Group: 6487 - Stochastic Modeling GWP 2

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Step 1

0.1 Data Collection

To analyze our Markov-regime switching model we need to get some data first and as we have been assigned to get data which is the 'new normals', we are going to collect data from pre-covid time i.e 2019 and run through pandemic and come up through the third quarter of 2022.

Each team member has collected different financial time series data for the modeling.

- Team member A has collected Bitcoin (BTC-USD) daily closing price data using Yahoo Finance [1] for the assigned period.
- Team member B has collected adjusted closing daily data for Vanguard ASX 300 ETF (ticker VAS.AX). This ETF seeks to track the return of the S&P/ASX 300 Index before taking into account fees, expenses and tax. Also, as a benchmark, the related index data (ticker ÂXKO) was collected using Yahoo Finance for the assigned period.
- Team member C has collected the weekly M2 US Money supply data via Federal Reserve Bank of St. Louis (FRED) [2].

Step 2

0.2 Data Visualization & Analysis

0.2.1 Bitcoin data Visualization and regime-change analysis

We need to visualize the data properly to understand the several regimes in the series, we will first calculate the returns as it normalizes the data and makes it easier to compare different periods. after that we will calculate the volatility of the returns this will help us in understanding the risk or uncertainty in the data over time. The most important visualization is calculating the rolling mean and standard deviation to identify shifts in these metrics over time. We will then plot histograms of these data to have an overview of the distribution of returns. But the normal return data analysis can have many outliers and we need to smooth that we will be using log returns which will reduce the impact of outliers and make the data more normally distributed, and plotting that we can have a clearer understanding of underlying distribution.

Now Let's see our analysis of BTC-USD daily closing data :

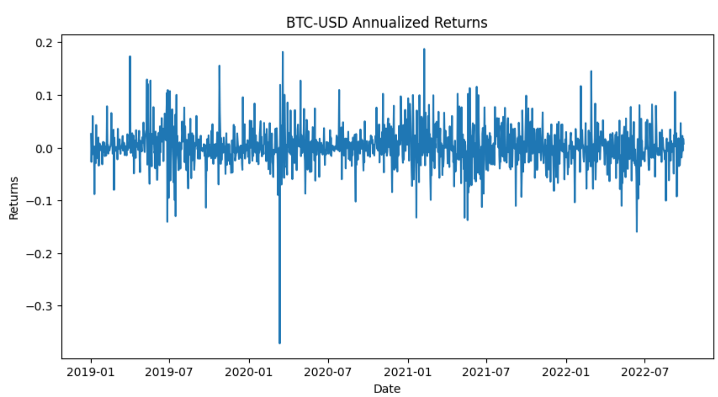

Figure 1: BTC annual returns

Here in the returns in Figure 1 and Volatility in Figure 2, we can see, there is a huge volatility spike near March 2020 resulting in a sharp drop in returns, this was definitely due to the global market shock of the Covid-19 pandemic.

To understand it further in a better way let's see its Rolling Statistic in Figure 3:

In this image, we can see the overall volatility in blue lines The orange line represents the 30-day rolling mean, The green line represents the 30-day rolling standard deviation (a measure of volatility).

We can see in the Figure 3 :

- Early 2020 (around March): There's a noticeable spike in volatility (green line) and a sharp drop in returns. This likely corresponds to the global market shock due to the COVID-19 pandemic.
- Late 2020 to Early 2021: We see an increase in both the rolling mean (orange line trending upward) and volatility (green line increasing). This could indicate a bullish regime with higher returns but also higher risk.

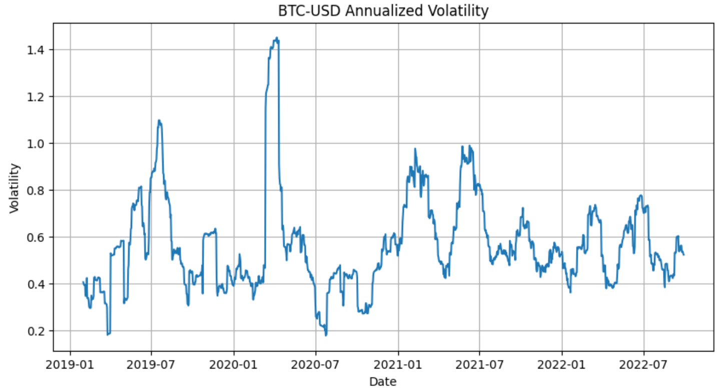

Figure 2: BTC-USD Volatility

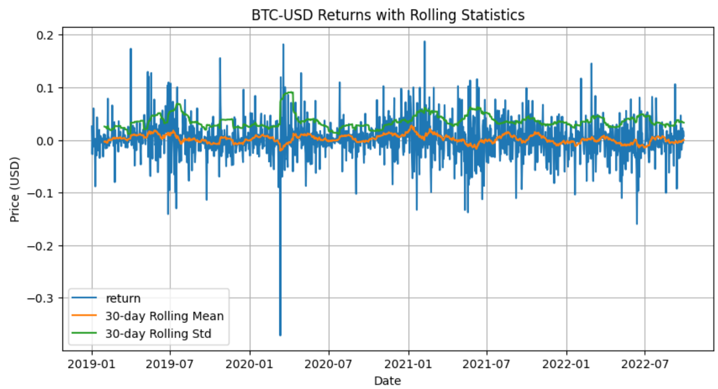

Figure 3: BTC-USD Rolling Statistics

- Mid-2021: Another spike in volatility is visible, followed by a period of relatively high volatility and fluctuating returns. This might represent a shift to a more uncertain or bearish regime.
- Late 2021 to 2022: The rolling mean seems to trend downward while volatility remains elevated. This could indicate a shift to a more bearish regime.

The rolling mean (orange line) and rolling standard deviation (green line) show sustained changes in their level. These are the indications of regime shifts.

We can also see the return distribution in the histogram in Figure 4 :

But as we know BTC is a highly volatile asset and it can have many untactful outliers which can affect our analysis so to tackle this we will convert our return to log returns :

Now our data is smoothed as we can see in Figure 5, it will not be affected by any impactful outliers, we can also see the underlying distribution in Figure 6.

