① Advances in Robotics & Control

Quiz-3

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(Q1) Sol:-

Given, EL system (class of robotics system) as

dynamics: $\ddot{q} + \theta_1 \dot{q} + \theta_2 q = \tau$. —— ①

Here, $q(t)$ is the generalized position.

let $q \in \mathbb{R}$ (a scalar)
(assume)

τ is the control input.

& θ_1, θ_2 are given as 'unknown' constants.

Objective : ~~Track~~ Track a given desired trajectory $q_d(t)$.

$q_d(t)$ is bounded.

$e = q - q_d$. (or) $e(t) = q(t) - q_d(t)$

→ Comparing this system with with the generalized Robotic system eqⁿ is $M\ddot{q} + h = \tau$. where m, h are unknown. —— ②

Assuming that we dont even know the bounds (upper bounds) of these parameter ie θ_1 & θ_2. These are the parameters & ~~we~~ we need to design them to achieve adaptive converge of tracking error.

→ In this situation, we can proceed with Adaptive Robust Control based on Filter Tracking error.

Comparing eqs ① & ②,

$$M\ddot{q} + h(q, \dot{q}) = \tau \quad \& \quad \ddot{q} + (\theta_1 \dot{q} + \theta_2 q) = \tau$$

we have $M = 1$ & $h = \theta_1 \dot{q} + \theta_2 q$.

In general case, we consider M & h are constants but parameterized (contains parameters) & they are completely unknown (in Adaptive Control). This is satisfied with the given system i.e. eq ②.

$\therefore M = 1 \rightarrow$ a fixed scalar constant ;

$h = \theta_1 \dot{q} + \theta_2 q \rightarrow$ also constant with parameters.

<u>Step 1</u> : choose an 'r' i.e. the filter tracking error variable

let $r = \dot{e} + \lambda e$, $\lambda > 0$ $(e = q - q^d)$

some 'λ' > 0

λ is a design parameter & is a constant throughout the design.

We need such a structure because; we need 'λ' because,

if $r = 0$, $\dot{e} = -\lambda e$.

The solution of this Differential Equation (i.e. error) goes down exponentially.

And,

If 'λ' were not there, i.e. say, $r = \dot{e} + e$, we will have 'r' basically as a constant & we cannot modulate 'r' in the design. Therefore a parameter 'λ' is introduced in the filter tracking error.

<u>Step 2</u>: Take $\dot{r}$ to get $\ddot{q}$ in the equations as we proceed. ($\because \ddot{q}$ is there in the given system i.e. eq ②)

$\therefore \dot{r} = \ddot{e} + \lambda \dot{e}$ & $\dot{e} = \dot{q} - \dot{q}^d$

$$\Rightarrow \dot{r} = \ddot{q} - \ddot{q}^{d} + \lambda \dot{e} \quad \text{③}$$

Usually, we multiply eq③ with 'm' (which is $M=1$).

is $m\dot{r} = m\ddot{q} - m\ddot{q}^{d} + m\lambda\dot{e}$

from eq②,

$$\ddot{q} + \theta_1 \dot{q} + \theta_2 q = \tau$$

$$\Rightarrow \ddot{q} = \tau - \theta_1 \dot{q} - \theta_2 q$$

& $M=1$

$$\Rightarrow \dot{r} = (\tau - \theta_1 \dot{q} - \theta_2 q) - \ddot{q}^{d} + \lambda \dot{e} \quad \text{④}$$

Step 3:

Let us consider a generalized lyapunov function to understand the structure of the unknown parameters & to see how we can get their estimates.

let $V = \frac{1}{2} r^T M r$, $M=1$

$$\therefore V = \frac{1}{2} r^T r.$$

$$\dot{V} = \frac{1}{2}\left[\dot{r}^T r + r^T \dot{r}\right] \Rightarrow \boxed{\dot{V} = r^T \dot{r}} \quad \text{⑤}$$

Goal: If $r \to 0$, then $e \to 0$ exponentially (we want to prove this) via lyapunov analysis.

If we prove asymptotic stability, It will ensure $r \to 0$ which in turn will ensure $e \to 0$.

