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Solutions to Homework 6.

Prob 1. You are given a tower of n disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger one onto a smaller. How many moves are necessary and sufficient to perform this task?

Find a formula for the necessary and sufficient number of moves and prove it by induction.

Solution. We claim that $2^n - 1$ are necessary and sufficient to accomplish this task.

Inductive proof of sufficiency.

Basis case: If $n = 1$, we simply move the only disk to another peg in 1 step.

Inductive step: Assume we can move a tower of $n-1$ disks to another peg in $2^{n-1} - 1$ moves. Consider a tower of n disks. Move the top $n-1$ disks to the second peg without touching the bottom disk on peg 1. By the inductive hypothesis, this can be done in $2^{n-1} - 1$ steps, as the bottom disk on peg 1 is not an obstacle to this process since any tower of smaller disks that may be necessitated by this transfer may sit on top of the largest disk per the rules of the game. Now, move the largest disk to peg 3. Then move all smaller disks from peg 2 to peg 3; as before, this can be done in $2^{n-1} - 1$ steps. All together, this gives

$$2(2^{-n-1} - 1) + 1 = 2^n - 1 \quad \text{steps.}$$

Inductive proof of necessity.

Basis case: if $n = 1$, at least one move is necessary.

Inductive step: Assume the transfer of a tower of $n-1$ disks to another peg requires at least $2^{n-1} - 1$ moves. Consider a tower of n disks. At some point, we must transfer the largest disk, which is originally on the bottom at peg 1, to another peg. At this moment, we must have that second peg unoccupied and a full tower of all smaller $n-1$ disks sitting on the third peg. By the inductive assumption, we need at least $2^{n-1} - 1$ moves to get to this configuration. Once we move the largest disk, we now need to transfer the tower of the remaining $n-1$ disks on top of that largest disk in decreasing size order. By the inductive hypothesis, this requires at least $2^{n-1} - 1$ moves as well. Thus, the minimal number of moves required to transfer n disks is again

$$2(2^{-n-1} - 1) + 1 = 2^n - 1.$$

This concludes the proof.

Prob 2. What is the maximum number L_n of regions obtained by drawing n lines in the plane?

(a) Find a formula for L_n and prove it by induction.

Solution: We can check directly that $L_0 = 1$, $L_1 = 2$, $L_2 = 4$, $L_3 = 7$, $L_4 = 11$, so we seem to observe that

$$L_0 = 1, \qquad L_n = L_{n-1} + n \qquad \text{for all } \; n \in \mathbb{N}. \tag{1}$$

Let us prove this recurrence. We already have the **base case** $L_0 = 1$ (zero lines give one planar region).

To prove the **inductive step**, consider adding an nth line to an existing configuration of $n-1$ lines. To maximize the resulting number of regions, the previous configuration should maximize its own number of planar regions, and when drawing the nth line, we should maximize the number of new regions we create.

To do so, the new line should not be parallel to any existing line or pass though any existing intersection point already present in the earlier configuration. Since there are only finitely many such points and lines, we can always satisfy that requirement. Thus, the new line can be made to intersect each of the previous lines at $n-1$ distinct points that are not among the earlier intersection points.

Now, $n-1$ distinct points divide our new line into n line segments. Each of these line segments corresponds to an earlier region that is now split into two. Thus, we created n new regions. This establishes (1).

Iterating (1) gives us $1 (= L_0)$ plus a sum of the following arithmetic progression:

$$L_n = L_{n-1} + n = L_{n-2} + n + (n-1) = \cdots = n + (n-1) + \cdots + 3 + 2 + 1 + 1 = \frac{n(n+1)}{2} + 1.$$

(b) Some of the regions defined by n lines are unbounded, while others are bounded. What is the maximum possible number of bounded regions created by n lines?

Solution: Denote the maximum number of bounded regions created by n lines by B_n. We check directly that $B_1 = 0$, $B_2 = 0$, $B_3 = 1$, $B_4 = 3$, so we seem to observe that

$$B_1 = 0, \qquad B_n = B_{n-1} + n - 2 \qquad \text{for all } \; n \in \mathbb{N}, \;\; n \geq 2. \tag{2}$$

Let us prove this recurrence. We already have the **base case** $B_0 = 1$ (one line gives zero bounded regions).

To prove the **inductive step**, consider adding an nth line to an existing configuration of $n-1$ lines. To maximize the resulting number of bounded regions, the previous configuration should maximize its own number of bounded regions, and when drawing the nth line, we should maximize the number of new bounded regions we create.

As before, we see that the new line should not be parallel to any existing line or pass though any existing intersection point already present in the earlier configuration. Since there are only finitely many such points and lines, we can always satisfy that requirement. Thus, the new line can be made to intersect each of the previous lines at $n-1$ distinct points that are not among the earlier intersection points.

Now, $n-1$ distinct points divide our new line into n line segments. Each of the interior $n-2$ line segments corresponds to an earlier region that is now split into two. If that was an unbounded region, between two (non-parallel) lines, it will now be split into a newly created bounded region and a new unbounded region. If it was already bounded, it will be split into two bounded regions. In either case, the number of bounded regions goes by 1. So, the total number of bounded regions goes up by $n-2$. This establishes (2).

Iterating (2) gives us a sum of the following arithmetic progression:

$$B_n = B_{n-1} + n - 2 = B_{n-2} + (n-2) + (n-3) = \cdots = (n-2) + (n-3) + \cdots + 2 + 1 = \frac{(n-1)(n-2)}{2}.$$

Prob 3. (a) How many pieces of cheese can you obtain from a single thick piece by making five cuts? (The cheese must stay in its original position while you do the cutting; each cut corresponds to a plane in 3D.)

Solution: For simplicity, we identify the piece of cheese with the entire $3D$ space.