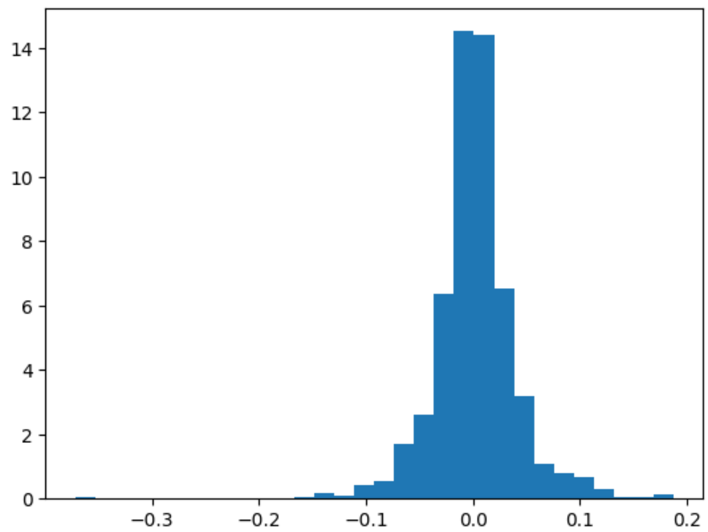

Figure 4: BTC-USD returns Histogram

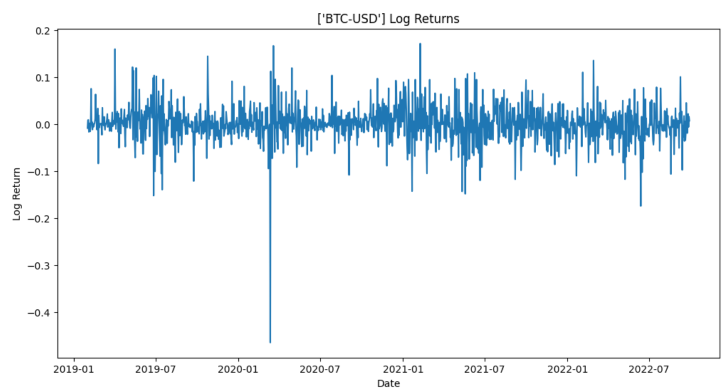

Figure 5: BTC-USD Log returns

0.2.2 ETF on ASX 300 (index on mid cap Australian companies)

Data from Yahoo finance was collected both for Vanguard ETF on Australian mid cap companies (ASX 300) and and the related S&P index. The main goal was not only understand price movements and volatility and possible periods of different regimes, but also understand how closely EFT follows the related synthetic index. Initially we plot a graph of these 2 time series (using left and right axis to maintain a similar scale).

Return

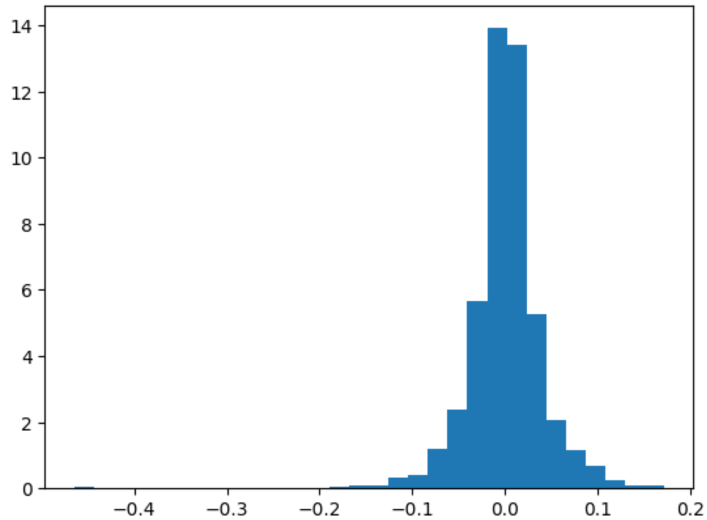

Figure 6: BTC-USD log histogram

0.2.3 M2 money supply data Visualization and regime-change analysis

The M2 Money Supply under the WM2NS ticker provided by the Federal Reserve Economic Data of St. Louis [2] is an economic indicator reflecting the total amount of money in circulation within the US economy. It covers various sources of money issuing and provides insights about liquidity and overall economic health.

Composition of M2

The M2 measure excludes large institutional cash deposits and is considered a broader measure than M1, as it accounts for assets that can be quickly converted into cash but are not typically used for everyday transactions. M2 includes the components of [3]:

- M1: or the most liquid form of money including:
  - Physical currency in terms of coins and paper money
  - Demand deposits or checking accounts
  - Other smaller liquid deposits
- Savings Deposits: These are funds held in savings accounts, including money market deposit accounts.
- Small-Denomination Time Deposits: These are time deposits (like certificates of deposit) smaller than $100,000.
- Retail Money Market Funds: These are investment funds that invest in short-term, low-risk securities, excluding balances from individual retirement accounts (IRAs) and Keogh accounts.

M2 Supply Charts

Next we plot a simple weekly chart 9 of M2 supply in billions of US $ in the period of interest between 2019-01-01 and 2022-12-31. For a better interpretation of events happening in this period and their effect on the money supply we also plot red vertical lines corresponding to dates of pivotal events that might signal a change of economic regime.

Focusing only on the M2 trends we can clearly split the period into 4 regimes:

- January 2019 to January 2020: A pre-pandemic period of gradual and controlled money supply growth. One can mention a transition regime in January and February of 2020 where a bullish trend in money supply changed to a short correction. It might be explained by a temporary slow down of money supply caused by cautious actions of US government evaluating the risks and probability of COVID spread affecting US economy.
- March 2020 to September 2020: An explosive growth of money supply issued by US FED to support halted economy and isolated people after announcing a world pandemic status of COVID infection and launching the country level lockdown. The weekly growth of money supply during this period reached more than 2% levels and had no corrections up to July 2020 when US government initiated preparations for lockdown termination
- September 2020 to April 2022: A more gentle supply growth period during which US government was gradually increasing the price of credit money by increasing interest rates. Money supply still had greater issuance rate compared to the pre-pandemic period, but demonstrated divergence in its momentum and momentum indicators signalling intentions of US government to reverse the trend and start decreasing the M2 supply.

