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Math 55, Homework 9.

Prob 1. If $X_1, \ldots, X_{2n}$ are mutually independent random variables with the same distribution and if α is any real number whatsoever, prove that

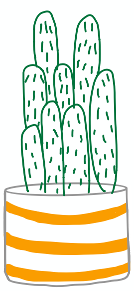
$$p\left(\left|\frac{X_1+\cdots+X_{2n}}{2n}-\alpha\right| \leq \left|\frac{X_1+\cdots+X_n}{n}-\alpha\right|\right) \geq \frac{1}{2}.$$

$x_1 \dots x_2$ are mutually independent,

$$\left|\frac{x_1+\dots+x_{2n}}{2n}-\alpha\right|$$

$$\left|\frac{x_1+\dots x_n+x_{n+1}+\dots+x_{2n}}{2n}-\alpha\right|$$

$$\left|\frac{x_1+\dots+x_n}{n}\frac{1}{2}+\frac{x_{n+1}+\dots+x_{2n}}{2n}-\alpha\right|$$



$$\frac{1}{2}\left|\frac{x_1+x_2+\dots x_n}{x_n}+\frac{x_{n+1}+\dots+x_{2n}}{n}-2\alpha\right|$$

$$\frac{1}{2}\left|\frac{x_1+\dots+x_n}{n}-\alpha+\frac{x_{n+1}+\dots x_{2n}}{n}-\alpha\right|$$

$$\geq \frac{1}{2}\left|\frac{x_1+\dots+x_n}{n}-\alpha\right|$$

$$\Rightarrow \left|\frac{x_1+\dots+x_{2n}}{2n}-\alpha\right| \geq \frac{1}{2}\left|\frac{x_1+\dots+x_{2n}}{n}-\alpha\right|$$

$$\Rightarrow \frac{\left|\frac{x_1+\dots+x_{2n}}{2n}-\alpha\right|}{\frac{x_1+\dots+x_n}{n}-\alpha} \geq \frac{1}{2} \quad \circledast$$

$$\therefore \frac{x_1+\dots x_{2n}}{2n} \leq \frac{x_1+\dots+x_n}{n}$$

$$\therefore \left|\frac{x_1+\dots+x_{2n}}{2n}-\alpha\right| \leq \left|\frac{x_1+\dots+x_n}{n}-\alpha\right|$$

From $\circledast$

$$p\left[\left|\frac{x_1+\dots+x_{2n}}{2n}-\alpha\right| \leq \left|\frac{x_1+\dots+x_n}{n}-\alpha\right|\right] \geq \frac{1}{2}$$

Prob 2. A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmission it sends a 1 one-third of the time and 0 two-thirds of the time. When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received as a 1 is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8 and the probability that it is received as a 0 is 0.2.

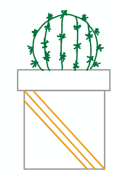
(a) Find the probability that a 0 is received.

P(0) = Probability that zero is recieved

P(0 is sent & recieved) = $\frac{2}{3} \times 0.9$

P(1 is sent and 0 is recieved) = $\frac{1}{3} \times 0.2$

P(0) = $\frac{2}{3} \times 0.9 + \frac{1}{3} \times 0.2 = \boxed{\frac{2}{3}}$



(b) Find the probability that a 0 was transmitted, given that a 0 was received.

P(0 is sent given that 0 is recieved) = P(0 is recieved given that 0 is sent) × $\frac{\text{P(0 is sent)}}{\text{P(0 is recieved)}}$

P(0 is recieved given that 0 is sent) = 0.9

P(0 is sent) = 2/3

P(0 is recieved) = $\frac{2}{3}$ (from part 1)

P(0 is sent given that 0 is recieved) = $0.9 \times \frac{2/3}{2/3} = \boxed{0.9}$

Prob 3. In a round-robin tournament with m players, every two players play one game in which one player wins and one player loses (i.e., there are no ties). Assume that when two players compete it is equally likely that either player wins that game, and that the oucomes of different games are independent. Let E be the event that for every set S of k players, where $k < m$, there is a player who has beaten all k players in S.

(a) Show that $p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j)$ where F_j is the event that there is no player who beats all players from the jth subset of k players.

$\overline{E}$ is the complementary of Event E

let F_j be the event that there is no player who beats all k players from the j^{th} set in a list of $\binom{m}{k}$ set of k players

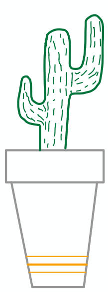
$$\left.\overline{E} = \bigcup_{j=1}^{\binom{m}{k}} F_j \right\} \text{ substitute Boole's inequality } \left\{ \boxed{p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j)} \right.$$

(b) Show that the probability of F_j is $(1 - 2^{-k})^{m-k}$.

the prob. that a particular player that is not in the j^{th} set beat all k players in the j^{th} set is $\frac{1}{2^k}$, therefore the probability that this player does not do so is $1-\frac{1}{2^k}$. therefore the prob that all $(m-k)$ of players not in j^{th} set are unable to keep a perfect record against everyone in j^{th} set is $(1-\frac{1}{2^k})^{(m-k)}$. this is the desired prob. $p(F_j)$

(c) Conclude from parts (a) and (b) that $p(\overline{E}) \leq \binom{m}{k}(1 - 2^{-k})^{m-k}$.

since there are $\binom{m}{k}$ summands and all the summands are same, the inequality $p(\overline{E}) \leq \binom{m}{k}(1-2^{-k})^{(m-k)}$ holds true. If this prob. is less than 1, then it must be possible that $\overline{E}$ fails, that is, E happens. therefore, there is a tournament that meets the conditions of the problems as long as the inequality $\binom{m}{k}(1-2^{-k})^{(m-k)} < 1$ holds true



(d) Use part (c) to find values of m such that there is a tournament with m players such that for every set S of two players, there is a player who has beaten both players in S. Repeat for sets of three players.

$m \geq 21$ for $k=2$

$m \geq 91$ for $k=3$

Prob 4. Suppose that n balls are tossed into $b \geq 2$ bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.

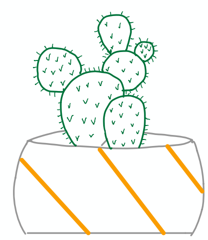
(a) Find the probability that a particular ball lands in a specified bin.

b= number of options independent of each other

If a particular ball is tossed into a specified bin then the rest of $(n-1)$ balls can be tossed into the bins in b^{n-1} ways, so the required probability is $\boxed{\frac{1}{b}}$

(b) Let E_j, $j = 1, \ldots, b$ be the event that bin j is not empty. What is the probability $p(E_1)$?

$$\boxed{\sum_{i=1}^{b} E[j_i] = \sum_{i=1}^{b} \frac{b}{b-i+1} = b\left(1+\frac{1}{2}+\cdots+\frac{1}{b}\right)}$$



(c) Are the events E_1 and E_2 independent? (Hint: explore the symmetry of the setup.)

$$\begin{aligned} P(E_1 \cap E_2) &= P(E_1)P(E_2) \\ &= P(E_1) - P(E_1 \cap F) \\ &= P(E_1) - P(E_1)P(E_2) \\ &= P(E_1)(1 - P(E_2)) \\ &= P(E_1)P(E_2^c) \end{aligned}$$

If E_1 & E_2 are independent, that means its also equivalent to

E_1 & E_2^c