Remarks

- Every topology is a basis.
- Intersection of basis is a basis.
- The more open sets we have, the harder it is to converge.
- If X has the discrete topology then every function $X \to Y$ is continuous.
- If Y has the indiscrete topology then every function $f : X \to Y$ is continuous.
- The preimage of a closed set is closed.
- $f[A] = \{f(x) : x \in A\}$.
- Every open in the product is open in the box
- In the $\mathbb{R}$, the order topology is metrizable. Does not mean that all order topologies are metrizable.
- Discrete topology is not connected
- If a set is included in the closed, then closure is included in the closed.

Starting

Recall that a subset U of $\mathbb{R}$ is open if every point $x \in U$ is an interior point of U, meaning that there exists some $\epsilon > 0$ such that

$$x \in (x - \epsilon, x + \epsilon) \subseteq U$$

Definition 0.1: Union of Open Set

If $\{U_\alpha, \alpha \in \Lambda\}$ are open subsets of $\mathbb{R}$, then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open as well.

If $\alpha \in \bigcup_{\alpha \in \Lambda} U_\alpha$, then $x \in U_\alpha$ for some $\alpha \in \Lambda$, so there is some $\epsilon > 0$ such that $x \in (x - \epsilon, x + \epsilon) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$.
This is not true for arbitrary intersections but is true for finite intersections.

Definition 0.2: Intersection of Open Sets

If $\{U_k, k \leq n\}$ is a finite collection of open sets then $\bigcap_{k \leq n} U_k$ is open.

Let $x \in \bigcap_{k \leq n} U_k$. So $x \in U_k$ for all $k \leq n$, Since each U_k is open there are $\epsilon_k, \epsilon_n > 0$ such that $x \in (x - \epsilon_k, x + \epsilon_k) \subseteq U_k$ for all $k \leq n$. We need to find some $\epsilon > 0$ such that $x \in (x - \epsilon, x + \epsilon) \subseteq \bigcap_{k \leq n} U_k$. Take

$$\epsilon = \min\{\epsilon_k\}_{k \leq n}$$

1 Topology and Topological Spaces

Definition 1.1: Topology

A topology τ on a set X is a collection of subsets of X having the following properties:

1. $\emptyset, X$ are elements of τ
2. The union of elements of τ is an element of τ
3. The intersection of finitely many (two) elements of τ is an element of τ

The last point means that if we have $U_1 \cap U_2 \cap U_3$, we can write this as $(U_1 \cap U_2) \cap U_3$.

- The pair (X, τ) is called a topological space.

- The elements of a topology are called open sets.
- The elements of a topological space are called points.

Examples

- $\tau = \{\emptyset, X\}$ is the trivial/indiscrete topology on X.
- $\tau = \{\emptyset, \{x\}, \{y\}, X\}$ is the discrete topology on X, where $X = \{x, y\}$. Meaning that every subset of X is a topology.
- $\tau = \{\emptyset, \{x\}, X\}$ is a topology on X. Also called the Sierpinski space.
- $\tau = \{\emptyset, \{y\}, X\}$ is a topology on X.

Exercise

Find all topologies on $X = \{x, y, z\}$.

Definition 1.2: Discrete/Indiscrete Topology

Given a set X, the topology $P(X)$ is called the discrete topology and the topology $\{\emptyset, X\}$ is called the indiscrete or trivial topology.

Example 1: Cofinite Topology

Let X be a set. The cofinite topology on X is the collection

$$\tau = \{U \subseteq X : X \setminus U \text{ is finite}\} \cup \{\emptyset\}$$

is a topology on X.

Proof

1. $\emptyset \in \tau$ is true
2. $X \in \tau$ is true because $\emptyset$ is finite.

3. Let $\{U_\alpha, \alpha \in \Lambda\}$ are open subsets of X. Since they are open, this means that the compliments are finite. So $X \setminus U_\alpha$ is finite. So

$$X \setminus \bigcup_{\alpha \in \Lambda} U_\alpha = \bigcap_{\alpha \in \Lambda} (X \setminus U_\alpha) \quad \leftarrow \text{finite}$$

4. If $\{U_k, k \leq n\}$ are open, this means that $X \setminus U_k$ are finite. So

$$X \setminus \bigcap_{k \leq n} U_k = \bigcup_{k \leq n} (X \setminus U_k) \quad \leftarrow \text{finite}$$

Definition 1.3: Coarser

Suppose that τ and τ' are topologies on a given set X. If $\tau \subseteq \tau'$, we say that τ is coarser than τ', or that τ' is finer than τ. If either $\tau \subset \tau'$ or $\tau' \subset \tau$, we say that these two topologies are comparable.

Definition 1.4: Basis of a topology

A basis for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that

1. Every point $x \in X$ is an elements of some $B \in \mathcal{B}$
2. If a point $x \in X$ belongs to the intersection of two basis elements $B_1, B_2 \in \mathcal{B}$, then there is some elements in B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

Lemma 1.1

Let $\mathcal{B}$ be a basis for a topology on a set X. Then the collection of all subsets U of X such that for every $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$ is a topology on X.

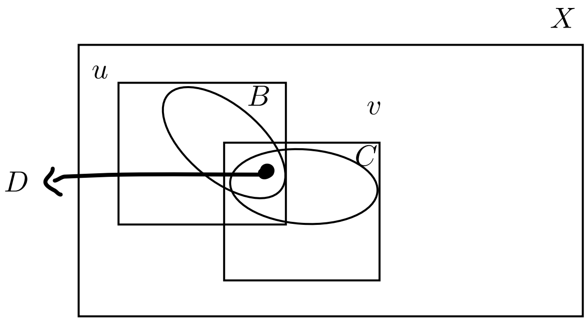
This topology is called the topology generated by $\mathcal{B}$.

Proof

1. $\emptyset \in \tau$
2. $X \in \tau$.

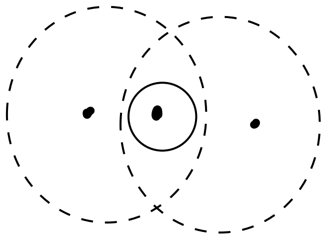
3. Let $\{U_\alpha, \alpha \in \Lambda\} \subseteq \tau$. Let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$. So there is some $\alpha \in \Lambda$ such that $x \in U_\alpha$. Let $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$.

4. Let $U, V \in \tau$. We need to show that $U \cap V \in \tau$. Let $x \in U \cap V$.



Example

1. Sets of the form



Here take the minimum of the distance between the intersection point and the centers of the circles, and divide it by 2, and this is the radius inside of the intersection of the circle.

> **Theorem 1.1**
>
> For a subset $\mathcal{B}$ of a topology τ on X the following are equivalent:
>
> 1. $\mathcal{B}$ is a basis generating τ.
> 2. Every element of τ is a union of elements of $\mathcal{B}$.

Proof

Let (X, τ) be a topological space.

- $1 \implies 2$: Let $U \in \tau$ and let $x \in U$. Since τ is generated by $\mathcal{B}$, there is some $B_x \in \mathcal{B}$

such that $x \in B_x \subseteq U$. So

$$U = \bigcup_{x \in U} B_x$$

- 2 $\implies$ 1: We need to prove that B is a basis and it generates τ. For generating τ, it is simple as we have already shown that every element of τ is a union of elements of $\mathcal{B}$.

  Now we need to show that B is a basis.

  1. Let $x \in X$. Since $X \in \tau$, then $X = \bigcup_{\alpha \in \Lambda} B_\alpha$. with $B_\alpha \in \mathcal{B}$ and $x \in B_\alpha$ for some $\alpha \in \Lambda$.
  2. Let $B_1, B_2 \in \mathcal{B}$. Let $x \in B_1 \cap B_2$. Since $\mathcal{B} \subseteq \tau$, $B_1 \cap B_2$ is open and there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

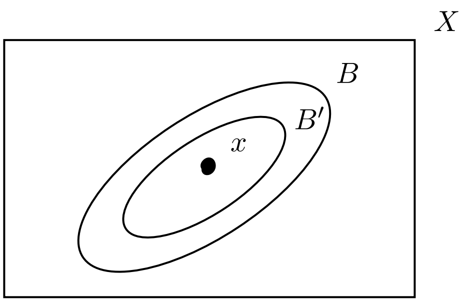
Proposition 1.1

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies τ and τ', respectively, on X. Then the following are equivalent:

- τ' is finer than τ.
- For all $x \in X$ and all $B \in \mathcal{B}$ with $x \in B$, there is some $B' \in \mathcal{B}'$ such that

$$x \in B' \subseteq B.$$

What point 2 is saying



$\forall B \in \mathcal{B}$, $\exists B' \in \mathcal{B}'$

Proof

- $1 \implies 2$: Let $x \in X$ and $x \in B \in \mathcal{B} \subseteq \tau$. So B is open with respect to τ. So $B \in \tau'$ and therefore some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
- $2 \implies 1$: We want to prove that if $U \in \tau$, then $U \in \tau'$. Since $U \in \tau$, then

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha$$

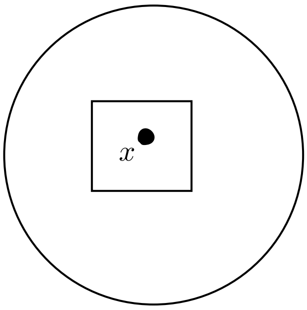
for some collection of $B_\alpha \in \mathcal{B}$. Then by 2, each

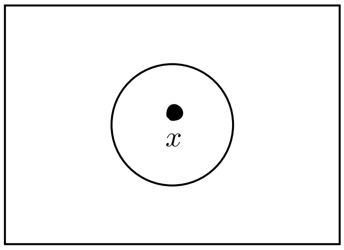
$$B_\alpha = \bigcup_{\gamma \in \Lambda_\alpha} B'_\gamma$$

where $B'_\beta \in \mathcal{B}'$.

$$U = \bigcup_{\alpha \in \Lambda} \bigcup_{\gamma \in \Lambda_\alpha} B'_\gamma$$

Example





Topologies of circles and rectangles are the same. The first part proves the topology created by circle is finite and the second part proves the topology created by the rectangle is finite.

Example 2: The line and Sorgenfrey line

The line: The set is $\mathbb{R}$ and τ is generated by open intervals (x, y) where $x, y \in \mathbb{R}$.

The Sorgenfrey line: The set is $\mathbb{R}$ and τ is generated by the intervals $[x, y)$ where $x, y \in \mathbb{R}$.

Example 3: The K-topology

The set $\mathbb{R}$ and τ is generated by 2 kinds of open sets

1. Open intervals (x, y), $x, y \in \mathbb{R}$.
2. Open intervals but by removing a set. $(x, y) \setminus K$ where $x, y \in \mathbb{R}$ and

$$K = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$$

Exercise: Prove that the collection form a basis for a topology.

Definition 1.5: subbasis

A subbasis for a topology on X is a collection $\mathcal{S}$ of subsets of X such that for all $x \in X$, there exists some $S \in \mathcal{S}$ such that $x \in S$.

Lemma 1.2

The set of finite intersections of elements of a subbasis $\mathcal{S}$ for a topology on X is a basis for a topology called the basis generated by $\mathcal{S}$.

Proof

Let $\mathcal{B}$ be the set collection of finite intersections of elements of $\mathcal{S}$. Prove that $\mathcal{B}$ is a basis for a topology on X.

1. Follows from the definition of a subbasis.

2. We want to prove that given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
   There are $S_1, \cdots, S_k, T_1, \cdots, T_n \in \mathcal{S}$ such that

   $$B_1 = \bigcap_{i \leq k} S_i$$
   $$B_2 = \bigcap_{i \leq n} T_i$$

   This implies that

   $$B_1 \cap B_2 = S_1 \cap \cdots \cap S_k \cap T_1 \cap \cdots \cap T_n$$

   So this means that

   $$B_3 = B_1 \cap B_2$$

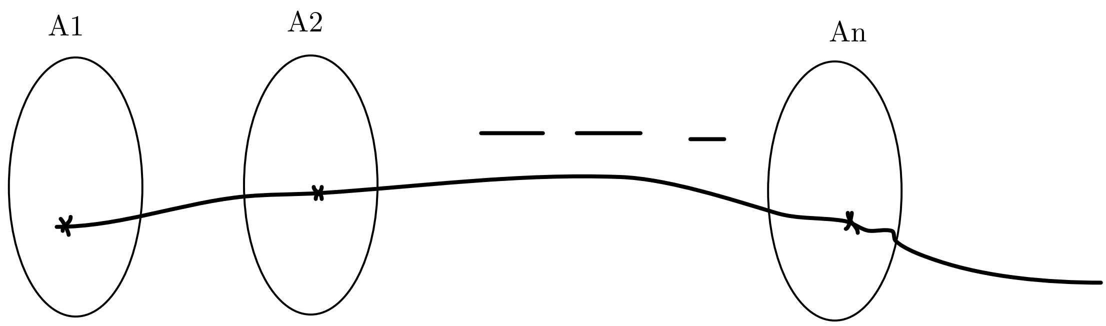
2 The Axiom of Choice

Definition 2.1: Axiom of Choice

If $\{A_\alpha : \alpha \in \Lambda\}$ is a collection of nonempty sets, then their product

$$\prod_{\alpha \in \Lambda} A_\alpha$$

is non-empty as well.



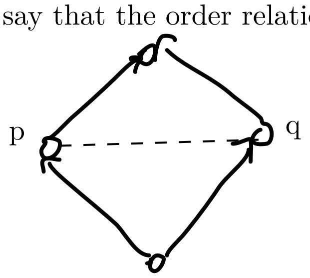
The function looks like $f : \Lambda \to \bigcup_{\alpha \in \Lambda} A_\alpha$ such that $f(\alpha) \in A_\alpha$ for each $\alpha \in \Lambda$.
Note that $\mathbb{R}^3$ is not an axiom of choice as $(0, 0, 0) \in \mathbb{R}^3$.

Definition 2.2: Partial Order

A partial order on a set P is a binary relation $\leq$ on P such that

1. For all $p \in P, p \leq p$.
2. For all $p, q \in P$, if $p \leq q$ and $q \leq p$, then $p = q$.
3. For all $p, q, r \in P$, if $p \leq q$ and $q \leq r$, then $p \leq r$.

If any two elements of P are comparable, i.e, for all $p, q \in P$, either $p \leq q$ or $q \leq p$, then we say that the order relation $\leq$ is total or linear.

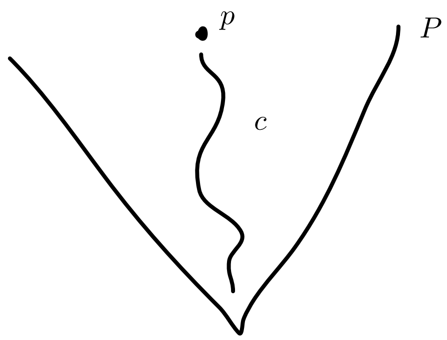


The above is not total as p and q are not comparable.
Let $(P, \leq)$ be a partially ordered set:

- $p \in P$ is maximal if there is no $q \in P$ such that $p \lneq q$.
- $q \in P$ is minimal if there is no $p \in P$ such that $p \lneq q$.
- $Q \subseteq P$ is bounded if there is some $r \in P$ such that $q \leq r$ for all $q \in Q$.
- $C \subseteq P$ is a chain if the restriction of the order relation $\leq$ to C is total.

Lemma 2.1: Zorn's Lemma

Every nonempty partially ordered set in which every chain is bounded has a maximal element.



$p \in P$ but p might not be an element of C.

Definition 2.3: well-order

A total (linear) ordered set is a well-order if all of its nonempty subsets have a minimal element.

Proposition 2.1

In a well-ordered set $(S, \leq)$, every element which is not maximal x has an immediate successor

Immediate successor means that a $y \in S$ such that $x \lneq y$ and if $x \leq z$ then $y \leq z$.

Proof

If $x \in S$ is not maximal, then $A = \{z \in S : x \lneq z\}$ is nonempty. Let y be the minimum of A. Then $x \lneq y$ and if $x \leq z$ then $y \leq z$.

Proposition 2.2: Well-ordering Principle

Every nonempty set S is well-orderable.

Definition 2.4: Ordered-Topology

Given an ordered set $(X, <)$ with at least two elements, the order topology on X is the topology generated by the basis consisting of intervals:

- (x, y) with $x, y \in X$. $(x, y) = \{z \in X : x \lneq z \text{ and } z \lneq y\}$
- $[x_{\min}, y)$ if $x_{\min}$ is the minimal element of X. $[x_{\min}, y) = \{z \in X : x_{\min} \leq z \text{ and } z \lneq y\}$
- $(x, y_{\max}]$ if $y_{\max}$ is the maximal element of X. $(x, y_{\max}] = \{z \in X : x \lneq z \text{ and } z \leq y_{\max}\}$

Recall that a set X is countable if there exists a surjection $f : \mathbb{N} \to X$.

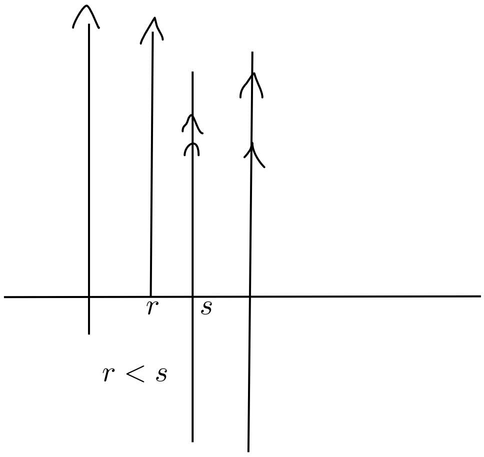
Example

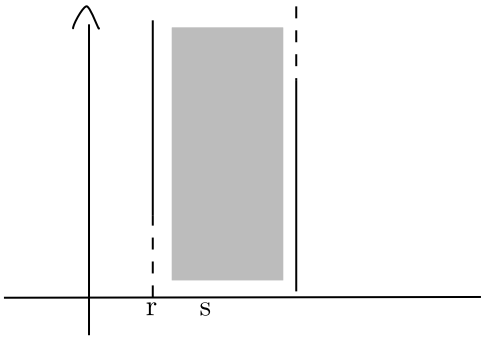
The line, the lexicographic plane, $\mathbb{N}$, two copies of $\mathbb{N}$, ω_1.

1. The line: In $(\mathbb{R}, \leq)$ the order topology is generated by open intervals.
2. Lexicographic plane: The set $\mathbb{R}^2$ and the order is given by:

$$\langle a, b\rangle \leq_{\text{lex}} \langle c, d\rangle$$

   if either $a \leq c$ or $a = c$ and $b \leq d$.





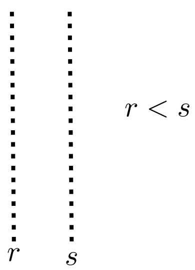
$(\langle a, b\rangle, \langle c, d\rangle)$, For intervals of the first kind $a = c$

3. The set is $\mathbb{N}$ and the order as usual. If $n \in \mathbb{N}$

$$\begin{cases} 0 \in [0,1) = \{0\} & \text{if } n = 0 \\ n \in (n-1, n+1) = \{n\} & \text{if } n > 0 \end{cases}$$

   So in $\mathbb{N}$, order topoology is discrete.

4. Two copies of $\mathbb{N}$: The set $\{0,1\} \times \mathbb{N}$. So $\langle i, n\rangle$ where $i \in \{0,1\}$ and $n \in \mathbb{N}$. And order it lexographically.



If we take the point $(1, 0)$ then there are infinitely many points below it so we can't take $(n-1, n+1)$ and so this order topology is not discrete.

5. $\omega_1(S_\Omega)$: Is an uncountable well-ordered set such that for all $\alpha \in \omega_1$, the set

$$\{\gamma \in \omega_1 : \gamma < \alpha\}$$

is countable.

Theorem 2.1

ω_1 exists

Proof

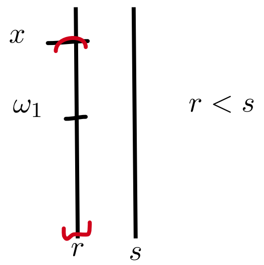
Let X be an uncountable set and let $\leq$ be a well-order of X.

- Consider the lexicographic order in $\{0, 1\} \times X$ such that

$$x \in \{0,1\} \times X : y \leq_{\text{lex}} x \text{ is uncountable}$$

is non-empty.

- There is a minimal x with uncountably many predecessors.



