Name:	GSI:	DISC #:

Solutions to Homework 12.

Prob 1. Prove the principle of inclusion-exclusion using mathematical induction.

Proof. We will use strong induction on the number of sets.

Induction base. The includion-inclusion holds trivially for $n = 1$ and, non-trivially, for $n = 2$.

Induction step. Assume the inclusion-exclusion formula holds for any $n \geq 2$ of fewer sets. Take $n + 1$ sets

$$A_1, \ \ldots, \ A_n, \ A_{n+1}.$$

Denote $A = A_1 \cup \ldots \cup A_n$. By the inductive hypothesis applied to two sets A and A_{n+1}, we get

$$|A \cup A_1| = |A| + |A_{n+1}| - |A \cap A_{n+1}|. \tag{1}$$

By the inductive hypothesis applied to n sets $A_1, \ldots, A_n$, we get

$$|A| = \sum_{j=1}^{n} |A_j| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap \ldots \cap A_n|. \tag{2}$$

Now let us denote $A_j \cap A_{n+1}$ by B_j for all $j = 1, \ldots, n$. We see that

$$A \cap A_{n+1} = B_1 \cup \ldots \cup B_n.$$

By the inductive hypothesis applied to n sets $B_1, \ldots, B_n$, we get

$$|B_1 \cup \ldots \cup B_n| = \sum_{j=1}^{n} |B_j| - \sum_{i<j} |B_i \cap B_j| + \sum_{i<j<k} |B_i \cap B_j \cap B_k| + \cdots + (-1)^n |B_1 \cap \ldots \cap B_n|.$$

But each intersection $B_i \cap B_j \cap \ldots \cap B_m$ is actually the intersection $A_i \cap A_j \cap \ldots \cap A_m \cap A_{n+1}$. So

$$|A \cap A_{n+1}| = \sum_{j=1}^{n} |A_j \cap A_{n+1}| - \sum_{i<j\leq n} |A_i \cap A_j \cap A_{n+1}| + \cdots + (-1)^n |A_1 \cap \ldots \cap A_n \cap A_{n+1}|. \tag{3}$$

Now, plugging (2) and (3) into (1) yields the inclusion-exclusion formula for the sets $A_1, A_2, \ldots, A_{n+1}$.

Prob 2. How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish, rat* or *bird*?

Solution. Within the universe U of all permutations, we must find the cardinality of the $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$, where A denotes all permutations that contain the string *fish*; A denotes all permutations that contain the string *rat*; C denotes all permutations that contain the string *bird*.

We already know that $|U| = 26!$ To count $|A|$, B, and $|C|$, note that we can treat the string that defines the corresponding set as one symbol, along with the other letters unused in that string. In case of the set A, this leaves 22 unused letters in addition to the one 'letter' that stands for the entire word 'fish'. Therefore, $|A| = 23!$ By the same token, $|B| = 24!$ and $|C| = 23!$

Note that $B \cap C = \emptyset$ because the letter *'r'* occurs in both strings *rat* and *bird,* hence there is no permutation of the 26 letters that contains both of these strings. By the same reasoning applied to the letter *'i'*, we see that $A \cap C = \emptyset$. The set $A \cap B$ is nonempty, and to count its cardinality we adopt the earlier method of treating strings *fish* and *rat* as one new symbol each. That way we need to permute a total of 19 letters plus the two new symbols, giving us the count $|A \cap B| = 21!$ Finally, the triple intersection $A \cap B \cap C$ is empty, since at least one of the pairwise intersections is already empty.

Now applying the alternative form of Inclusion-Exclusion, we obtain

$$|\overline{A \cup B \cup C}| = 26! - 23! - 24! - 23! + 21! = 26! - 24! - 2 \cdot 23! + 21!$$

Answer: $26! - 24! - 2 \cdot 23! + 21!$

Prob 3. Use a combinatorial argument to show that the sequence $\{D_n\}$, where D_n denotes the number of derangements of n objects, satisfies the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2}). \tag{4}$$

Proof. Consider a derangement of n numbers 1, 2, …, n. Then n is in position j for some j, $1 \leq j < n$.

Case 1. Suppose j is in position n, i.e., j and n are swapped. Removing both n and j leaves us with a derangement of $n-2$ objects. There are $n-1$ choices for such a j and then there are D_{n-2} derangements of the remaining $n-2$ numbers/objects.

Case 2. Otherwise j and n are not swapped, i.e., some other number $k \neq j$ is in position n. Swap n and k, which will put n in position n. Then remove n. Since $k \neq j$ and since we have not moved any other numbers, we have a derangement of the remaining $n-1$ numbers.

There are $n-1$ choices for j, then D_{n-1} derangements for each j. Note that k ends up in position j in the derangement of $n-1$ objects, so we do not need to separately account for k.

Conclusion. Cases 1 and 2 are mutually exclusive and exhaustive. They give us $(n-1)D_{n-2}$ and $(n-1)D_{n-1}$ choices, respectively. Hence (4) holds.

Prob 4. Euler's totient function $\phi(n)$ counts the positive integers up to n that are relatively prime to n. Use the principle of inclusion-exclusion to derive a formula for $\phi(n)$ when the prime factorization of n is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}.$$

Solution. By the complementary form of inclusion-exclusion principle, we can get $\phi(n)$ by first including all n numbers not exceeding n, then removing all numbers divisible by p_j for $j = 1, \ldots, m$, then adding back all numbers divisible by both p_i and p_j for $i \neq j$, etc.

Note also that, for any factor ℓ of n, there are n/ℓ positive integers divisible by ℓ and not exceeding n.

Combining these two facts, we obtain

$$\begin{aligned}
\phi(n) &= n - \sum_{j=1}^{m} \frac{n}{p_j} + \sum_{i<j} \frac{n}{p_i p_j} + \sum_{i<j<k} \frac{n}{p_i p_j p_k} + \cdots + (-1)^m \frac{n}{p_1 p_2 \cdots p_m} \\
&= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right) \\
&= n \prod_{j=1}^{m} \left(1 - \frac{1}{p_j}\right).
\end{aligned}$$

Prob 5. (a) Show that every connected graph with n vertices has at least $n-1$ edges.

Proof. First consider the graph consisting of n vertices and no edges. Each of its vertices is a separate connected component, so this graph has n connected components.

Now begin adding edges to this graph. At any stage of this process, adding an edge can reduce the number of connected components by at most 1 if that edges connects two previously disconneced components. So the number of connected components to go down from n to 1, at least $n-1$ edges need to be added.

So, any connected graph with n vertices must have at least $n-1$ edges.

(b) If a connected graph with n vertices has exactly $n-1$ edges, what kind of graph is it?

Proof. By the reasoning in part (a), removing any edge from such a graph would disconnect it. Removing any edge from a circuit would still keep the graph connected. Therefore, such a graph cannot contain a curcuit. A connected graph without circuits is a tree.

Answer: it is a tree.