TFA - Assignment 2

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(1) Given,

$$s(t) = \left(\frac{2}{\pi}\right)^{1/4} \cdot e^{\left(-10t^2 + j5t^2 + j100t\right)}$$

To find $\langle \omega \rangle$, $\langle \omega^2 \rangle$ & σ_ω^2.

$$E_t = \left(\frac{2}{\pi}\right)^{1/2} \int e^{-20t^2} dt$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \cdot \sqrt{\frac{\pi}{20}}$$

$$\boxed{E_t = 1/\sqrt{10}}$$

w.k.T

$$\langle \omega \rangle = \int_{-\infty}^{\infty} \omega |S(\omega)|^2 d\omega \Big/ E(\omega)$$

from eq 1.29 in Cohen TB,

$$\langle \omega \rangle = \int s^*(t) \frac{1}{j} \frac{d}{dt} s(t)\, dt \Big/ E_t \quad \text{— (1)}$$

Now, To calculate

$$\frac{1}{j}\frac{d}{dt}(s(t)) = \frac{1}{j}\left[\left(\frac{2}{\pi}\right)^{1/4} \cdot e^{-10t^2 + j5t^2 \ldots + j100t} \cdot \left(-10 \cdot 2 \cdot t + j5 \cdot 2 \cdot t + j \cdot 100\right)\right]$$

$$= \frac{1}{j}\left[-20t + 10jt + j100\right] \cdot s(t)$$

$$= \left(20tj + 10t + 100\right) s(t)$$

$$\therefore \langle \omega \rangle = \frac{\int s^*(t) \cdot \left[20tj + 10t + 100\right] s(t)\, dt}{E_t} \quad \text{— (2)}$$

Now,

$$s^*(t) \cdot s(t) = \left(\frac{2}{\pi}\right)^{1/2} \cdot e^{-10t^2(2) + \ldots \cancel{j5t^2} - \cancel{j5t^2} + \cancel{j100t} - \cancel{j100t}}$$

$\therefore$ (2) becomes,

$$\underset{\epsilon t}{\langle w \rangle} = \sqrt{10}\int_{-\infty}^{\infty}\left(\frac{2}{\pi}\right)^{1/2} e^{-20t^2}\left[20\overset{\alpha}{t}j + \overset{\beta}{10t} + 100\right] dt$$

$$= \left[\left(\frac{2}{\pi}\right)^{1/2}(j20)\int_{-\infty}^{\infty} e^{-20t^2}\cdot t\, dt \; + \right.$$ $= 0$ (odd f^n)

$$\left(\frac{2}{\pi}\right)^{1/2}\cdot(10)\int_{-\infty}^{\infty} e^{-20t^2}\cdot t\, dt \; +$$ $= 0$ (odd f^n)

$$\left.\left(\frac{2}{\pi}\right)^{1/2}(100)\int_{-\infty}^{\infty} e^{-20t^2}\cdot dt\right]\sqrt{10}$$

$$\Rightarrow \left(\frac{2}{\pi}\right)^{1/2} 100 \cdot \sqrt{10}\left[\frac{\sqrt{\pi}}{4\sqrt{5}}\,\mathrm{erf}\left(\sqrt{20}\,x\right) + c\,\Big|_{-\infty}^{\infty}\right]$$

(Indefinite Integral of $\int e^{-20t^2} dt$)

Here, x → a complex variable

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$$

erf → Complex f^n

$$\Rightarrow \left(\frac{2}{\pi}\right)^{1/2}\cdot 100\left[\frac{\sqrt{\pi}}{4\sqrt{5}} + \frac{\sqrt{\pi}}{4\sqrt{5}}\right]\cdot\sqrt{10}$$

$$= \left(\frac{2}{\pi}\right)^{1/2}\cdot 100\cdot\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{5}}\cdot\sqrt{10} = 10\cdot\sqrt{10}\cdot\sqrt{10}$$

~~$\langle w \rangle = 31.623$~~ $\Leftarrow$ $\boxed{\therefore \langle w \rangle = 100}$

$$\langle \omega^2 \rangle = \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 d\omega \Big/ E_\omega$$

$$= \frac{1}{E_t} \int \left| \frac{d}{dt} s(t) \right|^2 dt$$ please note (applied in the end)

$$\therefore \langle \omega^2 \rangle^* = \frac{\int s^*(t) \left(\frac{1}{j} \frac{d}{dt} \right)^2 s(t)\, dt}{E_t}$$

$$= - \int s^*(t) \cdot \frac{d^2}{dt^2} s(t) \cdot dt$$

$$= \int \left| \frac{d}{dt} s(t) \right|^2 dt$$

$$\cancel{\therefore \langle \omega^2 \rangle = \int \left| -20t + 10jt + j100 \right| (s(t))^2 dt}$$

$$\therefore \langle \omega^2 \rangle \cdot E_t = \int \left| (-20t + 10jt + j100)\, s(t) \right|^2 dt$$

$$= \int \left| -20t + 10jt + j100 \right|^2 \cdot \left| e^{-10t^2 + \cancel{j5t^2} + \cancel{j100t}} \right|^2 dt \cdot \left(\frac{2}{\pi}\right)^{1/2}$$

$$(\because e^x > 0\ \forall x)$$

$$= \int \left| -20t + 10jt + j100 \right|^2 \cdot e^{-20t^2} dt \cdot \left(\frac{2}{\pi}\right)^{1/2}$$

$$= \int \left| (-20t) + j(10t + 100) \right|^2 e^{-20t^2} dt \left(\frac{2}{\pi}\right)^{1/2}$$

$$= \int \left[(-20t)^2 + (10t + 100)^2 \right] e^{-20t^2} dt \left(\frac{2}{\pi}\right)^{1/2}$$

$$= \left[\int 400t^2 e^{-20t^2} dt + \int \left(100t^2 + 10^4 + 2000t\right) e^{-20t^2} dt\right]\left(\frac{2}{\pi}\right)^{1/2}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \Big[400\int t^2 \cdot e^{-20t^2} dt + 100\int t^2 e^{-20t^2} dt + 10^4 \underbrace{\int e^{-20t^2} dt}_{\frac{\sqrt{\pi}}{\sqrt{20}}} + 2000 \underbrace{\int \cancel{t}\, e^{-20t^2} dt}_{=0 \text{ (odd fn)}}\Big] \quad \text{——— (3)}$$

$$\left\{ \int_{-\infty}^{\infty} t^2 e^{-20t^2} dt = \sqrt{5}\left[-\frac{\sqrt{5}t}{200} + \frac{\sqrt{\pi} \cdot e^{20t^2} \cdot \operatorname{erf}(2\sqrt{5}t)}{800}\right] e^{-20t^2} \right\}$$

$$= \frac{\sqrt{5} \cdot \sqrt{\pi}}{400}$$

$\therefore$ (3) becomes,

$$\Rightarrow 500 \cdot \frac{\sqrt{5}\cdot\sqrt{\pi}}{400} = \frac{5 \cdot \cancel{\sqrt{5}}\sqrt{\pi}}{4} + \frac{10^4 \cdot \sqrt{\pi}}{\cancel{\sqrt{20}}\, 2}$$

$$\boxed{\langle w^2 \rangle = \left[\frac{5\sqrt{\pi}}{4} + \frac{10^4\sqrt{\pi}}{2}\right] \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sqrt{10}}$$

$$\boxed{\Rightarrow \langle w^2 \rangle = 3166.2 \cdot \sqrt{10}} \Rightarrow \underline{1.001255 \times 10^4}$$

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$$\sigma_\omega^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2$$

$$= \left[\frac{5\sqrt{\pi}}{4} + \frac{10^4 \sqrt{\pi}}{2} \right] \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sqrt{10} - \left(\frac{100}{[illegible]} \right)^2$$

$$1.00125 \times 10^4 \qquad\qquad 0.00125 \times 10^4$$

$$= \underset{A}{([illegible])} - 10^4 = [illegible]$$

$$\cancel{\sigma_\omega^2 = [illegible]\ 12.5}$$

$$\cancel{\Rightarrow \sigma_\omega = 46.5425}$$

$$\boxed{\langle \omega^2 \rangle = 1.00125 \times 10^4}$$

$$\boxed{\langle \omega \rangle = 100}$$

$$\therefore \sigma_\omega^2 = 1.00125 \times 10^4 - 10^4$$

$$= 0.00125 \times 10^4 = 12.5$$

$$\boxed{\therefore \sigma_\omega^2 = 12.5} \quad \& \quad \sigma_\omega = (12.5)^{1/2}$$

$$\boxed{\sigma_\omega = \cancel{[illegible]}\ 3.5355}$$

(2) Given

$$s(t) = \left[\left(\frac{3}{\pi}\right)^{1/4} \cdot e^{jm\sin\omega_m t - \frac{3t^2}{2}} \right] \cdot$$

$$e^{j\beta\frac{t^2}{2} + j\omega_0 t}$$. To find $\langle\omega\rangle$, $\langle\omega^2\rangle$ & σ_ω^2.

$$\boxed{\mathcal{E}(\omega) = 2\pi}$$

$$\boxed{\& \ \mathcal{E}_t = 1}$$

$$\langle\omega\rangle \cdot \mathcal{E}_t = \int_{-\infty}^{\infty} \omega |S(\omega)|^2 d\omega \Big/ \mathcal{E}(\omega)$$

$$= \int s^*(t) \cdot \frac{1}{j}\frac{d}{dt} s(t)\, dt \quad \text{—— (1)}$$

$\mathcal{E}_t = 1$

$$\frac{1}{j}\frac{d}{dt} s(t) = \frac{1}{j}\frac{d}{dt}\left[\left(\frac{3}{\pi}\right)^{1/4} e^{\left(jm\sin\omega_m t - \frac{3t^2}{2} + j\beta\frac{t^2}{2} + j\omega_0 t\right)} \right]$$

$$\Rightarrow \frac{1}{j} \cdot \left(\frac{3}{\pi}\right)^{1/4} \cdot e^{jm\sin\omega_m t - \frac{3t^2}{2} + j\beta\frac{t^2}{2} + j\omega_0 t}$$

$$\left[jm\cos\omega_m t \cdot (\omega_m) - 3t + j\beta t + j\omega_0 \right]$$

$$= \frac{1}{j} \cdot s(t)\left[jm\omega_m \cos(\omega_m t) - 3t + j(\beta t + \omega_0) \right]$$

