Name:	**GSI:**	**DISC #:**

Solutions to Homework 1.

Prob 1. (a) Find a formula of two propositional variables p and q that involves only negation and disjunction that is equivalent to the implication $\neg p \to q$ and provide its truth table to see it indeed coincides with the truth table for $\neg p \to q$.

Solution. We know that $\neg p \to q$ takes the value T when $\neg p = F$ (i.e., $p = T$) or $q = T$ and takes the value F otherwise. So it should be equivalent to $p \vee q$. And indeed, the two truth tables are the same:

p	q	$\neg p \to q$	$p \vee q$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

(b) Find a formula of two propositional variables p and q that involves only negation, conjunction, and disjunction that is equivalent to $p \oplus q$. Likewise, provide its truth table to see it indeed coincides with the truth table for $p \oplus q$.

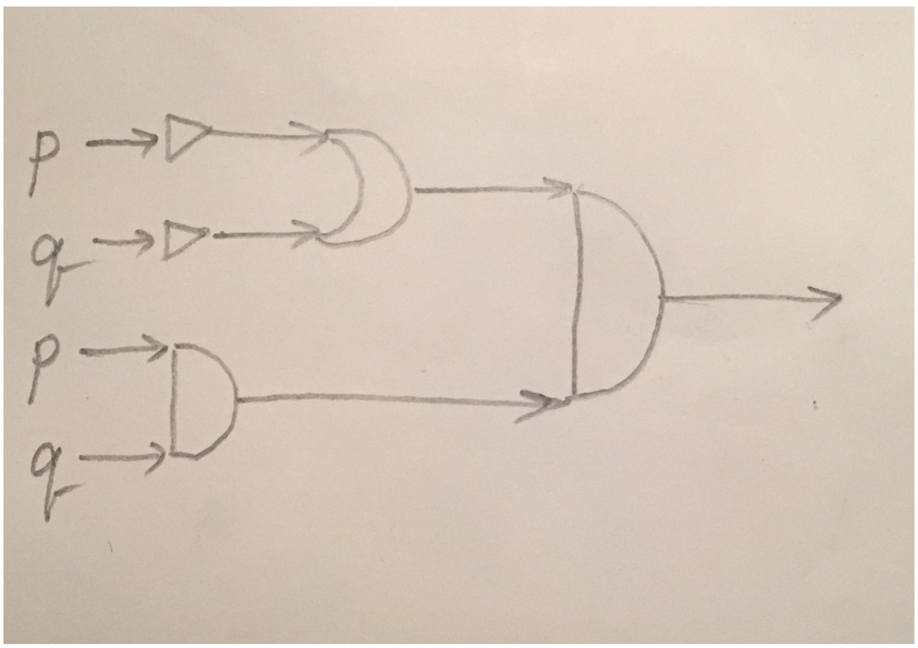
Solution. Here $p \oplus q$ means p and $\neg q$ are true of false simultaneously, so $(p \wedge \neg q) \vee (\neg p \wedge q)$ (a cleaner formula would be $p \leftrightarrow \neg q$ but we are not allowed implications per the stipulation of the problem). And the truth table confirms:

p	q	$p \oplus q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
F	F	F	F	F	F
F	T	T	F	T	T
T	F	T	T	F	T
T	T	F	F	F	F

Prob 2. Given the combinatorial circuit for the formula $\neg p \vee \neg q$, expand it to a larger one taking only propositional variables p and q as inputs so that the larger circuit outputs the value F (false) no matter the truth values of p and q.

Solution. We need to augment the given formula so as to get another formula depending on p and q only which is a contradiction. We can add a conjunction with the negation of the original formula to get:

$$(\neg p \vee \neg q) \wedge (p \wedge q).$$



Prob 3. The Island of Knights and Knaves. Mr. Smullyan visited a famous island where knights always tell the truth and knaves always lie and every inhabitant is either a knight or a knave. He was introduced to three inhabitants A, B, and C of which at least one was a knave and one a knight. One of them had a prize that Mr. Smullyan could win (if and) only if he could determine correctly which one had it. The three spoke to him.

A said: B does not have the prize.

B said: I don't have the prize.

C said: I have the prize.

Formalize the problem using six propositional variables, one for each of the locals being a knight (or not) and one for each having the prize or not. Write down a compound proposition that takes the value T (true) exactly when the values of all propositional variables describe the solution to this puzzle.

Which local has the prize? Can you argue formally using your formula?

Solution. Let variables t_A, t_B, t_C indicate whether each of A, B, C, respectively, is a knight, i.e., tells the truth. In other words, $t_A = T$ if and only if A is a knight etc. Likewise, let p_A, p_B, p_C indicate whether A, B, C, respectively, has the prize. In other words, $p_A = T$ if and only if A has the prize etc. Now let us translate into logic what they said.

A said: "$\neg p_B$"

B said: "$\neg p_B$".

C said: "p_C".

