Time Frequency Analysis

Assignment - 4 : Wavelets

1A) Wavelets equal to second derivative of gaussian are called Mexican hats. Normalized mexican hat wavelet is

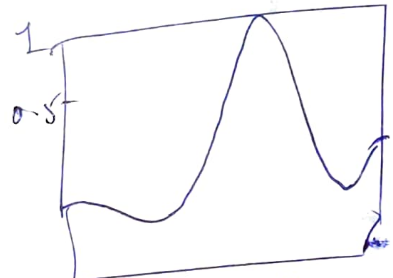
$$\psi(t) = \frac{2}{\pi^{1/4}\sqrt{3\sigma}}\left(\frac{t^2}{\sigma^2} - 1\right) e^{-t^2/2\sigma^2}$$

For $\sigma = 1$, its fourier transform is

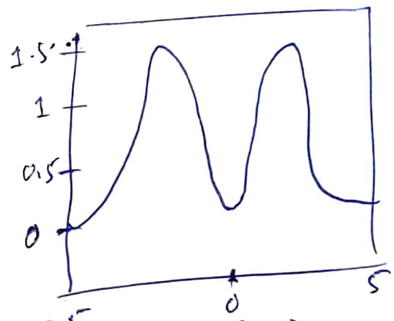
$$\hat{\psi}(\omega) = \frac{-\sqrt{8}\,\sigma^{5/2}\,\pi^{1/4}}{\sqrt{3}}\, \omega^2 e^{\frac{-\sigma^2\omega^2}{2}}$$

Mexican hat wavelet for

$\sigma = 1$



$-\psi(t)$



$-\hat{\psi}(\omega)$

For $\sigma = 1$ $\quad \psi(t) = \frac{2}{\sqrt{3}} \pi^{-1/4} (t^2 - 1) e^{-t^2/2}$

2A) a) To check absolute integrability

we know that

$$\int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx > \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x|\, dx$$

The value of the integral on RHS is $\frac{2}{(n+1)\pi}$

$\sum_{0}^{\infty} \frac{2}{(n+1)\pi}$ is a diverging series to ∞

Hence by comparing LHS & RHS

$$\lim_{B \to \infty} \int_0^B \left|\frac{\sin x}{x}\right| dx \to \infty$$

$\Rightarrow$ sinc(t) is not absolutely integrable.

To check square integrability,

Consider $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$

We know that F.T of sinc function is a box function.

So using Parseval's theorem and solving,

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$$

$\Rightarrow$ sinc(t) is square integrable

(b) $\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$, $\therefore \operatorname{sinc}(-x) = \frac{-\sin \pi x}{-\pi x}$

$$g(t) = \sum_{n\in\mathbb{Z}} \operatorname{sinc}(t-0.2n) = \sum_{n\in\mathbb{Z}} \operatorname{sinc}(0.2n-t)$$

$$g(t) = \sum_{n\in\mathbb{Z}} \operatorname{sinc}(0.2n-t) = \sum_{n\in\mathbb{Z}} \frac{\sin(\pi(0.2n-t))}{\pi(0.2n-t)}$$

Define $f(x) = \frac{\sin \pi x}{\pi x} \Rightarrow g(t) = \sum_{n\in\mathbb{Z}} f(0.2n-t)$

$g(t)$ is periodic with period 1 $\Rightarrow$ fourier series coeffs of g are given by $g_v = \int_0^1 dt\, g(t)\, e^{-j2\pi vt} = \int_0^1 dt \sum_{n\in\mathbb{Z}} f(0.2n-t)\, e^{-j2\pi vt}$

Let $x = 0.2n-t \Rightarrow \sum_{n\in\mathbb{Z}} \int_{0.2n}^{0.2n-1} dx\, f(x)\, e^{-j2\pi v(0.2n-x)}$

$$= \int_{-\infty}^{\infty} dx\, f(x)\, e^{j2\pi vx} = \tilde{f}(-v)$$

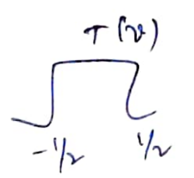
where $\tilde{f}$ is fourier transform of f.

