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Math 55, Homework 11.

Prob 1. Find all solutions to the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

(a) $F(n) = (n+1)2^n$

Homogeneous: characterstic polynomial

$x^3 = 6x^2 \cdot 12x + 8$

Guess: 2 is a root: $(x-2)(x^2-4x+4)$

$(x-2)^3 = 0 \longrightarrow$ so 2 multiple 3 is a root

$\boxed{a_n = (c_0 + c_1 n^2 + c_2 n^2)2^n}$ ← homogenous solution

<u>particular solution</u>: theorem 6: $F(n) = (b_t n^t + \dots + b_1 n + b_0)s^n$

$s=2$ $t=1$, $b_1 = 1$ $b_0 = 1$

$n^m(p_1 n^t + p_{t-1}n^{t-1} + \dots p_1 n + p_0)s^n$

$n^9(p_1 n + p_0)2^n$

solve by plugging in

(b) $F(n) = n^2(-2)^n$

From (a) homogenous solution: $a_n = (c_0 + c_1 n^2 + c_2 n^2)2^n$

theorem 6 the term is $n^3(p_t n^t + p_{t-1}n^{t-1} + p_1 n + p_0)2^n$?

Prob 2. Recall that a **partition** of a positive integer is a way to write this integer as the sum of positive integers where repetition is allowed and the order of summands does not matter.

(a) Let $p(n)$ denote the number of partitions of n. Show that the generating function for the sequence $\{p(n)\}$ is the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

m = positive integer

$p^{(m)}(n) \rightarrow$ # of positions of n in which each part is one of the integers 1, 2, ... m. Then $p^{(m)}(n)$ is the # of ways of expressing n as a sum

$$n = s_1 + s_2 + \cdots + s_m$$

s_k $(1 < k < m)$ is itself a sum of #s of k's. which is the same as number of ways of choosing for k = 1, 2, ... m a term x^{sx} from each of the power series $(1-x_k)^{-1}$ in such a way that the product of the terms is x^n.

the generating function for the numbers $p^{(m)}(n)$ is

$$p^{(m)}(n) = (1-x)^{-1}(1-x^2)^{-1} \cdots (1-x^m)^{-1}$$

For any given value of n we have $p(n) = p^{(n)}(n)$ since a position of n cannot contain any parts greater than n, this corresponds to the fact that the terms $(1-x^k)^{-1} = 1 + x^k + x^{2k} + \cdots$ with $k > n$ do not contribute the coefficient of x^n in the infinite product $p(x!)$. so $P(x)$ is the generating function for the sequence $\{p(n)\}$

(b) Find the generating function for $\{p_o(n)\}$ where $p_o(n)$ denotes the number of partitions of n into odd parts (where, as in (a), the order does not matter and repetitions are allowed).

$$p^{(0)}(n) = (1 + x + x^3 + x^5 + x^7)\ (1 + x^2 + x^4 + x^6 \cdots)$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Prob 3. Suppose X is a random variable on a sample space S such that $X(s)$ is a nonnegative integer for all $s \in S$. The **probability generating function** for X is defined as

$$G_X(x) = \sum_{k=0}^{\infty} p(X(s) = k)x^k.$$

(a) Prove that $E(X) = G'_X(1)$.

$$G_X'(x) = \sum_{k=1}^{\infty} k\,p(X(s)=k)\,\underbrace{x^{k-1}}_{\text{equals } 1} \longrightarrow G_X'(1) = \underbrace{\sum_{k=1}^{\infty} k\,p(X(s)=k)}_{E(X)}$$

(b) Let X be the random variable whose value is n if the first success occurs on the nth trial when independent Bernoulli trials are performed, each with probability of success p. Find a closed formula for the probability generating function G_X.

$$G_{X+Y}(x) = \sum_{k=0}^{\infty} p(X+Y=k)\,x^k$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} p(X=i \text{ and } Y=k-i)\right)x^k$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} p(X=i)\cdot p(Y=k-i)\right)x^k$$

$$= \boxed{G_X(x)\cdot G_Y(x)}$$

(c) Using parts (a) and (b), find the expected value of the random variable from (b).

$$E(X) = G'_X(1)$$

$$= \left(\frac{px}{1-x+px}\right)'\Bigg|_{x=1}$$

$$= \left(\frac{p(1-x+px)-(p-1)px}{(1-x+px)^2}\right)\Bigg|_{x=1}$$

$$= \frac{p(p)-p(p-1)}{p^2} = \boxed{\frac{1}{p}}$$