Quiz - ARC

Chepuri Shivani
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Given class of robotic system:

$$M(q)\ddot{q} + H(q,\dot{q}) = \tau \quad \text{--- (1)}$$

$q \in \mathbb{R}^n \rightarrow$ generalized position.

$M \in \mathbb{R}^{n \times n} \rightarrow$ mass matrix invertible.

$\tau \in \mathbb{R}^n \rightarrow$ Control i/p.

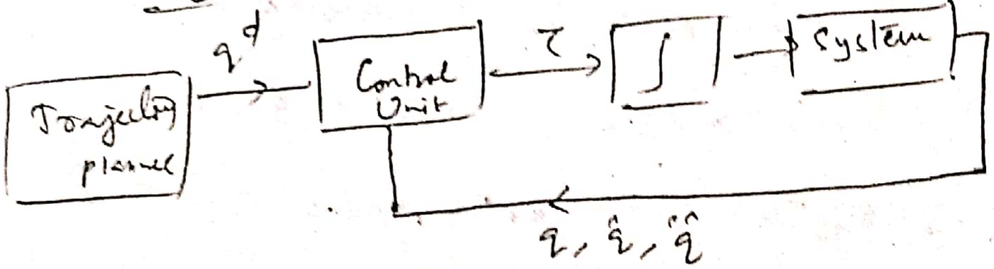
→ (i) Inverse Dynamic Controller to track q^d.

The idea of inverse dynamics is to get a non-linear feedback Control law

$$\tau = f(q, \dot{q}, t) \quad \text{--- (2)}$$

~~which~~ (2) substituted in (1) will result in a linear closed loop system.

Rough Sketch:



We can define error at any given time t as $e(t)$,

ie $e = q - q^d$ $\qquad \ddot{e} = \ddot{q} - \ddot{q}^d$

$\dot{e} = \dot{q} - \dot{q}^d$

Our objective is to find q as close to q^d

$$\Rightarrow q \rightarrow q^d$$

$$\Rightarrow \ddot{q} \rightarrow \ddot{q}^d$$

We can choose τ such that,

$$M\ddot{q} + H = \tau \quad \text{--- (3)}$$

$$\& \quad \tau = M\ddot{q}^d + H. \quad \text{--- (4)}$$

This is because we want perfect tracking.

This τ above is an ideal control input.

Putting (4) in (3),

$$M\ddot{q} + H = M\ddot{q}^d + H$$

M is invertible (given)

$$\Rightarrow M\ddot{q} = M\ddot{q}^d$$

$$\Rightarrow \ddot{q} = \ddot{q}^d$$

$$\ddot{e} = 0$$

$$\Rightarrow \dot{e} = c$$

$$e = ct$$

~~But~~ we see the error is increasing with time.

So, let choose some 'u' instead of $\ddot{q}^d$ to make the system stable & to make $e \rightarrow 0$.

(This method is called the Inverse Dynamic Control or Computed Torque Method)

$\therefore$ we have eq^n as follows now,

$$M\ddot{q} + H = \tau$$

$\tau = Mu + H \rightarrow$ Inverse dynamic Control

This 'u' represents the new input to the system, which is yet to be chosen.

ie $\ddot{q} = u$

$\hookrightarrow$ Double Integrator System.

$\Rightarrow$ This procedure will result in a 'new' system that is linear & decoupled.

we can u to be the fn of q & its derivatives;

we can set
$$u = \ddot{q}^d - K_d \dot{e} - K_p e$$

To correct proportional & derivative errors.

where $e = q - q^d$

$\dot{e} = \dot{q} - \dot{q}^d$

& K_p, K_D are 2 positive definite matrices.