Now put eq④ in eq⑤.

we have,

$$\dot{V} = r^T\left[\tau - \theta_1 \dot{q} - \theta_2 q - \ddot{q}^d + \lambda \dot{e}\right] \quad (6)$$

Note :

for exponential Converge in (of tracking error) a robust Controller, we can just take $\tau = +\theta_1\dot{q} + \theta_2 q + \ddot{q}^d - \lambda\dot{e} - kr$. (7)

($\because$ we know θ_1, θ_2 bounds) for some $k > 0$ a parametric Constant.

Then we would get

$\dot{V} \leq -kr^2 \Rightarrow$ lyapunov stability (atleast).

But in Adaptive Control we do not know the bounds of θ_1 & θ_2 also. (we don't if they are bounded or etc.)

Such a τ in (7) would cancel out the partially known parameters' effect.

$\therefore$ we Consider the estimates of θ_1 & θ_2 as $\hat{\theta}_1$ & $\hat{\theta}_2$ respectively. We still do not know anything about $\hat{\theta}_1$ & $\hat{\theta}_2$ but we want to know their structure atleast so that we can find their expressions & update rules.

$\Rightarrow$ We want to put all the uncertainities (or) unknowns together & look for the structure as a whole, as in Adaptive Robust Control.

$\rightarrow$ Step 4 :

$\therefore \phi = h + m\ddot{q}^d + m\lambda\dot{e}$ (in general when $m\ddot{q} + h = \tau$).

~~But~~ $M=1$, but we will still consider the terms $\ddot{q}^d$ & $\lambda\dot{e}$ in the uncertainity structure to allow for a more generalized case.

(we see that in further analysis, as $M=1$, these terms get cancelled out $\therefore$ the assumption of eq of ϕ is not wrong). (To be able to put in $\dot{V}$)

Not unknown

$$\therefore \phi = \theta_1\dot{q} + \theta_2 q + \ddot{q}^d - \lambda\dot{e} \quad \text{--- (8)}$$

or we can take $\boxed{\phi = \theta_1\dot{q} + \theta_2 q}$ itself we will get the same answers in the end

Now, we try to upper bound to this eq (8)

$$\Rightarrow |\phi| \leq \theta_1^*\dot{q} + \theta_2^* q + \ddot{q}_b^d - \lambda|\dot{e}| \quad \text{--- (9)}.$$

Note: In eq (8) only $\theta_1\dot{q} + \theta_2 q$ is 'h' is the uncertain term for this particular system with $M=1$.

θ_1^*, θ_2^* are the corresponding upper bounds we dont know them but supposing that θ_1 & θ_2 must be bounded ($\because$ they are constants & design parameters)

$\ddot{q}_b^d$ is the upper bound on $\ddot{q}^d$

λ is the design parameter & is constant.

step 5:

Consider eq (6) is

$$\dot{V} = r^T(\tau - \theta_1\dot{q} - \theta_2 q - \ddot{q}^d + \lambda\dot{e})$$

Note:

$\tau = -Kr + \phi$ is sufficient if we knew ϕ.

But here ϕ is unknown.

$\therefore$ let $\tau = -Kr + \hat{\phi}$ estimate of ϕ.

$$\hat{\phi} = \hat{\theta}_1 \dot{q} + \hat{\theta}_2 q + \ddot{q}^d - \lambda \dot{e} \quad \text{--- (10)}$$

(from eq (8))

~~$\dot{V} = \mathcal{I}($~~

$\therefore \dot{V} = r^T(\tau - \phi)$ & τ we wanted to keep as $\tau = -Kr + \hat{\phi}$.

$$\Rightarrow \dot{V} \leq r^T(-Kr + \hat{\phi} - \phi).$$

$$= -Kr^2 + r^T\hat{\phi} - r^T\phi. \quad \text{--- (11)}$$

(Note: e & r may not be (or) need not be scalar)

$$\dot{V} \leq -Kr^2 + r^T\hat{\phi} + |r|(\theta_1^* \dot{q} + \theta_2^* q + \ddot{q}_b^d - \lambda|\dot{e}|). \quad \text{--- (12)}$$

(from eq (9))

(i.e. by using the upper bound on ϕ).

Note:

In Robust Control,

$$r^T\hat{\phi} = -|r|(\underbrace{\theta_1^*}_{\text{known}} \dot{q} + \theta_2^* q + \ddot{q}_b^d - \lambda|\dot{e}|)$$

$$\Rightarrow \hat{\phi} = -\mathrm{sgn}(r)(\theta_1^* \dot{q} + \theta_2^* q + \ddot{q}_b^d - \lambda|\dot{e}|).$$

But in Adaptive Control, we only have estimates of these $\underbrace{\theta_1^* \text{ \& } \theta_2^*}_{\text{not known here.}}$ parameters

$$\therefore \hat{\phi} = -\mathrm{sgn}(r)(\hat{\theta}_1 \dot{q} + \hat{\theta}_2 q + \ddot{q}_b^d - \lambda|\dot{e}|) \quad \text{--- (13)}$$

(PTO)

(7)

Step 6 :

Consider the lyapunov f^n which takes cares of the errors between θ_1^* & $\hat{\theta}_1$ and θ_2^* & $\hat{\theta}_2$.

$$\therefore \text{ let } V = \frac{1}{2}r^T r + \frac{1}{2}(\hat{\theta}_1 - \theta_1^*)^2 + \frac{1}{2}(\hat{\theta}_2 - \theta_2^*)^2.$$

$(\because M = 1)$

$$\text{Now, } \dot{V} \leq (-Kr^2 + r^T\hat{\phi} - r^T\phi) + (\hat{\theta}_1 - \theta_1^*)\dot{\hat{\theta}}_1 + (\hat{\theta}_2 - \theta_2^*) \cdot \dot{\hat{\theta}}_2 \quad (\because \text{eq (11) \& (12)}).$$

Now, from eq (13), we have

$$\hat{\phi} = -\text{sgn}(r) \cdot (\hat{\theta}_1 \dot{q} + \hat{\theta}_2 q + \ddot{q}_b^d - \lambda|\dot{e}|)$$

Put this in the above eq^n

$$\Rightarrow \dot{V} \leq -Kr^2 - \underbrace{r^T \text{sgn}(r)}_{|r|}(\hat{\theta}_1 \dot{q} + \hat{\theta}_2 q + \cancel{\ddot{q}_b^d} - \cancel{\lambda|\dot{e}|}) + |r|(\theta_1^* \dot{q} + \theta_2^* q + \cancel{\ddot{q}_b^d} - \cancel{\lambda|\dot{e}|}) + (\hat{\theta}_1 - \theta_1^*)\dot{\hat{\theta}}_1 + (\hat{\theta}_2 - \theta_2^*) \cdot \dot{\hat{\theta}}_2$$

(we can see the known terms i.e. $\ddot{q}_b^d$ & $\lambda|\dot{e}|$ are getting cancelled out) $\Rightarrow$ ($\phi = \theta_1 \dot{q} + \theta_2 q$ will also give the same).

$$\Rightarrow \dot{V} \leq -Kr^2 - |r|(\hat{\theta}_1 \dot{q} + \hat{\theta}_2 q) + |r|(\theta_1^* \dot{q} + \theta_2^* q) + (\hat{\theta}_1 - \theta_1^*)\dot{\hat{\theta}}_1 + (\hat{\theta}_2 - \theta_2^*)\dot{\hat{\theta}}_2$$

$$= -Kr^2 - (\hat{\theta}_1 - \theta_1^*)|r|\dot{q} - (\hat{\theta}_2 - \theta_2^*)|r|q + (\hat{\theta}_1 - \theta_1^*)\dot{\hat{\theta}}_1 + (\hat{\theta}_2 - \theta_2^*)\dot{\hat{\theta}}_2$$

—— (14)

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from eqⁿ (14), we can see that,

if we want $\dot{V} \leq -kr^2$, we can make our update rules as follows.

$$\dot{\hat{\theta}}_1 = |r|\dot{q}, \quad \dot{\hat{\theta}}_2 = |r|q.$$

(They will cancel out the terms in the $\dot{V}$ inequality on RHS).

$\therefore$ If $\dot{\hat{\theta}}_1 = |r|\dot{q}$ & $\dot{\hat{\theta}}_2 = |r|q$, what happens to the stability & tracking error?

$\Rightarrow$ $\dot{V} \leq -kr^2$.

as $k > 0$ & $r^2 \geq 0$ we can say that

$\dot{V} \leq 0$. i.e. $-kr^2$ is NSD (can be zero when only one of k & r is zero).

from the lyapunov stability ~~Cont~~ inferences, we have if $\dot{V} \leq 0 \Rightarrow$ the system is lyapunov stable...

i.e. we can assure that the function is not diverging at atleast.

$\dot{V} < 0 \Rightarrow$ asymptotic stability i.e. the function will converge as $t \to \infty$.