By defn of fourier series,

$$g(t) = \sum_{n\in\mathbb{Z}} \operatorname{sinc}(0.2n-t) = \sum_{v=-\infty}^{\infty} e^{j2\pi vt} g_v \Rightarrow \sum_{n\in\mathbb{Z}} f(0.2n-t) = \sum_{v=-\infty}^{\infty} e^{-j2\pi vt} \tilde{f}(v)$$

$$= \sum_{v} \tilde{f}(v)\, e^{-j2\pi vt}$$

we know



$\rightarrow \frac{\sin \pi v}{\pi v}$ $\tilde{f}(0)$ v: freq domain

$$g(t) = \sum_{v=-\infty}^{\infty} T(v)\, e^{-j2\pi vt} = T(0)\, e^0 = 1e^0 = 1$$ only at $v=0$, the f^n exists

$\boxed{g(t) = 1}$

$\sum_{n\in\mathbb{Z}} \operatorname{sinc}(t-0.2n) = 1$, No matter how much you shift your sample points on a sinc funcn, sum of those samples is constant

4A) $g(t) \longleftrightarrow \left[\frac{-\pi}{\varepsilon_0}, \frac{\pi}{\varepsilon_0}\right]$

Time shift by $nu_0 \Rightarrow$ the support for this becomes

$$g(t-nu_0)f(t) \longrightarrow \left[nu_0 - \frac{\pi}{\varepsilon_0}, \; nu_0 + \frac{\pi}{\varepsilon_0}\right] \rightarrow 1$$

We know that $\{e^{ik\varepsilon_0 t}\}_{k \in \mathbb{Z}}$ is an orthogonal basis of this space, by property of orthogonal basis,

$$\|f(t)\|^2 = \sum_n |\langle f, \phi_n\rangle|^2$$

$$A\|f\|^2 = \sum_n |\langle f, \phi_n\rangle|^2$$

$$\|f\|^2 = \int |f(t)|^2 = \sum_n |\langle f, \phi_n\rangle|^2 \rightarrow 2 \quad (A=1)$$

Using 1, if we integrate

$$\int_{-\infty}^{\infty} |g(t-nu_0)|^2 |f(t)|^2 dt = \int_{nu_0-\pi/\varepsilon_0}^{nu_0+\pi/\varepsilon_0} |g(t-nu_0)|^2 |f(t)|^2 dt \rightarrow 3$$

Using 2,

$$\int_{nu_0-\pi/\varepsilon_0}^{nu_0+\pi/\varepsilon_0} |g(t-nu_0)f(t)|^2 = \sum_{k=-\infty}^{\infty} |\langle g(t-nu_0 f(t), e^{ik\varepsilon t}\rangle|^2 \rightarrow 4$$

Putting 4 in 3

$$= \frac{\varepsilon_0}{2\pi} \sum_{k=-\infty}^{\infty} |\langle g(u-nu_0)f(u), e^{ik\varepsilon_0 u}\rangle|^2$$

Given $g_{n,k}(t) = g(t-nu_0)\, e^{ik\varepsilon_0 t}$

$$\int_{-\infty}^{\infty} |g(t-nu_0)|^2 |f(t)|^2 dt = \frac{\varepsilon_0}{2\pi} \sum_{k=-\infty}^{\infty} |\langle f, g_{n,k}\rangle|^2$$

Summing over "n"

$$\int_{-\infty}^{\infty} \underbrace{\left[\sum_{n=-\infty} |g(t-nu_0)|^2\right]}_{\text{given } \frac{\varepsilon_0 A}{2\pi}} |f(t)|^2 dt = \frac{\varepsilon_0}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\langle f, g_{n,k}\rangle|^2$$

$$\|f(t)\|^2 \frac{\varepsilon_0 A}{2\pi} = \frac{\varepsilon_0}{2\pi} \sum_n \sum_k |\langle f, g_{n,k}\rangle|^2$$

$$A\|f\|^2 = \sum_n \sum_k | f, g_{n,k}|^2$$

$\therefore$ $\{g_{n,k}\}_{(n,k)\in Z^2}$ is a tight frame of $L^2(R)$

9A) $\phi(t) = \sum_{n\in Z} h(n)\,\phi(2t-n)$

a) $$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$\Phi(\omega) = \int_{-\infty}^{\infty} \phi(t) e^{-j\omega t}\, dt$$

$$\Phi(\omega) = \frac{1}{\sqrt{2}} H(\omega/2)\, \Phi(\omega/2)$$

$$\phi(x) = \sum_n h(n)\sqrt{2}\,\phi(2x-n)$$

Replacing $2x$ by y and integrating on both sides ($2x=y \Rightarrow dx = dy/2$)
(integral independent of translation)

$\phi \in L^1$ To allow sum & integral interchange and $\int \phi(x)\,dx \neq 0$

gives $\sum_n h(n) = \sqrt{2}$

This proof doesn't assume orthogonality nor any specific normalization of $\phi(t)$.

