TFA - Assg 4
wavelets

2018122004
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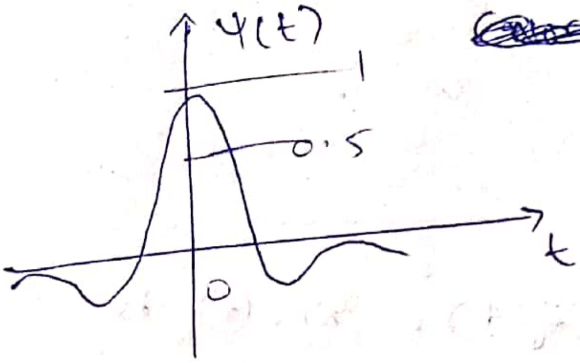
(1) Mexican hat wavelet:

$$\psi(t) = -K \cdot \frac{d^2}{dt^2}\left(e^{-\frac{t^2}{2\sigma^2}}\right) \quad \text{—} ①$$

$$= +K \cdot \left(e^{-\frac{t^2}{2\sigma^2}}\right) \cdot \left(1 - \frac{t^2}{\sigma^2}\right) \cdot \frac{1}{\sigma^2}$$

ie $\psi(t) = \frac{K}{\sigma^2} e^{-t^2/2\sigma^2}\left(1 - \frac{t^2}{\sigma^2}\right)$



ie $K=1, \sigma^2=1$.
The shown figure

To find the fourier transform of ①, ie $\hat{\psi}(\omega)$?

WKT if $f(t) \longleftrightarrow F(\omega)$

$$\frac{d^2}{dt^2} f(t) \longleftrightarrow (j\omega)^2 \cdot F(\omega)$$
$$= j^2 \omega^2 F(\omega)$$
$$= -\omega^2 F(\omega)$$

$$\therefore -K \cdot \frac{d^2}{dt^2} f(t) = K \cdot \omega^2 F(\omega).$$

here, $f(t) = e^{-t^2/2\sigma^2}$

$$F(\omega) = \sqrt{2\pi} \cdot \sigma \cdot e^{-2\sigma^2\frac{\omega^2}{4}}$$

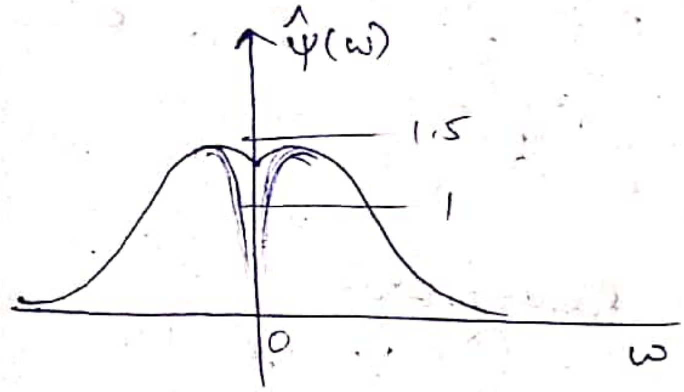
$$\therefore K\omega^2 F(\omega) = K\omega^2 \sqrt{2\pi}\, \sigma\, e^{-2\sigma^2\omega^2/4} = \hat{\psi}(\omega)$$

If $K=1$ & $\sigma=1$,

$$\therefore \hat{\psi}(\omega) = K\omega^2 \sqrt{2\pi}\,\sigma\, e^{-2\sigma^2 \frac{\omega^2}{4}}$$

$$= \omega^2 \sqrt{2\pi} \cdot e^{-\frac{2\omega^2}{4}}$$

$$\hat{\psi}(\omega) = \omega^2 \sqrt{2\pi}\, e^{-\omega^2/2}.$$



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(2) (a) To show that $f(t) = \operatorname{sinc}(t)$ is Not absolutely integrable but Square integrable.

→ Absolute integrable ⇒ $\int_{-\infty}^{\infty} |f(t)|\, dt \neq$ or $\not\to \infty$.