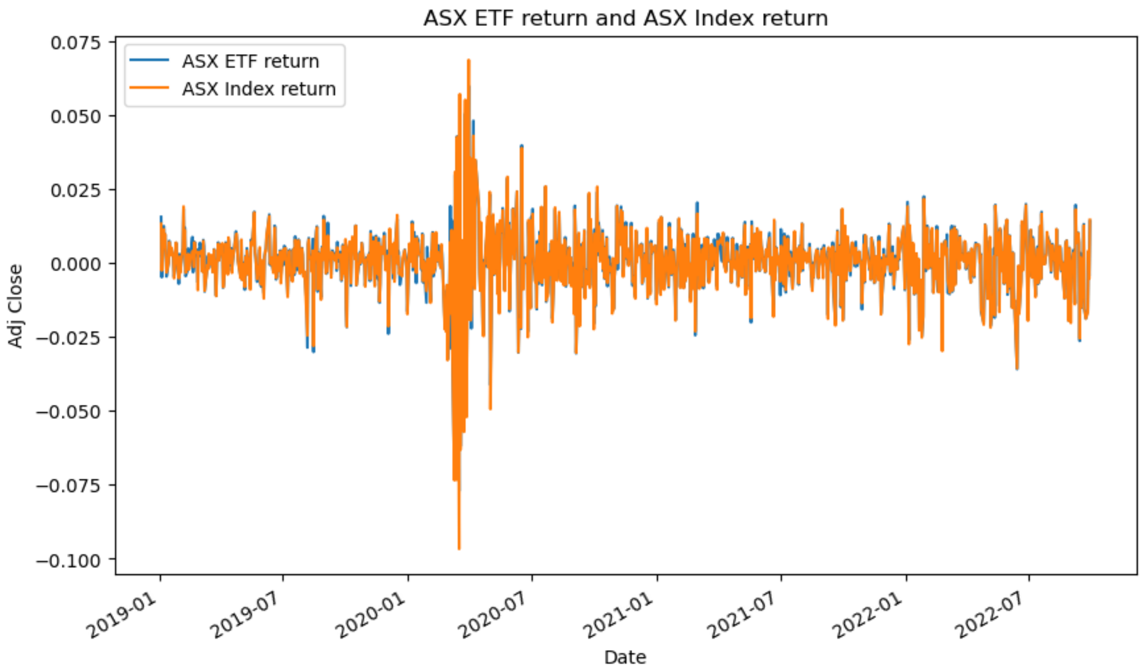

Figure 7: ASX 300 ETF and index price

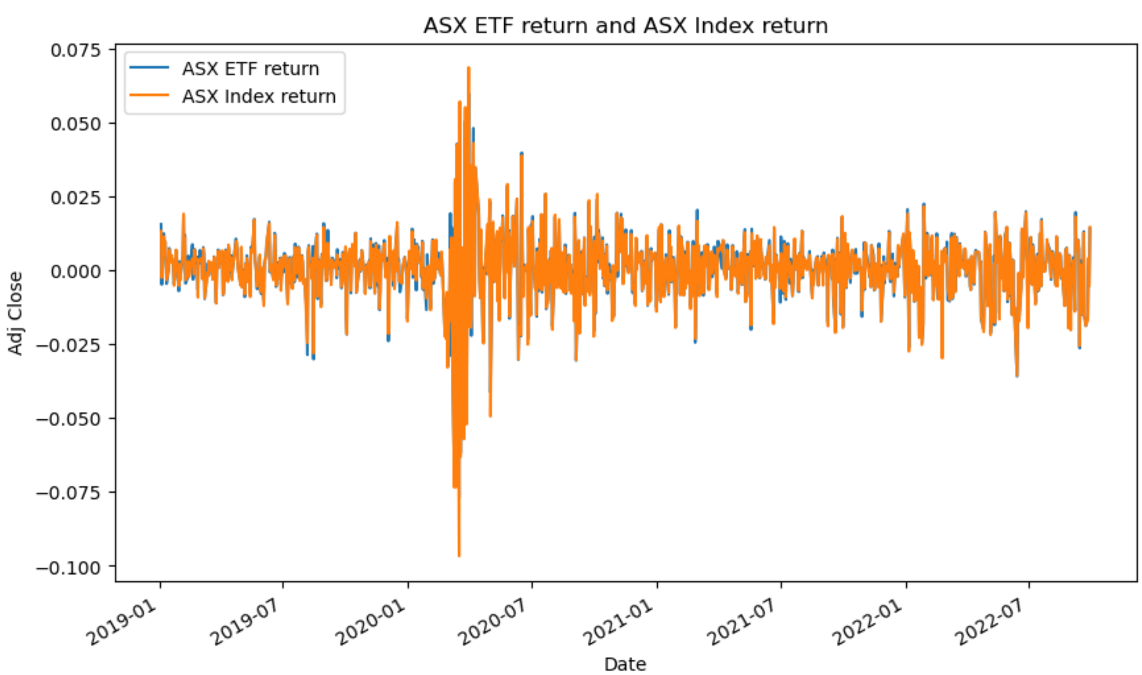

Figure 8: Daily return

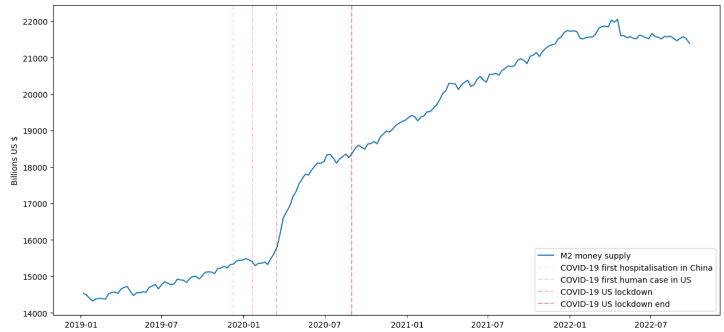

Figure 9: Weekly M2 Money Supply Chart

- May 2022 and after: US FED managed to halt the M2 bullish trend and and signalled a beginning of a flat market period of high interest rates where the US stock market started a deep correction caused by increasing price of US dollars and investors' intention to derisk their portfolios.

