TFA assignment - 3

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(3) TBWP calculation.

(a) $e^{-|t|} \cdot e^{j\omega_0 t}$.

$$s(t) = \begin{cases} e^{-t} \cdot e^{j\omega_0 t} = e^{(-1+j\omega_0)t} & \text{if } t \geq 0. \\ e^{t} \cdot e^{-j\omega_0 t} = e^{(1-j\omega_0)t} & \text{if } t < 0. \end{cases}$$

To calculate E_t,

$$E_t = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

$$= \int_{-\infty}^{0} \left| e^{t} \cdot e^{-j\omega_0 t} \right|^2 dt + \int_{0}^{\infty} \left| e^{-t} \cdot e^{j\omega_0 t} \right|^2 dt$$

$\& \; (e^{t} > 0 \;\; \forall t)$

$$\Rightarrow \int_{-\infty}^{0} e^{2t} \, dt + \int_{0}^{\infty} e^{-2t} \, dt.$$

$$= \left. \frac{e^{2t}}{2} \right|_{-\infty}^{0} + \left. \frac{e^{-2t}}{-2} \right|_{0}^{\infty} = \frac{1}{2}(1-0) - \frac{1}{2}(0-1)$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1 = E_t}$$

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The signal is not real $\Rightarrow$ ω_0 may not ~~[illegible]~~ be zero

not symmetric $\Rightarrow$ time centre may not ~~[illegible]~~ be zero.

$$\therefore \langle t \rangle = \frac{1}{E_t} \int_{-\infty}^{\infty} t\, |s(t)|^2\, dt$$

$$= \int_{-\infty}^{0} t \cdot e^{2t}\, dt + \int_{0}^{\infty} t e^{-2t} \cdot dt$$

The formula for $\int t \cdot e^{2t}\, dt$ indefinite integral

is, let $t = -p$

$$\Rightarrow dt = -dp.$$

$$\Rightarrow \int_{-\infty}^{0} t\, e^{2t}\, dt + \int_{+\infty}^{0} (-p)\, e^{2p} (-dp)$$

$$= \int_{-\infty}^{0} t\, e^{2t}\, dt - \int_{-\infty}^{0} p \cdot e^{2p}\, dp = 0$$

$$\therefore \boxed{\langle t \rangle = 0}$$

Now $\langle t^2 \rangle = \int_{-\infty}^{\infty} t^2 \cdot |s(t)|^2 dt \cdot \frac{1}{E_t}$.

$$= \int_{-\infty}^{0} t^2 e^{2t} dt + \int_{0}^{\infty} t^2 e^{-2t} dt .$$

Let $t = -p$
$-dt = dp$.

$$\Rightarrow \int_{-\infty}^{0} t^2 e^{2t} dt + \int_{0}^{-\infty} p^2 e^{2p} (-dp)$$

$$= 2 \int_{-\infty}^{0} t^2 e^{2t} dt .$$

$$= 2 \left[\frac{1}{4} \right] = \frac{1}{2} .$$

$$\sigma_t^2 = \langle t^2 \rangle - \langle t \rangle^2 = \frac{1}{2} - 0 .$$

$$\therefore \sigma_t^2 = \frac{1}{2} \text{ \& } \sigma_t = \frac{1}{\sqrt{2}} .$$

$\langle t \rangle = 0$,

Now, we know that $f(t) e^{j\omega_0 t} \xleftrightarrow{FT} F(\omega - \omega_0)$ —— (1)

$$\langle \omega \rangle = \int_{-\infty}^{\infty} \omega |s(\omega)|^2 d\omega \cdot \frac{1}{E_\omega} .$$

$$= \int_{-\infty}^{\infty} s^*(t) \cdot \frac{1}{j} \frac{d}{dt} s(t) \, dt \cdot \frac{1}{E_t} .$$

from ① we can see that $\langle\omega\rangle = \omega_0$.

