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Math 55, Handout 19.

LINEAR HOMOGENEOUS RECURRENCES WITH CONSTANT COEFFICIENTS.

	RECURRENCES	ODEs
Equation	$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$	$y^{(k)} =$ $C_1 y^{k-1} + \ldots + C_k y^{k_1}$
Solution Ansatz	$a_n =$ $a_1 r_1^n + a_2 r_2^n$	$y(x) = e^{rx}$
Ansatz plugged in	$c_1 a_{n-1} + c_2 a_{n-2}$	$r^k e^{rx} = c_1 r^{k-1} e^{rx} + c_2 r^{k-2} e^{rx} + \cdots + c_k e^{rx}$
Char. polynomial	$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$	$c_1 r^{k-1} + \ldots + c_k r$
Linearity	r_1, r_2 solutions $\Rightarrow c_1 r_1 + c_2 r_2$ solution	y_1, y_2 solutions $\Longrightarrow C_1 y_1 + C_2 y_2$ solution
Distinct real roots	$a_n = C_1 r_1^n + C_2 r_2^n + \cdots + C_k r_k^n$	$r^{k-1} e^{rx} \ldots e^{rx}$
Initial conditions	met by solving for C_1 through C_k	met by solving for C_1 through C_k
Complex roots	$a_n = r^n$ still a solution for $r \in \mathbb{C}$	$a_n = r^n$ still a solution for $r \in C$
Multiple roots	$a_1 r_1, \cdots, a_{n-1} r_{n-1}$	$e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are solutions

NB: The uppercase letters C_j and the lowercase letters c_j here are not to be confused!

Q1. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

start from the left. 2×2 piece, then there are $2 \times (n-2)$ ways. If instead we start with a 1×2 piece, then there are $2 \times (n-1)$ ways.

Q2. What is the general form of the solution of a linear homogeneous recurrence relation if its characteristic polynomial has precisely these roots: $1, 1, 1, 1, -2, -2, -2, 3, 3, -4$?

characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ has t distinct roots $r_1, \ldots, r_t$ with multiplicities $m_1 \ldots m_t$ for $i = 1, \ldots t$ and $m_1 + \ldots + m_t = k$. then, the seuence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$ if and only if

$$a_n = (a_{1,0} + a_{1,1} n + \ldots + a_{1,m_1-1} n^{m_1-1} + \ldots + (a_{t,0} + a_{t,1} n + \ldots + a_{1,m_1-1} n^{m_1-1}) r_t^n.$$

LINEAR INHOMOGENEOUS RECURRENCES WITH CONSTANT COEFFICIENTS.

	RECURRENCES	ODEs
Equation	$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n)$	$y^{(k)} = C_1 y(k-1) \ldots C_k y = F(x)$
Linearity yields	$a_n = a_n^{hom} + a_n^{part}$	$y = y_{hom} + y_{part}$
Special cases	$F(n) = p_t(n)s^n$	$y(x) = p_t(x)s^x$
s not char. root	$a^{part}(n) = q_t(n)e^{sn}$	$y^{part}(x) = q_t(x)e^{sx}$
s root; mult. m	$a^{part}(n) = n^m q_t(n)e^{sn}$	$y^{part}(x) = x^m q_t(x)e^{sx}$

Q3. What is the general form of the particular solution – guaranteed to exist by the above results – of the linear inhomogeneneous recurrence $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

(a) $F(n) = n^2$?

$f(n) = (b_t n^t + \cdots + b_1 n + b_0)s^n$, $s=1$, $t=2$

solution: $p_2 \cdot n^2 + p_1 \cdot n + p_0$

(b) $F(n) = 2$?

$f(n) = (b_t n^t + \cdots + b_1 n + b_0)s^n$, $s=1$, $t=0$

solution: p_0

(c) $F(n) = n^4 2^n$?

$f(n) = b_t n^t + \ldots + b_1 n + b_0)s^n$, $s=2$, $t=4$

solution: $n^2(p_4 \cdot n^4 + p_3 \cdot n^3 + p_2 \cdot n^2 + p_1 \cdot n + p_0)(2)^n$

Q4. Find all solutions to the recurrence $a_{n+2} = -a_n + 5 \cdot 2^n$ subject to the initial conditions $a_0 = 2$, $a_1 = 3$.

The associated differential equation has the form $y'' + y = 5 \cdot 2^n$

The auxilirary equation is $r^2+1=0$ so $r = \pm i$. So the complementary function: $a_n = C_1 \cos(\frac{n\pi}{2}) + C_2 \sin(\frac{n\pi}{2})$

The particular solution: $a_p = 5 \cdot \frac{2^n}{2^2+1} = 2^n$

so the general solution: $a_n = C_1 \cos(\frac{n\pi}{2}) + C_2 \sin(\frac{n\pi}{2}) + 2^n$

Apply the inital conditions $a_0 = 2, a_1 = 3 \longrightarrow C_1 + 0 + 1 = 2$, $0 + C_2 + 2 = 3$, $C_1 = C_2 = 1$

solution: $a_n = \cos(\frac{n\pi}{2}) + \sin(\frac{n\pi}{2}) + 2^n$