$f(t) = \operatorname{sinc} t$.

$$\int_{-\infty}^{\infty} |\operatorname{sinc}(t)|\, dt = \int_{-\infty}^{\infty} \left|\frac{\sin(t)}{(t)}\right| dt$$

$$= \sum_{K=-\infty}^{\infty} \int_{K\pi}^{(K+1)\pi} \left|\frac{\sin(t)}{t}\right| dt$$

Also, this term $> \int_{K\pi}^{(K+1)\pi} \left|\frac{\sin t}{(K+\frac{1}{2})\pi}\right| dt.$

$$\Rightarrow \int_{-\infty}^{\infty} |\operatorname{sinc} t|\, dt = \sum_{K=-\infty}^{\infty} \int_{K\pi}^{(K+1)\pi} \left|\frac{\sin t}{t}\right| dt$$

$$> \sum_{K=-\infty}^{\infty} \int_{K\pi}^{(K+1)\pi} \left|\frac{\sin t}{\left(K+\frac{1}{2}\right)\pi}\right| dt$$

$$> \frac{2}{\pi} \cdot \sum_{K=-\infty}^{\infty} \frac{1}{\left(K+\frac{1}{2}\right)} \qquad \left(\because \int_{K\pi}^{(K+1)\pi} \sin t\, dt = \frac{2}{\pi}\right)$$

Note: A H.P. → Harmonic progression. (sum)

∴ this expression → ∞

It does not converge.

⇒ $\operatorname{sinc}(t) = f(t)$ is Not Absolute Integrable.

Now to show that $f(t) = \operatorname{sinc}(t)$ is Square integrable, we can use parsevals theorem to do the integration

i.e. $$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{2} \left|\operatorname{Rect}\left(\frac{\omega}{2}\right)\right|^2 d\omega$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} \left|\operatorname{Rect}\left(\frac{\omega}{2}\right)\right|^2 d\omega \quad \text{—— (a)}$$

↳ we know that this Rectangular window function is Band-

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limited

(sinc spreads in the entire time domain

∴ Due to the uncertainty principle & TBWP, Rectangle window which is the Fourier Transform of sinc function will be Band limited in Freq domain)

$$\therefore \int_{-\infty}^{\infty} |f(t)|^2 dt \not\to \infty$$

or $< \infty$.

$\Rightarrow f(t) = \text{sinc}(t)$ is Square Integrable

$$= \frac{1}{4} \int_{-\infty}^{\infty} \left|\text{Rect}\left(\frac{w}{2}\right)\right|^2 dw$$

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(5) Given

$$\text{LHS} = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt$$

↳ complex Conjugate.

(this theorem basically implies that the integral over the absolute square of a function does not change after FT)

Using Inverse FT for $g(t)$ & $f(t)$

$g(t) = \frac{1}{2\pi} \int_R \hat{g}(\omega) \cdot e^{j\omega t} \, d\omega$

& $f(t) = \frac{1}{2\pi} \int_R \hat{f}(\omega) \, e^{j\omega t} \, d\omega.$

Here $\hat{g}(\omega)$ & $\hat{f}(\omega)$ are FTs of $g(\omega)$ & $f(\omega)$ respectively.

$\therefore \int_{-\infty}^{\infty} f(t) \, g^*(t) = \int_R \left(\frac{1}{2\pi} \int_R \hat{f}(\omega) e^{j\omega t} \right) d\omega.$

$\left(\frac{1}{2\pi} \int_R \hat{g}(\omega_1) \cdot e^{j\omega_1 t} \, d\omega_1 \right)$

(using a different integration variable

$= \left(\frac{1}{2\pi}\right)^2 \iint \hat{f}(\omega) \, \hat{g}^*(\omega_1) \left(\int_R e^{j(\omega - \omega_1)t} \, dt \right)$

$d\omega \, d\omega_1$.

[W.K.T,

If $\delta(t) = \frac{1}{2\pi} \int_R e^{j\omega t} \, d\omega.$

$\longleftrightarrow \delta(\omega) = \delta(\omega - \omega_1)$]

$= (1/2\pi) \iint \hat{f}(\omega) \cdot \hat{g}^*(\omega_1) \cdot \delta(\omega - \omega_1) \, d\omega \, d\omega_1.$

$= (1/2\pi) \int \hat{f}(\omega) \cdot \hat{g}^*(\omega) \, d\omega$

$(\because \delta(\omega - \omega_1) = 1$

when $\omega = \omega_1)$

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$$\Rightarrow \int f(t) \cdot g^*(t) = \frac{1}{2\pi} \int \hat{f}(\omega) \cdot \hat{g}^*(\omega) \quad \text{——— ①}$$

