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Math 55, Handout 19.

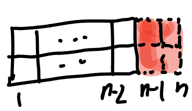
LINEAR HOMOGENEOUS RECURRENCES WITH CONSTANT COEFFICIENTS.

	RECURRENCES	ODEs
Equation	$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$	$y^{(k)} = C_1 \cdot y^{(k-1)} + C_2 \cdot y^{(k-2)} + \cdots + C_k y$
Solution Ansatz	$a_n = r^n$	$y(x) = e^{rx}$
Ansatz plugged in	$r^n = C_1 \cdot r^{n-1} + C_2 r^{n-2} + \cdots + C_k \cdot r^{n-k}$	$r^k e^{rx} = c_1 r^{k-1} e^{rx} + c_2 r^{k-2} e^{rx} + \cdots + c_k e^{rx}$
Char. polynomial	$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$	$r^k = C_1 \cdot r^{k-1} + \cdots + C_k$
Linearity	linear combination of sols is still solution	y_1, y_2 solutions $\Longrightarrow C_1 y_1 + C_2 y_2$ solution
Distinct real roots	$a_n = C_1 r_1^n + C_2 r_2^n + \cdots + C_k r_k^n$	$y = C_1 \cdot e^{r_1 x} + \cdots + C_k \cdot e^{r_k x}$
Initial conditions	Given $a_0 \cdots a_{k-1}$ to get $C_1, \cdots, C_k$.	met by solving for C_1 through C_k
Complex roots	$a_n = r^n$ still a solution for $r \in \mathbb{C}$	$y = e^{(a+bi)x} = e^{ax}(\cos bx + i \cdot \sin bx)$ is still solution.
Multiple roots	If r is of multiplicity m, $r^n, n \cdot r^n, n^2 r^n, \ldots, n^{m-1} \cdot r^n$	e^{rx}, xe^{rx}, …, $x^{m-1}e^{rx}$ are solutions

NB: The uppercase letters C_j and the lowercase letters c_j here are not to be confused!

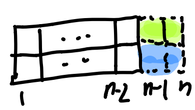
Q1. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

case 1:



$F(n) = F(n-1) + 2F(n-2)$

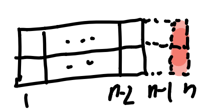
case 2:



$F(n) = C_1 \cdot 2^n + C_2 \cdot (-1)^n$

$F(1) = 1 \quad F(2) = 3$

Case 3.



$F(n) = \dfrac{2^{n+1} + (-1)^n}{3}$

Q2. What is the general form of the solution of a linear homogeneous recurrence relation if its characteristic polynomial has precisely these roots: $1, 1, 1, 1, -2, -2, -2, 3, 3, -4$?

$$a_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3 + C_5 (-2)^n + C_6 \cdot n \cdot (-2)^n + C_7 \cdot n^2 (-2)^n + C_8 \cdot 3^n + C_9 n \cdot 3^n + C_{10} (-4)^n$$

LINEAR INHOMOGENEOUS RECURRENCES WITH CONSTANT COEFFICIENTS.

	RECURRENCES	ODEs
Equation	$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n)$	$y^{(k)} = c_1 \cdot y^{(k-1)} + \cdots + c_k \cdot y + F(x)$
Linearity yields	$a_n = a_n^{hom} + a_n^{part}$	$y = y^{hom} + y^{part}$
Special cases	$F(n) = p_t(n)s^n$	$F(x) = p_t(x) e^{sx}$
s not char. root	$a_n^{part} = q_t(n) \cdot s^n$	$y^{part}(x) = q_t(x)e^{sx}$
s root; mult. m	$a_n^{part} = n^m \cdot q_t(n) \cdot s^n$	$y^{part}(x) = x^m q_t(x)e^{sx}$

Q3. What is the general form of the particular solution – guaranteed to exist by the above results – of the linear inhomogeneneous recurrence $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if $n^4 - 8n^2 + 16 = 0 \rightarrow 2, 2, -2, -2$

(a) $F(n) = n^2$? $s=1$ not root, $a_n = c_1 2^n + c_2 \cdot n \cdot 2^n + c_3 (-2)^n + c_4 n \cdot (-2)^n + \frac{1}{9}n^2 + \frac{32}{27}n + \frac{32}{27}$

(b) $F(n) = 2$? $s=1$ not root, $a_n = c_1 2^n + c_2 \cdot n \cdot 2^n + c_3 (-2)^n + c_4 n \cdot (-2)^n + \frac{2}{9}$

(c) $F(n) = n^4 2^n$? $s=2$ is the root, $a_n = c_1 2^n + c_2 \cdot n \cdot 2^n + c_3 (-2)^n + c_4 n \cdot (-2)^n + n^2 (an^4 + bn^3 + cn^2 + dn + e) \cdot 2^n$

Q4. Find all solutions to the recurrence $a_{n+2} = -a_n + 5 \cdot 2^n$ subject to the initial conditions $a_0 = 2$, $a_1 = 3$.

characristic equation: $n^2 + 1 = 0$.

$n = \pm i$

2 is not the root

$a_n^{part} = c \cdot 2^n$. plug into it: $c \cdot 2^{n+2} = -c \cdot 2^n + 5 \cdot 2^n$

$(4c + c - 5) \cdot 2^n = 0$

$c = 1$

$a_n = c_1 (i)^n + c_2 \cdot (-i)^n + 2^n$

$a_0 = 2 \quad a_1 = 3$

$\rightarrow c_1 = \frac{1-i}{2} \quad c_2 = \frac{1+i}{2}$

$a_n = \frac{1-i}{2} (i)^n + \frac{1+i}{2} (-i)^n + 2^n.$