27

Differentiation of Complex Function

$$f'(x) \quad \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

$$\lim_{x\to a^+} \frac{f(x+h)-f(x)}{h} = \lim_{x\to a^-} \frac{f(x+h)-f(x)}{h}$$

$w = f(z)$

$z = x + iy$

$z \to z_0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

when, $\Delta z \to 0 \Rightarrow \Delta x + i\Delta y \to 0$

First along x axis $\to$ then y axis

Q: Show that the function $f(z) = x + i4y$ is not differentiable at any point.

$$f(z+\Delta z) = (x+\Delta x) + i4(y+\Delta y)$$

$$\therefore \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\Delta x + i4\Delta y}{\Delta x + i\Delta y}$$

Assume $\Delta z \to 0$ parallel to the x axis $\Rightarrow$ consequently $\Delta y = 0$.

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\Delta x}{\Delta x} = 1.$$

Now $\Delta z \to 0$ parallel to y axis $\Rightarrow$ consequently $\Delta x = 0$

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{i4\Delta y}{i\Delta y} = 4.$$

As both values are different, the function is not differentiable.

Example: $f(z) = z^2 - 5z$.
$f'(z)$, if possible.

$$f'(z) = \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f(z+\Delta z) = (z+\Delta z)^2 - 5(z+\Delta z)$$
$$= z^2 + (\Delta z)^2 + 2z\Delta z - 5z - 5\Delta z$$

$$f(z+\Delta z) - f(z) = (\Delta z)^2 + 2z(\Delta z) - 5\Delta z$$

Q. $f(z) = 3z^4 - 5z^3 + 2z$

Q. $f(z) = z^2/4z+1$

Q. $f(z) = (z^2+3z)^5$

Analytic Function

A complex function $w = f(z)$ is said to be analytic at a point z_0 if it is differentiable there and it is also differentiable at every point in neighborhood of z_0.

Entire Function

A function which is analytic at every point in the complex plane, then it is known as entire function.

Singular Point

A singular point is a point at which the complex function $w = f(z)$ fails to be analytic.

Analyticity implies:- or

Every analytic function is differentiable but the converse is not true.

Note 1:

Analyticity at a point is not same as differentiability at a point.

Example: $f(z) = |z|^2$

$z = 0$

Here the given function is differentiable but it is not analytic at $z = 0$.

Every polynomial function is a entire function.
Every exponential function is also an entire function.
Every rational function $f(z) = \frac{P(z)}{Q(z)}$ if $Q(z) \neq 0$.

Example:

$f(z) = \frac{z}{z^2+1}$ - if the function analytic or not.
- YES
- NO — find singular points.

How to find S Points?

$Q(z) = 0 \rightarrow z^2+1 = 0$

$z = \pm i$

1. The algebraic operation of two analytic function is also analytic.

2. Division of two analytic function is also analytic if the denominator function is nonzero.

3. If a function f is differentiable at a point z_0 in a domain D then f is continuous there. but the converse may not be true.

Example-1 $f(z) = \frac{1}{z}$. Check whether the function is differentiable or not. If it is, find its derievative also.

Example-2 $f(z) = |z|^2$.

(1) Diff at $z=0$.

(2) Not Diff at $z \neq 0$.

87

Ans-1. As nothing is mentioned about z, $f(z)$ is not differentiable.

$\Delta z = 0$, not differentiable.

$\Delta z \neq 0$, Differentiable. → then Rational → analytic

↓

differentiable ✓.

Ans-2: (i) $f(z) = |z|^2 = z \cdot \bar{z}$

$f(z+\Delta z) = |z+\Delta z|^2 = z^2 + \Delta z$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}.$$

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$z = x + iy \quad z + \Delta z = (x + \Delta x) + i(y + \Delta y)$.

$$f(z + \Delta z) = [(x+\Delta x) + i(y+\Delta y)]\,[(x+\Delta x) - i(y+\Delta y)]$$

$$= (x+\Delta x)^2 - i(x+\Delta x)(y+\Delta y) + i(y+\Delta y)(x+\Delta x) + (y+\Delta y)^2$$

$$f(z) = (x+iy)(x-iy)$$
$$= x^2 + y^2.$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(x+\Delta x)^2 + (y+\Delta y)^2}{x^2+y^2}$$

Ans: $f(z) = |z|^2$

$$f(z+\Delta z) = |z+\Delta z|^2 = |z|^2 + 2|z||\Delta z| + (|\Delta z|)^2$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{2|z||\Delta z| + |\Delta z|^2}{\Delta z} = 2|z| + |\Delta z| = 2|z|$$

Not differentiable at $z \neq$

differentiable at z

Cauchy-Riemann Equation (C-R Eqn):

Let $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x$

Then the first order differentiation of u and v exists and satisfy

CR equation.

First order differentiation exists, it implies that:

$\frac{\partial u}{\partial x}$ exist, $\frac{\partial u}{\partial y}$ exist, $\frac{\partial V}{\partial x}$ exist, $\frac{\partial V}{\partial y}$ exist.

Relation between them is given as:

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial V}{\partial x}$$

$$f = u + iV$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial V}{\partial x}$$

If any complex function satisfies the Cauchy Riemann Equation then it is known as a analytic function.

$$f = z = x + iy$$

$$u(x,y) \quad V(x,y)$$

Q. Verify CR equation for the function $f(z) = z^2 + z$.

$$f(z) = (x+iy)^2 + (x+iy)$$

$$= x^2 - y^2 + 2xyi + x + iy.$$

$$u = x^2 - y^2 + x \qquad V = 2xy + y.$$

$\frac{\partial u}{\partial x} = 2x+1$ $\frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$ $\frac{\partial v}{\partial y} = 2x+1$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

As it satisfies CR eq, given function is analytic

The necessary and sufficient condition for analyticity:

If any complex function f satisfies C-R eq, then the function is said to be analytic. If it fails, the function is not analytic.

Q. $f(z) = (2x^2+y) + i(y^2-x)$. Check whether the given function is analytic or not.

$u(x,y) = 2x^2+y$ $v(x,y) = y^2-x$ $\frac{\partial v}{\partial x} = -1$

$\frac{\partial u}{\partial x} = 4x$

$\frac{\partial v}{\partial y} = 2y$

$\frac{\partial u}{\partial y} = 1$

As $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, the function is not analytic.

Q. $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$. The given function is analytic e[illegible] the point $z = 0$. Prove it.

When $z = 0$ $(x=0, y=0)$ function undefined $\rightarrow$ not analytic.

When $z \neq 0$

$$\frac{\partial u}{\partial x} = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

Differentiation of analytic functions is also derievable and hence continuous.

Harmonic Equation

$f(z) = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$u(x,y)$, $v(x,y)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \; , \; \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

equal as f is continuous.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; ; \; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

If for any Complex function, then we can say the function is harmonic.

Every analytic function is harmonic.

If any function f is harmonic, then u and v are called conjugate to each other.