$\Rightarrow$ Hence proved

given $\displaystyle\int_R \frac{1}{(t^2+a^2)} \frac{1}{(t^2+b^2)}$ → this be $g^*(t)$

let this be $f(t)$

$\left[\because g(t) = g^*(t) \text{ (Real signal)}\right]$

Using ①,

$$f(t) = \frac{1}{t^2+a^2}$$

$$\hat{f}(\omega) = \frac{\pi}{a} \cdot e^{-a|\omega|} \text{ (standard signal)}$$

similarly, $\displaystyle\frac{1}{t^2+b^2} \longleftrightarrow \frac{\pi}{b} e^{-b|\omega|}$

$\Rightarrow g^*(t)$

$$\therefore \int_R \frac{1}{(t^2+a^2)(t^2+b^2)} = \frac{1}{2\pi} \int_R \frac{\pi^2}{ab} e^{-(a+b)|\omega|} d\omega$$

→ PTO

$= \cancel{\frac{\pi}{ab}\left[\frac{2}{a+b}\right]}$

$\cancel{= \ldots}$

$= \cancel{\frac{2\pi}{ab(a+b)}}$

$$= \frac{\pi^2}{2\pi ab} \int_R e^{-(a+b)|w|} \, dw$$

$$= \frac{\pi}{2ab}\left[\int_{-\infty}^{0} e^{w(a+b)} \, dw + \int_0^{\infty} e^{-w(a+b)} \, dw \right]$$

$w \to -w$

$$= \frac{\pi}{2ab}\left[\frac{e^{w(a+b)}}{(a+b)} \Big]_{-\infty}^{0} + \frac{e^{-w(a+b)}}{a+b} \Big|_{\infty}^{0} \right]$$

$$= \frac{\pi}{2ab(a+b)} \cancel{[(0+1)}\left[(1) + (1) \right]$$

$$= \frac{2\pi}{2ab(a+b)} = \frac{\pi}{ab(a+b)}$$

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(4) Given $g(t) \to \left[-\frac{\pi}{\Omega_0}, \frac{\pi}{\Omega_0}\right]$ — ⓐ

To show that,

if $\frac{2\pi}{\Omega_0} \sum_{n \in Z} |g(t - nu_0)|^2 = A > 0$,

$\forall$ ~~8~~ $t \in R$, then,

$\{ g_{n,k}(t) = g(t - nu_0) e^{jk\Omega_0 t}, (n,k) \in Z \}$

is a tight frame of $L^2(R)$.

Time shift ⓐ by nu_0.

$\therefore$ support now ~~beau~~ becomes,

$g(t-nu_0)f(t) \longrightarrow \left[nu_0 - \frac{\pi}{\varepsilon_0}, nu_0 + \frac{\pi}{\varepsilon_0}\right]$ — (1)

an orthogonal basis of this space is,

$$\{e^{jk\varepsilon_0 t}\}_{k\in \mathbb{Z}}.$$

By the property of orthogonal basis,

$$A\cdot\|f(t)\|^2 = \sum_n |\langle f(t)\cdot \phi_n\rangle|^2$$

$$A\|f\|^2 = \sum_n |\langle f, \phi_n\rangle|^2 \quad \text{— (2)}$$

Using (1),

$$\int_{-\infty}^{\infty} |g(t-nu_0)|^2 |f(t)|^2 dt$$

$$= \int_{nu_0 - \frac{\pi}{\varepsilon_0}}^{nu_0+\pi/\varepsilon_0} |g(t-nu_0)|^2 |f(t)|^2 dt \quad \text{— (3)}$$

Using (2), the above eqn becomes,

i.e.,
$$\int_{nu_0-\frac{\pi}{\varepsilon_0}}^{nu_0+\pi/\varepsilon_0} |g(t-nu_0)f(t)|^2 = \sum_{k=-\infty}^{\infty} |\langle g(t-nu_0)f(t), e^{jk\varepsilon_0 t}\rangle|^2$$

— (4)