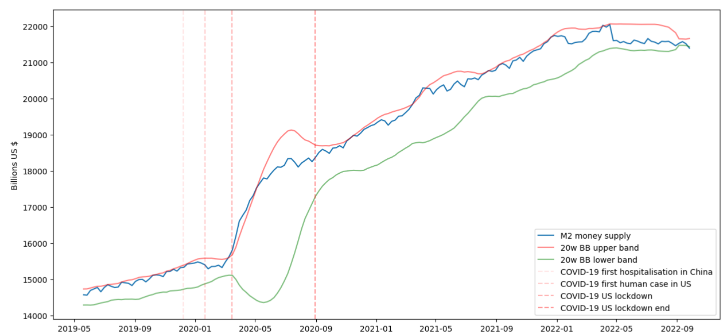

Figure 10: Weekly M2 Money Supply Chart with Bollinger Bands

In order to observe potential regime changes on M2 volatility, we add the Bollinger Bands [4] to the supply chart. The Bollinger Bands, calculated as 2 rolling standard deviations plotted below and above of rolling moving average for bottom and upper bands respectively with 20-step window periods, give a representation of market volatility and its reactions to explosive moves or entering into new regimes.

Figure 10 demonstrates the Bollinger Bands applied to weekly M2 Supply chart. Following the chart, one can see that supply volatility also transitions across 4 regimes.

- January 2019 to January 2020: A consistent low volatility pre-pandemic period without any shocks or strong fluctuations.
- March 2020 to September 2020: A volatility explosion where the supply growth was strong enough to break above the upper band and stay there for 6 weeks straight.
- September 2020 to April 2022: A volatility normalisation where M2 supply was gaining consistency in its growth and the market was certain in its expectation of higher interest rates and higher cost of money each quarter.
- May 2022 and after: An unusual decrease of volatility during prolonged flat money supply period and deep correction of US stock market.

0.2.4 Data Selection for analysis

Now that we have understood the different regime changes and seen proper visualization of the data in our data, we have decided to select the Bitcoin data because :

- We have seen proper evidence of changing volatility and return patterns over time, and to model these non-linear dynamics BTC will perfectly fit.
- BTC data have periods of high volatility followed by calmer periods, as well as shifts in the mean return. Modeling this data can help us identify the different regimes of the market more formally.
- BTC data is highly risky and by modeling this data we can better understand how to deal with these highly volatile risky assets in real-life scenarios, which can help use forecast better returns.

0.2.5 Markov-regime switching model: Base Model

We will develop a base model for our estimation, employing a regime-switching model to analyze the daily fluctuations of BTC-USD returns.

Let's consider the observable BTC-USD process as $y_t = \mu_t + \varepsilon_t$, where ε_t is sampled from a normal distribution with zero mean and variance σ_t^2. Both the conditional mean and variance of the process at each time point are dependent on the outcome of a Markov process with N possible states, represented by a tuple $\{(\mu_i, \sigma_i)_{i=1}^N\}$ and an associated transition matrix P.

We will implement the Hamilton filter process [5], augmented by a smoothing technique that allows us to infer the probability of each state, conditioned on the entire observable process $\mathcal{Y}_T$:

$$\xi_{t|T}(j) = \xi_{t|t}(j) \sum_{i=1}^{N} p_{ji} \frac{\xi_{t+1|T}(i)}{\xi_{t+1|t}(i)} \tag{1}$$

These smoothed probabilities are computed in reverse order, starting with $\xi_{T|T}(j)$ from the forward recursion. This procedure is known as the Kim smoother, as developed by Kim (1993) [6].

Consequently, we can express the log-likelihood function as:

$$\mathcal{L}(\theta) = \sum_{t=1}^{T} log\ f(y_t|\mathcal{Y}_{t-1};\theta) = \sum_{t=1}^{T} log\ \sum_{i=1}^{N} \xi_{t|T}(i) \times \phi_t(i) \tag{2}$$

where $\phi\_t(i) = \phi\left(\frac{y_t - \mu_i}{\sigma_i}\right)$.

Next, we'll introduce a method to reduce the computational complexity of this type of problem: the Expectation-Maximization (EM) algorithm [7].

In the Expectation (E) step, we assume knowledge of the actual path of the unobservable Markov process $\mathcal{S}_t$. Under this assumption:

$$\mathbb{P}(y_t, s_t = i|\mathcal{Y}_{t-1}, \mathcal{S}_{t-1};\theta) = \mathbb{P}(y_t|\mathcal{S}_t;\theta)\mathbb{P}(s_t = i|\mathcal{S}_{t-1};\theta) \tag{3}$$

$$= \phi_t(i) \prod_{j=1}^{N} \mathbb{P}(s_t = i|s_{t-1} = j;\theta)^{\mathbb{I}(s_{t-1}=j;\theta)} \tag{4}$$

$$= \phi_t(i) \prod_{j=1}^{N} p_{ji}^{\mathbb{I}(s_{t-1}=j;\theta)} \tag{5}$$

Here, $\mathbb{I}(s_{t-1} = j;\theta)$ is an indicator variable denoting if state j occurred at $t-1$. The joint log-likelihood of the observations can be expressed as:

$$log\ \mathbb{P}(y_t, s_t = i|\mathcal{Y}_{t-1}, \mathcal{S}_{t-1};\theta) = log\ \phi_t(i) + \sum_{j=1}^{M} log\ p_{ji} \times \mathbb{I}(s_{t-1} = j;\theta) \tag{6}$$

