TFA - Assignment - 1

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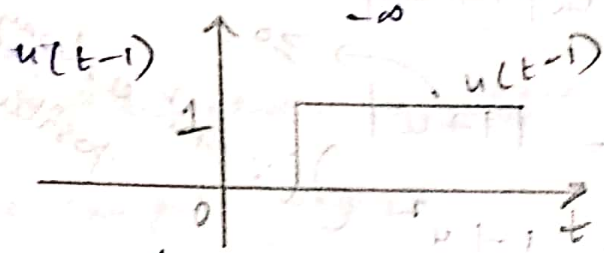
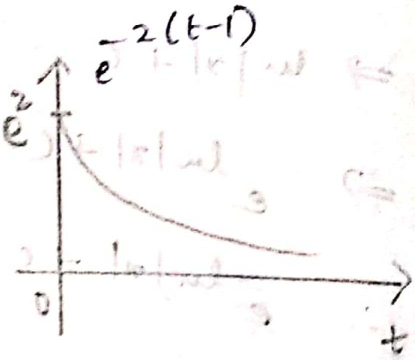
(1) To find the fourier transform of the given signals

(i) $e^{-2(t-1)} \cdot u(t-1) = f(t)$

W.K.T, $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$





$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

But,
$u(t-1) = \begin{cases} 1, & t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$

$\therefore \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-2(t-1)}\, u(t-1)\, e^{-j\omega t}\, dt$

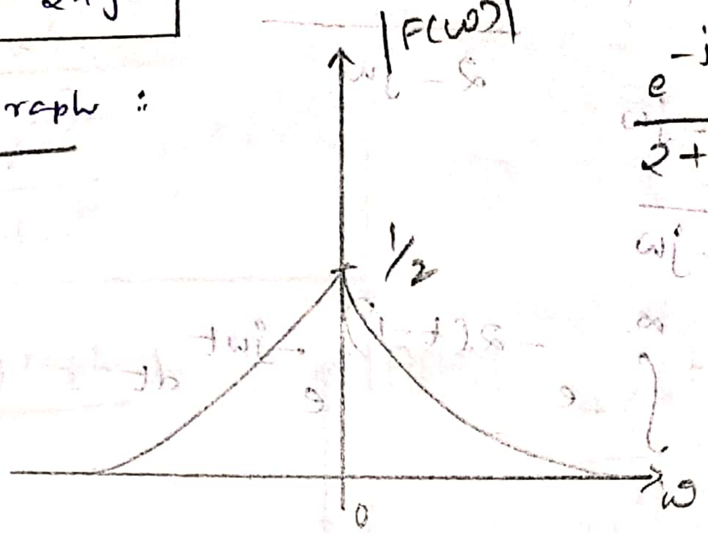
$= \int_{1}^{\infty} e^{-2(t-1)} \cdot e^{-j\omega t}\, dt$

$= \int_{1}^{\infty} e^{-2t - j\omega t + 2}\, dt = \left. \dfrac{e^{-2t - j\omega t + 2}}{-2 - j\omega} \right|_{1}^{\infty}$

$= 0 - \dfrac{e^{-\cancel{2} - j\omega + \cancel{2}}}{-2 - j\omega} = \dfrac{e^{-j\omega}}{2 + j\omega}$

$$\therefore \boxed{F(\omega) = \frac{e^{-j\omega}}{2+j\omega}}$$

$|F(\omega)|$ graph :



$$\frac{e^{-j(0)}}{2+j(0)} = \frac{1}{2}$$

(ii) $e^{-2|t-1|} = f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-2|t-1|} \cdot e^{-j\omega t} \, dt$$

W.K.T,

$|t-1| = t-1$ if $(t-1) > 0$ i.e $t > 1$

$|t-1| = -(t-1)$ if $(t-1) \leq 0$ i.e $t \leq 1$

$$F(\omega) = F_1(\omega) + F_2(\omega)$$

$$F_1(\omega) = \int_{-\infty}^{1} e^{-2(-(t-1))} e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{1} e^{2(t-1)} e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{1} e^{2t - j\omega t - 2} \, dt = \left[\frac{e^{2t - j\omega t - 2}}{2 - j\omega} \right]_{-\infty}^{1}$$

$$= -0 + \frac{e^{2-j\omega-2}}{2-j\omega}$$

$$= +\frac{e^{-j\omega}}{2-j\omega}$$

$$F_2(\omega) = \int_1^{\infty} e^{-2(t-1)} e^{-j\omega t}\, dt$$

$$= \int_1^{\infty} e^{-2t+2-j\omega t}\, dt$$

$$= \left. \frac{e^{-2t+2-j\omega t}}{-2-j\omega} \right|_1^{\infty}$$

$$= 0 - \frac{e^{-2+2-j\omega}}{-2-j\omega}$$

$$= \frac{e^{-j\omega}}{2+j\omega}$$

$$\therefore F(\omega) = F_1(\omega) + F_2(\omega)$$

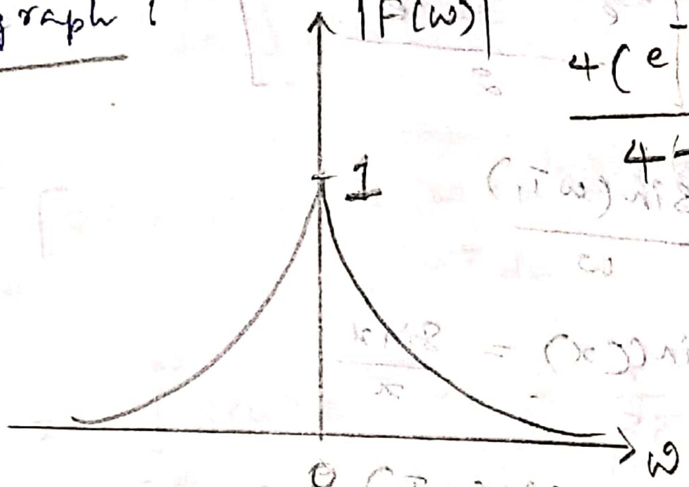
$$= +\frac{e^{-j\omega}}{2-j\omega} + \frac{e^{-j\omega}}{2+j\omega}$$