Cartesian product

Definition 2.5: Cartesian product

Let (X, τ) and (Y, μ) be the topological spaces. The product topology on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \in \tau \text{ and } V \in \mu\}$$

Take $X \times Y = \{\langle x, y\rangle : x \in X, y \in Y\}$. This is the cartesian product of X and Y. The product topology is generated by

$$\mathcal{B} = \{U \times V : U \in \tau \text{ and } V \in \mu\}$$

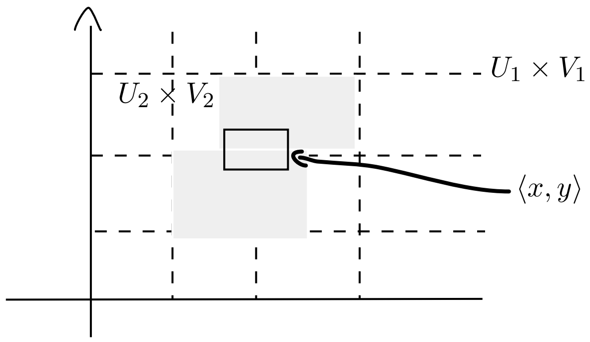
Proposition 2.3

This $\mathcal{B}$ is a basis for a topology on $X \times Y$.

Proof

1. Let $\langle x, y\rangle \in \overline{X} \times \overline{Y}$ where $\overline{X} \in \tau$ and $\overline{Y} \in \mu$.

2. Let $B_1, B_2 \in \mathcal{B}$ such that $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$. Let $\langle x, y\rangle \in (U_1 \times V_1) \cap (U_2 \times V_2)$.



This means that $\langle x, y\rangle \in (U_1 \cap U_2) \times (V_1 \cap V_2)$.

Theorem 2.2

If $\mathcal{B}$ and $\mathcal{C}$ are bases for topologies on X and Y respectively, then the set

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof

1. Let $\langle x, y\rangle \in X \times Y$. Since $\mathcal{B}$ is a basis, some $B \in \mathcal{B}$, satisfies that $x \in B$. Also there is some $c \in \mathcal{C}$ such that $y \in C$ and therefore $\langle x, y\rangle \in B \times C \in \mathcal{D}$.

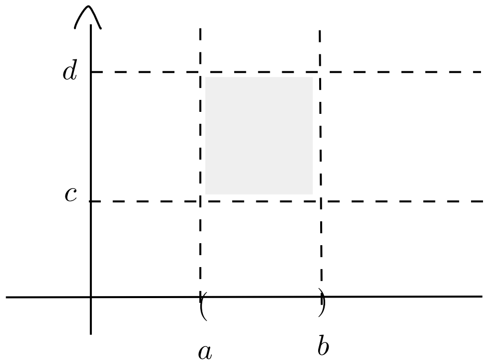
2. Let $B_1 \times C_1, B_2 \times C_2 \in \mathcal{D}$ and let $\langle x, y\rangle \in (B_1 \times C_1) \cap (B_2 \times C_2)$.

   - We might be tempted to say that $\langle x, y\rangle \in (B_1 \cap B_2) \times (C_1 \cap C_2) \subseteq (B_1 \times C_1) \cap (B_2 \times C_2)$. But note that basis are not generally closed under intersection so we can't say that $B_1 \cap B_2 \in \mathcal{B}$.
   - But note that since $\mathcal{B}$ is a basis, so there is some $B_3 \subseteq B_1 \cap B_2$ for some $B_3 \in \mathcal{B}$ such that $x \in B_3$. Similarly, $C_3 \subseteq C_1 \cap C_2$ for some $C_3 \in \mathcal{C}$ such that $y \in C_3$.
   - This means that we can say that $\langle x, y\rangle \in B_3 \times C_3 \subseteq (B_1 \times C_1) \cap (B_2 \times C_2)$.

3. If U is open in X then
$$U = \bigcup_{\alpha \in \Lambda} B_\alpha \text{ for some } B_\alpha \in \mathcal{B}$$
   If V is open in Y then
$$V = \bigcup_{\beta \in \Gamma} C_\beta \text{ for some } C_\beta \in \mathcal{C}$$
   Thus this means that
$$\begin{aligned} U \times V &= \left(\bigcup_{\alpha \in \Lambda} B_\alpha\right) \times \left(\bigcup_{\beta \in \Gamma} C_\beta\right) \\ &= \bigcup_{\alpha \in \Lambda} \bigcup_{\beta \in \Gamma} B_\alpha \times C_\beta \end{aligned}$$

Example 4

In $\mathbb{R}^2$, the product topology is generated by rectangles (interior of a rectangular region without boundary).



Definition 2.6: Projection functions

The functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given by

$$\pi_1\langle x, y\rangle = x \quad \text{and} \quad \pi_2\langle x, y\rangle = y$$

are called the projections of $X \times Y$ onto X and Y respectively.

π_1 and π_2 are sujective functions.

Proposition 2.4

Given topological spaces (X, τ) and (Y, μ), the collection

$$\mathcal{S} := \{\pi^{-1}(U), \pi^{-1}(V) : U \in \tau \text{ and } V \in \mu\}$$

form a subbasis for the product topology on $X \times Y$.

Proof

1. Prove that $\mathcal{S}$ is a subbasis for some topology.

$$\pi_1^{-1}(X) = X \times Y$$

2. Show that $\mathcal{S}$ generates the product topology. Let $U \in \tau, V \in \mu$. We want to express $U \times V$ as a finite intersection of elements of $\mathcal{S}$.

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

Note that

$$\pi_1^{-1}(U) = \{\langle x, y\rangle : \pi_1\langle x, y\rangle \in U\}$$
$$\pi_2^{-1}(V) = \{\langle x, y\rangle : \pi_2\langle x, y\rangle \in V\}$$

3 Subspaces

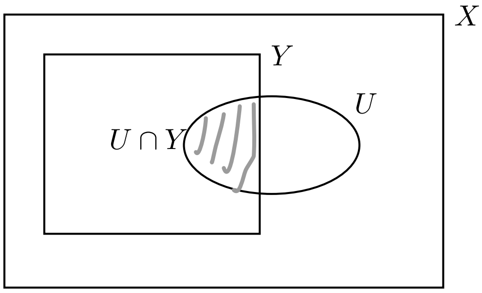
Definition 3.1: Subspace

Let (X, τ) be a topological space and let $Y \subseteq X$. The collection

$$\tau_Y = \{U \cap Y : U \in \tau\}$$

is called the **subspace topology** on Y.

Endowed with the subspace topology, a subset of X is called a subspace of X.



Proposition 3.1: Basis for the subspace topology

If $\mathcal{B}$ is a basis for the topology of X, and $Y \subseteq X$, then the collection

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof

Let X be a topological space and let $\mathcal{B}$ be a basis for τ.

1. $\mathcal{B}_Y \subseteq \tau_Y$
2. Let $U \cap Y$ be an open susbet of y. Since B is a basis for τ, let $\{B_\alpha : \alpha \in \Lambda\}$ be such that $U = \bigcup_{\alpha \in \Lambda} B_\alpha$. This implies that

$$U \cap Y = \left[\bigcup_{\alpha \in \Lambda} B_\alpha\right] \cap Y = \bigcup_{\alpha \in \Lambda} (B_\alpha \cap Y)$$

What does open mean?

Examples:

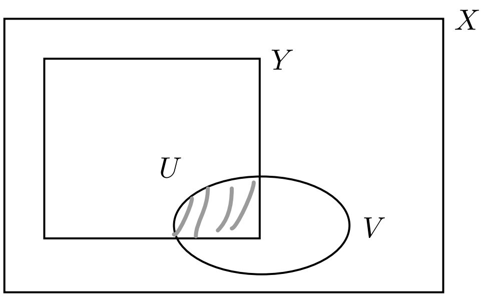
- $[0,1)$ is an open subset of $[0,1)$.

$$[0,1) = [0,1) \cap \left(\frac{-1}{2}, 2\right)$$

  Note that this is not open in $\mathbb{R}$.

Proposition 3.2

Let Y be a subspace of X. If U is an open subset of Y and Y is an open subset of X, then U is open on X as well.



Theorem 3.1

If A and B are subspaces of X and Y, respectively, then the product topology on $A \times B$ is equal to the topology on $A \times B$ as subspaces of $X \times Y$.

Proof

Let $\mathcal{B}$ be a basis for the topology of X and let $\mathcal{C}$ be a basis for the topology of Y.

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

$U \in \mathcal{B}$, and $V \in \mathcal{C}$

$=$

product then subspace

subspace then product

Note that $(U \cap A)$ is a basic element of A and $(V \cap B)$ is a basic element of B. Also note that order and subspace topologies don't interact as nicely.

Examples

1. In $\mathbb{R}$ consider $[0,1) \cup \{2\}$. In $\mathbb{R}$ we have order topology and in $[0,1) \cup \{2\}$, we have the subspace topology. Then $\{2\}$ is an open subset of $[0,1) \cup \{2\}$

$$\{2\} = \Big([0,1) \cup \{2\}\Big) \cap \left(\frac{3}{2}, 5\right)$$

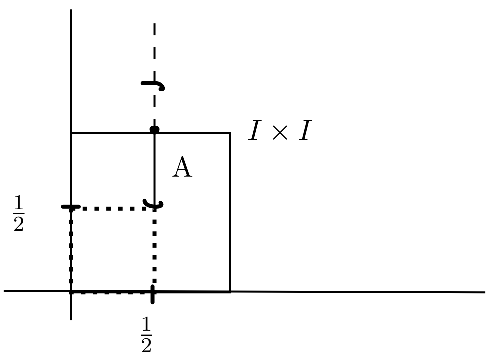
2. Now take $[0,1) \cup \{2\} \subseteq \mathbb{R}$. Now give $[0,1) \cup \{2\}$ the order topology. Note that the order topology is generated by intervals of the form:

   - (a, b)
   - $[a_{\text{min}}, b)$, if a_{min} is the minimum.
   - $(a, b_{\text{max}}]$, if b_{max} is the maximum.

   So if U is open and $2 \in U$, then there is some $x \in [0,1)$ such that $2 \in (x, 2] \subseteq U$.

3. In $\mathbb{R}^2$, the lexicographic order is given by

$$\langle x, y\rangle \leq_{\text{lex}} \langle z, w\rangle \iff \text{either } x \leq z \text{ or } (x = z \text{ and } y \leq w)$$



   So note that A is open in the subspace topology. But if you were to order it lexicographically, then A is not open as there is no next point after A for which an open set could be made.

Closed Sets

Definition 3.2: Closed

A subset C of a topological space X is called **closed** if its complement $X \setminus C$ is open.

Examples

- The cofinite topology. Remember that in X, the cofinite topology

$$\tau = \{U \subseteq X : X \setminus U \text{ is finite } \cup \emptyset\}$$

  Closed sets are X and finite subsets of X.

- The discrete topology. In the discrete topology, every subset is open and closed.
- In indiscrete topology, the only closed sets are X and $\emptyset$.
- $[0, 1) \cup \{2\}$ as a subspace of $\mathbb{R}$. Note that $\{2\}$ is open and also $[0, 1)$ is open.

Theorem 3.2

The collection of closed subset of a topological space X satisfies:

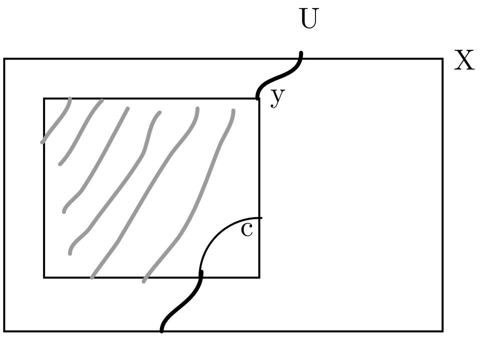
1. $\emptyset$ and X are in it.
2. It is closed under finite unions.
3. It is closed under arbitrary intersections.

Proposition 3.3

Let Y be a subspace of X. Then a subset $A \subseteq Y$ is closed in Y if and only if it is the intersection of a closed subset of X with Y.

Proof

Prove $\Rightarrow$

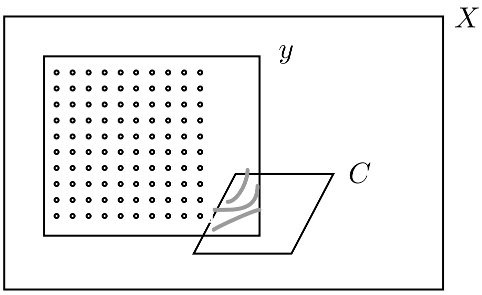


If $c \subseteq y$ is closed then $y \setminus c$ is open in y, so $y \setminus c = U \cap y$, where U is open in X.

$$c = y \cap (X \setminus U)$$

Where $X \setminus U$ is closed subset of X.

Prove $\Leftarrow$



Let $c \subseteq X$ be closed. We want to prove that $y \cap c$ is closeed in y. Note that

$$y \setminus C = y \cap (X \setminus C)$$

Where $X \setminus C$ is open subset of X.

Interiors and Closures

Definition 3.3: Interior and Closure

Let A be a susbet of topological space (X, τ):

- The interior of A is defined as

$$\text{Int}(A) = A^\circ := \bigcup \{U \subseteq X : U \text{ is open and } U \subseteq A\}$$

- The closure of A is defined as

$$\text{Cl}(A) = \overline{A} := \bigcap \{C \subseteq X : C \text{ is closed and } A \subseteq C\}$$

So then we see that A° is the largest open subset of X that includes A and $\overline{A}$ is the smallest closed subset of X including A. Also A° is open and $\overline{A}$ is closed and

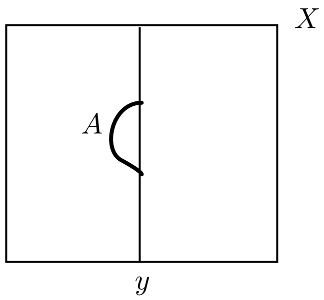
$$A^\circ \subseteq A \subseteq \overline{A}$$

Proposition 3.4

Let Y be a subspace of X and let $A \subseteq Y$. Then

$$\text{Cl}_Y(A) = \text{Cl}_X(A) \cap Y$$

Proof



Note that

$$\begin{aligned}\mathrm{Cl}_Y(A) &= \bigcap\{C \subseteq Y : C \text{ is closed and } A \subseteq C\} \\ &= \bigcap\{C \cap y : C \text{ is closed in } X \text{ and } A \subseteq C\} \\ &= \bigcap\{C \subseteq X : C \text{ is closed in } X \text{ and } A \subseteq C\} \cap Y \\ &= \mathrm{Cl}_X(A) \cap Y\end{aligned}$$

Definition 3.4: Open neighbourhood

An open neighbourhood of a point x in a topological space X is an open set U such that $x \in U$.

Proposition 3.5

Let X be a topological space and let $\mathcal{B}$ be a basis for the topology of X. If $A \subseteq X$, and $x \in X$, then the following are equivalent:

1. $x \in \overline{A}$
2. Every open neighbourhood of x intersects A.
3. Every basic open neighbourhood of x intersects A.

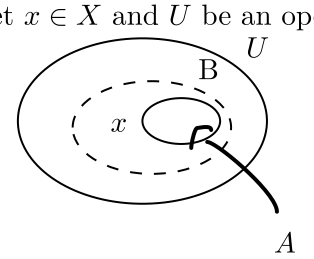
Proof

Prove 2 $\implies$ 3

Because every basic open neighbourhood is an open neighbourhood.

Prove 3 $\implies$ 2

Let $x \in X$ and U be an open neighbourhood of x. Let $B \in \mathcal{B}$ such that $x \in B \subseteq U$.



1 $\implies$ 2

Let $x \notin \overline{A}$. Then $x \in X \setminus \overline{A}$, which is open. So since $A \subseteq \overline{A}$, $(X \setminus \overline{A}) \cap A = \emptyset$.

2 $\implies$ 1

Let U be open, $x \in U$ and $U \cap A = \emptyset$. Note that $X \setminus U$ is closed and includes A. So $\overline{A} \subseteq X \setminus U$. So $x \notin \overline{A}$.

Examples

- What is the $\mathrm{Cl}_{\mathbb{R}}(0,1)$? It is $[0,1]$.
- What is the closure of the naturals in the reals? It is the naturals.
- What is the closure of the rationals in the reals? It is the reals.

Definition 3.5: Limit point

A point x in a topological space X is a limit point of the set A if every open neighbourhood of x intersects $A \setminus \{x\}$.

Proposition 3.6

Let A be a subset of a topological space X. Then

$$\overline{A} = A \cup A'$$

where A' is the set of limit points of A.

Proof

Prove $\supseteq$

We know that $A \subseteq \overline{A}$. We need to show that $A' \subseteq \overline{A}$. Let $x \in A'$. Then for all open neighbourhoods U of x, $U \cap A \neq \emptyset$. Since $A \subseteq \overline{A}$, we have $U \cap \overline{A} \neq \emptyset$. Thus $x \in \overline{A}$.

Prove $\subseteq$

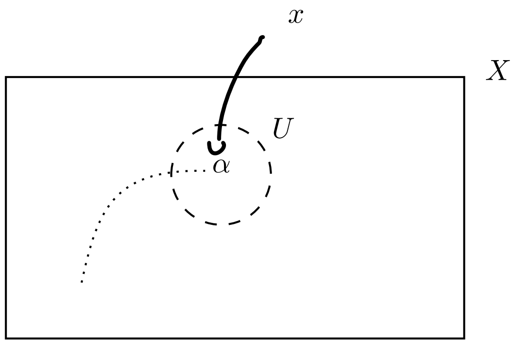
Let $x \in \overline{A}$. If $x \in A$, we are done. Otherwise, in any case, every open neighbourhood of x intersects A. Even more, since $x \notin A$, $U \cap (A \setminus \{x\}) \neq \emptyset$. Thus $x \in A'$.

Collary 3.1

A subset of a topological space is closed if and only if, it includes all of its limits points.

Definition 3.6

A sequence $(x_n : n \in \mathbb{N})$ on a topological space X converges to a point $x \in X$ if for every open neighbourhood U of x, there is some $N \in \mathbb{N}$ such that $x_n \in U$ whenever $n \geq N$.



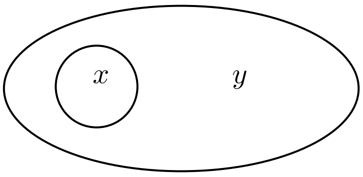
$(x_n : n \in \mathbb{N})$

$\forall U \exists N$ such that if $n \geq N$, $x_n \in U$.
If $\mathcal{B}$ is a basis for the topology of X, it is enough to ask that the condition above holds for basic open sets.

Example

1. In $\mathbb{R}, x_n \to x \forall \epsilon < 0 \exists N$ such that $\forall n \geq N, x_n \in (x-\epsilon, x+\epsilon) \iff |x_n - x| < \epsilon$.

2. Take the sequence $(x, x, x, \ldots)$.

   This sequence converges to x and also to y as the sequence of x is contained in the open set of y.

3. X, indiscrete $\tau = \{\emptyset, X\}$. Every sequence converges to every point.

4. If X has the discrete topology, then

$$\tau = \mathcal{P}(X)$$

   If a sequence is convergent, then it is eventually constant.

5. The sorgenfrey line $\mathbb{R}_l$ is the real numbers with the topology generated by $[x, y)$. Now consider 2 sequences,

   - $\overline{x} = \left(\frac{1}{n} : n \in \mathbb{N}\right)$
   - $\overline{y} = \left(-\frac{1}{n} : n \in \mathbb{N}\right)$

So note that $x_n \to 0$ because x_n has a lower limit of 0 so everything is above 0 but y_n diverges because

- If $x < 0, [x, \frac{x}{2})$
- If $x > 0, [x, x+1)$

6. Take $\mathbb{R}_K$ which is the reals with the topology generated by

$$\{(x,y) : x, y \in \mathbb{R}\} \cup \{(x,y) \setminus k : x, y \in \mathbb{R}\}$$

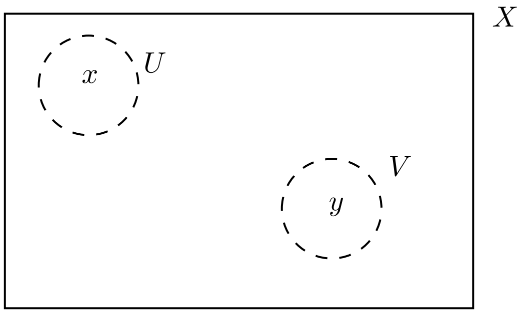
where $k = \{\frac{1}{n} : n \in \mathbb{N}\}$. Now consider the sequence

$$\frac{1}{n}$$

This diverges because $(-1,1) \setminus k$ has 0 but no other point of the sequence.

