Name:	**GSI:**	**DISC #:**

Solutions to Homework 4.

Prob 1. Let f be a function from A to B, and let S and T be subsets of B. Show that

$$(a) \qquad f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$$

Proof.

$$\begin{gathered} x \in f^{-1}(S \cup T) \\ \Updownarrow \\ f(x) \in S \cup T \\ \Updownarrow \\ f(x) \in S \ \bigvee \ f(x) \in T \\ \Updownarrow \\ x \in f^{-1}(S) \ \bigvee \ x \in f^{-1}(T) \\ \Updownarrow \\ x \in f^{-1}(S) \cup f^{-1}(T). \end{gathered}$$

$$(b) \qquad f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T).$$

Proof.

$$\begin{gathered} x \in f^{-1}(S \cap T) \\ \Updownarrow \\ f(x) \in S \cap T \\ \Updownarrow \\ f(x) \in S \ \bigwedge \ f(x) \in T \\ \Updownarrow \\ x \in f^{-1}(S) \ \bigwedge \ x \in f^{-1}(T) \\ \Updownarrow \\ x \in f^{-1}(S) \cap f^{-1}(T). \end{gathered}$$

Prob 2. Prove that a set S is infinite if and only if there is a proper subset A of S and a bijection between A and S.

Proof. Suppose S is finite and A is its proper subset. Then $|A|$ is a nonnegative integer, say, m, and likewise, $|S| = n$, and since $A \subset S$, $m < n$. For any function $f : A \to S$, the number of elements in the image of the set cannot exceed the number of elements in the original set, so $|f(A)| \leq |A|$. Now, if f is a bijection, then f is onto S, and hence $f(A) = S$. But then $n = |S| = |f(A)| \leq |A| = m$, in contradiction with $m < n$. So no bijection is possible between S and its proper subset A if S is finite. The contrapositive to this statement is one direction we needed to prove.

Now let S be infinite. By the Lemma from the solution to Problem 3, there is an injective map $g : \mathbb{N} \to S$. Let $A = S \setminus \{g(1)\}$. Then $A \subset S$. Define a map $f : S \to A$ by

$$f : s \mapsto \begin{cases} g(g^{-1}(s) + 1) & \text{if } \ s \in g(\mathbb{N}) \\ s & \text{otherwise} \end{cases}$$

This map sends all elements outside the range of g back to themselves, and it sends each element $g(j)$ to the next element $g(j+1)$ for any $j \in \mathbb{N}$. Hence f is a bijection between S and A, which finishes the proof.

Prob 3. Prove that there is no infinite set whose cardinality is smaller than $\aleph_0 = |\mathbb{N}|$.

Proof. Let us prove a Lemma first.

Lemma. Let S be an infinite set. Then there exists a map $g : \mathbb{N} \to S$ which is one-to-one.

Proof. The set S is nonempty (being infinite), so it contains an element s_1. The set S is infinite, so the set $S \setminus \{s_1\}$ is non-empty, hence contains some element s_2, and so on: an element s_j can be chosen from the necessarily non-empty set $S \setminus \{s_1, \ldots, s_{j-1}\}$ for any $j \in \mathbb{N}$. Now simply define $g : j \mapsto s_j$ for any $j \in \mathbb{N}$. By construction, this map is one-to-one. This finishes the proof of the Lemma.

NB: This proof requires the so-called Axiom of Choice: given any nonempty set, we can always choose an element of that set.

Back to main proof. By the Lemma we just proved, given an infinite set S, there exists an injection $g : \mathbb{N} \to S$. Hence by the definition of cardinality, $|\mathbb{N}| \leq |S|$. This finishes the proof.

Prob 4. Prove that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Proof. It was proved in class that $|[0,1]| = |\mathbb{R}|$, so if we establish that $|\mathcal{P}(N)| = |[0,1]|$, we will be done.

We first construct an injection $f : [0,1] \to \mathcal{P}(\mathbb{N})$. Any $x \in [0,1]$ has a binary expansion (if x has two different expansions, just choose one, does not matter which one). The binary string listing the digits of the binary expansion of x encodes some subset of $\mathbb{N}$ by specifying which elements of $\mathbb{N}$ should be out (digit 0) or in (digit 1). Let $f(x)$ be that subset of $\mathbb{N}$. Different numbers in $[0,1]$ have different binary expansions, hence f is injective. This establishes that $|[0,1]| \leq |\mathcal{P}(\mathbb{N})|$.

