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Math 55, Handout 21.

INCLUSION-EXCLUSION.

The Principle of Inclusion-Exclusion. Let A_1, A_2, ..., A_n be finite sets. Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Q1. Find the cardinality of $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and

(a) the sets are pairwise disjoint:

$A_1 \cap A_2 = \emptyset \quad A_1 \cap A_3 = \emptyset \quad A_2 \cap A_3 = \emptyset \Rightarrow A_1 \cap A_2 \cap A_3 = \emptyset$

$|A_1| + |A_2| + |A_3| - 0 - 0 - 0 + 0 = 100 + 100 + 100 = 300$

(b) there are 50 common elements in each pair of sets and no elements in all three sets:

$|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$= 100 + 100 + 100 - 50 - 50 - 50 + 0 = 300 - 150 = 150$

Q2. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, all are odd, all are divisible by 3, or all are divisible by 5.

Let A_1 = set of all odd numbers $\binom{50}{4}$

A_2 = set of all numbers divisible by 3 (multiples of 3) $\lfloor \frac{100}{3} \rfloor = 33 \Rightarrow \binom{33}{4}$

A_3 = set of all numbers divisible by 5 (multiples of 5) $\lfloor \frac{100}{5} \rfloor = 20 \Rightarrow \binom{20}{4}$

S = set of all 4 numbers picked at random from $1 \sim 100$ $\binom{100}{4}$

round up b/c the first element is odd (3, 9, ...)

$|A_1 \cap A_2| = \lceil \frac{33}{2} \rceil = 17 \quad \binom{17}{4}$

division rule (round down)

$|A_2 \cap A_3| = \lfloor \frac{100}{15} \rfloor = 6 \quad \binom{6}{4}$

$|A_1 \cap A_3| = \lceil \frac{20}{2} \rceil = 10 \quad \binom{10}{4}$

even/odd alternate, so half the elements divisible by 5 are odd

$|A_1 \cap A_2 \cap A_3| = \lceil \frac{50}{15} \rceil = \frac{|A_2 \cap A_3|}{2} = \frac{6}{2} = 3 \quad \binom{3}{4}$

$$\frac{|A_1 \cup A_2 \cup A_3|}{|S|} = \frac{|A_1|+|A_2|+|A_3|-|A_1 \cap A_2|-|A_1 \cap A_3|-|A_2 \cap A_3|+|A_1 \cap A_2 \cap A_3|}{|S|} = \frac{\binom{50}{4}+\binom{33}{4}+\binom{20}{4}-\binom{17}{4}-\binom{10}{4}-\binom{6}{4}+\binom{3}{4}}{\binom{100}{4}} = 0.0697$$

Alternative Form of Inclusion-Exclusion.

Let U be a universal finite set of cardinality N, and let A_i denote its subset of elements satisfying

Let $N(A_{i_1}, A_{i_2}, \ldots, A_{i_k})$ denote the number of elements of U satisfying properties P_{i_1}, P_{i_2}. ..., P_{i_k}.

Let $N(A'_{i_1}, A'_{i_2}, \ldots, A'_{i_k})$ denote the number of elements of U satisfying none of the properties P_{i_1}, P_{i_2}. ..., P_{i_k}.

Then $N(P'_1 \ldots P'_n) = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_i P_j) - \sum_{1\le i<j<k\le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n)$

Q3. How many primes are there not exceeding 120?

$\sqrt{120} = 10.95\ldots$ Primes not exceeding $10.95\ldots$ are $2, 3, 5, 7$. Let P_1 be the property that an integer is divisible by 2

P_2 be the property that an integer is divisible by 3, P_3 be the property that an integer is divisible by 5

P_4 be the property that an integer is divisible by 7. # of primes not exceeding 120: $4 + N(P_1'P_2'P_3'P_4')$ s.t $4 + N(P_1'P_2'P_3'P_4') =$

$4 + 119 - N(P_1) - N(P_2) - N(P_3) - N(P_4) + N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4)$

$-N(P_2P_3P_4) + N(P_1P_2P_3P_4) = 4 + 119 - \lfloor\frac{120}{2}\rfloor - \lfloor\frac{120}{3}\rfloor - \lfloor\frac{120}{5}\rfloor - \lfloor\frac{120}{7}\rfloor + \lfloor\frac{120}{2\cdot3}\rfloor + \lfloor\frac{120}{2\cdot5}\rfloor + \lfloor\frac{120}{2\cdot7}\rfloor + \lfloor\frac{120}{3\cdot5}\rfloor + \lfloor\frac{120}{3\cdot7}\rfloor + \lfloor\frac{120}{5\cdot7}\rfloor - \lfloor\frac{120}{2\cdot3\cdot5}\rfloor - \lfloor\frac{120}{2\cdot3\cdot7}\rfloor$

$- \lfloor\frac{120}{3\cdot7\cdot5}\rfloor + \lfloor\frac{120}{2\cdot3\cdot5\cdot7}\rfloor = 4 + 119 - 60 - 40 - 24 - 17 + 20 + 12 + 8 + 8 + 5 + 3 - 4 - 2 - 1 - 1 + 0 = 30$. So there are 30 primes not exceeding 120.

Q4. How many onto functions are there from a set with 5 elements to a set with 3 elements?

Suppose that the elements in the codomain are b_1, b_2, b_3. Let P_1, P_2, P_3 be the properties that b_1, b_2, b_3 are not in the range of the function. $N(P_1'P_2'P_3') = N - (N(P_1) + N(P_2) + N(P_3)) + (N(P_1P_2) + N(P_1P_3) + N(P_2P_3)) - N(P_1P_2P_3)$. $N = 3^5$ (total # of functions from a set w/ 5 elements to one w/ 3 elements. $N(P_i) = 2^5$ (# of functions that do not have b_i in their range. $N(P_iP_j) = 1^5 = 1$. $N(P_1P_2P_3) = 0$ since this term is the # of functions that have none of b_1, b_2, b_3 in their range

$\Rightarrow N(P_1'P_2'P_3') = 3^5 - 3\cdot 2^5 + 3\cdot 1^5 = 150$

Theorem. Let n, $m \in \mathbb{N}$ with $m \geq n$. Then there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)\cdot 1^m$$

onto functions from a set with m element to a set with n elements.

DERANGEMENTS.

A **derangement** is.a permutation of objects that

leaves no object in its original position.

Q5. What is the probability that none of the 10 employees receives the correct hat if a hatcheck person hands their hats back randomly?

$$\frac{D_{10}}{10!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}$$

$$\Rightarrow D_{10} = 10!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\right)$$

$$= 0.368$$

The number of derangements. The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right]$$