Name:	**GSI:**	**DISC #:**

Solutions to Homework 3.

Prob 1. Prove that there are no integer solutions to the equation

$$2x^2 + 5y^2 = 14.$$

Proof. Any real pair (x, y) satisfying this equation also satisfies $x^2 \geq 0$, $y^2 \geq 0$, hence $2x^2 = 14 - 5y^2 \leq 14$ and $5y^2 = 14 - 5y^2 \leq 14$. These inequalities imply

$$|x| \leq \sqrt{7}, \qquad |y| \leq \sqrt{14/5}.$$

The only integers y satisfying the second inequality above are $y = 0, \pm 1$. But setting $y = 0$ leads to the equation $x^2 = 7$ whose solutions $\pm\sqrt{7}$ are not integer, and setting $y = \pm 1$ leads to the equation $2x^2 = 9$ whose solutions $\pm 3/\sqrt{2}$ are not integer either. Thus, the original equation has no integer solutions.

Prob 2. (a) Prove or disprove that you can use dominoes to tile a standard checkerboard with all four corners removed.

Proof of possibility. A full row of length 8 squares can be tiled using 4 dominoes placed horizontally. The first and the last rows with corners removed each consist of 6 squares, so can be tiled using 3 dominoes placed horizontally as well. So the entire checkerboard with all four corners removed can be tiled with (horizontally placed) dominoes.

(b) Prove or disprove that you can tile a 10×10 checkerboard using straight tetrominoes.

Proof of impossibility. Use four colors to paint diagonals of the checkerboard cyclically. In other words, let the lower left corner be painted color 1, let the next diagonal consisting of the two neighbors of that corner be painted color 2, the next diagonal (consisting of 3 squares) color 3, the next diagonal color 4, the next diagonal color 1 again, and so on, until the upper right corner is painted.

If a straight domino is placed on the board so colored, it will cover four squares all whose colors will be distinct. So, for a tiling of the entire 10×10 board to exist it is necessary that the total number of squares of each color be equal exactly to the quarter of the total number (100) of squares, i.e., 25.

However, color 2 is assigned to diagonals number 2, 6, 10, 14, 18, which are comprised all together of $2 + 6 + 10 + 6 + 2 = 26 \neq 25$ squares.

So the condition necessary for the existence of a tiling does not hold, which implies that no tiling is possible.

Prob 3 [Russell's Paradox]. Let S be the set of all sets that do not contain themselves: $S = \{x : x \notin x\}$. Show that both $S \in S$ and $S \notin S$ lead to contradictions. This paradox shows inherent problems with Naïve Set Theory.

Proof. Let $P(x)$ be the predicate "$x \notin x$". Russell's definition of the set S is then given by

$$S = \{x : P(x)\}.$$

Consider the statement $P(S)$="$S \notin S$". This statement is either true or false.

Case 1: If $P(S)$ is true, i.e., $S \notin S$, then $x = S$ is included as an element of the set S, i.e., $S \in S$, i.e., $P(S)$ is false. Hence $P(S) \to \neg P(S)$.

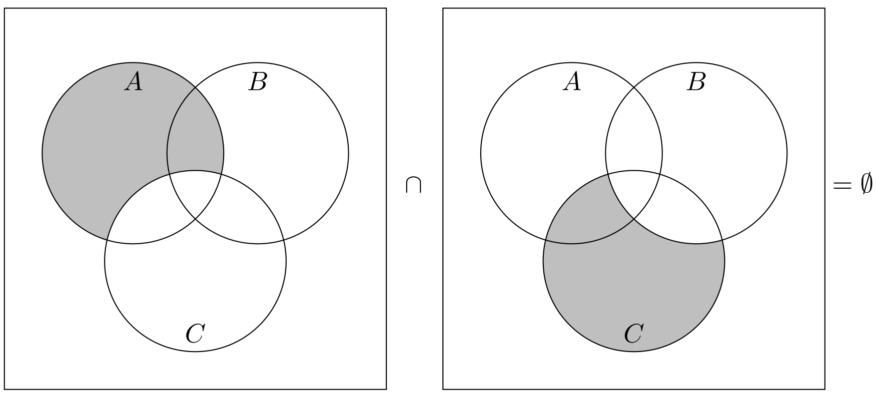
Case 2: If $P(S)$ is false, i.e., $S \in S$, then $x = S$ is not included as an element of the set S, i.e., $S \notin S$, i.e., $P(S)$ is true. Hence $\neg P(S) \to P(S)$.

Together, this gives $P(S) \leftrightarrow \neg P(S)$, which is not satisfiable.

Prob 4. Prove the following set inclusions using membership tables and illustrate them using Venn diagrams. What are the logical analogues of these formulas? (a) $(A \setminus C) \cap (C \setminus B) = \emptyset$.

Solution: Here are the membership table and the Venn diagram:

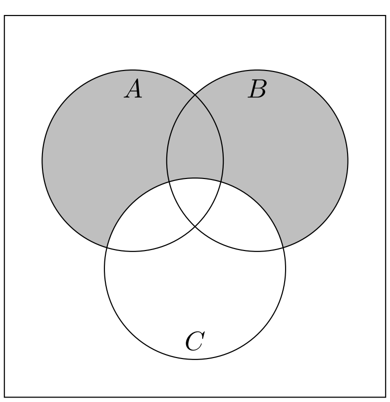
$x \in A$	$x \in B$	$x \in C$	$x \in A \setminus C$	$x \in C \setminus B$	$x \in (A \setminus C) \cap (C \setminus B)$
F	F	F	F	F	F
F	F	T	F	T	F
F	T	F	F	F	F
F	T	T	F	F	F
T	F	F	T	F	F
T	F	T	F	T	F
T	T	F	T	F	F
T	T	T	F	F	F



The logic(al) analogue is $(a \wedge \neg c) \wedge (c \wedge \neg b) \equiv F$.

(b) $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$. **Solution:**

$x \in A$	$x \in B$	$x \in C$	$x \in A \setminus C$	$x \in B \setminus C$	$x \in (A \setminus C) \cup (B \setminus C)$	$x \in (A \cup B) \setminus C$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	F	T	T	T
F	T	T	F	F	F	F
T	F	F	T	F	T	T
T	F	T	F	F	F	F
T	T	F	T	T	T	T
T	T	T	F	F	F	F



The logic(al) analogue is $(a \wedge \neg c) \vee (b \wedge \neg c) \equiv (a \vee b) \wedge \neg c$.

Prob 5. The **symmetric difference** of sets A and B denoted $A \oplus B$ is the set containing all elements in either A or B but not in both. Is the symmetric difference associative, i.e., is it always the case that

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C?$$

Solution. Yes. Here is a proof via a membership table:

$x \in A$	$x \in B$	$x \in C$	$x \in A \oplus B$	$x \in B \oplus C$	$x \in A \oplus (B \oplus C)$	$x \in (A \oplus B) \oplus C$
F	F	F	F	F	F	F
F	F	T	F	T	T	T
F	T	F	T	T	T	T
F	T	T	T	F	F	F
T	F	F	T	F	T	T
T	F	T	T	T	F	F
T	T	F	F	T	F	F
T	T	T	F	F	T	T

And the corresponding Venn diagram is symmetric with respect to A, B and C, which proves the same: