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Math 55, Handout 21.

INCLUSION-EXCLUSION.

The Principle of Inclusion-Exclusion. Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A \cap A_j| + \ldots (-1)^{m-1} \sum_{i_1 < i_2 < \ldots m} |A_{i1} \cap A_{i2} \cap \ldots \cap A_{im}|$

Q1. Find the cardinality of $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and

(a) the sets are pairwise disjoint: $100+100+100=300$

(b) there are 50 common elements in each pair of sets and no elements in all three sets:

$100+100+100-50-50-50 = 150$

Q2. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, all are odd, all are divisible by 3, or all are divisible by 5.

A_1 = set of all odd numbers = $\binom{50}{4}$

A_2 = set of all multiples of 3 = $\binom{33}{4}$

A_3 = set of all multiples of 5 = $\binom{20}{4}$

S = the set of all 4 element sets consisting of #s from 1-100 = $\binom{100}{4}$

$$\frac{|A_1 \cup A_2 \cup A_3|}{|S|} = \frac{\binom{50}{4} + \binom{33}{4} + \binom{20}{4} - \binom{17}{4} - \binom{10}{4} + 0}{\binom{100}{4}} = \boxed{0.07}$$

Alternative Form of Inclusion-Exclusion.

Let U be a universal finite set of cardinality N, and let A_i denote its subset of elements satisfying

Let $N(A_{i1}, A_{i1} \cdots A_{ik})$ denote the number of elements of U satisfying properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$.

Let $N(A_{i1}', A_{i1}' \cdots A_{ik}')$ denote the number of elements of U satisfying none of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$.

Then $N(P_1' \ldots P_n') = N - \sum_i N(A_i) + \sum_{i<j} N(A_i A_j) - \ldots + (-1)^n N(A_1, A_2 \ldots A_n)$

Q3. How many primes are there not exceeding 120?

$\lfloor \sqrt{120} \rfloor = 10$

P_1 = divisible by 2

P_2 = divisible by 3

P_3 = divisible by 5

P_4 = divisible by 7

$$= 120 - \left\lfloor \frac{120}{2} \right\rfloor - \left\lfloor \frac{120}{3} \right\rfloor - \left\lfloor \frac{120}{5} \right\rfloor - \left\lfloor \frac{120}{7} \right\rfloor + \left\lfloor \frac{120}{6} \right\rfloor + \left\lfloor \frac{120}{10} \right\rfloor + \left\lfloor \frac{120}{14} \right\rfloor + \left\lfloor \frac{120}{15} \right\rfloor + \left\lfloor \frac{120}{21} \right\rfloor$$

$$= \left\lfloor \frac{120}{35} \right\rfloor - \left\lfloor \frac{120}{30} \right\rfloor - \left\lfloor \frac{120}{42} \right\rfloor - \left\lfloor \frac{120}{70} \right\rfloor - \left\lfloor \frac{120}{105} \right\rfloor + \left\lfloor \frac{120}{490} \right\rfloor$$

$$= \boxed{30}$$

Q4. How many onto functions are there from a set with 5 elements to a set with 3 elements?

$n = 3$

$m = 5$

$$3^5 - \binom{3}{1}(2^5) + \binom{3}{2}(1^5) = \boxed{150}$$

Theorem. Let n, $m \in \mathbb{N}$ with $m \geq n$. Then there are n^m

onto functions from a set with m element to a set with n elements.

DERANGEMENTS.

A **derangement** is.a permutation of objects that leave no object in place

Q5. What is the probability that none of the 10 employees receives the correct hat if a hatcheck person hands their hats back randomly?

$$D_{10} = 10!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \ldots 1/10!\right) = 1334961$$

$$\frac{1334961}{10!} = \frac{1334961}{3628800} = 0.37$$

The number of derangements. The number of derangements of a set with n elements is

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}\right)$$