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Math 55, Handout 18.

THE EXPECTED VALUE.

1.1. The **expected value,** also called the expectation or mean , of the random variable X on the sample space S is defined by

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Q1. A coin biased so that heads are three times more likely than tails is flipped five times. What is the expected total number of heads?

$p(H) = 3p(T)$

$\hookrightarrow 3/4 \quad \downarrow 1/4$

1 2 3 4 5

$$\boxed{\left(\tfrac{3}{4}\right)(1) + \left(\tfrac{3}{4}\right)^2(2) + \left(\tfrac{3}{4}\right)^3(3) + \left(\tfrac{3}{4}\right)^4(4) + \left(\tfrac{3}{4}\right)^5(5)}$$

1.2. [Theorem.] The expected value of a random variable X can be computed by an equivalent formula

$$E(X) = \sum_{r \in X(S)} p(X=r) r$$

1.3. [Theorem.] The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial , is np

1.4. [Theorem.] Let X_i, $i = 1, \ldots, n$, be random variables on S, and let a, $b \in \mathbb{R}$. Then

(*i*) $E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$

(*ii*) $E(aX + b) = aE(X) + b$

Q2. Find the expected value of the sum of the numbers that appear when 100 fair dice are rolled.

$\Sigma(X) = \sum x\, p(x)$

$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6} = \frac{21}{6} n \qquad \frac{21}{6}(100) = \frac{2100}{6} = \boxed{350}$

THE GEOMETRIC DISTRIBUTION.

2.1. A random variable X has a **geometric distribution with parameter** p if $p(X=k) = (1-p)^{k-1}$ for $k = 1, 2, 3 \ldots$, where p is a real number with $0 \le p \le 1$

2.2. [Theorem.] If the random variable X has a geometric distribution with parameter p, then $E(X) = 1/p$.

INDEPENDENT RANDOM VARIABLES.

3.1. Two random variables X and Y on a sample space S are called **independent** if

$p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \cdot p(Y = r_2)$,

or in words, if the probability that $X = r_1$ and $Y = r_2$ equals the product of the probabilities that $X = r_1$ and $Y = r_2$, for all real numbers r_1 and r_2

3.2. [Theorem.] If X and Y are independent random variables on a sample space S, then $E(XY) = E(X)E(Y)$

Q3. Suppose we roll a fair die four times. Give a non-trivial example of two pairs of random variables on this sample space, one pair dependent, the other independent.

sample space contains $6^4 = 1296$ outcomes

non-trival example for dependent pair

- same spaces appear on 4 rolls of a die
ie. (1,1,1,1), (2,2,2,2), (3,3,3,3), (4,4,4,4), (5,5,5,5), (6,6,6,6)
$\Rightarrow$ outcome of each fair on a single roll depends on a outcome of a each fair on another roll

non-trivial example for independent pair

- sum of the uppermost faces is 20 when we roll a fair die four times & sum of the uppermost faces is 15 when we roll a fair die four times

VARIANCE.

4.1. Let X be a random variable on a sample space S. The **variance** of X, denoted $V(X)$, is defined by

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 \, p(s)$$

4.2. [Theorem.] For any random variable X, $V(X) = E(X^2) - E(X)^2$

4.3. [Corollary.] If X is a random variable X with $E(X) = \mu$, then $V(X) = E((X-\mu)^2)$

4.4. [Bienaymé's formula.] If X and Y are two independent random variables on a sample spaces S, then $V(X+Y) = V(X) + V(Y)$. Furthermore, if X_j, $j = 1, \ldots, n$, are pairwise independent random variables on S, then

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$$

Q4. What is the variance of the combined value when a pair of fair dice is tossed once?

X_i	x_1 2	x_2 3	x_3 4	x_4 5	x_5 6	x_6 7	x_7 8	x_8 9	x_9 10	x_{10} 11	x_{11} 12
$PX(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

variance $= E(x^2) - (E(x)^2)$

$E(X) = \sum_{i=1}^{11} x_i P(x_i)$

$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36}$

$= \frac{252}{36} = \boxed{E(X) = 7}$

$E(x^2) = \sum_{i=1}^{11} x_i^2 P(x_i)$

$= \frac{4}{36} + \frac{9(2)}{36} + \frac{16(3)}{36} + \frac{25(4)}{36} + \frac{36(5)}{36} + \frac{49(6)}{36} + \frac{64(5)}{36} + \frac{81(4)}{36} + \frac{100(3)}{36} + \frac{121(2)}{36} + \frac{164}{36}$

$= E(x^2) = 54.833$

variance $= 54.833 - 49$

$\boxed{\text{variance} = 5.833}$

CHEBYSHEV'S AND MARKOV'S INEQUALITIES.

5.1. [Chebyshev's Inequality.] Let X be a random variable on a sample space S with probability distribution p. If $r > 0$, then

$$P(|X(s) - E(X)| \geq r) \leq V(X)/r^2$$

5.2. [Markov's Inequality.] Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Then

$$P(X(s) \geq a) \leq E(X)/a \text{ for every positive real number } a.$$

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Q5. Prove Markov's inequality.

$$E(X) = \sum_{s=\Omega} P(s)\, X(s)$$

$$= \sum_{s \in A} P(s) X(s) + \sum_{s \notin A} P(s)\, X(s)$$

$$\geq \sum_{s \in A} P(s)\, X(s)$$

$$\geq \sum_{s \in A} P(s) a$$

$$= a \sum_{s \in A} P(s) = a \cdot P(A)$$