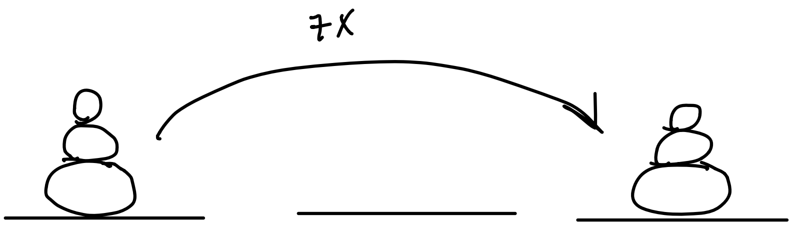
Name: Shivani Patel GSI: Madeline Brandt DISC #: 103

Math 55, Homework 6.

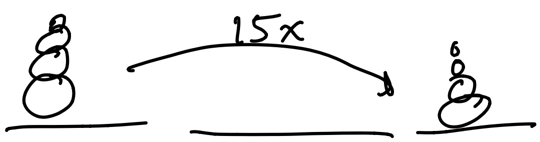
Prob 1. You are given a tower of n disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger one onto a smaller. How many moves are necessary and sufficient to perform this task?

Find a formula for the necessary and sufficient number of moves and prove it by induction.



$2^n - 1$

$2^2 - 1$



(P_k)

Base case : 1

P_k = assume T for $S_n = 2^k - 1$

prove $S_{k+1} = 2^{k+1} - 1$

(k is previous # of rack)

$$S_{k+1} = 2S_k + 1$$
$$= 2(2^k - 1) + 1$$
$$= 2 \cdot 2^k - 2 + 1$$
$$= 2^{k+1} - 1 = S_{k+1}$$

$$\boxed{\#\text{ of moves} = 2^{\#\text{ of racks}} - 1}$$

Prob 2. What is the maximum number L_n of regions obtained by drawing n lines in the plane?

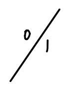
(a) Find a formula for L_n and prove it by induction.

$\frac{n(n+1)}{2}+1$

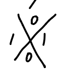
formula: $\frac{n(n+1)}{2}+1$

$\sum_{k=1}^{n} \frac{k(k+1)}{2}+1$

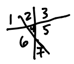
first case: $\frac{1(1+1)}{2}+1=2$



second case: $\frac{2(2+1)}{2}+1=4$

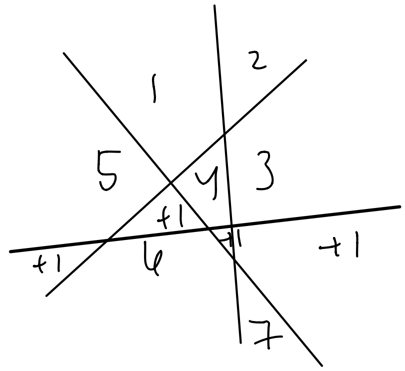


third case: $\frac{3(3+1)}{2}+1=7$



$\sum_{k=1}^{n+1}$

$(k+1)((k+1)+1)$



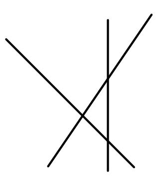
$(n+1)^{st}$ line:

regions from n lines + # new region

$1+\frac{(n+1)(n+2)}{2}=\frac{n(n+1)}{2}+1+n+1$

n^{th} line makes n new regions

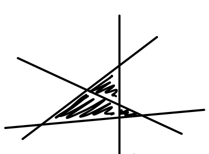
(b) Some of the regions defined by n lines are unbounded, while others are bounded. What is the maximum possible number of bounded regions created by n lines?



= 6 unbounded
1 bounded

unbounded equation: $2n$

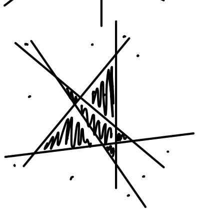
bounded equation: $\frac{n(n+1)}{2}+1-2n$



