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Math 55, Handout 22.

GRAPHS.

1.1. A **graph** $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

1.2. Any graph is either **infinite** or **finite**. The latter means that both sets V and E are finite.

1.3. A **directed graph** (or **digraph**) $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of directed edges. Each directed edge is associated with an ordered pair of vertices (u, v) is said to start at u and end at v.

1.4. All graphs are **undirected**, **directed**, or mixed.

1.5. A **loop** is an edge that connects a vertex to itself

1.6. Graphs with **multiple edges** are called multigraphs

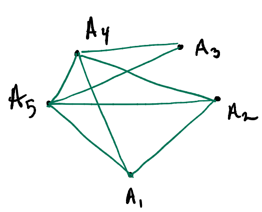
1.7. Directed graphs with **multiple directed edges** are called directed multigraphs

1.8. A graph without loops or multiple edges is called simple.

1.9. Graphs can be used to model social networks, information networks, communication networks, road networks, acquaintanceship, collaboration, influence, citations, dependencies, airlines routes, protein interaction, tournaments, etc. etc.

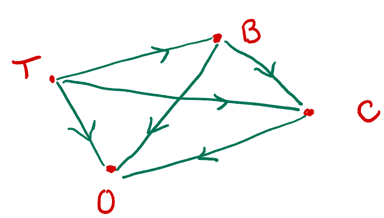
Q1. The **intersection graph** of a collection of sets $A_1, A_2, \ldots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets:

$$A_1 = \{x : x < 0\},\ \ A_2 = \{x : -1 < x < 0\}.\ A_3 = \{x : 0 < x < 1\},\ \ A_4 = \{x : -1 < x < 1\},\ \ A_5 = \mathbb{R}.$$



Q2. In a round-robin tournament the Tigers beat the Blue Jays, the Tigers beat the Cardinals, the Tigers beat the Orioles, the Blue Jays beat the Cardinals, the Blue Jays beat the Orioles, and the Cardinals beat the Orioles. Model this outcome with a directed graph.

T = Tigers, B = Blue Jays, C = Cardinals, O = Orioles



Winner → loser

GRAPH TERMINOLOGY.

2.1. Two vertices u and v in an undirected graph are called **adjacent** if there is at least one edge connecting u and v.

2.2. The **degree** $\deg(v)$ of a vertex v in an undirected graph is the number of edges with endpoint v

2.3. [Handshake Lemma] Let $G = (V, E)$ be an undirected graph. Then

$$2\,|E| = \sum_{v \in V} \deg(v)\,.$$

2.4. [Corollary] An undirected graph has an even number of vertices of odd degree.

Q3. Can a simple graph exist with 15 vertices each of degree 5?

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No, since each vertice has an odd degree, there must be an even number of these vertices - and there's 15

2.5. When an edge (u, v) is directed, u is said to be adjacent to v and v is said to be adjacent from u. The **initial vertex** of (u, v) is u and the **terminal vertex** of (u, v) is v.

2.6. The in-degree $\deg^-(v)$ of a vertex v is the number of edges with terminal vertex v.

2.7. The out-degree $\deg^+(v)$ of a vertex v is the number of edges with initial vertex v.

2.8. [Directed Handshake Lemma] Let $G = (V, E)$ be a directed graph. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)\,.$$

SPECIAL SIMPLE GRAPHS.

Complete graphs K_n. Cycles C_n. Wheels W_n. n-cubes Q_n.

BIPARTITE GRAPHS.

3.1. A simple graph G is called **bipartite** if its vertex set can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

3.2. [Theorem] A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color

Q4. For which values of n are these graphs bipartite?

K_n: 2 $\quad C_n$: $\quad W_n$: $\quad Q_n$: 1

3.3. A **matching** M in a simple graph $G = (V, E)$ is a subset of E such that no 2 edges are incident with the same vertex

3.4. A matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching from** V_1 to V_2 if $|M| = |V_1|$.

3.5. [Hall's Marriage Theorem] The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1, where $N(A)$ denotes the set of all neighbours of A.

NEW GRAPHS FROM OLD.

Removing or adding edges to a graph. Subgraphs. Induced subgraphs. Edge contractions.

Removing vertices from a graph. Graph unions.