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Math 55, Handout 24.

HAMILTONIAN PATHS AND CIRCUITS.

1.1. A **Hamiltonian circuit** in a graph G is a simple circuit that reaches every vertex exactly once.

1.2. A **Hamiltonian path** in a graph G is a simple path that reaches every vertex exactly once.

1.3. Unlike the Eulerian path/circuit problem, the Hamiltonian path/circuit problem is **NP-complete**.

Q1. Which trees have a Hamiltonian path?
Linear trees: because if any vertex has degree 3, moving through that vertex will disconnect the graph.

Q2. For which values of n do the following graphs have a Hamiltonian circuit?
$K_n : n > 2$, C_n : All n , W_n: All n, $Q_n : n > 1$

1.3. [Dirac's Theorem] If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex is at least $n/2$, then G has a Hamiltonian circuit.

1.4. [Ore's Theorem] If G is a simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamiltonian circuit.

Q3. One of these theorems is a corollary of the other. Which one? Why?
Direac's theorem: if each vertex has degree at least $n/2$,then the sum of two vertices's degrees is at least n.

Q4. Construct a proof of Ore's Theorem following the following outline. Suppose that the assumptions of Ore's Theorem hold for a graph G but that G does not have a Hamiltonian circuit.

(a) Show that by adding edges to G if necessary, we can construct another simple graph H with the same vertices as G such that H does not have a Hamiltonian circuit but the addition of a single edge to H would produce a Hamiltonian circuit in H.
We can add edges until every edge that doesn't exist will make a Hamiltonian circuit. This method terminates since K_n has a circuit.

(b) Show that there is a Hamiltonian path in H.
We can simply add an edge, find the Hamiltonian circuit and then remove the edge added recently (and it becomes the Hamiltonian path).

(c) v_1, v_2, …, v_n be a Hamiltonian path in H. Show that $\deg(v_1) + \deg(v_n) \geq n$ and that there are at most $\deg(v_1)$ vertices not adjacent to v_n (including v_n itself).
The first half can be proven using the assumption: since adding edges maintains the property and v_1, v_n are not adjacent; the second half can be proven henceforth, because we know that there are a total of n vertices.

(d) Let S be the set of vertices preceding each neighbor of v_1 in our Hamiltonian path. Show that S contains $\deg(v_1)$ vertices and $v_n \notin S$.
Each vertex except v_1 has exactly one preceding vertex, therefore, they form a set of $deg(v_1)$ vertices (the group of preceding vertices of v_1's neighbors). Since there is no succceding vertex of v_n, it is not in S.

(e) Show that S contains a vertex v_k which is adjacent to v_n, implying that there are edges connecting v_1 to v_{k+1} and v_k to v_n.
We know from the previous questions that v_n is not in S and at most $deg(v_1) - 1$ vertices in S are not adjacent to v_n (this is because at most $deg(v_1)$ vertices are not adjacent to v_n), we can conclude that at least on vertex in S is adjacent to v_n.

(f) Show that part (e) implies that $v_1, v_2, \ldots, v_{k-1}, v_k, v_n, v_{n-1}, \ldots, v_{k+1}$ is a Hamiltonian circuit in H. Conclude from this contradiction that Ore's Theorem holds.
v_{k+1} is adjacent to v_1 and so v_k is adjacent to v_n. Since the reverse of a path is still a path, it's a circuit. Since it also goes through every vertex, it's a Hamiltonian circuit.