Q. $f(z) = x^2 - y^2 + i\,2xy$.

$\frac{\partial U}{\partial x} = 2x$ $\qquad$ $\frac{\partial U}{\partial y} = -2y$.

$\frac{\partial^2 U}{\partial x^2} = 2$ $\qquad$ $\frac{\partial^2 U}{\partial y^2} = -2$.

$\frac{\partial V}{\partial x} = 2y$ $\qquad$ $\frac{\partial V}{\partial y} = 2x$

$\frac{\partial^2 V}{\partial x^2} = 0$. $\qquad$ $\frac{\partial^2 V}{\partial y^2} = 0$

↳ Harmonic, also analytic.

Q. $f(z) = U + iV$

$U = x^2 - y^2$

$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$; $\frac{\partial V}{\partial y} = 2x$, $V = 2xy + c$.

$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$; $\frac{\partial V}{\partial x} = -(-2y)$ $V = 2xy + c$

Q. $U(x,y) = x^3 - 3xy^2 - 5y$.

F. Verify the function u is harmonic and find its harmonic conjugate

A: $\frac{\partial U}{\partial x} = 3x^2 - 3y^2$ $\qquad$ $\frac{\partial U}{\partial y} = -6xy - 5$

$\frac{\partial^2 U}{\partial x^2} = 6x$ $\qquad$ $\frac{\partial^2 U}{\partial y^2} = -6x$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \checkmark \text{ Harmonic.}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad ; \quad \partial v = 3x^2y - \frac{3y^3}{3}.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6xy - 5 \qquad \partial v = \frac{6x^2y}{2} + 5x.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2\,\partial V = (3x^2 - 3y^2)\,\partial y + (6xy + 5)\,\partial x$$

$$V = \frac{1}{2}\left[3x^2y - \cancel{3}\frac{y^3}{3} + 3x^2y + 5x\right]$$

$$V = 3x^2y - \frac{y^3}{3\times\cancel{2}} + \frac{5}{2}x$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -(-6xy - 5)\,dx + 3(x^2 - y^2)\,dy$$

$$V = 3x^2y + 5x - y^3 + C.$$

$$\partial V = \underbrace{(3x^2 - 3y^2)\,\partial y}_{\sim dy} + \underbrace{(6xy + 5)\,\partial x}_{-M dx}$$

$$V = 3x^2y + 5x - y^3 + C.$$

Orthogonal Family of Curves:

Suppose $f = u + iv = u(x,y) + iv(x,y)$ is analytic in a domain D. Then the real and imaginary part of f can be used to define two families of curves. One is $u(x,y) = c_1$ and $v(x,y) = c_2$

where c_1 and c_2 are arbitrary constants, known as level curve of U and V respectively.

If the two are ~~orgono~~ orthogonal, complex combination of these will also be orthogonal.

(*)

Harmonic Function

$f(z) = u + iv$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$u(x,y)$
(harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

* Show that $U(x,y) = x^2 + y$ cannot be the real part of any analytic function. Q1.(c)

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = 1$ $\quad \frac{\partial^2 u}{\partial y^2} = 0$

$\times$ harmonic.

As every analytic function is harmonic, it is not analytic.

→ $f(x,y) = 2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4$.
Show that the function is harmonic and find its conjugate harmonic function.

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$2y^3 - 6x^2y + 4x^2 - 7xy - 4y^2 + 3x + 4y - 4$.

$\frac{\partial f}{\partial x} = -12xy + 8x - 7y + 3$. $\frac{\partial f}{\partial y} = 6y^2 - 6x^2 - 7x - 8y + 4$.

$\frac{\partial^2 f}{\partial x^2} = -12y + 8$ $\frac{\partial^2 f}{\partial y^2} = 12y - 8$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \rightarrow$ Hormonic.

$\frac{\partial F}{\partial y} = -12xy + 8x - 7y + 3$

$\frac{\partial F}{\partial x} = -6y^2 + 6x^2 + 7x + 8y - 4$.

$\partial F = (-12xy + 8x - 7y + 3)\,\partial y - (-6y^2 + 6x^2 + 7x + 8y - 4)\,\partial x$.

$\int M\,dx + \int N\,dy$

$F = -\frac{12x^2y}{2} + \frac{8x^2}{2} - 7yx + 3x + \frac{6y^3}{3} + \frac{8y^2}{2} + C$

$V = -6x^2y + 4x^2 - 7xy + 3x + 2y^3 + 4y^2 + C$

$F = 6xy^2 - \frac{6x^3}{3} - \frac{7x^2}{2} - 8xy + 4x - \frac{7y^2}{2} + 3y$.

[Ans $f = 2x^3 + \frac{7x^2}{2} - 6xy^2 + 8xy - 4x - \frac{7}{2}y^2 + 3y$] ✓

$$dV = \int\left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy\right] + C$$

assuming y constant. (x-free)

Method 2: obtain $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}$

Now Let $V(x,y)$ be the conjugate:

$$\frac{\partial V}{\partial x} = -\frac{\partial f}{\partial y} = -6y^2+6x^2+7x+8y-4.$$

$$V = -6y^2x + \frac{6x^3}{3} + \frac{7x^2}{2} + 8xy - 4x + \phi(x)$$

$$V = -6xy^2 + 2x^3 + \frac{7}{2}x^2 + 8xy - 4x + \phi(y)$$

where ϕ is an arbitrary function depending on y.

(integration w.r.t x, get ϕ as func. of y).

Now $\frac{\partial V}{\partial y} = -12xy + 8x + \phi'(y)$

Now $\frac{\partial V}{\partial y} = -12xy + 8x - 7y + 3$

$\therefore \phi'(y) = -7y + 3$

$\phi(y) = -\frac{7y^2}{2} + 3y$

$\therefore V = -6xy^2 + 2x^3 + \frac{7}{2}x^2 + 8xy - 4x - \frac{7}{2}y^2 + 3y$ Ans.

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Ex-3. $U = xy$.

$$\frac{\partial U}{\partial x} = y. \qquad \frac{\partial U}{\partial y} = x$$

$$\frac{\partial^2 U}{\partial x^2} = 0. \qquad \frac{\partial^2 U}{\partial y^2} = 0.$$

Harmonic.

Now $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = y$. $\qquad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} = -x$.

$$\therefore V = \frac{y^2}{2} + \phi(x).$$

$$\frac{\partial V}{\partial x} = 0 + \phi'(x).$$

$$\therefore \phi'(x) = -x$$

$$\phi(x) = \frac{-x^2}{2}.$$

$$\therefore V = \frac{y^2}{2} + \left(\frac{-x^2}{2}\right) = \frac{1}{2}(y^2 - x^2) + C$$

Ex-4. $U = x^3 - 3xy^2$.

$(x-iy)^3$

$x^3 - i^3y^3$

$-3xyi(x-$

$$\frac{\partial U}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial U}{\partial y} = -6xy$$

$$\frac{\partial^2 U}{\partial x^2} = 6x \qquad \frac{\partial^2 U}{\partial y^2} = -6x$$

Harmonic.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2$. $\qquad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 6xy$.