$$= s(t)\left[m\omega_m \cos(\omega_m t) + 3jt + \beta t + \omega_0 \right]$$

Using (1),

$$\langle\omega\rangle = \int \left(\frac{3}{\pi}\right)^{1/4} \cdot e^{-jm\sin\omega_m t - \frac{3t^2}{2} - j\beta\frac{t^2}{2} - j\omega_0 t}$$

$$\left(\frac{3}{\pi}\right)^{1/4} \cdot e^{jm\sin\omega_m t - \frac{3}{2}t^2 + j\beta\frac{t^2}{2} + j\omega_0 t} \left[m\omega_m \cos(\omega_m t) + 3jt + \beta t + \omega_0\right] dt$$

$$\cancel{\langle\omega\rangle = \int_{-\infty} \left(\frac{3}{\pi}\right)^{1/2} e^{-3t} dt}$$

$$\cancel{= \left(\frac{3}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-3t^2} dt}$$

$$= \left(\frac{3}{\pi}\right)^{1/2} \cdot \frac{\sqrt{\pi}}{\sqrt{3}} = 1$$

$$E(\omega) = \int_{-\infty}^{\infty} 2\pi |s(t)|^2 dt$$

$$= \left(\frac{3}{\pi}\right)^{1/2} 2\pi \int_{-\infty}^{\infty} e^{-3t^2}$$

$$\left(\frac{3}{\pi}\right)^{1/2} \cdot 2\pi \cdot \frac{\sqrt{\pi}}{\sqrt{3}}$$

$$= 2\pi$$

$$\boxed{\therefore\ E_t = 1}$$

$$\langle\omega\rangle = \int \left(\frac{3}{\pi}\right)^{1/2} \cdot e^{-3t^2} \cdot \left[m\omega_m \cos(\omega_m t) + 3jt + \beta t + \omega_0\right] dt$$

$$= \left(\frac{3}{\pi}\right)^{1/2} \cdot \omega_0 \int_{-\infty}^{\infty} e^{-3t^2} dt + \left(\frac{3}{\pi}\right)^{1/2} \cdot m\omega_m \int_{-\infty} e^{-3t^2} \cdot \cos(\omega_m t)\, dt + \left(\frac{3}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-3t^2} (\beta + 3j) t\, dt$$

$= 0$ (odd function)

$$= \left(\frac{3}{\pi}\right)^{1/2} \omega_0 \cdot \frac{\sqrt{\pi}}{\sqrt{3}} + \left(3/\pi\right)^{1/2} m\omega_m \int_{-\infty}^{\infty} e^{-3t^2} \cos\omega_m t\, dt.$$

$$= \omega_0 + \left(\frac{3}{\pi}\right)^{1/2} \cdot m \cdot \omega_m \left[\frac{\sqrt{\pi}}{\sqrt{3}} \cdot e^{-\frac{\omega_m^2}{12}}\right]$$

$$= \boxed{\omega_0 + m\omega_m \cdot e^{-\omega_m^2/12} = \langle\omega\rangle}$$

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$$\langle \omega^2 \rangle = \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 d\omega \Big/ E(\omega)$$

$$= \frac{1}{E_t} \int \left| \frac{d}{dt} s(t) \right|^2 dt$$

$$\boxed{E_t = 1}$$

$$= \int \left| s(t) \left[jm\omega_m \cos\omega_m t - 3t + j(\beta t + \omega_0) \right] \right|^2 dt$$

$$= \int \left| \left(\frac{3}{\pi}\right)^{1/4} \cdot e^{jm \sin\omega_m t - \frac{3t^2}{2} + j\beta \frac{t^2}{2} + j\omega_0 t} \cdot \left[jm\omega_m \cos\omega_m t - 3t + j(\beta t + \omega_0) \right] \right|^2 dt$$

$$= \int \left(\frac{3}{\pi}\right)^{1/2} \cdot e^{-3t^2} \cdot \left| \left[jm\omega_m \cos\omega_m t - 3t + j(\beta t + \omega_0) \right] \right|^2 dt$$

$$= \left(\frac{3}{\pi}\right)^{1/2} \int e^{-3t^2} \left| -3t + j\left[\beta t + \omega_0 + m\omega_m \cos\omega_m t\right] \right|^2 dt$$

$$= \left(\frac{3}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-3t^2} \left(9t^2 + (\beta t + \omega_0 + m\omega_m \cos\omega_m t)^2 \right) dt$$

$$= \left(\frac{3}{\pi}\right)^{1/2} 9 \int e^{-3t^2} \cdot t^2 dt + \left(\frac{3}{\pi}\right)^{1/2} \int e^{-3t^2} \cdot$$

