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Math 55, Lecture 20.

GENERATING FUNCTIONS.

The **(ordinary) generating function** $\mathcal{G}(x)$ for the sequence $\{a_n\}$, $n \in \mathbb{Z}_+$ is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

The coefficient of x^n in $\mathcal{G}(x)$ is the n^{th} object in the sequence $\{a_n\}$

[Theorem.] Let $\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n$ and $\mathcal{H}(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$\mathcal{G}(x) + \mathcal{H}(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n \qquad \mathcal{G}(x) \cdot \mathcal{H}(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{j-n} \right) x^n$$

Q1. If $\mathcal{G}(x)$ is the generating function for the sequence $\{a_n\}$, what is the generating function for each of the following sequences?

(a) $0, a_0, a_1, a_2, \ldots$: $x\,G(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$

(b) $a_1, 2a_2, 3a_3, \ldots$: $G'(x) = \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n$

(c) $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$: $\frac{G(x)}{1-x} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j \right) x^n$

(d) $a_0^2, 2a_0a_1, 2a_0a_2 + a_1^2, 2a_0a_3 + 2a_1a_2, 2a_0a_4 + 2a_1a_3 + 2a_2^2, \ldots$: $(G(x))^2 = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j a_{j-n} \right) x^n$

THE EXTENDED BINOMIAL THEOREM.

Let $u \in \mathbb{R}$ and let $k \in \mathbb{Z}_+$. The **extended binomial coefficient** $\binom{u}{k}$ is defined as

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

In particular, for $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, the following equality holds:

$$\binom{-n}{r} = (-1)^r \, C(n+r-1, r)$$

[Extended Binomial Theorem.] Let $u \in \mathbb{R}$ and let $x \in \mathbb{C}$ with $|x| < 1$. Then

$$(1+x)^u = \sum_{n=0}^{\infty} \binom{u}{n} x^n$$

Q2. Find a closed form of the generating function for the sequence $\{a_n\}$ where

(a) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}, 0, 0, 0, \ldots$:

$$\sum_{n=0}^{7} \binom{7}{n} (2x)^n = (1+2x)^7$$

(b) $a_n = \binom{10}{n+1}$, $n \in \mathbb{Z}_+$:

$$\frac{1}{x}\left(\sum_{n=0}^{10} \binom{10}{n} x^n - 1 \right) = \frac{1}{x}\left((1+x)^{10} - 1 \right)$$

(c) $a_n = \binom{n+4}{n}$, $n \in \mathbb{Z}_+$:

$$\sum_{n=0}^{\infty} \binom{5+n-1}{n} x^n = \frac{1}{(1-x)^5}$$

COUNTING VIA GENERATING FUNCTIONS.

Q3. What generating functions can be used to find the number of ways in which postage of r cents can be pasted on an envelope using 1-cent, 3-cent, and 20-cent stamps?

(a) Assume that the order the stamps are pasted on does not matter.

$$\frac{1}{(1-x)(1-x^3)(1-x^{20})}$$

(b) Assume that the stamps are pasted in a row and their order matters.

$$\frac{1}{1-(x+x^3+x^{20})}$$

RECURRENCES VIA GENERATING FUNCTIONS.

Q4. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$, $a_1 = 5$.

$$A(x) = \sum_{k=0}^{\infty} a_k x^k \Rightarrow A(x) - 2 - 5x = \sum_{k=2}^{\infty} 4a_{k-1}x^k - \sum_{k=2}^{\infty} 4a_{k-2}x^k + \sum_{k=0}^{\infty} k^2 x^k$$

$$A(x) - 2 - 5x = 4(x)(A(x) - 2) - 4x^2 A(x) + \frac{x(1+x)}{(1-x)^3} - 1 - x$$

$$A(x) = \frac{1-4x}{(1-2x)^2} + \frac{x(1+x)}{(1-2x)^2(1-x)^3} = \sum_{k=0}^{\infty}\left(-24\cdot 2^k + 5(k+1)2^k + 13\cdot(-1)^k x^4 + 5(k+1) + (k^2+k)\right)x^k$$

$$\Rightarrow a_k = -24\cdot 2^k + 5(k+1)2^k + 13(-1)^k + 5(k+1) + (k^2+k)$$

IDENTITIES VIA GENERATING FUNCTIONS.

Q5. (a) Show that $(x^2+x)/(1-x)^4$ is the generating function for the sequence $\{1^2+2^2+\cdots+n^2\}_{n\in\mathbb{Z}_+}$.

Use $a_n = a_{n-1} + n^2$ to represent sum of first n squares $a_0 = 0$

$$A(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + \sum_{k=1}^{\infty} a_k x^k = x\sum_{k=1}^{\infty} a_{k-1}x^{k-1} + \sum_{k=0}^{\infty} k^2 x^k = x A(x) + \frac{x(x+1)}{(1-x)^3}$$

$$(1-x)A(x) = \frac{x^2+x}{(1-x)^3} \Rightarrow A(x) = \frac{x^2+x}{(1-x)^4}$$ thus $A(x)$ is a generating function for $\{1^2+2^2+\cdots+n^2\}$

(b) Use (a) to find an explicit formula for $1^2+2^2+\cdots+n^2$.

$$(x^2+x)(1+(-x))^{-4} = (x^2+x)\cdot\sum_{k=0}^{\infty}\binom{-4}{k}x^k = x^2\sum_{k=0}^{\infty}\binom{-4}{k}x^k + x\sum_{k=0}^{\infty}\binom{-4}{k}x^k$$

$$= \sum_{k=0}^{\infty}\binom{-4}{k}x^{k+2} + \sum_{k=0}^{\infty}\binom{-4}{k}x^{k+1} = \sum_{k=2}^{\infty}\binom{-4}{k-2}x^k + \sum_{k=1}^{\infty}\binom{-4}{k-1}x^k$$

$$= x + \sum_{m=2}^{\infty}\left(\binom{-4}{k-2} + \binom{-4}{k-1}\right)x^k \Rightarrow$$

$a_0 = 0, a_1 = 1$

$$a_k = \binom{-4}{k-2} + \binom{-4}{k-1} \quad m \geq 2$$

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