If we have $\dot{V} < -(\cdot)V$ form, we can also ensure exponential stability of the system.

$\Rightarrow$ our system is lyapunov stable only so far.

$\Rightarrow$ All signals (or) variables in the lyapunov function remain bounded but it does not ensure that either one or any of them will go to zero.

⑨

To ensure $r \to 0$ (ie tracking error $\to 0$), we can use lyapunov like lemma.

$\dot{V} \leq -K r^2 \Rightarrow$ lyapunov stability

$\Rightarrow V \in L_\infty$ (bounded ie absolutely integrable)

$\Rightarrow r, \hat{\theta}_2, \hat{\theta}_1 \in L_\infty$.

$\Rightarrow$ Tracking error 'r' is bounded.

Lyapunov like lemma: Then as $t \to \infty$, $\dot{V} \to 0$.

Conditions are

① V should be lower bounded.

② $\dot{V} \leq 0$ (which also implies V is lower bounded)

③ $\dot{V}$ is uniformly continuous.

$\Rightarrow$ prove $\ddot{V}$ is bounded.

$\rightarrow$ ① is satisfied $\because V \geq 0$ (positive Definite matrix) zero only when all arguments are zero.

ie from Definition (or) characteristics of a lyapunov fn are

ie ① V is Positive Definite

② V has to be a Continuous function

③ V is radially unbounded.

The two lyapunov functions we considered ie

$V = \frac{1}{2} r^T r$ (or) $V = \frac{1}{2} r^2$ and $V = \frac{1}{2} r^2 + \frac{1}{2}(\Delta\theta_1)^2 + \frac{1}{2}(\Delta\theta_2)^2$

will satisfy all these constraints

$\rightarrow$ square fn is continuous.

$\rightarrow$ The terms are all ≥ 0 & no semidefiniteness.

& they all increase readially $\Rightarrow$ unbounded.

$\rightarrow$ ② is satisfied since $\dot{V} \leq -K r^2$ (NSD).

(10)

→ To check (3),

$\ddot{V} \leq -2K r \dot{r}$ ~~$K\dot{e}$~~

$\ddot{V} \in L_\infty$ if $r, \dot{r} \in L_\infty$
(Bounded)
(K is a constant)
∴ bounded.

$r \in L_\infty$ from Lyapunov stability.

$\dot{r} \in L_\infty$?
we have, $\dot{r}$ expression from eq (4) as,

$$\dot{r} = (\underbrace{\tau}_{\substack{\text{constant} \\ (\therefore \text{bounded})}} - \theta_1 \dot{q} - \underbrace{\theta_2 q}_{\text{constant}}) - \underbrace{\ddot{q}^d}_{\text{a constant}} + \underbrace{\lambda \dot{e}}_{\text{constant}}$$

Now to check for τ & $\dot{e}$,

$\tau = -Kr + \hat{\phi}$

& $\hat{\phi} = -\,\mathrm{sgn}(r)\left[\hat{\theta}_1 \dot{q} + \hat{\theta}_2 q + \ddot{q}_b^d - \lambda|\dot{e}|\right]$

∴ All terms in τ expression are bounded.

∴ τ is bounded

$\boxed{\therefore \dot{r} \in L_\infty}$

$\boxed{\Rightarrow \ddot{V} \in L_\infty}$

Lyapunov like lemma is satisfied.

$r = \dot{e} + \lambda e$.

$r \in L_\infty \Rightarrow \dot{e} \in L_\infty$.

because if $r = 0$
An LTI ← $\dot{e} = -\lambda e$
system sol^n is exponentially decaying.

Bounded input
⇒ Bound output
(BIBO)

ie $e = \underbrace{e^{-\lambda t} e(0)}_{\text{goes to zero}} + \int_{t-\psi}^{\psi} \overbrace{r(\psi)}^{\text{Bounded}}\, d\psi$.

⇒ e is also bounded.

$\boxed{\therefore \text{If } r \in L_\infty,\ e, \dot{e} \in L_\infty}$

$\therefore \lim_{t\to\infty} \dot{V} \to 0 \Rightarrow r \to 0$ as $t \to \infty$. $(\because \dot{V} \leq -kr^2)$

from Lyapunov like lemma.

Hence, the adaptive convergence of tracking error 'r' is achieved.