Using ④ in ③,

$$\Rightarrow \frac{\Omega_0}{2\pi} \sum_{K=-\infty}^{\infty} \left| \langle g(u-nu_0)f(u) \cdot e^{jK\Omega_0 u} \rangle \right|^2$$

Also given $g_{n,K}(t) = g(t-nu_0)e^{jK\Omega_0 t}$

$$\int_{-\infty}^{\infty} |g(t-nu_0)|^2 |f(t)|^2 dt$$

$$= \frac{\Omega_0}{2\pi} \sum_{K=-\infty}^{\infty} |\langle f, g_{n,K} \rangle|^2$$

$$\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |g(t-nu_0)|^2 |f(t)|^2 dt$$

$$= \frac{\Omega_0}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} |\langle f, g_{n,K} \rangle|^2$$

$$\Rightarrow \|f(t)\|^2 \cdot \frac{\Omega_0 A}{2\pi} = \frac{\Omega_0}{2\pi} \cdot \sum_n \sum_K |\langle f, g_{n,K} \rangle|^2$$

$$\Rightarrow A\|f\|^2 = \sum_n \sum_K |\langle f, g_{n,K} \rangle|^2$$

$\therefore \{g_{n,K}\}_{n,K \in Z^2}$ is a tight frame of $L^2(R)$

Hence, proved.

(9) Given, $\phi(t) = \sum_{n \in \mathbb{Z}} h(n) \cdot \phi(2t-n)$.

To prove

(a) If $\int \phi(t)\, dt \neq 0$, then $\sum_{n \in \mathbb{Z}} h(n) = \sqrt{2}$.

WKT

$$\phi(t) = \sum_{n \in \mathbb{Z}} h(n) \cdot \sqrt{2} \cdot \phi(2t-n).$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\omega n} \quad (\text{DTFT})$$

$$\Phi(\omega) = \int_{-\infty}^{\infty} \phi(t) \cdot e^{-j\omega t}\, dt.$$

$$\Phi(\omega) = \frac{1}{\sqrt{2}} \cdot H\left(\frac{\omega}{2}\right) \cdot \Phi\left(\frac{\omega}{2}\right).$$

Now, $\phi(t) = \sum_{n \in \mathbb{Z}} h(n) \sqrt{2}\, \phi(2t-n)$.

let $\gamma = 2t$ & integration on b.s.

$d\gamma = 2\,dt$

$$\Rightarrow \int \phi(t)\, dt = \int \sum_{n \in \mathbb{Z}} h(n) \sqrt{2}\, \phi(2t-n)\, dt$$

$$= \sum_{n \in \mathbb{Z}} h(n) \int \sqrt{2}\, \phi(2t-n)\, dt$$

$$= \left[\sum_{n \in \mathbb{Z}} h(n)\right] \sqrt{2} \int \phi(\gamma) \cdot \frac{d\gamma}{2}.$$

this integrand is made independent of translation.

$\Rightarrow \int \phi(t)\,dt = \frac{1}{\sqrt{2}} \cdot \sum_{n \in \mathbb{Z}} h(n) \cdot \int \phi(\tau)\,d\tau$

$\Rightarrow \sum_{n \in \mathbb{Z}} h(n) = \sqrt{2}$. (cancellation is possible since given that $\int \phi(t)\,dt \neq 0$).

Hence proved. (No assumption of orthogonality & or $\phi(t)$ normalization).

(7)(b)

If $\{\phi(t-k), k \in \mathbb{Z}\}$ is an orthonormal set, then $\rightarrow$ (a) $\sum_n h(n)\, h(n-2k) = \delta(k)$

& (b) $\sum_n h(2n) = \sum_n h(2n+1) = \frac{1}{\sqrt{2}}$.