For $t = 1$, where no past history is observed, we estimate or predetermine an initial probability π_j:

$$log\ \mathbb{P}(y_1, s_1 = i;\theta) = log\ \phi_t(i) + log\ \pi_i \tag{7}$$

We can now construct the following log-likelihood function that accounts for all potential realizations of s_t (the "complete" likelihood):

$$\mathcal{L}(\theta;\mathcal{Y}_T,\mathcal{S}_T)=\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbb{I}(s_t=i;\theta)log\ \phi_t(i)+ \tag{8}$$

$$+\sum_{t=2}^{T}\sum_{i=1,j=1}^{N}\mathbb{I}(s_t=i;\theta)\mathbb{I}(s_{t-1}=j;\theta)log\ p_{ji}+ \tag{9}$$

$$+\sum_{i=1}^{N}\mathbb{I}(s_1=i;\theta)log\ \pi_{ij} \tag{10}$$

The crucial step now is to take the expectation of the above expression by replacing the indicators $\mathbb{I}(s_t=j;\theta)$ with the corresponding probabilities estimated earlier, $\xi_{t|T}(j)$. The terms $\mathbb{I}(s_t=j;\theta)\mathbb{I}(s_{t-1}=i;\theta)$ can be substituted with $p_{ij}\xi_{t|t}(i)\xi_{t+1|T}(j)/\xi_{t+1|t}(j)$, derived from each specific component of the Kim filter arising from each state j.

This completes the E step, given an initial guess for the parameter vector $\theta=\{(\mu_i,\sigma_i)_{i=1}^N,(p_{ij})_{i,j=1}^N\}$.

In the Maximization (M) step, instead of optimizing in a single step, we generate a series of parameter estimates $\theta^{(k)}$, such that:

$$\theta^{(k)}=\arg\max_{\theta}E\left[\mathcal{L}(\theta;\mathcal{Y}_T,\mathcal{S}_T)|\mathcal{Y}_T;\theta^{(k-1)}\right] \tag{11}$$

We repeat this process until convergence is achieved.

The assumption of normality allows us to derive analytical expressions for a sequence of estimates. At step k, with the guessed parameter vector $\theta^{(k-1)}$, differentiation of the log-likelihood function with respect to σ_i yields:

$$\frac{\partial\sum_{t=1}^{T}\xi_{t|T}^{(k-1)}(i)\ log\ \phi_{it}}{\partial\sigma_i}=\sum_{t=1}^{T}\xi_{t|T}^{(k-1)}(i)\ \left(\frac{(y_t-\mu_i)^2}{\sigma_i^2}-1\right)=0 \tag{12}$$

Resulting in:

$$\sigma_i^{(k)}=\sqrt{\frac{\sum_{t=1}^{T}\xi_{t|T}^{(k-1)}(i)\left(y_t-\mu_{-1}^{(k)}\right)^2}{\sum_{t=1}^{T}\xi_{t|T}^{(k-1)}(i)}} \tag{13}$$

This expression is the square root of a weighted average of the squared deviations of the process, with weights being the relative estimated probability of state i occurring. Differentiation with respect to $\mu_i^{(k)}$ yields:

$$\sum_{t=1}^{T}\xi_{t|T}^{(k-1)}(i)\frac{y_t-\mu_i^{(k)}}{\left(\sigma_i^{(k)}\right)^2}=0 \tag{14}$$

The corresponding estimates $\mu_i^{(k)}$ are averages weighted by the estimated probability that the process is in state i at each date t:

$$\mu_i^{(k)}=\frac{\sum_{t=1}^{T}\xi_{t|T}^{(k-1)}(i)y_t}{\sum_{i=1}^{M}\xi_{t|T}^{(k-1)}(i)} \tag{15}$$

Lastly, we can demonstrate that the estimates for the transition matrix and the initial distribution are given by:

$$p_{ij}^{(k)}=\frac{\sum_{t=2}^{T}p_{ij}^{(k-1)}\xi_{t-1|t-1}(i)\xi_{t|T}(j)/\xi_{t|t}(j)}{\sum_{t=2}^{T}\xi_{t-1|T}^{(k-1)}(i)} \tag{16}$$

$$\pi_i^{(k)}=\xi_{1|T}^{(k-1)}(i) \tag{17}$$

Note the similarity of the solution $p_{ij}^{(k)}$ to the MLE estimate. With these components in place, we can now establish the optimization algorithm and estimate the parameters.

Base Model result

Using the Expectation-Maximization (EM) algorithm to estimate the parameters, we calculated the probability of being in each state at each time point, given the current parameter estimates.

Let's look at our result of the base model in Figure 11:

```
Iteration:  37
Log-Likelihood:  2576.4965 Change:  0.0001
Final Estimates
Log-Likelihood:  2576.4965 Akaike:  -5132.993 Schwarz:  -5081.0037
Mu:  [0.0002 0.0015]
Sigma:  [0.0686 0.0315]
Transation Matrix:
[0.7664 0.2336]
[0.0379 0.9621]
Intial Probabilities:  [0.0593 0.9407]
```

Figure 11: results: Base Model

Model Convergence & Fit:

Iterations	Log-Likelihood	AIC	BIC
37	2576.4965	-5260.8239	-5208.6202

Table 1: Results with Base Model

- Mu: [0.0001 0.0014]
- Sigma: [0.0682 0.0313]
- Initial Probabilities: [0.0672 0.9328]

Transition Matrix:

$$\begin{bmatrix} 0.7659 & 0.2341 \\ 0.0381 & 0.9619 \end{bmatrix}$$

- State 0: $\mu_0 = 0.0001$, $\sigma_0 = 0.0682$ (High volatility, low return)
- State 1: $\mu_1 = 0.0014$, $\sigma_1 = 0.0313$ (Low volatility, high return)

The transition matrix shows:

- If in State 0 $=$ 76.59% chance of staying in State 0
- If in State 1 $=$ 96.19% chance of staying in State 1

This suggests State 1 (the "good" state) is more persistent.

We use AIC and BIC to assess model fit:

- $AIC = 2k - 2\ln(L)$
- $BIC = k\ln(n) - 2\ln(L)$

Where k is the number of parameters, n is the number of observations, and L is the likelihood. We can the values of AIC and BIC in Table 1, The negative values suggest a good fit, as we're trying to minimize these criteria.

This model suggests that the Bitcoin market alternates between two regimes:

1. A volatile, low-return regime (possibly during market stress or bearish periods)
2. A stable, higher-return regime (possibly during bullish or growth periods)

The high probability of staying in State 1 (96.19%) suggests that the stable, high-return periods tend to persist, but there's always a chance of switching to the more volatile regime.