$v = 3x^2 y - \frac{3y^3}{3} + \phi(x)$

$= 3x^2 y - y^3 + \phi(x)$.

$\frac{\partial v}{\partial x} = 6xy + \phi'(x)$.

$\therefore \phi'(x) = 0$

$\phi(x) = C$.

$\therefore v = \underline{3x^2 y - y^3 + C}$.

Q Prove that $u = x^3 - 3xy^2$ is a harmonic function. Determine its harmonic conjugate and then find the corresponding analytic function. Write the function $f(z)$ as a function of z.

$f(z) = u + iv$

$= (x^3 - 3xy^2) + i[3x^2 y] - iy^3 + C_1 \rightarrow$ imag. constant

$= (x+iy)^3 + C_1$

$= z^3 + C_1$

$\hookrightarrow$ is terms of z.

Q. Write in terms of z.

$f(z) = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right)$

$= \underline{2xy} \pm i(y^2 - x^2) = \frac{x^2 - y^2 + 2xyi}{2i} = \frac{(x+iy)^2}{2i} = \frac{z^2}{2i} = -i\frac{z^2}{2}$

Method 3: Millnésr - Thompson Method

$U_x = 3x^2 - 3y^2$

$U_y = -6xy$

Consider $u = x^3 - 3xy^2$.

Let V be the conjugate harmonic function
dynamic function
and $f(z) = u + iv$

$f'(z) = U_x + iV_x = U_x + i(-U_y) = U_x - iU_y$ [CR Theorem]

$U_x = V_y$

Let $\phi_1(x,y) = U_x$

$\phi_2(x,y) = -U_y$

$U_x + iV_x$ $V_y + iV_x$

$U_x + i(-U_y) \cdot x = z$, $y = 0$

$f'(z) = \phi_1(z,0) + \phi_2(z,0) = 3z^2 + i \cdot 0$

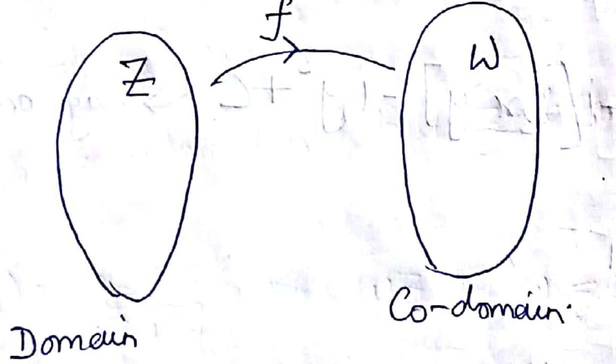
$f(z) = z^3 + c$

$3x^2 - 6xy$

$\frac{3z^3}{3} = z^3$

Complex Mapping

$w = f(z)$

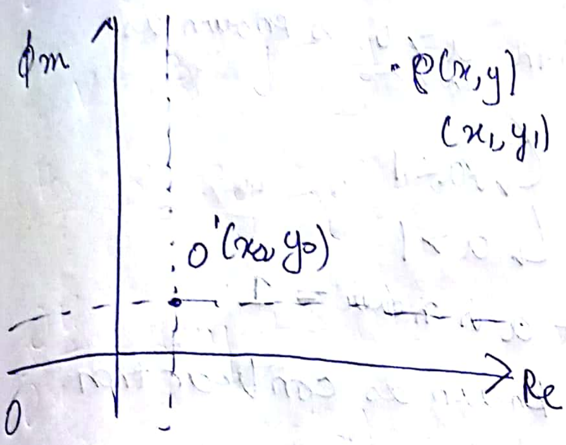


$(z, f(z))$
↳ 4 dimensions.

Linear mapping:

- ↳ Translation (T)
- ↳ Rotation (R)
- ↳ Magnification (M)

Magnification (M)



$x = x_1 + x_0$.
$y = y_1 + y_0$.
$x + iy = (x_0 + iy_0) + (x_1 + iy_1)$.

$\hookrightarrow T(z) = a + z \quad a \neq 0$.

Q. Find the image S' described of the square S with vertices

$1+i$
$2+i$
$2+2i$
$1+2i$ under the linear mapping $T(z) = z + 2 - i$

A: $\begin{array}{r} 1+i \\ +2-i \\ \hline 3 \end{array} \quad \begin{array}{r} 2+i \\ +2-i \\ \hline 4 \end{array} \quad \begin{array}{r} 2+2i \\ 2-i \\ \hline 4+i \end{array} \quad \begin{array}{r} 1+2i \\ 2-i \\ \hline 3+i \end{array}$

→ A complex linear function of the form $R(z) = az$ with $|a| = 1$ is known as rotation linear mapping.

Q Find the image for the real axis under the real mapping:

$$R(z) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) z.$$

Here $|a| = 1$.

Now find the argument of a:

$\theta = \tan^{-1}(1) = \pi/4$.

→ A complex linear function $f(z) = az$ with $a > 0$ is known as magnification.

$\hookrightarrow 0 < a < 1$

$\hookrightarrow a > 1$

but a is never $= 1$.

When $0 < a < 1$, then the magnification is known as contraction.

Bilinear Transformation

Mapping linear in both z and w → bilinear transformation.

685

$w = f(z)$

$$T(z) = \frac{az+b}{cz+d}.$$

If a, b, c, d are complex constants with $ad - bc \neq 0$ then the complex mapping $T(z) = \dfrac{az+b}{cz+d}$ is known as linear fractional transformation.

Mobias transformation / Bilinear transformation / Linear — Transfor

$$w = \frac{az+b}{cz+d} \Rightarrow cwz + dw = az + b$$

$ad - bc \neq 0$.

Q. Show that the unit circle $|z| = 1$ has a image under the mapping $f(z) = \dfrac{2z+1}{z-i}$

A: $a=2,\ b=1$
$c=1,\ d=-i$. $\quad ad-bc = 2i-1 = -2i-1$

$$w = T(z) = \frac{2z+1}{z-i}.$$

$$w^2 = \frac{4z^2+4|z|+1}{z^2-2|z|i-1} = \frac{9}{-2i}$$

$$u^2+v^2 = -\frac{9}{2i}.$$

Imp

Ex: Show that under $w = \frac{1}{z}$, the circle given by $|z-3| = 5$ is mapped into the circle $\left|w+\frac{3}{16}\right| = \frac{5}{16}$.