Definition 3.7: Hausdorff space

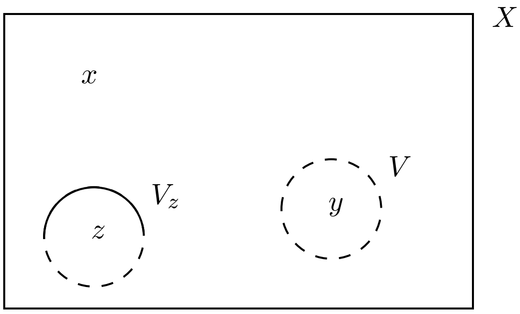
topological space X is a Hausdorff space if for every pair of different points $x, y \in X$, there are disjoint open neighbourhoods U and V of x and y respectively.



Theorem 3.3

Points are closed in a Hausdorff space.

Proof



Define $V = \bigcup_{x \neq y} V_y$. Note that this is the same as $V = X \setminus \{x\}$. This is closed because X is Hausdorff. Thus $X \setminus V$ is open. Thus V is closed. Thus $\{x\}$ is closed.

Example 5: A space with closed points which is not Hausdorff

In X infinite, the cofinite topology. In the cofinite topology,

- The points are closed.
- If U and V are non-empty opens, then $U \cap V \neq \emptyset$. This is because

$$(U \cap V)^c = \underbrace{U^c \cup V^c}_{\text{finite}} \neq X$$

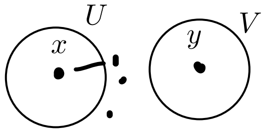
They are finite because U and V are finite. This means that they cannot equal X

Theorem 3.4

If X is Hausdorff and $(x_n : n \in \mathbb{N})$ is a convergent sequence on X then its limit is unique.

Proof

If not, let $x \neq y$ be points in X such that $x_n \to x$ and $x_n \to y$.



4 Functions

Definition 4.1: Continuous Functions

A function $X \to Y$ is continuous if

$$f^{-1}(V) := \{x \in X : f(x) \in V\}$$

is open for every open subset V of Y.

Proposition 4.1

It is enough to check continuity for basic (even subbasic) open sets.

If $\mathcal{B}$ is a basis for the topology of Y,

- then for basic open sets if $V \subseteq Y$ is open then

$$V = \bigcup_{\alpha \in \Lambda} B_\alpha \quad [B_\alpha \in \mathcal{B}]$$

$$f^{-1}(V) = f^{-1}(\bigcup_{\alpha \in \Lambda} B_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha)$$

  is open.

- For subbasic open sets, If $\mathcal{S}$ is a subbasis for the topology of Y. Let $B \subseteq Y$ be basic then

$$B = S_1 \cap S_2 \cap \cdots \cap S_n \quad [S_k \in \mathcal{S}]$$
$$f^{-1}(B) = f^{-1}(\bigcap_{k \le n} S_k) = \bigcap_{k \le n} f^{-1}(S_k)$$

Example

1. If X has the discrete topology then every function $X \to Y$ is continuous.
2. If Y has the indiscrete topology then every function $f : X \to Y$ is continuous.
3. If $f : \mathbb{R} \to \mathbb{R}$ is a function then f is continuous if and only if for all $x \in \mathbb{R}$ and all $\varepsilon > 0, \exists \delta > 0$ such that
$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$
4. Let X and Y be topological spaces then the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.
5. Let id be the identity function $(id(x) = x)$ as such $id : \mathbb{R} \to \mathbb{R}_l$. Note that $[x, y)$ is not open in $\mathbb{R}$ so then id is not continuous.
6. Let $id : \mathbb{R}_l \to \mathbb{R}$ be the identity function. Then note that since $(x, y) = \bigcup_{x<z} [z, y)$ where $[z, y)$ is open in $\mathbb{R}_l$ then (x, y) are open so id is continuous.

Proposition 4.2

Let τ and τ' be topologies on a set X. Then τ is finer than τ' if and only if the identity $id : (X, \tau) \to (X, \tau')$ is continuous.

Note that τ is finer than τ' if $\tau' \subseteq \tau$
$\iff$ Every open $U \in \tau'$ is open in τ
$\iff$ $id : (X, \tau) \to (X, \tau')$ is continuous.

Theorem 4.1

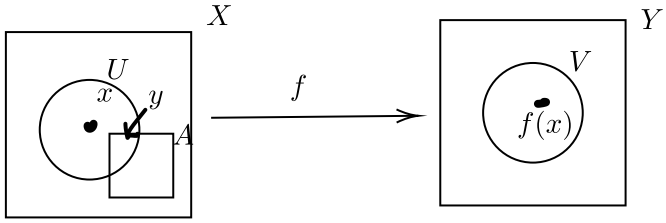
For a function $f : X \to Y$ the following are equivalent:

1. f is continuous.
2. For each $x \in X$ and each open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $f[U] \subseteq V$.
3. For every $A \subseteq X$, $f[\overline{A}] \subseteq \overline{f[A]}$.
4. The preimage of a closed set is closed.

Proof

1 $\implies$ 3

Let $x \in \overline{A}$ and we want to prove that $f(x) \in f[A]$. Let V be an open neighborhood of $f(x)$ and let $U = f^{-1}(V)$. Since f is continuous, U is open and $x \in U$.



Let $y \in U \cap A$ then $f(y) \in f[A] \cap V$, thus $f(x) \in \overline{f[A]}$.
$f[U] = f[f^{-1}(V)] \subseteq V$

3 $\implies$ 4

Let $B \subseteq Y$ be closed and let $A = f^{-1}(B)$. We want to prove that A is closed ($\overline{A} = A$). We already know that $A \subseteq \overline{A}$ so we need to prove that $\overline{A} \subseteq A$.
Let $x \in \overline{A}$, we want to prove that $x \in A$. Note that $f(x) \in f[\overline{A}]$. By 3, $f[\overline{A}] \subseteq \overline{f[A]} \subseteq B = f^{-1}(A)$. So $f(x) \in B$ therefore $x \in A = f^{-1}(B)$.
Note that $f(A) = f(f^{-1}(B)) \subseteq B$.

Prove 4 $\implies$ 1

Follows from $f^{-1}(U^c) = (f^{-1}(U))^c$.

Prove $1 \implies 2$

If $x \in X$ and V is an open neighborhood of $f(x)$. By continuity, $U := f^{-1}(V)$ is open neighborhood of x and $f[U] \subseteq V$. $[f[U] = f[f^{-1}(V)] \subseteq V]$.

Prove $2 \implies 1$

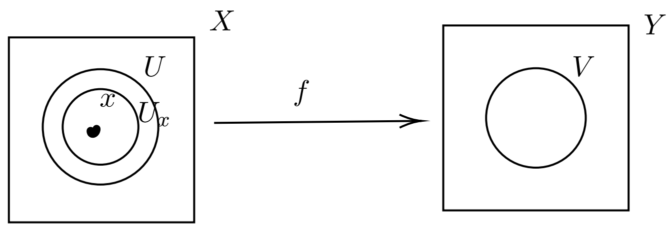
Let $V \subseteq Y$ be open. We want to prove that $U := f^{-1}(V)$ is open. For each $x \in U$, and V (which is a neighborhood of $f(x)$) there is some open set $U_x \subseteq X$ such that $x \in U_x$ and

$$f[U_x] \subseteq V$$

Claim:

$$U = \bigcup_{x \in U} U_x$$

- $U_x \subseteq U = f^{-1}(V), (\supseteq)$
- If $x \in U$ then $x \in U_x, (\subseteq)$

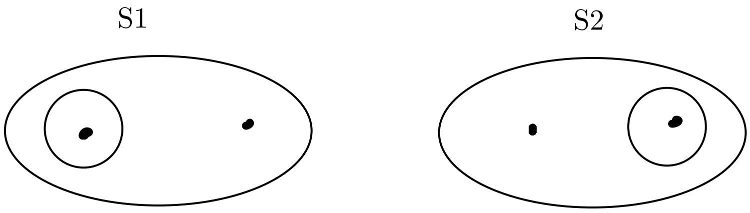


Definition 4.2: Homeomorphism
homeomorphism is a continuous function with continuous inverse.

An imbedding is a homeomorphism onto its image.

Example

- Take $f : (a, b) \to (x, y)$. We can go from an interval to an interval by rescaling and translating. They are homeomorphic.
- Open intervals are homeomorphic to $\mathbb{R}$. Take $\tan\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a homeomorphism.

- 

  Take $h : S1 \to S2$ where $h(x) = 1$ and $h(y) = 0$. This is a homeomorphism.

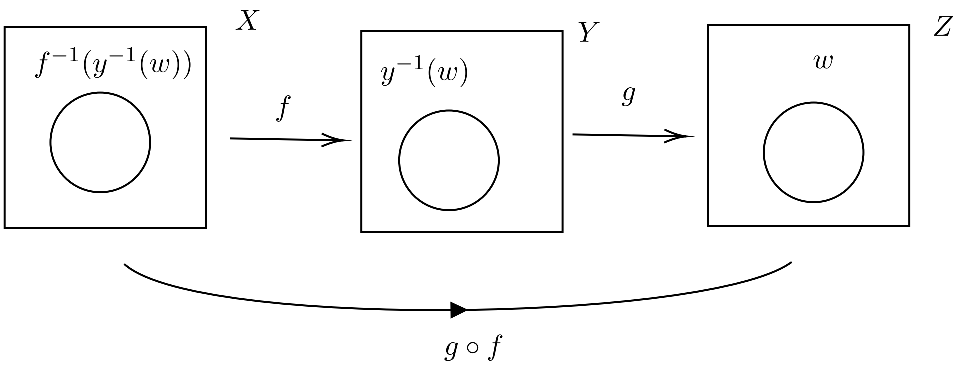
- If x and y are topological spaces and $x \in X$ is fixed and map $f_x : Y \to X \times Y$ defined by $y \mapsto \langle x, y \rangle$.

Example

Prove that f_x is continuous. Note that $f_x^{-1} = \pi_2 \mid \{x\} \times Y$

Theorem 4.2

- Constant functions are continuous.
- The inclusion is continuous.
- Composition of continuous functions is continuous.
- Restricting the domain or range of a function, or expanding its range, preserves its continuity.



Theorem 4.3: Pasting Lemma

Let $X = A \cup B$ where A and B are closed and let

$$f : A \to Y \quad \textit{and} \quad g : B \to Y$$

be continuous functions that agree on $A \cap B$. Then the function $h : X \to Y$ given by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.

Proof

If C is a closed subset of Y then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Then this means that $h^{-1}(C)$ is closed.

Theorem 4.4

Let $f : Z \to X \times Y$ be given by

$$f(z) = \langle f_1(z), f_2(z) \rangle$$

for some functions $f_1 : Z \to X$ and $f_2 : Z \to Y$. Then f is continuous if and only if both f_1 and f_2 are continuous.

Proof

Prove $\implies$

If f is continuous then

$$f_1 = \pi_1 \circ f \quad \text{and} \quad f_2 = \pi_2 \circ f$$

are continuous as well.

Prove $\Longleftarrow$

Let $U \times V$ be a basic open set in $X \times Y$. Then

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

If both f_1 and f_2 are continuous then f_1 is open and f_2 is open. Thus $f^{-1}(U \times V)$ is open.

Last Time

We proved that in Hausdorff space, sequences converge to at most one point. But note that the converse is not true. The counterexample is in $\mathbb{R}$, consider the cocountable topology.

- The reason its not Hausdorff is because if $U, V \subseteq \mathbb{R}$ is open and non-empty, then $(U \cap V)^c = U^c \cup V^c$ is countable. Thus this means that $U \cap V$ is non-empty.
- If $x_n \to x$ and $y \neq x$, then $x_n \mapsto y$.
  1. Only finitely many $x_n = y$ $[\mathbb{R} \setminus \{y\}]$.
  2. $\exists N \in \mathbb{N}$ such that if $n \geq N$, $x \neq y$. Consider $[\mathbb{R} \setminus \{x_n : n \geq N\}]$.

Initial Topology

Definition 4.3

Let X be a set and let $\{Y_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. Let $\mathcal{F} := \{f_\alpha : X \to Y_\alpha : \alpha \in \Lambda\}$ be a family of functions. The initial topology of $\mathcal{F}$ is defined as

$$\bigcap \{\tau : \tau \text{ is a topology on } X \text{ and every element of } \mathcal{F} \text{ is } \tau - \text{continuous}\}$$

Theorem 4.5

The initial topology of $\mathcal{F}$ is generated by the subbasis

$$\mathcal{F} := \bigcup_{\alpha \in \Lambda} \{f_\alpha^{-1}(V) : V \text{ is an open subset of } Y_\alpha\}$$

Example

If X is a topological space and $A \subseteq X$, $i : A \to X$ given by $a \mapsto a$ is continuous.

Proof

Let $\tau_{\mathcal{S}}$ be the topology generated by $\mathcal{S}$ and τ be the initial topology of $\mathcal{F}$.

- Show that $\tau \subseteq \tau_{\mathcal{S}}$. Because every element of $\mathcal{F}$ is $\tau_{\mathcal{S}}$-continuous, $\tau \subseteq \tau_{\mathcal{S}}$.

- $\tau_{\mathcal{S}} \subseteq \tau$. If $\tau_{\mathcal{S}} \subseteq \mu$ for every topology μ for which every f_α is continuous, then $\tau_{\mathcal{S}} \subseteq \tau$.

  1. First note that $\mathcal{S} \subseteq \mu$ that make the f_α's continuous.

  Since $\mathcal{S} \subseteq \mu$ then this implies $\tau_{\mathcal{S}} \subseteq \mu$.

The Product Topology

Definition 4.4: Product Topology

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces. The product topology on the cartesian product

$$X := \prod_{\alpha \in \Lambda} X_\alpha$$

is the initial topology of the projections $\{\pi_\alpha : X \to X_\alpha : \alpha \in \Lambda\}$.

This can also be written as:

$$\prod_{\alpha \in \Lambda} A_\alpha = \{f : \Lambda \to \bigcup_{\alpha \in \Lambda} A_\alpha : f(\alpha) \in A_\alpha\}$$

Observation

A basic open set in the product topology is of the form

$$U = \prod_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \prod_{\alpha_n}^{-1}(U_{\alpha_n})$$

This means that the open sets in the product topology are the finite intersections of the basic open sets. In the rest of the spaces, the product can go anywhere. It is only restricted for the finite intersections.

Definition 4.5: Box Topology

The box topology on X is the topology generated by the basis:

$$\mathcal{B} := \left\{ \prod_{\alpha \in \Lambda} U_\alpha : U_\alpha \text{ is an open subset of } X_\alpha \right\}$$

There is no restriction.

Theorem 4.6

If for each $\alpha \in \Lambda$, $\mathcal{B}_\alpha$ is a basis for the topology of X_α, and

$$X := \prod_{\alpha \in \Lambda} X_\alpha$$

then:

1. The box topology on X has a basis all sets of the form

$$\prod_{\alpha \in \Lambda} U_\alpha$$

   where each $U_\alpha \in \mathcal{B}_\alpha$.

2. The product topology on X has a basis all sets of the form

$$\prod_{\alpha \in \Lambda} U_\alpha$$

   where each $U_\alpha \in \mathcal{B}_\alpha$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$.

Theorem 4.7

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces, and let $A_\alpha \subseteq X_\alpha$ for each $\alpha \in \Lambda$ if

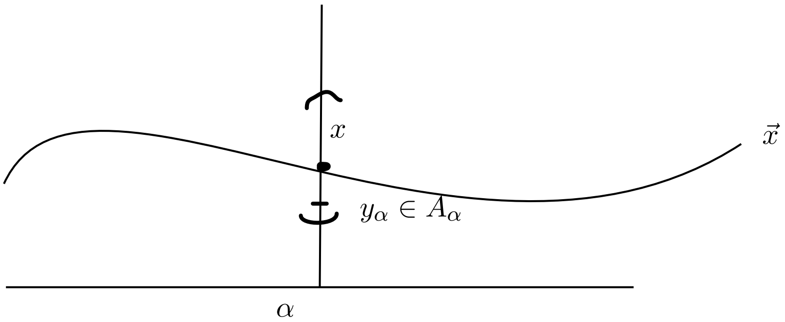
$$X := \prod_{\alpha \in \Lambda} X_\alpha$$

is given either the product or box topology, then

$$\prod_{\alpha \in \Lambda} \overline{A_\alpha} = \overline{\prod_{\alpha \in \Lambda} A_\alpha}$$

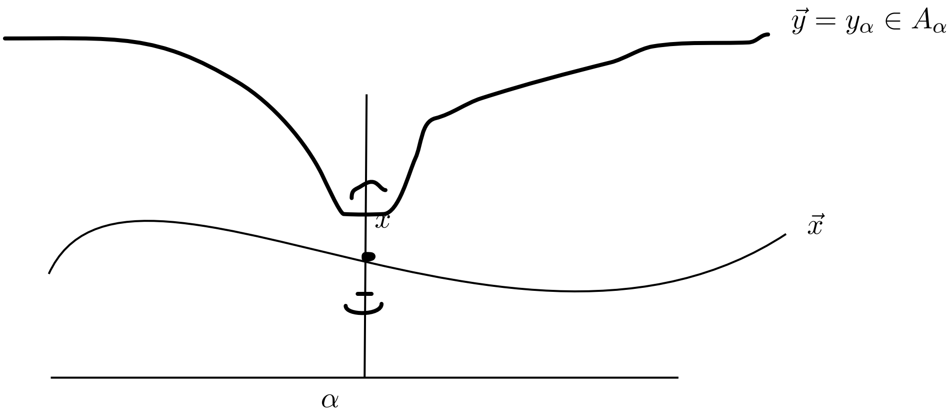
Proof

$\subseteq$ Let $\vec{x} = (x_\alpha : \alpha \in \Lambda) \in \prod_{\alpha \in \Lambda} \overline{A_\alpha}$. Let U be an open neighborhood around $\vec{x}$



So $(y_\alpha : \alpha \in \Lambda) \in \prod_{\alpha\in\Lambda} A_\alpha$. and an element of U so $\vec{x} \in \overline{\prod_{\alpha\in\Lambda} A_\alpha}$.

$\supseteq$ Let $\vec{x} = (x_\alpha : \alpha \in \Lambda) \in \overline{\prod_{\alpha\in\Lambda} A_\alpha}$. For each $\alpha \in \Lambda$, $x_\alpha \in \overline{A_\alpha}$.



So $x_\alpha \in \overline{A_\alpha}$ and therefore $\vec{x} \in \prod_{\alpha\in\Lambda} \overline{A_\alpha}$.

Theorem 4.8

Let $f : Z \to X \times Y$ be given by

$$f(z) = \langle f_1(z), f_2(z) \rangle$$

for some functions $f_1 : Z \to X$ and $f_2 : Z \to Y$. Then f is continuous if and only if both f_1 and f_2 are continuous.

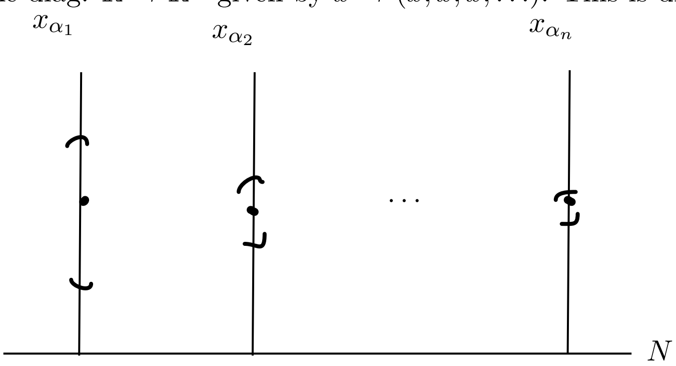
Take $f : Z \to \prod_{\alpha\in\Lambda} X_\alpha$ given by

$$f(z) = \langle f_\alpha(z) : \alpha \in \Lambda \rangle$$
$$\implies f_\alpha = \pi_\alpha \circ f$$

Exercise: Yes for the product, replicate the proof from wednesday for the product topology.

Example

Take diag: $\mathbb{R} \to \mathbb{R}^w$ given by $x \mapsto (x, x, x, \ldots)$. This is discontinuous in the box topology.



