Name:	GSI:	DISC #:

Solutions to Homework 10.

Prob 1. Suppose that the number of cans of soda pop filled in a day at a bottling plant is a random variable with an expected value of $10,000$ and a variance of $1,000$.

(a) Use Markov's inequality to obtain an upper bound on the probability that the plant will fill more than $11,000$ cans on a particular day.

Solution. Using this inequality with $a = 11000$, we get

$$p(X \geq a) \leq E(X)/a = 10/11.$$

(b) Use Chebyshev's inequality to obtain a lower bound on the probability that the plant will fill between $9,000$ and $11,000$ cans on a particular day.

Solution. By Chebyshev's inequality, we get

$$p\left(|X - 10000| \geq 1000\right) \leq \frac{1000}{1000^2} = \frac{1}{1000}.$$

So, the complementary probability is

$$p\left(|X - 10000| < 1000\right) \geq 1 - \frac{1}{1000} = .999$$

Prob 2. This problem has been posted on Midterm 2 in 2017; see its solution there.

Prob 3. (a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.

Solution. Denote the sought number of specified ternary strings by A_n. Note that exactly a third of such strings start with each of the three different symbols. Consider any such string of length n. If its first symbol is followed by the same one, then the remaining symbols can be filled in any of the possible 3^{n-2} ways. Otherwise its first symbol is followed by a different one, which can be done in 2/3 times A_{n-1} ways because then the remaining string of $n-1$ must contain two consecutive symbols that are the same. This gives

$$A_n = 3\left(\frac{2A_{n-1}}{3} + 3^{n-2}\right) = 2A_{n-1} + 3^{n-1}.$$

(b) What are the initial conditions?

Solution. $A_0 = A_1 = 0$.

(c) How many ternary strings of length seven contain consecutive symbols that are the same?

Solution. We have

$$\begin{array}{rcccl} A_2 & = & 2 \cdot A_1 + 3 & = & 3 \\ A_3 & = & 2 \cdot A_2 + 3^2 & = & 15 \\ A_4 & = & 2 \cdot A_3 + 3^3 & = & 57 \\ A_5 & = & 2 \cdot A_4 + 3^4 & = & 15 \\ A_6 & = & 2 \cdot A_5 + 3^5 & = & 195 \\ A_7 & = & 2 \cdot A_6 + 3^6 & = & 1995. \end{array}$$

Prob 4. Let $\{a_n\}$ be a sequence of real numbers. The **backward differences** of this sequence are defined recursively as follows: The **first difference** ∇a_n is defined as

$$\nabla a_n = a_n - a_{n-1}.$$

The $(k+1)$**st difference** $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}.$$

(a) Prove that a_{n-k} can be expressed in terms of a_n, ∇a_n, …, $\nabla^k a_n$.

Proof. We will prove the binomial formula for backward differences:

$$a_{n-k} = \sum_{j=0}^{k} \binom{k}{j} (-\nabla)^j a_n \qquad \text{for all } n \in \mathbb{Z}, \quad k \in \mathbb{Z}_+. \tag{1}$$

The proof will be by induction on k, and we will think of n as fixed.

Induction base: the case $k = 0$ is vacuously true.

Induction step. Assume (1) holds for some k, and prove it for $k + 1$. We have

$$\begin{aligned}
a_{n-k-1} &= a_{n-k} - \nabla a_{n-k} \\
&= \sum_{j=0}^{k} \binom{k}{j} (-\nabla)^j a_n + \sum_{j=0}^{k} \binom{k}{j} (-\nabla)^{j+1} a_n \\
&= \sum_{j=0}^{k} \binom{k}{j} (-\nabla)^j a_n + \sum_{j=1}^{k+1} \binom{k}{j-1} (-\nabla)^j a_n \\
&= a_n + \sum_{j=1}^{k} \left\{ \binom{k}{j} + \binom{k}{j-1} \right\} (-\nabla)^j a_n + (-\nabla)^{k+1} a_n \\
&= a_n + \sum_{j=1}^{k} \binom{k+1}{j} (-\nabla)^j a_n + (-\nabla)^{k+1} a_n \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} (-\nabla)^j a_n.
\end{aligned}$$

(Here the transition from line 4 to line 5 uses Pascal's identity.) That's exactly (1) with k replaced by $k+1$.

(b) Show that any recurrence relation for the sequence $\{a_n\}$ can be written in terms of a_n, ∇a_n, $\nabla^2 a_n, \ldots$. The resulting equation is called a **difference equation**.

Proof. By the result of part (a), each of the terms a_{n-k} in a recurrence is a linear combination of differences a_n, ∇a_n, $\nabla^2 a_n, \ldots$. Hence any recurrence relation for $\{a_n\}$ can be rewritten as a difference equation.

Prob 5. Solve the simultaneous recurrence relations

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1} \\ b_n &= a_{n-1} + 2b_{n-1} \end{aligned}$$

with initial values $a_0 = 1$ and $b_0 = 2$.

Solution. Shifting the index in the second equation by 1 and plugging the resulting expression into the first equation, we get

$$a_n = 3a_{n-1} + 2(a_{n-2} + 2b_{n-2}).$$

But $2b_{n-2} = a_{n-1} - 3a_{n-2}$ from the first equation. Plugging that in, we get

$$a_n = 3a_{n-1} + 2(a_{n-2} + a_{n-1} - 3a_{n-2}) = 5a_{n-1} - 4a_{n-2}.$$

The characteristic equation corresponding to this recurrence is

$$r^2 = 5r - 4,$$

which has roots 1 and 4. So $a_n = C_1 + C_2 \cdot 4^n$, and plugging this into $2b_n = a_{n+1} - 3a_n$ gives $b_n = -C_1 + \frac{1}{2}C_2 4^n$. Now use the initial values $a_0 = 1$ and $b_0 = 2$ to obtain

$$\begin{aligned} C_1 + C_2 &= 1 \\ -C_1 + \frac{1}{2}C_2 &= 2, \end{aligned}$$

which implies $C_1 = -1$, $C_2 = 2$.

Answer: $a_n = -1 + 2 \cdot 4^n$, $b_n = 1 + 4^n$.