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# Math 55, Handout 10.

## RECURSIVE DEFINITIONS.

1.1. **Recursively defined functions.** A function $f$ on $\mathbb{Z}_+$ can be **recursively defined** using the following two steps:

**Basis step:** specify the value of the function at zero

**Recursive step:** Give a rule for finding its value at an integer from its values at smaller integers

Q1. Is this a valid recursive definition of a function $f : \mathbb{Z}_+ \to \mathbb{Z}$?

$$f(0) = 2, \quad f(n) = \begin{cases} f(n-1) & \text{if } n \text{ is odd and } n \geq 1 \\ 2f(n-2) & \text{if } n \geq 2. \end{cases}$$

$n=0$ $f(n)=2$ closed form solution:
$n=1$ $f(n)=2$
$n=2$ $f(n)=4$ $\lfloor 2n \rfloor$? NO
$n=3$ $f(n)=4$ $2^n$? NO
$2^{(n/2)+1}$? yes

this is a valid recursive defintion as the function is well-defined

1.2. Properties of recursively defined functions can be typically proved by

Q2. Prove that the Fibonacci sequence $\{f_n\}$ satisfies $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$.

Basis: $n=1$, $\sum_{j=1}^{n} f_j^2 = f_1^2 = 1^2 = 1$ and $f_1 f_2 = 1 \cdot 1 = 1$ ✓

inductive step $\sum_{j=1}^{n+1} f_j^2 = \underbrace{\sum_{j=1}^{n} f_j^2}_{} + f_{n+1}^2$

$= f_n f_{n+1} + f^2 \rightarrow$ by inductive hypothesis

$= f_{n+1}(f_n + f_n + 1) \rightarrow f_{n+2}$ by the r.r for fibonacci

$= f_{n+1} + {}_{n+2}$ V I

this shows the equivalence for n+1 whenever it holds for n. this proves the inductive step, hence proving the equivalence for any any $n \in \mathbb{N}$

1.3. **Recursively defined sets.** A set $S$ can be **recursively defined** using the following two steps:

**Basis step:** $\lambda \in \Sigma^*$ (where $\lambda$ is the empty string containing no symbols)

**Recursive step:** If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in^*$

Q3. Show that the set $S$ defined by $1 \in S$ and $s + t \in S$ whenever $s, t \in S$ is the set $\mathbb{N}$.

$1 \in S$ $s+t \in S$ whenever $s \in S$ and $t \in S$

proof: $n \in S$ for every positive integer

Basis step $n=1$

$1 \in S$ thus $P(1)$ is true

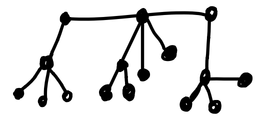
Inductive step $P(k)$ is true $k \in S$. prove that $P(k+1)$ is also true since $1 \in S$ and $k \in S$ by the def of $S$: $k+1 \in S$

thus $P(k+1)$ is true. $P(n)$ is true when $n$ is a positive integer □

Q4. Give a recursive definition of the set of positive integer powers of 5.

basis $5 \in S$ and inductive step $s \in S \longrightarrow 5s \in S$

Rooted trees 

Recursively defined graphs 

1.4. Besides functions and sets, other structures can be recursively defined, such as strings & rooted trees

**STRUCTURAL AND GENERALIZED INDUCTION.**

3.1. **Structural induction principle.** Results about recursively defined sets (and other structures) can be proved using the following two steps:

**Basis step:** show that the result holds for all elements specified in the basis step of the recursive definition to be in the set

**Recursive step:** show that if the statement is true for each of the element used to construct new elements in the recursive step of the definition, the result holds for these new elements.

3.2. **Generalized induction** is using (weak or strong) induction to prove results about sets that have the well-ordering property. The lexicographic **ordering** is often used for that purpose.

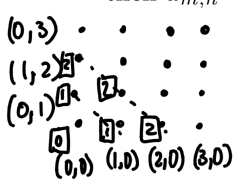
3.3. One can extend generalized induction even further to **partially ordered** sets.

3.4. **Transfinite induction** can be used for ordinals larger than $\mathbb{N}$. This is beyond the scope of Math 55.

Q5. Use generalized induction to show that if $a_{m,n}$ is defined recursively by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + 1 & \text{if } n > 0, \end{cases}$$

then $a_{m,n} = m + n$ for all $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$.




conjecture: $a_{m,n} = m+n$

Induction on $m+n$

Basis: $m+n=0 \longrightarrow m=n=0$ and $a_{m,n} = m+n = 0+0 = 0$ ✓

Inductive step: Assume $a_{m,n} = m+n$ for $m+n=k$

show that $m+n=k+1$

case 1: $n=0$, $m=k+1$. then $a_{m,n} = a_{m-1,n}+1$ $(m-1)+n = k+1-1 = k$. the inductive hypothesis applies to $a_{m-1,n}$ and gives $a_{m-1,n} = m-1+n = k$. $a_{m,n} = k+1$ ✓

case 2: $n>0$, $m+n=k+1$. then $a_{m,n} = a_{m,n-1}-1$. And $m+(n+1) = k+1-k = k$. so the inductive hypothesis applies to $a_{m,n-1}$ and gives $a_{m,n-1} = m+(n-1) = k$. so $a_{m,n} = a_{m,n-1}+1 = k+1$ ✓

Both cases the inductive step is proven. □