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$$(\beta t + w_0 + m w_m \cos w_m t)^2 \, dt.$$

$$= \frac{\sqrt{3}\cdot\sqrt{\pi}}{18} + \left(\frac{3}{\pi}\right)^{1/2} \Big[\int e^{-3t^2} \cdot \beta^2 t^2 \, dt \; +$$

$$\int e^{-3t^2} \cdot w_0^2 \cdot dt + \int e^{-3t^2} \cdot m^2 w_m^2 \cos^2 w_m t \, dt$$

$$+ \int e^{-3t^2} \cdot \underbrace{(\cancel{2}\beta t \, w_0)}_{=0} \, dt \; (\text{odd } f^n) + \int \underbrace{e^{-3t^2} \cdot \beta t \cdot 2}_{=0} \cdot$$

$$m w_m \cos w_m t \, dt + \int e^{-3t^2} \cdot w_0 \cdot 2 \cdot m w_m \cdot$$

$$\cos w_m t \, dt \Big]$$

$$= \frac{\cancel{\sqrt{3}}\,\cancel{\sqrt{\pi}}}{18} + \left(\frac{3}{\pi}\right)^{1/2} \left[\frac{\sqrt{3}\cdot\sqrt{\pi}}{18} \cdot \beta^2 + w_0^2 \cdot \frac{\sqrt{\pi}}{\sqrt{3}} + \right.$$

$$m^2 w_m^2 \int e^{-3t^2} \cdot \cos^2 w_m t \, dt + 2 w_0 m w_m \int e^{-3t^2} \cdot$$

$$\left. \cos w_m t \, dt \right]$$

$$= \frac{\cancel{\sqrt{3}}\,\cancel{\sqrt{\pi}}}{\cancel{18}} + \frac{\sqrt{3}}{\sqrt{\pi}} \left[\frac{\sqrt{3}\,\sqrt{\pi}}{18} \beta^2 + w_0^2 \frac{\sqrt{\pi}}{\sqrt{3}} + m^2 w_m^2 \cdot \right.$$

$$\boxed{\int e^{-3t^2} \left(\frac{1 + \cos 2 w_m t}{2} \right) dt} \Rightarrow A \; + 2 w_0 m w_m \left[\right.$$

$$\left. \frac{\sqrt{3}\cdot\sqrt{\pi}\cdot e^{-w_m^2/12}}{3} \right] \quad \text{—— (2)}$$

(A) $\Rightarrow$

$$\int e^{-3t^2}\cdot\frac{1}{2}\,dt+\frac{1}{2}\int e^{-3t^2}\cdot\cos(2\omega_m t)\,dt$$

$$=\frac{1}{2}\cdot\frac{\sqrt{\pi}}{\sqrt{3}}+\frac{1}{2}\cdot\frac{\sqrt{3}\cdot\sqrt{\pi}\cdot e^{-\omega_M^2/3}}{3}$$

$$A=\frac{\sqrt{\pi}}{2\sqrt{3}}\left[1+e^{-\omega_M^2/3}\right]$$

$\therefore$ (2) $\Rightarrow$

$$\langle\omega^2\rangle=\frac{\overset{\frac{3}{18}\cdot 9}{\cancel{\sqrt{3}\sqrt{\pi}}}}{\cancel{18}}+\frac{\cancel{\sqrt{3}}}{\cancel{\sqrt{\pi}}}\left[\frac{\overset{\cdot\sqrt{3}}{\sqrt{3}\sqrt{\pi}}}{18}\beta^2+\frac{\sqrt{\pi}}{\sqrt{3}}\omega_0^2+m^2\omega_m^2\left(\frac{\sqrt{\pi}}{2\sqrt{3}}\right)\left(1+e^{-\omega_M^2/3}\right)\right.$$

$$\left.+\,2\omega_0 m\omega_m\left(\frac{\sqrt{\pi}}{\sqrt{3}}e^{-\omega_m^2/12}\right)\right]$$

$$=\frac{3}{18}(9+\beta^2)+\omega_0^2+\frac{m^2\omega_m^2}{2}\left(1+e^{-\omega_m^2/3}\right)+2\omega_0 m\omega_m\cdot e^{-\omega_m^2/12}$$

$$\boxed{=\frac{9+\beta^2}{6}+\omega_0^2+\frac{m^2\omega_m^2}{2}\left(1+e^{-\omega_m^2/3}\right)+2m\omega_0\omega_m\cdot e^{-\omega_m^2/12}=\langle\omega^2\rangle}$$

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$$\sigma_\omega^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2$$

$$= \frac{9+\beta^2}{6} + \cancel{\omega_0^2} + \frac{m^2 \omega_m^2}{2}\left(1+e^{-\omega_m^2/3}\right)$$

$$+ \cancel{2m\omega_0\omega_m \cdot e^{-\omega_m^2/12}} - \underbrace{\left(\omega_0 + m\omega_m \cdot e^{-\omega_m^2/12}\right)^2}$$

$$\left(\cancel{\omega_0^2} + m^2\omega_m^2 \cdot e^{\frac{-\omega_m^2}{6}} + \cancel{2\omega_0 m\omega_m \cdot e^{-\omega_m^2/12}}\right)$$

$$= \frac{9+\beta^2}{6} + \frac{m^2\omega_m^2}{2}\left[1+e^{-\omega_m^2/3} - 2e^{\frac{-\omega_m^2}{6}}\right]$$

$$= \boxed{\frac{9+\beta^2}{6} + \frac{m^2\omega_m^2}{2}\left[1 - e^{\frac{-\omega_m^2}{6}}\right]^2 = \sigma_\omega^2}$$

$$\therefore \sigma_\omega = \left[\frac{9+\beta^2}{6} + \frac{m^2\omega_m^2}{2}\left(1-e^{-\omega_m^2/6}\right)^2\right]^{1/2}$$

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(3) Given, $s(t) = \sqrt{t} \cdot e^{\underbrace{j\phi(t)}}$

Print not clear in the question. Therefore, assuming the best & most general case possible

To find $\langle t \rangle$, $\langle t^2 \rangle$ & σ_t.