Now construct an injection $g : \mathcal{P}(\mathbb{R}) \to [0,1]$. Given $S \subseteq \mathbb{N}$, consider the binary string $d_1 d_2 d_3 \ldots$ that corresponds to it as in the paragraph above. Define $g : \mathcal{P}(\mathbb{N}) \to [0,1]$ by

$$g(S) = \frac{d_1}{3} + \frac{d_2}{3^2} + \frac{d_3}{3^3} + \cdots = \sum_{n=1}^{\infty} \frac{d_n}{3^n}.$$

Notice that any such $g(S)$ is between 0 and 1, i.e., indeed lands in $[0,1]$. Finally, no two subsets of $\mathbb{N}$ can be mapped by g to the same number in $[0,1]$ since the only ambiguity in base-3 expansions could arise from expansions containing digit 2 – but we do not have that digit among the d_n's (which are 0s and 1s).

Thus, $g : \mathcal{P}(\mathbb{N}) \to [0,1]$ is injective, proving $|\mathcal{P}(\mathbb{N})| \leq |[0,1]|$.

Now invoke the Schröder-Bernstein theorem to conclude $|\mathcal{P}(\mathbb{N})| = |[0,1]| = |\mathbb{R}|$.

Prob 5. Determine if the following sets are countable or uncountable. For those that are countably infinite, provide a bijection between that set and the set $\mathbb{N}$:

(a) the set S_1 all integers not divisible by 4:

Solution: S_1 is countably infinite and $f : \mathbb{N} \to S_1$ can be defined as follows. Any positive integer can be written uniquely in the form $6n + m$ for some $n \in \mathbb{Z}_+$, $m \in \{0, \ldots, 6\}$. Now define

$$f(6n+m) = \begin{cases} 4n+m & \text{if } m \leq 3 \\ -4(n+1)+m-3 & \text{if } m > 3. \end{cases}$$

This map is a bijection as can be seem from direct inversion. Indeed, any number in the set S_1 can be uniquely written as $4k + \ell$, with $k \in \mathbb{Z}$ and $\ell \in \{1, 2, 3\}$. Then the formula

$$f^{-1}(4k+\ell) = \begin{cases} 6k+\ell & \text{if } k \geq 0 \\ 6|k+1|+\ell+3 & \text{if } k < 0 \end{cases}$$

provides the inverse function for f.

(b) all bit strings not containing the bit 0;

Solution: This set is countably infinite and a bijection with $\mathbb{N}$ is rather straightforward:

Bit strings not containing the bit 0 contain only bit 1. There is one infinite string consisting of 1's; all others are finite of some finite length $n \in \mathbb{N}$. Map the number 1 to the infinite string consisting of all 1's. Map any other number $n \in \mathbb{N} \setminus \{1\}$ to the string of 1's of length $n - 1$. (Incidentally, if you want to include the empty string, which has length 0, you can send $n \in \mathbb{N} \setminus \{1\}$ to the string of 1's of length $n - 2$.) Different numbers get mapped to strings of different lengths, and all lengths occur, so this is a bijection.

(c) the set S_2 of all real numbers containing only a finite number of 1s in their decimal representation.

Solution: This set is uncountable.

In fact even its subset T_2 of all real numbers in the interval $[0, 1]$ *with* 0*s and* 2*s only* in their decimal representations is uncountable. Indeed, we can construct an injection from $\mathcal{P}(\mathbb{N})$ to T_2 by taking a binary string $d_1 d_2 d_3 \ldots$ that encodes a subset $S \subseteq \mathbb{N}$ as discussed in our solution to Problem 4 and mapping S to

$$f(S) = \sum_{n=1}^{\infty} \frac{2d_n}{10^n}.$$

Then $f(S)$ is between 0 and 1 and has only 0s and 2s (digits $2d_n$) in its decimal representation, i.e., $f(S) \in T_2$. No two bit strings are mapped to the same number since we avoid numbers with 9s in their decimal representation. Thus $f : \mathcal{P}(\mathbb{N}) \to T_2$ is an injection, hence

$$|\mathcal{P}(\mathbb{N})| \leq |T_2| \leq |S_2|.$$

By the result of Problem 4, $\mathcal{P}(\mathbb{N})$ is uncountable, and hence so is S_2.