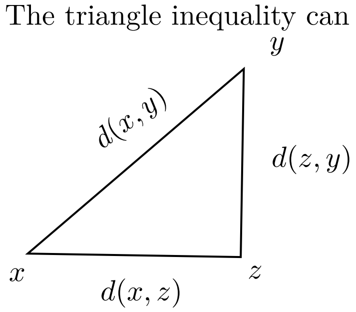
5 Metric Topology

Definition 5.1: Metric Topology

A function $d : X \times X \to \mathbb{R}$ is a metric if and only if

1. $d(x, y) \geq 0$ for all $x, y \in X$. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The triangle inequality can be interpreted as follows:



Definition 5.2

If (X, d) is a metric space then the metric topology on X is the topology generated by

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

Where $x \in X$ and $\varepsilon > 0$.

Note that $B(x, \varepsilon)$ is the open ball of radius ε centered at x.

Definition 5.3: Metrizable

(X, τ) is metrizable if τ is a metric topology.

Examples

1. $\mathbb{R}$ with the usual topology is metrizable. Consider $d(x, y) = |x - y|$, then $B(x, \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} = (x - \varepsilon, x + \varepsilon)$.

2. A set X with the discrete topology. Consider

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Note that it is always positive and $d(x, y) = d(y, x)$. Also, $d(x, z) \leq d(x, y) + d(y, z)$ as if $x \neq z$ then atleast $d(x, y) = 1$ or $d(y, z) = 1$. Now take the open ball

$$B(x, 1/2) = \{y \in X : d(x, y) < 1/2\} = \{x\}$$

Since singletons are open, the discrete topology is metrizable.

3. A set X with the indiscrete topology. This is not metrizable.

Proposition 5.1: Subspaces of metrizable spaces are metrizable

If X is metrizable and y is a subspace of X. If $d : X \times X \to \mathbb{R}$ is a metric then $d|_{y\times y} : y \times y \to \mathbb{R}$ defined as $d|_{y\times y}(y_1, y_2) = d(y_1, y_2)$ is a metric on y.

Lemma 5.1

Metrizable spaces are Hausdorff.

Proof

Let $d : X \times X \to \mathbb{R}$ be a metric inducing the topology of X. Let $x, y \in X$ such that $x \neq y$. Let $\varepsilon = \frac{d(x,y)}{2} > 0$. If $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$ then

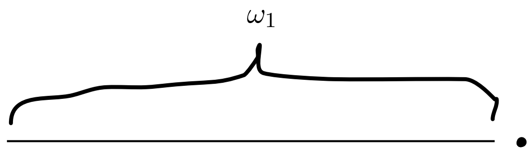
$$d(x, y) \leq d(x, z) + d(z, y) < \varepsilon + \varepsilon = d(x, y)$$

But this is a contradiction as it says $d(x, y) < d(x, y)$, which is not true. Thus, $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$.

Order Topologies

1. $\mathbb{R}$ is metrizable.

2. $\alpha\omega_1$



This space is not metrizable.

Definition 5.4: Sequential

A topological space X is sequential if for every subset $A \subseteq X$, and $x \in \overline{A}$, there exists a sequence $(x_n : n \in \mathbb{N}) \subseteq A$ such that $x_n \to x$.

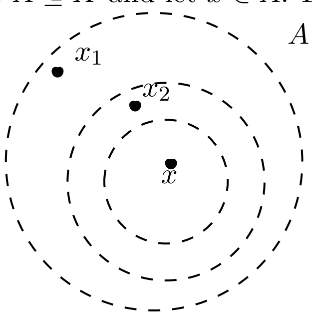
Note: If there is a sequence $(x_n : n \in \mathbb{N}) \subseteq A$ such that $x_n \to x$ then $x \in \overline{A}$.

Proposition 5.2

Metrizable spaces are sequential. Meaning metrizable $\implies$ sequential.

Proof

Let $A \subseteq X$ and let $x \in \overline{A}$. For each $n \in \mathbb{N}$, let $x_n \in A \cap B(x, 1/n)$.



Products

Let (X, d_X) and (Y, d_Y) be metric spaces. If we consider $X \times Y$ with the product topology, is it metrizable? We will prove that it is metrizable.
Consider

$$P : (X \times Y) \times (X \times Y) \to \mathbb{R}$$

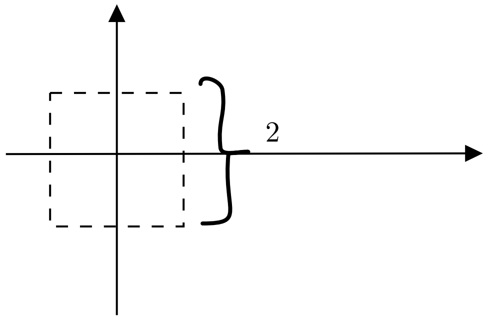
defined as

$$P(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

Example

In $\mathbb{R}^2$, let's consider $B_P(\vec{0}, 1)$. This means

$$\begin{aligned} B_P(\vec{0}, 1) &= \{\vec{x} \in \mathbb{R}^2 : P(\vec{0}, \vec{x}) < 1\} \\ &= \{\vec{x} \in \mathbb{R}^2 : \max\{d_X(\vec{0}, x_1), d_Y(\vec{0}, x_2)\} < 1\} \\ &= \{\vec{x} \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} < 1\} \end{aligned}$$



Proposition 5.3

If (X, d_X) and (Y, d_Y) are metric spaces then the product topology on $X \times Y$ is metrizable by the metric P.

Note that this is for finite products. Infinite products are not necessarily metrizable, they can be.

Proof

1. First see that P is a metric.
    (a) Since d_X and d_Y are metrics, $P(\vec{x}, \vec{y}) \geq 0$. $P(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$.
    (b) Note that

$$\begin{aligned} P(\vec{x}, \vec{y}) &= \max\{d_X(x_1, y_1), d_Y(x_2, y_2)\} \\ &= \max\{d_X(y_1, x_1), d_Y(y_2, x_2)\} \\ &= P(\vec{y}, \vec{x}) \end{aligned}$$

(c) Take the following

$$\begin{aligned}
P(\vec{x},\vec{y}) &= \max\{d_X(x_1,y_1), d_Y(x_2,y_2)\} \\
&\le \max\{d_X(x_1,z_1)+d_X(z_1,y_1), d_Y(x_2,z_2)+d_Y(z_2,y_2)\} \\
&\le \max\{d_X(x_1,z_1), d_Y(x_2,z_2)\} + \max\{d_X(z_1,y_1), d_Y(z_2,y_2)\} \\
&= P(\vec{x},\vec{z}) + P(\vec{z},\vec{y})
\end{aligned}$$

2. Now prove that P induces the product topology.

$$\begin{aligned}
y \in B_P(\vec{x},\varepsilon) &\iff P(\vec{x},\vec{y}) < \varepsilon \\
&\iff \max\{d_X(x_1,y_1), d_Y(x_2,y_2)\} < \varepsilon \\
&\iff d_X(x_1,y_1) < \varepsilon \text{ and } d_Y(x_2,y_2) < \varepsilon \\
&\iff y \in B_{d_X}(x_1,\varepsilon) \times B_{d_Y}(x_2,\varepsilon)
\end{aligned}$$

Infinite product topology

Assume that all the X_α's are metric with metric d_α ($\alpha \in \Lambda$). We can do something of the following:

$$\begin{aligned}
\vec{x} &= (x_\alpha : \alpha \in \Lambda) \\
\vec{y} &= (y_\alpha : \alpha \in \Lambda) \\
P(\vec{x},\vec{y}) &= \max\{d_\alpha(x_\alpha,y_\alpha) : \alpha \in \Lambda\}
\end{aligned}$$

But note that the maximum might not exist as the sequence d_α keeps going. So the supreminum may exist if the set is bounded. Thus, we can define

$$P(\vec{x},\vec{y}) = \sup\{d_\alpha(x_\alpha,y_\alpha) : \alpha \in \Lambda\}$$

Using the lemma below, we can take the supremum of $\overline{d}$ as such

$$P(\vec{x},\vec{y}) = \sup\{\overline{d_\alpha}(x_\alpha,y_\alpha) : \alpha \in \Lambda\}$$

which solves the problem of the supremum not existing.

Lemma 5.2

If (X, d) is a metric space, then $\overline{d}(x, y) = \min\{d(x, y), 1\}$ is a metric on X inducing the same topology as d.

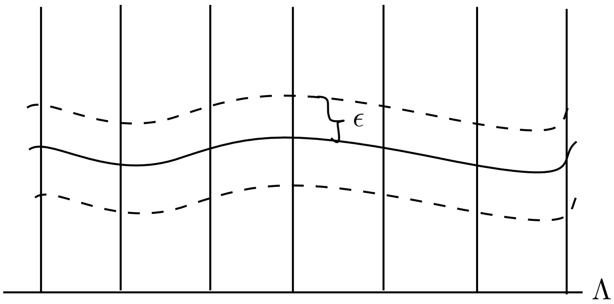
Exercise: Prove this

Definition 5.5: Uniform topology

The topology on $\prod_{\alpha\in\Lambda} X_\alpha$ induced by the metric P is called the uniform topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

Uniform topology only works if the X_α's are metrizable.

Uniform topology



Note that it is infinite in the Λ direction.

Proposition 5.4: I

$\{X_\alpha : \alpha \in \Lambda\}$ is a collection of metrizable spaces, then

$$\tau_{\text{prod}} \subsetneq \tau_{\text{uniform}} \subsetneq \tau_{\text{box}}$$

Note that τ_{box} is the box topology, $\prod_{\alpha\in\Lambda} X_\alpha$.

Definition 5.6

Let $\{(X_\alpha, d_\alpha) : \alpha \in \Lambda\}$ be a collection of metric spaces. The uniform topology on the cartesian product

$$X := \prod_{\alpha \in \Lambda} X_\alpha$$

is the metric topology induced by the metric topology induced by the metric $p : X \times X \to \mathbb{R}$ given by

$$p(\overline{x}, \overline{y}) := \sup_{\alpha \in \Lambda} \overline{d_\alpha}(x_\alpha, y_\alpha)$$

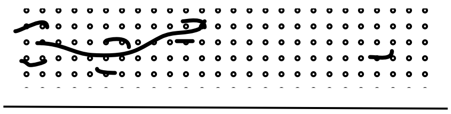
where $\overline{x} = (x_\alpha)_{\alpha \in \Lambda}$ and $\overline{y} = (y_\alpha)_{\alpha \in \Lambda}$.

Warning: The box topology might not be metrizable.

Example

In $\{0, 1\}$, we have the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$



Λ

If $(x_\alpha)_{\alpha \in \Lambda} \in \{0, 1\}^\Lambda$, then the open set $\prod_{\alpha \in \Lambda} \{x_\alpha\}$ only includes $\{x_\alpha : \alpha \in \Lambda\}$

Theorem 5.1

If X is not discrete, then $X^w_\square$ is not metrizable

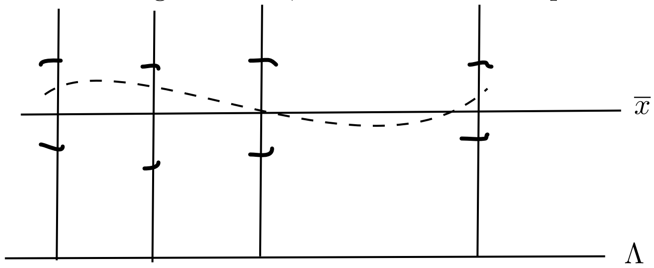
Proof

If X is not discrete, then $X^w_\square$ is not sequential.
There exist some $x \in X$ such that $\{x\}$ is not open. This means that $\{x\} \in \overline{X \setminus \{x\}}$. Let $\overline{x} \in X^\omega$ be a sequence that is

$$\overline{x} = (x, x, x, \dots)$$

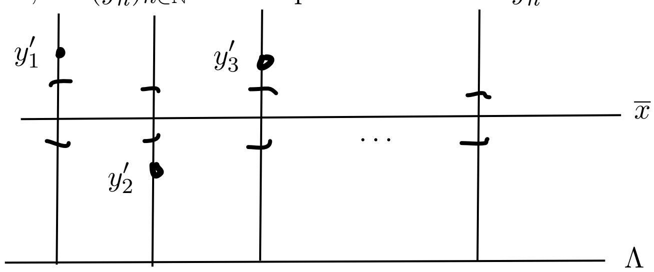
and let $A \subseteq X^\omega$ be given by $\overline{y} \in A$ if and only if no coordinate of $\overline{y}$ equals x.
Claim: $\overline{x} \in \overline{A}$.

If you take the line A with vertical lines on it. Then take $\overline{x}$ to be a horizontal line and so if we take a neighborhood, there is another sequence in that neighborhood that is not $\overline{x}$.



Claim: No sequence on A converges to $\overline{x}$.

If not, let $(\overline{y}_n)_{n\in\mathbb{N}}$ be a sequence such that $\overline{y}_n \to \overline{x}$.



So then there is a sequence which is not in the neighborhood of $\overline{x}$ which is a contradiction.

Warning

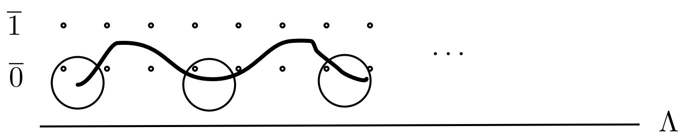
The product topology might not be metrizable.

> **Theorem 5.2**
>
> $\{0,1\}^\Lambda$ is not metrixable if Λ is uncountable.

Proof

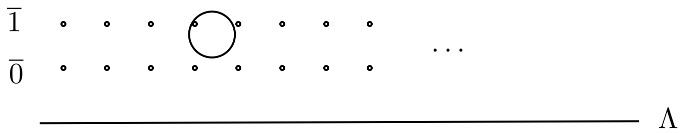
Let $A = \{\overline{x} \in \{0,1\}^\Lambda : x_\alpha = 0$ for only finitely many $\alpha \in \Lambda\}$, and let $\overline{0} = (0,0,0,\ldots)$.

Claim 1: $\overline{0} \in \overline{A}$.



Set some finitely many coordinates to be 0 and the rest to be 1.

Claim 2: No sequence in A converges to $\overline{0}$. If not, let $\overline{x_n} \to \overline{0}$. For each $n \in \mathbb{N}$, let $\Lambda_n = \{\alpha \in \Lambda : \overline{x_n}(\alpha) = 0\}$. Note that $\bigcup_{n\in\mathbb{N}} \Lambda_n$ is countable. Thus let $\alpha \in \Lambda \backslash \bigcup_{n\in\mathbb{N}} \Lambda_n$.



> **Theorem 5.3**
>
> The product of countably many metrizable spaces is metrizable.

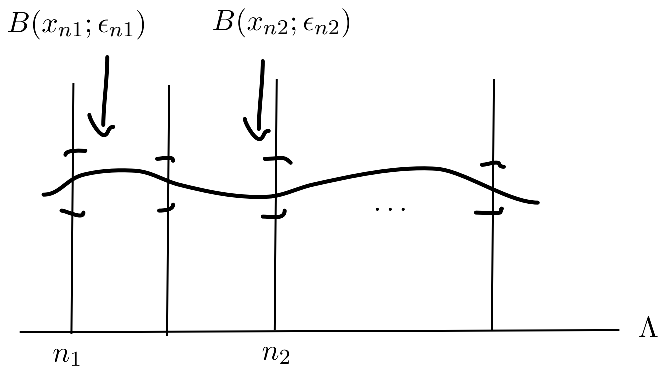
Proof

Let (X_n, d_n) be metric spaces. Let $D : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n \to \mathbb{R}$ be given by

$$D(\overline{x}, \overline{y}) = \sup_{n\in\mathbb{N}} \frac{\overline{d_n}(x_n, y_n)}{n}$$

Prove that this is metric is left as an exercise.

1. Let $x \in \prod_{n\in\mathbb{N}} X_n$ and let U be a basic open neighborhood of $\overline{x}$ in the product topology.



6 Path connectedness

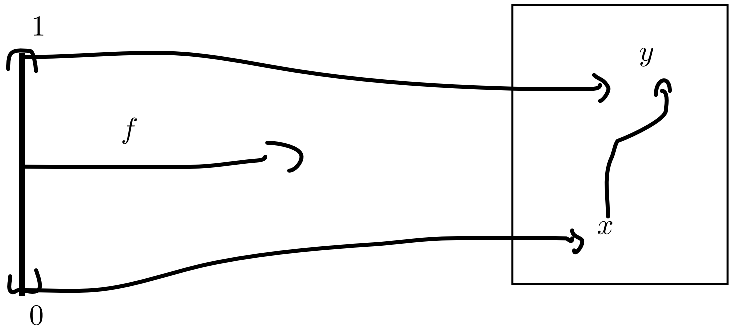
Definition 6.1: Partition

A seperation of a topological space X is a partition $X = A \cup B$ into two open, non-empty sets. The space X is connected if and only if, it has no seperation if and only if, it has no non-trivial clopen subsets.

Note that clopen means both closed and open. Discrete topology is not connected.

Definition 6.2: Path-connected

A path from x to y, both points in the topological space X, is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. The space X is path-connected if a path exists joining any two points of X.

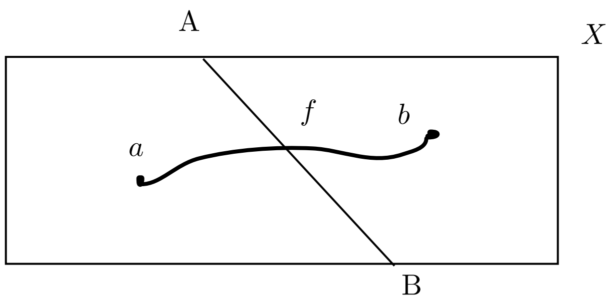


Proposition 6.1

Path-connected spaces are connected.

Proof

Let X be a path-connected, not connected.



This means that $f^{-1}(A)$ and $f^{-1}(B)$ are open and closed and seperates $[0, 1]$.

Definition 6.3: Totally disconnected

X is totally disconnected if the only connected subspaces of X are singletons.

Take $\mathbb{R}$. In $\mathbb{R}$ take the following subset $X = [(-\infty, x) \cap X] \cup [(x, \infty) \cap X]$ Since both are open and closed, X is totally disconnected.

Definition 6.4: Convex

A subset $X \subseteq \mathbb{R}$ is convex if for all $x, y \in X$, $[x, y] \subseteq X$.

Theorem 6.1

A subset of $\mathbb{R}$ is connected if and only if it is convex.

Proof $\implies$

If X is not convex, let $x, y \in X$ and $z \in \mathbb{R}$ such that $x < z < y$ and $z \notin X$. Then

$$X = [(-\infty, z) \cap X] \cup [(z, \infty) \cap X]$$

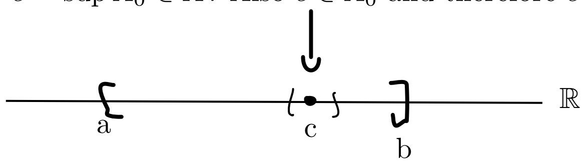
The two sets are open and non-empty and X is not connected. Thus a contradiction.

$\impliedby$

Let $X \subseteq \mathbb{R}$ be convex and assume that $X = A \cup B$ is a seperation. Fix $a \in A$ and $b \in B$. Assume that $a < b$. Since X is convex, $[a, b] \subseteq X$. Then $[a, b]$ can be expressed as

$$[a, b] = \underbrace{(A \cap [a, b])}_{A_0} \cup \underbrace{(B \cap [a, b])}_{B_0}$$

Let $c = \sup A_0 \in X$. Also $c \in A_0$ and therefore $c \neq b$.



Now note that since A_0 is open and $c \in A_0$, there exists $\epsilon > 0$ such that $c + \epsilon \in A_0$. Meaning that since it is open, a small interval can be found around c. This means that it cannot be the supremum of A_0, and thus a contradiction. Thus X is connected.

Proposition 6.2

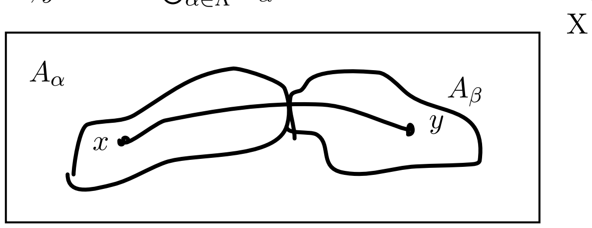
The union of an arbitrary collection of (path)-connected subspaces of the topological space X with a common point is (path-)connected.