By not cutting, we have 1 piece, by cutting once (say by the plane $x = 0$), we obtain 2 pieces, cutting again by the plane $y = 0$ makes it 4 pieces, the third cut (say, by the plane $z = 0$) produces 8 pieces, the fourth cut by the plane $x+y+z = 1$ produces 15 pieces, and the fifth cut by any plane that is not parallel to the existing ones and does not pass through the existing intersection lines or points will produce 26 pieces.

(b) Find a recurrence relation for P_n, the maximum number of 3D regions defined by n different planes.

Solution: From what we saw in (a), we seem to observe

$$P_0 = 1, \qquad P_n = P_{n-1} + L_{n-1} \qquad \text{for all} \quad n \in \mathbb{N}. \tag{3}$$

with the sequence (L_n) from Problem 2 (a).

Let us prove this recurrence. We already have the **base case** $P_0 = 1$ (zero planes give one $3D$ region).

To prove the **inductive step**, consider adding an nth plane to an existing configuration of $n-1$ planes. To maximize the number of $3D$ regions, the previous configuration should maximize its own number of regions, and when using the nth plane, we should maximize the number of new $3D$ regions we create.

As in Prob.2, we see that the new plane should not be parallel to any existing plane or line or contain any existing intersection point or an intersection line already present in the earlier configuration. Since there are only finitely many such points, lines, and planes, we can always satisfy that requirement. Thus, the new plane can be made to intersect each of the previous planes at $n-1$ brand-new distinct lines.

Now, the maximum number of segments those $n-1$ lines divide our plane into is L_{n-1}. Each of these segments corresponds to an earlier $3D$ region that is now split into two. So, the total number of $3D$ regions goes up by L_{n-1}. This establishes (3).

Prob 4. The well-ordering principle can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let

$$S = \{as + bt \ : \ s, t \in \mathbb{Z}\} \cap \mathbb{N}.$$

(a) Show that S is non-empty.

Proof. $a = a \cdot 1 + b \cdot 0 \in S$ (and $b = a \cdot 1 + b \cdot 1 \in S$), so S is non-empty.

(b) Use the well-ordering property to show that S has a smallest element c.

Proof. By the well-ordering principle and part (a), S has a smallest element c.

(c) Show that if d is a common divisor of a and b, then d is a divisor of c.

Proof. By the definition of S, $c = as^* + bt^*$ for some integers s^* and t^*. Any divisor of a and b also divides as^* and bt^*, hence it divides their sum c.

(d) Show that $c|a$ and $c|b$.

Proof. Suppose $c \not| a$. Then $a = qc + r$ where $q, r \in \mathbb{Z}_+$, $0 < r < c$. The number $r = a - qc$ is in S since

$$0 < r = a - qc = a - q(as^* + bt^*) = a(1 - qs^*) + b(-qt^*).$$

But $r < c$ so c is not the minimum element of S. This contradicts the definition of c. So our assumption $c \not| a$ fails, i.e., $c|a$. Since the roles of a and b can be interchanged, this proof also shows that c must divide b.

(e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing $\gcd(a, b)$ is unique.

Proof. By (d), the number c is a common divisor of a and b. By (c), any other common divisor of a and b must divide c. Therefore c is larger than any other common divisor of a and b.

The greatest common divisor $\gcd(a, b)$ is unique since, given two distinct common divisors of a and b, one is always strictly larger than the other. So both cannot be the $\gcd(a, b)$.

Prob 5. A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or one space vertically (in either direction) and two spaces horizontally (in either direction). Suppose you have an infinite chessboard, made up of all squares (m, n) where m and n are nonnegative integers that denote the row and column of the square, respectively. Use induction on $m+n$ to show that a knight starting at $(0,0)$ can visit every square using a finite sequence of moves.

Proof. We will prove by induction on $m+n$ that any square with coordinates (m, n) where $m, n \in \mathbb{Z}_+$ can be reached by the knight in finitely many moves starting from the origin $(0,0)$.

Inductive bases: If $m+n=0$ for m, $n \in \mathbb{Z}_+$, then $m=n=0$, and it requires zero (so finitely many) steps to remain at the origin, so this basis case is vacuously true.

If $m+n=1$ for m, $n \in \mathbb{Z}_+$, then $(m,n)=(1,0)$ or $(m,n)=(0,1)$. The knight can get to the square $(0,1)$ in 3 steps by moving to $(1,2)$, then $(2,0)$, then $(0,1)$. Likewise, it can get to the square $(1,0)$ in 3 steps by moving to to $(2,1)$, then $(0,2)$, then $(0,1)$.

Inductive step: Let $m+n \geq 2$ and suppose the inductive hypothesis holds for all (k,l) such $k+l=m+n-1$.

Case 1: If $m \geq 2$, consider the square with coordinates $(m-2, n+1)$. As $m+1+n-2=m+n-1$, the inductive hypothesis applies. Hence the knight can reach that square in finitely many steps. Once at that squence, the knight can move to (m,n) by the rules of knight movement, so it can reach our square (m,n) in finitely steps as well.

Case 2: If $n \geq 2$, consider the square with coordinates $(m+1, n-2)$. As $m+1+n-2=m+n-1$, the inductive hypothesis applies. Hence the knight can reach that square in finitely many steps. Once at that squence, the knight can move to (m,n) by the rules of knight movement, so it can reach our square (m,n) in finitely steps as well.

Case 3: If $m<2$ and $n<2$, then $m+n \geq 2$ implies $m=1$, $n=1$, and the knight can get to the square $(1,1)$ in four moves $(0,0) \to (1,2) \to (2,4) \to (0,3) \to (1,1)$.

Cases 1 through 3 exhaust all possibilities we can encounter in the inductive step, so this concludes the proof of the inductive step, and of the whole statement.