(PTO)

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(3) ie,

Contin. $s(t) = \sqrt{t} \cdot e^{j\phi(t)}$ $t_1 \le t \le t_2$

To find $\langle t \rangle$, $\langle t^2 \rangle$, σ_t^2

$$E_t = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |\sqrt{t}|^2 dt = \int_{-\infty}^{\infty} t\, dt$$

$$= \int_{t_1}^{t_2} t\, dt$$

$$= \boxed{\frac{t_2^2 - t_1^2}{2} = E_t}$$

$$\langle t \rangle = \int_{-\infty}^{\infty} t\, |s(t)|^2 dt \Big/ E_t$$

Nr

$$\Rightarrow \int_{-\infty}^{\infty} t\, |\sqrt{t}|^2 dt = \int_{t_1}^{t_2} t^2 dt = \frac{t_2^3 - t_1^3}{3}$$

$$\therefore \langle t \rangle = \frac{\frac{t_2^3 - t_1^3}{3}}{\frac{t_2^2 - t_1^2}{2}}$$

$$= \frac{2}{3} \cdot \frac{\cancel{(t_2 - t_1)}(t_2^2 + t_2 t_1 + t_1^2)}{\cancel{(t_2 - t_1)}(t_2 + t_1)}$$

$$\therefore \boxed{\langle t \rangle = \frac{2}{3} \cdot \left(\frac{t_2^2 + t_2 t_1 + t_1^2}{t_2 + t_1} \right)}$$

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$$\langle t^2 \rangle = \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \Big/ E_t$$

Nr

$$= \int_{-\infty}^{\infty} t^2 \cdot |\sqrt{t}|^2 dt$$

$$= \int_{t_1}^{t_2} t^3 \, dt = \frac{t_2^4 - t_1^4}{4}$$

$$\therefore \langle t^2 \rangle = \frac{\frac{t_2^4 - t_1^4}{4\,2}}{\frac{t_2^2 - t_1^2}{\cancel{2}}}$$

$$= \frac{1}{2}\left[\frac{(t_2^2 + t_1^2)\cancel{(t_2^2 - t_1^2)}}{\cancel{(t_2^2 - t_1^2)}}\right]$$

$$\therefore \boxed{\langle t^2 \rangle = \frac{t_1^2 + t_2^2}{2}}$$

$$\sigma_t^2 = \langle t^2 \rangle - \langle t \rangle^2$$

$$= \frac{t_1^2 + t_2^2}{2} - \frac{4}{9}\left(\frac{t_2^2 + t_2 t_1 + t_1^2}{t_2 + t_1}\right)^2$$

$$= \frac{(9t_1^2 + 9t_2^2)(t_2 + t_1)^2 - 8(t_1^2 + t_2^2 + t_1 t_2)^2}{18(t_1 + t_2)^2}$$

Nr $\Rightarrow$

$$= 9(t_1^2 + t_2^2)(t_1^2 + t_2^2 + 2t_1 t_2) -$$

$$8(t_1^4 + t_2^4 + t_1^2 t_2^2 + 2t_1^2 t_2^2 + 2t_1^3 t_2 + 2t_2^3 t_1)$$

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$$Nr \Rightarrow 9(t_1^4 + t_1^2t_2^2 + 2t_1^3t_2 + t_1^2t_2^2 + t_2^4 + 2t_1t_2^3) -$$

$$8(t_1^4 + t_2^4 + 3t_1^2t_2^2 + 2t_1^3t_2 + 2t_2^3t_1)$$

$$\Rightarrow \frac{t_1^4 + t_2^4 + 18t_1^2t_2^2 - 24t_1^2t_2^2 + 2t_1^3t_2 + 2t_2^3t_1}{18(t_1+t_2)^2} \leftarrow Dr$$

$$\boxed{\sigma_t^2 = \frac{t_1^4 + t_2^4 - 6t_1^2t_2^2 + 2t_1^3t_2 + 2t_2^3t_1}{18(t_1+t_2)^2}}$$

$$\therefore \sigma_t = \left[\frac{t_1^4 + t_2^4 - 6t_1^2t_2^2 + 2t_1^3t_2 + 2t_2^3t_1}{18(t_1+t_2)^2}\right]^{1/2}$$

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(4) Given

(a) $s(t) = t \cdot e^{j\phi(t) + j10t + j\sin\omega_0 t}$ $1 \le t \le 4$.

To find $\langle t \rangle = t_0$ & σ_t^2.

W.K.T, $\langle t \rangle = \int_{-\infty}^{\infty} t\,|s(t)|^2\,dt \,/\, E_t$.

Nr $\Rightarrow$

$= \int t \left| t \cdot e^{j\phi(t)} \cdot e^{j10t} \cdot e^{j\sin\omega_0 t} \right|^2 dt$

$= \int_{-\infty}^{\infty} t\,|t|^2\,dt = \int_1^4 t \cdot t^2\,dt$

($\because |t| = t$ for $t > 0$ & $1 \le t \le 4$)

$= \int_1^4 t^3\,dt = \left.\frac{t^4}{4}\right|_1^4 = \frac{4^4}{4} - \frac{1^4}{4}$

$= \frac{255}{4}$

$\therefore \langle t \rangle = \frac{255}{4} / E_t$

$\langle t^2 \rangle = \int_{-\infty}^{\infty} t^2\,|s(t)|^2\,dt \,/\, E_t$

$= \int_1^4 t^2\,|t|^2\,dt = \int_1^4 t^4\,dt$

$= \frac{4^5}{5} - \frac{1^5}{5} = \frac{1023}{5} / E_t$.

