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Math 55, Handout 12.

PERMUTATIONS AND COMBINATIONS.

1.1. A **permutation** of a set of distinct objects is an ordered arrangement of these objects.
An r-**permutation** is an ordered arrangement of r elements of a set

1.2. [**Theorem**]. The number $P(n, r)$ of r-permutations of a set with n elements is an integer $1 \le r \le n$, then there are

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

r-permutations of a set with n distinct elements

Q1. A group contains n men and n women. How many ways are there to arrange these people in a row for picture so that the men and the women alternate?

women
$n=n$
$r=n$
$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

men
$n=n$
$r=n$
$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

WMWMW...M
MWMWM...W
2 possible combos

use product rule!
$n! * n! = (n!)^2$ for both combos

$(n!)^2 + (n!)^2 = \boxed{2(n!)^2}$ ways to arrange men & women

2.1. An r-**combination** of elements of a set is an unordered selection of r elements from the set

2.2. [**Theorem**]. The number $C(n, r)$ of r-combinations of a set with n elements is a nonnegative integer and r is an integer with $0 \le r \le n$, equals:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Q2. A coin is flipped eight times, and each flip comes up heads or tails. How many possible outcomes

(a) are there in total? 2 possible outcomes $\longrightarrow 2^8 = \boxed{256}$

(b) contain at most three tails?

0 tails: $C(8,0) = 1$
1 tails: $C(8,1) = 8$
2 tails: $C(8,2) = 28$
3 tails: $C(8,3) = 56$

$1+8+28+56 = \boxed{93}$

THE BINOMIAL THEOREM.

3.1. [**Binomial Theorem**]. Let x and y be variables and let $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

3.2. [**Corollary**]. Let $n \in \mathbb{N}$. Then

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

3.3. [**Corollary**]. Let $n \in \mathbb{N}$. Then

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

3.4. [**Corollary**]. Let $n \in \mathbb{N}$. Then

$$3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

BINOMIAL IDENTITES.

4.1. **[Pascal's Identity].** Let n, $k \in \mathbb{N}$, with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Q3. Derive Pascal's identity from the formula for $C(n, r)$.

Pascal's Identity states:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

If $k>n$ then $\binom{n}{k} = 0 = \binom{n-1}{k-1} + \binom{n-1}{k}$ ⟶ result is trivial

If $k \leq n$ then:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$$

$$= (n-1)!\left(\frac{k}{k!(n-k)!} + \frac{n-k}{k!(n-k)!}\right)$$

$$= (n-1)!\left(\frac{n!}{k!(n-k)!}\right)$$

$$= \binom{n}{k} \quad \square$$

4.2. **[Vandermonde's Identity].** Let n, m, $k \in \mathbb{N}$, with $k \leq m$, $k \leq n$. Then

$$\binom{m+n}{k} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

4.3. **[Corollary].** Let $n \in \mathbb{N}$. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Q4. For $n \in \mathbb{N}$, derive the following identity using algebra and (basic) calculus:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

$$S = \sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n} (n-k)\binom{n}{n-k}$$

$$2S = \sum_{k=0}^{n} \left(k\binom{n}{k} + (n-k)\binom{n}{n-k}\right)$$

$$= \sum_{k=0}^{n} \left(k\binom{n}{k} + (n-r)\binom{n}{r}\right)$$

$$= \sum_{k=0}^{n} \left(n\binom{n}{r}\right)$$

$$= n\sum_{k=0}^{n} \binom{n}{k}$$

$$= n2^n \quad \square$$

4.4. **[Unnamed Identity].** Let n, $k \in \mathbb{N}$, with $k \leq n$. Then

$$\binom{n+1}{k+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Q5. Draw a diagram within Pascal's triangle as a memo for the last identity.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1