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Math 55, Homework 10.

Prob 1. Suppose that the number of cans of soda pop filled in a day at a bottling plant is a random variable with an expected value of 10,000 and a variance of 1,000.

(a) Use Markov's inequality to obtain an upper bound on the probability that the plant will fill more than 11,000 cans on a particular day.

$E(X) = 10{,}000$
$V(X) = 1{,}000$
Markov's inequality with $a = 11000$

$$P(X(s) \geq 11{,}000 \leq \frac{E(x)}{11{,}000} = \frac{10{,}000}{11{,}000} = \frac{10}{11} \approx \boxed{0.9091}$$

(b) Use Chebyshev's inequality to obtain a lower bound on the probability that the plant will fill between 9,000 and 11,000 cans on a particular day.

Chebyshev's inequality with $r = 1000$

$$\begin{aligned}
P(9000 < X < 11000) &= P(-1000 < X - 10{,}000 < 1000) \\
&= P(-100 < X - E(x) < 1000) \\
&= P(|X(s) - E(x)| < 1000) \\
&= 1 - P(|X(s) - E(x)| \geq 1000) \\
&\geq 1 - \frac{V(x)}{1000^2} \\
&= 1 - \frac{1000}{1000^2} \\
&= 1 - \frac{1}{1000} \\
&= \frac{999}{1000} \\
&= \boxed{0.999}
\end{aligned}$$

Prob 2. You have a bag with 5 marbles. 4 are black and 1 is red. You draw marbles without replacement and stop once you get the red marble.

a) What is the expected number of draws?

$E(x) = \sum_{s \in S} p(s)\, x(s)$

$X(s) = \text{tries}$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1/5 & 4/5 \cdot \frac{1}{4} & 1/5 & 1/5 & 1/5 \\ & = 1/5 & & & \end{array}$$

$\left(\frac{1}{5} \cdot 1\right) + \left(\frac{1}{5} \cdot 2\right) + \left(\frac{1}{5} \cdot 3\right) + \left(\frac{1}{5} \cdot 4\right) + \left(\frac{1}{5} \cdot 5\right)$

$= \frac{15}{5} = \boxed{3}$

b) What is the variance in the number of draws?

$V(X) = E\left((X - E(X))^2\right)$

$= E\left((X-3)^2\right)$

$= \frac{2}{5} \cdot 4 + \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 0$

$= \boxed{2}$

Prob 3. (a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.

a_n = ternary strings of length n that contain 0's or 1's

First case:

- starts w/ 0 & does not start with two consecutive symbols, then the remaining ternary string length n-1 can start with 1 or 2
- there are a_{n-1} strings w/o a pair of consecutive integer & two thirds of these strings will start with a 1 or 2. $\Rightarrow \underline{2/3 \cdot a_{n-1} \text{ strings}}$

Second case:

- string starts with 1 & does not start with two consecutive symbols, the remaining ternary string length n-1 can start with 0 or 2.
- there are a_{n-1} strings without a pair of consecutive integer & 2/3 will start with 0 or 2 $\Rightarrow \underline{2/3 a_{n-1} \text{ strings}}$

Third case

- string starts with 2 & does not start with two consecutive symbols, then the remaining ternary string length n-1 can start with 0 or 1.
- there are a_{n-1} strings w/o a pair of consecutive integer & 2/3 will start with 0 or 1 $\Longrightarrow 2/3 \cdot a_{n-1}$ strings

Fourth case:

- starts with 00 then the remaining ternary string length n-2 can be any string
- there are 3^{n-2} strings of length n-2

Fifth case:

- starts with 11 then the remaining ternary string length n-2 can be any string.
- there are 3^{n-2} strings of length n-2

Sixth case:

- starts with 22 then the remaining ternary string length n-2 can be any string
- there are 3^{n-2} strings of length n-2

Add all number sequences together to get

$\frac{2}{3}a_{n-1} + \frac{2}{3}a_{n-1} + \frac{2}{3}a_{n-1} + 3^{n-2} + 3^{n-2} + 3^{n-2}$

$= \boxed{2a_{n-1} + 3^{n-1}}$

(b) What are the initial conditions?

- when n=0, there exactly 0 ternary strings with a pair of consecutive integers: $a_0 = 0$
- when n=1, there is exactly 0 ternary strings with a pair of consecutive ints: $a_1 = 0$
- when n=2, there are exactly 3 ternary strings with a pair ~~of~~ of consecutive integers (00, 11 22). $a_2 = 2a_1 + 3^{2-1} = 3 \longrightarrow$ the recurrence relation holds for n=2 & thus n=2 does not need to be the inital condition

(c) How many ternary strings of length seven contain consecutive symbols that are the same?

