ integers smaller than n that are divisible by q

In total, there are then $p-1+q-1 = p+q-2$ integers divisible by p or q.

☆ there are no ints less than n that are divisible by p & q (since $n=pq$)

there are $n-1$ integers less than n, while $p+q-2$ integers of the $n-1$ integers less than n are divisible by p or q

$$P(\text{Divisible by } p \text{ or } q) = \frac{\#\text{ of favorable outcomes}}{\#\text{ of possible outcomes}} = \frac{p+q-2}{n-1} = \boxed{\frac{p+q-2}{pq-1}}$$

Prob 5. What is the probability that a five-card poker hand contains cards of five different values but does not contain a flush or a straight?

Given: A deck has 52 cards; 26 red, 26 black; 13 spades, diamonds, hearts & clubs

Formulas: $P(n,r) = \frac{n!}{(n-r)!}$ $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

① # of hands containing 5 different cards

$C(13,5) = 1287$
$C(4,1) = 4$
" $= 4$
" $= 4$
" $= 4$
" $= 4$

product rule:
$1287 \cdot 4^5 = \boxed{1{,}317{,}888}$

② # of hands containing a flush

$C(4,1) = 4$
$C(13,5) = 1287$

product rule:
$1287 \cdot 4 = \boxed{5148}$

③ # of hands containing a straight

10 possible ways: select 1 of the 4 suits of 5 cards.

- A2345
- 23456
- 34567
- 45678
- 56789
- 678910
- 78910J
- 8910JQ
- 910JQK
- 10JQKA

$C(4,1) = 4$
" $= 4$
" $= 4$
" $= 4$
" $= 4$

product rule:
$10 \cdot 4^5 = \boxed{10{,}240}$

④ # of hands containing a straight flush

10 straights | of the 4 suits
$C(4,1) = 4$

Product Rule: $10 \cdot 4 = \boxed{40}$

⑤ # of hands containing straight or flush

subtraction rule

|straight or flush| = |straight| + |flush| − |straight flush|
= 10,240 + 5148 − 40
= 15,348

⑥ # of hands containing 5 different kinds & no flush nor straight

= |Five diff kinds & no flush nor straight|
= |Five different kinds| − |straight flush|
= 1,317,888 − 15,348
= 1,302,540

⑦ Probability

1,302,540 of the 2,598,960 will contain no Ace

P(Five diff kinds & no flush/straight) = $\frac{\text{\# of favorable outcomes}}{\text{\# of possible outcomes}} = \frac{1{,}302{,}540}{2{,}598{,}960} = \boxed{\frac{1277}{2548} \approx 0.5012}$