Solved Problems - III

HCU 2012

1. *[Q13]* If 1 gm of ice at 0°C is melted and converted into water at the same temperature, then the change in entropy is? (assuming heat of fusion=79.7 cal/g)

$$\Delta S = \int_R \frac{dq}{T} = \frac{Q}{T} \ \{ \text{as T is constant}$$

Heat is absorbed. So Q is positive.

$$\therefore \Delta S = \frac{79.7 cal/g \times 1g}{273.15K} = \boxed{0.29\ calK^{-1}}$$

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2. *[Q52]* A refrigerator is operated between two reservoirs maintained at 250 K and 300 K respectively. If its coefificient of performance is one-third that of a Carnot engine and it absorbs 500 J from the low temperature reservoir, then the heat lost to the high temperature reservoir is?

$$\text{C.O.P, } \alpha = \frac{Q_L}{W} = \frac{Q_L}{Q_H - Q_L} = \left\{ \frac{Q_H}{Q_L} - 1 \right\}^{-1} \quad ...(1)$$

$$\text{For Carnot Engine, } \frac{Q_H}{Q_L} = \frac{T_H}{T_L} = \frac{300}{250} = \frac{6}{5}$$

$$\therefore \alpha_c = \left\{ \frac{6}{5} - 1 \right\}^{-1} = 5$$

$$\therefore \quad \text{As given,} \quad \alpha_R = \frac{\alpha_c}{3} = \frac{5}{3}$$

$$\text{i.e. } \left\{ \frac{Q_H}{Q_L} - 1 \right\}^{-1} = \frac{5}{3} \Rightarrow \frac{Q_H}{Q_L} = \frac{3}{5} + 1 = \frac{8}{5}$$

$$\therefore Q_H = \frac{8}{5} \times Q_L = \frac{8}{5} \times 500 = \boxed{800\ J}$$

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3. *[Q55]* The fraction of kinetic energy in the total energy of a simple harmonic oscillator, when its displacement is half of its amplitude, is given by?

$$\text{Total energy, } E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$x = \frac{A}{2} \Rightarrow U = \frac{1}{2}k\frac{A^2}{4} = \frac{E}{4}$$

$$\therefore K = E - \frac{E}{4} = \frac{3E}{4} \Rightarrow \frac{K}{E} = \frac{3}{4} = \boxed{0.75}$$

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4. *[Q60]* The resonance frequency of an LCR circuit was found to be 1200 Hz. If another capacitor of equal value is connected in parallel to the existing one in the circuit, then the new resonance frequency is?

$$\text{Resonance frequency, } f = \frac{1}{2\pi\sqrt{LC}} \Rightarrow \frac{f'}{f} = \frac{2\pi\sqrt{LC}}{2\pi\sqrt{LC'}} = \sqrt{\frac{C}{C'}}$$

$$\text{Here, } C' = 2C \Rightarrow f' = f\sqrt{\frac{1}{2}} = \frac{1200}{\sqrt{2}} = \boxed{849\ Hz}$$

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5. *[Q73]* The output of light from a laser source of 1 mW power passing through two crossed polarizers is zero. If another polarizer is introduced between these two at an angle of 45° with respect to the optic axis of one of them, the output power at the detector is?

$$I = I_0 \cos^2(45^\circ) \cdot \cos^2(45^\circ) = 1mW \times \frac{1}{2} \times \frac{1}{2} = \boxed{0.25\ mW}$$

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6. *[Q10]* A certain weight of liquid has a volume of 100 cc at 0°C. If the volume expansion coefficient of the liquid is 0.00112/°C, then its volume at 50°C is approximately?

$$\text{Coefficient of volume expansion, } \alpha_v = \frac{dv}{V}\frac{1}{dT}$$

$$\therefore \int_{V_1}^{V_2} \frac{dV}{V} = \int_{T_1}^{T_2} \alpha_v dT \Rightarrow \ln\left(\frac{V_2}{V_1}\right) = \alpha_v(T_2 - T_1)$$

$$\Rightarrow V_2 = V_1 e^{(T_2 - T_1)\alpha_v} = 100e^{50 \times 0.00112} = \boxed{105.76 \text{ cc}}$$

7. *[Q75]* If the actual wavelength of light emitted from a luminous body is 6000 Å, but is measured to be 6030 Å, then its reiative velocity with respect to the observer is?