Each statement is true if and only if its author tells the truth. In other words, we must have

$$(t_A \leftrightarrow \neg p_B) \wedge (t_B \leftrightarrow \neg p_B) \wedge (t_C \leftrightarrow p_C).$$

Furthermore, we know only one has the prize, that is,

$$(p_A \leftrightarrow \neg p_B \wedge \neg p_C) \wedge (p_B \leftrightarrow \neg p_A \wedge \neg p_C) \wedge (p_C \leftrightarrow \neg p_A \wedge \neg p_B).$$

Furthermore, we know at least one lies and one tells the truth, i.e.,

$$\neg(t_A \wedge t_B \wedge t_C) \wedge (t_A \vee t_B \vee t_C).$$

And a formula that completely describes the setup of our problem is a conjunction of the three obtained formulas:

$$\begin{aligned}&(t_A \leftrightarrow \neg p_B) \wedge (t_B \leftrightarrow \neg p_B) \wedge (t_C \leftrightarrow p_C) \wedge \neg(t_A \wedge t_B \wedge t_C) \wedge (t_A \vee t_B \vee t_C) \\ &\wedge (p_A \leftrightarrow \neg p_B \wedge \neg p_C) \wedge (p_B \leftrightarrow \neg p_A \wedge \neg p_C) \wedge (p_C \leftrightarrow \neg p_A \wedge \neg p_B).\end{aligned}$$

Note that the beginning of our formula implies $t_A \leftrightarrow t_B (\leftrightarrow \neg p_B)$, so if $t_C = T$, then $t_A = t_B = F$, and $p_B = T$. But then also $p_C = T$, while p_B and p_C cannot both be true. So $t_C = F$, hence $p_C = F$ and $t_A = t_B = T$, hence $p_B = F$. Thus $p_A = T$, meaning A has the prize.

Prob 4. A harrowing adventure. Having just won his prize, Mr. Smullyan was enjoying his visit - that is, until he got captured by a ferocious gang of local brigands! They grabbed his prize too, and promised to kill him unless he could solve another puzzle: Among three gang members A, B, C, exactly one was a witch doctor. Mr. Smullyan had to point at one of the three. If he pointed at the witch doctor, he would be killed! If he pointed at one that was not a witch doctor, he would go free. But he would have his prize back only if he could give a rigorous explanation. Here is what those three said to Mr. Smullyan:

A: I am a witch doctor.
B: I am not a witch doctor.
C: At most one of us is a knight.

Can you do it? Perform formal analysis as in Prob 3.

Solution. Let variables t_A, t_B, t_C indicate whether each of A, B, C, respectively, is a knight, i.e., tells the truth. Let w_A, w_B, w_C indicate whether A, B, C, respectively, is a witch doctor. Translate what A, B, C said:

A: w_A
B: $\neg w_B$
C: $(\neg t_A \vee \neg t_B) \wedge (\neg t_A \vee \neg t_C) \wedge (\neg t_B \vee \neg t_C)$

So we have
$(t_A \leftrightarrow w_A) \wedge (t_B \leftrightarrow \neg w_B) \wedge (t_C \leftrightarrow (\neg t_A \vee \neg t_B) \wedge (\neg t_A \vee \neg t_C) \wedge (\neg t_B \vee \neg t_C))$.

Finally, exactly one is a witch doctor, i.e.,

$$(w_A \vee w_B \vee w_C) \wedge ((\neg w_A \wedge \neg w_B) \vee (\neg w_A \wedge \neg w_C) \vee (\neg w_B \vee \neg w_C)).$$

The conjunction of these statements completely describes out setup:

$$(t_A \leftrightarrow w_A) \wedge (t_B \leftrightarrow \neg w_B) \wedge (t_C \leftrightarrow (\neg t_A \vee \neg t_B) \wedge (\neg t_A \vee \neg t_C) \wedge (\neg t_B \vee \neg t_C))$$
$$\wedge (w_A \vee w_B \vee w_C) \wedge ((\neg w_A \wedge \neg w_B) \vee (\neg w_A \wedge \neg w_C) \vee (\neg w_B \vee \neg w_C)).$$

Now, if $w_C = T$, then $w_A = w_B = F$, so $t_A = F$, $t_B = T$, and the clause

$$t_C \leftrightarrow (\neg t_A \vee \neg t_B) \wedge (\neg t_A \vee \neg t_C) \wedge (\neg t_B \vee \neg t_C))$$

turns into $t_C \leftrightarrow \neg t_C$, which is not satisfiable! So $w_C = F$, i.e., C cannot be a witch doctor, and Mr. Smullyan can safely point at him (and recover his prize).

Prob 5. The Island of Musica. Stressed out by his last adventure, Mr. Smullyan left the Island of Knights and Knaves and visited another island instead. There, every inhabitant is either a pianist or a violinist but no inhabitant is both (and, likewise, either male or female but not both). Moreover, it so happens that female pianists there always tell the truth and male pianists always lie, and for violinists it is the opposite. Mr. Smullyan was invited to the home or Mr. and Mrs. Smith. They owned a nice piano and Mr. Smullyan asked the wife whether it was a Steinway. She said, "I am actually a violinist but this piano is not a Steinway."

Was this piano a Steinway or not? Perform formal analysis as in Prob 3.

Solution. We need only two variables here, say, v for Mrs. Smith's profession and s for the piano. Namely, v takes the value T if Mrs. Smith is a violinist and F otherwise, and s takes the value T if the piano is a Steinway and F otherwise.

Mrs. Smith's statement, translated into logic, means: "$v \wedge \neg s$". That statement is true if and only if Mrs. Smith tells the truth if and only if she is not a violinist. So, the whole setup of this problem boils down to one logical formula:

$$\neg v \longleftrightarrow v \wedge \neg s.$$

Now, if $s = F$, then $\neg s = T$, and this formula turns into

$$v \longleftrightarrow \neg v,$$

which is not satisfiable. Hence, $s = T$, i.e., the piano is a Steinway.