Checking

$$\langle\omega\rangle = \int_{-\infty}^{\infty}\left(e^{-|t|}\cdot e^{j\omega_0 t}\right)^* \frac{1}{j}\left[\frac{d}{dt}\left(e^{-|t|+j\omega_0 t}\right)\right] dt \cdot \frac{1}{\varepsilon t}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(e^{-|t|}\cdot e^{j\omega_0 t}\right)^* \frac{1}{j}\left[\frac{d}{dt}\left(e^{-|t|+j\omega_0 t}\right)\right] dt$$

$$\Rightarrow \int_{-\infty}^{0} e^{(1+j\omega_0)t} \cdot \frac{1}{j}\left[\frac{d}{dt}\left(e^{+t-j\omega_0 t}\right)\right] dt$$

$$+ \int_{0}^{\infty} e^{t}\cdot e^{-j\omega_0 t} \cdot \frac{1}{j}\left[\frac{d}{dt}\left(e^{+t+j\omega_0 t}\right)\right] dt$$

$$\Rightarrow \int_{-\infty}^{0} e^{(1-j\omega_0)t} \cdot \frac{1}{j} \cdot e^{(1+j\omega_0)t} (1+j\omega_0)\, dt$$

$$+ \int_{0}^{\infty} e^{-(1+j\omega_0)t} \cdot \frac{1}{j}\, e^{(-1+j\omega_0)t} (-1+j\omega_0)\, dt.$$

$$= \int_{-\infty}^{0} e^{2t}\cdot\frac{1}{j}(1+j\omega_0)\, dt + \int_{0}^{\infty} e^{-2t}\cdot\frac{1}{j}(-1+j\omega_0)\, dt$$

$$= \frac{e^{2t}}{2j}(1+j\omega_0)\Big|_{-\infty}^{0} + \frac{e^{-2t}}{-2j}(-1+j\omega_0)\Big|_{0}^{\infty}$$

$$= \left(\frac{1+j\omega_0}{2j} - 0\right) + \frac{1-j\omega_0}{2j}(0-1)$$

$$= \frac{1+j\omega_0}{2j} + \frac{-1+j\omega_0}{2j} = \frac{2j\omega_0}{2j} = \omega_0$$

$\therefore \langle \omega \rangle = \omega_0$

$$\cancel{\langle \omega^2 \rangle = \int_{-\infty}^{\infty} \left| 1+j\omega_0 \, e^{t(-1+j\omega_0)} \right|^2 dt}$$

$$\text{Now } \langle \omega^2 \rangle = \int_{-\infty}^{\infty} \left| \frac{d}{dt}(s(t)) \right|^2 dt$$

& $t \geq 0$

$$\frac{d}{dt}\left(e^{-t} \cdot e^{j\omega_0 t}\right)$$

$$\frac{d}{dt}\left(e^{(-1+j\omega_0)t}\right) = e^{(-1+j\omega_0)t}(-1+j\omega_0)$$

$t<0$

$$\frac{d}{dt}\left(e^{t} \cdot e^{-j\omega_0 t}\right) = e^{(1-j\omega_0)t}(1-j\omega_0)$$

$$\langle \omega^2 \rangle = \int_{-\infty}^{0} \left| e^{2(1-j\omega_0)t} (1-j\omega_0)^2 \right| dt +$$

$$\int_{0}^{\infty} \left| e^{2(-1+j\omega_0)t} (-1+j\omega_0)^2 \right| dt$$

$$\Rightarrow \int_{-\infty}^{0} 1 + \omega_0^2 e^{2t} \, dt + \int_{0}^{\infty} 1 + \omega_0^2 e^{-2t} \, dt$$

$$\langle \omega^2 \rangle = \left(1 + \omega_0^2\right).$$

Now,

$$\sigma_w^2 = \langle \omega^2 \rangle - \langle \omega \rangle^2$$

$$= 1 + \omega_0^2 - (\omega_0^2) = 1$$

$$\therefore \sigma_\omega^2 = 1$$

$$TBWP = \sigma_t^2 \cdot \sigma_\omega^2 = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

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(3)(b) $x(t) = \frac{1}{1+t^2} \longrightarrow$ Symmetric

$\therefore \langle t \rangle = 0$

W.K.T $\quad FT(e^{-|t|}) = \frac{2}{1+\omega^2}$

(Standard Signal)

$$E_t = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} \frac{1}{(1+t^2)^2} dt$$

$= \frac{\pi}{2}$. (After Simplification).

$$\langle t \rangle = \frac{1}{E_t} \cdot \int_{-\infty}^{\infty} t.|s(t)|^2 dt \quad \text{(Check)}$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{t}{(1+t^2)^2} dt$$

let $t^2 = p \Rightarrow 2t\,dt = dp$.