Now,
$$\begin{cases} M\ddot{q} + H = \tau \\ \tau = M(\ddot{q}^d - K_d\dot{e} - K_p e) + H \end{cases}$$

from these 2 eqn's,

$$M\ddot{q} = Mu = M(\ddot{q}^d - K_d\dot{e} - K_p e)$$

$$\Rightarrow \ddot{q} = \ddot{q}^d - K_d\dot{e} - K_p e.$$

$$\Rightarrow \boxed{\ddot{e} + K_d\dot{e} + K_p e = 0}$$

Required error dynamics.

Here a simple choices for gain matrices can be

$$K_p = \text{diag}\{\omega_1^2, \ldots \omega_n^2\}.$$

$$K_d = \text{diag}\{2\omega_1, \ldots 2\omega_n\}$$

$\Rightarrow$ Results in a globally decoupled closed loop system.

(ω_i is the natural frequency)

If uncertainity is considered in. Mass matrix & H matrix (as given it is generalized), error dynamics will be as follows.

$$M = \hat{M} + \Delta M$$

$\hat{M}$ → nominal value

ΔM → uncertainity

$$H = \hat{H} + \Delta H.$$

$$\tau = Mu + H.$$

$$\tau = \hat{M}u + \hat{H}.$$

$$\& \quad u = \ddot{q}^d - K_d\dot{e} - K_p e$$

(chosen)

$$M\ddot{q} + H = \tau$$

$$= \hat{M}u + \hat{H}$$

$$\Rightarrow M\ddot{q} = \hat{M}u + (\hat{H} - H)$$

$$\Rightarrow \ddot{q} = M^{-1}\hat{M}u + M^{-1}(\hat{H} - H)$$

Scanned with CamScanner

($\because$ M is invertible) add & sub 'u'

$$\Rightarrow \ddot{q} = u + (M^{-1}\hat{M} - I)u + M^{-1}(\hat{H} - H)$$

$$= \ddot{q}^{d} - K_D\dot{e} - K_P e$$

$$\Rightarrow \ddot{e} + K_D\dot{e} + K_P e = (M^{-1}\hat{M} - I)u + M^{-1}(\hat{H} - H).$$

$$s_{1,2} = \frac{-K_D \pm \sqrt{K_D^2 - 4K_P}}{2}$$

If this part is zero, the solⁿ to $e(t)$ looks like follows,

① $\rightarrow e(t) = e^{-K_D/2} \cdot (1+t) \rightarrow$ decaying

② $e(t) = y(0) \cdot e^{-s_1 t} + y_1(0) e^{-s_2(t)}$

③ $e(t) = e^{-K_D/2\, t}(\cos\omega t + j\sin\omega t)$

decaying & oscillation

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when

~~$s_{1,2} = -\frac{K_D}{2}$ if $K_D^2 = 4K_P$.~~

~~$K_D^2 > 4K_P$~~

$s_{1,2}$

$K_D^2 < 4K_P$

&

(2) To get the stability of the closed loop system via Lyapunov,

from error dynamics,

$$\ddot{e} + k_D\dot{e} + K_p e \neq 0 \quad \because \text{generalized system}$$

$$\ddot{q} - \ddot{q}^d + k_D(\dot{q} - \dot{q}^d) + k_p(q - q^d) \neq 0$$

$$= \delta a$$

let $\ddot{e} + k_D\dot{e} + K_p e = \delta a$ control i/p due to uncertainty ~~term~~

$$\Rightarrow \ddot{q} = \ddot{q}^d + k_D(\dot{q}^d - \dot{q}) + k_p(q^d - q) + \delta a$$

~~In terms of tracking error~~, (redefined in next page)

let $x_1 = e$, $x_2 = \dot{e}$ & $x = [x_1, x_2]$

Then, $\dot{x}_1 = \dot{e} = x_2$

$$\dot{x}_2 = \ddot{e} = -K_d\dot{e} - K_p e + \delta a$$

$$= -K_d x_2 - K_p x_1 + \delta a$$

$$\dot{x} = [\dot{x}_1, \dot{x}_2]$$

$$= \begin{bmatrix} 0 & I \\ -K_p & -K_D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta a \end{bmatrix}$$

$$\dot{x} = Ax + B\delta a$$

where $A = \begin{bmatrix} 0 & I \\ -K_p & -K_D \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Here δa is the addition controlip to ~~as~~ compensate for the effect of uncertainty

say 'η'