Now,

$$E_t = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} \left| t e^{j\phi(t) + j10t + j\sin\omega_0 t} \right|^2 dt$$

$$= \int_1^4 t^2 \, dt = \frac{4^3}{3} - \frac{1^3}{3} = \frac{63}{3} = \underline{\underline{21}}$$

$$\therefore \langle t \rangle = \frac{255}{4 \cdot 21} = \frac{255}{84} = \boxed{\underline{\underline{3.0357}} = \langle t \rangle}$$

$$\langle t^2 \rangle = \frac{1023}{5 \cdot 21} = \frac{1023}{105} = \boxed{\underline{\underline{9.7428}} = \langle t^2 \rangle}$$

$$\sigma_t^2 = \langle t^2 \rangle - \langle t \rangle^2$$

$$= (9.7428) - (3.0357)^2$$

$$= \boxed{0.5273 = \sigma_t^2}$$

$$\therefore \sigma_t = (0.5273)^{1/2} = \underline{\underline{0.7262}}$$

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4(b) Given $s'(t) = s(t-2)$

$\therefore$ If t_0 is $\langle t \rangle$ for $s(t)$, the mean of $s(t-2)$ (right shifted $s(t)$) will be $t_0 + 2$ $= 3.0357 + 2$

$= 5.0357$. & $\sigma_t^2 = 0.5273$

same as 4(a)

Also checking again with integration,

we have
$$s'(t) = s(t-2)$$
where $s(t) = t e^{j\phi(t) + j10t + j\sin\omega_0 t}$ $1 \le t \le 4$

$$\therefore s'(t) = s(t-2)$$
$$= (t-2)e^{j\phi(t-2) + j10(t-2) + j\sin\omega_0 (t-2)}.$$

(To find $\langle t \rangle$, σ_t^2)

$1 \le t \le 4$

$1-2 \le t-2 \le 4-2$

$-1 \le t-2 \le 2$.

$$\therefore E_t = \int_{-\infty}^{\infty} |s'(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} |t-2|^2 dt = \cancel{\int_{-1}^{2} (t-2)^2 dt}.$$
$$\cancel{= \int_{-1}^{2} t^2 + 4 - 4t = \frac{t^3}{3} + 4t - \frac{4t^2}{2}\Big|_{-1}^{2}}$$
$$\cancel{= \left(\frac{8}{3} + 8 - 8\right) - \left(\frac{-1}{3} - 4 - 2\right) = 3 + 6}$$
$$\cancel{= 9}$$

$$= \int_{-\infty}^{\infty} |s(t-2)|^2 \, dt.$$

$$= \int_{-\infty}^{\infty} |s(t')|^2 \, dt' = \mathcal{E}_t = 21$$

$$\begin{pmatrix} t-2 = t' \\ dt = dt' \end{pmatrix}$$ (from previous Question)

Note → The energy wont change if we shift the ~~graph~~ signal $s(t)$.

$$\langle t \rangle = \frac{1}{\mathcal{E}_t} \int_{-\infty}^{\infty} t \, |s(t-2)|^2 \, dt$$

$$= \frac{1}{21} \int_{-\infty}^{\infty} t \, |(t-2)|^2 \, dt$$

let $t-2 = t'$

(or) $dt = dt'$.

$$\therefore \Rightarrow \frac{1}{21} \int_{-\infty}^{\infty} (t'+2) \, |t'|^2 \, dt'$$

$$= \frac{1}{21} \int_{-\infty}^{\infty} (t'+2) \, t'^2 \, dt'$$

$$= \frac{1}{21} \int_{1}^{4} \left(t'^3 + 2t'^2\right) dt$$

$$= \frac{1}{21} \left[\frac{t'^4}{4} + \frac{2t'^3}{3} \right]_1^4 = \frac{1}{21} \left[\frac{4^4}{4} + \frac{2 \cdot 4^3}{3} - \left(\frac{1}{4} + \frac{2}{3} \right) \right]$$

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$$\frac{1}{21}\left[4^3 + \frac{2}{3}4^3 - \frac{1}{4} - \frac{2}{3}\right]$$

$$= \boxed{5.0357 = \langle t \rangle}$$

= mean in part(a) + 2. ✓

$$\langle t^2 \rangle = \frac{1}{21}\int_{-\infty}^{\infty} t^2 \left|s(t-2)\right|^2 dt$$

let $t-2 = t'$

$dt = dt'$

$$= \frac{1}{21}\int_{-\infty}^{\infty} (t'+2)^2 |t'|^2 dt'$$

$$= \frac{1}{21}\int_{-\infty}^{\infty} (t+2)^2 t^2 dt \qquad t^2+4+4t$$

$$= \frac{1}{21}\left[\int_{-\infty}^{\infty} (t^4 + 4t^2 + 4t^3) dt\right]$$

$$= \frac{1}{21}\left[\frac{t^5}{5} + \frac{4t^3}{3} + \frac{4t^4}{4}\right]_1^4$$