$\rightarrow$ (a) From orthogonality condition,

$$\int \phi(x) \cdot \phi(x-m)\,dx = \delta(m)$$

w.r.t, $\phi(x) = \sum_{n \in \mathbb{Z}} h(n) \sqrt{2}\, \phi(2x-n)$ (scaling function)

putting this eq^n in the above eq^n, we have,

$$\Rightarrow \int \left[\sum_{n \in \mathbb{Z}} h(n) \sqrt{2}\, \phi(2x-n)\right]\left[\sum_K h(K) \cdot \sqrt{2} \cdot \phi(2x-2m-k)\right] dx = \delta(m).$$

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let $2x = y \Rightarrow dx = \frac{dy}{2}$.

$$\Rightarrow \sum_n \sum_k h(n)\, h(k) \int \phi(y-n) \cdot \phi(y-2m-k)\, dy$$

$$= \delta(m) . \quad \text{——— ①}$$

To evaluate this integral,

let $y - n = l$

$dy = dl$.

$$\Rightarrow \int \phi(l)\, \phi(l + n - 2m - k)\, dl$$

$l - (-n + 2m + k)$.

$= \delta(2m + k - n)$ ~~for~~

($\because$ orthogonality). $\Downarrow$

$\delta(2m + k - n) = 1$ only

when $k = n - 2m$.

① $\Rightarrow$

$$\therefore \sum_n \sum_k h(n)\, h(k)\, \delta(2m - n + k) = \delta m$$

$$= \sum_n h(n) . h(n - 2m) . = \delta m$$

$\longrightarrow$ Hence proved

keeping $k = n - 2m$ in this eqn

→ (b) we got $\sum_{n \in \mathbb{Z}} h(n) = \sqrt{2}$.

any signal can be split into even & odd terms

form: $2K \to$ even, $2K+1 \to$ odd

$$\Rightarrow \underbrace{\sum_n h(n)}_{\text{all } n} = \underbrace{\sum_K h(2K) + \sum_K h(2K+1)}_{\text{all } K}.$$

$$\Rightarrow \sqrt{2} = \sum_K h(2K) + \sum_K h(2K+1) \quad —— (2)$$

& we got

$$\delta(m) = \sum_n h(n) \cdot h(n-2m)$$

⇓

can be written as,

$$\sum_n \sum_K h(K) \cdot h(K+2n) = 1$$

↳ split this fn in even & odd terms.

$$= \sum_n \left[\sum_K h(2K+2n)\, h(2K) + \sum_K h(2K+2n+1)\, h(2K+1) \right]$$

(Basically replacing K with 2K (even) & K with 2K+1 (odd))

let (2) eqn be represent with K_0 & K_1

~~where~~ ie $\sqrt{2} = K_0 + K_1$;

here, $K_0 = \sum_K h(2K)$ & $K_1 = \sum_K h(2K+1)$.

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$$\Rightarrow \sum_K \sum_n h(2K+2n)\, h(2K) +$$

$$\sum_K \sum_n h(2K+2n+1)\, h(2K+1)$$

$$\Rightarrow \sum_K \cdot K_0 \cdot h(2K) + \sum_K K_1 \cdot h(2K+1)$$

$\because$ n is the variable of summation & by ~~wr~~ rewritting, & similarly 'K_1'.

we can see that

$$\sum_n h(2K+2n) = \sum_n h(2K)$$

(By laws of summation)

$$\Rightarrow \sum_K K_0\, h(2K) + \sum_K K_1\, h(2K+1)$$

$$= K_0 \cdot \underbrace{\sum_K h(2K)} + K_1 \underbrace{\sum_K h(2K+1)}$$

again this is K_0 & this is K_1

$$\Rightarrow K_0^2 + K_1^2 \Rightarrow \text{LHS}$$

& RHS was 1.

$$\therefore K_0^2 + K_1^2 = 1 \leftarrow (3)$$

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we also have $k_0 + k_1 = \sqrt{2}$ — ⓐ

& $k_0^2 + k_1^2 = 1$ — ⓑ

To solve for k_0 & k_1 using the above 2 eq's,

ⓐ $\Rightarrow k_0^2 + k_1^2 + 2k_0 k_1 = 2$

squaring on b.s

& $k_0^2 + k_1^2 = 1$

$\Rightarrow 2k_0 k_1 = 2 - 1$

$= 1$

$\Rightarrow k_0 k_1 = \frac{1}{2}$

$k_1 + k_0 = \sqrt{2}$

$\Rightarrow k_1 + \frac{1}{2k_1} = \sqrt{2}$

$2k_1^2 + 1 - 2\sqrt{2}k_1 = 0$

$$k_1 = \frac{2\sqrt{2} \pm \sqrt{8 - 4(2)(1)}}{4}$$

$$= \frac{1}{\sqrt{2}} \pm 0$$

$\therefore \boxed{k_1 = \frac{1}{\sqrt{2}}}$, $k_0 = \frac{1}{2k_1} = \frac{1}{\sqrt{2}\,\cancel{2}\left(\frac{1}{\cancel{\sqrt{2}}}\right)} = \boxed{\frac{1}{\sqrt{2}} = k_0}$