Now, $V = f(r, \tilde{\theta}_1, \tilde{\theta}_2)$

$\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1^* = \Delta\theta_1$

$\tilde{\theta}_2 = \hat{\theta}_2 - \theta_2^* = \Delta\theta_2$.

only $r \to 0$, not other variables.

$\therefore$ we cannot say V is asymptotically stable.

V is only Lyapunov stable with adaptive tracking error convergence.

($\because$ for Asymptotic stability, we need that $V \to 0$ i.e. stable + asymptotic convergence of the Lyapunov function which will happen only when all variables are going to zero as $t \to \infty$).

Advantages of this Adaptive Robust Control : (ARC)

→ Taking upperbounds (even though unknown) will reduce the # parameters ~~design~~ in the control design. i.e. design is independent (or) less dependent on the system's complexity.

→ # parameters = # update rules in general adaptive control. In ARC, we club the parameters together & squeeze the parameter space from N-dim to min. possible dimension.

(12)

Disadvantage:

→ The update rules i.e, $\hat{\theta}_0, \hat{\theta}_1$ are $\mathcal{L}_2 \cap \mathcal{L}_\infty$ &

$\mathcal{L}_2 \cap \mathcal{L}_\infty$. That is, only as $t \to \infty$, $r \to 0$ but (error)

$t \to \infty$ ~~does~~ never comes.

Practically, $\hat{\theta}_0, \hat{\theta}_1$ are monotonically increasing (Never decreasing)

∴ It has issues if we run for longer times.

Then, we can't prove $r \to 0$, we can only prove

UB ⇒ ~~Always~~ Trade off

Stating lyapunov like lemma: (Important for Non-autonomous systems)

If V satisfies

1. V is lower bounded
2. $\dot{V} \leq 0$, i.e, $V \in \mathcal{L}_\infty$ (i.e $\dot{V}$ is NSD).
3. $\dot{V}$ is uniformly continuous i.e, $\ddot{V}$ is $\mathcal{L}_\infty$.

Then, we can say that as $t \to \infty$, $\dot{V} \to 0$.

$$\lim_{t \to \infty} \dot{V} \to 0$$

here $\dot{V} \leq -kr^2$

$$\Rightarrow \lim_{t \to \infty} -kr^2 \to 0$$

i.e, $r \to 0$ as $t \to \infty$ (∵ k is constant).

(a)
(Q2) Given System is $\dot{x} = u$.

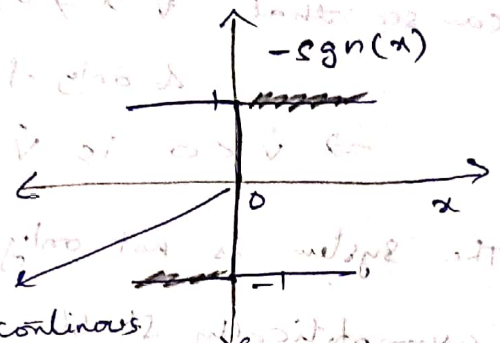
(i) $\boxed{u = -\operatorname{sgn}(x)}$ → Signum (or) Sign function

ie $\boxed{\dot{x} = -\operatorname{sgn}(x)}$ —— ①

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

→ $\boxed{|x| = x \cdot \operatorname{sgn}(x)}$ —— ②

(This is a sliding mode system)



Discontinous at '0'.

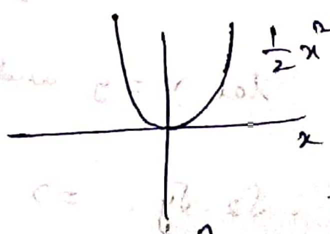
We need to comment on the stability of this system.

→ Consider a Lyapunov function.

(Assuming 'x' $\in \mathbb{R}$ ie 'x' is scalar).

$V = \frac{1}{2}x^2$ (or) $V = \frac{1}{2}x^T x$.

→ V is Positive definite ie for $\forall x \neq 0$, $V > 0$ & for $x = 0$ only $V = 0$. 'V' has only one argument 'x'.



→ V is Continuous & differentiable also.

$\dot{V} = x \cdot \dot{x}$.

→ V is also radially unbounded (from here)

ie $x \to \infty$, $V \to \infty$.

the only argument.