Eliminate z from the given equation:

$w = 1/z \Rightarrow z = 1/w.$

$|z-3| = 5$

$\left|\frac{1}{w} - 3\right| = 5$

$|1-3w| = 5|w|$

$|(1-3u) - 3iv| = 5|u+iv|$

$(1-3u)^2 + (3v)^2 = 25(u^2+v^2)$

$(1-3u)^2 + 9v^2 = 25u^2 + 25v^2$

$1+9u^2-6u+9v^2$

$= 16u^2+16v^2$ $\quad \left|w+\frac{3}{16}\right| = 5/16.$

$1-6u$

$= u^2+v^2.$ $\quad \cancel{2\times\frac{3}{16}\times u}$

$\frac{1}{16} - \frac{6u}{16}$

$u^2 - \frac{1}{16} + \frac{6u}{16} + v^2 = 0. \Rightarrow \left(u+\frac{3}{16}\right)^2 + v^2 = \left(\frac{5}{16}\right)^2$

$\left|w+\frac{3}{16}\right|^2 = (5/16)^2$

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Ex. Find the image of the circle $|z-3i|=3$; $w = 1/z$

$z = 1/w$

$|z-3i| = 3$

$\left|\frac{1}{w} - 3i\right| = 3.$

$|1-3iw| = 3|w|$

$|1-3i(u+iv)| = 3|w|$

$1-3iu+3v = 3|u+iv|$

$(1+3v)^2 + (3u)^2 = 9(u^2+v^2)$

$1+\cancel{9v^2}+6v+\cancel{9u^2} = \cancel{9u^2}+\cancel{9v^2}$

$1+6v=0$

Ex. Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $z\bar{z} - 2(z+\bar{z}) = 0$ into a straight line $4u+3=0$.

$\rightarrow \quad w = \frac{2z+3}{z-4}$

$wz - 4w = 2z+3$

$wz - 2z = 4w+3$

$z(w-2) = 4w+3$

$z = \frac{4w+3}{w-2}$

$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2}$

$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right) - 2\left[\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right]$

$\frac{16|w|^2+12(w+\bar{w})+9}{|w|^2+4-2(\bar{w}+w)} - 2\left[\frac{4|w|^2-8w+3\bar{w}-6}{|w|^2+4-2(\bar{w}+w)}\right.$

$+\,4|w|^2+3w-8\bar{w}-6$

$$\frac{16w\bar{w}+12(w+\bar{w})+9}{w\bar{w}+4-2(w+\bar{w})} - 2\left[\frac{8w\bar{w}-5w-5\bar{w}-12}{w\bar{w}+4-2(w+\bar{w})}\right] = 0$$

$$16w\bar{w}+12(w+\bar{w})+9 - 16w\bar{w}+10w+10\bar{w}+24$$

$w = u+iv$

$\bar{w} = u-iv$

$$= w\bar{w} - 2(w+\bar{w}) \pm 4$$

q. Show that $w = \frac{z-1}{z+1}$ maps the imaginary axis in z plane to a circle $|w| = 1$ in w plane. [Take $x = 0$]

reverse q.

q. Show that the mapping $w = \frac{i-iz}{1+z}$ maps the unit circle $|z| = 1$ to $u = 0$

Cross Ratio:

Let z_1, z_2, z_3, z_4 be any 4 points in complex plane. The cross ratio of these four points is denoted by (z_1, z_2, z_3, z_4) and defined by:

$$\frac{z_1-z_2}{z_2-z_3} \times \frac{z_3-z_4}{z_4-z_1}$$

- All in z plane
- Order is to be maintained.

$$\frac{w_1-w_2}{w_2-w_3} \times \frac{w_3-w_4}{w_4-w_1}.$$

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Cross Ratio & linear Fractional Transformation:

if $w = T(z)$ is a linear fractional transformation that maps distinct points z_1, z_2, z_3, z_4 into the points w_1, w_2, w_3 and w_4 then

$$\frac{z_1 - z_2}{z_2 - z_3} \times \frac{z_3 - z_4}{z_4 - z_1} = \frac{w_1 - w_2}{w_2 - w_3} \times \frac{w_3 - w_4}{w_4 - w_1}$$

Bilinear transformation preserves the cross ratio property.

Q. Prove the binomial transformation preserves the cross ratio property.

Take $w = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$.

Hence $w_i = \dfrac{az_i + b}{cz_i + d}$ where $i = 1, 2, 3, 4$.

Now $w_1 - w_2 = \dfrac{ad(z_1 - z_2) + bc(z_2 - z_1)}{(cz_1 + d)(cz_2 + d)}$

$w_2 - w_3 = \dfrac{ad(z_2 - z_3) + bc(z_3 - z_2)}{(cz_2 + d)(cz_3 + d)}$

$w_3 - w_4 = \dfrac{az_3 + b}{cz_3 + d} - \dfrac{az_4 + b}{cz_4 + d} = \dfrac{a\cancel{cz_3z_4} + az_3d + bcz_4 + \cancel{bd} - a\cancel{cz_3z_4} - az_4d - bcz_3 - \cancel{bd}}{(cz_3 + d)(cz_4 + d)}$

$$\frac{ad(z_3 - z_4) + bc(z_4 - z_3)}{(cz_3 + d)(cz_4 + d)} = \frac{az_3d - az_4d + bz_4c - bz_3c}{(cz_3 + d)(cz_4 + d)}.$$

w_3

$$w_4 - w_1 = \frac{az_4d - az_1d + bz_1c - bz_4c}{(cz_4+d)(cz_1+d)}$$

$$= \frac{ad(z_4-z_1) + bc(z_1-z_4)}{(cz_4+d)(cz_1+d)}.$$

$\therefore$ Cross ratio (w_1, w_2, w_3, w_4)

$$\frac{(ad-bc)(z_1-z_2)}{} \quad \text{simplify}$$

Q. Find the bilinear transformation which maps the point

$z_1 = 2,\ z_2 = i$

$z_3 = -2$

onto $w_1 = 1,\ w_2 = i$ and $w_3 = -1$ respectively.

→ As we know cross ratio property is preserved by B.L. transformation.

Take z_4 & w_4 as unknown and solve.

$$\frac{1-i}{i+1} \cdot \frac{-1-w_4}{w_4-1} = \frac{2-i}{i+2} \cdot \frac{-2-z_4}{z_4-2}$$

$$\frac{-1-w_4}{w_4-1} = \frac{(2-i)(i+1)}{(i+2)(1-i)} \cdot \frac{-2-z_4}{z_4-2}$$

$$\frac{-1-w_4}{w_4-1} = \frac{i+3}{3-i} \cdot \frac{-2-z_4}{z_4-2}$$

$$\frac{-1-w_4}{w_4-1} = \frac{(i+3)(3+i)}{(3-i)(3+i)} \cdot \frac{-2-z_4}{z_4-2} = \frac{8+6i}{10} \cdot \frac{-2-z_4}{z_4-2}$$

$_4 + bd$

$z_3 - bd$

$z_3 c$

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$$\frac{-1-w_4}{w_4-1} = \frac{4+3i}{5} \cdot \frac{-2-z_4}{z_4-2}$$

$-5z_4+10$

$-5w_4z_4+10w_4$

$$5(-1-w_4)(z_4-2) = (w_4-1)(4+3i)(-2-z_4)$$

$$= (w_4-1)[-8-4z_4-6i-3iz_4]$$

$$-5z_4+10+5w_4(2-z_4) = w_4(-8-4z_4-6i-3iz_4] + 8+4z_4+6i+3iz_4$$

$$w_4[10-5z_4] - w_4[-8-4z_4-6i-3iz_4] = 8+4z_4+6i+3iz_4 +5z_4-10$$

$$= \frac{-2+9z_4+6i+3iz_4}{10-5z_4+8+4z_4+6i+3iz_4}$$

$$w_4 = \frac{-2+9z_4+3i[z_4+2]}{-z_4+18+3i(z_4+2)}$$

$$\frac{z(9-3i)+(2+6i)}{z(1+3i)+(18-6i)}$$

Ex: Find the b.L. Trans. that maps $0, 1, i$ → $1+i, -i, 2-i$

Ex. DTLFT that sends the point $z = 0, -i, 2i$ into the point $w = 5i, \infty, -i/3$ respectively.