$$= \frac{(2+j\omega)(+e^{-j\omega}) + (e^{-j\omega})(2-j\omega)}{(2-j\omega)(2+j\omega)}$$

$$= \frac{e^{-j\omega}}{4+\omega^2} \cdot (2 - j\omega + 2 + j\omega)$$

$$= \boxed{\frac{4 e^{-j\omega}}{4+\omega^2} = F(\omega)}$$

$|F(\omega)|$ graph :



$$\frac{4(e^{-j(0)})}{4+0} = 1$$

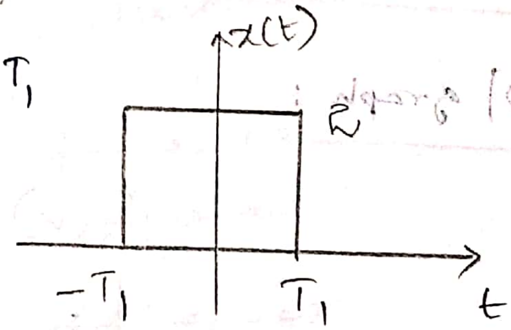
(iii) $x(t) = \begin{cases} 2, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$

$x(t) \rightarrow$ A Rectangular pulse

$|t| < T_1 \Rightarrow -T_1 < t < T_1$

$|t| > T_1 \Rightarrow t > T_1$ &

$t < -T_1$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T} 2 \cdot e^{-j\omega t}\, dt = 2 \int_{-T_1}^{T} e^{-j\omega t}\, dt$$

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$$= 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1} = \frac{2}{-j\omega}\left[e^{-j\omega T_1} - e^{+j\omega T_1}\right]$$

$$= \frac{2}{-j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

$$= \frac{4}{j\omega}\left[\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2}\right]$$

$$= 4 \cdot \frac{\sin(\omega T_1)}{\omega}$$

w.k.T $\operatorname{sinc}(x) = \frac{\sin x}{x}$

$$\therefore \Rightarrow 4 \cdot \frac{\sin(\omega T_1)}{(\omega \cdot T_1)} \cdot (T_1)$$

$$\Rightarrow 4T_1 \operatorname{sinc}(\omega T_1)$$

Comparing with

$$\therefore \boxed{X(\omega) = 4T_1 \operatorname{sinc}(\omega T_1)} = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

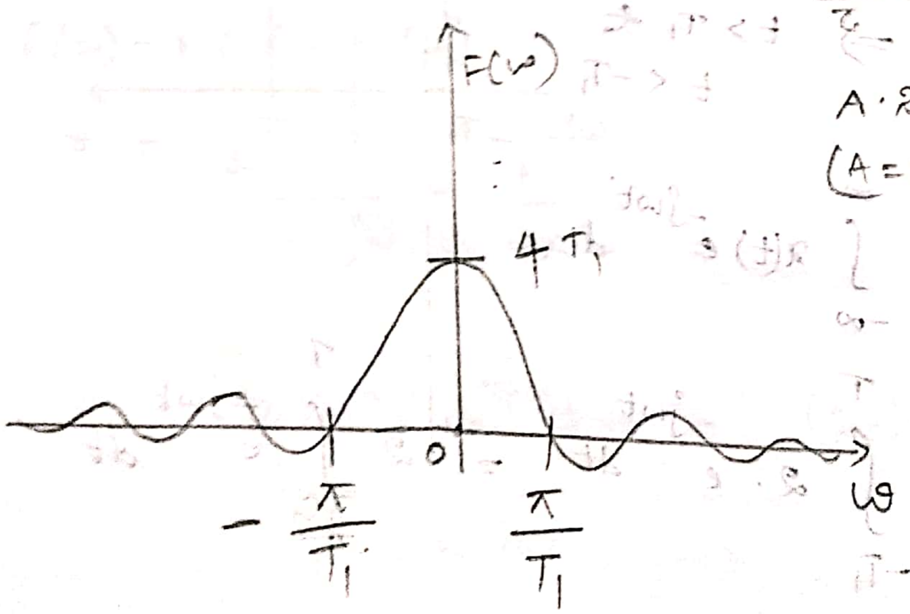
$T_1 = \frac{\tau}{2}$

$\frac{2\pi}{\tau} = \frac{2\pi}{2T_1}$

$A \cdot 2T_1$

$(A = 2)$

<u>$|X(\omega)|$ graph :</u>



2) Find inverse fourier transforms of the given signals.

(a) $X(jw) = 2\pi\delta(w) + \pi\delta(w-4\pi) + \pi\delta(w+4\pi)$

w.k.T

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[2\pi\delta(w) + \pi\delta(w-4\pi) + \pi\delta(w+4\pi)\right] e^{jwt}\, dw$$

$$= \frac{1}{2\pi}\cdot 2\pi\int_{-\infty}^{\infty} \delta(w)\, e^{jwt}\, dw + \frac{1}{2\pi}\cdot\pi\int_{-\infty}^{\infty}\delta(w-4\pi)\, e^{jwt}\, dw + \frac{1}{2\pi}\cdot\pi\int_{-\infty}^{\infty}\delta(w+4\pi)\, e^{jwt}\, dw \quad \text{--- (1)}$$

w.k.T,

$$\delta(w) = \begin{cases} 1, & \text{for } w = 0 \\ 0, & \text{else where} \end{cases}$$

$$\delta(w-4\pi) = \begin{cases} 1, & \text{for } w = 4\pi \\ 0, & \text{else where} \end{cases}$$

$$\delta(w+\pi\cdot 4) = \begin{cases} 1, & \text{for } w = -4\pi \\ 0, & \text{elsewhere} \end{cases}$$

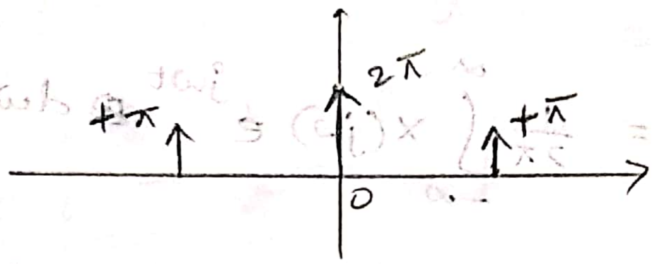
$\therefore$ (1) $\Rightarrow$

$$x(t) = \int_{-\infty}^{\infty} \delta(w)\, e^{jwt}\, dw + \frac{1}{2}\int_{-\infty}^{\infty} e^{jwt}\cdot\delta(w-4\pi)\, dw + \frac{1}{2}\int_{-\infty}^{\infty} e^{jwt}\,\delta(w+4\pi)\, dw$$

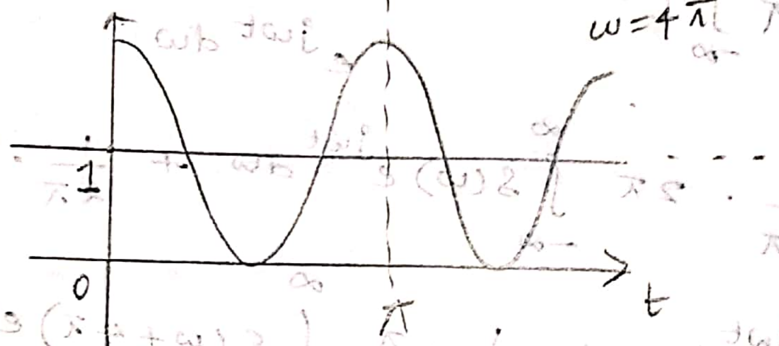
$$x(t) = 1 + \frac{1}{2} e^{4\pi j t} + \frac{1}{2} \cdot e^{-4\pi j t}$$

$$\therefore \boxed{x(t) = 1 + \cos(4\pi t)}$$

$X(j\omega)$

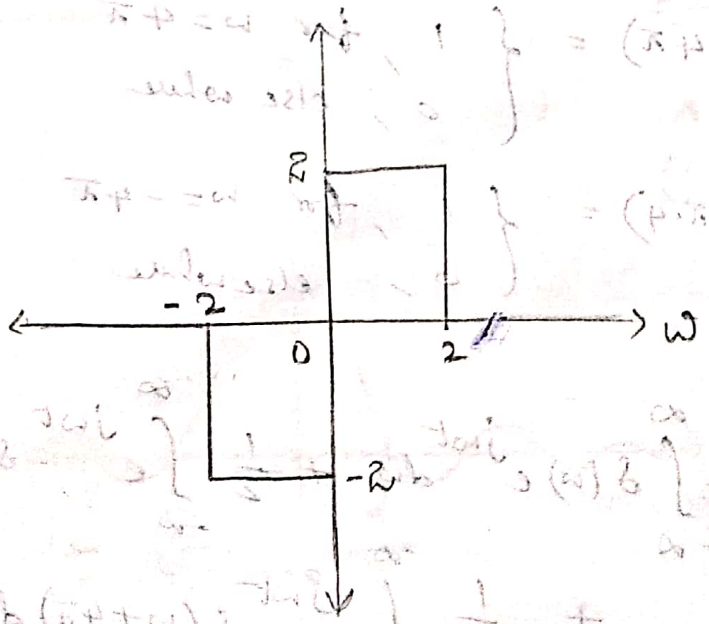


$x(t)$



(b) $X(j\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 2 \end{cases}$

$X(j\omega)$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-2}^{0} -2\, e^{j\omega t}\, d\omega + \int_{0}^{2} 2\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{\pi}\left[-\left.\frac{e^{j\omega t}}{jt}\right|_{-2}^{0} + \left.\frac{e^{j\omega t}}{jt}\right|_{0}^{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{jt}\left(1 - e^{-2jt}\right) + \frac{1}{jt}\left(e^{2jt} - 1\right)\right]$$

$$= \frac{1}{j\pi t}\left[-1 + e^{-2jt} + e^{2jt} - 1\right]$$

$$= \frac{2}{2j\pi t}\left[e^{-2jt} + e^{2jt} - 2\right]$$

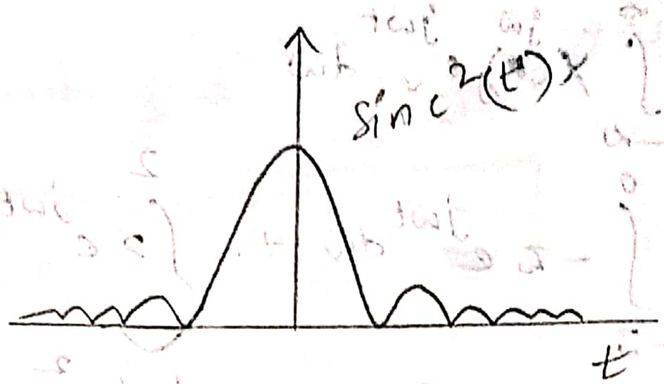
$$= \frac{2}{j\pi t}\left[\frac{e^{-2jt} + e^{2jt}}{2} - 1\right]$$

$$= \frac{2}{j\pi t}\left[\cos 2t - 1\right]$$

$$= \frac{2(-2\sin^2 t)}{j\pi t} = \frac{-4\sin^2 t}{j\pi t \cdot t}\cdot t$$

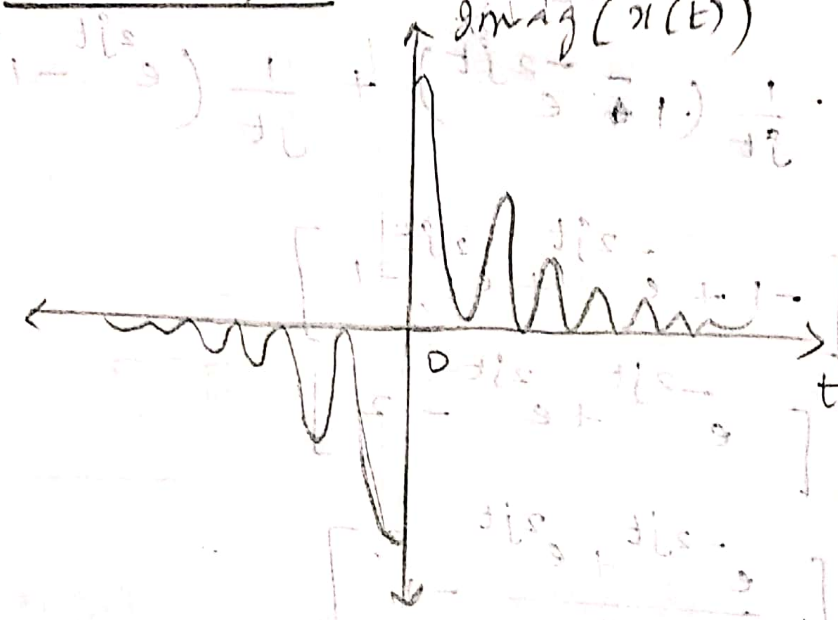
$$= \frac{4jt}{\pi}\cdot\frac{\sin^2 t}{t^2}$$

$$\boxed{x(t) = \frac{4jt}{\pi}\cdot \operatorname{sinc}^2 t}$$



Real(x(t)) = 0

Img(x(t)) plot :



(3) To find the fourier transform of the signal,

$$x(t) = \frac{2}{1+t^2}$$

Solving this with direct formula makes integration difficult. But the structure looks like properties of Fourier transform can be used.

W.K.T, Duality property is

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi \cdot x(-\omega)$$

W.K.T, $F.T\left(e^{-a|t|}\right) = \frac{2a}{a^2+\omega^2}$

let $a=1$

$$F.T\left(e^{-|t|}\right) = \frac{2}{1+\omega^2}$$

$|t| = t \quad t>0$

$|t| = -t \quad t<0$

$$\int e^{-|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{+t-j\omega t} dt + \int e^{-t-j\omega t} dt$$

$$= \left.\frac{e^{+t-j\omega t}}{+1-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-t-j\omega t}}{-1-j\omega}\right|_{0}^{\infty}$$

$$= -0 + \frac{1}{+1-j\omega} + 0 - \frac{1}{-1-j\omega}$$

$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$$

let $g(t) = e^{-|t|}$

$$\xleftrightarrow{F.T} G(\omega) = \frac{2}{1+\omega^2}$$

By duality,

$$G(t) = \frac{2}{1+t^2}$$

$$\xleftrightarrow{I.F.T} 2\pi \cdot g(-\omega)$$

$$= 2\pi \cdot e^{-|-\omega|}$$

$$= 2\pi e^{-|\omega|}$$

$$\therefore \boxed{X(\omega) = 2\pi e^{-|\omega|}}$$

(4) To prove the linearity property of F.T

<u>Linearity property:</u>

let $x(t) \longleftrightarrow X(\omega) = FT\{x(t)\}$

$y(t) \longleftrightarrow Y(\omega) = FT\{y(t)\}$

Then, $\alpha x(t) + \beta y(t) \longleftrightarrow \alpha X(\omega) + \beta Y(\omega)$ —— (a)

α, β are scalars & constants.

ie superposition $x(t) + y(t) \longleftrightarrow X(\omega) + Y(\omega)$

& homogenity $\alpha x(t) \longleftrightarrow \alpha X(\omega)$

should be satisfied.

let $z(t) = \alpha x(t) + \beta y(t)$.

~~Applying~~ To find $F.T\{z(t)\} = Z(\omega)$

$$= \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$Z(\omega) = \int_{-\infty}^{\infty} \left(\alpha x(t) + \beta y(t)\right) e^{-j\omega t} dt$$

$$= \alpha \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt + \beta \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} dt$$ —— (1)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$ —— (2)

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$ —— (3)

Using (2) & (3) in (1), we have,

$$Z(\omega) = \alpha\,(X(\omega)) + \beta\,(Y(\omega))$$

$\Longrightarrow$ RHS.

$$\therefore Z(\omega) = \alpha X(\omega) + \beta Y(\omega).$$

$$\& \quad z(t) = \alpha x(t) + \beta y(t).$$

Hence proved @

(5) To find F.T using properties

(a) $x(t) = \dfrac{d^2}{dt^2} x(t-1)$.

From the differentiation property of the F.T,

$$\text{let} \quad x(t) \xleftrightarrow{F.T} X(\omega)$$

$$\frac{d}{dt}\{x(t)\} \longleftrightarrow j\omega X(\omega) = X_1(\omega)$$

$$\frac{d^2}{dt^2}\{x(t)\} \longleftrightarrow j\omega X_1(\omega)$$

$$= j\omega \cdot j\omega X(\omega)$$

$$= j^2\omega^2 X(\omega)$$

$$= -\omega^2 X(\omega)$$

$$\therefore \frac{d^2}{dt^2} x(t) \xleftrightarrow{F.T} -\omega^2 X(\omega)$$

from time scaling property, we have that

$$\text{let} \quad x(t) \longleftrightarrow X(\omega)$$

$$\longleftrightarrow X(\omega) \cdot e^{-j\omega t_0}$$

$x(t-t_0)$

Here $t_0 = 1$.

$$\therefore x(t-1) \longleftrightarrow X(\omega) \cdot e^{-j\omega} = X_1(\omega)$$

from differentiation property of the F.T,

$$\frac{d}{dt}\{x(t-1)\} \longleftrightarrow j\omega\, X_1(\omega)$$
$$= j\omega \cdot X(\omega) e^{-j\omega} = X_2(\omega)$$

$$\frac{d^2}{dt^2}\{x(t-1)\} \longleftrightarrow j\omega \cdot X_2(\omega)$$
$$= j\omega \cdot j\omega \cdot X(\omega) e^{-j\omega}$$
$$= j^2 \omega^2 X(\omega) e^{-j\omega}$$
$$= -\omega^2 X(\omega) e^{-j\omega}$$

$$\boxed{\therefore \frac{d^2}{dt^2}(x(t-1)) = -\omega^2 X(\omega) e^{-j\omega}}$$

(b) $x(t) = x(3t-6)$

We can shift first & Scale next.

let $x(t) \longleftrightarrow X(\omega)$

$$x(t-6) \longleftrightarrow X(\omega) \cdot e^{-j\omega 6}$$

from the time Scaling property we have,

$$x(t) \longleftrightarrow X(\omega)$$
$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

let $x_1(t) = x(t-6)$
$$X_1(\omega) = X(\omega) \cdot e^{-j\omega 6}$$

Scaling, $x_1(3t) \longleftrightarrow \frac{1}{3} \cdot X_1\left(\frac{\omega}{3}\right)$
$$= \frac{1}{3} X\left(\frac{\omega}{3}\right) \cdot e^{-j\frac{\omega}{3} \cdot 6}$$

$(\because x_1(3t) = x(3t-6))$

$$\Rightarrow \boxed{\frac{1}{3} X\left(\frac{\omega}{3}\right) e^{-2j\omega} = X(\omega) \quad \text{required}}$$

(c) . $x(t) = x(1-t) + x(-t-1)$

By using linearity, time shifting & time scaling properties,

let $x(t) \longleftrightarrow X(\omega)$

To find $F.T\{x(1-t)\}$

ie $x(-t+1)$,

$x(t+1) \longleftrightarrow e^{+j\omega} \cdot X(\omega)$.

let $x_1(t) = x(t+1)$; $X_1(\omega) = e^{j\omega} X(\omega)$.

$x_1(-t) = x(-t+1) = x(1-t)$

Now,

$$x_1(-t) \longleftrightarrow \frac{1}{|-1|} \cdot X_1\left(\frac{\omega}{-1}\right) = X_1(-\omega)$$

$$= e^{-j\omega} \cdot X(-\omega).$$

$\therefore x(1-t) \longleftrightarrow e^{-j\omega} X(-\omega)$ —— (1)

To find $F.T\{x(-t-1)\}$

$x(t-1) \longleftrightarrow e^{-j\omega} \cdot X(\omega)$.

let $x_2(t) = x(t-1)$

& $X_2(\omega) = e^{-j\omega} X(\omega)$

$x_2(-t) = x(-t-1)$

$$x_2(-t) \longleftrightarrow \frac{1}{|-1|} \cdot X_2\left(\frac{\omega}{-1}\right) = X_2(-\omega)$$

$$= e^{j\omega}(X(-\omega))$$

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$$\therefore x(-1-t) \longleftrightarrow e^{j\omega} X(-\omega) \quad \text{—— (2)}$$

from linearity property

$$\alpha x(t) + \beta y(t) \longleftrightarrow \alpha X(\omega) + \beta Y(\omega)$$

here α, β are 1

$$\therefore F.T\{ x(1-t) + x(-t-1) \}$$

$$= e^{-j\omega} X(-\omega) + e^{j\omega} X(-\omega)$$

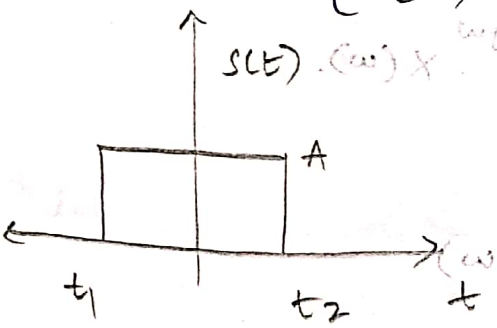
$$= 2X(-\omega)\left[\frac{e^{-j\omega} + e^{j\omega}}{2}\right]$$

$$= 2X(-\omega) \cdot \cos(\omega)$$

$$\therefore \boxed{\text{Ans: } 2\cos\omega \cdot X(-\omega)}$$

(Q1) To find t_0 or $\langle t \rangle$, σ_t^2, ω_0 or $\langle \omega \rangle$, σ_ω^2 for the signal

$$s(t) = \begin{cases} A, & t_1 \le t \le t_2 \\ 0, & \text{otherwise} \end{cases}$$



$$E = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{t_1}^{t_2} A^2 \, dt$$

$$= A^2(t_2 - t_1)$$

Now,

$$\langle t\rangle = t_0 = \frac{1}{E}\int_{-\infty}^{\infty} t\cdot|s(t)|^2\,dt$$

$$= \frac{1}{A^2(t_2-t_1)}\left[\int_{t_1}^{t_2} t\cdot A^2\,dt\right]$$

$$= \frac{1}{\cancel{A^2}(t_2-t_1)}\left[\cancel{A^2}\cdot \frac{t^2}{2}\Big|_{t_1}^{t_2}\right]$$

$$= \frac{1}{(t_2-t_1)}\cdot\frac{t_2^2-t_1^2}{2} = \frac{t_2+t_1}{2}.$$

$$\sigma_t^2 = \langle t^2\rangle - \langle t\rangle^2 = \langle t^2\rangle - t_0^2$$

$$\langle t^2\rangle = \frac{1}{E}\int_{-\infty}^{\infty} t^2\,|s(t)|^2\,dt$$

$$= \frac{1}{\cancel{A^2}(t_2-t_1)}\left[\int_{t_1}^{t_2}\cancel{A^2}\,t^2\,dt\right]$$

$$= \frac{1}{t_2-t_1}\cdot\frac{t_2^3-t_1^3}{3}$$

$$= \frac{t_2^2+t_1^2+t_1t_2}{3}$$

$$\sigma_t^2 = \frac{t_2^2+t_1^2+t_1t_2}{3} - \frac{(t_2^2+t_1^2+2t_1t_2)}{4}$$

$$= \frac{t_2^2+t_1^2-2t_1t_2}{12} = \frac{(t_2-t_1)^2}{12}$$

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$$\therefore \sigma_t = \frac{t_2-t_1}{\sqrt{12}}$$

$$S(\omega) = \int_{-\infty}^{\infty} s(t) \cdot e^{-j\omega t} dt$$

$$= \int_{t_1}^{t_2} A \cdot e^{-j\omega t} dt$$

$$= A \cdot \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{t_1}^{t_2}$$

$$= \frac{A}{-j\omega}\left[e^{-j\omega t_2} - e^{-j\omega t_1} \right]$$

$$S(\omega) = \frac{Aj}{\omega}\left[e^{-j\omega t_2} - e^{-j\omega t_1} \right]$$

To find ω_0 or $\langle \omega \rangle$,

$$\langle \omega \rangle = \omega_0 = \frac{1}{E} \int_{-\infty}^{\infty} \omega \, |S(\omega)|^2 d\omega.$$

To find

$$|S(\omega)|^2 = \left| \frac{Aj}{\omega} \left(e^{-j\omega t_2} - e^{-j\omega t_1} \right) \right|^2$$

$$= \left| \frac{Aj}{\omega} \left(\cos(\omega t_2) - j\sin(\omega t_2) - \cos(\omega t_1) + j\sin(\omega t_1) \right) \right|^2$$

$$= \left(\frac{A}{\omega}\right)^2 \left| j\left(\cos\omega t_2 - \cos\omega t_1 \right) - j^2 \left(\sin\omega t_2 - \sin\omega t_1 \right) \right|^2$$

$$= \frac{A^2}{\omega^2}\left|(\sin\omega t_2 - \sin\omega t_1) + j(\cos\omega t_2 - \cos\omega t_1)\right|^2$$

A complex no.

$$= \frac{A^2}{\omega^2}\left[(\sin\omega t_2 - \sin\omega t_1)^2 + (\cos\omega t_2 - \cos\omega t_1)^2\right]$$

$$= -\frac{A^2}{\omega^2}\left[2\sin\omega t_2 \sin\omega t_1 + 2\cos\omega t_2 \cdot \cos\omega t_1 - 2\right]$$

$$= -\frac{A^2}{\omega^2}\left[2(\cos(\omega t_2 - \omega t_1)) - 2\right]$$

$$= \frac{2A^2}{\omega^2}\left[1 - \cos\omega(t_2 - t_1)\right]$$

$$= \frac{2A^2}{\omega^2} \cdot 2\sin^2\left(\frac{\omega t_2 - \omega t_1}{2}\right)$$

$$= \frac{4A^2}{\omega^2}\left[\sin^2\left(\frac{\omega(t_2 - t_1)}{2}\right)\right]$$

$$\therefore \omega_0 = \frac{1}{\cancel{A^2}(t_2 - t_1)} \int_{-\infty}^{\infty} \cancel{\omega} \cdot \frac{4\cancel{A^2}}{\omega^{\cancel{2}}} \sin^2\left(\frac{\omega(t_2 - t_1)}{2}\right) d\omega$$

$$= \frac{4}{(t_2 - t_1)} \int_{-\infty}^{\infty} \frac{1}{\omega} \sin^2\left(\frac{\omega(t_2 - t_1)}{2}\right) d\omega$$

goes to zero

function is odd.

$$\Rightarrow \boxed{\omega_0 = \langle \omega \rangle = 0}$$

$$(\because E = A^2(t_2 - t_1))$$

mean freq is zero

PTO.

$$\langle \omega^2 \rangle = \frac{1}{E} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 \, d\omega$$

$$= \frac{1}{A^2 (t_2 - t_1)} \int_{-\infty}^{\infty} \cancel{\omega^2} \cdot \frac{4 A^2}{\cancel{\omega^2}} \sin^2\left(\frac{\omega(t_2 - t_1)}{2}\right) d\omega$$

$$= \frac{4}{(t_2 - t_1)} \int_{-\infty}^{\infty} \sin^2\left(\frac{\omega t_2 - \omega t_1}{2}\right) d\omega$$

$$= \frac{2}{(t_2 - t_1)} \int_{-\infty}^{\infty} \left[1 - \cos(\omega t_2 - \omega t_1)\right] d\omega$$

$$\longrightarrow \infty$$

$$\sigma_\omega^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2$$

$$= \infty - 0 \longrightarrow \infty$$

$$\boxed{\sigma_\omega \to \infty}$$

$\therefore$ The spread of a sinc function is infinity.

(Q2) $s(t) = \left(\frac{\alpha}{\pi}\right)^{1/4} \cdot e^{-\frac{\alpha(t-t_0)^2}{2} + j\phi(t)}$

To find t_0, σ_t, ω_0, σ_ω.

$$E = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

$$|s(t)| = \left| \left(\frac{\alpha}{\pi}\right)^{1/4} \cdot e^{-\frac{\alpha(t-t_0)^2}{2}} \cdot e^{j\phi(t)} \right|$$

WKT $\quad |e^{j\phi(t)}| = 1$

$$\therefore \; E = \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot e^{-\alpha \frac{(t-t_0)^2 \cdot 2}{2}} dt$$

$(\because |e^{x}| = e^{x} \;\because e^{x} > 0)$

$$\therefore \; E = \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot e^{-\alpha (t-t_0)^2} dt$$

let $t - t_0 = P, \; dt = dp$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot e^{-\alpha p^2} dp$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\alpha p^2} dp}_{= \sqrt{\frac{\pi}{\alpha}}}$$

$$= \sqrt{\frac{\alpha}{\pi}} \cdot \sqrt{\frac{\pi}{\alpha}} = 1 \qquad \boxed{E = 1}$$

$$\langle t \rangle = \frac{1}{1} \int_{-\infty}^{\infty} t \, |s(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} t \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha (t-t_0)^2} dt$$

Put $t - t_0 = p \;;\; dt = dp$

$$= \int_{-\infty}^{\infty} (p + t_0) \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot e^{-\alpha p^2} dp$$

$$= \int_{-\infty}^{\infty} p \cdot \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha p^2} dp + \underbrace{t_0 \left(\frac{\alpha}{\pi}\right)^{1/2} \int e^{-\alpha p^2} dp}_{= t_0}$$

~~Let $p^2 = q$~~.

$$\langle t \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2}\left[\underbrace{\int_{-\infty}^{\infty} p \cdot e^{-\alpha p^2}\, dp}_{=0 \text{ (odd function)}}\right] + t_0$$

$$= t_0 + \left(\frac{\alpha}{\pi}\right)^{1/2}\left[\int_{-\infty}^{0} p \cdot e^{-\alpha p^2}\, dp + \int_{0}^{\infty} p e^{-\alpha p^2}\, dp\right]$$

$$= t_0 + \left(\frac{\alpha}{\pi}\right)^{1/2}\left[\cancel{-\int_{0}^{\infty} p e^{-\alpha p^2}\, dp} + \cancel{\int_{0}^{\infty} e^{-\alpha p^2} p\, dp}\right]$$

$$= \boxed{t_0 = \langle t \rangle}$$

$$\langle t^2 \rangle = \int_{-\infty}^{\infty} t^2 |s(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} t^2 \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha (t-t_0)^2}\, dt$$

put $t - t_0 = p \Rightarrow dt = dp$

$t^2 + t_0^2 - 2 t t_0 = p^2$

$$\left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \left(p^2 - t_0^2 + 2 t t_0\right) e^{-\alpha p^2}\, dp.$$

$$\cancel{= \left(\frac{\alpha}{\pi}\right)^{1/2}\left[\int_{-\infty}^{\infty} p^2 e^{-\alpha p^2}\, dp - t_0^2 \int_{-\infty}^{\infty} e^{-\alpha p^2}\, dp\right.}$$

$$\cancel{+\, 2(p+t_0)t_0}$$

here, $p^2 - t_0^2 + 2(p+t_0)t_0$

$= p^2 - t_0^2 + 2p t_0 + 2 t_0^2$

$= p^2 + t_0^2 + 2p t_0.$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \left[\int_{-\infty}^{\infty} p^2 e^{-\alpha p^2} dp + 2t_0 \underbrace{\int_{-\infty}^{\infty} p\, e^{-\alpha p^2} dp}_{=0} + t_0^2 \int_{-\infty}^{\infty} e^{-\alpha p^2} dp \right]$$

(odd function)