Proof

Let $\{A_\alpha : \alpha \in \Lambda\}$ be subsets of X with a common point p

(path-connected)

Let $x, y \in A := \bigcup_{\alpha \in \Lambda} A_\alpha$. Then this means $x \in A_\alpha$ and $y \in A_\beta$



Let $f : [0, 1] \to A_\alpha$ be a path from x to p and $y : [1, 2] \to A_\beta$ be a path from p to y. Then $h : [0, 2] \to A$ given by $h(t)$ is

$$h(t) = \begin{cases} f(t) & \text{if } t \in [0, 1] \\ g(t) & \text{if } t \in [1, 2] \end{cases}$$

(Connected)

Let $A = C \cup D$ be a seperation. Assume that $p \in C$. For $\alpha \in \Lambda$, A_α intersects C. Since A_α is connected, $A_\alpha \subseteq C$.

Proposition 6.3

If A is connected subspace of X and

$$A \subseteq B \subseteq \overline{A}$$

then B is connected as well.

Proof

Express $B = C \cup D$ of two disjoint open sets. Since A is connected, then $A \subseteq C$. This means that $B \subseteq \overline{A} \subseteq \overline{C}$. If a set is included in the closed, then closure is included in the closed.

Theorem 6.2

Continuous images of (path-)connected spaces are (path-)connected.

Let $f : X \to Y$ with X connected, f continuous. We want to prove that $f[X]$ is also connected.

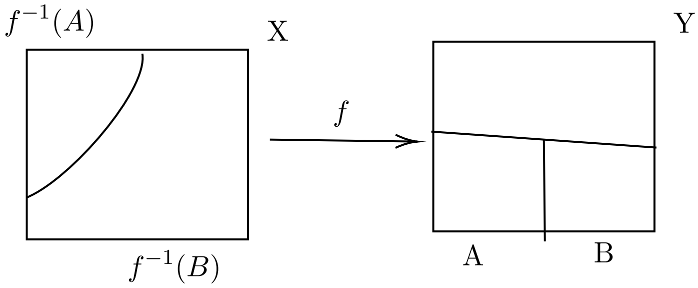


Image of a connected space is connected.

Collary 6.1: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < y < f(b)$, then there exists some $x \in [a, b]$ such that $f(x) = y$.

Proof

1. $[a, b]$ is connected

2. $f[a, b]$ is connected

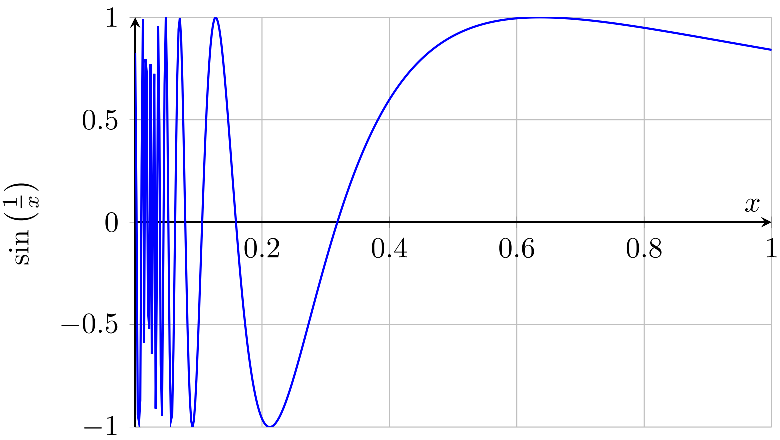
3. $f[a,b]$ is convex
4. $f(a), f(b) \in f[a,b]$
5. If $f(a) < y < f(b)$, then $y \in f[a,b]$

Example

Consider the function

$$\sin\left(\frac{1}{x}\right) : (0,1] \to \mathbb{R}$$

Let S be the graph of the function.



So $\overline{S}$ is connected but not path-connected. Now let's see why it is not path connected. Let $x \in S$ and consider $\langle 0,0\rangle$. If $f : [0,1] \to \overline{S}$ is a path from x to $\langle 0,0\rangle$, the set $T = \{t \in [0,1] : f(t) \in \{0\} \times [-1,1]\}$ is closed. This would be because if we take the set $\{0\} \times [-1,1]$, it is a closed subset of $\overline{S}$. Also see that $\inf T \in T$.
We can assume that $f(0) \in \{0\} \times [-1,1]$ and if $t > 0$

$$f(t) = \left(x(t), \sin\left(\frac{1}{x(t)}\right)\right)$$

Sub-proof

Now we will prove that the function defined above is continuous. For $n \in \mathbb{N}$, let $0 < u < x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u}\right) = (-1)^n$. If f is continuous by the Intermediate Value Theorem, let $0 < t_n < \frac{1}{n}$ such that

$$x(t_n) = u$$

- $\overline{S}$ is connected: Take $\ln(0,1]$ and consider the graph S of $\sin(1/x)$. Compute $\overline{S}$
- $\overline{S}$ is not path-connected: Let $f:[0,1]\to\overline{S}$ be a path from a point in the vertical segment (V) to a point in S.

  1. $T=\{t\in[0,1]:f(t)\in V\}$ is closed in $[0,1]$. Now take $c=\sup T\in T$. Now take $f:[c,1]\to\overline{S}$. But note that c is on the vertical segment so take $f:[0,1]\to\overline{S}$. now this means that
     - $f(0)\in V$
     - If $t>0$ then $f(t)\in S$ where $f(t)=\langle x(t),\sin\left(\frac{1}{x(t)}\right)\rangle$
- By contradiction, For $n\in\mathbb{N}$, consider $x\left(\frac{1}{n}\right)$. By the oscilating property of sin there exists $0<u<x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u}\right)=(-1)^n$. By the intermediate value theorem, there exists $0<t<\frac{1}{n}$ such that $x(t_n)=u$. Note that the sequence $t_n\to 0$. Nevertheless, the sequence $f(t_n)=\langle x(t_n),(-1)^n\rangle$ diverges. Thus f is not path connected.

Theorem 6.3

A finite product of (path-)connected spaces is (path-)connected.

Proof

Path-connected

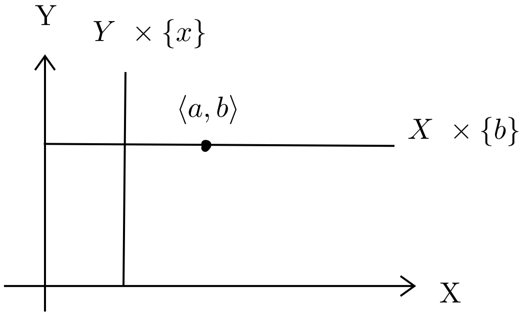
Assume that $(X_\alpha,\alpha\in\Lambda)$ are path-connected spaces. We want to prove that

$$X:=\prod_{\alpha\in\Lambda}X_\alpha$$

is path-connected. Let $\vec{x}=(x_\alpha:\alpha\in\Lambda)$ and $\vec{y}=(y_\alpha:\alpha\in\Lambda)$ be elements of X.
For $\alpha\in\Lambda$, $x_\alpha,y_\alpha\in X_\alpha$, let $f_\alpha:[0,1]\to X_\alpha$ be a path from x_α to y_α. Then $f:[0,1]\to X$ given by $f(t)=\langle f_\alpha(t):\alpha\in\Lambda\rangle$

Connected

1. Let X and Y be connected. If we consider $X\times\{b\}$, it is connected. If $x\in X$ then $Y\times\{x\}$ is connected.



The set $T_x := (X \times \{b\}) \cup (Y \times \{x\})$ is connected as the two spaces have a point in common. Also

$$X \times Y = \bigcup_{x \in X} T_x$$

is connected since all the T_x's have the point $\langle a, b\rangle$ in common.

Theorem 6.4

Any product of (path-)connected spaces is (path-)connected.

Proof for arbitrary product

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of connected spaces. We will prove that

$$X := \prod_{\alpha \in \Lambda} X_\alpha$$

is connected.
Fix $\langle a_\alpha : \alpha \in \Lambda\rangle \in X$. If $F \subseteq \Lambda$ is finite then

$$X_F := \{\vec{x} \in X : \text{ if } x_\alpha \neq a_\alpha \text{ then } \alpha \in F\}$$

An element of X_F is of the form $\langle x_\alpha : \alpha \in \Lambda\rangle$. The places where x_α is in F, we have the freedom to choose anything but outside of F, we have to choose a_α.

1. For each F, X_F is connected.

   $$X_F \cong \prod_{\alpha \in F} X_\alpha$$

   is connected. (left as an exercise)

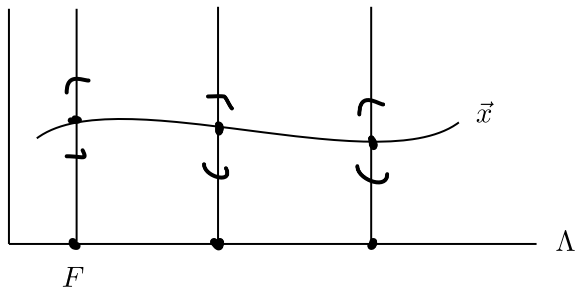
2. Take

$$Z = \bigcup_{F \subseteq \Lambda \text{ finite}} X_F$$

Z is connected as a is in all the X_F's.

3. Take

$$\overline{Z} = X$$



The tuple

$$\vec{z} = \begin{cases} x_\alpha & \text{if } \alpha \in F \\ a_\alpha & \text{if } \alpha \notin F \end{cases} \in X_F$$

When $\overline{Z} = X$ then we say Z is dense in X.

Example

Take $\mathbb{R}^\omega_\square$. We will show that it is disconnected. Introduce a new space

$$l^\infty = \left\{ \vec{x} \in \mathbb{R}^\omega : \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}$$

this is a bounded sequence. To prove that $\mathbb{R}^\omega_\square$ is disconnected, we will show that l^∞ is a open and closed. If you take a sequence in $\mathbb{R}^\omega_\square$, and take open neighbourhoods of radius 1, then if a sequence is inside bounded, then any other sequence is also bounded. But if the sequence is not bounded, then there exists a sequence that is not bounded.

Definition 6.5

On the topological space X, consider the equivalence relation $x \sim_c y$ if there exists a connected subspace $A \subseteq X$ such that $x, y \in A$ and $x \sim_{pc} y$ if there exists a path from x to y.

- Equivalence classes modulo $\sim_c$ are called **components**,
- Equivalence classes modulo $\sim_{pc}$ are called **path components**.

Proposition 6.4

1. (Path) components form a partition of X.
2. (Path) components are (path-)connected.
3. Path components are included in components.
4. (Path-)connected subspaces of X intersect at most one (path) component.
5. Components are closed.

- **3** If $x \sim_{pc} y$ then $x \sim_c y$. If $x \sim_{pc} y$ there is a continuous function

$$f : [0,1] \to X \quad \text{such that} \quad f(0) = x \quad \text{and} \quad f(1) = y$$

  so $x, y \in f([0,1])$ which is connected and implies $x \sim_c y$.

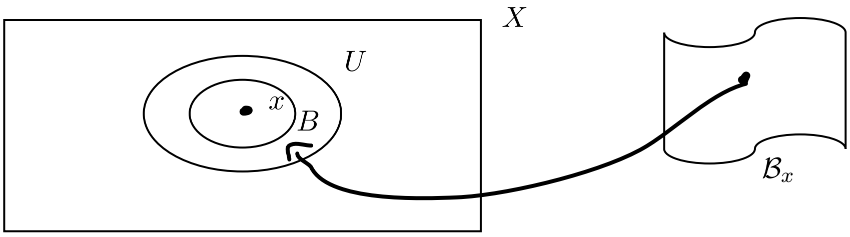
- **4** Let $A \subseteq X$ connected and $x \in A$ then $A \cap [x]_{\sim_c} \neq \emptyset$. If $y \in A$ then $x \sim_c y$, then $y \in [x]_{\sim_c}$ and $A \subseteq [x]_{\sim_c}$.

- **5** If $[x]_{\sim_c}$ is a component, then
  - $\overline{[x]_{\sim_c}}$ is connected. Not true for path connected.
  - Since $\overline{[x]_{\sim_c}} \cap [x]_{\sim_c} \neq \emptyset$, then $\overline{[x]_{\sim_c}} \subseteq [x]_{\sim_c}$.

Definition 6.6

A local basis around a point $x \in X$ is a collection $\mathcal{B}_x$ of open neighbourhoods of x such that for every open neighbouhood U of x there exists $B \in \mathcal{B}_x$ such that $B \subseteq U$. A topological space is locally (path-)connected if every point has a local basis consisting of (path-)connected sets.

Local basis



Proposition 6.5

If X is locally path-connected, if and only if for every open subset $U \subseteq X$, all (path) componenets of U are open in X.

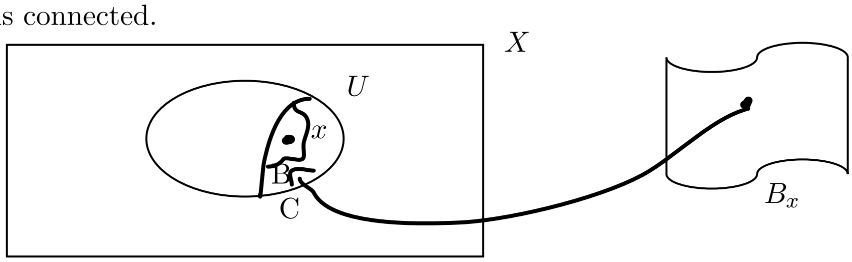
Collary 6.2: I

X is locally path-connected, then components = path components

Proof for proposition

$\Longrightarrow$

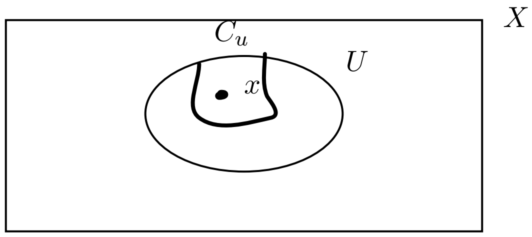
Let X be locally connected and $U \subseteq X$ be open. Now let $x \in U$ and let C be the component of x in U. We will prove that x is an interior point of C.
Since X is locally connected, there exists some open set $B \subseteq X$ such that $x \in B \subseteq U$. And B is connected.



Since $B \subseteq C$ then C is open.

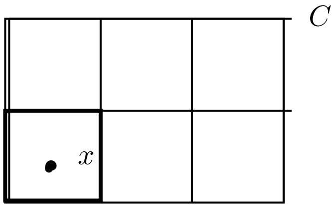
$\Longleftarrow$

Let $x \in X$. For each open neighbouhood U of x, let C_u be the component of x in U.



Claim: $\{C_u\colon U$ is an open neighbouhood of $x\}$ is a local basis around x consisting of connected sets.

Proof for corollary



This gives us

$$C = [x]_{\sim_{pc}} \cup Q$$

where Q is the union of path-components. Then note that the component that x is in is open and closed. Thus C is a union of path-components and C is open and closed. Thus C is a path-component.

7 Filters and Ultrafilters

Definition 7.1

A collection $\mathcal{F} \subseteq \mathbb{P}(\mathbb{N})$ is a filter if and only if

- $\mathbb{N} \in \mathcal{F}$, and $\emptyset \notin \mathcal{F}$,
- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,
- If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

If in addition we have that

- given $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A \in \mathcal{F}$

then $\mathcal{F}$ is an ultrafilter.

Examples

1. $\{\mathbb{N}\}$ form a filter.
2. Take $A \subseteq \mathbb{N}$, and let
$$\mathcal{F}_A = \{B \subseteq \mathbb{N} : A \subseteq B\}$$
3. Take
$$\mathcal{F}_r = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$$
4. $U_n = \{A \subseteq \mathbb{N} : n \in A\}$ where $n \in \mathbb{N}$ is fixed. Also note that U_n is an ultrafilter. The $U_n's$ are called parincipal ultrafilters.

Theorem 7.1

Every filter is included in an ultrafilter.

Let $\mathcal{F}$ be a filter. Consider

$$\mathbb{P}_{\mathcal{F}} = \{G \subseteq \mathbb{P}(\mathbb{N}) : G \text{ is a filter and } \mathcal{F} \subseteq G\}$$

We will order $\mathbb{P}_{\mathcal{F}}$ by inclusion.

1. $\mathbb{P}_{\mathcal{F}}$ is non-empty because $\mathcal{F} \in \mathbb{P}_{\mathcal{F}}$.

2. Let $\mathcal{C} \subseteq \mathbb{P}_{\mathcal{F}}$ be a chain and let $H = \bigcup \mathcal{C} = \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$.
   Claim: H is a filter.

   (a) $\mathbb{N} \in H$ because $\mathbb{N} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.

   (b) Let $A, B \in H$, then this means that there are $\mathcal{F}_A, \mathcal{F}_B \in \mathcal{C}$ such that $A \in \mathcal{F}_A$ and $B \in \mathcal{F}_B$. Since $\mathcal{C}$ is a chain, then $A, B \in \mathcal{F}_B$ and $A \cap B \in \mathcal{F}_B \subseteq H$.

3. Take $A \in H$ and $A \subseteq B$. Since $A \in H$ then this means $A \in \mathcal{F}_A$ for some $\mathcal{F}_A \in \mathcal{C}$ then $B \in \mathcal{F}_A \subseteq H$. Then by Zorn's lemma, there exists some maximal $U \in \mathbb{P}_{\mathcal{F}}$.
   Claim: U has to be an ultrafilter.
   Otherwise, let $A \subseteq \mathbb{N}$ such that $A \notin U$ and $X \setminus A \notin U$. If these are true then $U \cup \{A\}$ is included in a filter. For it to be included we do not need axiom of choice.
   The main thing is that $X \setminus A \notin U$ means $X \setminus A$ doesn't include any element of U.

Example

Take the following set

$$\beta\mathbb{N} = \{U \subseteq \mathbb{P}(\mathbb{N}) : U \text{ is an ultrafilter}\}$$

and its topology is generated by sets of the form:

- Fix $A \subseteq \mathbb{N}$ and consider
$$[A] = \{U \in \beta\mathbb{N} : A \in U\}$$

Let's see that $[A]$'s for $A \subseteq \mathbb{N}$ is a basis for a topology on $\beta\mathbb{N}$.

1. $[\mathbb{N}] = \beta\mathbb{N}$, $[\emptyset] = \emptyset$.

2. Let $[A]$ and $[B]$ be fixed and let $u \in [A] \cap [B]$. This means that $u \in [A \cap B] \subseteq [A] \cap [B]$.

Also note that $\beta\mathbb{N}$ is zero-dimensional. Means that it has a basis consisting of clopen sets.

$$\beta\mathbb{N} \setminus [A] = [\mathbb{N} \setminus A]$$

8 Compactness

Definition 8.1: Open Cover

Let X be a topological space then a collection $\mathcal{O}$ where

$$\mathcal{O} = \{U_\alpha\}_{\alpha \in \Lambda}$$

is an open cover if

$$X = \bigcup_{\alpha \in \Lambda} U_\alpha.$$

and each U_α is open in X. A subset of $\mathcal{O}$ that still covers X is called a subcover of X.

Definition 8.2: Compact

A topological space X is called compact if every open cover of X has a finite subcover.

Examples

- Clearly every finite set is compact
- $\mathbb{R}$ is not compact, since the open cover

$$\mathcal{O} = \{(-\infty, n) : n \in \mathbb{N}\}$$

  has no finite subcover.
- The set $K \cup \{0\}$ is compact. Let $\mathcal{O}$ be an open cover of $K \cup \{0\}$. Then there is some $V \in \mathcal{O}$ such that $0 \in V$. Since $V_n \to 0$, infinitely many terms of $\frac{1}{n}$ are in V, so that finitely many are not covered by V. For the finitely many terms not in V, we just take the finitely many sets from $\mathcal{O}$ that covers the remaining terms.

Definition 8.3

A collection $\mathcal{C} = \{C_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of the topological space X has the finite intersection proporty (FIP) if and only if, for every finite subset $F \subseteq \Lambda$, we have that

$$\bigcap_{\alpha\in F} C_\alpha \neq \emptyset.$$

Theorem 8.1

A topological space X is compact if and only if, every collection $\mathcal{C}$ of closed subsets of X with the FIP has a nonempty intersection.

Proof

Compactness: For every open cover $\mathcal{O} = \{U_\alpha\}_{\alpha\in\Lambda}$ there are $U_{\alpha_1}, \cdots, U_{\alpha_n} \in \mathcal{O}$ such that $X = \bigcup_{i\leq n} U_{\alpha_i}$ If we take complements.

