Name:	**GSI:**	**DISC #:**

Solutions to Homework 9.

Prob 1. If X_1, …, X_{2n} are mutually independent random variables with the same distribution and if α is any real number whatsoever, prove that

$$p\left(\left|\frac{X_1+\cdots+X_{2n}}{2n}-\alpha\right| \leq \left|\frac{X_1+\cdots+X_n}{n}-\alpha\right|\right) \geq \frac{1}{2}. \tag{1}$$

Proof. Consider two random variables X and Y, independent and identically distributed. By the tirangle inequality, $|X+Y| \leq |X|+|Y|$ holds always, hence

$$p\left(\frac{1}{2}|X+Y| \leq |X|\right) \geq p\left(\frac{1}{2}(|X|+|Y|) \leq |Y|\right) = p\left(|X| \leq |Y|\right), \tag{2}$$

since replacing $\frac{1}{2}|X+Y|$ by a larger quantity on the left-hand side of an inequality $\leq$ makes that inequality less likely to hold.

Now, X and Y being identically distributed, we have $p(|X|<|Y|) = p(|X|>|Y|)$, so

$$1 = p(|X|<|Y|)+p(|X|=|Y|)+p(|X|>|Y|) = 2p(|X|<|Y|)+p(|X|=|Y|),$$

which implies

$$p(|X| \leq |Y|) = p(|X|<|Y|)+p(|X|+|Y|) = \frac{1}{2}+\frac{1}{2}p\left(|X|=|Y|\right) \geq \frac{1}{2}.$$

(The last inequality is actually strict but never mind.) Now (2) implies

$$p\left(\frac{1}{2}|X+Y| \leq |X|\right) \geq 1/2. \tag{3}$$

Now set

$$X := \frac{X_1+\cdots X_n}{n}-\alpha, \qquad Y := \frac{X_{n+1}+\cdots X_{2n}}{n}-\alpha$$

and plug them into (3) to obtain (1).

Prob 2. A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmission it sends a 1 one-third of the time and 0 two-thirds of the time. When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received as a 1 is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8 and the probability that it is received as a 0 is 0.2.

(a) Find the probability that a 0 is received.

Solution.

$$\begin{aligned}
P(0 \text{ received}) &= P(0 \text{ received} \cap 0 \text{ sent}) + \mathrm{P}(0 \text{ received} \cap 1 \text{ sent}) \\
&= P(0 \text{ received} \mid 0 \text{ sent})\mathrm{p}(0 \text{ sent}) + \mathrm{P}(0 \text{ received} \mid 1 \text{ sent})\mathrm{p}(1 \text{ sent}) \\
&= \frac{9}{10} \cdot \frac{2}{3} + \frac{2}{10} \cdot \frac{1}{3} \\
&= \frac{2}{3}.
\end{aligned}$$

(b) Find the probability that a 0 was transmitted, given that a 0 was received.

Solution.

$$\begin{aligned}
p(0 \text{ sent} \mid 0 \text{ received}) &= \frac{p(0 \text{ sent} \cap 0 \text{ received})}{\mathrm{p}(0 \text{ received})} \\
&= \frac{p(0 \text{ received} \mid 0 \text{ sent})\mathrm{p}(0 \text{ sent})}{\mathrm{p}(0 \text{ received})} \\
&= \frac{\frac{9}{10} \cdot \frac{2}{3}}{\frac{2}{3}} \\
&= \frac{9}{10}.
\end{aligned}$$

Prob 3. In a round-robin tournament with m players, every two players play one game in which one player wins and one player loses (i.e., there are no ties). Assume that when two players compete it is equally likely that either player wins that game, and that the oucomes of different games are independent. Let E be the event that for every set S of k players, where $k < m$, there is a player who has beaten all k players in S.

(a) Show that $p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j)$ where F_j is the event that there is no player who beats all players from the jth subset of k players.

Proof. By our choice of the events F_j, we have $\overline{E} = \cup_j F_j$. There are $\binom{m}{k}$ events F_j in total because there are so many subsets of players of size k. By Boole's inequality, we therefore get $p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j)$

(b) Show that the probability of F_j is $(1-2^{-k})^{m-k}$.

Proof. Consider the jth subset of players. For any player outside that subset, there are 2^k equally likely outcomes in this player's games with those chosen players. In only one case does the outside player beat everyone from the jth subset, so there is a $1-2^{-k}$ chance of the outside player failing to beat all k players in the jth subset. Now, there are $m-k$ players outside the jth subset, so the probability $p(F_j)$ that all of them fail to beat the entire jth subset of players is $(1-2^{-k})^{m-k}$.

(c) Conclude from parts (a) and (b) that $p(\overline{E}) \leq \binom{m}{k}(1-2^{-k})^{m-k}$.

Proof.

$$p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j) = \binom{m}{k}(1-2^{-k})^{m-k}.$$

(d) Use part (c) to find values of m such that there is a tournament with m players such that for every set S of two players, there is a player who has beaten both players in S. Repeat for sets of three players.

Solution. Given $k=2$ or $k=3$, we need to find a suitable $m = m(k)$ such that the inequality $p(\overline{E}) < 1$ would follow from the bound in (c).

If $k=2$, we need to ascertain that $\binom{m}{2}(3/4)^{m-2} < 1$. It can be calculated directly that this inequality holds for $m = 21$. Next, $\binom{m+1}{2}(3/4)^{m-1} = \binom{m}{2}(3/4)^{m-2} \cdot \frac{m+1}{m-1} \cdot \frac{3}{4}$ and $\frac{m+1}{m-1} \cdot \frac{3}{4} < 1$ whenever $m > 7$. So, the inequality $\binom{m}{2}(3/4)^{m-2} < 1$ holds for every $m \geq 21$ by induction.

If $k=3$, we need to ascertain that $\binom{m}{3}(7/8)^{m-3} < 1$. It can be shown by induction, similarly to the case $k=2$, that this inequality holds for all $m \geq 90$.

Prob 4. This problem has been posted on Midterm 2 in 2017; see its solution there.