Fixed Point:

Let $w = f(z) = \frac{az+b}{cz+d}$ with $ad-bc \neq 0$.

Now the point (z_0) is said to be a fixed point if

$$z_0 = f(z_0) = \frac{az_0+b}{cz_0+d}$$

Cross Ratio:

that maps a point

$z = 0, i, -2i$

$w = 5i, \infty, -i/3$

$z_1 = 0 \quad w_1 = 5i$

$z_2 = i \quad w_2 = \infty$

$z_3 = -2i \quad w_3 = -i/3$

$$\frac{z_1-z_2}{z_2-z_3} \times \frac{z_3-z_4}{z_4-z_1} = \frac{w_1-w_2}{w_2-w_3} \quad \frac{w_3-w_4}{w_4-w_1}$$

$$\frac{0-i}{i+2i} \times \frac{-2i-z_4}{z_4-0} = \frac{5i-\infty}{\infty+i/3} \times \frac{-i/3-w_4}{w_4-5i}$$

$\frac{\infty(0-1)}{\infty(1+0)}$

$$\frac{-i}{3i} \times \frac{-2i-z_4}{z_4} = (-1) \cdot \frac{-\frac{i}{3}-w_4}{w_4-5i}$$

$$\frac{1}{3} \times \frac{-2i-z_4}{z_4} = \frac{-\frac{i}{3}-w_4}{w_4-5i}$$

$$\frac{2i+z_4}{z_4} = \frac{i+3w_4}{w_4-5i}$$

$\Rightarrow z_4 i + 3w_4 z_4 = 2iw_4 + z_4 w_4 + 10 - 5iz_4$

$3w_4z_4 - 2iw_4 - z_4w_4 = 10 - 6iz_4$

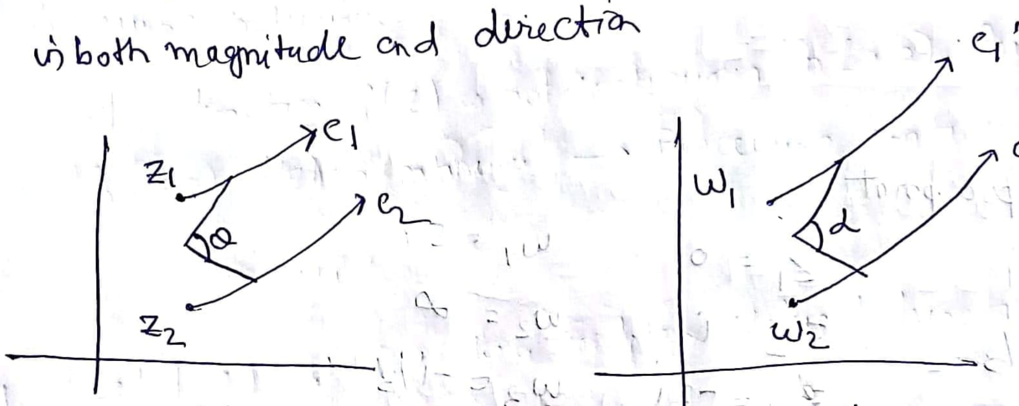
$$w_4 = \frac{10-6iz_4}{(2z_4-2i)}$$

$-3iz_4$

Ans. $\frac{-3z+5i}{-1-iz}$

(*) Conformal Mapping

In conformal mapping, that is both magnitude and direction



z-plane. $\theta = \alpha$ w-plane

The angle b/w the curves must remain constant.

For any transformation, the angle remains unchanged but the direction is not necessarily changed $\rightarrow$ Isogonal mapping.

Theorem:

Let $f(z)$ be an analytic function in a domain D and z_0 be any point in D then the mapping $w = f(z)$ is said to be conformal if $f'(z_0) \neq 0$.

Example: Consider the mapping $w = f(z) = z^2$. Check whether the mapping is conformal or not.

$z^2 = (x+iy)^2$

$= x^2 + i^2y^2 + 2xyi$

$= x^2 - y^2 + i(2xy)$

$u = x^2 - y^2 \qquad V = 2xy.$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial V}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial V}{\partial y} = 2x.$

↳ Analytic.

∴ The given mapping is analytic.

$f'(z) = 2z.$

$f'(z) \neq 0$ for $z \neq 0$

∴ The mapping is conformal except at $z = 0$.

Ex: $w = f(z) = e^z.$

$e^z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i\sin y).$

$= e^x \cos y + ie^x \sin y.$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial V}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial V}{\partial y} = e^x \cos y$

↓

Analytic.

$f'(z)$.

Now $f'(z) = e^z$ → Never zero.

$\forall z$ → given mapping is conformal for all z.

Let $f(z)$ be an analytic function defined in domain D and $z_0 \in D$ then z_0 is said to be critical point if $f'(z_0) = 0$.

The entire function $f(z) = e^z$ in the complex plane

Complex Integration

Piecewise continuous

If a function $f(z)$ is not continuous in $[a,b]$. But in some sub-interval of $[a,b]$, $f(z)$ is continuous $\rightarrow$ Piecewise continuous.

$f(z) = u + iv$

Def 1: Let $f(z) = u + iv$ be a complex valued function defined on ~~closed~~ $[a,b]$: $\int_a^b f(z)dz = \int_a^b (u+iv)dz$

$$= \int_a^b u\,dz + i\int_a^b v\,dz.$$

Real integration: 1 fixed direction.

Complex integration: No fixed direction.

Def 2. Let C be a piecewise differentiable curve, given by the equation

$C: z = z(t)$, $a \le t \le b$

Let $f(z)$ be a complex continuous function defined in a region containing C.

$$\int_C f(z)\,dz = \int_a^b f(z)\,z'(t)\,dt$$

Example:

$$\int_C f(z)\,dz \qquad f(z) = \frac{1}{z} \qquad C: |z| = r.$$

$C: |z| = r \Rightarrow z = re^{i\theta}$. $\{0 \leq \theta \leq 2\pi\}$.