$$= \frac{1}{21}\left[\frac{4^5}{5} + \frac{4.4^3}{3} + \frac{4.4^4}{4} - \frac{1}{5} - \frac{4}{3} - 1\right]$$

$$= \boxed{25.8857 = \langle t^2 \rangle}$$

$$\therefore \sigma_t^2 = \langle t^2 \rangle - \langle t \rangle^2$$

$$= 25.8857 - (5.0357)^2$$

$$= 0.5274$$

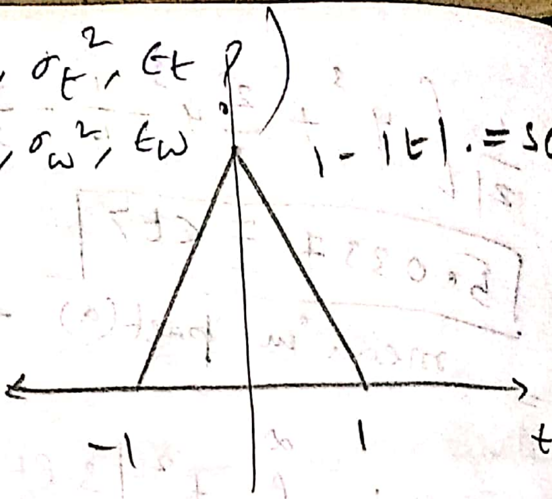
$$\boxed{\begin{aligned} &\therefore \sigma_t^2 = 0.5274 \\ &\therefore \sigma_t = 0.7262 \end{aligned}}$$

same as part(a)

(5) Given $\begin{pmatrix} \langle t \rangle, \sigma_t^2, \epsilon_t \\ \langle \omega \rangle, \sigma_\omega^2, \epsilon_\omega \end{pmatrix}$

$s(t) = 1 - |t|$.

$-\frac{T}{3} \le t \le \frac{T}{3}$.



$$\epsilon_t = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} \big| 1 - |t| \big|^2 \, dt \qquad \because |t| = -t \text{ if } t < 0$$

$$\& \ |t| = t \text{ if } t > 0$$

$$= \int_{-\infty}^{0} |1+t|^2 \, dt + \int_{0}^{\infty} |1-t|^2 \, dt.$$

$$= \int_{-T/3}^{0} (1 + t^2 + 2t) \, dt + \int_{0}^{T/3} (1 + t^2 - 2t) \, dt$$

$$= t + \frac{t^3}{3} + t^2 \Big|_{-T/3}^{0} + t + \frac{t^3}{3} - t^2 \Big|_{0}^{T/3}$$

$$= -\left(-\frac{T}{3} - \frac{T^3}{3 \cdot 3^3} + \frac{T^2}{3^2} \right) + \left(\frac{T}{3} + \frac{T^3}{3 \cdot 3^3} - \frac{T^2}{3^2} \right)$$

$$= \boxed{2 \left(\frac{T}{3} + \frac{T^3}{3^4} - \frac{T^2}{9} \right) = \epsilon_t}$$

$\boxed{\epsilon_\omega = 2\pi \cdot \epsilon_t}$ (parseval's theorem)

$$\boxed{\therefore \ \epsilon_\omega = 4\pi \left(\frac{T}{3} + \frac{T^3}{3^4} - \frac{T^2}{9} \right)}$$

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$$\langle t \rangle = \frac{1}{E_t} \int t \, |s(t)|^2 \, dt$$

from the graph $\langle t \rangle = 0$.

<u>Check by integration:</u>

$$\int_{-T/3}^{T/3} t \, \big| 1 - |t| \big|^2 \, dt \Rightarrow ?$$

$$\Rightarrow \int_{-T/3}^{0} t \, |1+t|^2 \, dt + \int_{0}^{T/3} t \, |1-t|^2 \, dt$$

$$= \int_{-T/3}^{0} (t + t^3 + 2t^2) \, dt + \int_{0}^{T/3} (t + t^3 - 2t^2) \, dt$$

$$= \left. \frac{t^2}{2} + \frac{t^4}{4} + \frac{2t^3}{3} \right|_{-T/3}^{0} + \left. \frac{t^2}{2} + \frac{t^4}{4} - \frac{2t^3}{3} \right|_{0}^{T/3}$$

$$= -\frac{T^2}{3^2 \cdot 2} - \frac{T^4}{3^4 \cdot 4} + \frac{2}{3} \cdot \frac{T^3}{3^3} + \frac{T^2}{3^2 \cdot 2} + \frac{T^4}{3^4 \cdot 4}$$

$$- \frac{2T^3}{3 \cdot 3^3} = \underline{\underline{0}}$$

$$0 \cdot \frac{1}{E_t} = 0 = \underline{\underline{\langle t \rangle}}$$

$$\boxed{\therefore \langle t \rangle = 0}$$

$$\langle t^2 \rangle = \frac{1}{E_t} \int t^2 \, |s(t)|^2 \, dt$$

$$= \int_{-T/3}^{0} t^2 + t^4 + 2t^3 \, dt + \int_{0}^{T/3} t^2 + t^4 - 2t^3 \, dt$$

$$= \left. \frac{t^3}{3} + \frac{t^5}{5} + \frac{2t^4}{4} \right|_{-T/3}^{0} + \left. \frac{t^3}{3} + \frac{t^5}{5} - \frac{2t^4}{4} \right|_{0}^{T/3}$$

