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Math 55, Handout 6.

FUNCTIONS.

Blanket assumption: A, B are nonempty sets.

1.1. A **function** $f : A \to B$ (also called a map or transformation) is a rule which assigns a unique output in B for every input in A.

1.2. The domain of a function $f : A \to B$ is A, the **codomain** is B.

If $f(a) = b$, we say b is an image of a and a is a preimage of b.

The range of $f : A \to B$ is $B = \{x : x = f(a)a \in A\}$, or the set of all possible outputs of the function.

1.3. Let $f : A \to B$, let $C \subseteq A$, and let $D \subseteq B$.

The **(full) image** of C is a subset of B, and is $\{i_c : c \in C \wedge i_c = f(c)\}$.

The **(full) pre-image** of D is a subset of A, and is $\{p_d : f(p_d) \in D\}$.

Q1. Let $f : \mathbb{Q} \to \mathbb{Q}$ be the function $f(x) = x^2$. What is the full pre-image of the set $\mathbb{N}$ and why?

f maps the rationals to the rationals, and since we know that the square-roots of non-perfect squares are irrational, they have no pre-image in this mapping. The set $\mathbb{Z} \setminus \{0\}$ consists of every rational whose square is a natural number, so $f^{-1}(\mathbb{N}) = \mathbb{Z} \setminus \{0\}$, although $\exists x \in \mathbb{N}\, \forall k \in \mathbb{Z} \setminus \{0\} \quad k^2 \neq x$.

ARITHMETIC FOR REAL-VALUED FUNCTIONS.

1.4. Let f_1, $f_2 : A \to \mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are **defined by**$f_1 + f_2(x) = f_1(x) + f_2(x)$ and $f_1 f_2(x) = f_1(x) f_2(x)$ respectively.

Q2. Let f_1, $f_2 : \mathbb{R} \to \mathbb{R}$ be the functions $f_1(x) = x^2 - 1$, $f_2(x) = x + 1$. Write down formulas for the functions $f_1 + f_2$ and $f_1 f_2$.

$$f_1 + f_2(x) = x^2 + x$$
$$f_1 f_2(x) = (x^2 - 1)(x + 1)$$

PROPERTIES OF FUNCTIONS.

1.5. A function $f : A \to B$ is called **one-to-one**, or **injective**, if there is a unique input for a given output in the image of the function.

1.6. A function $f : A \to B$ is called **onto**, or **surjective**, if the image of the function is the co-domain.

1.7. If a function $f : A \to B$ is both one-to-one and onto, it is called a **bijective** function, or a **bijection**.

1.8. A function $f : A \to B$ is **invertible** means it has an onto function $f^{-1} : B \to B$. A function is invertible if and only if it is a **bijection**.

1.9. Let $f : A \to B$, $g : B \to C$. The composition $g \circ f : A \to C$ is defined by: $g \circ f(x) = g(f(x))$.

Q3. What are the functions $f_1 \circ f_2$ and $f_2 \circ f_1$ for the functions f_1 and f_2 from Q2?
$f_1 \circ f_2 = (x+1)^2 - 1$
$f_2 \circ f_1 = x^2 - 1 + 1 = x^2$

CARDINALITY OF SETS.

2.1. Two sets A and B have the same cardinality (written $|A| = |B|$) if there is a bijection that maps $A \to B$.

2.2. A set is called **countable** if it is finite or has the same cardinality as the set of all natural numbers. If a set is not countable, it is called uncountable.

Q4. Underline all of the following sets that are countable:

$$\underline{2\mathbb{N} - 1 (= \{1, 3, 5, \ldots\})}, \qquad \underline{\mathbb{Z}}, \qquad \underline{\mathbb{Q}}, \qquad \mathbb{R}, \qquad \mathbb{C}, \qquad [-1, 1].$$

Q5. Prove that the set of all finite subsets of $\mathbb{N}$ is countable.

An enumeration of the set of all finite subsets of $\mathbb{N}$ suffices to show that it is countable. A possible enumeration is as follows:
Let F be the set of all finite subsets of $\mathbb{N}$. Let $E(n)$ give the n^{th} element of F (or, let $E : \mathbb{N} \to F$, such that $F = \{E(n) : n \in \mathbb{N}\}$. Let $L(X)$ give the largest element in the set X. Let $E(1) = \varnothing$ and $E(2) = \{1\}$. The element $E(n+1)$ is given as the set of all elements with the following properties:

$$\exists m \; E(x) = \{x : x \leq m\} \to E(n+1) = \{m+1\}$$

and

$$E(n) \neq \{x : x \leq n\} \to E(n+1) = \{x : \sum_{\forall x \in E(n+1)} x = (\sum_{\forall x \in E(n)} x) + 1 \wedge L(E(n+1)) = L(E(n))\}.$$

In plain language, if the previous set is a list of numbers from 1 to some number m, e.g. $\{1, 2, 3, \ldots, m\}$, then the set will contain only the smallest number larger than any other number in the previous set. If it is not such a list, then the set is the set whose sum is one larger than the previous set, but whose largest element is the same as that of the previous set, e.g. $E(n) = \{1, 3, 4\} \to E(n+1) = \{2, 3, 4\} \to E(n+2) = \{1, 2, 3, 4\}$. The enumeration is as follows:

$$\{\varnothing, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}, \{1,4\}, \{2,4\}, \{1,2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}, \ldots\}.$$

This enumeration has the property that a function $i_E : F \to \mathbb{N}$ can be defined in such a way:

$$i_E(E(n)) = (\sum_{\forall x \in E(n)} 2^{x-1}) + 1 = n.$$

This demonstrates the invertibility of $E(n)$, thus showing $|F| = |\mathbb{N}|$. ■