Therefore
$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{1}{\cancel{r}\cancel{e^{i\theta}}}\, \cancel{r} i \cancel{e^{i\theta}}\, d\theta$$

$$= 2\pi i$$

Theorems:

1. $$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

2. $$\int_C [af(z) + b\,g(z)]\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz$$

3. If $C = C_1 + C_2 + \cancel{C_3} + \cdots$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots$$

Length of the curve:

Length of the curve: $C: z = z(t)$, $a \leq t \leq b$ is defined by

$$l = \int_a^b |z'(t)|\,dt.$$

Example:

Find the length of the circle $|z| = r$.

$$Z = re^{i\theta}$$

$$l = \int_0^{2\pi} r i e^{i\theta} d\theta = ri \int_0^{2\pi} (\cos\theta + i\sin\theta) d\theta$$

$$|z'(t)| = |rie^{i\theta}| = |ri(\cos\theta + i\sin\theta)| = \sqrt{r^2 i^2 [\cos^2\theta + \sin^2\theta]} = ri$$

$$l = \int_0^{2\pi} ri\, d\theta = ri(2\pi) = 2\pi ri$$

Solution:

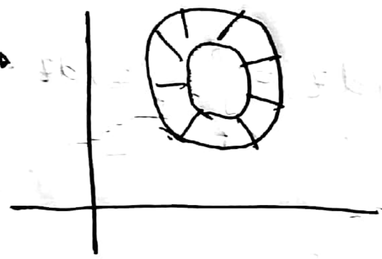
$$l = \int_0^{2\pi} |rie^{i\theta}| d\theta = r \int_0^{2\pi} d\theta = 2\pi r.$$

Simple, Multiple Connected Region:

1. If the region can be represented by a single curve - Simple curve
2. If the region is represented by intersection of two or more curves - Multiple connected.

1. →

2 →



(*) $\int_c f(z)dz$; where $f(z) = y - x - 3ix^2$

c: $z = 0$ to $z = 1+i$

and between the line segment

if we transform into other variables (say y or x), integration becomes easier.

Solution:

Substitute $x = y$.

$f(z) = x \cdot x - 3ix^2 = -3ix^2$

$z = x + iy$

$dz = dx + i\,dy = (1+i)dx$

$$\int_C f(z)dz = \int_0^1 (-3ix^2)(1+i)dx$$

$$= \int_0^1 -3ix^2 - 3i^2x^2\,dx = \int_0^1 -3ix^2 + 3x^2\,dx$$

$$= -3i\left[\frac{x^3}{3}\right]_0^1 + 3\left[\frac{x^3}{3}\right]_0^1$$

$$= 1 - i \quad \text{Ans}$$

12/10/2017

1. $r + 2s + 5t + p - 2q - 3u = 0$

2. $y^2 r + x^2 t = 0$

Ans 1. $\left(\frac{dy}{dx}\right)^2 - 2\left(\frac{dy}{dx}\right) + 5 = 0$

roots: $\frac{+2 \pm \sqrt{4-20}}{2} = +1 \pm 2i$

$\frac{dy}{dx} = (1 \pm 2i)$

$\xi = y - x - 2ix$

$\eta = y - x + 2ix$

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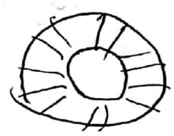
Formulaes for Complex Integration:

1. Cauchy Integral Theorem : State and prove.
2. Cauchy Intl. Formulae.
3. Generalised Cauchy' Intl. Formula

Transform x into another var. > if no relation b/w x and y is
y into another var. given.

Curve may be bounded by single region — Simply curve. 
bounded by >1 curve — Multiply curve 

Formula:

$$\oint_C f(z)\,dz = \int_a^b f(z(t))\,|z'(t)|\,dt.$$

For obtaining length of a curve:

$$l = \int_a^b |z'(t)|\,dt$$

1. Circle: Find the length of the circle $|z| = r$.

Here parameter is θ. $z = re^{i\theta}$.

$$l = \int_0^{2\pi} r\theta e^{i\theta}\,d\theta = r\,\frac{\theta e^{i\theta}}{\theta}$$

$$= r\left[e^{i\theta}\right]_0^{2\pi} = r\left[e^{i(2\pi)} - e^{0}\right]$$

$$= r[2\pi]$$

: Cauchy's integral theorem (Not formula)

Let $f(z)$ be a function which is analytic at all points on and a simply closed curve then $\oint_C f(z)\,dz = 0$

Proof:

Using Green's Theorem:

Smooth function → continuous, differentiable everywhere

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

N, M - functions of x, y

Region changed

$f(z) = u + iv$

$z = x + iy$

$dz = dx + i\,dy$

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx + i\,dy) = \oint_C u\,dx - v\,dy + i(v\,dx + u\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy)$$

Similar $M\,dx + N\,dy$ $\quad$ $M\,dx + N\,dy$

Now using Green's Theorem:

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Now applying Cauchy Riemann Equation, we have

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\iint_D \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + \iint_D 0 \, dx\,dy$$

$= 0$

2. Cauchy integral formula

Let $f(z)$ be a function which is analytic inside and on a simply closed curve c and let $f(0)$ be any point in the interior of c. then

$$\frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} \, dz = f(z_0)$$

$$\Rightarrow \boxed{\oint_c \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0)}$$

↳ for Q. identity what is z_0

. then check for analyticity

↓ YES

apply cauchy integral formula

→ z_0 must lie inside the curve.

Q. $\oint_c \frac{z^2+5}{z-3} dz$ without defining c, the question is incomplete.

↳ Now $c: |z| = 4$.

A. Comparing we get $z_0 = 3$.

$z_0 = 3 \rightarrow$ lies inside the circle $|z| = 4$.

Here $f(z) = z^2 + 5$.

$\therefore$ Ans: $2\pi i\, f(3)$

$= 2\pi i\,(3^2 + 5)$

$= 2\pi i\,(14)$

$= 28\pi i$

Values of z for which denominator becomes 0

↳ pole.

↓

All these points lie inside $|z| = 2 \rightarrow$ Apply C.I.F for all parts

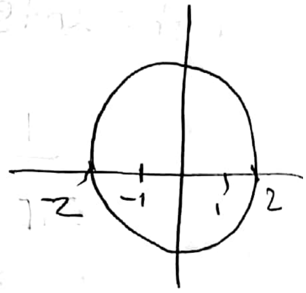
↳ No of poles = No of parts of separation.

Q: $\displaystyle\oint_C \frac{z\,dz}{z^2 - 1}$. $C: |z| = 2$.

A: $= \displaystyle\oint_C \frac{z\,dz}{(z+1)(z-1)} = \oint_C \frac{z/z+1}{z-1}\,dz + \oint_C \frac{z/z-1}{z+1}\,dz$.

$2\pi i \oint\left(\frac{1}{1+1}\right) + 2\pi i\left(\frac{-1}{-1-1}\right)$

$= 2\pi i\left(\frac{1}{2}\right) + 2\pi i\left(\frac{1}{2}\right) = 2\pi i$



$C = C_1 + C_2 + C_3 + \cdots$

$\displaystyle\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots$

n points (here 1, −1)

↓

n parts.