This model assumes only two regimes. In reality, there might be more. It assumes returns within each regime are normally distributed, which might not always be true for cryptocurrency returns.

To further validate the model, we will:

1. Compare it with a model with a different number of states
2. Allowing the expected realization of the time series to differ across states (different "mus"), but with constant variance (same "sigma").
3. Allowing the variance of the time series to change across states (different "sigmas"), but with constant expectation (same "mu").
4. Allowing for different expectations and variances across states.

By understanding these concepts, you can now interpret Markov regime-switching models applied to various financial time series, not just Bitcoin returns.

i. Different number of states

Now we will change our parameters of states to N = 3 and run the model to see how our model has performed. we can see the results in Figure 12 and the important values in Table 2

```
Iteration:  25
Log-Likelihood:  2577.7089 Change:  0.0001
Final Estimates
Log-Likelihood:  2577.7089 Akaike:  -5119.4177 Schwarz:  -5025.837
Mu:  [0.0008 0.0015 0.0001]
Sigma:  [0.0572 0.0314 0.0698]
Transation Matrix:
[0.0893 0.5242 0.3865]
[0.0069 0.9608 0.0323]
[0.039  0.2137 0.7473]
Intial Probabilities:  [0.008  0.9403 0.0517]
```

Figure 12: Results: Model with different states

Our Results are:

Model Convergence & Fit:

Iterations	Log-Likelihood	AIC	BIC
25	2577.7089	-5119.4177	-5025.837

Table 2: Results with different states

Estimated Parameters:

State-Specific Means (μ):

1. State 1: 0.0008
2. State 2: 0.0015
3. State 3: 0.0001

State-Specific Volatilities (σ):

1. State 1: 0.0572
2. State 2: 0.0314
3. State 3: 0.0698

Transition Probability Matrix (P):

$$\begin{bmatrix} 0.0893 & 0.5242 & 0.3865 \\ 0.0069 & 0.9608 & 0.0323 \\ 0.039 & 0.2137 & 0.7473 \end{bmatrix}$$

Initial State Probabilities (π):

$$\begin{bmatrix} 0.0086 \\ 0.9403 \\ 0.0517 \end{bmatrix}$$

Interpretation:

States Characteristics:

- State 1: Moderate positive mean (0.0008) with moderate volatility (0.0572)
- State 2: Highest positive mean (0.0015) with the lowest volatility (0.0314)
- State 3: Slight positive mean (0.0001) with the highest volatility (0.0698)

Transition Probabilities:

- State 1 is least persistent (9.33% chance of staying)
- State 2 is most persistent (96.08% chance of staying)
- State 3 has moderate persistence (74.48% chance of staying)

Initial Probabilities:

- The model suggests that State 2 is the most likely initial state (93.32% probability)

Model Fit:

- The negative AIC and BIC suggest a good fit, with the model explaining the data well while accounting for its complexity.

This model identifies three distinct regimes in the financial time series, potentially corresponding to periods of moderate growth with moderate risk, high growth with low risk, and slight decline with high risk.

You can see more details in the Colab notebook provided with the report.

Now let's move and analyze a different model that allows the expected realization of the time series to differ across states (different "μs"), but with constant variance (same "σ").

ii. different mus but constant variance sigma

when σ is constant, we have the following FOC:

$$\frac{\partial \sum_{t=1}^{T} \xi_{t|T}^{(k-1)}(i)\ log\ \phi_{it}}{\partial \sigma_i} = \sum_{i=1}^{N} \sum_{t=1}^{T} \xi_{t|T}^{(k-1)}(i)\ \left(\frac{(y_t - \mu_i)^2}{\sigma_i^2} - 1 \right) = 0 \tag{18}$$

Rearranging and nothing that $\sum_{i=1}^{N} \xi_{t|T}^{(k-1)}(i)\ = 1$ we get :

$$\sum_{i=1}^{N} \sum_{t=1}^{T} \xi_{t|T}^{(k-1)}(i)\ (y_t - \mu_i)^2 = \sigma_i^2 \sum_{i=1}^{N} \sum_{t=1}^{T} \xi_{t|T}^{(k-1)}(i) = \sigma_i^2 T \tag{19}$$

The new estimate for σ^k will be:

$$\sigma_i^{(k)} = \sqrt{\frac{\sum_{i=1}^{N} \sum_{t=1}^{T} \xi_{t|T}^{(k-1)}(i) \left(y_t - \mu_i^{(k)} \right)^2}{T}} \tag{20}$$

```
Iteration:  3
Log-Likelihood:  2450.0612 Change:  -0.0
Final Estimates
Log-Likelihood:  2450.0612 Akaike:  -4882.1225 Schwarz:  -4835.3321
Mu:  [0.0013 0.0013]
Sigma:  0.0388
Transation Matrix:
[0.0175 0.9825]
[0.0068 0.9932]
Intial Probabilities:  [0.007 0.993]
```

Figure 13: Results: Model with different mus but same Sigma

Iterations	Log-Likelihood	AIC	BIC
3	2450.0612	-4882.1225	-4835.3321

Table 3: Results of Different same sigma

with the new estimates, we will again define our estimate function and change the necessary parameters i.e different "mus" we will keep mus [-0.3, 0.2] and sigma = 0.1

Now let's analyze our result from Figure 13 and Table 3:

Estimated Parameters:

State-Specific Means (μ):

- **State 1**: 0.0013
- **State 2**: 0.0013

Shared Volatility (σ):

- σ: 0.0388

Transition Probability Matrix (P):

$$\begin{bmatrix} 0.0175 & 0.9825 \\ 0.0068 & 0.9932 \end{bmatrix}$$

Initial State Probabilities (π):

$$\begin{bmatrix} 0.007 \\ 0.993 \end{bmatrix}$$

Interpretation:

1. **States Characteristics**:
    - **State 1**: Mean return (0.0013)
    - **State 2**: Mean return (0.0013)
    - **Volatility**: Both states share the same volatility (0.0388)
2. **Transition Probabilities**:
    - **State 1**: 1.75% chance of staying in the same state
    - **State 2**: 99.32% chance of staying in the same state
    - Very high persistence in State 2.
3. **Initial Probabilities**:
    - Very high probability (99.3%) of starting in State 2.
4. **Model Fit**:
    - The negative AIC and BIC suggest a good fit.