b) (i) $\sum_n h(n)\, h(n-2k) = \delta(k)$

By condition of orthonormality

$$\int \phi(x)\, \phi(x-m)\, dx = \delta(m)$$

Substituting scaling functions,

$$\int \left[\sum_n h(n)\sqrt{2}\,\phi(2x-n)\right]\left[\sum_k h(k)\sqrt{2}\,\phi(2x-2m-k)\right] dx = \delta(m)$$

changing variable $y = 2x$ & reordering

$$\sum_n \sum_k h(n) h(k) \int \phi(y-n)\,\phi(y-2m-k)\, dy = \delta(m)$$

lets evaluate $\int \phi(y-n)\phi(y-2m-k)\,dy$

Let $y-n=l \Rightarrow dy=dl$

$\int \phi(l)\,\phi(l+n-2m-k)\,dl$

Using orthogonality

$\int \phi(l)\,\phi(l-(k+2m-n))\,dl = \delta(2m+k-n)$

$\sum_n \sum_k h(n)h(k)\underset{\text{only 1 when } k=n-2m}{\delta(2m+k-n)} = \delta(m)$

$\underline{\sum_n h(n)h(n-2m) = \delta(m)}$

(ii) $\sum_n h(2n) = \sum_n h(2n+1) = 1/\sqrt{2}$

~~From~~ From 9(a), $\sum_n h(n) = 2$

splitting odd & even terms,

$\sum_n h(n) = \sum_n h(2k) + \sum_n h(2k+1)$

$\Rightarrow \boxed{\sqrt{2} = k_0 + k_1}$ — (1)

from 9(b) (i), $\delta(m) = \sum_n h(n)h(n-2m)$

Rewriting & summing over n,

$\sum_n \sum_k h(k)h(k+2n) = 1$

Split into even & odd terms, reordering

$= \sum_n \left[\sum_k h(2k+2n)h(2k) + \sum_k h(2k+1+2n)h(2k+1) \right]$

$= \sum_k \left[\underbrace{\sum_n h(2k+2n)}_{k_0} \right] h(2k) + \sum_k \left[\underbrace{\sum_n h(2k+1+2n)}_{k_1} \right] h(2k+1)$

$= \sum_k k_0 h(2k) + \sum_k k_1 h(2k+1)$

$= k_0 \underbrace{\sum_k h(2k)}_{k_0} + k_1 \underbrace{\sum_k h(2k+1)}_{k_1}$

$\Rightarrow k_0^2 + k_1^2 = 1$ — (2)

Using (1), (2) $\quad 2k_1k_2 = 1 \Rightarrow k_1 = \frac{1}{\sqrt{2}},\ k_2 = \frac{1}{\sqrt{2}}$

$\therefore \sum_n h(2n) = \frac{1}{\sqrt{2}}$

$\sum_n h(2n+1) = \frac{1}{\sqrt{2}}$

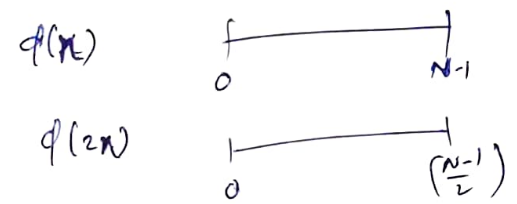
(c) ~~$\phi(t)$ Compact support on $0 \le t \le N-1$ $\{\phi(t-k), k \in \mathbb{Z}\}$~~

$\phi(t) = \sum_n h(n)\sqrt{2}\,\phi(2t-n)$ let $h(n)$ support as $[N_1, N_2]$

Given $\phi(t)$ has compact support

$$\underbrace{\phi(n)}_{[0,\,N-1]} = \sum_{n=N_1}^{N_2} \underbrace{h(n)}_{[N_1,N_2]} \sqrt{2}\, \underbrace{\phi(2x-n)}_{\text{support} = \left[\frac{N_1}{2},\, \left(\frac{N-1+N_2}{2}\right)\right]}$$

Since LHS & RHS are same, compact support should be equal



So, $\phi(n) \to [0, N-1]$

$\sum h(n)\sqrt{2}\,\phi(2n-n) \to \left[\frac{N_1}{2}, \left(\frac{N-1+N_2}{2}\right)\right]$

Limits of indices of non zero $h(n)$ are such that $N_1 = 0$ & $N_2 = N$

$\frac{N_1}{2} = 0 \Rightarrow N_1 = 0$, $\frac{N_1 - 1 + N_2}{2} = N-1 \Rightarrow N_2 = N-1$

