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Math 55, Handout 7.

SEQUENCES.

1.1. A **sequence** is a function from a subset of integers (either $\mathbb{Z}_+$ or $\mathbb{N}$) to a set S.

1.2. An **arithmetic progression** is a sequence of the form: $a_n = a + kd$ where $k \in \mathbb{Z}_+$, a is the first term of the sequence, and d is the difference.

1.3. The sum of an arithmetic progression is

$$\sum_{k=0}^{n} a + kd = \frac{n+1}{2}(2a + nd)$$

1.4. A **geometric progression** is a sequence of the form: $a_n = ar^k$ where $k \in \mathbb{Z}_+$, a is the first term of the sequence, and r is the ratio.

1.5. The sum of a geometric progression is

$$\sum_{k=0}^{n} ar^k = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Q1. Evaluate the sum $\sum_{k=1}^{n}(2k-1)$.

Answer:

$\sum_{k=1}^{n}(2k-1) = (\sum_{k=0}^{n}(2k-1)) - (-1) = n^2 - 1 + 1 = n^2$

RECURRENCE RELATIONS.

2.1. A **recurrence relation** for a sequence $\{a_n\}$ is an equation for a_n written in terms of its preceding terms.

Its **initial conditions** are a_{n-1}, a_{n-2}.

Q2. If a sequence satisfies a 3-term recurrence relation, say, $a_n = 3a_{n-1} + 4a_{n-2}$, how many initial conditions determine that sequence? **Answer:** 2.

Q3. Write down a closed formula for the nth term of a sequence defined recursively via

$$a_0 = 0, \quad a_1 = 3, \quad a_n = 2a_{n-1} - a_{n-2}.$$

Answer: $a_n = 3n$ for $n \in \mathbb{Z}_+$.

DIVISION.

3.1. Given $a, b \in \mathbb{Z}$, $a \neq 0$, we say that a **divides** b (and write $a|b$) if there is an integer c such that $ac = b$.

3.2. **The division algorithm.** Let $a \in \mathbb{Z}$ and let $d \in \mathbb{N}$. Then there exists $q \in \mathbb{Z}$ (called the quotient) and $r \in \{0, \ldots, d-1\}$ (called the remainder) such that $a = dq + r$.

In that case, we write

$$q = a \textbf{ div } d, \quad r = a \textbf{ mod } d.$$

ARITHMETIC MODULO m.

4.1. Given $m \in \mathbb{N}$, we can define **arithmetic operations on** $Z_m = \{0, \ldots, m-1\}$ as

$$\begin{aligned} a +_m b &= (a+b) \textbf{ mod } m \\ a \cdot_m b &= (a \cdot b) \textbf{ mod } m \end{aligned}$$

4.2. These operations satisfy many properties of ordinary addition and multiplication, e.g., closure, associativity, commutativity, identity elements, additive inverses, and distributivity.

Q4. Does multiplication modulo m satisfy the property of ordinary multiplication

$$\forall x, y \ [\, x \cdot y = 0 \longrightarrow x = 0 \ \vee \ y = 0 \,] \ ?$$

Answer: If $x \cdot y = 0$, then 0 **mod** $m = 0$. However, $x \cdot_m y = 0$ does not necessarily imply $x = 0$ or $y = 0$.

4.3. Let $a, b \in \mathbb{Z}$ and let $m \in \mathbb{N}$. The notation $a \equiv b \pmod{m}$ means that a is congruent to b modulo m.

Q5. Suppose $a, b, k, m \in \mathbb{N}$ and $ak \equiv bk \pmod{m}$. Does this imply $a \equiv b \pmod{m}$? Why or why not?

Answer:

$ak \equiv bk$ $(\mathbf{mod}\ m)$

$m|(ak - bk)$

$ak - bk = sm$

$k(a - b) = sm$

$a - b = \frac{s}{k}m$ where $\frac{s}{k}$ is not necessarily an integer.

However, the definition states that the fraction $\frac{s}{k}$ must always be an integer. Thus, $ak \equiv bk$ $(\mathbf{mod}\ \mathrm{m}) \not\Rightarrow a \equiv b$ $(\mathbf{mod}\ \mathrm{m})$.