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dp}{(1+p)^2} = \frac{1}{\pi} \left. \frac{-1}{(1+p)} \right|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi} \left[\frac{-1}{1+t^2} \right] \Bigg|_{-\infty}^{\infty} = 0.$$

Signal is real $\therefore \omega_0 = 0$. (freq. centre)

(Check)

$$\langle \omega \rangle = \frac{1}{E_t} \int_{-\infty}^{\infty} s^*(t) \cdot \frac{1}{j} \frac{d}{dt} s(t)\, dt.$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \cdot \frac{1}{j} \cdot \frac{-2t}{(1+t^2)^2} \, dt.$$

$$= \frac{2}{j\pi} \int_{-\infty}^{\infty} \frac{-2t}{(1+t^2)^3} \, dt \Rightarrow 0.$$

$\therefore \langle \omega \rangle = 0$.

For this case,

WKT $\sigma_\omega^2 = \dfrac{\text{energy in } \frac{d}{dt}(x(t))}{\text{energy in } x(t)}$.

$$= \frac{2}{\pi} \cdot \int_{-\infty}^{\infty} \frac{4t^2}{(1+t^2)^4} \, dt$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{4} = \frac{1}{2}.$$

$\left(= \frac{\pi}{4}. \right.$ after simplification$\left. \right)$

$\sigma_t^2 = \langle t^2 \rangle - 0.$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} t^2 |s(t)|^2 \, dt$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{t^2}{(1+t^2)^2}}_{\pi/2} \, dt = 1.$$

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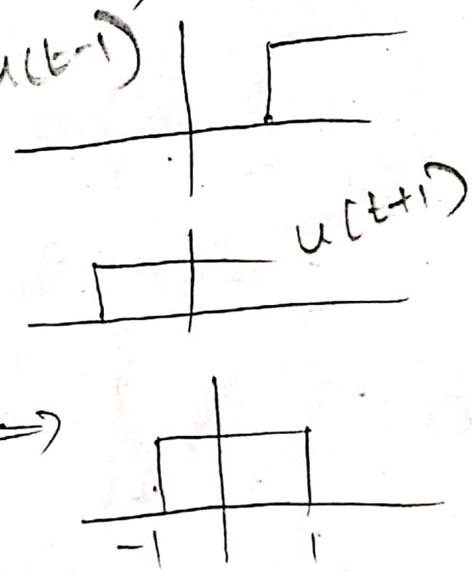
$\rightarrow TBWP = \frac{1}{2} \cdot 1 = \underline{\underline{\frac{1}{2}}}$

3 (c) $s(t) = (1-|t|)\left[u(t+1) - u(t-1)\right]$



$s(t) = 1-t \quad 0<t<1$

$= 1+t \quad -1<t\leq 0$

$$E_t = \int_{-1}^{0} [1+t]^2 dt + \int_0^1 [1-t]^2 dt$$

$$= \frac{t^3}{3} + t^2 \cdot \frac{2}{2} + t \Big|_{-1}^{0} + \frac{t^3}{3} - t^2 \cdot \frac{2}{2} + t \Big|_0^1$$

$$= -\left(\frac{-1}{3} + 1 - 1\right) + \frac{1}{3} - 1 + 1$$

$$= \frac{2}{3} - 1 + 1 - 1 + 1 = \boxed{\frac{2}{3} = E_t}$$

$$\langle t \rangle = \frac{1}{E_t} \int_{-\infty}^{\infty} t \cdot |s(t)|^2 dt$$

$$= \frac{3}{2} \int_{-1}^{0} t(1+t)^2 dt + \frac{3}{2} \int_0^1 t(1-t)^2 dt$$

$$\Rightarrow \frac{3}{2}\left(\frac{t^4}{4}+\frac{2t^3}{3}+\frac{t^2}{2}\right)_{-1}^{0} +$$

$$\frac{3}{2}\left(\cancel{\frac{t^4}{4}}-\cancel{\frac{2t^3}{3}}+\cancel{\frac{t^2}{2}}\right)_{0}^{1}$$

$$= \frac{3}{2}[0] = 0.$$

$$\therefore \langle t \rangle = 0$$

$$\langle t^2 \rangle = \frac{3}{2}\int_{-1}^{0} t^2(1+t)^2\,dt + \frac{3}{2}\int_{1}^{0} t^2(1-t)^2\,dt$$

$$= \frac{3}{2}\left(\frac{t^5}{5}+\frac{t^4}{2}+\frac{t^3}{3}\right)_{-1}^{0} + \frac{3}{2}\left(\frac{t^5}{5}-\frac{t^4}{2}+\frac{t^3}{3}\right)_{0}^{1}$$

$$\Rightarrow \frac{3}{2}\left(\frac{+1}{5}-\frac{1}{2}+\frac{1}{3}\right)+\frac{3}{2}\left(\frac{1}{5}-\frac{1}{2}+\frac{1}{3}\right)$$