$\dot{V} = \frac{1}{2} \cdot 2 \cdot x \cdot \dot{x} = x\dot{x}$;

from eq ①,

$\dot{x} = -\operatorname{sgn}(x)$

$$\therefore \dot{V} = x(-\operatorname{sgn}(x))$$
$$= -x\operatorname{sgn}(x)$$
$$= -|x| \quad (\because \text{eq } ②).$$

We can see that $\forall x \neq 0, \dot{V} < 0.$

& only for $x = 0$ $\dot{V} = 0.$

$\Rightarrow \dot{V} < 0$ ie $\dot{V}$ is Negative Definite

ie The System is not only lyapunov Stable but also asymptotically stable. ie as $t \to \infty$, $V \to 0$.

Now,

$V = \frac{1}{2}x^2$

$\Rightarrow 2V = x^2$

$\Rightarrow |x| = \sqrt{2V}$

$\dot{V} = -|x|$

$$\Rightarrow \boxed{\dot{V} = -\sqrt{2V}} \quad \text{——③}$$

Now, to see what happens to this system's solution as $t \to \infty$. (ie V)

from equation ③, we cannot say it is exponentially decaying as the solution for V won't be exponential.

Solving ③, let $V = y$ and $t = x$.

Then $\frac{dy}{dx} = -\sqrt{2}.\sqrt{y}$. $\Rightarrow \frac{dy}{\sqrt{y}} = -\sqrt{2}\,dx$

(variable separable).

$$\int \frac{dy}{\sqrt{y}} = \int -\sqrt{2}\,dx$$

$$\Rightarrow \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -\sqrt{2}.x + c \Rightarrow 2y^{1/2} = -\sqrt{2}x + c$$

$$\Rightarrow \sqrt{y} = \frac{-x}{\sqrt{2}} + c$$

ie $y = \left(\frac{-x}{\sqrt{2}} + c\right)^2 \to$ soln.

Scanned with CamScanner

$\therefore V = \left(-\frac{t}{\sqrt{2}} + c\right)^2$

~~we can see that the convergence is quadratic~~

$\Rightarrow V$ is quadratic.

Now, $\dot{x} = -1 \quad x > 0$ → given system.

$\quad\quad 0 \quad x = 0$

$\quad\quad 1 \quad x < 0$

$\therefore$ solution is $x = -t + c, \; x > 0$

$= c, \; x = 0$

$= t + c, \; x < 0.$

More precisely,

$$x(t) = \begin{cases} x(0) - t & \text{when } x(0) > 0 \text{ — case (1)} \\ x(0) + t & \text{when } x(0) < 0 \text{ — case (2)} \\ 0 & \text{at } t = |x(0)| \end{cases}$$

ie $x(t)$ reaches 0 at $t = |x(0)|$ in each of the 2 cases.

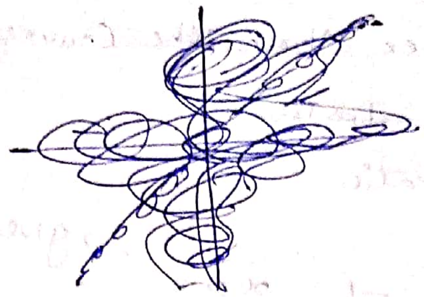
Finite Time Convergence Criteria :

if a system $\dot{x} = f(x,t)$, $f(t,0) = 0 \; \forall t$ (satisfied for this question), & if $\exists$ a lyapunov fun V & constant $c > 0$ & $\alpha \in (0,1)$ such that $\forall t$ & x $\dot{V}(x) \leq -c(V(x))^{\alpha}$, then we can say that all trajectories converge to the origin in finite time.

$\Rightarrow$ finite Time Stability

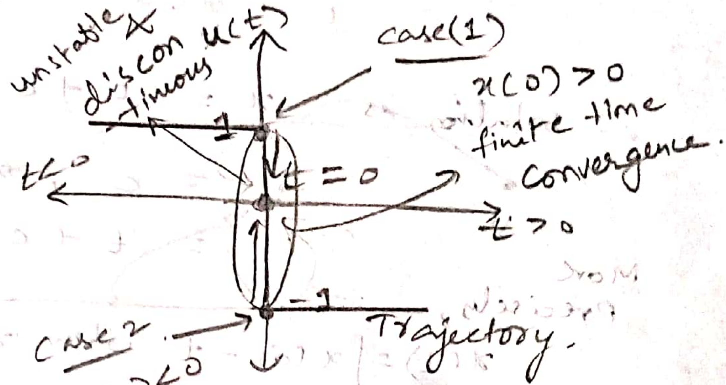
Here $\dot{V} \leq -\sqrt{2}\, V^{1/2}$ $\quad \therefore c = \sqrt{2}$ & $\alpha = 1/2$.

