Assignment-1 AE630A

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Problem 1 ***To Show:*** *$\exp(\hat{\omega}) = R$ where, $\hat{\omega}$ is an anti-symmetric matrix and R is a rotational matrix.*

Explanation. From our discussion in Class, we know Rotation matrix is of the form:

$$R = I + \sin\theta\hat{N} + (1-\cos\theta)\hat{N}^2$$

Also for anti-symmetric matrix,

$$\hat{N}^3 = -(n_1^2 + n_2^2 + n_3^2)\hat{N} = -n^2\hat{N}$$

$$\implies \hat{N}^{2m+1} = (-1)^m n^{2m}\hat{N} \text{ and } \hat{N}^{2m} = (-1)^{m-1}n^{2m-2}\hat{N}^2$$

Note: symbols have usual meanings(as defined in class).
Now, use Taylor Series expansion for $\exp(\hat{\omega})$ and separate into even and odd

$$\exp(\hat{\omega}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\hat{\omega})^n = I + \sum_{m=0}^{\infty}\frac{1}{(2m+1)!}(\hat{\omega})^{2m+1} + \sum_{m=1}^{\infty}\frac{1}{(2m)!}(\hat{\omega})^{2m}$$

put $\hat{w}^{2m+1} = (-1)^m n^{2m}\hat{w}$ and $\hat{w}^{2m} = (-1)^{m-1}n^{2m-2}\hat{w}^2$

$$= I + \sum_{m=0}^{\infty}\frac{(-1)^m n^{2m}}{(2m+1)!}(\hat{\omega}) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}n^{2m-2}}{(2m)!}(\hat{\omega})^2$$

Taylor Series expansion of sine and cosine function:

$$\sin(n) = n\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}n^{2m} \text{ and } 1-\cos(n) = n^2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m)!}n^{2m-2}$$

Substitute these in above equation to get:

$$\exp(\hat{\omega}) = I + \frac{\sin n}{n}\hat{w} + \frac{1-\cos n}{n^2}\hat{w}^2$$

replace $\dfrac{\hat{w}}{n}$ by $\hat{w}'$ to get:

$$\exp(\hat{\omega}) = I + (\sin n)\hat{w}' + (1-\cos n)\hat{w}'^2$$

So, we can see experession on RHS is a rotational matrix(put $\theta = n$) we wrote on the top. Therefore,

$$\exp(\hat{\omega}) = R$$

$\implies$ Exponential of $\exp(\hat{\omega})$ is a rotational matrix

Problem 2 *Difference between tangent and gradient.*

Explanation. The gradient is can be seen as derivative of a function for multiple variables. It is a vector that points in the same direction as the normal to the function at a given point and shows the rate of change depending on all the given variables of the function.
The tangent vector points along the surface of a function and can represent a plane(for 3D surface) that is parallel to point on that surface. The tangent vector is thus perpendicular to the gradient.

Mathematically,
Gradient of the function $(x_1, x_2, \ldots, x_n) \mapsto y$ is the vector $\left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}\right)$
The tangent to a curve $x \mapsto (y_1, y_2, \ldots, y_m)$ is the vector $\left(\frac{\mathrm{d}y_1}{\mathrm{d}x}, \frac{\mathrm{d}y_2}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}y_n}{\mathrm{d}x}\right)$

Reference:
(1). https://math.stackexchange.com/a/2790005
(2). https://math.stackexchange.com/a/290930
(3). http://math.mit.edu/classes/18.013A/HTML/chapter06/section05.html

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Problem 3 *To show that trace of product of symmetric and anti-symmetric 3×3 matrix is always zero.*

Explanation. Let A is $n \times n$ symmetric matrix.and B is $n \times n$ anti-symmetric matrix. So, we know by definition, $a_{ij} = a_{ji}$, $b_{ij} = -b_{ji}$ and $b_{ii} = 0$ for all i and j.

$$tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$$

We can split the summation as below: (note- $1 \leq i \leq n$ and $1 \leq j \leq n$)

$$\implies tr(AB) = \sum_{i>j}\sum a_{ij} b_{ji} + \sum_{i<j}\sum a_{ij} b_{ji} + \sum_{i=j}\sum a_{ij} b_{ji}$$

Observe that 3rd term becomes 0 since $b_{ii} = 0$

$$\implies tr(AB) = \sum_{i>j}\sum a_{ij} b_{ji} + \sum_{i<j}\sum a_{ij} b_{ji}$$

Now, in 2nd term put $a_{ij} = a_{ji}$, $b_{ij} = -b_{ji}$, we get:

$$\implies tr(AB) = \sum_{i>j}\sum a_{ij} b_{ji} - \sum_{i<j}\sum a_{ji} b_{ij}$$

Swap the role of i and j in second term, that is $i, j \xleftarrow{swap} j, i$

$$\implies tr(AB) = \sum_{i>j}\sum a_{ij}b_{ji} - \sum_{j<i}\sum a_{ij}b_{ji}$$

Observe that, both terms cancels out, since they are equal.

$$\implies tr(AB) = 0$$

It's a general derivation true for all positive n.

Problem 4 *Prove:* $a^T b = \langle a, b\rangle = -\frac{1}{2}trace(\hat{a}\hat{b})$

Explanation. $\langle a, b\rangle$ is a vector inner product, By the definition of inner product:

$$\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i$$

Now, calculate $a^T b$:

$$a^T b = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i$$

Hence proved, $a^T b = \langle a, b\rangle$
Now,

$$\hat{a} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad \hat{b} = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

$tr(\hat{a}\hat{b}) = (-a_3b_3 - a_2b_2) + (-a_3b_3 - a_1b_1) + (-a_2b_2 - a_1b_1) = -2(a_1b_1 + a_2b_2 + a_3b_3)$

$$\implies (a_1b_1 + a_2b_2 + a_3b_3) = -\frac{1}{2}trace(\hat{a}\hat{b})$$

$$\implies \langle a, b\rangle = -\frac{1}{2}trace(\hat{a}\hat{b})$$

Hence proved.

Problem 5

Explanation. The controller was tested with different values of **kr** and **kw**. But the it failed to converge to desired value of angle. The controller is unstable. The code to controller(implemented in python notebook) can be found here at this github link. `https://github.com/shivankgarg98/AE630-control-codes/blob/master/Assignment1.ipynb`

Plot of w against number of iterations. Ideally, it should have converged at (1,1,1)