$$= 3\left(\frac{+6-15+10}{30}\right) = \frac{1}{10}(-5+6)$$

$$\langle t^2 \rangle = \frac{1}{10} \Rightarrow \sigma_t^2 = \frac{1}{10}$$

$\langle \omega \rangle$ is zero since the signal is real

$(1+t)' \to 1$

$(1-t)' \to -1$

Check :

$$\langle \omega \rangle = \frac{1}{E_t}\int s^*(t)\,\frac{1}{j}\left(\frac{d}{dt}s(t)\right)dt$$

$$= \frac{3}{2j}\int_{-1}^{0}(1+t)\,dt - \frac{3}{2j}\int_{0}^{1}(1-t)\,dt$$

$$= \frac{3}{2j}\left(\frac{t^2}{2}+t\right)_{-1}^{0} + \frac{3}{2j}\left(\frac{-t^2}{2}+t\right)_{0}^{1}$$

$$= \frac{3}{2j}\left[-\frac{1}{2}+1\right] - \frac{3}{2j}\left[-\frac{1}{2}+1\right]$$

$$= \underline{\underline{0}}$$

$$\boxed{\langle w \rangle = 0}$$

$$\langle w^2 \rangle = \frac{3}{2}\int_{-1}^{1}\left|\frac{d\,s(t)}{dt}\right|^2 dt$$

$$= \frac{3}{2}\int_{-1}^{0}(1)^2 + \frac{3}{2}\int_{0}^{1}(-1)^2\,dt$$

$$= \frac{3}{2}(0+1) + \frac{3}{2}(1-0) = \underline{\underline{3}}$$

$$\boxed{\sigma_w^2 = 3}$$

$$TBWP = \sigma_t^2 \cdot \sigma_w^2 = \frac{1}{10} \cdot 3$$

$$\boxed{= \frac{3}{10}}$$

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(4) In 3(C) Question, the signal is limited in Time domain.

(i) $TBWP = \frac{3}{10} = 0.3$

from the uncertainity principle,

$$\sigma_t^2 \cdot \sigma_\omega^2 \geq \frac{1}{4} \rightarrow 0.25$$

& we see that

$$0.3 \geq 0.25$$

i.e when the signal is limited to some region in the T.D, we are not able to achieve min TBWP. i.e In option (i),

that is for all cases like this, Min. TBWP

$\left(\Rightarrow \sigma_t^2 \cdot \sigma_\omega^2 = \frac{1}{4} \right)$ is not possible.

$\hookrightarrow$ This is because, as the signal becomes limited in time, it's extent in freq. domain increases, i.e σ_ω^2 will become high, when σ_t^2 is less, (due to uncertainity principle).

4 (ii) This is the same case as with time limited signals.

e.g. of a freq. limited signal is

$x(t) = \sin\omega_1 t + \sin\omega_2 t$.

(or any sum of harmonics)

Its fourier transform has only impulses (#4).

$-\omega_1, \omega_1, -\omega_2, \omega_2$.

we can consider n

$$x(t) = \left(\frac{\sin t}{t}\right)^n \Rightarrow S(\omega) = \pi^2\left(\frac{2-|\omega|}{2}\right) \quad |\omega| \leq 2.$$

let $n = 2$.

$$E_t = \int_{-\infty}^{\infty} \left|\frac{\sin^2 t}{t^2}\right| dt$$

we can use parseval's theorem

$$E_\omega = \left(\frac{\pi^2}{2}\right)^2 \int |2-|\omega||^2 \, d\omega.$$

$$= \frac{\pi^4}{4}\left[\int_{-\infty}^{0}(2+\omega)^2 d\omega + \int_{0}^{\infty}(2-\omega)^2 d\omega\right]$$

$|\omega| \leq 2$

$$= \frac{\pi^4}{4}\left[2\omega + \frac{\omega^3}{3} + \omega^2\right]_{-2}^{0} + \frac{\pi^4}{4}\left[2\omega + \frac{\omega^3}{3} - \omega^2\right]_{0}^{2}$$

$$= \frac{\pi^4}{4}\left[4 + \frac{8}{3} - 4\right] + \frac{\pi^4}{4}\left[4 + \frac{8}{3} - 4\right]$$

$$E_\omega = \frac{4}{3}\pi^4$$

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$$E_t = \frac{\frac{4}{3}\pi^4}{2\pi} = \frac{2\pi^3}{3}$$