1. $\oint_C \frac{e^z}{z^2+4}\,dz \quad C: |z-i|=2$

2. $\oint_C \frac{dz}{z^2+4} \quad C: |z-i|=2$

3. $\oint_C \frac{z\,dz}{(9-z^2)(z+i)} \quad C: |z|=2$

4. $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz \quad C: |z|=5 \qquad \oint_C \frac{f(z)}{(z-z_0)^n}\,dz$

21/11/2017

About the point z_0:

$$f(z) = f(z_0) + \frac{z-z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$

In Taylor Series expansion if $z_0 = 0$ then the series is known as Maclaurin Series.

For Taylor Series order of convergence $|z-z_0| < r$.

Similarly, order of convergence for Maclaurin series $|z| < r$

Q. $f(z) = \frac{1}{z}$, $z_0 = 1$, Expand: $\quad f'(z) = \frac{-1}{z^2}, \quad f''(z) = +\frac{2}{z^3}$

A: $f(1) + \frac{z-1}{1!} f'(1) + \frac{(z-1)^2}{2!} f''(1) + \cdots$

$= 1 + \frac{z-1}{1}\left(\frac{-1}{1}\right) + \frac{(z-1)^2}{2}\left(\frac{2}{1}\right) + \cdots$

$= 1 + (z-1)(-1) + (z-1)^2 + \cdots$

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$= 1 - \{ (z-1) + (z-1)^2 + \dots \}$.

↳ Value less than 1.

$|z| < 1$.

Q. $f(z) = \frac{z-1}{z+1}$ about (i) $z=0$, (ii) $z=1$ and determine the region of convergence.

about $z=0$.

$f(z) = \cancel{f(0)+} (z-1) \underbrace{(z+1)^{-1}}_{g(z)}$

$= (z-1) [1 - z + z^2 - z^3 + \dots]$

$= z - z^2 + z^3 - z^4 + \dots -1 + z - z^2 + z^3$

$= -1 + 2z - 2z^2 + 2z^3 - 2z^4$

We take a neighborhood of z_0 such that it does not contain a point where function is analytic.

$f(z) = f(z_0) + (z - z_0)$

$|z - z_0| < r$

$|z| < r$. In most cases r is taken as 1.

about $z = 1$.

$f(z) = \frac{1}{2}(z-1) - \frac{1}{4}(z-1)^2 + \frac{1}{8}(z-1)^3 + \dots$

$\left| \frac{z-1}{2} \right| < 1$ (Take first term).

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$|z-1| < 2.$

$\downarrow$

contains no other integral points where f is analytic.

Laurent Series

The domain of analyticity of f is a multiply region.

Let C_1 and C_2 denote the concentric circle $|z-z_0| = r_1$ and $|z-z_0| = r_2$ with $r_1 < r_2$. Let $f(z)$ be analytic function within the domain $r_1 < |z-z_0| < r_2$ then $f(z)$ can be represented as a convergence series of -ve and +ve powers of $(z-z_0)$. That is:

$$f(z) = \underbrace{\sum_{n\geq 1} b_n (z-z_0)^{-n}}_{\text{Negative power part}} + \underbrace{\sum_{n\geq 1} a_n (z-z_0)^n}_{\text{Analytic part}}.$$

Principle part

where n is a positive integer.

Ex-1: $f(z) = \dfrac{1}{z(z-1)}$, about $z = 0$

$$= -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1+z+z^2+z^3+z^4+\dots)$$

$$= -\frac{1}{z}(1+z+z^2+z^3\dots).$$

$$= -\Big(\underbrace{\frac{1}{z}}_{\text{principal part}} + \underbrace{1+z+z^2+z^3}_{\text{Analytic part}}\dots\Big)$$

About some other point:

$$= \frac{1}{(z-1)} z^{-1} = \frac{1}{(z-1)}[1+(z-1)]^{-1} = \frac{1}{(z-1)}[1-(z-1)+(z-1)^2-(z-1)^3+ \ldots]$$

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \ldots$$

For region of convergence:

$$\left|\frac{1}{z-1}\right| < 1$$

$$1 < |z-1|$$

$$|z-1| > 1$$

⇒ Let $f(z)$ be a analytic function in domain D and let any point z_0 belongs to D is said to be zero of order r if $f(z)$ can be written as $f(z) = (z-z_0)^r \phi(r)$.

Order of a zero is no. of times it occurs.

Example: Find the zero and order of zeroes of:

① $\sin z$

② $f(z) = z^2 \sin z$.

③ $f(z) = \frac{(z^3-1)(z-1)}{z^3+1}$.

A.1: Zeroes: $0, \pi, 2\pi, 3\pi, \ldots = n\pi$. where $n \in Z$ (integers)

Order of each 1.

A.2 zeroes: $0, n\pi$

$0 \to 2$, $n\pi \to 1$ each

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P.3. Zeros: 1

$z^3-1=0$. $\quad z-1=0$

$z=1,\omega,\omega^2$ $\quad z=1$.

$-\frac{1+i\sqrt{3}}{2}$ $\quad \frac{-1-i\sqrt{3}}{2}$.

Order

$1 \rightarrow 2$

$\omega \rightarrow 1$

$\omega^2 \rightarrow 1$

Singularity

A point $z=z_0$ is said to be singular point or singularity of a function $f(z)$ if $f(z)$ is not analytic at that point but $f(z)$ is analytic in the deleted neighborhood of z_0.

Set denominator $=0$ and obtain.

Example: $f(z)=\frac{1}{z(z-1)}$.

Singular points: $0,1 \rightarrow$ Poles.

Regular Points

A point z_0 is said to be regular if the function $f(z)$ is analytic there.

Singular point $\times$ Regular point

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Classification of singularity

Let z_0 be an isolated singularity for a function $f(z)$ and let $r > 0$ be such that $f(z)$ be analytic

$$0 < |z - z_0| < r$$

$$f(z) = \sum_{n \geq 1} b_n (z - z_0)^{-n} + \sum_{n \geq 1} a_n (z - z_0)^{n}.$$

- Removable singularity:

If the principle part of Laurent series expansion has no terms → removable singularity

- Pole (a kind of singularity)
  - If the principle part has finite no of terms then z_0 is known as the pole.

- Essential singularity:

If the principle part has infinite no of terms, then z_0 is called essential singularity.

- Isolated Singularity

A singularity of complex function $f(z)$ is said to be isolated if

Cond ① The function is not analytic at that point.

Cond. ②. If we take any +ve no. r then the function is analytic between $0 < |z - z_0| < r$

In case of removable singularity:

(#) Removable Singularity:

$$\lim_{z \to z_0} (z - z_0) f(z) = 0$$

Ex: $f(z) = \frac{\sin z}{z}$. Find nature of singularity.

↳ Singular point $z = 0$.

$$\lim_{z \to 0} (z - 0) f(z) = \lim_{z \to 0} z \frac{\sin z}{z} = 0$$ ∴ $z = 0$ is removable singularity.