Suppose X is compact, i.e., any collection of open subsets that cover X has a finite collection that also cover X. Further, suppose $\{F_i\}_{i\in I}$ is an arbitrary collection of closed subsets with the finite intersection property. We claim that $\bigcap_{i\in I} F_i$ is non-empty. Suppose otherwise, i.e., suppose $\bigcap_{i\in I} F_i = \emptyset$. Then,

$$X = \left(\bigcap_{i\in I} F_i\right)^c = \bigcup_{i\in I} F_i^c.$$

(Here, the complement of a set A in X is written as A^c.) Since each F_i is closed, the collection $\{F_i^c\}_{i\in I}$ is an open cover for X. By compactness, there is a finite subset $J \subset I$ such that $X = \bigcup_{i\in J} F_i^c$. But then

$$X = \left(\bigcap_{i\in J} F_i\right)^c,$$

so $\bigcap_{i\in J} F_i = \emptyset$, which contradicts the finite intersection property of $\{F_i\}_{i\in I}$.

The proof in the other direction is analogous. Suppose X has the finite intersection property. To prove that X is compact, let $\{F_i\}_{i\in I}$ be a collection of open sets in X that cover X. We claim that this collection contains a finite subcollection of sets that also cover X. The proof is by contradiction. Suppose that $X \neq \bigcup_{i\in J} F_i$ holds for all finite $J \subset I$. Let us first show that the collection of closed subsets $\{F_i^c\}_{i\in I}$ has the finite intersection property. If J is a finite subset of I, then

$$\bigcap_{i\in J} F_i^c = \left(\bigcup_{i\in J} F_i\right)^c \neq \emptyset,$$

where the last assertion follows since J was finite. Then, since X has the finite intersection property,

$$\emptyset \neq \bigcap_{i\in I} F_i^c = \left(\bigcup_{i\in I} F_i\right)^c .$$

This contradicts the assumption that $\{F_i\}_{i\in I}$ is a cover for X.

Proposition 8.1

The subspace Y of the topological space X is compact if and only if, every open cover of Y by open subsets of X has a finite subcover of Y.

Proof

Let $\mathcal{O} = \{V_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of X such that $Y \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$. Let $\hat{\mathcal{O}} = \{V_\alpha \cap Y : \alpha \in \Lambda\}$. is an open cover of Y. Since Y is compact, let $\{V_{\alpha_1} \cap Y, \cdots, V_{\alpha_n} \cap Y\}$ be a finite subcover of $\hat{\mathcal{O}}$. So $\{V_{\alpha_1}, \cdots, V_{\alpha_n}\} \subseteq \mathcal{O}$ is finite and covers Y.

Proposition 8.2

A closed subset of a compact space is compact

Proof

Let X be compact and let $\mathcal{C} \subseteq X$ be closed. Let $\mathcal{O}$ be an open cover of $\mathcal{C}$ by opens in X. Let

$$\mathcal{O}' = \mathcal{O} \cup \{X \setminus \mathcal{C}\}$$

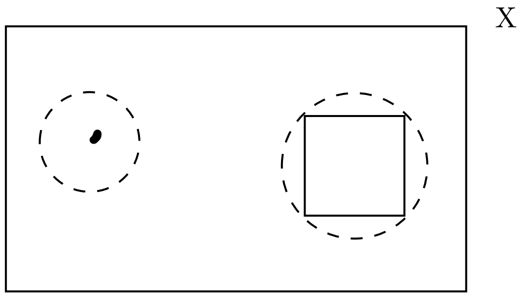
Since $\mathcal{O}$' covers X and X is compact, let $\{U_1, \cdots, U_n\} \in \mathcal{O}'$ such that $X = \bigcup_{i\leq n} U_i$. Let

$$\mathcal{Q} = \{U_i : U_i \neq X \setminus \mathcal{C}\}$$

Theorem 8.2

Let X be a compact Hausdorff space, let $Y \subseteq X$ be compact, and let $x \notin Y$. Then there are disjoint open subsets of U and V of X with

$$x \in U \quad \text{and} \quad Y \subseteq V.$$

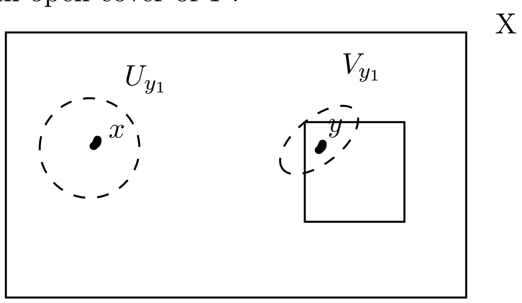


Proof

For each $y \in Y$, let U_x and V_y be disjoint, open and $x \in U_x$ and $y \in V_y$. The set

$$\mathcal{O} = \{V_y : y \in Y\}$$

is an open cover of Y.



Let $V_{y_1}, \cdots, V_{y_n}$ be a finite subcover of $\mathcal{O}$. Let

$$V = \bigcup_{i \leq n} V_{y_i} \quad \text{and} \quad U = \bigcap_{i \leq n} U_{y_i}.$$

Collary 8.1

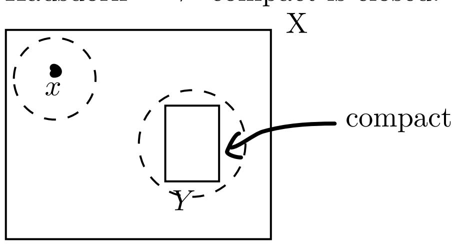
A compact subspace of a compact Hausdorff space is closed.

Last time

Proposition 8.3

A compact subspace of a Hausdorff space is closed

Hausdorff $\implies$ compact is closed.



Example

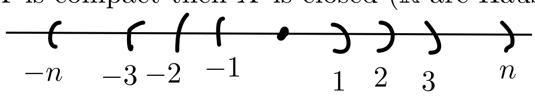
Find a counter example if X is not Hausdorff

Theorem 8.3: A

ubset X of $\mathbb{R}$ is compact if and only if X is closed and bounded with a usual metric.

$\implies$

If X is compact then X is closed ($\mathbb{R}$ are Hausdorff).



If X is compact then X is bounded

$\impliedby$

It is enough to check that the closed intervals $[a, b]$ are compact. Fix $[a, b]$ and let $\mathcal{O} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $[a, b]$. Let

$$A = \{x \in [a, b] : [a, x] \text{ has no finite subcover of } \mathcal{O}\}$$

1. A $\neq \emptyset$ ($a \in A$)
2. A is bounded above (by b)
3. By completeness, $c = \sup A$

(a) $c \in A$

(b) $c = b$

$\tilde{a}$

$c \in A$

- $\exists U_\alpha \in \mathcal{O}$ such that $c \in U_\alpha$
- $\exists \epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subseteq U_\alpha$
- $\exists x \in A$ such that $c - \epsilon < x < c$

Now note that

$$[a, c] = [a, x] \cup [x, c]$$

Where $[a, x]$ has finitely many elements of $\mathcal{O}$ cover and $[x, c]$ is cover by U_α.

$\tilde{b}$

$c = b$
If $c < b$, then note that $[a, x] = [a, c] \cup [c, x]$ and $[c, x]$ is covered by U_α. This is a contradiction to the fact that c is the least upper bound of A.

Definition 8.4: Isolated points

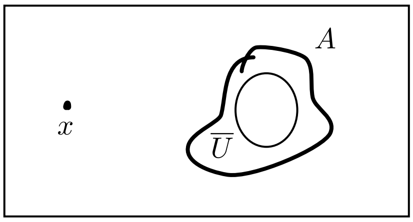
A point x in the topological space X is called **isolated** if and only if the singleton $\{x\}$ is open subset of X.

Theorem 8.4

A compact Hausdorff space with no isolated points is uncountable.

Proof

Observation: If $A \subseteq X$ and $x \notin X$, then there exists some open neighborhood U such that $x \notin \overline{U}$



If X is countable, let $X = \{x_1, x_2, \cdots, x_n\}$.

- Let $U_1 \subseteq X$ such that $x_1 \notin \overline{U}$
- If we already defined $U_1, \cdots, U_n$, let $U_{n+1} \subseteq U_n$ open such that $\overline{U}_{n+1} \subseteq \overline{U}_n$ and $x_{n+1} \notin \overline{U}_{n+1}$
- Since the family $\{\overline{U}_n : n \in \mathbb{N}\}$ of closed subsets of X has the finite intersection property, then

$$\bigcap_{n\in\mathbb{N}} \overline{U}_n \neq \emptyset$$

Theorem 8.5

The continuous image of a compact space is compact.

Let X be compact and let $\mathcal{O} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $f[X]$. For each $\alpha \in \Lambda$, let

$$U_\alpha = f^{-1}[V_\alpha]$$

The U_α's form an open cover of X. Since X is compact, there are $U_{\alpha_1}, \cdots, U_{\alpha_n}$ that cover X. Thus $V_{\alpha_1}, \cdots, V_{\alpha_n}$ is a finite subcover of $\mathcal{O}$.

Collary 8.2: EVT

If $A \subseteq \mathbb{R}$ is compact and $f : A \to \mathbb{R}$ is continuous, there are points $a, b \in A$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in A$.

Theorem 8.6

Let $f : X \to Y$ be a continuous bijection, where X is compact and Y is Hausdorff. Then f is a homeomorphism.

Proof

If $C \subseteq X$ is closed, then C is compact as X is compact, so $f[C] \subseteq Y$ is compact. So $f[C]$ is closed as Y is Hausdorff.

Collary 8.3

Let τ be a compact Hausdorff topology on the set X. Then:

1. Topologies finer than τ are not compact.
2. Topologies coarser than τ are not Hausdorff.

Proof

Let τ be compact Hausdorff. If $\tau \subsetneq \tau'$ then

$$id : (X, \tau') \to (X, \tau)$$

is continuous and bijective, but not a homeomorphism. Thus τ' is not compact.
If $\tau' \subsetneq \tau$ then

$$id : (X, \tau) \to (X, \tau')$$

is continuous and bijective, but not a homeomorphism. Thus τ' is not Hausdorff.

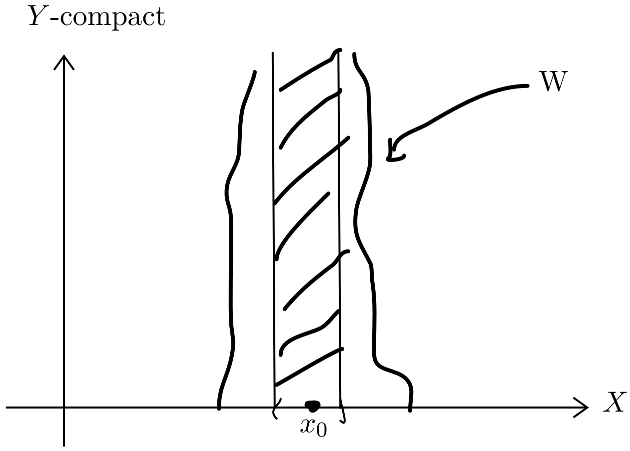
Theorem 8.7

The finite product of compact spaces is compact.

Lemma 8.1: Tube Lemma

Let X and Y be topological spaces with Y compact. If W is an open subset of $X \times Y$ including fiber $\{x_0\} \times Y$, then there exists some open neighbourhood U of x_0 such that W includes the strip $U \times Y$.

Let $W \subseteq X \times Y$ open with $\{x_0\} \times Y \subseteq W$.



Proof

Let $\{U_\alpha \times V_\alpha : \alpha \in \Lambda\}$ be an open cover of $\{x_0\} \times Y$ by basic open sets such that $U_\alpha \times V_\alpha \subseteq W$.

1. Note that $\{x_0\} \times Y$ is compact. So let $U_{\alpha_1}, \cdots, U_{\alpha_n}$ and $V_{\alpha_1}, \cdots, V_{\alpha_n}$ be such that $\mathcal{O} = \{U_{\alpha_1} \times V_{\alpha_1} : 1 \leq n\}$ covers $\{x_0\} \times Y$. Let

$$U = \bigcup_{i=1}^{n} U_{\alpha_i}$$

Now note that U is open neighborhood of x_0.
Claim: $U \times Y \subseteq W$. Let $\langle x, y\rangle \in U \times Y$, then $y \in V_{\alpha_i}$ ($\langle x_0, y\rangle \in U_{\alpha_i} \times V_{\alpha_i}$). Since $x \in U$, then $x \in U_{\alpha_i}$ for all $i \leq n$. So $\langle x, y\rangle \in U_{\alpha_i} \times V_{\alpha_i} \subseteq W$.

Proof

Let X and Y be compact. Let $\mathcal{O}$ be an open cover of $X \times Y$. For each $x \in X$, the fiber $\{x\} \times Y$ is compact. There are finitely many elements of $\mathcal{O}_x$ such that $\mathcal{O}_x$ covers $\{x\} \times Y$. For each $x \in X$,

$$\{x\} \times Y \subseteq \bigcup \mathcal{O}_x = \bigcup_{v \in \mathcal{O}_x} V$$

By the tube lemma, for each $x \in X$, there is some open $U_x \subseteq X$ with $x \in U_x$ such that $U_x \times Y \subseteq \bigcup \mathcal{O}_x$.

Since $\{U_x : x \in X\}$ covers X and X is compact, there are $x_1, \cdots, x_n$ such that

$$X = \bigcup_{i \leq n} U_{x_i}$$

Note: $U_{x_i} \times Y$ for $i \leq n$, this covers $X \times Y$.

Example

Compact subspaces of $\mathbb{R}^n$

Tychonoff's Theorem

Theorem 8.8: Tychonoff's Theorem

The arbitrary product of compact spaces is compact.

Note

Note the following 2 properties:

1. X is compact
2. If B is a basis for the topology of X, every open cover by elements of B has a finite subcover.
3. If $\mathcal{S}$ is a subbasis for X then every cover by elements of $\mathcal{S}$ has a finite subcover.

Lemma 8.2: Alexander's subbasis lemma

Let X be a topological space with a subbasis $\mathcal{S}$. If for every open cover of X by elements of $\mathcal{S}$, there exists a finite collection of elements of $\mathcal{S}$ that covers X, then X is compact.

Theorem 8.9

Tychonoff's theorem implies the axiom of choice

Proof of Alexander's subbasis lemma

If X satisfy condition 3, but is not compact, let

$$\mathcal{C} = \{\mathcal{O}\}$$

Where $\mathcal{O}$ is an open cover of X without finite subcover. Note that the order is inclusion meaning $\mathcal{O} \leq \mathcal{O}'$ if and only if $\mathcal{O} \subseteq \mathcal{O}'$.

1. $\mathcal{C}$ is non-empty.

2. We will prove that chains in this partially ordered set are bounded.

Let $K \subseteq \mathcal{C}$ be a chain and let

$$\mathcal{O}' = \bigcup_{\mathcal{O} \in K} \mathcal{O}$$

- $\mathcal{O}'$ is an open cover of X since every $\mathcal{O} \in K$ is an open cover.

- If $U_{\alpha_1}, \cdots, U_{\alpha_n} \in \mathcal{O}'$ is a finite cover of X then each $U_{\alpha_i} \in \mathcal{O}_i \in K$ and $\exists \mathcal{O} \in K[\mathcal{O}_n]$ such that

  $$U_{\alpha_1}, \cdots, U_{\alpha_n} \in \mathcal{O} \in K \subseteq \mathcal{C}$$

  By zorn's lemma, there exists a maximal (with respect to inclusion) open cover of X without a finite subcover.

Theorem 8.10

Tychonoff's Theorem $\implies$ axiom of choice.

Proof of Alexander's Lemma

Let S be a subbasis for X. Let $\mathscr{O}$ be an open cover of X with no finite subcover and $\mathscr{O} is \leq -$maximal.

1. The collection $\mathscr{O} \cap S$ cannot be a cover for X.
2. Let $x \in X \setminus \bigcup(\mathscr{O} \cap S)$.
3. There is some $U \in \mathscr{O}$ such that $x \in U$.
4. There are $B_1, \ldots, B_n \in S$ such that $x \in \bigcap_{i \leq n} B_i \subseteq U$.
5. No $B_i \in \mathscr{O}$ so let $\mathscr{O}_i = \mathscr{O} \cup \{B_i\}$
6. By maximally of $\mathscr{O}$, let $\tilde{\mathscr{O}}_i \cup \{B_i\} \subseteq \mathscr{O}_i$ be a finite subcover of X.
7. $\tilde{\mathscr{O}}_1 \cup \cdots \cup \tilde{\mathscr{O}}_n \cup (\bigcap_{i \leq n} B_i)$ covers X.
8. $\tilde{\mathscr{O}}_1 \cup \cdots \cup \tilde{\mathscr{O}}_n \cup \{U\}$ covers X.

This is a contradiction and so X is compact.

Proof of Tychonoff's Theorem

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of compact spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$ By alexander's lemma, let

$$\mathscr{O} = \left\{ \prod_{\alpha}^{-1}(U) : \text{ for some open } U \subseteq X_\alpha, \alpha \in \Lambda \right\}$$

be an open cover of X.

Claim: There is some $\beta \in \Lambda$ such that

$$\mathscr{O}_\beta = \left\{ \prod_{\beta}^{-1}(U) : U \subseteq X_\beta \right\}$$

this covers X. Otherwise for each $\alpha \in \Lambda$, there is some $x_\alpha \in X_\alpha$ such that x_α is not covered by $U \subseteq X_\alpha$ such that $\prod_\alpha^{-1}(U) \in \mathscr{O}$ and therefore the tuple

$$(x_\alpha)_{\alpha \in \Lambda}$$

is not covered by $\mathscr{O}$. Fix this β and note that the collection

$$\{U \subseteq X_\beta : \prod_\beta^{-1}(U) \in \mathscr{O}\}$$

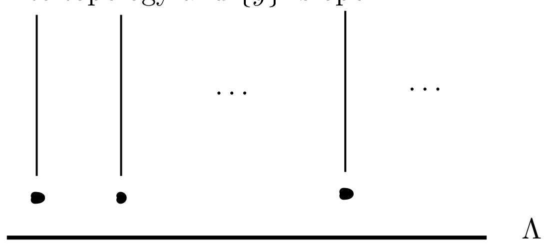
covers X_β. Since X_β is compact, finitely many of these opens $U_1, U_2, \ldots, U_n$ already cover X_β and therefore $\{\prod_\beta^{-1}(U_i) : i \leq n\}$ is a finite subcover of $\mathscr{O}$.

Proof

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of non-empty sets. we want to prove that

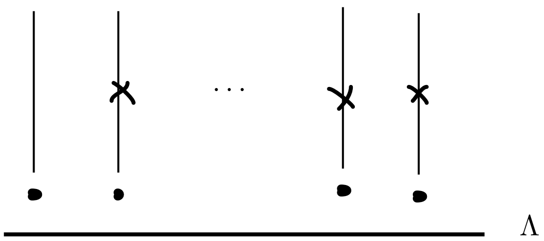
$$X = \prod_{\alpha \in \Lambda} X_\alpha$$

is non-empty. Let $y \notin \bigcup_{\alpha \in \Lambda} X_\alpha$ In each $X_\alpha \cup \{y\}$ define a new topology. Consider the cofinite topology and $\{y\}$ is open.



By Tychonoff, X is compact.
Claim: $\{\prod_\alpha^{-1}(X_\alpha) : \alpha \in \Lambda\}$ only has closed sets and has the finite intersection property.
Let $\alpha_1, \cdots, \alpha_n \in \Lambda$. Let's find some element in

$$\bigcap_{i \leq n} \prod_\alpha^{-1}(X_{\alpha_i})$$



Since X is compact, $\bigcap_{\alpha\in\Lambda}\prod_{\alpha}^{-1}(X_\alpha)\neq\emptyset$. So if $\vec{x}\in\bigcap_{\alpha\in\Lambda}\prod_{\alpha}^{-1}(X_\alpha)$, then for each $\alpha\in\Lambda$, $x_\alpha\neq y$, so $\vec{x}\in\prod_{\alpha\in\Lambda}X_\alpha$.

9 Locally Compact Spaces

Definition 9.1: Locally compact

A topological space X is locally compact at $x \in X$ if only if there are subsets U and K compact, such that $x \in U \subseteq K$.
If X is locally compact at every point, then X is locally compact.

Proposition 9.1

For a Hausdorff space X and $x \in X$, the following are equivalent:

1. X is locally compact at x.
2. There is open neighborhood V of x such that $\overline{V}$ is compact.
3. There is a local basis around x of open sets with compact closure.

