Name:	**GSI:**	**DISC #:**

Solutions to Homework 7.

Prob 1. (a) How many bit strings of length 10 contain five consecutive 0s or five consecutive 1s?

Solution. We first count the number of bit strings of length 10 with five consecutive zeros. Depending on when the earliest substring of five consecutive zeros begins within the whole bit string, our string will have exactly one of the following forms:

$00000d_6d_7d_8d_9d_{10},\ 100000d_7d_8d_9d_{10},\ d_1100000d_8d_9d_{10},\ d_1d_2100000d_9d_{10},\ d_1d_2d_3100000d_{10},\ d_1d_2d_3d_4100000.$

There are $2^5 + 5 \cdot 2^4 = 112$ such strings.

Since the roles of 0s and 1s are symmetric in our setting, there are also 112 strings containing five consecutive 1s. Now, the set of all bit strings with five consecutive 0 and the set of all bit strings with five consecutive 1s overlap at the strings

$$0000011111, \quad 1111100000$$

Hence the total number of bit strings containing five consecutive 0s or five consecutive 1s is equal to

$$112 + 112 - 2 = 222.$$

(b) How many bits strings of length 10 contain at least three 1s and at least four 0s?

Solution. Consider the set A of bit strings of length 1 with at most two 1s: there are $\binom{10}{0} + \binom{10}{1} + \binom{10}{2}$ of those. Similarly, the set B of bit strings with at most three 0s has $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}$ elements. Note that $A \cap B = \emptyset$, so that $|A \cup B| = |A| + |B|$ and that the problem is to determine the size of the set $\overline{A \cup B}$.

Our underlying set of all bit strings of length 10 has 2^{10} elements, so

$$|\overline{A \cup B}| = 2^{10} - |A| - |B| = 2^{10} - 2\left(\binom{10}{0} + \binom{10}{1} + \binom{10}{2}\right) - \binom{10}{3} = 792.$$

Prob 2. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Proof. Consider a set S of size $2n$ that consists of two disjoint subsets A and B, each with n elements. We count the number of subsets of S of size n that contain exactly one starred element from A, in two ways.

Th right-hand side count corresponds to first choosing the starred element from A, which can be done in n ways, then choosing the other $n-1$ elements from the remaining $2n-1$ elements of S, in $\binom{2n-1}{n-1}$ ways.

Each summand of the left-hand side corresponds to fixing the number k of all elements of A that are included into our subset. It can be anywhere between 1 and n. For any k, there are $\binom{n}{k}$ choices of elements in A to be included, and $\binom{n}{k}$ choices of elements of B to be *excluded* from our subset. And finally, there are k ways to give a star to one of the selected elements from A.

So, the two sides of our formula count the same objects in two different ways, and are therefore equal.

Prob 3. Let n, $k \in \mathbb{N}$ be such that $k \leq n$. Prove (in any way you like) that

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}.$$

Proof. Let us simplify the right-hand side first. We get

$$\begin{aligned} \frac{(2n+2)!}{2(n+1)!^2} - \frac{(2n)!}{n!^2} &= \frac{(2n+2)! - (2n)!2(n+1)^2}{2(n+1)!^2} = \frac{(2n)!2((n+1)(2n+1) - (n+1)^2)}{2(n+1)!^2} \\ &= \frac{(2n)!(n+1)n}{(n+1)!^2} = \frac{(2n)!}{(n-1)!(n+1)!} = \binom{2n}{n+1}. \end{aligned}$$

By the binomial theorem, $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum \binom{n}{j} z^{n-j}$. So

$$(1+z)^{2n} = \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{n}{j} z^{k+n-j}.$$

By the binomial theorem again, the coefficient of z^{n+1} on the left is equal to $\binom{2n}{n+1}$. The same coefficient on the right is given as a sum over all possible choices of j and k such that $k+n-j = n+1$, i.e., $k = j+1$. As both k and j are restricted to the range from 0 to n, this creates a further restriction $k \geq 1$, and we obtain

$$\binom{2n}{n+1} = \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1}.$$

But that is exactly what we needed to prove.

Prob 4. Suppose that 21 girls and 21 boys enter a mathematical competition, that each entrant solves at most six questions, and that for every boy-girl pair, there is at least one question that they both solved. Prove that there is a question that was solved by at least three girls and three boys.

Solution. Number the girls 1 through 21 and the boys 1 through 21, and consider a 21×21 matrix (table). Fill its (i, j) entry with the number of a question that girl i and boy j both solved (if both solved more than one, just choose one). This gives a 21×21 matrix filled with positive integers.

If we establish that there is an entry that appears in three columns and three rows, we will be done. Assume that an arrangement is possible with no integer in at least 3 rows and at least 3 columns.

Color a cell/square white if its entry appears in 3 or more rows and black if its integer appears in only 1 or 2 rows. We shall count the white and black squares.

Each row contains at most 6 different integers. As $6 \times 2 < 21$, each row includes an integer which appears 3 or more times and hence is in at most 2 rows per our assumption. So at most 5 different integers in the row appear in 3 or more rows. Each such integer can appear at most 2 times in the row, so there are at most $5 \times 2 = 10$ white cells in the row. This is true for every row, so there are at most 210 white cells in total.

Similarly, any given column has at most 6 different integers and hence at least one appears 3 or more times. So at most 5 different integers appear in 2 rows or less. Each such integer can occupy at most 2 cells in that column, so there are at most $5 \times 2 = 10$ black cells in the column. This is true for every column, so there are at most 210 black cells in total.

This gives a contradiction since any cell is either black or white but $210 + 210 < 441$.

Prob 5. (a) Let $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that the Ramsey numbers $R(m, n)$ and $R(n, m)$ are equal.

Proof. Let $R(m, n) = k$. Take an arbitrary party of k people. Consider its "dual" (or "Upside Down") party where all friendships are replaced by animosity and all animosity is replaced by friendship. That dual party still has k people. Since $R(m, n) = k$, there are at least m mutual "friends" or at least n mutual "enemies" at the dual party. But friendship at the dual party is actually animosity in the real world, and animosity is friendship in the real world. Hence the original party has m mutual enemies or n mutual friends. That original party was arbitrary, so we have proved that $R(n, m) \leq (k =) R(m, n)$.

But there is nothing special we used about n and m, so we can swap them too, to obtain $R(m, n) \leq R(n, m)$. Now we have inequalities both ways, hence $R(m, n) = R(n, m)$ for any m, $n \geq 2$.

(b) Prove that at a party with $m \geq 2$ people there are two who know the same number of other people.

Proof. For each person $j = 1, \ldots, m$ at the party, let K_j be the number of people at that party he or she knows, not counting oneself. That integer is between 0 and $m - 1$. Since there are m people and m possible values for the numbers F_j, $j = 1, \ldots, m$, the only way these numbers can be all different is when all possibilities, from 0 to $m - 1$, occur.

But that means there is a person who knows nobody and another person who knows everybody (including that first person). Since we assume that "knowing" is a symmetric relation, this cannot happen. This contradiction shows that there are at least two people at the party who know the same number of people.