$\therefore h(n) \to [0, N-1]$

7A) For $\phi(t-n)$ to be orthogonal, we need to have,

$$\int \phi(t-l)\,\phi(t-n) = \delta(l-n) \quad \forall\, l, n$$

(or)

let $\bar{\phi}(t) = \phi^*(-t)$ for any $(n,p) \in \mathbb{Z}^2$

$$\langle \phi(t-n)\, \phi(t-p) \rangle = \int_{-\infty}^{\infty} \phi(t-n)\,\phi^*(t-p)\, dt$$

$$= \phi * \bar{\phi}\,(p-n)$$

Thus, $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ is orthogonal if & only if $\phi * \bar{\phi}(n) = \delta[n]$

$a_{\phi\phi}(k) = |\hat{\phi}(\omega)|^2$ (taking fourier transform) (in non-sampled domain)

↳ correlation

As here $\delta[n]$, discrete samples (DTFT)

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega+2\pi k)|^2 = 1$$

$$\Rightarrow \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega+2\pi k)|^2 = 1$$

(using property that sampling a function periodizes its fourier transform)

Here given θ. hence

$$\sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega+2\pi k)|^2 = 1$$

5a) $\int f(t)\, g^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}(\omega)\, d\omega$

Let $h = f * \bar{g}$ where $\bar{g}(t) = g^*(t)$

$g(t) \leftrightarrow \hat{g}(\omega)$

$g^*(t) \leftrightarrow \hat{g}^*(\omega)$

$\Rightarrow \hat{h}(\omega) = \hat{f}(\omega)\, \hat{g}^*(\omega)$

but $t=0$

$$\int_{-\infty}^{\infty} f(t) \cdot g^*(t)\, dt = h(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(\omega)\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}^*(\omega)\, d\omega$$

$$\int_{\mathbb{R}} \frac{1}{(t^2+a^2)(t^2+b^2)}\, dt \rightarrow x(t) = \frac{1}{t^2+a^2}$$

$f(t) = e^{-a|t|}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \frac{2a}{a^2+\omega^2}$$

$$\Leftrightarrow \frac{1}{t^2+a^2} \longleftrightarrow \frac{\pi}{a} e^{-|\omega|a}$$

$$\int_{\mathbb{R}} \frac{1}{(t^2+a^2)} \cdot \frac{1}{(t^2+b^2)}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{a} e^{-a|\omega|} \cdot \frac{\pi}{b} e^{-b|\omega|}\, d\omega$$

$$= \frac{\pi}{2ab} \int_{-\infty}^{\infty} e^{-a|\omega|} e^{-b|\omega|}\, d\omega$$

$$= \frac{\pi}{2ab} \left[\int_{-\infty}^{0} e^{(a+b)\omega}\, d\omega + \int_{0}^{\infty} e^{-(a+b)\omega}\, d\omega \right]$$

$$= \frac{\pi}{2ab} \left[\frac{1}{(a+b)} - \left(\frac{-1}{a+b} \right) \right] = \frac{\pi}{ab(a+b)}$$

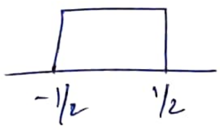
6A) $\|\phi\|^2 = \frac{1}{2\pi} \int |\hat{\phi}(\omega)|^2 d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{1}^{\infty} |\hat{\varphi}(s\omega)|^2 \frac{ds}{s}$$

$$= \frac{1}{2\pi} \int_{1}^{\infty} \left(\int |\hat{\varphi}(s\omega)|^2 d\omega \right) ds/s$$

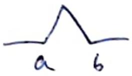
$s\omega \to s$, $\|\varphi\| = 1$

$$\|\phi\|^2 = \int_{1}^{\infty} \frac{1}{2\pi} |\hat{\varphi}(\omega)|^2 d\omega \, \frac{ds}{s^2} = \int_{1}^{\infty} \frac{ds}{s^2} = \underline{\underline{1}}$$

8A) We know β^0 

(a)

β^n (n^{th} spline) $\Rightarrow \underbrace{\beta^0 * \beta^0 * \cdots}_{n \text{ times}}$

We know Convolution of 

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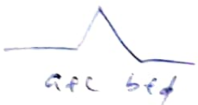
$\Rightarrow$ 

So, Convolution upto n+1 times,

Support will be from $\left[-\frac{1}{2}(n+1), \frac{1}{2}(n+1) \right]$

Riesz basis,

we derived condition for Riesz basis that,

$$A \leq \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 \leq B$$

$\hookrightarrow$ F.T ($a_{\phi\phi}(k)$)
(sampled)

