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Math 55, Lecture 20.

GENERATING FUNCTIONS.

The **(ordinary) generating function** $\mathcal{G}(x)$ for the sequence $\{a_n\}$, $n \in \mathbb{Z}_+$ is the infinite series

$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$

The coefficient of x^n in $\mathcal{G}(x)$ is a_n

[Theorem.] Let $\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n$ and $\mathcal{H}(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$\mathcal{G}(x) + \mathcal{H}(x) = \sum_{k=0}^{\infty} (a_k + b_k)\, x^k \qquad \mathcal{G}(x) \cdot \mathcal{H}(x) = \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} a_j b_{k-j}\Big) x^k$$

Q1. If $\mathcal{G}(x)$ is the generating function for the sequence $\{a_n\}$, what is the generating function for each of the following sequences?

(a) $0, a_0, a_1, a_2, \ldots$: $a_0 x + a_1 x^2 + a_2 x^3 + \cdots = x(a_0 + a_1 x + a_2 x^2) = xg(x)$

(b) $a_1, 2a_2, 3a_3, \ldots$: $G'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n$

(c) $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$: $a_0 + (a_0 + a_1)\, x + (a_0 + a_1 + a_2) x^2 = \frac{G(x)}{1-x}$

(d) $a_0^2, 2a_0a_1, 2a_0a_2 + a_1^2, 2a_0a_3 + 2a_1a_2, 2a_0a_4 + 2a_1a_3 + 2a_2^2, \ldots$: $G(x) \cdot G(x)$

THE EXTENDED BINOMIAL THEOREM.

Let $u \in \mathbb{R}$ and let $k \in \mathbb{Z}_+$. The **extended binomial coefficient** $\binom{u}{k}$ is defined as

$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k>0 \\ 1 & \text{if } k=0 \end{cases}$

In particular, for $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, the following equality holds:

$$\binom{-n}{r} = (-1)^r\, C(n+r-1, r)$$

[Extended Binomial Theorem.] Let $u \in \mathbb{R}$ and let $x \in \mathbb{C}$ with $|x| < 1$. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Q2. Find a closed form of the generating function for the sequence $\{a_n\}$ where

(a) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}, 0, 0, 0, \ldots$: $\binom{7}{0} + 2\binom{7}{1} x + 2^2\binom{7}{2} + \cdots$

$G(x) = (1+2x)^7$

(b) $a_n = \binom{10}{n+1}$, $n \in \mathbb{Z}_+$:

$\sum_{n=0}^{\infty} C(10, n+1)\, x^n = \sum_{n=1}^{\infty} C(10, n)\, x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} C(10, n)\, x^n = \frac{1}{x}((1+x)^{10} - 1)$

(c) $a_n = \binom{n+4}{n}$, $n \in \mathbb{Z}_+$:

$a_n = \binom{n+k-1}{n}$ has $G(x) = \frac{1}{(1-x)^k}$, given $k=5$

COUNTING VIA GENERATING FUNCTIONS.

Q3. What generating functions can be used to find the number of ways in which postage of r cents can be pasted on an envelope using 1-cent, 3-cent, and 20-cent stamps?

(a) Assume that the order the stamps are pasted on does not matter.

1 cent $= 1 + x + x^2 + x^3 \ldots = \frac{1}{1-x}$

3 cent $= 1 + x^3 + x^6 + x^9 \ldots = \frac{1}{1-x^3}$

20 cent $= 1 + x^{20} + x^{40} + x^{60} \ldots = \frac{1}{1-x^{20}}$

$$\boxed{[x^r]\left(\frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^{20}}\right)}$$

(b) Assume that the stamps are pasted in a row and their order matters.

$$\sum_{r=0}^{\infty} (x + x^3 + x^{20})^r = \frac{1}{1-x-x^3-x^{20}}$$

$$\boxed{[x^r] \frac{1}{1-x-x^3-x^{20}}}$$

RECURRENCES VIA GENERATING FUNCTIONS.

Q4. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$, $a_1 = 5$.

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) - 4x\,G(x) - 4x^2 G(x)$$

$$= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 4a_{k-1} x^k + \sum_{k=2}^{\infty} 4a_{k-2} x^k$$

$$G(x)(1-4x+4x) = 2 + 5x - 8x + \sum_{k=2}^{\infty} (a_k - 4a_{k-1} + 4a_{k-2}) x^k$$

$$G(x)(1-2x)^2 = 2 - 3x + \frac{x^2+x}{(1-x)^3} - x$$

$$= \frac{4x^4 - 14x^3 + 19x^2 - 9x + 2}{(1-x)^3}$$

$$G(x) = \frac{4x^4 - 14x^3 + 19x^2 - 9x + 2}{(1-x)^3 (1-2x^2)}$$

$$= \frac{13}{1-x} + \frac{1}{(1-x)^2} + \frac{2}{(1-x)^3} + \frac{24}{1-2x} + \frac{6}{(1-2x)^2}$$

$$= \sum_{0}^{\infty} \left[13x^k + 5(k+1)x^k + 2\,\frac{(k+2)(k+1)}{2} x^k - 24 \cdot 2^k x^k + 6(k+1)\,2^k x^k\right]$$

$$= \sum_{0}^{\infty} \left[k^2 + 8k + 20 + (16k - 18)\,2^k\right] x^k$$

$$a_k = k^2 + 8k + 20 + (6k - 18)2^k$$

IDENTITIES VIA GENERATING FUNCTIONS.

Q5. (a) Show that $(x^2+x)/(1-x)^4$ is the generating function for the sequence $\{1^2 + 2^2 + \cdots + n^2\}_{n \in \mathbb{Z}_+}$.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$x\,G(x) = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$G(x) - xG(x) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$G(x)(1-x) = a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1})\, x^n = \sum_{n=0}^{\infty} n^2 x^n$$

$G(x)$ for $\{n^2\}$ $= \frac{2}{(1-x^3)} - \frac{3}{(1-x)^2} + \frac{1}{1-x}$

$$= \frac{2 - 3(1-x) + (1-x)^2}{(1-x)^3} = \frac{x^2+x}{(1-x)^5}$$

so $G(x)(1-x) = \frac{x^2+x}{(1-x)^3}$

$$G(x) = \frac{x^2+x}{(1-x)^4}$$

(b) Use (a) to find an explicit formula for $1^2 + 2^2 + \cdots + n^2$.

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$G(x) = \frac{x^2+x}{(1-x)^2} = \frac{x^2}{(1-x)^4} + \frac{x}{(1-x)^4}$$

$$= x^2 \sum_{n=0}^{\infty} \binom{n+1}{3} x^{n+2} + x \sum_{n=0}^{\infty} \binom{n+3}{3} x^n$$

$$= \sum_{n=0}^{\infty} \binom{n+1}{3} x^n + \sum_{n=1}^{\infty} \binom{n+2}{2} x^n$$

$$= \sum_{n=0}^{\infty} \left[\binom{n+1}{3} + \binom{n+2}{3}\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{(n+1)n(n-1)}{6} + \frac{(n+2)(n+1)n}{6}\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{(n+1)n(2n+1)}{6}\right] x^n$$