Proof

If X is locally compact set at x, then there exists U which is open and $K \subseteq X$ which is compact such that

$$\overline{U} \subseteq K(\text{Hausdorff})$$
$$\overline{U} \text{ is compact}$$

2 $\implies$ 3

By (2), there exists V open neighborhood of x and $\overline{V}$ is compact. Now let

$$\mathcal{B}_x = \{U \cap V \text{ is an open neighborhood of } x\}$$

This means that $\mathcal{B}$ is a local basis around x. Also $\overline{U \cap V} \subseteq \overline{V}$ so $\overline{U \cap V}$ is compact.

Examples

Some examples are $\mathbb{N}, \mathbb{R}, \mathbb{Q}, \omega_1$.

- Discrete topology: $x \in \{x\}$ where $\{x\}$ is open and compact.

- If $x \in \mathbb{R}$ then $\forall \epsilon > 0$, $x \in (x-\epsilon, x+\epsilon) \subseteq [x-\epsilon, x+\epsilon]$.
- It is enough to check that sets of the form $[a,b] \cap \mathbb{Q}$ where $a, b \in \mathbb{R} \setminus \mathbb{Q}$ are not compact. Otherwise let U be open in $\mathbb{Q}$ and K be compact be such that $x \in U \subseteq K$. Let $\epsilon > 0$ be such that $(x-\epsilon, x+\epsilon) \cap \mathbb{Q} \subseteq U$. So $[x-\epsilon, x+\epsilon] \cap \mathbb{Q} \subseteq K$.
  Note that the collection $\mathscr{O}$

  $$\mathscr{O} = \{(a, r) : r \text{ is irrational and } a < r < b\}$$

  is an open cover of $[a,b] \cap \mathbb{Q}$ but has no finite subcover.
- A closed interval in ω_1 are not compact.

Theorem 9.1

For a topological space X, the following are equivalent:

1. X is Hausdorff locally compact
2. There exists a compact Hausdorff space Y and an imbedding

   $$i : X \to Y$$

   such that $Y \setminus i[X]$ has a single point.
3. X is homeomorphic to an open subspace of a compact Hausdorff space.

Also if

$$i : X \to Y \quad \text{and} \quad j : X \to Y'$$

are two spaces and imbeddings as in 2. above, then there exists a unique homeomorphism $f : Y \to Y'$ such that

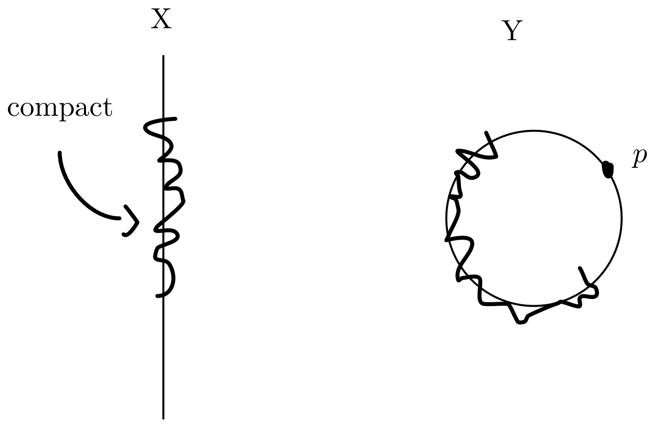
$$f \circ i = j.$$

Proof

1 $\implies$ 2

Let X be a locally compact and hausdorff and let $p \notin X$. As a set $Y = X \cup \{p\}$. We want to define a topology τ_p on Y.

- $\tau \subseteq \tau_p$.(old opens remain open)
- A neighborhood of p is a subset U of Y such that $X \setminus U$ is compact in X.



1. τ_p is a topology on Y.

   (a) $Y \in \tau_p$

   (b) $\emptyset \in \tau_p$

   Now we have cases. The first case is

   (a) If $U, V \subseteq X$ then $U \cap V$ is open in X and therefore open in Y.

   (b) If U and V are open neighborhoods of p then $X \setminus U = K_u$ is compact and $X \setminus V = K_v$ is compact. To check if $U \cap V$ is open, we need to check if the compliment is compact.

   $$X \setminus (U \cap V) = K_u \cup K_v$$

   is compact.

   (c) If $U \subseteq X$ and V is an open neighborhood of p. Then by definition $X \setminus V$ is compact so we have $X \setminus V = X \setminus (X \cap V)$. Now if we compute $U \cap V$ we have

   $$U \cap V = U \cap (X \cap V)$$

   is open in X

Now check if the topology is compact and hausdorff.

Compact

Let $\mathscr{O}$ be an open cover of Y and let $U \in \mathscr{O}$ such that $p \in U$. Since $X \setminus U$ is compact, it can be covered by finitely many $V_1, \ldots, V_n$ thus $\{U, V_1, \ldots, V_n\}$ is a finite subcover of $\mathscr{O}$.

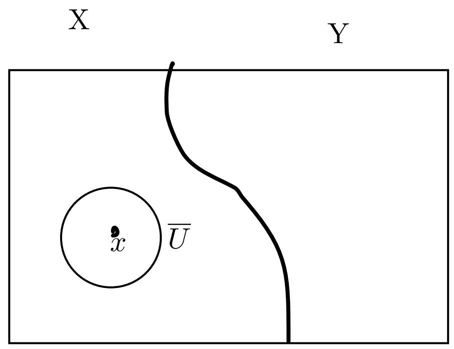
Hausdorff

Let $x, y \in Y$. We have 2 cases:

1. $x, y \in X$, follows from X being Hausdorff.

2. If $y = p$, let U be an open neighborhood of x and K compact such that $x \in U \subseteq K$.

3 $\implies$ 1



where Y is compact and hausdorff. Now note that there exists some U open neighborhood of x such that $\overline{U} \subseteq X$.

Definition 9.2: Compactification

A compactification of a hausdorff space X is a compact hausdorff space Y for which there exists a dense imbedding $i : X \to Y$. If, in addition, $Y \setminus i[X]$ has a single point, then Y is called the one-point compactification of X and denoted by αX.

Examples

- The one point compactification of $\alpha\mathbb{N}$ is homeomorphic to $\left\{\frac{1}{n} : n \in \mathbb{N} \cup \{0\}\right\}$

- $\alpha\mathbb{R}$ is homeomorphic to S^1.

- $\alpha\omega_1$. This means take ω_1 and add a point at the end of it. Now if you take away α you get ω_1. so it is the one point compactification of ω_1.

Countability Axioms

Definition 9.3

A topology space X is:

1. first-countable if every point has a countable local basis

2. second-countable if there is a countable basis for the topology of X.

Proposition 9.2

For a topology space X,

$$\text{second-countable} \implies \text{first-countable} \implies \text{squential}$$

1st countable $\implies$ sequential

X is sequential if $\forall A \subseteq X$ and $x \in \overline{A}$, $\exists (x_n : n \in \mathbb{N}) \subseteq A$ such that $x_n \to x$.
Let $\{U_n : n \in \mathbb{N}\}$ be a local basis around $x \in X$. Since $x \in \overline{A}$, for each $n \in \mathbb{N}$, there is some $x_n \in A \cap \bigcap_{j \leq n} U_j$. Then $x_n \to x$.

Examples

- $\mathbb{R}$

- $\mathbb{R}^{\omega}_{\text{unif}}$. This means $\mathbb{R}^{\omega}$ with the uniform topology. this is metrizable.
  Claim: If X is second-countable and $A \subseteq X$ is discrete, then A is countable.

- The rose

Proof

Let $A \subseteq X$ be discrete, and for each $a \in A$, let $U_a \in B$ (where B is fixed countable basis for X) such that $U_a \cap A = \{a\}$.
Consider $a \mapsto U_a$ is a injective function so A is countable. In $\mathbb{R}^{\omega}_{\text{unif}}$, $\{0,1\}^{\omega}$ is a discrete subspace. If $\vec{x}, vecy \in \{0,1\}^{\omega}$ and $\vec{x} \neq \vec{y}$, then $p(\vec{x}, \vec{y}) = 1$.

Proposition 9.3

If X is a second-countable space, then:

1. Every open cover of X has a countable subcover.
2. X has a countable dense subset.
3. Any collection of pairwise disjoint open sets of X is countable.

Proof of 2

Fix a countable basis B and for each $U \in B$, $U \neq \emptyset$, choose $x_0 \in U$. Now Consider

$$D = \{x_0 : U \in B\}$$

D is dense in X.

Definition 9.4

When a topological space satisfies:

1. is called a Lindelöf space.
2. is called a separable space.
3. is called a Countable Chain Condition.

Definition 9.5

1. **Seperable:** X has a countable dense subset ($\overline{Z} = X$)
2. **Lindelof Space:** Every open cover of X has a countable subcover.
3. **Countable Chain Condition:** Every collection of pairwise disjoint open sets is countable.

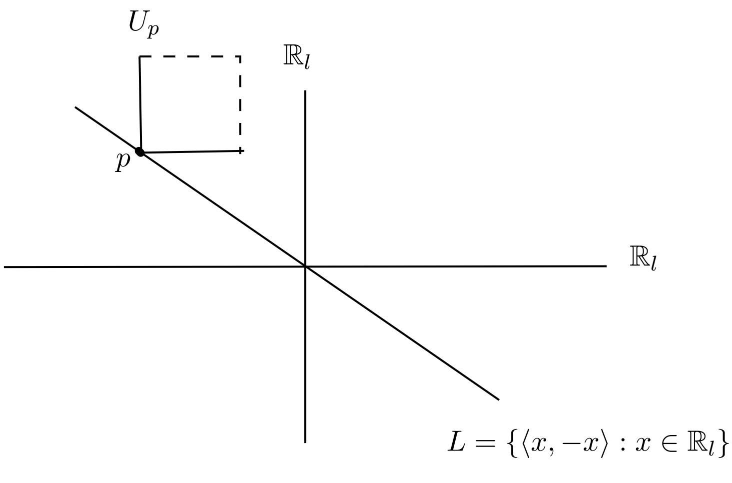
- Seperability implies countable chain condition. ccc does not imply seperable.
- Lindelof does not imply ccc.
- Lindelof does not imply seperable. Also seperable does not imply Lindelof and ccc does not imply Lindelof.

Take the following examples to understand the above points:

i Cocountable topology on $\mathbb{R}$. This is Lindelof. This is also ccc but it is not separable.

ii $\alpha\omega_1$. This is Lindelof as this is compact and linear. This is not ccc
Proof of not ccc: For each $\gamma \in \alpha\omega_1$, with an immediate predecessor, the set $\{\gamma\}$ is open. We have uncountably $\{\gamma\}$'s which are pairwise disjoint. so such a set is uncountable.
This is not seperable as it is not ccc.

iii $\mathbb{R}_l^2$. This is seperable, ccc but not Lindelof.
This is seperable because $\mathbb{Q}^2$ is dense in $\mathbb{R}_l^2$ ($\overline{\mathbb{Q}^2} = \mathbb{R}_l^2$).
It is not Lindelof



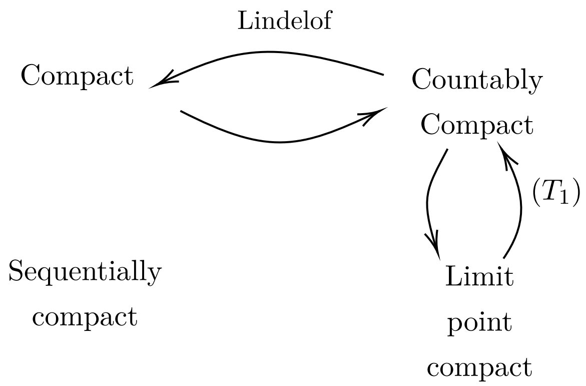
(a) Note that L is closed subset as it is the diagonal of the hausdorff space $\mathbb{R}_l$.

(b) Consider the cover $\{\mathbb{R}_l^2 \setminus L, U_p : p \in L\}$

(c) This cover has no countable subcover.

Definition 9.6: What it means for a topological space to be:

1. Limit point compact: Every infinite subset has a limit point.
2. Sequentially compact: Every sequence has a convergent subsequence.
3. Countably compact: Every countable cover has a finite subcover.



Take the following examples

i Divisor topology. As a set, $\mathbb{N}_{\geq 2}$ and the topology is generated by $U_n = \{m : m \mid n\}$.

ii ω_1

iii $\mathcal{B}\mathbb{N}$

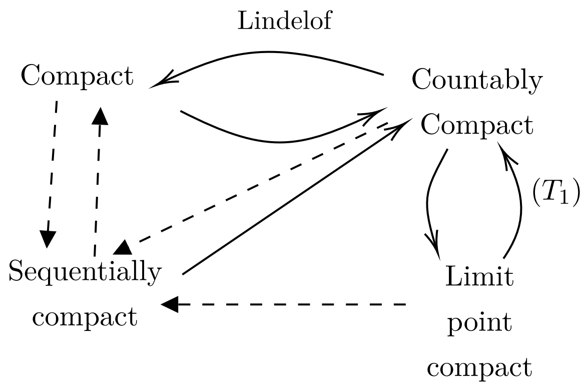
- **countably compact $\implies$ limit point compact:** Let $A \subseteq X$ be infinite with no limit point. Then this means that
  - A is closed.
  - For each $a \in A$ there exists some open $U_a \subseteq X$ such that $U_a \cap A = \{a\}$.

  Consider the cover $\{X \setminus A, U_a : a \in A\}$.

- **LPC + T_1 $\implies$ ccc:** Otherwise, let $\{U_n : n \in \mathbb{N}\}$ be an open cover of X without finite subcover. For each $n \in \mathbb{N}$, let $x_n \in X \setminus \bigcup_{j \leq n} U_j$. Let $A = \{x_n : n \in \mathbb{N}\}$. A is infinite.

Definition 9.7: A topological space is:

1. **Countably compact** if every countable open cover has a finite subcover.
2. **limit point compact** if every infinite subset has a limit point.
3. **sequentially compact** if every sequence has a convergent subsequence.



LPC + $T_1 \implies$ cc

Let $\{U_n : n \in \mathbb{N}\}$ be an open cover of X with no finite subcover. For each $n \in \mathbb{N}, x_n \in X \setminus \bigcup_{i \leq n} U_i$. This means that $A = \{x_n : n \in \mathbb{N}\}$ is infinite.
If $x \in X$, let U_n be such that $x \in U_n$. Then $U_n \cap A$ has to be finite.

- The sequence $(x_n : n \in \mathbb{N})$ doesn't have a convergent subsequence.
- Consider the open neighborhood U_n of x that is $U_n \setminus (A \setminus \{x\})$.

Examples

Divisor topology: As a set $\mathbb{N}_{\geq 2} = \{2, 3, 4, \ldots\}$, generated by basic open sets

$$U_n = \{m \in \mathbb{N} : m \mid n\}$$

- If $\gcd(m, n) = 1$, then $U_m \cap U_n = \emptyset$. Otherwise $U_m \cap U_n = U_{\mathrm{gcd(m,n)}}$.
- The divisor topology is not countably compact but is limit point compact. The reason why it is not countably compact is because the open cover $\{U_n : n \in \mathbb{N}\}$ has no finite

subcover.

- It is limit point compact. Let $A \subseteq \mathbb{N}_{\geq 2}$ be infinite. In particular, there are two different elements in A (let's say m and n).
- For every point $k \in \mathbb{N}_{\geq 2}$, there exists a minimal open set including k.

ω_1 **is not compact:** Take the following cover

$$\mathscr{O} = \{[\alpha_{\min}, \alpha) : \alpha\}$$

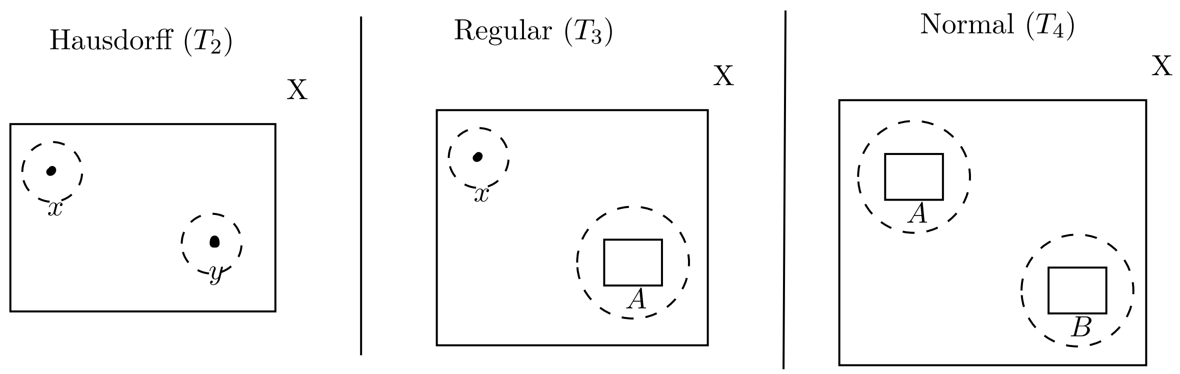
- ω_1 is not lindelof.
- ω_1 is limit point compact. Let $A \subseteq \omega_1$ be countable. Since A is countable, it is bounded above so let α be an upper bound for A. So $A \subseteq [\alpha_{\min}, \alpha^+]$. Since $[\alpha_{\min}, \alpha^+]$ is compact, A has a limit point in $[\alpha_{\min}, \alpha^+] \subseteq \omega_1$.

10 Seperation Axioms

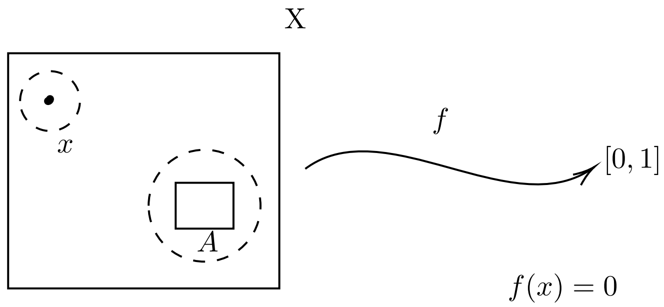
Definition 10.1

A topological space X is:

1. T_1 if points in X are closed.
2. T_2 (or Hausdorff) if points can be separated by open sets.
3. T_3 (or regular) if
   - X is T_1.
   - Points can be seperated from closed sets by open sets.
4. $T_{3\frac{1}{2}}$ (or completely regular) if
   - X is T_1.
   - Points can be seperated from closed sets by continuous functions.
5. T_4 (or normal) if
   - X is T_1.
   - Closed sets can be seperated by open sets.



Complete Regularity $T_{3\frac{1}{2}}$



$f(x) = 0$

$\forall y \in A, (f(y) = 1)$

Examples

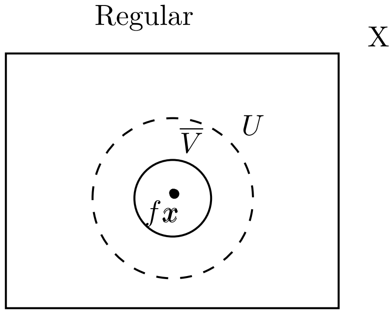
- **Cofinite topology:** The cofinite topology in $\mathbb{R}$ is T_1 but not Hausdorff.
- $\mathbb{R}_K$ is generated by (a, b) and $(a, b) \setminus K$ where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. It is Hausdorff because It is similar to the standard topology on $\mathbb{R}$.
  In $\mathbb{R}_K$, $\{0\}$ cannot be seperated from K. But it is not regular.

Proposition 10.1

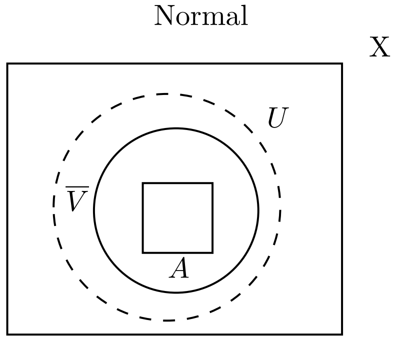
Let X be a topological space. Then:

1. X is regular if and only if, for every $x \in X$ and every open neighborhood U of x, there exists an open neighborhood V of x such that $\overline{V} \subseteq U$.
2. X is normal if and only if for every closed subset $A \subseteq X$ and every open neighborhood U of A, there is an open neighborhood V of A such that $\overline{V} \subseteq U$.

Regular



Normal



Proposition 10.2

1. Subspaces of (completely) regular spaces are (completely) regular.
2. Products of (completely) regular spaces are (completely) regular.

