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Math 55, Handout 23.

Q1. You are a boss of four employees: Ping, Quiggley, Ruiz, and Sitea. Ping is qualified to support hardware, networking, and wireless. Quiggley is qualified to support software and networking. Ruiz is qualified to support networking and wireless. Sitea is qualified to support hardware and software. Is there an assignment of employees so that each employee is assigned a unique area to support?

It is possible to create a bipartite graph with a matching for all employees in this mapping. In fact, there exists three such matchings that can work.
Ping-networking, Quiggley-software, Ruiz-wireless, Sitea-hardware
Ping-hardware, Sitea-software, Quiggley-network, Ruiz-wireless
Ping-wireless, Ruiz-networking, Quiggley-software, Sitea-hardware

PATHS IN GRAPHS.

1.1. A **path** of length n from a vertex u to a vertex v in an undirected graph $G = (V, E)$ is a sequence of n edges $e_1, \ldots, e_n$ of G such that there exists a sequence of $x_0 = u, x_1, x_2, \cdots, x_n = v$ of vertices such that e_i has for i from 1 to n the endpoints x_{i-1} and x_i

A path is a **circuit** if it begins and ends at begins and ends at the same vertex and has a non-zero length. The path or circuit is said to **pass through** the vertices $x_1, x_2, \cdots, x_{n-1}$ or **traverse** the edges $e_1, e_2, \cdots, e_n$. A path or circuit is **simple** if it does not contain the same edge more than once

1.2. All these notions generalize to directed graphs.

CONNECTEDNESS IN UNDIRECTED GRAPHS.

2.1. An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph.

2.2. An undirected graph that is not connected is called **disconnected**.

2.3. A **connected component** of a graph G is a connected subgraph of G.

2.4. A **cut edge** is an edge whose removal would produce more connected subgraphs than the original graph.

2.5. A **cut vertex** is a vertex whose removal would produce more connected subgraphs than the original graph.

Q2. (a) Explain why, in the collaboration graph of mathematicians, a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdős if and only if that mathematician has a finite Erdős number.

To have a finite Erdos number is to have a finite shortest path to Erdos. Given a finite path to Erdos, that would mean the mathematician is necessarily in the same connected component as Erdos.

(b) What does the Erdős number mean in that graph?

Erdos number in this graph is the shortest path from Erdos to the mathematician, each intermediary vertex being a different mathematician, and each edge representing a joint paper.

2.6. A **tree** is a connected graph not containing a simple circuit.

2.7. An undirected graph all whose connected components are trees is called a forest

Q3. If a forest has exactly 3 connected components, each with 100 vertices, how many edges does it have?

Given the number of edges on a tree is necessarily n-1, n being the number of vertices, each tree in the forest contains 99 edges.
Since there are 3 trees, that would make the total number of edges 297.

CONNECTEDNESS IN DIRECTED GRAPHS.

3.1. A directed graph is called **strongly connected** if there is a path from a to b and b to a given a b are vertices in the graph.

3.2. A directed graph is called **weakly connected** if there is a path between every two vertices in the underlying undirected graph.

Q4. Show that every strongly connected graph is weakly connected but not *vice versa*.

Given a strongly connected graph, namely that there is always a path from a to b and b to a, this would mean that in the underlying undirected graph, there is also always a path from a to b and b to a, given an undirected graph does not impose any additional restrictions on paths. As such, every strongly connected graph has a path between every two vertices in the underlying undirected graph, and so every strongly connected directed graph is also weakly connected.

The reverse is not true however, and this can be done with the simplest example.
A graph consisting of two vertices a b connected only by a single edge is weakly connected, as every two vertices (of which there are only two in this graph) have a path between them. Taking this nudirected graph and taking the case where the single edge in this graph is directed, it is not possible for this 2 vertex, single edge graph to be strongly connected. The underlying undirected graph however is the same as the undirected example from earlier in this paragraph. As such, the directed graph of two vertices with a directed edge between them is an example of a weakly connected graph that isn't strongly connected. By proof by example, it cannot be the case that every weakly connected graph is strongly connected.

EULERIAN PATHS AND CIRCUITS.

4.1. An **Eulerian circuit** in a graph G is a simple circuit containing every edge of G

4.2. An **Eulerian** path in a graph G is a simple path containing every edge of G

4.3. [Theorem] A connected multigraph with at least two vertices has an Eulerian circuit if and only if each of its vertices has an even degree

4.4. [Theorem] A connected multigraph with at least two vertices has an Eulerian path if and only if it has exactly two vertices of odd degree

Q5. For which values of n do the following graphs have an Eulerian circuit?

$$K_n : \{k \in \mathbb{Z}^+ ; n = 2k+1\}, \qquad C_n : n \geq 3, \qquad W_n : \emptyset, \qquad Q_n : \{k \in \mathbb{Z}^+ ; n = 2k\}$$