This two-state model identifies two distinct regimes with the same mean returns and shared volatility. The model shows very high persistence in State 2, with a much higher probability of starting in State 2.

iii. different sigma but constant mu

Another experiment would be to modify the model to have different variances for each state while maintaining a constant expectation across states. The main change made to allow it is that we now calculate a single sample mean for all states while estimating separate sample deviation values for each state. We pick $\hat{\mu} = 0.0$ as the starting guess value. Making $\hat{\mu}$ constant across the states means assuming that different regimes in the time series do not differ in generated returns but demonstrate different levels of volatility explained by price fluctuations, shocks, and rapid moves in either direction.

Let's see the results of this model from Figure 14:

```
Iteration:  36
Log-Likelihood:  2575.9751 Change:  0.0001
Final Estimates
Log-Likelihood:  2575.9751 Akaike:  -5133.9503 Schwarz:  -5087.1599
Mu:  0.0013
Sigmas:  [0.0315 0.0685]
Transition Matrix:
[0.9618 0.0382]
[0.2358 0.7642]
Initial Probabilities:  [0.9404 0.0596]
```

Figure 14: Results: Model with different sigma but constant mu

Table 4 describes AIC, BIC, and Log-Likelihood metrics after model convergence hits on the 36th iteration. Deeply negative AIC and BIC suggest a good fit.

Iterations	Log-Likelihood	AIC	BIC
36	2575.9751	-5133.95	-5087.15

Table 4: Results of fitting a model with constant expectation and dynamic variance

Estimated Parameters:

Shared mean (μ)

- μ: 0.0013

State-specific Volatility (σ):

- **State 1**: 0.0315
- **State 2**: 0.0685

Transition Probability Matrix (P):

$$\begin{bmatrix} 0.9618 & 0.0382 \\ 0.2358 & 0.7642 \end{bmatrix}$$

Initial State Probabilities (π):

$$\begin{bmatrix} 0.9404 \\ 0.0596 \end{bmatrix}$$

Observed parameters reveal two potential states of lower and higher volatility. The lower volatility state is much more likely to appear as an initial state with 94% probability, and the price returns tend to remain in this state longer with only a 3.8% chance of switching into a high volatility regime.

iv. Allowing for different expectations and variances across states

For the last model experiment, we return to an assumption of $\hat{\sigma}$ and $\hat{\mu}$ varying between the states. We are also increasing the potential number of states to $N = 4$ to challenge the existence of other market conditions, different from the ones detected in the previous steps. Figure 15 Table 5 shows an earlier convergence after 21 steps resulting in heavily negative AIC and BIC values.

```
Iteration:  21
Log-Likelihood:  2577.7793 Change:  0.0001
Final Estimates
Log-Likelihood:  2577.7793 Akaike:  -5099.5586 Schwarz:  -4953.9885
Mu:  [0.0011 0.0015 0.0012 0.0001]
Sigma:  [0.0453 0.0313 0.0433 0.0698]
Transation Matrix:
[0.0121 0.8092 0.1    0.0787]
[0.0023 0.9557 0.0115 0.0305]
[0.0043 0.5231 0.0709 0.4017]
[0.0045 0.2139 0.02   0.7615]
Intial Probabilities:  [0.0024 0.9325 0.0128 0.0523]
```

Figure 15: Results: Dynamic expectation and variance

Iterations	Log-Likelihood	AIC	BIC
21	2577.77	-5099.55	-4953.98

Table 5: Results of fitting a dynamic expectation and variance model with 4 steps

Estimated Parameters:

State-specific mean (μ)

- **State 1**: 0.0011
- **State 2**: 0.0015
- **State 3**: 0.0012
- **State 4**: 0.0001

State-specific Volatility (σ):

- **State 1**: 0.0453
- **State 2**: 0.0313
- **State 3**: 0.0433
- **State 4**: 0.0698

Transition Probability Matrix (P):

$$\begin{bmatrix} 0.0121 & 0.8092 & 0.1 & 0.0787 \\ 0.0023 & 0.9557 & 0.0115 & 0.0305 \\ 0.0043 & 0.5231 & 0.0709 & 0.4017 \\ 0.0045 & 0.2139 & 0.02 & 0.7615 \end{bmatrix}$$

Initial State Probabilities (π):

$$\begin{bmatrix} 0.0024 \\ 0.9325 \\ 0.0128 \\ 0.0523 \end{bmatrix}$$

Given that the returns process spends only the least time in states 1 and 3 and how unlikely they are to appear, we can simply the model and determine that the process has only two major states: state 2 describing high returns and low volatility, and state 4 supported by low log-returns and high volatility. States 1 and 3 are transition states and can be removed to keep our model simple and interpretable.

Step 3

a. The model with different mus and same variance

We implemented a Markov regime-switching approach where the mean (μ) can vary across different regimes while maintaining constant volatility (σ) throughout all states. This model configuration is beneficial for capturing shifts in the central tendency of the data without assuming changes in its variability. log-likelihood of 2450.0612 and the AIC-BIC -4882.1225, -4835.3321 indicates the model's goodness of fit to the observed data. This value quantifies how well the model explains the observed data patterns but if we compare it with a model that has dynamic expectations and dynamic variances it has not performed very well the model with dynamic expectations and dynamic variances log-likelihood function is 2577.77 and the AIC-BIC is -5099.55, -4953.98 shows that the model which has both dynamic expectations and dynamic variances work far better than the model which has only different expectations. by the results, we can interpret that our model might be effective in scenarios where the primary difference between regimes is a shift in levels and the underlying volatility of the system remains relatively stable across different states.