$a_0 = 0$

$a_1 = 0$

$a_n = 2a_{n-1} + 3^{n-1}$

$a_0 = 0$

$a_1 = 0$

$a_2 = 2a_{2-1} + 3^{2-1} = 3$

$a_3 = 2a_{3-1} + 3^{3-1} = 15$

$a_4 = 2a_{4-1} + 3^{4-1} = 57$

$a_5 = 2a_{5-1} + 3^{5-1} = 195$

$a_6 = 2a_{6-1} + 3^{6-1} = \boxed{633}$

Prob 4. Let $\{a_n\}$ be a sequence of real numbers. The **backward differences** of this sequence are defined recursively as follows: The **first difference** ∇a_n is defined as

$$\nabla a_n = a_n - a_{n-1}.$$

The $(k+1)$**st difference** $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}.$$

(a) Prove that a_{n-k} can be expressed in terms of a_n, ∇a_n, ..., $\nabla^k a_n$.

$\nabla^1 a_n = a_n - a_{n-1} \longrightarrow \nabla^{k+1} \quad a_n = \nabla^k a_n - \nabla^k a_{n-1}$

$\nabla^2 a_n = \nabla^1 a_n - \nabla^1 a_{n-1}$

$= a_n - a_{n-1} - (a_{n-1} - a_{n-2})$

$= a_n - 2a_{n-1} + a_{n-2}$

$\nabla^3 a_n = \nabla^2 a_n - \nabla^2 a_{n-1}$

$= (a_n - 2a_{n-1} + a_{n-2}) - (a_{n-1} - 2a_{n-2} + a_{n-3})$

$= a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3}$

Guess (lemma): $\nabla^k a_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^i a_{n-1}$

Base case: $k=1$

$\nabla^1 a_n = a_n - a_{n-1}$

$= \binom{1}{0}(-1)^0 a_{n-0} + \binom{1}{1}(-1)^1 a_{n-1}$

Induction step: $k+1$

$\nabla^k a_n - \nabla^k a_{n-1}$

$= \sum_{i=0}^{k} \binom{k}{i}(-1)^i a_{n-1} - \sum_{i=0}^{k} \binom{k}{i}(-1)^i a_{n-1-i}$

$= \sum_{i=0}^{k} \binom{k}{i}(-1)^i a_{n-i} - \sum_{i'=1}^{k+1} \binom{k}{i'-1}(-1)^{i'-1} a_{n-i'}$

$= \binom{k}{0}(-1)^0 a_{n-0} - \binom{k}{k}(-1)^k a_{n-k-1} + \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1}\right)(-1) a_{n-i}$

$= \binom{k+1}{0}(-1)^0 a_{n-0} + \binom{k+1}{k+1}(-1)^{k+1} a_{n-k-1} + \sum_{i=1}^{k} \binom{k+1}{i}(-1)^i a_{n-i}$

$= \sum_{i=0}^{k+1} \binom{k+1}{i}(-1)^i a_{n-i}$

(b) Show that any recurrence relation for the sequence $\{a_n\}$ can be written in terms of a_n, ∇a_n, $\nabla^2 a_n$, ... The resulting equation is called a **difference equation**.

$$a_{n+1} = \sum_{i=0}^{n} c_i a_{n-1}$$

Prob 5. Solve the simultaneous recurrence relations

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1} \\ b_n &= a_{n-1} + 2b_{n-1} \end{aligned}$$

with initial values $a_0 = 1$ and $b_0 = 2$.

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} b_{n-1} x^n$$

$$\sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} b_{n-1} x^n$$

$$f(x) - a_0 = 3xf(x) + 2xg(x)$$
$$g(x) - b_0 = xf(x) + 2xg(x)$$

so $f(x) = \dfrac{2xg(x)+1}{1-3x}$ and $g(x) = \dfrac{xf(x)+2}{1-2x}$

Isolate $f(x)$

$$f(x) = \frac{2x\frac{xf(x)+2}{1-2x} + 1}{1-3x}$$

$$= \frac{2x^2 f(x) + 2x + 1}{(1-2x)(1-30)}$$

so $f(x) = \dfrac{2x+1}{4x^2-5x+1} = \dfrac{2x+1}{(1-x)(1-4x)}$

likewise, $g(x) = x\dfrac{2x+1}{(1-x)(1-4x)} + 2$

$$= \frac{2-5x}{(1-x)(1-4x)}$$

sum of $\dfrac{A}{1-x} + \dfrac{B}{1-4x}$

$$f(x) = \frac{2}{1-4x} - \frac{1}{1-x}$$

$$g(x) = \frac{1}{1-4x} + \frac{1}{1-x}$$

$$f(x) = \frac{2}{1-4x} - \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} (4x)^n - \sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2 \cdot 4^n - 1) x^n$$

$$\boxed{a_n = 2 \cdot 4^n - 1}$$

like $g(x) = \dfrac{1}{1-4x} + \dfrac{1}{1-x}$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (0.4^n + 1) x^n$$

$$\boxed{b_n = 4^n + 1}$$