Name:	**GSI:**	**DISC #:**

Solutions to Homework 8.

Prob 1. Prove the **Multinomial Theorem**: If m, $n \in \mathbb{N}$, then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} \frac{n!}{n_1!n_2!\cdots n_m!}\, x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}.$$

Proof. When we expand the left-hand side, we will obtain m^n terms of the form $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ with $n_1 + n_2 + \cdots + n_m = n$, many of which will be repeated. When collecting similar terms, we will see the same monomial $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ whenever we pick x_1 from n_1 factors $(x_1 + x_2 + \cdots + x_m)$, x_2 from n_2 factors $(x_1 + x_2 + \cdots + x_m)$, etc., x_m from n_m factors $(x_1 + x_2 + \cdots + x_m)$. We now recall that the multinomial coefficient $\frac{n!}{n_1!n_2!\cdots n_m!}$ counts the number of permutations of m elements where each element j is repeated n_j times, $j = 1, \ldots, m$. Therefore, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ is exactly the multinomial coefficient $\frac{n!}{n_1!n_2!\cdots n_m!}$. This is true for each monomial in the right-hand side, so this completes the proof.

Remark. This is the no-work proof based on the meaning of multinomial coefficients, which we already established. An inductive proof and other, fancier proofs of this formula also exist.

Prob 2. Solve the problem posed by Chevalier de Méré to Blaise Pascal and Pierre de Fermat:

(a) Find the probability that a double six comes up at least once when a pair of fair dice is rolled 24 times. Is this probability greater than $1/2$?

Solution. The probability of getting a double six once is $1/36$, so the probability of not getting a double six once is $35/36$, so when we toss a pair of dice 24 times, the probability of not getting a double six even once is $(35/36)^{24}$, so the probability of getting a double six at least once is

$$1 - \left(\frac{35}{36}\right)^{24} \approx .4914 < \frac{1}{2}.$$

(b) Is it more likely that a six comes up at least once when a fair die is rolled four times or that a double six comes up at least once when a pair of fair dice is rolled 24 times?

Solution. Like in part (a), the probability that a six never comes up when a fair die is rolled four times is $(5/6)^4$, so the probability that a six comes up at least once is

$$1 - \frac{5}{6}^4 \approx .5177 > \frac{1}{2} > .4914.$$

Prob 3. A player in the Mega Millions lottery picks five different integers between 1 and 56, inclusive, and a sixth integer between 1 and 46, which may duplicate one of the earlier five integers. The player wins the jackpot of the first five numbers match the first five numbers drawn (irrespective of the order) and the sixth number matches the sixth number drawn.

(a) What is the probability a player wins the jackpot?

Solution. Only one combination of the 5 first numbers out of the $\binom{56}{5}$ numbers and only 1 number of the 46 numbers in position 6 wins the jackpot. So the probability of winning the jackpot is

$$\frac{1}{\binom{56}{5}} \cdot \frac{1}{46} \approx 0.000000005691.$$

(b) What is the probability a player wins \$250,000, the prize for matching only the first five numbers?

Solution. Still, there is one winning combination of the first 5 numbers and the sixth number must now be wrong, so 45 out of possible 46 choices exist. So this probability is 45 times greater than the previous one:

$$\frac{1}{\binom{56}{5}} \cdot \frac{45}{46} \approx 0.0000002561.$$

Prob 4. Suppose that p and q are primes and that $n = pq$. What is the probability that a randomly chosen positive integer less than n is not divisible by p or q?

Solution. Case 1: $p = q$. Then there are exactly $p - 1$ integers between 1 and $p^2 - 1$ that are divisible by p, namely p, $2p$, $\ldots$, $(p-1)p$. So the probability of randomly choosing a number that is not divisible by p is

$$1 - \frac{p-1}{p^2-1} = 1 - \frac{1}{p+1} = \frac{p}{p+1}.$$

Case 2: $p \neq q$. Then there are $p-1$ integers divisible by q between 1 and $pq-1$, namely q, $2q$, $\ldots$, $(p-1)q$. Likewise, there are $q-1$ numbers divisible by p between 1 and $pq-1$, namely p, $2p$, $\ldots$, $(q-1)p$. Notive that these sets of numbers do not overlap because the least common multiple of p and q is equal to pq, which is not reached. So the probability of randomly choosing a number which is not divisible by p or q is

$$1 - \frac{q-1}{pq-1} - \frac{p-1}{pq-1} = \frac{pq-q-p+1}{pq-1} = \frac{(p-1)(q-1)}{pq-1}.$$

Prob 5. What is the probability that a five-card poker hand contains cards of five different values but does not contain a flush or a straight?

Solution.

Step 1. We have to draw 5 cards of different ranks, which means 5 out of 13 choices for ranks times 4 possible choices of a suit for each card, so all together

$$|V| = 4^5 \binom{13}{5} \text{ choices.}$$

Here V denotes the set of all five-poker hands of five different values/ranks.

Step 2. Of those, we get a flush

$$|F| = 4\binom{13}{5} \text{ times}$$

because we still need to pick five different values of cards but now only one suit. Here F denotes the set of all flushes.

Step 3. We also get a straight

$$|S| = 10 \cdot 4^5 \text{ times}$$

because there are 10 possible highest values (A through 5) in a straight, and each card can be of any of the 4 suits. Here S denotes the set of all straights.

Step 4. There are $|S \cap F| = 40 = 4 \cdot 10$ straight flushes (determined by the highest value of the hand and the suit). Here $S \cap F$ denotes the set of all straight flushes.

Step 5. Finally, we must form the complement of the $S \cup F$ relative to the set V. That gives us

$$|V \setminus (S \cup F)| = |V| - |S \cup F| = |V| - (|F| + |S| - |F \cap S|) = |V| - |S| - |F| + |S \cap F|,$$

the latter by the inclusion-exclusion formula.

Plugging in all the earlier numbers, we obtain

$$|V \setminus (S \cup F)| = \binom{13}{5}4^5 - \binom{13}{5}4 - 10 \cdot 4^5 + 4 \cdot 10.$$

Step 6. We have a total of $|P| = \binom{52}{5}$ poker hands, so the probability that a five-card poker hand contains cards of five different values but does not contain a flush or a straight is

$$\frac{|V \setminus (S \cup F)|}{|P|} = \frac{\binom{13}{5}4^5 - \binom{13}{5}4 - 10 \cdot 4^5 + 4 \cdot 10}{\binom{52}{5}} = .5011.$$