→ needs to be upper bounded.

from the uncertainty eq^n

$$\delta a = \eta = (M^{-1}\hat{M} - I)u + M^{-1}(\hat{H} - H)$$

~~[illegible]~~ Also,

uncertainty in H is bounded is

$$\|\Delta H\| \le \theta_c^* \|\dot{q}\|^2 + \theta_F^* \|\dot{q}\| + \theta_G^*$$

(all kinds of other forces on the system are within H)

→ continuing in the robust controller format, we might arrive at a lyapunove eq^n with stability if $\delta a = \eta \dfrac{S}{\|S\|}$ $\quad S = B^T P . e$

PD matrix

ie as follows, ⊛

→ for the current question, let us consider (entire section)

$B = 0$.

ie $\dot{x} = Ax$. $A \in \mathbb{R}^{n \times n}$ $x \in \mathbb{R}^n$.

Consider $V = x^T P x$ P is some PD matrix

w.r.t $\lambda_{min}(P) \|x\|^2 \le V \le \lambda_{max}(P) \|x\|^2$

$$\therefore \dot{V} = \dot{x}^T P x + x^T \dot{P} x + x^T P \dot{x}$$

$$\Rightarrow x^T (A^T P + P A) x$$

we need $\dot{V} < 0$ for asymptotic stability

($\dot{V} \leq 0$ — stability

this matrix < 0

$\dot{V} \leq e^{-at} V(0)$

if $x^T (A^T P + P A) x < 0$ → exponential stability)

$\therefore \dot{V} < 0 \Rightarrow$ asymptotically stable & exponentially stable

$\dot{V} \leq -\frac{\lambda_{min}(Q)}{\lambda_{min} P} V$.

(*) Continued.

let $V = x^T P x$

$$\dot{V} = \dot{x}^T P x + \cancel{x^T \dot{P} x} + x^T P \dot{x}$$

$$= [Ax + B \cdot \delta a]^T P x +$$

$$x^T P [Ax + B \delta a]$$

$$\dot{V} = (Ax)^T P x + x^T P A x + (B \delta a)^T P x$$

$$+ x^T P (B \delta a).$$

$$= x^T (A^T P + P A) x + 2 x^T P B (\delta a).$$

let this be $-Q$ & Q is P.D.

if $A^T P + P A = -Q$.

$\Rightarrow$ negative definite

but $P > 0$ (PD)

$\Rightarrow A$ has to be negative definite.

But we cant say anything about $\dot{V}$ yet because,

$$\dot{V} = -x^T Q x + \underbrace{2x^T P B \delta a}$$ — ⓐ

Ideally $\dot{V} = -x^T Q x$ extra term is there.

ie $\dot{V} < 0$ ∴ we cant directly comment on stability.

ie, if $x^T Q x > 2x^T P B \delta a$

$$\Rightarrow \lambda_{min}(Q) \|x\|^2 > 2\|x\| \|\delta a\| \|PB\|$$

$$\Rightarrow \|x\| \left(\lambda_{min}(Q) \|x\| - 2\|\delta a\| \|PB\| \right) > 0$$

$$\Rightarrow \|x\| > \frac{2\|\delta a\| \|PB\|}{\lambda_{min}(Q)}$$

The system here is not ideal Based on our selection of δa, the stability depends.

if $\delta a = -\eta \dfrac{w}{\|w\|}$

some uncertainity coeffcient where $w = B^T P e$.

$\dot{V} < 0 \Rightarrow$ asymtotically stable.

Because of the ~~kind of~~ uncertainity, we cant directly comment but if δa & $B < 0$ in eqⓐ st $\dot{V} < 0$ is guaranteed, then the system is stable.