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$$= \left[\frac{T^3}{3 \cdot 3^3} + \frac{T^5}{3^5 \cdot 5} - \frac{2 \cdot T^4}{3^4 \cdot 4}\right] +$$

$$\left[\frac{T^3}{3 \cdot 3^3} + \frac{T^5}{3^5 \cdot 5} - \frac{2T^4}{3^4 \cdot 4}\right]$$

$$= 2\left[\frac{T^3}{3^4} + \frac{T^5}{3^5 \cdot 5} - \frac{2T^4}{3^4 \cdot 4}\right] \cdot \frac{1}{E_t}$$

$$\Rightarrow \boxed{\langle t^2 \rangle \; (= \sigma_t^2) = \frac{\left[\frac{T^3}{3^4} + \frac{T^5}{3^5 \cdot 5} - \frac{2T^4}{3^4 \cdot 4}\right]}{\left[\frac{T}{3} + \frac{T^3}{3^4} - \frac{T^2}{3^2}\right]}}$$

$$\sigma_t^2 = \langle t^2 \rangle - \underbrace{\langle t \rangle^2}_{=0}$$

$$\boxed{\sigma_t^2 = \langle t^2 \rangle}$$

$$\therefore \sigma_t = \left[\frac{\left[\frac{T^3}{3^4} + \frac{T^5}{3^5 \cdot 5} - \frac{2T^4}{3^4 \cdot 4}\right]}{\left[\frac{T}{3} + \frac{T^3}{3^4} - \frac{T^2}{3^2}\right]}\right]^{1/2}$$

→ from the graph of $S(\omega)$, we can see that ω_0 (frequency centre) $= 0$
$= \langle \omega \rangle$

<u>Checking with integration:</u>

$$\langle \omega \rangle = \frac{1}{E_\omega} \int \omega \cdot |S(\omega)|^2 \, d\omega$$

$$= \frac{1}{E_t} \int s^*(t) \frac{1}{j} \frac{d}{dt} s(t) \, dt$$

the signal $s(t)$ is real $\therefore s^*(t) = s(t)$

$1 - |t| = s(t)$

$t > 0 \Rightarrow \quad s(t) = 1 - t$

$$\therefore \frac{ds(t)}{dt} = -1.$$

$1 - |t| = s(t)$

$t < 0 \Rightarrow \quad s(t) = 1 + t \quad \therefore \frac{ds(t)}{dt} = 1.$

$$\langle \omega \rangle_A = \int_{-T/3}^{0} (1+t) \frac{1}{j} (1) \, dt$$

$$= \frac{1}{j} \int_{-T/3}^{0} 1 + t = \frac{1}{j} \left[t + \frac{t^2}{2} \right]_{-T/3}^{0}$$

$$= \frac{1}{j} \left[+ \frac{T}{3} - \frac{T^2}{3^2 \cdot 2} \right]$$

$$= j \left[\frac{T^2}{18} - \frac{T}{3} \right] \quad \text{—(A)}$$

$$\langle \omega \rangle_B = \int_0^{T/3} (1-t) \frac{1}{j} (-1) \, dt$$

$$= j \left[t - \frac{t^2}{2} \right]_0^{T/3} = j \left[\frac{T}{3} - \frac{T^2}{18} \right] \quad \text{—(B)}$$

$$\langle \omega \rangle = \left(\langle \omega \rangle_A + \langle \omega \rangle_B \right) \frac{1}{E_t} = 0$$

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$$\boxed{\therefore \langle \omega \rangle = 0}$$

$$\langle \omega^2 \rangle = \frac{1}{E_\omega} \int \omega^2 |S(\omega)|^2 d\omega$$

$$= \frac{1}{E_t} \int \left| \frac{d}{dt} s(t) \right|^2 dt$$

$$= \frac{1}{E_t} \left[\int_{-T/3}^{0} (1)^2 dt + \int_{0}^{T/3} (-1)^2 dt \right]$$

$$= \frac{1}{E_t} \left[t \Big|_{-T/3}^{0} + t \Big|_{0}^{T/3} \right]$$

$$= \frac{1}{E_t} \left[\frac{T}{3} + \frac{T}{3} \right] = \frac{2T}{3E_t}$$

where $E_t = 2\left(\frac{T}{3} + \frac{T^3}{34} - \frac{T^2}{9} \right)$

$$\therefore \langle \omega^2 \rangle = \frac{1}{\cancel{3}\left(\frac{1}{\cancel{3}} + \frac{T^2}{3 \times 3} - \frac{T}{9 \cancel{3}} \right)}$$

$$\boxed{\langle \omega^2 \rangle = \frac{1}{1 + \frac{T^2}{27} - \frac{T}{3}}}$$

$$\sigma_\omega^2 = \langle \omega^2 \rangle - \underbrace{\cancel{\langle \omega \rangle^2}}_{=0}$$

$$\boxed{\therefore \sigma_\omega^2 = \frac{1}{1 + \frac{T^2}{27} - \frac{T}{3}}} \qquad \boxed{\sigma_\omega = \left(1 + \frac{T^2}{27} - \frac{T}{3} \right)^{-1/2}}$$