= 8 unbounded
3 bounded

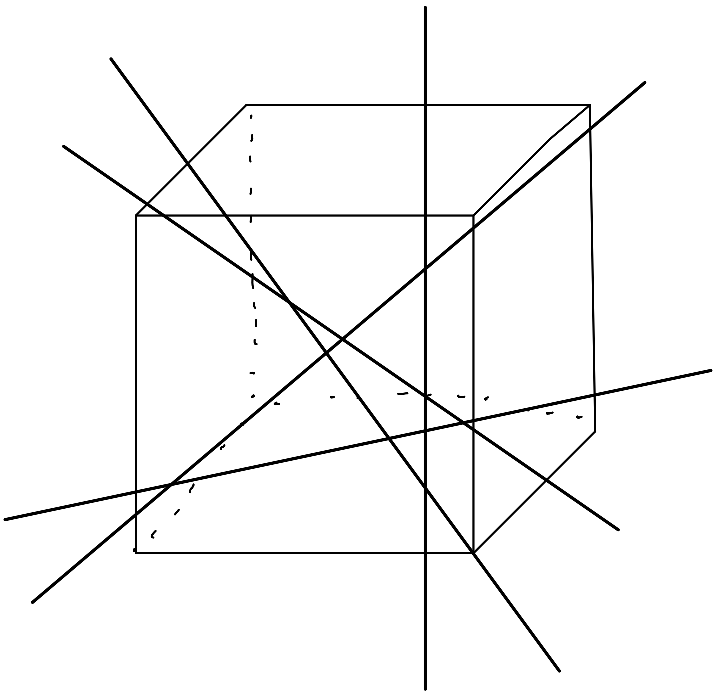
Inductive step: we assume that P(n) is true
this means we assume that

$\frac{n(n+1)}{2}+1-2n$ is true



10 unbounded
= 6 bounded

Prob 3. (a) How many pieces of cheese can you obtain from a single thick piece by making five cuts? (The cheese must stay in its original position while you do the cutting; each cut corresponds to a plane in 3D.)



for 5 cuts you get 26 cuts

(b) Find a recurrence relation for P_n, the maximum number of 3D regions defined by n different planes.

$P(n) = P(n-1) + 1 + \frac{n(n-1)}{2}$

$P(0) = 1$	$= 1$	$=$ N/A
$P(1) = 2$	$= 2$	$=$ N/A
$P(2) = 2+1+1$	$= 4$	$= (1 \cdot 2 \cdot 3)/6 + 2 + 1$
$P(3) = 4+3+1$	$= 8$	$= (2 \cdot 3 \cdot 4)/6 + 3 + 1$
$P(4) = 8+6+1$	$= 15$	$= (3 \cdot 4 \cdot 5)/6 + 4 + 1$
$P(5) = 15+10+1$	$= 26$	$= (4 \cdot 5 \cdot 6)/6 + 5 + 1$
$P(6) = 26+15+1$	$= 42$	$= (5 \cdot 6 \cdot 7)/6 + 6 + 1$

Proof by induction

Base: $P(0) = 1$

Induction: $P(n) = P(n-1) + n(n-1)/2 + 1$ then

$P(n) = (n-2)(n-1)(n)/6 + n + n(n-1)/2 + 1$

AND $P(n) = (n-1)n(n+1)/6 + n + 1$

Prob 4. The well-ordering principle can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let

$$S = \{as + bt \ : \ s, t \in \mathbb{Z}\} \cap \mathbb{N}.$$

(a) Show that S is non-empty.

$s = 1,\ t = 0$

$S = \{\emptyset, a \ldots\}$ → corresponds to different combos of s & t

S is nonempty

(b) Use the well-ordering property to show that S has a smallest element c.

$s = 1,\ t = 0 \rightarrow a$

$s = 2,\ t = 1 \rightarrow 2a + b$

$s = 3,\ t = 2 \rightarrow 3a + 2b$

If a & b are positive, then $c < e < d$

the 3 #s in the set S and they cant equal each other bc a & b are constants being multiplied by different #s & then being added so they cant be equal so 2 must be bigger than 1

(c) Show that if d is a common divisor of a and b, then d is a divisor of c.

★ c is composed of a and b

c belongs to the set S, meaning that it is some linear combo of $as + bt$

$c = as + bt$ d is a diviser of c

(d) Show that $c|a$ and $c|b$.

$c = as + tb$

so by Bezouts identity theorem, $c = \gcd(a, b)$

(e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing $\gcd(a, b)$ is unique.

Prob 5. A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or one space vertically (in either direction) and two spaces horizontally (in either direction). Suppose you have an infinite chessboard, made up of all squares (m, n) where m and n are nonnegative integers that denote the row and column of the square, respectively. Use induction on $m + n$ to show that a knight starting at $(0, 0)$ can visit every square using a finite sequence of moves.

Base case

P(2) for all m an n in $\mathbb{N}$ & $m+n \leq 2$ a knight can reach square(m, n)

you start at (0,0):

to reach (1,0)

1) move to (0,2)
2) move to (1,2)
3) move to (1,0)

<u>Inductive step</u>

$m+n \leq k+1 \therefore k+1 \geq 3$ which shows that at least one of m & n is greater than 2

$\longrightarrow$ many moves will get you to

$(m-2, n+1) \longrightarrow m-2+n+1 = m+n-1 = k$

from this square it is only a short move to (m, n) so $P(k) \rightarrow P(k+1)$