Here $\quad a_{\phi\phi}(k) = \phi * \bar{\phi}(n) \qquad \bar{\phi} = \phi(-t)$

$$= \beta_n * \bar{\beta}_n(n) \qquad \bar{\beta}_n = \beta_n(-t)$$

Computing fourier transform and sampling,

$$= |\hat{\beta}_n(\omega)|^2$$

$$\hookrightarrow \text{sampling} \rightarrow \sum_{k=-\infty}^{\infty} |\hat{\beta}_n(\omega + 2\pi k)|^2$$

Here $\hat{\beta}_0(\omega) = \text{sinc}(\omega)$

By property of F.T that (Conv) $\leftrightarrow$ multiplication,

$$\hat{\beta}_n(\omega) = \text{sinc}^{(n+1)}\omega$$

$$A \le \sum_{k=-\infty}^{\infty} |\text{sinc}^{(n+1)}(\omega + 2\pi k)|^2 \le B$$

By the range of this function, it is shown that this range is from $\frac{1}{2n+1}$ to 1

Hence Riesz basis bounds,

$$A = \frac{1}{2n+1}, \quad B = 1$$

b) $\sum_{k \in \mathbb{Z}} \beta^n(t-k)$

$\beta^n(t) \Rightarrow$

$-\frac{1}{2}(n+1) \qquad \frac{1}{2}(n+1)$

$\beta^n(t-1) \rightarrow$

$-\frac{1}{2}(n+1)+1 \qquad \frac{1}{2}(n+1)+1$

F.T of $\beta^n(t-k) = e^{-j\omega k}\hat{\beta}(\omega) \qquad \hat{\beta}^n(\omega) = \text{sinc}^{(n+1)}\omega$

Summation in fourier domain,

$$\sum_{k \in \mathbb{Z}} e^{-j\omega k}\hat{\beta}^n(\omega)$$

$$\hat{\beta}^n(\omega)\left[\sum_{k \in \mathbb{Z}} e^{-j\omega k}\right]$$

$$= \hat{\beta}^n(\omega)\left[1 + e^{-j\omega} + e^{j\omega} + e^{-2j\omega} + e^{2j\omega} + \dots\right]$$

$$= \hat{\beta}^n(\omega)\left[1 + 2\cos j\omega + 2\cos(2j\omega) + \dots + \right]$$

$$e^{-jw} + e^{-2jw} + \dots + \infty = \frac{a}{1-r} = \frac{e^{-jw}}{1-e^{-jw}}$$

$$\beta^n(w)\left[1 + \frac{e^{-jw}}{1 + e^{-jw}} + \frac{e^{jw}}{1+e^{jw}}\right]$$

$$\Rightarrow \beta^n(w)\left[1 + \frac{1}{e^{jw}-1} + \frac{1}{e^{-jw}+1}\right]$$

$$\Rightarrow \beta^n(w)\left[\frac{e^{jw}}{e^{jw}-1} + \frac{e^{jw}}{1+e^{jw}}\right] = \frac{2\beta^n(w)e^{2jw}}{(e^{2jw}-1)}$$

$$= 2\hat{\beta}^n(w)\left[\frac{1}{1-e^{-2jw}}\right]$$

$$= \hat{\beta}^n(w)\left[\frac{2}{1-e^{-2jw}}\right]$$

$$\sum_{k\in Z}\beta^n(t-k) = IFT\left(\beta^n(w)\left(\frac{2}{1-e^{-2jw}}\right)\right)$$

$$= IFT\left(\frac{sinc^{n+1}(w) \times 2}{1-e^{-2jw}}\right)$$

Method 2:

Using Poisson summation formula,

$$\sum_{k\in Z} f(k-t) = \sum_{w=-\infty}^{\infty} e^{-j2\pi wt}\tilde{\beta}(w)$$

We know $\beta^n(k-t) = \beta^n(t-k)$ as it is symmetric

$$\sum_{k\in Z}\beta^n(k-t) = \sum_{w=-\infty}^{\infty} e^{-j2\pi wt}\tilde{\beta}(w)$$

$$\tilde{\beta}(w) = FT(\beta^n(t)) = sinc^{n+1}(w)$$

$$\sum_{k\in Z}\beta^n(k-t) = \sum_{k\in Z} f(k-t) = \sum_{w=-\infty}^{\infty} e^{-j2\pi wt} sinc^{n+1}(w)$$

