Name: Sibo Ma	GSI: Theodore Zhu	DISC #: 106

Math 55, Handout 12.

PERMUTATIONS AND COMBINATIONS.

1.1. A **permutation** of a set of distinct objects is an ordered arrangement of these objects.
An r-**permutation** is an ordered arrangement of r elements of a sets

1.2. **[Theorem].** The number $P(n, r)$ of r-permutations of a set with n elements is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

Q1. A group contains n men and n women. How many ways are there to arrange these people in a row for picture so that the men and the women alternate?

$$n! \times n! \times 2 = 2(n!)^2$$

2.1. An r-**combination** of elements of a set is an unordered selection of r elements from the set.

2.2. **[Theorem].** The number $C(n, r)$ of r-combinations of a set with n elements is

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Q2. A coin is flipped eight times, and each flip comes up heads or tails. How many possible outcomes

(a) are there in total?

2^8

(b) contain at most three tails?

$$C_8^0 + C_8^1 + C_8^2 + C_8^3 = 1+8+28+56 = 93$$

THE BINOMIAL THEOREM.

3.1. **[Binomial Theorem].** Let x and y be variables and let $n \in \mathbb{N}$. Then

$$(x+y)^n = \sum_{i=0}^{n} C_n^i x^i y^{n-i}$$

3.2. **[Corollary].** Let $n \in \mathbb{N}$. Then

$$2^n = \sum_{i=0}^{n} C_n^i 1^i 1^{n-i} = \sum_{i=0}^{n} C_n^i$$

3.3. **[Corollary].** Let $n \in \mathbb{N}$. Then

$$0 = \sum_{i=0}^{n} C_n^i 1^i (-1)^{n-i} = \sum_{i=0}^{n} (-1)^{n-i} C_n^i = \sum_{i=0}^{n} (-1)^i C_n^i$$

3.4. **[Corollary].** Let $n \in \mathbb{N}$. Then

$$3^n = \sum_{i=0}^{n} C_n^i 2^i 1^{n-i} = \sum_{i=0}^{n} 2^i C_n^i$$

BINOMIAL IDENTITES.

4.1. **[Pascal's Identity].** Let n, $k \in \mathbb{N}$, with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Q3. Derive Pascal's identity from the formula for $C(n, r)$.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= n!\left[\frac{1}{(k-1)!\,(n-k)!\cdot(n-k+1)} + \frac{1}{(k-1)!\,(n-k)!\cdot k}\right]$$

$$= n!\,\frac{n-k+1+k}{k!\,(n-k+1)!} = n!\,\frac{n+1}{k!\,(n+1-k)!} = \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}$$

4.2. **[Vandermonde's Identity].** Let n, m, $k \in \mathbb{N}$, with $k \leq m$, $k \leq n$. Then

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{k-r}\binom{n}{r}$$

4.3. **[Corollary].** Let $n \in \mathbb{N}$. Then

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{n-r}\binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r}^2$$

Q4. For $n \in \mathbb{N}$, derive the following identity using algebra and (basic) calculus:

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

$\because 0\binom{n}{0} = 0$

$$\therefore = \sum_{k=1}^{n} k\,\frac{n!}{k!\,(n-k)!}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!}$$

$$= n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,[n-1-(k-1)]!}$$

$$= n\sum_{k=1}^{n} \binom{n-1}{k-1}$$

let $k' = k-1$

$$= n\sum_{k'=0}^{n-1} \binom{n-1}{k'}$$

4.4. **[Unnamed Identity].** Let n, $k \in \mathbb{N}$, with $k \leq n$. Then

$$\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$$

Q5. Draw a diagram within Pascal's triangle as a memo for the last identity.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

$\binom{8+1}{3+1}$

is the sum of