$\Rightarrow \sum_n h(2n) = \frac{1}{\sqrt{2}}$,

$\sum_n h(2n+1) = \frac{1}{\sqrt{2}}$ $\Rightarrow$ Hence, proved

9(c). Given

$\phi(t)$ has compact support on $0 \leq t \leq N-1$. &

To prove that, ~~if~~ If $\{\phi(t-k), k \in Z\}$ is linearly independent, then $h(n)$ has compact support over $0 \leq n \leq N-1$.

→ WKT, the expression for scaling f^n, is,

$$\phi(t) = \sum_n h(n) \cdot \sqrt{2}\,\phi(2t-n)$$

~~support of h(n).~~

Let $[N_1, N_2]$ be the support of $h(n)$

then ~~~~ $\phi(t)$ has support

$$\left[\frac{N_1}{2}, \frac{N-1+N_2}{2}\right]$$

& $\phi(n)$ has support $[0, N-1]$

$$\left(\because \begin{array}{l} \phi(n) \to [0, N-1] \\ \phi(2n) \to \left[0, \frac{N-1}{2}\right] \end{array}\right)$$

$\boxed{\text{equating LHS \& RHS compact support}}$

LHS → $[0, N-1]$

RHS → $\left[\frac{N_1}{2}, \frac{N-1+N_2}{2}\right]$

limits of indices of non-zero $h(n)$ are

~~sat~~ such that,

$$\frac{N_1}{2} = 0$$

$$\Rightarrow \boxed{N_1 = 0}$$

$$\frac{N-1+N_2}{2} = N-1$$

$$\Rightarrow \cancel{N} \cancel{-1} + N_2 = \cancel{2}N - \cancel{2} -1$$

$$\boxed{N_2 = N-1}$$

$$\therefore h(n) \to [0, N-1]$$ //

Hence proved

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(6) To S/T scaling function $\phi(t)$, obtained by aggregating wavelets at scales larger than unity satisfies $\|\phi(t)\| = 1$.

$$\|\phi\|^2 = \frac{1}{2\pi}\int |\hat{\phi}(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{1}^{\infty} |\hat{\psi}(s\omega)|^2 \frac{ds}{s}$$

$$= \frac{1}{2\pi}\int_{1}^{\infty}\left(\int |\hat{\psi}(s\omega)|^2 d\omega\right)\frac{ds}{s}$$

keep

$s\omega \to s \Rightarrow$

$\& \|\psi\| = 1$

$$\|\phi\|^2 = \int_{1}^{\infty}\left(\underbrace{\frac{1}{2\pi}\int |\hat{\psi}(\omega)|^2 d\omega}_{=1}\right)\frac{ds}{s^2}$$

$$= \int_{1}^{\infty}\frac{ds}{s^2} = 1 \; // \quad \left(\because -\frac{1}{s}\Big|_1^{\infty} = [-0 + \tfrac{1}{1}] = 1\right)$$

⑦ Integer shifted functions :

$\{\theta(t-n), n \in Z\}$.

If mutually orthogonal ? get a condition on θ

→ we need to have the following eqⁿ true

is $\int_k \phi(t-l)\phi(l-n) = \delta(l-n)$

(or) $\bar{\Phi}(t) = \phi^*(-t)$ for any $(n,p) \in Z^2$

$<\phi(t-n)\phi(t-p)> = \int_{-\infty}^{\infty} \phi(t-n)\,\phi^*(t-p)\,dt$

$= \phi * \bar{\phi}(p-n)$.

∴ $\{\phi(t-n)\}_{n\in Z}$ is orthogonal iff

$\phi * \bar{\phi}(n) = \delta(n)$.
discrete.

