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Math 55, Handout 11.

BASICS OF COUNTING.

1.1. **The product rule.** Suppose that a procedure can be broken down into a sequence of K tasks. If, for each $j = 1, \ldots, k$, there are n_j ways to perform task j, then there are $(n_1)(n_2)\cdots(n_K)$ ways to do the procedure.

Q1. How many vegetarian sandwiches can be made using 4 kinds of bread, 5 kinds of cheese, 3 kinds of salad leaves, 6 kinds of chopped veggies, and 7 kinds of dressing? Assume a vegetarian sandwich must be made of all these components, i.e., bread, cheese, etc.

$(4)(5)(3)(6)(7) = (20)(18)(7) = (360)(7) = 2100 + 420 = \boxed{2520 \text{ sandwiches}}$

1.2. **The sum rule.** Suppose that all ways to perform a task can be split into K non-overlapping groups where the jth group has size n_j for each $j = 1, \ldots, k$. Then there are $\sum n_j$ ways to perform a task.

Q2. Now solve a modified Q1, allowing for the possibilities that any of cheese, salad leaves, chopped veggies, and dressing (but not bread) may be omitted when making a sandwich.

4 kinds of bread
5 kinds of cheese or no cheese = 6 cheese options
3 " " salad or no salad = 4 salad options
6 " " veggies or no veggies = 7 veggie options
7 " " dressing or no dressing = 8 dressing options

$4(6)(4)(7)(8) = (24)(28)(8) = \boxed{5376 \text{ sandwiches}}$

1.3. **The subtraction rule (inclusion-exclusion for two sets).** Suppose that all ways to perform a task can be split into 2 overlapping groups where the jth group has size n_j for each $j = 1, 2$. Then there are $|A \cup B| = |A| + |B| - |A \cap B|$ ways to perform a task.

Q3. How many bit strings of length 100 start with 10 or end with 111?

2^{100} bit strings
2^{98} bit strings starting with 10
2^{97} bit strings ~~starting~~ ending with 111
2^{95} strings w/ either 10 or 111

Number of strings starting with 10 or ending with 111 $= 2^{98} + 2^{97} - 2^{95}$
$= 2^{95}(8+4-1)$
$= 11 \cdot 2^{95}$

1.4. **The division rule.** There are $\frac{n}{d}$ ways to do a task using a procedure that can be performed in n ways such that exactly d of the n ways to perform a procedure correspond to each way to do the task.

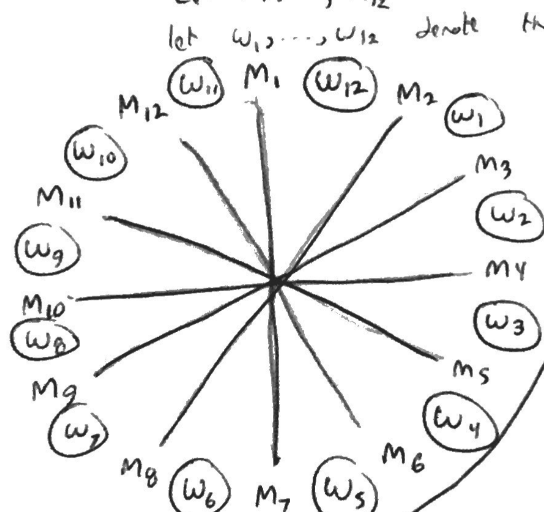
Q4. A butler at Downton Abbey is expecting 11 ladies and 11 gentlemen for dinner with the lord and lady of the house. How many seating plans at a round table are at his displosal if etiquette prescribes that no two gentlemen and no two ladies may sit next to each other? To the butler, two seatings may be considered the same if they differ only by some rotation of the table.

In total, 12 lords and 12 ladies. Lords and ladies occupy every other seat. Pick a first seat. One of the 24 people can sit there. WLOG assume its a lord. In the seat directly clockwise, there are 12 options for which lady. In the next seat, there are 11 options for which lord. There are 11 options for the next lady. In total, the amount of permutations becomes $2(12!)(12!)$. However, we have to divide the sets of 24 permutations that are rotationally symmetric, so it becomes $\frac{2(12!)^2}{24} = \frac{(12!)^2}{12} = \boxed{(11!)(12!)}$

Q5. By Jeeves! Our friend the butler was so baffled by his counting endeavors he almost forgot that the arriving guests are all married couples (of opposite gender, this being early 20th century England). And, needless to say, no married couple may be seated next to each other. How many seating plans does he now have?

[If the butler runs out of time before dinner trying to solve this, he can just exhibit a single arrangement that works and answer the same question for 2 or 3 couples of guests, in addition to the host couple.]

Let $m_1, \dots, m_{12}$ denote the 12 gentlemen (including host) and let $w_1, \dots, w_{12}$ denote the 12 ladies (including host)



This arrangement solves the butler's problem.

In the case with 2 couples and a host, a lady's husband must sit directly opposite of her, so we only need to count the ways of arranging 3 ladies around a table (accounting for rotation). There's only 2 distinct ways to arrange 3 ladies: $f_1 f_2 f_3$ and $f_1 f_3 f_2$; any other way is rotationally equivalent

$\boxed{\text{2 couples + host : 2 ways to arrange}}$

With 3 couples and a host, we have 8 seats to fill. Pick a first seat and we can choose to put a lord or lady (2 options). WLOG assume its a lord. Among the seats lords occupy, there are 4! ways of arranging them. Take the lord in the first seat; there are 2 seats his wife can occupy. Once one of those seats is selected, there's only a single place each of the other ladies can sit. Thus, accounting for rotation, there are

$\frac{2 \cdot 4! \cdot 2}{8} = \boxed{\text{12 arrangements for 3 couples + host}}$