Pole:

Let $z = z_0$ be an isolated singularity of a complex function $f(z)$ then the singularity is said to be a pole of order n if

$$\lim_{z \to z_0} (z - z_0)^n f(z) \neq 0 \text{ (is non zero)}$$

↳ +ve → -ve. but non zero.

Then Singular point $(z_0) \Rightarrow$ Pole.

A pole of order 1 - Simple pole

Ex: $f(z) = \frac{e^z}{z}$.

Singular point: $z = 0$ of order $n = 1$.

So $\lim_{z \to 0} (z - 0)^1 \frac{e^z}{z} = e^0 = 1 \; (\neq 0)$

↳ Pole of order 1.

In case of essential singularity, the limit of complex function $f(z)$ at the singular point does not exist.

$= 0$ Removable

$\neq 0$ Pole

DNE essential

Ex-1. $f(z) = \frac{z - \sin z}{z^3}$

$(z-0)z -$

Singular point: $z = 0$ $n = 3$

$\frac{1-\cos z}{2z}$

$\lim_{z \to 0} (z-0)^3 \frac{z - \sin z}{z^3} = 0 - 0$

$\frac{\sin z}{1}$

L Hospital → Removable singularity.

Ex-2. $f(z) = \frac{\cos z}{z^2}$

Singular point: $z = 0$ $n = 2$

$\lim_{z \to 0} (z-0)^2 \frac{\cos z}{z^2} = 1$

→ Pole.

Ex-3. $f(z) = \frac{z}{e^{z}-1}$

Singular point $z = 0$ $n = 1$

$\lim_{z \to 0} (z-0) \frac{z}{e^{z}-1}$ $\lim_{z \to 0} \frac{2z}{e^{z}} \to \lim_{z \to 0}$ → ~~Pole~~

Removable

4) $\lim_{z \to 0} z \cdot \frac{1}{\lim_{z \to 0} \frac{e^z - 1}{z}}$

Theorem 1: If an analytic function $f(z)$ has a pole of order m at $z = z_0$ then $z = z_0$ is also known as the zero of the function $\frac{1}{f(z)}$ of order m.

Theorem 2: The limit point of the set of all poles of the function $f(z)$ is a non isolated essential singularity.

For any $f(z)$ if the singular points do not satisfy → Cond 1. Cond 2.

Theorem 3: The limit point of the zeroes of a function $f(z)$ is an isolated essential singularity.

Poles (End)

Residue

Definition: Let α be an isolated singularity of an analytic function $f(z)$ then the coefficients of $\frac{1}{z - \alpha}$ in the Laurent series expansion of $f(z)$ about $z = \alpha$ is defined as residue of the function at $z = \alpha$.

Representing in Laurent series: b_n → Residue of the function (in principle part).

Representation: $\underset{z=\alpha}{\text{Res}}\, f(z) = \frac{1}{2\pi i} \oint f(z)\, dz$

Residue Theorem:

Let α be a pole of $f(z)$ of order m then

$$\underset{z=\alpha}{\text{Res}}\, f(z) = \text{Res}(\alpha) = \frac{1}{(m-1)!} \lim_{z \to \alpha} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-\alpha)^m f(z) \right\}.$$

Let $m=1$

$$\text{Res}(\alpha) = \lim_{z \to \alpha} \left\{ (z-\alpha) f(z) \right\}$$

Let $m=2$

$$\text{Res}(\alpha) = \frac{1}{1!} \times \lim_{z \to \alpha} \left\{ \frac{d}{dz} (z-\alpha)^2 f(z) \right\}$$

obtain pole and order to finish.

Ex: Find the residue of $f(z) = \frac{1}{(z+1)^2(z-2)}$.

Pole: -1, 2
(2) (1).

$$\text{Res}(-1) = \lim_{z \to -1} \left\{ \frac{d}{dz} (z+1)^2 \times \frac{1}{(z+1)^2(z-2)} \right\} = \lim_{z \to -1} \left\{ \frac{d}{dz} \frac{1}{(z-2)} \right\}$$

$$\lim_{z \to -1} - \cancel{(z-2)}^{-2} \frac{-1}{(z-2)^2} = \frac{-1}{9}$$

$$\text{Res}(2) = \lim_{z \to 2} (z-2) \frac{1}{(z+1)^2(z-2)} = \lim_{z \to 2} \frac{1}{(z+1)^2} = \frac{1}{9}$$

Cauchy's Residue Theorem

Let $f(z)$ be analytic within and on a closed contour C, ~~and~~ except at a finite no of singularities : $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ then

$$\oint_C f(z)\,dz = 2\pi i \times \{Res(\alpha_1) + Res(\alpha_2) + \ldots Res(\alpha_n)\}.$$

Contour? It is a piecewise smooth curve.

↳ ✓ Continuous

✓ Differentiable.

Q. Use ~~Cauchy~~ State Cauchy's Residue theorem and using this find the integration of $f(z) = \dfrac{z+1}{z^2-2z}$; $C \equiv |z| = 5$. [2+4]

1. Find poles
2. Check if inside C.

A: Poles : $z^2 - 2z = 0 \rightarrow z = 0, \; z = 2.$

(1) (1) → Order.

$$Res(0) = \lim_{z\to 0} z \frac{(z+1)}{z(z-2)} = \lim_{z\to 0} \frac{z+1}{z-2} = \frac{-1}{2}$$

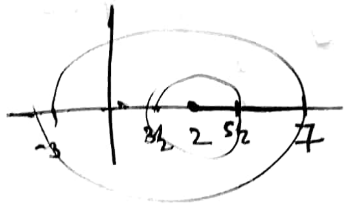
$$Res(2) = \lim_{z\to 2} \frac{(z-2)(z+1)}{z(z-2)} = \lim_{z\to 2} \frac{3}{2}$$

$$\therefore \oint_C f(z)\,dz = 2\pi i \left\{\frac{-1}{2} + \frac{3}{2}\right\} = 2\pi i$$

{By Cauchy's Residue Theorem}.

Ex $\oint \frac{z\,dz}{(z-1)(z-2)^2}$ $C_1 : |z-2| = 5$

$C_2 : |z-2| = 1/2$.



Poles: $z = 1$ (1) order
$z = 2$ (2)

C_1.

$$Res(1) = \lim_{z \to 1} \left\{ (z-1) \frac{z}{(z-1)(z-2)^2} \right\} = \lim_{z \to 1} \frac{z}{(z-2)^2} = \frac{1}{1} = 1$$

$$Res(2) = \lim_{z \to 2} \left\{ \frac{d}{dz} \frac{z}{z-1} \right\} = \lim_{z \to 2} \frac{(z-1)(1) - z(1)}{(z-1)^2} = \frac{-1}{(1)^2} = -1.$$

$$\therefore \oint_{C_1} f(z)\,dz = 2\pi i \{ Res(1) + Res(2) \} = 2\pi i \{ 1 - 1 \} = 0.$$

$$\oint_{C_2} f(z)\,dz = 2\pi i \{ Res(2) = 2\pi i(-1) = -2\pi i$$

Ex: