① (i) $x(t) = e^{-2(t-1)} u(t-1)$ $\qquad X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$\Rightarrow X(j\omega) = \int_{-\infty}^{\infty} e^{-2(t-1)} u(t-1) e^{-j\omega t} dt$$

Let $t-1 = K \Rightarrow t = K+1$

$dt = dK$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} e^{-2K} u(K) e^{-j\omega(K+1)} dK$$

$$= \int_{0}^{\infty} e^{-2K-j\omega K} \cdot e^{-j\omega} dK \qquad \left[\because u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}\right]$$

$$= e^{-j\omega} \int_{0}^{\infty} e^{-K(2+j\omega)} dK$$

$$= e^{-j\omega} \left[\frac{e^{-K(2+j\omega)}}{-(2+j\omega)} \right]_0^{\infty}$$

$$= e^{-j\omega} \left[0 + \frac{1}{2+j\omega} \right]$$

$$\boxed{X(j\omega) = \frac{e^{-j\omega}}{2+j\omega}}$$

(ii) $x(t) = e^{-2|t-1|}$

$$x(t) = \begin{cases} e^{-2(t-1)} & t \geq 1 \\ e^{2(t-1)} & t < 1 \end{cases}$$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{1} e^{2(t-1)} e^{-j\omega t} dt + \int_{1}^{\infty} e^{-2(t-1)} e^{-j\omega t} dt$$

Let $t-1 = K$ & when $t=1 \Rightarrow K=0$

$dt = dK$.

$$\therefore X(j\omega) = \int_{-\infty}^{0} e^{2K - j\omega(K+1)} dK + \int_{0}^{\infty} e^{-2K - j\omega(K+1)} dK$$

$$= e^{-j\omega} \int_{-\infty}^{0} e^{K(2-j\omega)} dK + e^{-j\omega} \int_{0}^{\infty} e^{-K(2+j\omega)} dK.$$

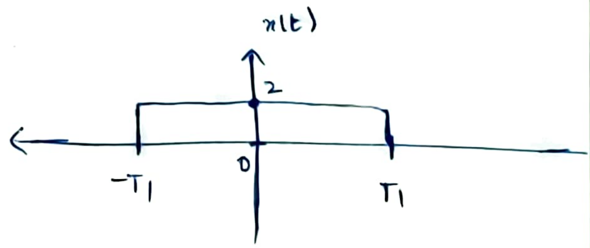
$$= e^{-j\omega} \left[\frac{e^{K(2-j\omega)}}{2-j\omega} \right]_{-\infty}^{0} + e^{-j\omega} \left[\frac{e^{-K(2+j\omega)}}{-(2+j\omega)} \right]_{0}^{\infty}$$

$$= e^{-j\omega} \left[\frac{1}{2-j\omega} \right] + e^{-j\omega} \left[\frac{1}{2+j\omega} \right]$$

$$= e^{-j\omega} \left[\frac{2+j\omega+2-j\omega}{4+\omega^2} \right]$$

$$\boxed{X(j\omega) = e^{-j\omega} \frac{4}{4+\omega^2}}$$

(iii) $x(t) = \begin{cases} 2 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T_1}^{T_1} 2 e^{-j\omega t} dt$$

$$X(j\omega) = 2\int_{-T_1}^{T_1} e^{-j\omega t}\,dt$$

$$= 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1}$$

$$= \frac{2}{-j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

$$\boxed{X(j\omega) = \frac{2\left[e^{j\omega T_1} - e^{-j\omega T_1}\right]}{+j\omega}}$$

(2)

a) $X(j\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)\right)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)\, e^{+j\omega t}\, d\omega + \frac{\pi}{2\pi}\int_{-\infty}^{\infty}\delta(\omega - 4\pi)\, e^{j\omega t}\, d\omega + \frac{\pi}{2\pi}\int_{-\infty}^{\infty}\delta(\omega + 4\pi)\, e^{j\omega t}\, d\omega$$

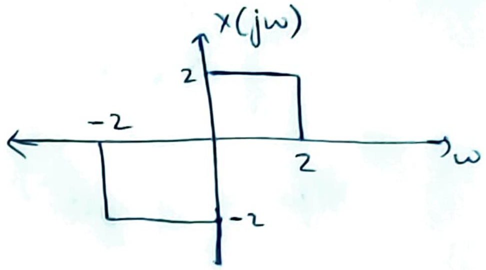
$$= \int_{-\infty}^{\infty}\delta(\omega)\, e^{j\omega t}\, d\omega + \frac{1}{2}\left[\int_{-\infty}^{\infty}\delta(\omega - 4\pi)\, e^{j\omega t}\, d\omega + \int_{-\infty}^{\infty}\delta(\omega + 4\pi)\, e^{j\omega t}\, d\omega\right]$$

$\because$ $\delta(\omega) = \begin{cases} 1 & \omega = 0 \\ 0 & \text{otherwise} \end{cases}$ also $\delta(\omega - \omega_0) = \begin{cases} 1 & \omega = \omega_0 \\ 0 & \text{otherwise} \end{cases}$

$$\therefore x(t) = 1 + \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] \quad \left[\because \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = 1\right]$$

$$\boxed{x(t) = 1 + \cos 4\pi t} \quad \left[\because \cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]$$

b) $$X(j\omega) = \begin{cases} 2 & 0 \le \omega \le 2 \\ -2 & -2 \le \omega < 0 \\ 0 & |\omega| > 2 \end{cases}$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{-2} (0) e^{j\omega t} d\omega + \int_{-2}^{0} (-2) e^{j\omega t} d\omega + \int_{0}^{2} 2 e^{j\omega t} d\omega + \int_{2}^{\infty} (0) e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi}\left[(-2)\int_{-2}^{0} e^{j\omega t} d\omega + 2\int_{0}^{2} e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi}\left[(-2)\left[\frac{e^{j\omega t}}{jt}\right]_{-2}^{0} + (2)\left[\frac{e^{j\omega t}}{jt}\right]_{0}^{2}\right]$$

