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Math 55, Handout 10.

RECURSIVE DEFINITIONS.

1.1. **Recursively defined functions.** A function f on $\mathbb{Z}_+$ can be **recursively defined** using the following two steps:

Basis step: f(0) = some value

Recursive step: f(n) = in terms of some $f(k)$'s where $k < n$

Q1. Is this a valid recursive definition of a function $f : \mathbb{Z}_+ \to \mathbb{Z}$?

$$f(0) = 2, \quad f(n) = \begin{cases} f(n-1) & \text{if } n \text{ is odd and } n \geq 1 \\ 2f(n-2) & \text{if } n \geq 2. \end{cases}$$

Yes the function can be rewritten as $2^{\lfloor \frac{n}{2} \rfloor}$.

1.2. Properties of recursively defined functions can be typically proved by induction.

Q2. Prove that the Fibonacci sequence $\{f_n\}$ satisfies $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$.
It holds true for $n = 1$, which will be our basis step.
$\sum_{j=1}^{n+1} f_j^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1} f_{n+2}$, which works out to be our induction step.
Since both hold true, the statement is proven.

1.3. **Recursively defined sets.** A set S can be **recursively defined** using the following two steps:

Basis step: specify initial elements of set.

Recursive step: specify a recursive rule that tells you how to add new elements to the set.

Q3. Show that the set S defined by $1 \in S$ and $s + t \in S$ whenever $s, t \in S$ is the set $\mathbb{N}$.
$\forall s(s \in S \to s + 1 \in S)$, since $1 \in S$
this is the construction of the natural numbers.

Q4. Give a recursive definition of the set of positive integer powers of 5.
Basis: $5 \in S$
$s \in S \rightarrow 5s \in S$

1.4. Besides functions and sets, other structures can be recursively defined, such as strings and binary trees.

STRUCTURAL AND GENERALIZED INDUCTION.

3.1. **Structural induction principle.** Results about recursively defined sets (and other structures) can be proved using the following two steps:

Basis step: show that the result holds for all elements specified in the basis step of the set's recursive definition

Recursive step: show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

3.2. **Generalized induction** is using (weak or strong) induction to prove results about sets that have the well-ordering property. The well **ordering** is often used for that purpose.

3.3. One can extend generalized induction even further to **partially ordered** sets.

3.4. **Transfinite induction** can be used for ordinals larger than $\mathbb{N}$. This is beyond the scope of Math 55.

Q5. Use generalized induction to show that if $a_{m,n}$ is defined recursively by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + 1 & \text{if } n > 0, \end{cases}$$

then $a_{m,n} = m + n$ for all $(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$.
Basis: $a_{0,0} = 0 + 0 = 0$
Inductive Step:
$a_{m,n} = a_{m-1,n} + 1 = (m - 1 + n) + 1 = m + n$
or
$a_{m,n} = a_{m,n-1} + 1 = (m + n - 1) + 1 = m + n$
Basis and inductive step hold true, so $a_{m,n} = m + n$ is proven.