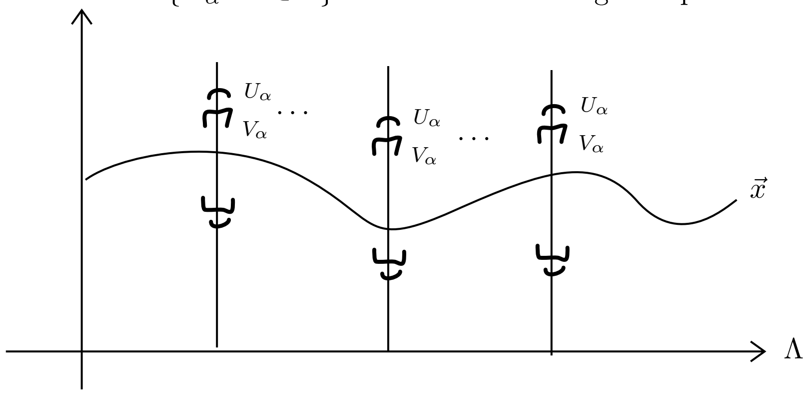
Regularity

Subspaces: Let X be regular and $Y \subseteq X$. Let $y \in Y$ and $A \subseteq Y$ be closed. By definition of the subspace topology $\exists B \subseteq X$ closed such that $A = B \cap Y$.
Since $y \notin B$ and X is regular, then $\exists U, V$ open such that $x \in U$ and $B \subseteq V$ with $U \cap V = \emptyset$. So

$$y \in U \cap Y, \quad A \subseteq V \cap Y$$

Where $U \cap Y$ and $V \cap Y$ are disjoint so Y is regular.

Products: Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of regular spaces.



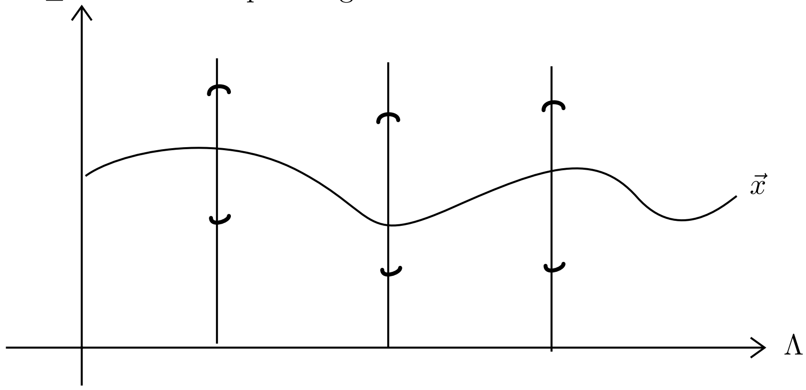
$\vec{x} \in \prod_{\alpha \in \Lambda} V_\alpha$ so we get

$$\overline{\prod_{\alpha \in \Lambda} V_\alpha} \subseteq \prod_{\alpha \in \Lambda} \overline{V_\alpha} \subseteq \prod_{\alpha \in \Lambda} U_\alpha$$

Complete Regularity

Subspaces: Let X be $T_{3\frac{1}{2}}$ and $Y \subseteq X$. Let $y \in Y$ and $A \subseteq Y$ be closed such that $y \notin A$. Let $B \subseteq X$ be closed with $A = B \cap Y$ (in particular, $y \notin B$).
Since X is $T_{3\frac{1}{2}}$, $\exists f : X \to [0,1]$ continuous such that $f(y) = 0$ and $f(B) = 1$. Then $f|_Y$ is continuous and $f|_Y(y) = 0$ and $f|_Y(A) = 1$.

Products: Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of completely regular spaces. We want to prove that $\prod_{\alpha\in\Lambda} X_\alpha$ is completely regular. Let $\vec{x} \in X$ and $A \subseteq X$ be closed with $\vec{x} \notin A$. Let $U \subseteq X$ be a bsic open neighborhood of $\vec{x}$ such that $U \cap A = \emptyset$.



For each α_i, let $f_i : X_{\alpha_i} \to [0,1]$ such that $f_i(x_{\alpha_i}) = 1$ and $f_i\big|_{X_{\alpha_i}\setminus U_{\alpha_i}} = 0$.
Let $g_i : X \to [0,1]$ such that $g_i := f_i \circ \pi_{\alpha_i}$. Let $g : X \to [0,1]$ defined as $g = \prod_{i\leq n} g_i$ So this function is continuous and $g(\vec{x}) = 1$ and $g(A) = 0$ so X is completely regular.

Examples

$\mathbb{R}_l^2$ and the tychonoff plank

Examples

Products in $\mathbb{R}_l^2$

- $\mathbb{R}_l$ is normal. Let $A, B \subseteq \mathbb{R}_l$ be closed and disjoint. For each $a \in A$, let $[a, x_a) \subseteq \mathbb{R}_l$ be disjoint from B. If $b \in B$, let $[b, x_b) \subseteq \mathbb{R}_l$ be disjoint from A.

  $$A \subseteq \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad B \subseteq \bigcup_{b \in B} [b, x_b)$$

  Lets say that these 2 overlap so that means that there is some $a \in A$ and $b \in B$ such that $[a, x_a) \cap [b, x_b) \neq \emptyset$. This means that $x_a \geq b$ and $x_b \geq x_a$. This is a contradiction as A and B are disjoint. Therefore $x_a < b$. This means that $[a, x_a) \cap [b, x_b) = \emptyset$. This shows that $\mathbb{R}_l$ is normal.

- $\mathbb{R}_l^2$ is not normal.

  1. The subset $L = \{\langle x, -x \rangle : x \in \mathbb{R}_l\}$ is closed in $\mathbb{R}_l^2$.
  2. L is discrete. Every subset of L is a closed subset of $\mathbb{R}_l^2$. For every $A \subseteq L$, there are open sets U_A and V_A such that $A \subseteq U_A$ and $L \setminus A \subseteq V_A$ and $U_A \cap V_A = \emptyset$.
  3. Now consider the following function.

     $$f : \mathbb{P}(L) \to \mathbb{P}(\mathbb{Q}^2)$$

     Where $\mathbb{Q}^2$ is countable.
  4. We have some properties of f.
     (a) $f(\emptyset) = \emptyset$
     (b) $f(L) = \mathbb{Q}^2$
     (c) If $\emptyset \subsetneq A \subsetneq L$ then $f(A) = U_A \cap \mathbb{Q}^2$.
  5. **Claim:** f is injective. Let $A \subseteq L$
     - If $A \neq \emptyset$ then $U_A \neq \emptyset$, so $f(A) \neq \emptyset$.
     - If $A \neq L$ then $L \setminus A \neq \emptyset$, so $V_A \neq \emptyset$ then $f(A) \neq \mathbb{Q}^2$.

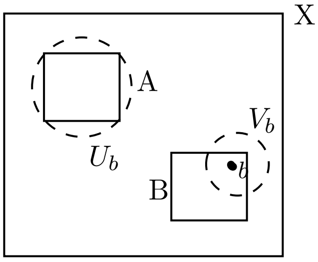
     Let $A, B \subseteq L$ such that $A \neq B$. Want to prove that $f(A) \neq f(B)$. If $A \neq \emptyset$ then there is some point $x \in A \setminus B$. So $x \in U_A \cap V_B$ open non-empty. So in $U_A \cap V_B$ there is some $q \in \mathbb{Q}^2$. Then $q \in f(A) \setminus f(B)$. So $f(A) \neq f(B)$.

Proposition 10.3

The following classes of spaces are normal:

1. Regular, Lindelöf spaces.
2. Compact Hausdorff spaces.
3. Metrizable spaces.
4. Linear orders (with the order topology).

Proof of 2



The family

$$\mathscr{O} = \{V_b : b \in B\}$$

covers B so let $V_{b_1}, V_{b_2}, \dots, V_{b_n}$ be a finite subcover. So

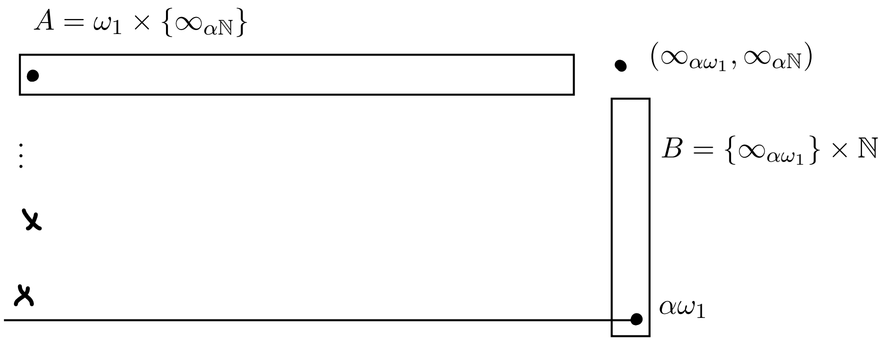
$$A \subseteq \bigcap_{i \leq n} U_{b_i} \quad \text{and} \quad B \subseteq \bigcup_{i \leq n} V_{b_i}$$

But note that $U_{b_i} \cap V_{b_i} = \emptyset$ for all $i \leq n$. So A and B are separated by open sets. Therefore X is normal.

Examples of normal subspaces

- Tychonoff's plank is normal.
- The product space $\alpha\omega_1 \times \alpha\mathbb{N}$ is normal.

So when you remove the point at the corner, it is not normal.



Let $\alpha = \sup(\alpha_n)$. Now this supremum is a point in B such that the α line of that point is included in every other point in B and it's α line. Now consider the successor of α

Proof not complete

Lemma 10.1: Urysohn's Lemma

Let X be a normal space and let $A, B \subseteq X$ be closed. Then there exists a continuous function $f : X \to [0,1]$ such that

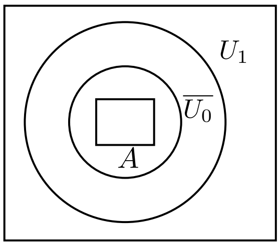
$$f\big|_A = 0 \quad \text{and} \quad f\big|_B = 1$$

Proof

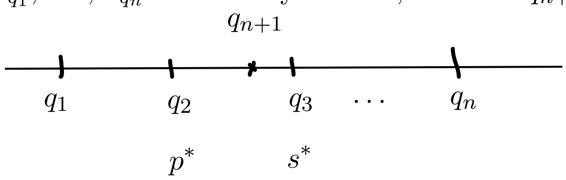
We will find a family of open sets $\{U_p : p \in \mathbb{Q} \cap [0,1]\}$ such that if $p < q$ then $\overline{U_p} \subseteq U_q$. Fix enumeration of $\mathbb{Q} \cap [0,1] = \{q_n : n \in \mathbb{N}\}$ with $q_2 = 0$. Let $U_1 = X \setminus B$ and note that $A \subseteq U_1$.

By normality, there exists some open U_0 such that $A \subseteq U_0$ and $\overline{U_0} \subseteq U_1$.

X



If $U_{q_1}, \cdots, U_{q_n}$ are already defined, consider q_{n+1}

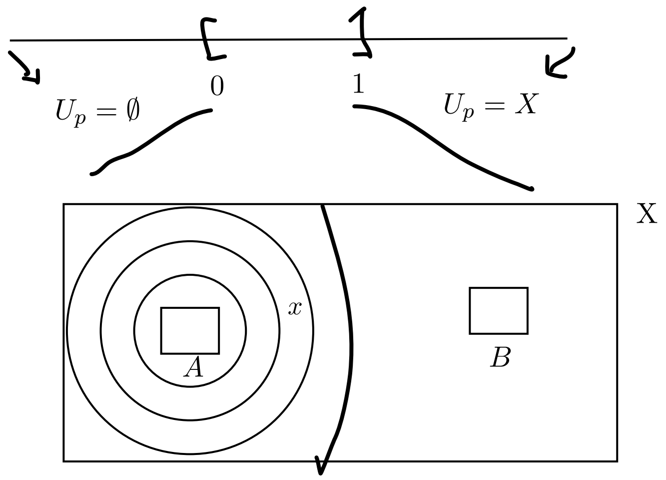


p^* = predecessor and s^* = successor. We know that $\overline{U_{p^*}} \subseteq U_{r^*}$. Let $U_{q_{n+1}}$ be an open neighborhood of $\overline{U_{p^*}}$ such that $\overline{U_{q_{n+1}}} \subseteq U_{r^*}$.

If $p > 1$ $(\in \mathbb{Q})$, let $U_p = X$. If $p < 0$ $(\in \mathbb{Q})$, let $U_p = \emptyset$. Let

$$f : X \to \mathbb{R} \quad ([0,1])$$
$$f(x) = \inf\{p : x \in U_p\}$$

If $x \in A$, then $x \in U_p$ for all $p \geq 0$ but $x \notin U_p$ for any $p < 0$. Therefore $f(x) = 0$.



$f(x) = \inf\{p : x \in U_p\}$. Also if $x \in B$ then $x \in U_p$ for $p > 1$. Both $x \notin U_p$ for $p \le 1$, so $f(x) = 1$.

Claim: f is continuous. We have the following facts.

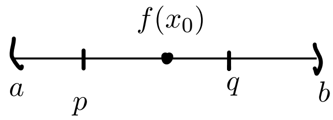
1. $x \in \overline{U_r}$ then $f(x) \le r$

2. If $x \notin U_r$ then $f(x) \ge r$

   - If $x \in \overline{U_r}$ then $x \in U_p$ for all $p > r$. so $f(x) \le r$.

3. If $x \notin U_r$ then $x \notin \overline{U_p}$ for any $p \le r$ so $f(x) \ge r$.

We will check that if V is a neighborhood of $f(x_0)$, there exists a neighborhood U of x_0 such that $f[U] \subseteq V$. Let $x_0 \in X$ and



Consider the neighborhood $U = U_q \setminus \overline{U_p}$ of x_0. From fact (1), $x \in \overline{U_r}$ then $f(x) \le r$. The negation is: if $f(x) > p$, then $x \notin \overline{U_p}$. So

$$U = U_q \setminus \overline{U_p}$$

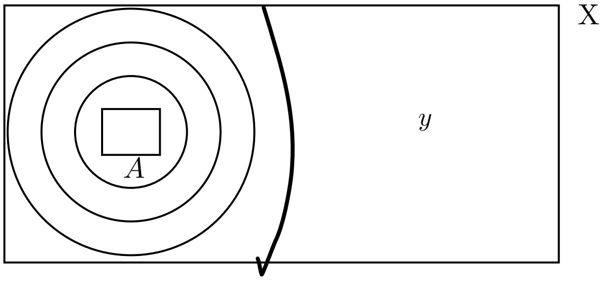
Proof

If we prove that $f[U] \subseteq (a, b)$ then we are done. So let $x \in U$ so

* $x \in U_q \subseteq \overline{U_q}$

** $x \notin \overline{U_p} \supseteq U_p$

By (*), $f(x) \leq q$ and by (**), $f(x) \geq p$.



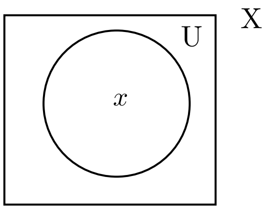
Lemma 10.2: Imbedding

et X be T_1 space and let $\{f_\alpha : \alpha \in \Lambda\}$ be family of continuous functions from X into $[0,1]$. such that for every $x \in X$ and every neighborhood U of x, there exists some $\alpha \in \Lambda$ such that

$$f_\alpha(x) = 1 \text{ and } f_\alpha\big|_{X \setminus U} = 0$$

Then the map:

$$F : X \to [0,1]^\Lambda \text{ given by}$$
$$F(x) = \{f_\alpha(x) : \alpha \in \Lambda\} \text{ is a topological imbedding.}$$



$\{f_\alpha : \alpha \in \Lambda\}$ is a family of continuous functions from X into $[0,1]$. such that

$$f_\alpha(x) = 1$$
$$f_\alpha\big|_{X \setminus U} = 0$$
$$F : X \to [0,1]^\Lambda$$
$$x \mapsto \langle f_\alpha : \alpha \in \Lambda \rangle$$

Collary 10.1

For a topological space X, the following are equivalent:

1. X is completely regular.

2. X is homeomorphic to a subspace of $[0,1]^\Lambda$ for some Λ.

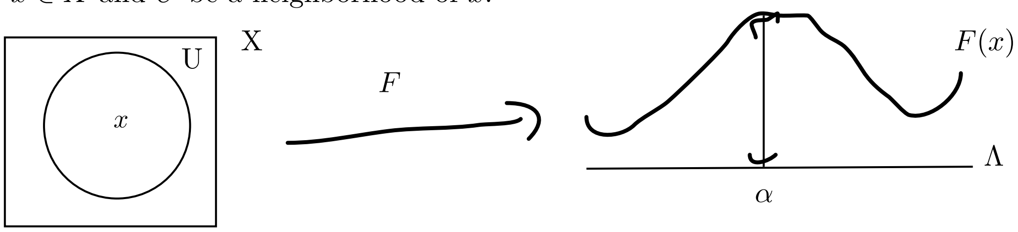
Proof

1. F is continuous.

2. F is injective.

3. F is open.

Let $x \in X$ and U be a neighborhood of x.



Let $V = \prod_\alpha^{-1}(0,1]$.
Claim $F(x) \in F[X] \cap V \subseteq F[U]$. If $F(y) \in F[X] \cap V$ then $f_\alpha(y) > 0$ then $y \in U$. So $F(y) \in F[U]$.

Theorem 10.1: Urysohn's Metrization

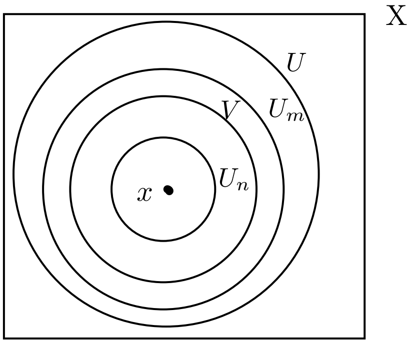
If X is a regular, second-countable space, then X is metrizable.

Proof

1. Regular + second-countable space $\implies$ normal.

2. We will find a countable family of functions separating points from closed sets. Take $F : X \to [0,1]^{\mathbb{N}}$ defined by $x \mapsto \langle f_n(x) : n \in \mathbb{N} \rangle$. Let $B = \{U_n : n \in \mathbb{N}\}$ be a countable basis for X. Define the functions as follows:

   - Always that $\overline{U_n} \subseteq U_m$, let $f_{n,m} : X \to [0,1]$ such that $f_{n,m}\big|_{\overline{U_n}} = 1$ and $f_{n,m}\big|_{X \setminus U_m} = 0$ where $n, m \in \mathbb{N}$.



Define X to be as such

- $x \in U$
- $x \in U_m \subseteq U$
- $x \in V \subseteq \overline{V} \subseteq U_m$
- $x \in U_n \subseteq V$

So we have $\overline{U_n} \subseteq U_m$

11 Stone-Čech Compactification

Theorem 11.1

For a topological space X, the following are equivalent:

1. X is completely regular.
2. X is homeomorphic to a subspace of $[0,1]^{\Lambda}$ for some set Λ.
3. There exists a compactification Y of X such that for every compact Hausdorff space K, every continuous function $f : X \to K$ has a unique continuous extension $g : Y \to K$.

The compactification of X described in item 3 above is called the Stone-Čech compactification of X and is denoted by βX.

$$
\begin{tikzcd}
\beta\mathbb{N} \arrow[dr, dashed, "\beta f := \lim_{n\to u} f(n)"] & \\
\mathbb{N} \arrow[u, "n \mapsto U_n"] \arrow[r, "f"'] & K
\end{tikzcd}
$$

Where K is compact Hausdorff.

Proof

Let X be $T_{3\frac{1}{2}}$ and let $\{f_\alpha : \alpha \in \Lambda\}$ be all the continuous functions from X into $[0,1]$. By the imbedding lemma, the function $F : X \to [0,1]^\Lambda$ is an imbedding $x \mapsto \langle f_\alpha(x) : \alpha \in \Lambda\rangle$. Let $Y = \overleftarrow{F[X]}$. We have the following:

1. Y is compact + Hausdorff "X" ($F[X]$) is dense in Y.

2. For the extension property, let's start with $K = [0,1]$. Let $f : X \to [0,1]$ be continuous. Then there is some $\alpha \in \Lambda$ such that $f = f_\alpha$. Consider $\pi_\beta : Y \to [0,1]$. We have that

   - π_β is continuous.
   - $\pi_\beta(F(x)) = \pi_\beta\langle f_\alpha(x) : \alpha \in \Lambda\rangle = f_\beta(x) = f(x)$.