$$= \frac{1}{\pi}\left[(-1)\left[\frac{1}{jt} - \frac{e^{-j2t}}{jt}\right] + 1\left[\frac{e^{j2t}}{jt} - \frac{1}{jt}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{jt}\right]\left[e^{j2t} - 1 + e^{-j2t} - 1\right]$$

$$= \frac{1}{\pi j t}\left(e^{j2t} + e^{-j2t}\right) - \frac{2}{\pi j t} = \frac{2[\cos 2t - 1]}{\pi j t}$$

③ $x(t) = \frac{2}{1+t^2}$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{2}{1+t^2} e^{-j\omega t} dt$$

Solving this directly is very difficult. Lets use the property of duality. of fourier transform.

Consider, $x(t) = e^{-a|t|}$. fourier Transform of above signal is $X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{t(a-j\omega)} dt + \int_{0}^{\infty} e^{-t(a+j\omega)} dt$

$$= \left[\frac{e^{t(a-j\omega)}}{a-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-t(a+j\omega)}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{a-j\omega}\right] + \left[\frac{1}{a+j\omega}\right]$$

$$X(j\omega) = \frac{2a}{a^2+\omega^2} \quad \text{--- ①}$$

$\therefore$ $x(t) \xrightarrow{F.T} X(j\omega)$

$X(jt) \longleftrightarrow 2\pi x(\omega)$

$\therefore$ $e^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2}$. [From obtained result ①]

$\therefore$ $\frac{2}{1+t^2} \longleftrightarrow (2\pi) e^{-|\omega|}$

$\therefore$ Fourier Transform $\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}$

④ Linearity property of Fourier Transform :-

Consider 2 signals $x_1(t)$ & $x_2(t)$ and their fourier transforms as $X_1(j\omega)$ & $X_2(j\omega)$.

Let $y(t) = a x_1(t) + b x_2(t)$ & $y(t) \xrightarrow{F.T} Y(j\omega)$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} (a x_1(t) + b x_2(t)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} a x_1(t) e^{-j\omega t} + b x_2(t) e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$\because X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt$ & $X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt.$

$\therefore Y(j\omega) = a X_1(j\omega) + b X_2(j\omega)$

$\therefore$ Hence proved

⑤

a) $y(t) = \frac{d^2}{dt^2} x(t-1)$

$$\frac{d^2}{dt^2} x(t-1) = \frac{d}{dt}\left(\frac{d}{dt} x(t-1)\right)$$

$x(t) \longleftrightarrow X(j\omega)$

$\frac{d}{dt} x(t) \longleftrightarrow j\omega X(j\omega)$ & $x(t-1) \longleftrightarrow X(j\omega) e^{-j\omega}$

$\because y(t) = \frac{d}{dt}\left(\frac{d}{dt}(x(t-1))\right)$

$Y(j\omega) = j\omega \; F.T\left\{\frac{d}{dt}(x(t-1))\right\}$

$= j\omega\,(j\omega \cdot X(j\omega) e^{-j\omega})$

$= -\omega^2 X(j\omega) e^{-j\omega}$

$\therefore \boxed{Y(j\omega) = -\omega^2 e^{-j\omega} X(j\omega)}$

b) $y(t) = x(3t-6)$

$x(t) \longleftrightarrow X(j\omega)$

$x(t-t_0) \longleftrightarrow X(j\omega) e^{-j\omega t_0}$ & $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

$\therefore x(3t-6) = x(3(t-2))$

$Y(j\omega) = F.T\{x(3(t-2))\}$ $\left[\because FT\{x(3(t-2))\} = \frac{1}{|3|} F.T\left\{X\left(\frac{t-2}{3}\right)\right\}\right]$

$= \frac{1}{|3|} X\left(\frac{j\omega}{3}\right) e^{-j\omega 2}$

$\therefore Y(j\omega) = F.T\{x(3t-6)\} = \frac{1}{3} e^{-2j\omega} X\left(\frac{j\omega}{3}\right)$

c) $x(t) = x(1-t) + x(-1-t)$

Fourier Transform $\{x(t)\} = F.T\{x(-t+1) + x(-t-1)\}$.

$$x(t) \longleftrightarrow X(j\omega)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} \cdot X\left(\frac{j\omega}{a}\right)$$

$$x(t-t_0) \longleftrightarrow X(j\omega)\, e^{-j\omega t_0}$$

$\therefore$ Using above 3 properties $\Rightarrow$

$$x(t-1) \longleftrightarrow X(j\omega) e^{-j\omega}$$

$$x(-(t-1)) \longleftrightarrow X\left(\frac{j\omega}{-1}\right) e^{-j\omega} \cdot \frac{1}{|-1|}$$

$$x(-(t-1)) \longleftrightarrow e^{-j\omega} X(-j\omega)$$

$$x(t+1) \longleftrightarrow X(j\omega) e^{j\omega}$$

$$x(-(t+1)) \longleftrightarrow \frac{1}{|-1|} \cdot X\left(\frac{j\omega}{-1}\right) e^{j\omega}.$$

$\therefore$ Using linearity property of Fourier Transform,

$$F.T\{x(-t-1) + x(-t+1)\} = X(-j\omega)\left[e^{-j\omega} + e^{j\omega}\right]$$

$$= \underline{\underline{2\cos\omega\, X(-j\omega)}}$$