Name:	**GSI:**	**DISC #:**

Solutions to Homework 2.

Prob 1. Establish these logical equivalences where x does not occur as a free variable in A. Assume that the domain is nonempty.

(a) $\forall x\ (A \to P(x)) \equiv A \to \forall x\ P(x)$.

Proof. Case 1: $A = F$. Then $A \to P(x)$ is true regardless of the value of x, so the left-hand side takes the value T. And $A \to \forall x\ P(x)$ also takes the value T. So both sides take the same value.

Case 2: $A = T$. Then $A \to P(x)$ takes the same value as $P(x)$ so the whole left-hand side takes the same value as $\forall x P(x)$. Now, the right-hand side also takes the same value as $\forall x P(x)$. So both sides take the same value.

Thus the left-hand side and the right-hand side are equivalent.

(b) $\exists x(P(x) \to A) \equiv \forall x P(x) \to A$.

Proof. Case 1: $A = F$. Then $P(x) \to A$ takes the same value as $\neg P(x)$, and the whole left-hand side takes the value $\exists x\ \neg P(x)$. Now, the right-hand side takes the same value as $\neg\ \forall x P(x)$. Finally, by negation laws, $\exists x \neg P(x) \equiv \neg \forall x P(x)$.

Case 2: $A = T$. Then $P(x) \to A$ is true regardless of x, so the whole left-hand side takes the value T. And the right-hand side takes the value T too.

Thus the left-hand side and the right-hand side are equivalent.

Prob 2. Use predicates, quantifiers, logical connectives, and mathematical operators to express the Twin Prime Conjecture.

Solution. The Twin Prime Conjecture is the statement that there are infinitely many pairs of primes whose difference is equal to 2. A positive integer $p(> 1)$ is prime if $p = kl$ implies $k = 1$ or $l = 1$. Our domain of discourse is $\mathbb{N}$. The existence of infinitely many elements of $\mathbb{N}$ with some property is equivalent to the existence, for any given $n \in \mathbb{N}$, of an element greater than n with that property. Collecting all those parts, we obtain the following:

$$\forall n\ \exists p > n\ [\forall k\ \forall l\ (p = kl \to (k = 1 \lor l = 1))] \land [\forall k\ \forall l\ (p+2 = kl \to (k = 1 \lor l = 1))].$$

This formula can be also rewritten with all quantifiers placed in front:

$$\forall n\ \exists p > n\ \forall i\ \forall j\ \forall k\ \forall l\ [p = ij \to (i = 1 \lor j = 1)] \land [p+2 = kl \to (k = 1 \lor l = 1)].$$

[NB: Notice what happens when we pull quantifiers forward!]

Prob 3. This argument supposedly shows that if $\forall x\,(P(x) \vee Q(x))$ is true, then $\forall x\, P(x) \vee \forall x\, Q(x)$ is true. What is wrong with it?

1. $\forall x\,(P(x) \vee Q(x))$ Premise
2. $P(c) \vee Q(c)$ Universal instantiation from (1)
3. $P(c)$ Simplification from (2)
4. $\forall x\, P(x)$ Universal generalization from (3)
5. $Q(c)$ Simplification from (2)
6. $\forall x\, Q(x)$ Universal generalization from (5)
7. $\forall x\, P(x) \vee \forall x\, Q(x)$ Conjunction from (4) and (6)

Solution. There are three mistakes here:

- simplification on line 3 is not valid from a disjunction,
- simplification on line 5 is not valid from a disjunction,
- line 7 is not a conjunction.

Prob 4. Show that the argument form with premises

$$p \wedge t \to r \vee s, \quad q \to u \wedge t, \quad u \to p, \quad \neg s$$

and conclusion

$$q \to r$$

is valid using rules of inference from Table 1 in Section 1.6.

Solution 1. Since $a \wedge b \to c \equiv a \to (b \to c)$, showing that the argument form with premises a and conclusion $b \to c$ is valid is equivalent to showing that the argument form with premises a and b and conclusion c is valid.

So, our task is equivalent to showing that the argument form with premises $q, p \wedge t \to r \vee s$, $q \to u \wedge t$, $u \to p$, $\neg s$ and conclusion r is valid.

1. q (new) premise
2. $q \to u \wedge t$ premise
3. $u \wedge t$ modus ponens from lines 1 and 2
4. u simplification from line 3
5. $u \to p$ premise
6. p modus ponens from lines 4 and 5
7. t simplification from line 3
8. $p \wedge t$ conjuncton from lines 6 and 7
9. $p \wedge t \to r \vee s$ premise
10. $r \vee s$ modus ponens from lines 8 and 9
11. $\neg s$ premise
12. r resolution from lines 10 and 11

Solution 2. Alternatively, we can use just the original premises and *resolution only* if we recall that $a \to b \equiv \neg a \vee b$ ($\equiv b \vee \neg a$) and manipulate a few negations:

1. $\neg(p \wedge t) \vee (r \vee s)$ ($\equiv \neg p \vee \neg t \vee r \vee s$) premise
2. $\neg s$ premise
3. $\neg p \vee \neg t \vee r$ resolution from lines 1 and 2
4. $p \vee \neg u$ premise
5. $\neg t \vee r \neg u$ ($\equiv r \vee \neg(u \wedge t)$) resolution from lines 3 and 4
6. $\neg q \vee (u \wedge t)$ premise
7. $r \vee \neg q$ resolution from lines 5 and 6

The last formula is equivalent to $q \to r$, so we are done.

Prob 5. Let the integers $1, 2, \ldots, 10$ be placed around a circle, in any order. Show that there are 3 integers in consecutive locations whose sum is at least 17.

Solution 1. The sum of all integers around the circle is $1 + 2 + \cdots + 10 = 55$. For any location on the circle, consider the sum of the integer at that location and its two neighbors (on either side). Adding all these 10 sums amounts to adding all integers around the circle three times, which equals $55 \cdot 3 = 165$. So, the average value of such sums is 16.5. However, each sum must be an integer since it is obtained by adding some integers. Hence at least one of these sums must be greater than 16.5, i.e., at least 17.

Solution 2. In fact, one can prove a better bound 18 easily. Cross out the number 1 on the circle. The remaining numbers can be organized into 3 groups of 3 consecutive integers. The sum of all remaining numbers is $55 - 1 = 54$. So the total in at least one of the groups is at least $54/3 = 18$. Voilà.

[NB: The bound 18 is tight as the arrangement $1, 9, 7, 2, 8, 6, 4, 3, 10, 5$ shows, where 18 is the maximum sum of any three consecutive integers.]