3) a) $e_k = \left(\cos\frac{2\pi k}{N}, \sin\frac{2\pi k}{N}\right)$

(i) orthonormal basis,

lets consider $s_n = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n$

$e_k e_k^T = 1 \qquad \cos^2\frac{2\pi k}{N} + \sin^2\frac{2\pi k}{N} = 1$

$\|s_n\|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \dots + |\alpha_n|^2 = \sigma_n$

Then for only $n > m$

$\|s_n - s_m\|^2 = \|\alpha_{m+1} e_{m+1} + \dots + \alpha_n e_n\|^2$

$= |\alpha_{m+1}|^2 + \dots + |\alpha_n|^2$

$= \sigma_n - \sigma_m$

s_n is cauchy if and only if σ_n is cauchy

$\therefore$ series converges only if $\sum_{k=0}^{N-1} |\alpha_k|^2$ converges $\rightarrow$ ①

$c\|f\|^2 = \sum_{k=0}^{N-1} |\langle f, e_k\rangle|^2$ (for orthogonality)

$e_k e_k^T = 1$ & we should have $e_a e_b^T = 0$ for $a \neq b$

$\therefore \cos\frac{2\pi a}{N}\cos\frac{2\pi b}{N} + \sin\frac{2\pi a}{N}\sin\frac{2\pi b}{N} = 0$ for $a \neq b$ $\rightarrow$ ②

①, ② should satisfy for orthogonality of this basis

(ii) Riesz basis

$A\|f\|^2 \leq \sum_{k=0}^{N-1} |\langle f, e_k\rangle|^2 \leq B\|f\|^2 \qquad B \geq A > 0$

Basis linearly independent

$\sum_{k=0}^{N-1} \langle f, e_k\rangle e_k = 0$ iff $\langle f, e_k\rangle = 0$ in entire support

$f = \alpha_0 e_0 + \dots + \alpha_{N-1} e_{N-1}$

$\langle f, e_k\rangle = \alpha_0 e_0 e_k^T + \dots + |\alpha_k|^2 + \dots + \alpha_{N-1} e_{N-1} e_k^T$

$\langle f, e_k\rangle e_k = \alpha_0 e_0 + \dots + |\alpha_k|^2 e_k + \dots + \alpha_{N-1} e_{N-1}$

To get condition satisfied, $\underline{\underline{A = B = 1}}$

(iii) Frame

This will be a tight frame with $A = B = 1$

Since $\|f\|^2 = |\alpha_0|^2 + \dots + |\alpha_{N-1}|^2$

$\sum_n |\langle f, e_n \rangle|^2 = |\alpha_0|^2 + |\alpha_1|^2 + \dots + |\alpha_{N-1}|^2$

assuming $(e_a e_b^T = 0 \quad a \neq b)$

b) For a frame

$$A\|f\|^2 \leq \sum_{n \in T} |\langle f, \phi_n \rangle|^2 \leq B\|f\|^2 \rightarrow (1)$$

Φ is called a frame analysis operator

Also (1) can be rewritten as

$$A\|f\|^2 \leq \|\Phi f\|^2 = \langle \Phi^* \Phi f, f \rangle \leq B\|f\|^2$$

with $\Phi^* \Phi f = \sum_{n \in T} \langle f, \phi_n \rangle \phi_n$

A and B are "infimum" and "supremum" values of spectrum of symmetric operator $\Phi^* \Phi$, which corresponds to "smallest" and "largest" eigen values in finite dimension.

$\therefore$ Eigen values of $\phi^* \phi$ are between A & B. and $\|\phi_n\| = 1$

$\therefore$ Trace of $\phi^* \phi$ satisfies $AN \leq tr(\phi^* \phi) \leq BN$

$(tr(matrix))$ = sum of eigen values

$$tr(AB) = tr(BA)$$

$$AN \leq tr(\phi^* \phi) = tr(\phi \phi^*) = \underbrace{\sum_{n=1}^{P} |\langle \phi_n, \phi_n \rangle|^2}_{trace} = P \leq BN$$

$$AN \leq P \leq BN$$

$$A \leq \frac{P}{N} \leq B$$

10 A) a) Theorem 1:-

Let ψ and ϕ be a wavelet and a scaling function that generates an orthogonal basis. Suppose that $|\psi(t)| = O((1+t^2)^{-p/2-1})$ and $|\phi(t)| = O((1+t^2)^{-p/2-1})$. Then following statements are equivalent:

1) ψ has p vanishing moments
2) $\hat{\psi}(w)$ and its first $p-1$ derivatives are zero at $w=0$
3) $\hat{h}(w)$ and " " " " " " " $w=\pi$
4) for any $0 \le k < p$
$$q_k(t) = \sum_{-\infty}^{\infty} n^k \phi(t-n) \Rightarrow \text{polynomial of degree } k$$

Theorem 2:-

ϕ has compact support if and only if h has a compact support and their supports are equal. If support of h and ϕ is $[N_1, N_2]$ then support of ψ is $[(N_1-N_2+1)/2, (N_2-N_1+1)/2]$.

