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Math 55, Handout ~~10~~ 9.

BACKGROUND.

Q1. When is a set countably infinite?
A set is countably infinite if its elements can be put in one-to-one correspondence with the set of natural numbers. In other words, one can count off all elements in the set in such a way that even though the counting will take forever, you will get to any particular element in a finite amount of time

INDUCTION.

1.1. **Principle of mathematical induction:** To prove that $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ is a propositional function, we complete two steps:

Basis step: we verify $P(1)$ is true

Inductive step: we show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k

1.2. This principle can be expressed as the following rule of inference:
$(P(1) \wedge \forall k\, (P(k) \longrightarrow P(k+1))) \to \forall_n P(n)$ where the domain is the set of positive integers

1.3. The basis step does not have to start at one (i.e. $P(1)$) it can be negative, zero, or positive

Q2. Prove by induction that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

$(n+1)(n+1)! = (n+2-1)(n+1)!$
$= (n+2)(n+1)! - (n+1)!$
$= (n+2)! - (n+1)! \quad \forall n \in \mathbb{N}$
therefore $P(n)$ implies
$1 \cdot 1! + \cdots + n \cdot n! + (n+1)(n+1)!$
$= (n+1)! - 1 + (n+2)! - (n+1)!$
$= (n+2)! - 1$
from this we see $\forall_n (P(n) \to P(n+1))$

CAVEATS IN INDUCTIVE PROOFS.

Q3. Here is a supposed 'proof' that all whiteboard markers have the same color. Let $P(n)$ denote the proposition that all markers in any set of n markers have the same color. Clearly, $P(1)$ is true. Now assume $P(k)$ is true and take a set of $k+1$ markers. Then the first k markers have the same color by the inductive hypothesis, and so do the last k markets. So, all $k+1$ markers must be of the same color! What is wrong with this 'proof'?

The issue is the transition from 1 to 2.
P(1) is true, but P(2) is not

$n \longrightarrow n+1$ overlap   no overlap

STRONG INDUCTION.

2.1. **Principle of strong induction:** To prove that $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ is a propositional function, we complete two steps:

2.2. **Basis step:** we verify that the proposition P(1) is true

2.3. **Inductive step:** we show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \longrightarrow P(k+1)$ is true for all positive integers k

Q4. Let $P(n)$ be the statement that a postage of n cents can be formed using only 3-cent stamps and 5-cent stamps. $P(n)$ can be proved by strong induction.

What is the correct basis step?
P(8): one 3-cent & one 5-cent stamp ✓
P(9): three 3-cent stamps ✓
P(10): two 5-cent stamps ✓

What is the inductive hypothesis?
P(n) is true for $8 \leq n \leq k$, where $k \geq 10$

Complete the inductive step:
If $k \geq 10$, then $k+1 = (k-2)+3$ since $k-2 \geq 8$, by the induction hypothesis we have that $P(k-2)$ is true. In other words a postage of k-2 cents can be paid by using 3-cent and 5-cents stamps. By adding one 3-cent stamp, we can pay a postage of k+1 cents (i.e. P(k+1) is true)

WELL-ORDERING PROPERTY.

3.1. **The well-ordering property of $\mathbb{N}$:** Every non-empty subset of $\mathbb{N}$ contains
a least element

Q5. Does the well-ordering principle hold in $\mathbb{Z}$? $\underline{2\mathbb{N}}$? $\mathbb{Q}$? $\underline{3\mathbb{N}-2}$? $\mathbb{R}$? $[0,1]$?

Underline those sets where it holds (per their **natural order** $<$).