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Math 55, Handout 9.

BACKGROUND.

Q1. When is a set countably infinite?

Solution: A set is countably infinite if it has the same cardinality as $\mathbb{N}$, the set of natural numbers. In other words, there exists a bijection from this set to $\mathbb{N}$.

INDUCTION.

1.1. **Principle of mathematical induction:** To prove that $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ is a propositional function, we complete two steps:

Basis step: We verify that $P(1)$ is true.

Inductive step: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

1.2. This principle can be expressed as the following rule of inference:

$$(P(1) \wedge \forall k(P(k) \to P(k+1))) \to \forall n P(n)$$

1.3. The basis step does not have to be 1 (it could be any integer larger or smaller than 1).

Q2. Prove by induction that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for all $n \in \mathbb{N}$.
Solution:Let $P(n)$ be the proposition that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.
Basis Step: P(1) is true because $1 \cdot 1! = 1 = (1+1)! - 1$.
Inductive Step: Suppose P(k) holds for an arbitrary positive integer k. That is, we assume that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$$

Under this assumption, we need to show P(k+1) is true.

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)! = (k+2)(k+1)! - 1 = (k+2)! - 1,$$

where the first equality comes from our inductive hypothesis. So $P(k+1)$ holds, completing the inductive step.
By mathematical induction, $P(n)$ is true for all integers n with $n1$, as desired.

CAVEATS IN INDUCTIVE PROOFS.

Q3. Here is a supposed 'proof' that all whiteboard markers have the same color. Let $P(n)$ denote the proposition that all markers in any set of n markers have the same color. Clearly, $P(1)$ is true. Now assume $P(k)$ is true and take a set of $k+1$ markers. Then the first k markers have the same color by the inductive hypothesis, and so do the last k markets. So, all $k+1$ markers must be of the same color! What is wrong with this 'proof'?
Solution: the step $P(1) \to P(2)$ fails. Because the first one marker and the last one (the second) marker have no intersection so they could have different colors, say white and blue. They satisfy P(1) but P(2) fails. Therefore this "proof" is not valid.

STRONG INDUCTION.

2.1. **Principle of strong induction:** To prove that $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ is a propositional function, we complete two steps:

2.2. **Basis step:** We verify that the proposition $P(1)$ is true.

2.3. **Inductive step:** We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for all positive integers k.

Q4. Let $P(n)$ be the statement that a postage of n cents can be formed using only 3-cent stamps and 5-cent stamps. $P(n)$ can be proved by strong induction.

What is the correct basis step?

P(8), P(9), P(10) are true.

What is the inductive hypothesis?

$P(j)$ is true for all integers j with $8 \leq j \leq k$, where k is an integer with $k \geq 10$.

Complete the inductive step:

We are supposed to show that $P(k+1)$ is true. Since $P(k-2)$ is supposed to be true by our inductive hypothesis, we know that $k-2$ cents could be formed by some 3 and 5 cent stamps. So we could form k+1 cents by adding one more 3-cent stamp. So $P(k+1)$ holds, completing the inductive step.

WELL-ORDERING PROPERTY.

3.1. **The well-ordering property of $\mathbb{N}$:** Every non-empty subset of $\mathbb{N}$ contains a least element.

Q5. Does the well-ordering principle hold in $\mathbb{Z}$? $\underline{2\mathbb{N}}$? $\mathbb{Q}$? $\underline{3\mathbb{N}-2}$? $\mathbb{R}$? $[0,1]$?

Underline those sets where it holds (per their **natural order** $<$).

Solution: the well-ordering principle holds in $\underline{2\mathbb{N}}$, $\underline{3\mathbb{N}-2}$