Theorem 2 proves that wavelets of compact support are computed with finite impulse response conjugate mirror filters h.

Here we consider "real causal filters $h[n]$", which implies $\hat{h}$ is a trigonometric polynomial.
$$\hat{h}(w) = \sum_{n=0}^{N-1} h[n]\, e^{-inw}$$

Daubechies wavelets have a support of minimum size for given number p of vanishing moments.

To ensure ψ has p vanishing moments, theorem 1 shows that $\hat{h}$ must have a zero at order p at $w=\pi$

* To construct a trigonometric polynomial of min size, we factor $(1+e^{-i\omega})^p$, which is a min. size polynomial having p zeroes at $\omega = \pi$:

$$\hat{h}(\omega) = \sqrt{2}\left(\frac{1+e^{-j\omega}}{2}\right)^p R(e^{-j\omega}) \rightarrow ①$$

* We need to design a polynomial $R(e^{-j\omega})$ of min. degree m such that $\hat{h}$ satisfy

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2 \rightarrow ②$$

As a result, h has $N = m+p+1$ non-zero coefficients $\rightarrow ③$

Since $h[n]$ is real, $|\hat{h}(\omega)|^2$ is an even function and can be written as polynomial in $\cos\omega$.

$|R e^{-j\omega}|^2$ is a polynomial in $\cos\omega$ that we can also write as a polynomial $P(\sin^2(\omega/2))$.

$$② \Rightarrow (1-y)^p P(y) + y^p(P(1-y)) = 1 \text{ for any } y = \sin^2(\omega/2) \in [0,1].$$

Theorem 3 :- A real conjugate mirror filter h such that $\hat{h}(\omega)$ has p zeroes at $\omega = \pi$ has atleast $2p$ non-zero coefficients. Daubechies filters have $2p$ non-zero coefficients.

Using this theorem, we can see that min. degree of R is $m = p-1$ $\rightarrow ④$

③ $\Rightarrow$ ~~Dau~~ Daubechies compactly represented wavelets with only $2p$ length filter [In general $\geq 2p$]

b) $P=2$ vanishing moments, scaling and wavelet filters

(db2) wave	$h[4]$	$h[5]$	$h[6]$	$h[7]$
	$h[0]$	$h[1]$	$h[2]$	$h[3]$

LP filter $h[n] = [0.4830 \quad 0.8365 \quad 0.2241 \quad -0.1294]$

HP filter $g[n] = [-0.1294 \quad -0.2241 \quad 0.8365 \quad -0.4830]$

Design any N db(N) wavelet : [10(a)]

[Daubechies wavelet p van ⇒ filter length 2p (smallest wavelet with p vanishing ment)

1) LPF characteristics

$$\sum_{n=0}^{2N-1} h[n] = \sqrt{2}$$

2) orthonormality of $\phi(t)$

$$\sum_{n=0}^{2N-1} h[n]\, h[n+2k] = \delta[k] \qquad k = 0, 1, \ldots N-1$$

3) vanishing moments

$$\sum_{n=0}^{2N-1} (-1)^n n^k h[n] = 0 \quad , k = 0, 1, \ldots N-1$$

(2N+1) linearly dependent equations for 2N unknowns

Here $N = 2$ $\quad h[0] + h[1] + h[2] + h[3] = \sqrt{2} \rightarrow (1)$

$k=0 \quad h[0]^2 + h[1]^2 + h[2]^2 + h[3]^2 = 1 \rightarrow (2)$

$k=1 \quad h[0]h[2] + h[1]h[3] + h[2]h[4] + h[3]h[5] = 0 \rightarrow (3)$

$k=0 \quad \sum_{n=0}^{3} (-1)^n h[n] = 0 \rightarrow (4)$

$k=1 \quad \sum_{n=0}^{3} (-1)^n h[n] = 0 \rightarrow (5)$

α: angle parameter

Using (1), (2), (3), (4), (5), we got

LP filter h[n] coeffs

$g[n] = (-1)^{1-n} h[n]$

↳ g[n] coeffs

$$h(0) = \frac{1-\cos\alpha+\sin\alpha}{2\sqrt{2}}$$

$$h(1) = \frac{1+\cos\alpha+\sin\alpha}{2\sqrt{2}}$$

$$h(2) = \frac{1+\cos\alpha-\sin\alpha}{2\sqrt{2}}$$

$$h(3) = \frac{1-\cos\alpha-\sin\alpha}{2\sqrt{2}}$$

$$h_{D_4} = \left[\frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}}\right]$$