∴ The given system is finite time stable (ie $\dot{x} = -\mathrm{sgn}(x)$)



ie in a finite time the solutions of this asymptotic system will reach an equilibrium point (ie the origin). This time is called the settling time.

→ $t = |x(0)|$ is the settling time for this system. The solution reaches the origin in this time & remains there for all later time instants.



problem: we have very good convergence but we have chattering. It is due to the discontinuity in $u(x)$-function at $x=0$, due to high frequency oscillations (sensors & actuators may cause chattering), using the large gain in a controller & also due to the input control signal switching.

→ To avoid it many measures, like to avoid the $\mathrm{sgn}(x)$ discontinuity at $x=0$ (next question). Basically chattering comes through unmodelled dynamics, ie the motion in the vicinity of $x=0$ being unstable. (ie local instability)

Actuator / sensor dynamics.

chattering will wear them out

(PTO).

(Q2) (b) $\dot{x} = u$

$$u \begin{cases} = -\text{sgn}(x) & \text{when } |x| > \varepsilon. \\ = -\dfrac{x}{\varepsilon} & \text{when } |x| \le \varepsilon. \end{cases}$$

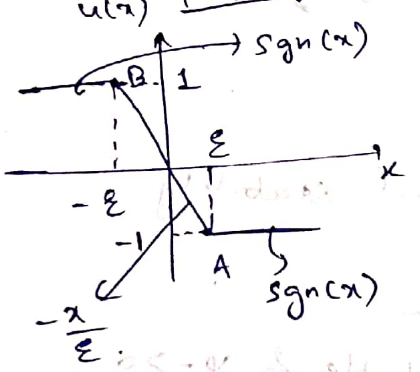
$\varepsilon > 0$ is a constant.

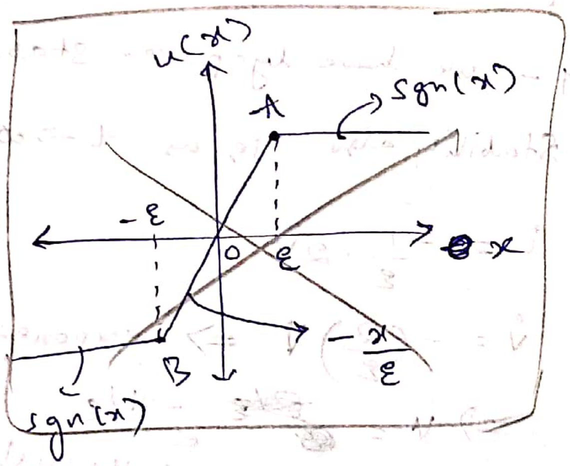
This kind of function is used for 'u' when we want to avoid the discontinuity at the origin due to the sgn function.

$$\therefore \dot{x} = \begin{cases} -\text{sgn}(x) & \text{if } |x| > \varepsilon \text{ i.e. } x < -\varepsilon \text{ or } x > \varepsilon. \\ -\dfrac{x}{\varepsilon} & \text{if } |x| \le \varepsilon \text{ i.e. } -\varepsilon \le x \le \varepsilon. \end{cases}$$

We can consider ε to be arbitrarily small, it is still valid.

u(x) Plot for 'u'





Now,
To analyse the stability of this system,

for $|x| > \varepsilon$, $u = -\text{sgn}(x)$ & $\dot{x} = -\text{sgn}(x)$, we have looked at its stability in Part (a).

→ Let us analyse the stability of the system in $|x| \le \varepsilon$

i.e. $\dot{x} = -\dfrac{x}{\varepsilon}$.

Consider the standard Lyapunov function

$V = \frac{1}{2}x^2$ (previously in other question, shown how it can be a valid Lyapunov function).

Now, $\dot{V} = x\dot{x}$

we have $\dot{x} = \frac{-x}{\varepsilon}$

$$\Rightarrow \dot{V} = x \cdot \left(\frac{-x}{\varepsilon}\right) = -\frac{x^2}{\varepsilon} = -\left(\underbrace{\frac{\overbrace{x^2}^{x \geq 0}}{\varepsilon}}_{>0}\right)$$

$\Rightarrow \dot{V}$ is Negative definite.

ie $\dot{V} < 0 \;\forall\; x \neq 0 \;\&\; -\varepsilon \leq x \leq \varepsilon$.

