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Math 55, Handout 18.

THE EXPECTED VALUE.

1.1. The **expected value,** also called expectation or mean, of the random variable X on the sample space S is defined by

$$E(X) = \sum_{s \in S} p(s)X(s)$$

Q1. A coin biased so that heads are three times more likely than tails is flipped five times. What is the expected total number of heads?

$p(H) = \frac{3}{4}, p(T) = \frac{1}{4}$
$E(X) = 0 \cdot (\frac{1}{4})^5 + 1 \cdot \binom{5}{1} \cdot (\frac{1}{4})^4 \cdot \frac{3}{4} + 2 \cdot \binom{5}{2} \cdot (\frac{1}{4})^3 (\frac{3}{4})^2 + 3 \cdot \binom{5}{3} \cdot (\frac{1}{4})^2 \cdot (\frac{3}{4})^3 + 4 \cdot \binom{5}{4} \cdot \frac{1}{4} \cdot (\frac{3}{4})^4 + 5 \cdot (\frac{3}{4})^5 \approx 4.014$

1.2. [Theorem.] The expected value of a random variable X can be computed by an equivalent formula

$$E(X) = \sum_{r \in X(s)} p(X = r)r.$$

1.3. [Theorem.] The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial, is np.

1.4. [Theorem.] Let X_i, $i = 1, \ldots, n$, be random variables on S, and let a, $b \in R$. Then

$$\begin{array}{rl} (i) & E(X_1 + X_2 + \ldots + X_n) = E(X_1) + \ldots + E(X_n) \\ (ii) & E(aX + b) = aE(X) + b \end{array}$$

Q2. Find the expected value of the sum of the numbers that appear when 100 fair dice are rolled.

Let X_i be be the expected value for the ith die, where $1 \leq i \leq 100$. We know for an individual die, the expected value is $\frac{7}{2} = \frac{1}{6} \cdot (1+2+3+4+5+6)$. Therefore, $E(X_1 + \ldots + X_{100}) = E(X_1) + \ldots + E(X_{100}) = 100 \cdot \frac{7}{2} = 350$.

THE GEOMETRIC DISTRIBUTION.

2.1. A random variable X has a **geometric distribution with parameter** p if $p(X = k) = (1 - p)^{k-1}$ for $k = 1, 2, 3, \ldots$, where p is a real number $0 \leq p \leq 1$.

2.2. [Theorem.] If the random variable X has a geometric distribution with parameter p, then $E(X) = \frac{1}{p}$.

INDEPENDENT RANDOM VARIABLES.

3.1. Two random variables X and Y on a sample space S are called **independent** if

$$p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \cdot p(Y = r_2).$$

3.2. [Theorem.] If X and Y are independent random variables on a sample space S, then $E(XY) = E(X)E(Y)$.

Q3. Suppose we roll a fair die four times. Give a non-trivial example of two pairs of random variables on this sample space, one pair dependent, the other independent.

Let X record the value of the first die and let Y record the value on the second die. We know that $p(X = i) = p(Y = j) = \frac{1}{6}$ for all $i, j \in \{1, 2, 3, 4, 5, 6\}$. There are 36 outcomes for rolling a pair of dice, so $p(X = i, Y = j) = \frac{1}{36} = \frac{1}{6^2} = p(X = i) \cdot p(Y = j)$. Therefore, X and Y are independent.
Let X record the value of the first die and let W record the sum of the two dice. We see that $p(X = 1) = \frac{1}{6}$ and $p(W = 4) = \frac{3}{36} = \frac{1}{12}$. For $p(X = 1, Y = 4) = \frac{1}{36}$. Clearly, $\frac{1}{12} \cdot \frac{1}{6} \neq \frac{1}{36}$, therefore X and W are not independent.

VARIANCE.

4.1. Let X be a random variable on a sample space S. The **variance** of X, denoted $V(X)$, is defined by

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

4.2. [Theorem.] For any random variable X, $V(X) = E(X^2) - E(X)^2$

4.3. [Corollary.] If X is a random variable X with $E(X) = \mu$, then $V(X) = E((X - \mu)^2)$.

4.4. [Bienaymé's formula.] If X and Y are two independent random variables on a sample spaces S, then $V(X + Y) = V(X) + V(Y)$. Furthermore, if X_j, $j = 1, \ldots, n$, are pairwise independent random variables on S, then $V(X_1 + X_2 + \ldots + X_n) = V(X_1) + V(X_2) + \ldots + V(X_n)$.

Q4. What is the variance of the combined value when a pair of fair dice is tossed once?

Since the we are looking at two random events, X the value of die 1 and Y the value of die 2, then $V(X+Y) = V(X) + V(Y) = E(X^2) - E(X)^2 + E(Y^2) + E(Y)^2 = 2 \cdot (\frac{91}{6} - (\frac{7}{2})^2) = \frac{35}{6}$

CHEBYSHEV'S AND MARKOV'S INEQUALITIES.

5.1. [Chebyshev's Inequality.] Let X be a random variable on a sample space S with probability distribution p. If $r > 0$, then $p(|X(s) - E(X)| \geq r) \leq \frac{V(X)}{r^2}$.

5.2. [Markov's Inequality.] Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Then $p(X(s) \geq a) \leq \frac{E(X)}{a}$, for $a \in R_{>0}$

Q5. Prove Markov's inequality.

Let $A = \{s \in S : X(s) \geq a\}$. When looking at $E(X) = \sum_{s \in S} p(s)X(s)$, we can rewrite this as $E(X) = \sum_{s \in A} p(s)X(s) + \sum_{s \notin A} p(s)X(s)$. Clearly, $E(X) \geq \sum_{s \in A} p(s)X(s)$. For all $s \in A$, we know $X(s) \geq a$, so $E(X) \geq \sum_{s \in A} p(s)X(s) \geq \sum_{s \in A} p(s) \cdot a$. Lastly, we see that $\sum_{s \in A} p(s) \cdot a = a \cdot \sum_{s \in A} p(s) = a \cdot p(A)$, therefore

$$E(X) \geq a \cdot p(A).$$