Name: Shivani Patel GSI: Madeline Brandt DISC #: 103

Math 55, Handout 24.

HAMILTONIAN PATHS AND CIRCUITS.

1.1. A **Hamiltonian circuit** in a graph G is a simple circuit that passes through every vertex exactly once

1.2. A **Hamiltonian path** in a graph G is a simple path passes through every vertex exactly once

1.3. Unlike the Eulerian path/circuit problem, the Hamiltonian path/circuit problem is **NP-complete**.

Q1. Which trees have a Hamiltonian path?

linear trees
If any vertex has degree 3, moving through that vertex will disconnect the graph

Q2. For which values of n do the following graphs have a Hamiltonian circuit?

K_n : All $n>2$ by going around the outside C_n : All n W_n : All n Q_n : All $n>1$

the last can be achieved by dividing the graph into two graphs of Q_{n-1}, and making a Hamiltonian path along the first, traveling to the second, making a reverse-order corresponding path along the second, and returning to the first.

1.3. [Dirac's Theorem] If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex is at least $n/2$, then G has a Hamiltonian circuit.

1.4. [Ore's Theorem] If G is a simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamiltonian circuit.

Q3. One of these theorems is a corollary of the other. Which one? Why?

Dirac's theorem; if each vertex has degree at least $n/2$, then the sum of two vertices's degree is at least n.

Q4. Construct a proof of Ore's Theorem following the following outline. Suppose that the assumptions of Ore's Theorem hold for a graph G but that G does not have a Hamiltonian circuit.

(a) Show that by adding edges to G if necessary, we can construct another simple graph H with the same vertices as G such that H does not have a Hamiltonian circuit but the addition of a single edge to H would produce a Hamiltonian circuit in H.

Add edges until every edge that doesnt exist will create a Hamiltonian circuit. this will terminate as K_n has a circuit.

(b) Show that there is a Hamiltonian path in H.

If an edge is removed from a Hamiltonian circuit, it becomes a Hamiltonian path, simply add an edge, find the Hamiltonian circuit, and remove the edge you just added.

(c) $v_1, v_2, \ldots, v_n$ be a Hamiltonian path in H. Show that $\deg(v_1) + \deg(v_n) \geq n$ and that there are at most $\deg(v_1)$ vertices not adjacent to v_n (including v_n itself).

The former stems from the initial assumption as adding lines maintains the property, and V_1 and V_n are not adjacent, otherwise there is a Hamiltonian circuit. thus, the latter statement is true, since there are a total of n vertices.

(d) Let S be the set of vertices preceding each neighbor of v_1 in our Hamiltonian path. Show that S contains $\deg(v_1)$ vertices and $v_n \notin S$.

Each vertex except V_1 has exactly 1 vertex preceding it. Thus, the preceding vertices of each neighbor of V_1 form a set of $\deg(v_1)$ vertices. Additionally, since nothing succeeds V_n, it cannot be in S.

(e) Show that S contains a vertex v_k which is adjacent to v_n, implying that there are edges connecting v_1 to v_{k+1} and v_k to v_n.

since $V_n \notin S$ and at most $\deg(V_1)$ vertices are not adjacent to V_n and V_n is not adjacent to V_n at most $\deg(V_1)-1$ vertices in S are not adjacent to V_n. Thus, at least one vertex in S is adjacent to V_n.

(f) Show that part (e) implies that $v_1, v_2, \ldots, v_{k-1}, v_k, v_n, v_{n-1}, \ldots, v_{k+1}$ is a Hamiltonian circuit in H. Conclude from this contradiction that Ore's Theorem holds.

By the definitions of S, V_{k+1} is adjacent to V_1 and by the def of k, V_k is adjacent to V_n, the reverse of a path is still a circuit. since it goes through every point, it is a Hamiltonian circuit.