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Math 55, Handout 13.

THE PIGEONHOLE PRINCIPLE.

1.1. **The pigeonhole principle.** Let $k \in \mathbb{N}$. Suppose k+1 objects are placed into k boxes. Then atleast 1 box will have 2 objects

1.2. **Equivalent formulation.** A function from a set of k+1 elements to a set of k elements cannot be injective

Q1. Show that for every $n \in \mathbb{N}$ there is a multiple of n that has only 0s and 7s in its decimal expansion.

Consider the n+1 numbers: 7 , 77 , ... , 77...77 (n+1 7s)

There are n+1 numbers and only n remainders mod n, thus 2 distinct numbers have the same remainder mod n, let them be 'a' and 'b'.

WLOG suppose b¿a, then b-a is 0 (mod n) and is of the form 7...70...0

Hence proved

THE GENERALIZED PIGEONHOLE PRINCIPLE.

2.1. **The generalized pigeonhole principle.** Let $n, k \in \mathbb{N}$. Suppose n objects are placed into k boxes. Then atleast one box has $\lceil \frac{n}{k} \rceil$ objects

Q2. How many cards must be selected from a standard deck of 52 cards to guarantee that at least four cards of the same suit are chosen?

picking 13 cards guarantees this since (by the generalized pigeonhole principle): $\lceil \frac{13}{4} \rceil = 4$

We notice 12 is not enough since we can have 3 of each suit

Q3. Suppose that a CS lab has 16 workstations and 12 servers. A cable can directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 12 or fewer workstations can simultaneously access different servers via direct connections. What is the minimum number of direct connections needed?

First we note that if any server is connected to less 5 workstations, then any 12 workstation subset of the remaining workstation cannot connect simultaneously to different servers since none of them connect to the above server.

Therefore each server has atleast 4 workstations connected to it, and the minimum number of connections is 5*12 = 60

This is achievable by connecting workstations 1-12 to one server each and the remaining stations to all servers giving a total of 60.

We see that this is achievable

Hence Proved

APPLICATIONS OF THE PIGEONHOLE PRINCIPLE.

3.1. **[Theorem].** Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Q4. Prove or disprove that every sequence of 9 distinct real numbers contains a subsequence of length 4 that is either strictly increasing or strictly decreasing.

Consider the sequence 3,2,1,6,5,4,9,8,7

Hence the claim is false

RAMSEY THEORY.

4.1. **The party problem.** At any party with at least six people, there are three people who either all know each other or all do not know each other.

4.2. **The Ramsey number** $R(m, n)$, where $m, n \in \mathbb{N}$, denotes the minimum number of guests needed such that for any relations between the guests there are atleast m people who know each other or atleast n people who don't know each other

Q5. Let $n \in \mathbb{N}$, $n \geq 2$. Show that the Ramsey number $R(2, n)$ equals n.

We show that for any given party of n people, the desired property is present:

Case 1: No-one knows anyone:

Thus we have n people that don't know each other and the property holds.

Case 2: There is atleast one pair of people that know each other

Thus we have a group of 2 people that know each other.

We now show that the property cannot hold any number less than n by construction:

Consider m¡n and call a party of m people who don't know each other.

Thus there is neither a group of 2 people that know each other nor a group of n that don't