we can see that

$$<t> = 0 \ \& \ <\omega> = 0.$$

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} \left| \frac{d}{d\omega} S(\omega) \right|^2 d\omega}{E_\omega}$$

$$= \frac{\int_{-\infty}^{\infty} \left(-\frac{\pi}{2}\right)^2 d\omega}{6} = \frac{3}{4}$$

$$\sigma_\omega^2 = \frac{\int_{-2}^{2} \omega^2 \times 3\left(1 - |\omega| + \frac{\omega^2}{4}\right) d\omega}{4}$$

$$= \frac{2}{5}$$

$$TBW = \frac{3}{\cancel{4}_2} \cdot \frac{\cancel{2}}{5} = \frac{3}{10} \geq \frac{1}{4}$$

∴ In this case also, due to similar reasons, mentioned in (i) case, we wont get min TBWP.

(iii) $\sigma_t^2 \cdot \sigma_w^2 = \frac{1}{4}$ is possible only for Gaussian signals (as they have weak confinement & almost band limited in both the domains)

→ This is due to FT of gaussian is also gaussian.

ie, $x(t) = e^{-\frac{t^2}{2}}$

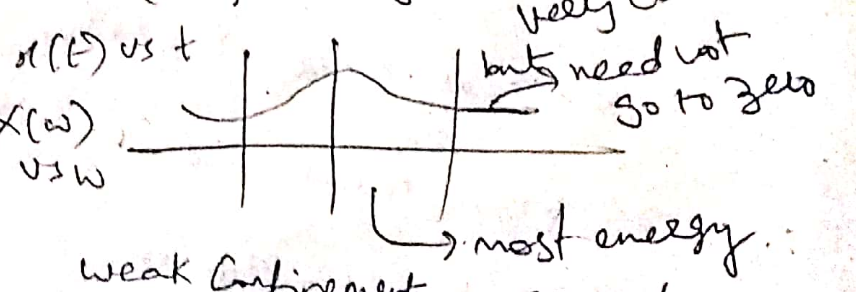
or $e^{-\frac{(t+t_0)^2}{2}}$

(with $-t_0$ as time centre).

Here,

$X(\omega) = e^{-\omega^2/2}$.

Here, Gaussians in both the domains are not exactly band limited (very strictly)



weak confinement.

5) $x(t) = \sin(\omega t + \theta)$

(a) $x_a(t) = x(t) + j\hat{x}(t)$.

$H.T\{\sin(\omega t + \theta)\} = \sin(\omega t + \theta - \frac{\pi}{2})$.

Now, $\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega \tau + \theta)\, d\tau}{t - \tau}$

($x(t)$ & $\hat{x}(t)$ are orthogonal in phase) $= \sin(\omega t + \theta - \frac{\pi}{2})$.

($\because$ Standard Signals)

$x_a(t) = \sin(\omega t + \theta) + j \sin(\omega t + \theta - \frac{\pi}{2})$

$= \sin(\omega t + \theta) - j\cos(\omega t + \theta)$

(This is because HT shifts the phase angle of the signal by $\pm \frac{\pi}{2}$)

5) (b) $x(t) = \cos(\omega t + \theta)$

Hilbert Transform is

$\hat{x}(t) = \cos(\omega t + \theta - \frac{\pi}{2})$

$\therefore x_a(t) = x(t) + j\hat{x}(t)$

$= \cos(\omega t + \theta) + j\sin(\omega t + \theta)$

$x_a(t) = e^{(\omega t + \theta)}$.

5) (c) $x(t) = 2.e^{-j\omega t}$

WKT $e^{-j\omega t} = \cos\omega t - j\sin\omega t$.

$x(t) = 2\cos\omega t - 2j\sin\omega t$

~~$\hat{x}(t) = 2(\cos\omega t + j\sin\omega t)$~~

~~from 5(a)~~ $\quad 2 j(\cancel{\sin\omega t - j\cos\omega t}))$

~~& 5(b)~~

~~$\Rightarrow 2\cos\omega t + 2j\sin$~~

Hilbert transform of $\cos\omega t = \sin\omega t$

$\sin\omega t = -\cos\omega t$.

$\therefore \hat{x}(t) = 2\sin\omega t - 2j(-\cos\omega t)$

$= 2\sin\omega t + 2j\cos\omega t$.

Now,

$x_a(t) = x(t) + j\,\hat{x}(t)$

$= 2\cos\omega t - 2j\sin\omega t + j2\sin\omega t - 2\cos\omega t$

$= 0$