Name:	**GSI:**	**DISC #:**

Solutions to Homework 11.

Prob 1. Find all solutions to the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

(a) $F(n) = (n+1)2^n$

Solution. First, solve the homogeneous recurrence. The characteristic polynomial is

$$r^3 - 6r^2 + 12r - 8 = (r-2)^3,$$

so 2 is its only triple root. Thus the homogeneous recurrence has the solution $a_n^{hom} = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n$.

A particular solution to the inhomogeneous recurrence must therefore have the form $a_n^{part} = n^3(An+B)2^n$ where A and B are coefficients to be determined. Plugging that expression into our inhomogeneous recurrence relation and dividing both sides by 2^n, we obtain

$$n^3(An+B) = 3(n-1)^3[A(n-1)+B] - 3(n-2)^3[A(n-2)+B] + (n-3)^3[A(n-3)+B] + (n+1).$$

If we set $n = 1$, the latter expression simplifies to $6A - 3B = -1$. If we set $n = 2$, we get $4A + 2B = 1$. This linear system has solutions $A = 1/24$, $B = 5/12$. Finally, our general solution is $a_n^{part} + a_n^{hom}$.

Answer: $a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + (n/24 + 5/12)n^3 2^n$ where C_1, C_2, C_3 are arbitrary constants.

(b) $F(n) = n^2(-2)^n$

Solution. Unlike in (a) above, -2 is not a characteristic root, so a_n^{part} has the form $(An^2 + Bn + C)(-2)^n$, with constants A, B, C to be determined. Plugging that in and dividing by $(-2)^n$ gives

$$An^2 + Bn + C = -3[A(n-1)^2 + B(n-1) + C] - 3[A(n-2)^2 + B(n-2) + C] - [A(n-3)^2 + B(n-3) + C] + n^2.$$

Equating coefficients of n^2, n, 1 on both sides, we get the linear system

$$\begin{aligned} A &= -7A + 1 \\ B &= 24A - 7B \\ C &= -24A + 12B - 7C \end{aligned}$$

whose solution is $A = 1/8$, $B = 3/8$, $C = 3/16$.

Answer: $a_n = C_1 2^n + C_2 n 2^n + C_3 n^2 2^n + (\frac{n^2}{8} + \frac{3n}{8} + \frac{3}{16})(-2)^n$ where C_1, C_2, C_3 are arbitrary constants.

Prob 2. Recall that a **partition** of a positive integer is a way to write this integer as the sum of positive integers where repetition is allowed and the order of summands does not matter.

(a) Let $p(n)$ denote the number of partitions of n. Show that the generating function for the sequence $\{p(n)\}$ is the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

Proof. Recall that each geometric series expands as

$$1/(1-x^k) = 1 + x^k + x^{2\cdot k} + x^{3\cdot k} + \cdots. \tag{1}$$

Multiplying all these series for all values of k produces the sum of all possible monomials of the form

$$x^n \quad \text{where} \qquad n = m_1k_1 + \cdots + m_\ell k_\ell = \underbrace{k_1 + \cdots + k_1}_{m_1 \ times} + \cdots + \underbrace{k_\ell + \cdots + k_\ell}_{m_\ell \ times}. \tag{2}$$

Such a sum is a partition of n with k_1 repeated m_1 times, k_2 repeated m_2 times, etc. Hence there will be exactly the same number of monomials x^n as the number of ways $p(n)$ to partition n in that manner. So

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

(b) Find the generating function for $\{p_o(n)\}$ where $p_o(n)$ denotes the number of partitions of n into odd parts (where, as in (a), the order does not matter and repetitions are allowed).

Solution. Now we are looking to partition each n into as a sum of odd integers. In other words, each summand k_1 through k_ℓ in the partition (2) must be odd. Therefore, we should consider only the geometric series (1) for odd numbers k. This yields

$$\sum_{n=0}^{\infty} p_o(n)x^n = \prod_{k\geq 1 \ odd}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}.$$

Prob 3. Suppose X is a random variable on a sample space S such that $X(s)$ is a nonnegative integer for all $s \in S$. The **probability generating function** for X is defined as

$$G_X(x) = \sum_{k=0}^{\infty} p(X(s) = k)x^k.$$

(a) Prove that $E(X) = G'_X(1)$.

Proof.

$$G'_X(x) = \sum_{k=0}^{\infty} p(X(s) = k)kx^{k-1}, \qquad \text{so} \quad G'_X(1) = \sum_{k=0}^{\infty} kp(X(s) = k) = E(X).$$

(b) Let X be the random variable whose value is n if the first success occurs on the nth trial when independent Bernoulli trials are performed, each with probability of success p. Find a closed formula for the probability generating function G_X.

Solution. The probability of the first success occuring on the nth trial is $(1-p)^{n-1}p$ since we need to initially fail $n-1$ times, then succeed on the nth attempt. There is no zeroth trial. Thus

$$G_X(x) = \sum_{k=1}^{\infty}(1-p)^{k-1}px^k = px\sum_{k=0}^{\infty}((1-p)x)^k = \frac{px}{1-(1-p)x}.$$

(c) Using parts (a) and (b), find the expected value of the random variable from (b).

Solution. Differentiating the probability geenrating function from part (b), we get

$$G'_X(x) = \frac{p}{(1-(1-p)x)^2}.$$

So, by part (a),

$$E(X) = G'_X(1) = \frac{1}{p}.$$