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Math 55, Homework 7.

Prob 1. (a) How many bit strings of length 10 contain five consecutive 0s or five consecutive 1s?

$k \in \{1,2,3,4,5,6\}$

$k=1 \longrightarrow 2^5$ strings (00000*****)

$k>1 \longrightarrow 2^4$ strings (*100000*** for $k=3$)

grouping & exclusive

$2^5 + 5 * 2^4 = 112$

since we have to account for symmetry the total is $112 + 112 - 2 = \boxed{222}$

(b) How many bits strings of length 10 contain at least three 1s and at least four 0s?

at least 3 1s   6 ones is the max # of 1's because you need at least 4 0's

4 1s

5 1s

6 1s

All possibilities:

the total number of bit strings of length 10 that contain at least three 1's and at least 4 0s:

$= C(10,3) + C(10,4) + C(10,5) + C(10,6)$

$= 120 + 210 + 252 + 210$

$= \boxed{792}$

Prob 2. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

we count in two ways to select a committee, consisting n members, from n math professor and n physics professor such that the chairman is a math professor

(1). The number of ways to choose a chairman from n math professors is n. & choosing (n-1) committee members from the remaining (2n-1) professors has $\binom{2n-1}{n-1}$ possibilities. so the total number of ways such committee can be formed is $n\binom{2n-1}{n-1}$

(2) Another way to form such committee:
if the committee contains k math professors, the # of ways to chose k math prof.: $\binom{n}{k}$
- The # of ways to select a chairman from these k math prof.: k
- the # of ways to choose (n-k) physics prof.: $\binom{n}{n-k} = \binom{n}{k}$
- By product rule, the # of ways to form such a committee: $\sum_{k=0}^{n} k \cdot \binom{n}{k}\binom{n}{k}$

$$= \sum_{k=0}^{n} k\binom{n}{k}^2$$

$$= \sum_{k=1}^{n} k\binom{n}{k}^2$$

From (1) & (2), we get that

$$\boxed{\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}}$$

Prob 3. Let n, $k \in \mathbb{N}$ be such that $k \leq n$. Prove (in any way you like) that

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}.$$

First, break down the term $\binom{2n+2}{n+1}/2$.

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{(2n)!}{(n+1)!(2n-(n+1))!} + \frac{(2n)!}{n!(2n-n)!}$$

$$= \frac{(2n)!}{(n+1)!(n-1)!} + \frac{(2n)!}{n!\,n!}$$

$$= (2n)!\left[\frac{n+(n+1)}{(n+1)!\,n!}\right]$$

$$= \frac{2(n+1)(2n+1)!}{2(n+1)(n+1)!\,n!}$$

$$= \frac{1}{2}\,\frac{(2n+2)!}{(n+1)!((2n+2)-(n+1))!} = \binom{2n+2}{n+1}/2$$

the RHS $= \binom{2n+2}{n+1}/2 - \binom{2n}{n} = \binom{2n}{n+1} + \binom{2n}{n} - \binom{2n}{n} = \binom{2n}{n+1}$

1. the # of ways to select $(n+1)$ person from $(2n)$ person, where there are one more men than women : $\binom{2n}{n+1}$

2. the # of ways to choose k men from n men and $(k-1)$ women from $(n-1)$ women: $\binom{n}{k}\binom{n}{k-1}$

total # of ways : $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}$

thus 1 & 2, we get $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n}{n+1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}$

Prob 4. Suppose that 21 girls and 21 boys enter a mathematical competition, that each entrant solves at most six questions, and that for every boy-girl pair, there is at least one question that they both solved. Prove that there is a question that was solved by at least three girls and three boys.

the table has $21 \times 21 = 441$ boxes

each box with a letter showing the question solved by this pair is marked. At least one question is solved by every pair, each box will have a letter. Because each entrant solves at most 6 questions, any row (column) can contain at most 6 letters. so 6 different letters can be in a row if and only if at least 11 of the boxes contain letters appearing 3 or more times in a row. thus, the # of boxes in the rows that contain 3 or more identical letters: $21 \times 11 = 231$

Also the # of boxes in the columns that contain 3 or more identitical letters in a column: $21 \times 11 = 231$

$231 + 231 = 462 > 441$, there is at least one box in the rows that contain 3 or more identical letters and in the columns that contain 3 or more identical letters.

Therefore, there was a question that was solved by at least three girls and three boys

Prob 5. (a) Let $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that the Ramsey numbers $R(m, n)$ and $R(n, m)$ are equal.

If a group of size k, which is $\{a_1, \ldots a_k\}$ and consider a subgroup of size m where they do not know each other.

it is false that m of them know each other, and it is also false that n of them do not know each other. therefore $R(m,n) = k$.

Now if we switch the #s m,n for friends & enemies. If a group of size k, $\{b_1 \ldots b_k\}$, such that if a_i and a_j are friends, b_i and b_j are enemies; if a_i and a_j are enemies, b_i and b_j are friends.

since $R(n,m) = k$ then $R(m,n) = R(n,m)$

(b) Prove that at a party with $m \geq 2$ people there are two who know the same number of other people.

If $m = 2$ then there are two options:

1. they know each other
2. they do not know each other

If $m > 2$, then suppose there are not two people who know the same # of other people. $\longrightarrow$ we can list all these people such that the 1st person knows 0 person. the second person knows 1 person... the m^{th} person knows $(m-1)$ person. Now the m^{th} person knows all other people, so he knows the 1st person which contradicts with the fact that the 1st person knows 0 people. Thus, there are two people who know the same # of other people