b. The model with different Sigmas and the same mus

In this model, we have used a different approach where our variance (σ) can vary across the different regimes but our mean(μ) will be constant, this configuration will not capture the shift of the central tendency but it will capture the variability of data, this is why it is our best-performing model it results are log-likelihood of 2575.9751 and the AIC-BIC -5133.95, -5087.15 which signifies a model's goodness of fit, and if we compare it with our model which also has different values of sigma σ with different mean μ which have results log-likelihood of 2577.77 and the AIC-BIC -5099.55, -4953.98, we can interpret by these results that the model with different sigmas and same mu perform better and can be effect under volatile data like BTC-USD.

c. The model with dynamic expectations and dynamic variances

As we mentioned in the previous section, the model utilizing dynamic expectations along with dynamic variances is superior compared to alternatives of keeping static expectations and static variances, dynamic expectations and static variances, and static expectations and dynamic variances. The model reaches the highest Log-Likelihood of 2577.8 and the AIC-BIC -5099.55, -4953.98, compared to alternative models with the same amount of states. That means that the underlying process of BTC-USD returns undergoes the regime change in both the means and volatility. The only alternative surpassing this model is the dynamic expectations and dynamic variances model with 3 states from section 2. a.

d. Model Ranking: Best to Worst

Now we will rank all the models from best to worst performing in Table 6:

Model	Log-likelihood	AIC	BIC	AIC Rank	BIC Rank
Model 2: Different σ, same μ	2575.9751	-5133.95	-5087.15	1	1
Model 3: Dynamic μ and σ	2577.77	-5099.55	-4953.98	2	3
Model 1: Different μ, same σ	2450.0612	-4882.1225	-4835.3321	3	2

Table 6: Ranking of Markov Regime Switching Models

Model Descriptions:

- **Model 2 (Best overall):** This model assumes different volatilities (σ) across regimes but the same mean (μ). It performs best according to both AIC and BIC, suggesting it provides the best balance between model fit and complexity.
- **Model 3:** This model allows both mean (μ) and volatility (σ) to vary across regimes. It has the highest log-likelihood, indicating the best fit to the data, but is penalized more heavily by AIC and especially BIC due to its higher complexity.
- **Model 1:** This model assumes different means (μ) but the same volatility (σ) across regimes. It performs worst in terms of AIC but ranks second in BIC, suggesting it might be preferred over Model 3 when prioritizing model simplicity.

Final Conclusion: Model 2, with different volatilities but the same mean across regimes, appears to be the best choice based on both AIC and BIC criteria. This suggests that capturing regime-dependent volatility is more important than the regime-dependent mean for this particular dataset.

Step 4

In a Markov-regime switching model with autoregressive coefficient changes, the autoregressive parameter of the time series process is allowed to vary depending on the underlying state or regime. This extension allows for more flexible modeling of time series data that exhibit different dynamic behaviors across different economic or market conditions.

0.2.6 Formulation

Following Hamilton [5] (1994, section 22.4), we can formulate the model as follows:

$$y_t = \mu_{s_t} + \phi_{s_t}(y_{t-1} - \mu_{s_{t-1}}) + \epsilon_t \tag{21}$$

where:

- y_t is the observed time series at time t
- s_t is the unobserved state at time t
- μ_{s_t} is the mean of the process in state s_t
- ϕ_{s_t} is the autoregressive coefficient in state s_t
- $\epsilon_t \sim N(0, \sigma^2_{s_t})$ is the error term with state-dependent variance

And the state s_t follows a Markov chain process with transition probabilities of:

$$P(s_t = j | s_{t-1} = i) = p_{ij} \tag{22}$$

Compared to the previous Markov regime-switching model, it allows for different dynamic behaviors in each state, capturing more complex patterns in the data, while changes in autoregressive coefficients can introduce changes in the reversion speed of the process across different economic regimes. On the other hand, the model has increased complexity in terms of a higher amount of parameters to estimate, which can lead to overfitting and high variance of predictions. While the model assumes linear autoregressive dynamics within each regime, it may not be always the case.

0.2.7 Model parameters estimation

Based on our previous conclusions we have decided to estimate parameters for a 3-state autoregressive Markov regime-switching model with dynamic expectations and variances.

Using the Expectation-Maximization (EM) algorithm to estimate the parameters, we calculated the probability of being in each state at each time point, given the current parameter estimates. The table 7 shows the final metrics over parameters estimation.

Model Convergence & Fit:

Iterations	Log-Likelihood	AIC	BIC
23	2577.0021	-5118.0042	-5024.4234

Table 7: Results with Autoregressive dynamic Markov regime switching model

0.2.8 Results

- Phi (AR coefficients): [-0.0203 -0.0762 -0.1315]
- Sigma: [0.0314 0.051 0.07]
- Initial Probabilities: [0.9321 0.019 0.0489]

Transition Matrix (P):

$$\begin{bmatrix} 0.956 & 0.0137 & 0.0303 \\ 0.4528 & 0.3524 & 0.1948 \\ 0.2089 & 0.0461 & 0.7451 \end{bmatrix}$$

The model demonstrates a very high probability of 93.2% of starting in State 1 with persisting in it with the probability of 95.6%. The States 2 and 3 are unlikely to be initial states with π equal to 1.9% and 4.9% respectively, with only the later one to be persistent with the probability of 74.5%, while State 2 appears to be a transition state with a higher probability of transition into States 1 or 3 instead of persisting. Figure 16 demonstrates probabilities of being in each of the three states during the overall returns process.

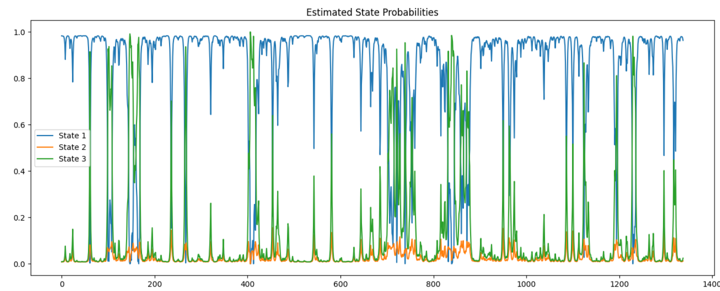

Figure 16: Results: Markov regime-switching model with autoregressive coefficient changes states probabilities

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