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Math 55, Handout 17.

THE PROBABILISTIC METHOD.

1.1. If the probability that an element chosen at random from a set S **does not** have a particular property is less than 1, then there exists an element in S with this property

Q1. Does the underlying probability distribution have to be uniform for the probabilistic method to work? Explain your answer.

NO, the underlying distribution does not matter. there just needs to be some appropriately defined set present

1.2. [Theorem.] If $k \in \mathbb{N} \setminus \{1\}$, then $R(k,k) \geq 2^{k/2}$

Q2. Sketch the main steps of the proof given for this Theorem.

- What probability distribution is used?

  uniform distribution

- What are the events E_i? How many of them are there?

  - you know someone
  - you do not know someone
  - everyone knows each other

- What is the probability of each E_i?

  - equally likely to know someone as not, so $1/2$

- What is the initial upper bound on $p(\cup_i E_i)$?

  $\sum_j p(E_j)$

- Why is that upper bound strictly less than 1?

  if it equals 1, then probability complement is $\emptyset$

- So what if it is strictly less than 1?

  then we are done

- What is the punchline?

  there exists an element in S with property p

- What did you learn from this proof?

  There exists an element s with property p.

BAYES' THEOREM.

2.1. [Bayes' Theorem.] Suppose E and F are events from a sample space S such that . Then

$$p(F|E) = \frac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|\bar{F})\,P(\bar{F})}$$

Q3. You are handed two boxes. You cannot see what is inside but you have reliable information that one of the boxes contains two green and ten red balls and the other contains eight green and four red balls. You select a ball from one of the boxes at random and take it out. It is red. Which of the two boxes is it more likely to have come from? What is that probability?

E = choosing a red ball
F = choosing box 1
$\bar{F}$ = choosing box 2

$P(E|F) = \frac{10}{12}$
$P(F) = \frac{1}{2}$
$P(\bar{F}) = \frac{1}{2}$
$P(E|\bar{F}) = \frac{4}{12}$

$$P(F|E) = \frac{5/6 \cdot 1/2}{5/6 \cdot 1/2 + 1/3 \cdot 1/2} = \frac{5/6}{7/6} = \boxed{\frac{5}{7}}$$

$\boxed{\text{the first box is more likely}}$

2.1. [Generalized Bayes' Theorem.] Suppose that $E \subseteq S$ and $F_1, \ldots, F_n$ are mutually exclusive events from a sample space S such that $\bigcup_{i=1}^{n} F_i = S$. Assume that $P(E) \neq 0$ and $P(F_i) \neq 0$ for $i = 1, 2, \ldots, n$. Then

$$p(F_j|E) = \frac{P(E|F_j)\,P(F_j)}{\sum_{i=1}^{n} P(E|F_i)\,P(F_i)}$$

E = test is positive
F = has disease
$P(F) = 8\% = 0.08$
$P(E|F) = 98\% = 0.98$
$P(E|\bar{F}) = 3\% = 0.03$

Q4. Suppose that 8% of the patients tested in a particular clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

(a) a patient testing positive for HIV with this test is infected with it?

complement rule

$P(\bar{F}) = 1 - P(F) = 1 - 0.08 = 0.92$

Bayes theorem

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|\bar{F})\,P(\bar{F})} = \frac{0.98 \cdot 0.08}{0.98 \cdot 0.08 + 0.03 \cdot 0.92} = \frac{0.0784}{0.106} \approx \boxed{0.7396}$$

(b) a patient testing positive for HIV with this test is not infected by it?

complement rule

$P(\bar{F}) = 1 - P(F) = 1 - 0.08 = 0.92$

Bayes theorem

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|\bar{F})\,P(\bar{F})} \qquad \frac{0.03 \cdot 0.92}{0.03 \cdot 0.92 + 0.98 \cdot 0.08} = \frac{0.0276}{0.106} \approx \boxed{0.2604}$$

(c) a patient testing negative for HIV with this test is infected with it?

complement rule

$P(\bar{F}) = 1 - P(F) = 1 - 0.08 = 0.92$
$P(\bar{E}|F) = 1 - P(E|F) = 1 - 0.98 = 0.02$
$P(\bar{E}|\bar{F}) = 1 - P(E|\bar{F}) = 1 - 0.03 = 0.97$

Bayes theorem

$$P(F|\bar{E}) = \frac{P(E|F)\,P(F)}{P(E|F)\,P(F) + P(E|\bar{F})\,P(\bar{F})}$$

$$P(F|\bar{E}) = \frac{0.02 \cdot 0.08}{0.02 \cdot 0.08 + 0.97 \cdot 0.92} = \frac{0.0016}{0.894} \approx \boxed{0.00179}$$

(d) a patient testing negative for HIV with this test is not infected with it?

complement rule

$P(\bar{F}) = 1 - P(F) = 1 - 0.08 = 0.92$
$P(\bar{E}|F) = 1 - P(E|F) = 1 - 0.98 = 0.02$
$P(\bar{E}|\bar{F}) = 1 - P(E|\bar{F}) = 1 - 0.03 = 0.97$

Bayes Theorem

$$P(\bar{F}|\bar{E}) = \frac{P(\bar{E}|\bar{F})\,P(\bar{F})}{P(\bar{E}|F)\,P(F) + P(\bar{E}|\bar{F})\,P(\bar{F})} = \frac{0.97 \cdot 0.92}{0.02 \cdot 0.08 + 0.97 \cdot 0.92} = \frac{0.8924}{0.894} \approx \boxed{0.99821}$$