Consider autocorrelation fn

$a_{\phi\phi}(k) = |\hat{\Phi}(\omega)|^2$
FT in non-sampled domain

$\delta(n)$ is discrete

$\Rightarrow \sum_{k=-\infty}^{\infty} |\hat{\Phi}(\omega+2\pi k)|^2 = 1$ (Taking DTFT & periodicity of FT of sampled functions)

$\Rightarrow \sum_{k=-\infty}^{\infty} |\theta(\omega+2\pi k)|^2 = 1$

(3)(b) $A\|f\|^2 \le \sum_{n \in T} |\langle f, \phi_n \rangle|^2 \le B\|f\|^2$

for a frame

let Φ be framed analysis operator.

$A\|f\|^2 \le \|\Phi f\|^2 = \langle \Phi^* \Phi f \rangle$

$\le B\|f\|^2$

with $\Phi^* \Phi f = \sum \langle f, \Phi_n \rangle \phi_n$.

here $A \to$ infimum, $B \to$ supremum

of the fn: $\Phi^* \Phi$

$\therefore$ $A \to$ smallest eigen val & B largest eigen value.

WKT $\|\phi_n\| = 1$

$\therefore$ $Tr(\Phi^* \Phi)$ follows,

$AN \le Tr(\Phi^* \Phi) \le BN$.

WKT $Tr(\text{Matrix}) = \text{Sum}(\text{eig. values})$.

$\therefore AN \le Tr(\Phi^* \Phi)$

$= Tr(\Phi \cdot \Phi^*)$

$= \sum |\langle \phi_n, \phi_n \rangle|^2 = P$.

$\le BN$

$\therefore AN \le P \le BN$

$\Rightarrow A \le \frac{P}{N} \le B$ Hence, proved

(2b) Given

$$\sum_n \operatorname{sinc}(t - 0.2n)$$

let $x(t) = \dfrac{\sin \pi(t)}{\pi(t)}$

& $y(t) = \sum_n x(t - 0.2n)$.

$Y(\omega) = \int_0^1 y(t)\, e^{-i\omega t}\, dt$ (Note $\omega = 2\pi f$)

or

$Y(f)$

$$= \int_0^1 \sum x(t - 0.2n)\, e^{-i\omega t}\, dt$$

$$= \sum \int_{-0.2n}^{1+0.2n} x(s) \cdot e^{-i2\pi f(s+0.2n)}\, ds.$$

(shifting)

$$= \int_{-\infty}^{\infty} x(s) \cdot e^{-i2\pi f s}\, ds.$$

$$= \hat{x}(\omega) \text{ or } \hat{x}(f)$$

(FT)

$\Rightarrow$ $y(t) = \sum_{n=-\infty}^{\infty} x(n-r)$ —— (1)

$$= \sum_{n=-\infty}^{\infty} \hat{x}(f) \cdot e^{-i2\pi f t}.$$

Using the fn,

$$T(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

we have its FT as,

$$\hat{T}(f) = \text{sinc}(ct)$$

Now using this function in ①,

& duality,

$$y(t) = \sum_{f=-\infty}^{\infty} \hat{T}(f) \cdot e^{-i2\pi ft}$$

$$= 1 \cdot e^{0}$$

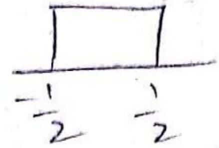
$$\therefore y(t) = 1 = \sum_{n} x(t - 0.2n)$$

$$= \sum_{n} \frac{\sin . \pi (t - 0.2n)}{\pi (t - 0.2n)}$$

$$\Rightarrow \boxed{\sum_{n} \text{sinc}(t - 0.2n) = 1}$$

PTO

(8) (a) To Compute the support of n^{th} order Spline & its Riesz bounds

$\beta^0 \rightarrow$ 

β^n will be β^0 convolved $(n+1)$ times.

$\therefore$ Support after $(n+1)$ convolutions of β_0 will be $\left[-\frac{(n+1)}{2}, \frac{n+1}{2}\right)$

W.K.T the Condition for Riesz basis is

$$A \leq \sum_K |\hat{\phi}(\omega + 2\pi k)|^2 \leq B$$

$$a_{\phi\phi}(K) = \phi * \bar{\phi}(n), \quad \bar{\phi}(t) = \phi^*(-t)$$

$$= \beta_n * \bar{\beta}_n(n).$$

$$= |\hat{\beta}_n(\omega)|^2.$$

& $\hat{\beta}_0(\omega) = \text{sinc}(\omega)$.

