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Math 55, Homework 12.

Prob 1. Prove the principle of inclusion-exclusion using mathematical induction.

formula is true for $n=2$

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

true for $(n-1)$

$$\left|\bigcup_{i=1}^{n-1} A_i\right| = \sum_{k=1}^{n-1} \sum_{1<i_1 \leq i_2 < \cdots i_k \leq n-1} (-1)^{k+1} \left|\bigcap_{j=1}^{k} A_{ij}\right|$$

let's test: n sets

$B_n = \bigcup_{i=1}^{n-1} A_i$

$$\left|\bigcup_{i=1}^{n} A_i\right| = |B_n \cup A_n| = |B_n| + |A_n| - |B_n \cap A_n|$$

whereas $|B_n \cap A_n| = |(A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cap A_n| = \left|\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right|$

$$|B_n \cap A_n| = \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n-1} (-1)^{k+1} \left|\bigcap_{j=1}^{k} (A_{ij} \cap A_n)\right|$$

then substitute

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n-1} (-1)^{k+1} \left|\bigcap_{j=1}^{k} A_{ij}\right| + |A_n| - \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n-1} (-1)^{k+1} \left|\bigcap_{j=1}^{k} (A_{ij} \cap A_n)\right|$$

Final result

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} \sum_{1<i_1<i_2<\cdots<i_k \leq n} (-1)^{k+1} \left|\bigcap_{j=1}^{k} A_{ij}\right|$$

Prob 2. How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat* or *bird*?

Formulas:

no rep. : $P(n,r) = \frac{n!}{(n-r)!}$, rep: n^r } permutation

No rep: $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$, rep. : $C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$ } combinations

Principle of Inclusion-exclusion

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Legend:

U: perm. of alphabet (26 letters) = 26!

A_1: strings with "fish"

A_2: strings with "rat"

A_3: strings with "bird"

"fish"

$$|A_1| = P(23,23) = \frac{23!}{(23-23)!} = 23!$$

"rat"

$$|A_2| = P(24,24) = \frac{24!}{(24-24)!} = 24!$$

"bird"

$$|A_3| = P(23,23) = \frac{23!}{(23-23)!} = 23!$$

Both "fish" and "bird"

$$|A_1 \cap A_3| = 0$$

Both "rat" and "bird"

$$|A_2 \cap A_3| = 0$$

All can't happen at the same time

$$|A_1 \cap A_2 \cap A_3| = 0$$

Principle of inclusion-exclusion

$$|A_1 \cup A_2 \cup A_3| = 23! + 24! + 23! - 21! - 0 - 0 + 0$$
$$= 23! + 24! + 23 - 21!$$

$$|(A_1 \cup A_2 \cup A_3)^c| = 26! - 23! - 24! - 23! + 21!$$

There are $(26! - 23! - 24! - 23! + 21!)$ permutations not containing the strings "fish", "rat", "bird"

Prob 3. Use a combinatorial argument to show that the sequence $\{D_n\}$, where D_n denotes the number of derangements of n objects, satisfies the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

suppose $S = \{a_1, a_2, \ldots, a_{n-1}\}$ is an $(n-1)$ element set. If we include a new element a_n to the set then we see the cases

case 1: when a_n and $a_i \in S$ for some $1 \le i \le n-1$. swap their places. can be done for all $i = 1, 2, \cdots, n-1$

Therefore the total number of derangements obtained in this way is $(n-1)D_{n-2}$

case 2: when a_n does not swap with any a_i's & a_j $(1 \le j \le n-1)$ goes to the nth place after deangement while a_n might go to the first $(n-1)$ place but not the jth place.
The total number of derangements is $(n-1)D_{n-1}$

If you combine both cases you get $D_n = (n-1)(D_{n-1} + D_{n-2})$

Prob 4. Euler's totient function $\phi(n)$ counts the positive integers up to n that are relatively prime to n. Use the principle of inclusion-exclusion to derive a formula for $\phi(n)$ when the prime factorization of n is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}.$$

let $A_i \{ x \in [n] : p_i \text{ divides } x \}$

If $S < [n]$, let $A_S = \{ x \in [n] : p_i, \text{ for } i \in S, \text{ divides } x \}$

$$\phi(n) = \left| \bigcap_{i=1}^{n} \overline{A_i} \right| = \left| \overline{\bigcup_{i=1}^{n} A_i} \right| = n - \left| \bigcup_{i=1}^{n} A_i \right|$$

$$\phi(n) = n - \left| \bigcup_{i=1}^{n} A_i \right| = n - \sum_{\emptyset \leq S \subset [M]} \left[(-1)^{|S|+1} \right] |A_S|$$

$$n - \sum_{\emptyset \leq S \subset [M]} (-1)^{|S|+1} \frac{n}{\prod_{i \in S} p_i} = n\left(1 - \sum_{\emptyset \leq S \subset [M]} (-1)^{|S|+1} \frac{1}{\prod_{j \in S} p_j} \right)$$

$$\longrightarrow n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right) = \phi(n)$$

a path exists between every pair of vertices

Prob 5. (a) Show that every connected graph with n vertices has at least $n-1$ edges

Proof by Induction:

let P(n) be "Every connected graph with n vertices has at least n-1 edges

Basis step: $n=1$

A connected graph cannot have any edges

so $P(1)$ is true $\Longrightarrow n-1=1-1=0$

Inductive step: $k-1$ edges

since $P(k)$ is true then the connected graph has at least $k-1$ edges. Therefore, V_{k+1} needs to be connected to at least 1 vertices. which means that the connected graphs,(G and GR) G contains 1 more edge than GR. $\Longrightarrow k-1+1=k$ edges

so $P(k+1)$ = True

(b) If a connected graph with n vertices has exactly $n-1$ edges, what kind of graph is it?

This graph is a digraph

Base step: $n=2$

$|V|=2$ $|E|=1$. this shows a diagraph because e_1 is associated with (V_1, V_2)

Inductive step: $G(n+1, n)$

for every new V_i there is a new e_k connecting to $V \in V$. so if the graphs are connected, there is a new e_k at V_i and ends at $V \in V$. from the $G(n+1, n)$