For getting scalar relation

$$\phi(t/2) = \sqrt{2} \sum h(n)\, \phi(t-n) \quad [L \to \text{length of filter}]$$

We can get $\phi(t)$ by solving iteratively.

$$\phi(t/4) = \sqrt{2} \sum_{n=0}^{L-1} h[n]\, \phi\left(\frac{t}{2} - n\right)$$

$$= \sqrt{2} \sum_{p=0}^{2(L-1)} \underbrace{h[p/2]}_{\text{upsampled } h}\, \phi\left(\frac{t}{2} - p/2\right)$$

$\phi_{m,t}$ is convolution of $\phi_{m-1}(t)$ with upsampled $h[\cdot]$.

Iterative algorithm:

1. Initialize $\phi(t)$ with a vector of 1 (guess that is non-zero average)
2. choose associated LP filter $h[n]$
3. Upsample $h[n]$ to $h_u[n]$ by inserting 0s
4. Convolve $\phi^{(i)}t$ with $h_u[n]$ to get $\phi^{(i+1)}(t)$
5. Repeat steps 3-4 until convergence

We get $\phi(t)$ coefficients.

To get wavelet after getting $\phi(t)$

$$\psi(t) = \sum g(n)\, \phi(t-n)$$

Upsample $h(n)$

$\Rightarrow [1\ 1\ 1\ 1] * [0.4830 \quad 0 \quad 0.8365 \quad 0 \quad 0.2241 \quad 0 \quad -1.1294]$

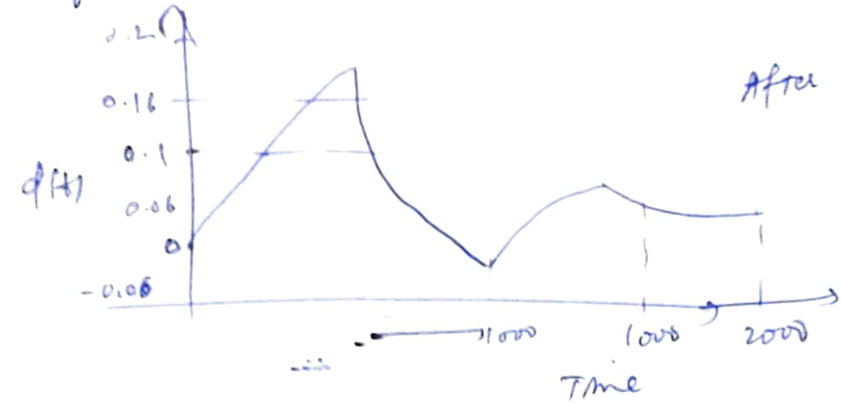
$\phi^{(1)}t * h_1[n]$

$$\boxed{\phi^{(2)}t \Rightarrow \phi^{(1)}t * h_1[n]}$$

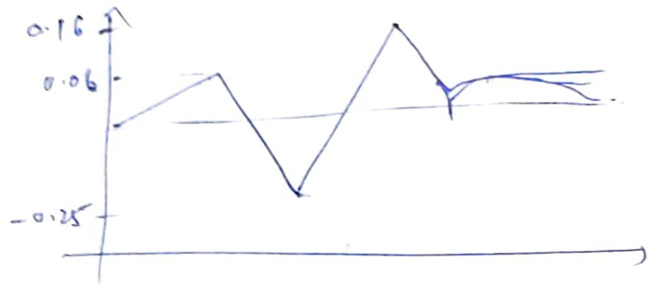
$\hookrightarrow$ again do $\phi^{(3)}t \Rightarrow \phi^{(2)}t * h_2[n]$

⋮

Using matlab to solve, we get



After 9 iterations



After 9 iterations

$\sum_{n=0}^{L-1} g(n)\,\phi(t-n)$

k	First moment	Zero moment
0	0	1.414
1	0	0.89
2	1.224744	0.56
3	6.572012	-0.86

10(b) here shows detailed description on how to construct compactly supported wavelets.

10(a) → pure theory

Flow chart:

get $h[n]$ & $g[n]$ using 3 conditions

↓

get $\phi(t)$ & $\psi(t)$ using iterative algorithm mentioned