$\Rightarrow \hat{\beta}_n(\omega) = \text{sinc}^{(n+1)}(\omega)$ → power

($*$ in time is $\cdot$ in freq domain)

$$\therefore A \leq \sum_K \left| \text{sinc}^{n+1}(\omega + 2\pi k)^2 \right| \leq B$$

we need to calculate the range of this fn.

W.K.T max is $B = 1$ & $\left(A = \frac{1}{2n+1}\right)$.

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8(b) To Compute

$$\sum_{K \in Z} \beta^n (t-K)$$

using poisson summation formula,

$$\sum_{K} f(K-t) = \sum_{\omega} e^{-2j\omega t \pi} \tilde{f}(\omega).$$

$$\Rightarrow \sum_{K} \beta^n (K-t) = \sum_{-\infty}^{\infty} e^{-j2\pi\omega t} \tilde{f}(\omega)$$

$$= \sum_{K} \beta^n (t-K) \qquad = Sinc^{(n+1)}(\omega).$$

(symmetry)

$$\therefore \sum_{K \in Z} \beta^n (\cancel{t}-\cancel{K}) = \sum_{N=-\infty}^{\infty} e^{-j2\pi\omega t} \, Sinc^{(n+1)}(\omega)$$

3(a). $e_K = \left(\cos \frac{2\pi K}{N}, \sin \frac{2\pi K}{N} \right)$

(i) orthonormal Basis:

let $S_n = \alpha_1 e_1 + \alpha_2 e_2 + \dots \alpha_n e_n.$

$\therefore e_K \cdot e_K^T = 1 \Rightarrow \cos^2 \frac{2\pi k}{N} + \sin^2 \frac{2\pi K}{N} = 1.$

$\therefore \|S_n\|^2 = \|\alpha_1\|^2 + \dots \|\alpha_n\|^2 = \sigma_n.$

for any $n > m$,

$\|S_n - S_m\|^2 = \|\alpha_{m+1} \cdot e_{m+1} + \dots \alpha_n e_n\|^2$

$= |\alpha_{n+1}|^2 + \cdots + |\alpha_n|^2$

$= \sigma_n - \sigma_m$

$\Rightarrow S_n$ is Cauchy iff σ_n is Cauchy

if $\sum_{K=0}^{N-1} |\alpha_K|^2 \longrightarrow$ converges — ①

then the series will also Converge.

$c\|f\|^2 = \sum_{K=0}^{N-1} |\langle e_K, e_K \rangle|^2$

(orthogonality)

$e_a \cdot e_b^T = 0$ for $a \neq b$.

$\therefore \cos\frac{2\pi a}{N} \cos\frac{2\pi b}{N} + \sin\frac{2\pi a}{N} \sin\frac{2\pi b}{N} = 0$

— ② for $a \neq b$.

① & ② should satisfy
for orthogonality of this basis.

(ii) Reisz basis.

$A\|f\|^2 \leq \sum_{K=0}^{N-1} |\langle f, e_K \rangle|^2 \leq B\|f\|^2$

$B \geq A > 0$

linearly independent
bias

$\sum_{K=0}^{N-1} \langle f, e_K \rangle e_K = 0$ iff $\langle f, e_K \rangle = 0$

for entire support.

$f = \alpha_0 e_0 + \dots + \alpha_{N-1} e_{N-1}$

$< f, e_k > = \alpha_0 e_0 e_k^T + \dots + |\alpha_k|^2 + \alpha_{N-1} e_{N-1} e_k^T$

$< f, e_k > e_k = \alpha_0 e_0 + \dots + |\alpha_k|^2 e_k + \dots \alpha_{N-1} e_{N-1}.$

$A = B = 1$

for conditions to satisfy.

(iii) frame

This will be a tight frame with $A = B = 1$

Since $\|f\|^2 = |\alpha_0|^2 + \dots + |\alpha_{N-1}|^2$

~~Taking~~

$\sum_n |< f, e_n >|^2 = |\alpha_0|^2 + \dots + |\alpha_{N-1}|^2$

Taking $e_a e_b^T = 0$ for $a \neq b$.