& $\varepsilon > 0$ always.

$\dot{V} = 0$ only for $x = 0$.

$\Rightarrow$ asymptotic convergence of the variable to the equilibrium point (ie origin)

Stability $\rightarrow$ we have Lyapunov stability & asymptotic Stability also. ie as $t \rightarrow \infty$, $V \rightarrow 0$.

Also, $\dot{V} = -\frac{1}{\varepsilon} \cdot 2V$.

$\dot{V} = -\left(\frac{2}{\varepsilon}\right)V \Rightarrow$ exponential stability.

$\Rightarrow V = e^{-2t/\varepsilon}$

$\therefore$ within the $|x| \leq \varepsilon$ region the system is stable & $V \rightarrow 0$ as $t \rightarrow \infty$.

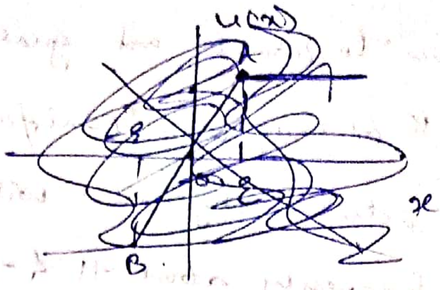
Note: We dont have finite time stability in $|x| \leq \varepsilon$.

$\therefore$ The trajectory (for $x > 0$) will come an settle at 'A' and those with $x < 0$, will settle at 'B' in a finite time because of $-\text{sgn}(x)$. From A or B, we cannot ensure that these trajectories will go to zero. (atleast not in a finite time, since $t \rightarrow \infty$ is never achieved)

Consider

$x > \varepsilon$.

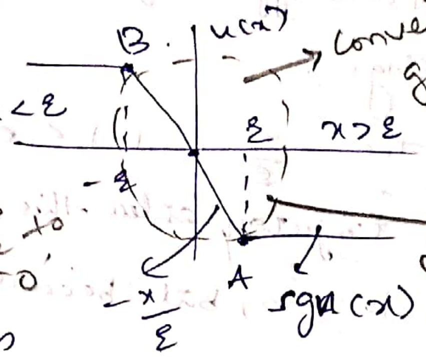
Then $\dot{x} = -\mathrm{sgn}(x)$



& we have seen with $V = \frac{1}{2}x^2$;

we have $\dot{V} = -\sqrt{2V} \Rightarrow$ finite time stability

$\therefore$ we can measure this settling time. If it was $-\mathrm{sgn}(x)$, we can guarantee that this trajectory will converge to zero (or) origin in finite time.



ε is close to 0

convergence guaranteed in this x region.

called as 'Boundary layer' with Linear Saturation fn.

But Now, because the function is truncated at $x = \varepsilon$, the new settling point is A.

If we can assure finite time stability in $|x| \le \varepsilon$ also, we can still say the trajectory will go to origin in finite time. But it is not assured.

$\therefore$ when $t \to \infty$ only, in the region $x \le |\varepsilon|$, the trajectory will go to zero. It is expontially stable in there but wont reach zero (in finite time).

$\therefore$ This Continuous approximation eliminates chattering (no discontinuities) but guarantees convergence to a small set (we make ε very close to zero for max. Convergence to the origin) around the origin rather than to the origin itself.

→ How is chattering avoided? & other advantages

→ We got a Continuous Controller.

→ Disadvantage is that from the Boundary Layer, further

Convergence to zero is not guarenteed. It is a part of the class ". Globally UUB "(uniform ~~ultl~~ ultimate Boundedness)"

→ The actuator dynamics will no longer switch with high frequency about +1 & -1 around the origin. Their durability is also increased.

→ with trade off origin convergence, we could reduce chattering.

→ ~~Ano~~ The $-\frac{x}{\varepsilon}$ ($|x| \le e$) part is called the boundary & ~~beg~~ after the trajectory enters this, we cant guarentee when it will go to zero, but because of exp. stability we can say as $t \to \infty$. ($\therefore$ No guarentee). This is a linear saturation fn and it might still have problems due to switching because of sharp ends i.e A & B points.

→ Because of no discontinuity, chattering is reduced.