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Math 55, Handout 16.

PROBABILITY THEORY.

1.1. Let S be a sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome $s \in S$ so that two conditions are met:

$$0 \leq p(s) \leq 1, \forall s \in S$$
$$\sum_{s \in S} p(s) = 1$$

The function p is then called probability distribution.

1.2. Let $|S| = n$. The **uniform distribution** assigns the probability $1/n$ to each element of S.

1.3. **Selecting** an element of S **at random** means select an element from a sample space with a uniform distribution.

1.4. The **probability of an event** $E \subseteq S$ is then defined as the sum of the probabilities of the outcomes in E.

Q1. Suppose that a die is biased (loaded) so each odd number is twice as likely to appear as each even number. What is the probability that an odd and an even number appear when we roll this die twice?
p(an even and an odd)=p(1 even, 2 odd)+p(1 odd, 2 even)=p(even)p(odd)+p(odd)p(even)=$\frac{4}{9}$

UNIONS AND COMPLEMENTS.

2.1. The following formulæ continue to hold in the non-uniform case:

$$\begin{aligned} p(\overline{E}) &= 1 - p(E) \\ p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \end{aligned}$$

2.2. [Theorem.] Let E_1, …, E_k be pairwise disjoint events in a sample space S. Then

$$p(\cup_{j=1}^{k} E_j) = \sum_i p(E_i)$$

Q2. Prove this Theorem.
proof: By induction.
If E_1 and E_2 disjoint, from

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

we know that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2)$$

If E_1, …, E_{k+1} be pairwise disjoint, then $\cup_{j=1}^{k} E_j$ and E_{k+1} are disjoint, thus

$$p(\cup_{j=1}^{k} E_j \cup E_{k+1}) = p(\cup_{j=1}^{k} E_j) + p(E_{k+1}) = \sum_i^{k+1} p(E_i)$$

2.3. [Boole's Inequality.] Let E_1, ..., E_k be not necessarily disjoint events in a sample space S. Then

$$p(\cup_{j=1}^{k} E_j) \leq \sum_{j=1}^{k} p(E_i)$$

Q3. Prove Boole's Inequality.
proof: For two events A, B, $A \cup B = A \cap (B - A)$ where A and $(B - A)$ are disjoint. Thus $p(A \cup B) = p(A) + p(B - A) \leq p(A) + p(B)$, where $B - A \subset B$ thus $p(B - A) \leq p(B)$.
By induction, $p(E_1 \cup E_2) \leq p(E_1) + p(E_2)$
If $p(\cup_{j=1}^{k} E_j) \leq \sum_{j=1}^{k} p(E_i)$,
then

$$p(\cup_{j=1}^{k+1} E_j) \leq \sum_{j=1}^{k} p(E_i) + p(E_{k+1}) = \sum_{j=1}^{k+1} p(E_i)$$

CONDITIONAL PROBABILITY.

3.1. Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F is defined by

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

3.2. If two events E and F are **independent**, then $p(E|F) = p(E)$

Q3. Are the events E, that a family with three children has children of both sexes, and F, that this family has at most one girl, independent? Assume that the child gender at birth is unambiguous and that the resulting eight ways a family can have three children are equally likely.
No.$p(E|F) = p(1 * girl, 2 * boy) = \frac{3}{8}$, but $p(E) = p(1 * girl, 2 * boy) + p(2 * girl, 1 * boy) = \frac{3}{4}$ thus not equal.
Thus E,F not independent.

3.2. Events E_1, ..., E_k are **pairwise independent** if $p(E_i \cap E_j) = p(E_i)p(E_j)$ for all pairs of integers i and j with $1 \leq i < j \leq n$.

3.3. Events E_1, ..., E_k are **mutually independent** if $p(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \ldots p(E_{i_m})$

BERNOULLI TRIALS AND THE BINOMIAL DISTRIBUTION.

Q4. A coin biased so that the probability of heads is 3/4 is tossed 10 times. What is the probability that exactly five heads come up assuming that the flips are mutually independent?
p(5 heads come up)= $C(10, 5)(\frac{3}{4})^5(\frac{1}{4})^5$

4.1. [Theorem.] The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is equal to $C(n, k)p^k(1 - p)^{n-k}$

RANDOM VARIABLES.

5.1. A **random variable** is a function from the sample space of an experiment to the set of real numbers.

5.2. [NB.] A random variable is neither random nor a variable.

5.3. The **distribution** of a random variable X on a sample space S is the set of pairs (r,p(X=r)) for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r.

Q5. What is the distribution of the total number of heads when a fair coin is flipped four times? $(0, \frac{1}{16}), (1, \frac{1}{4}), (2, \frac{3}{8}), (3, \frac{1}{4}), (4, \frac{1}{16})$