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \left[\int_{-\infty}^{\infty} p^2 e^{-\alpha p^2} dp + t_0^2 \left(\frac{\pi}{\alpha}\right)^{1/2} \right]$$

let $\sqrt{\alpha} \cdot p = q \Rightarrow \sqrt{\alpha} \cdot dp = dq$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \left[\int_{-\infty}^{\infty} \frac{q^2}{\alpha} e^{-q^2} \frac{dq}{\sqrt{\alpha}} + t_0^2 \left(\frac{\pi}{\alpha}\right)^{1/2} \right]$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \left[\frac{1}{\alpha\sqrt{\alpha}} \underbrace{\int q^2 e^{-q^2} dq}_{= \frac{\sqrt{\pi}}{2}} + t_0^2 \left(\frac{\pi}{\alpha}\right)^{1/2} \right]$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \left[\frac{\sqrt{\pi}}{2\alpha\sqrt{\alpha}} + t_0^2 \left(\frac{\pi}{\alpha}\right)^{1/2} \right]$$

$$= t_0^2 + \frac{1}{2\alpha} = \langle t^2 \rangle$$

$$\sigma_t^2 = \langle t^2 \rangle - \langle t \rangle^2$$

$$= t_0^2 + \frac{1}{2\alpha} - t_0^2 = \frac{1}{2\alpha}$$

$$\boxed{\therefore \sigma_t = \sqrt{\frac{1}{2\alpha}}}$$

To find $\langle \omega \rangle$, σ_ω

$$\langle \omega \rangle = \omega_0 = \frac{1}{\epsilon} \int_{-\infty}^{\infty} \omega \, |s(\omega)|^2 d\omega$$

$$S(\omega) = \int_{-\infty}^{\infty} s(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha}{2}(t-t_0)^2 + j\phi t - j\omega t} dt$$

let $t - t_0 = p \Rightarrow t = p + t_0 \quad dt = dp$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/4} \cdot e^{-\frac{\alpha}{2}p^2 + (j\phi - j\omega)(p + t_0)} dp$$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{1/4} \cdot e^{-\frac{\alpha}{2}p^2 + j t_0(\phi - \omega) + pj(\phi - \omega)} dp$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/4} \cdot e^{j t_0(\phi - \omega)} \cdot \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}p^2 + pj(\phi - \omega)} dp.$$

We can use the equation 1.29 from Cohen Text book, that is

$$\int \omega \, |S(\omega)|^2 d\omega = \int s^*(t) \cdot \frac{1}{j} \frac{d}{dt} s(t) \, dt$$

$$= \langle \omega \rangle = \omega_0$$

To find $\frac{1}{j} \frac{ds(t)}{dt}$

$$s(t) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha}{2}(t-t_0)^2 + j\phi(t)}$$

$$\frac{ds(t)}{dt} = \left(\frac{\alpha}{\pi}\right)^{1/4} \cdot e^{-\frac{\alpha(t-t_0)^2}{2} + j\phi(t)} \left(-\frac{\alpha}{2} \cdot 2(t-t_0) + j\cancel{\frac{d\phi(t)}{dt}}\right)$$

$= 0$ ($\phi(t)$ is const or linear)

$$= s(t)\cdot\left(-\alpha(t-t_0)\right)$$

$$\frac{1}{j}\frac{ds(t)}{dt} = \frac{-s(t)\,\alpha\,(t-t_0)}{j} = j\alpha\, s(t)(t-t_0).$$

Now,

$$\langle\omega\rangle = \int \omega\,|s(\omega)|^2\,d\omega = \int s^*(t)\cdot j\alpha\, s(t)\cdot(t-t_0)\,dt.$$

$$s^*(t)\cdot s(t) = |s(t)|^2$$

$$\therefore \langle\omega\rangle = \int_{-\infty}^{\infty} j\alpha\,|s(t)|^2\,(t-t_0)\,dt.$$

$$|s(t)|^2 = \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha(t-t_0)^2}$$

$$\langle\omega\rangle = \int_{-\infty}^{\infty} j\alpha\left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha(t-t_0)^2}(t-t_0)\,dt.$$

put $t-t_0=p$, $dt=dp$.

$$\therefore \langle\omega\rangle = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty}\alpha j p\, e^{-\alpha p^2}\,dp.$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2}\cdot\alpha j\,\cancel{\int_{-\infty}^{\infty} p e^{-\alpha p^2}\,dp}$$

$= 0$ (odd function)

$$\boxed{\therefore \langle\omega\rangle = \omega_0 = 0}$$

$$\langle\omega^2\rangle = \int\left|\frac{d}{dt}s(t)\right|^2 dt$$ (1.32 from Cohen TB)

$$= \left(\frac{\alpha}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}\alpha^2(t-t_0)^2 e^{-\alpha(t-t_0)^2}\,dt,$$

let $t - t_0 = p \quad dt = dp$

$$\langle \omega^2 \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \alpha^2 p^2 e^{-\alpha p^2} dp$$

let $\sqrt{\alpha}\, p = q, \quad dp = \frac{dq}{\sqrt{\alpha}}$

$$\langle \omega^2 \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \alpha^2 \cdot \frac{q^2}{\alpha} e^{-q^2} \frac{dq}{\sqrt{\alpha}}$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot \sqrt{\alpha} \underbrace{\int_{-\infty}^{\infty} q^2 e^{-q^2} dq}_{\frac{\sqrt{\pi}}{2}}$$

$$= \frac{\alpha}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{\alpha}{2}$$

$$\sigma_\omega^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2$$

$$= \left(\frac{\alpha}{2}\right) - 0 = \frac{\alpha}{2}$$

$$\therefore \boxed{\sigma_\omega = \sqrt{\frac{\alpha}{2}}}$$

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