For small velocities, equation for Doppler effect is: $\lambda' = \lambda\left\{1 + \frac{v}{c}\right\}$

$$\Rightarrow v = \left\{\frac{\lambda'}{\lambda} - 1\right\}c = \left\{\frac{6030}{6000} - 1\right\} \cdot 3 \times 10^8 = \boxed{1.5 \times 10^6\ ms^{-1}}$$

8. *[Q51]* The outer surface of a brass sheet at 100°C is kept in contact with the outer surfaceof a steel sheet at 0°C. If the thickness and cross-sectional areas of both the sheets are equal and the ratio of their thermal conductivities is 2:1, then the temperature of the interface at equilibrium is?

THERMAL PROPERTIES OF MATTER

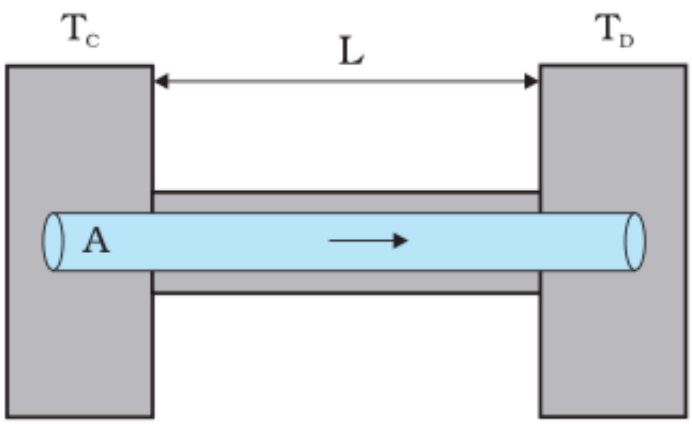

Fig. 11.14 *Steady state heat flow by conduction in a bar with its two ends maintained at temperatures T_C and T_D; ($T_C > T_D$).*

experimentally that in this steady state, the rate of flow of heat (or heat current) H is proportional to the temperature difference ($T_C - T_D$) and the area of cross section A and is inversely proportional to the length L :

$$H = KA\frac{T_C - T_D}{L} \qquad (11.14)$$

The constant of proportionality K is called the **thermal conductivity** of the material. The

At equilibrium, heat flow in both directions will be same. Hence,

$$H_1 = H_2 \Rightarrow K_1 A \frac{100^\circ C - T_{eq}}{l} = K_2 A \frac{T_{eq} - 0^\circ C}{l}$$

$$\Rightarrow \frac{K_2}{K_1} = \frac{100 - T}{T} = \frac{100}{T} - 1 \Rightarrow \frac{100}{T} = 1 + \frac{K_2}{K_1}$$

$$\therefore T = 100\left[1 + \frac{K_2}{K_1}\right]^{-1} = 100\left[1 + \frac{1}{2}\right]^{-1} = \boxed{66.67\,^\circ C}$$

HCU 2013

1. *[Q72]* A body executes simple harmonic motion about the origin with amplitude A = 13 cm and period T = 12 sec. At time t = 0 it is at x = 13 cm. The shortest time of passage from x = 6.5 cm to x = -6.5cm is ?

$$\text{At } t = 0,\ x = 13cm = A \Rightarrow x = 13\cos\omega t$$

$$\therefore x_1 = 6.5 = 13\cos\omega t_1 \Rightarrow \omega t_1 = \cos^{-1}\left(\frac{6.5}{13}\right) = 60^\circ$$

$$\text{Similarly, } x_2 = -6.5 = 13\cos\omega t_2 \Rightarrow \omega t_2 = \cos^{-1}\left(-\frac{6.5}{13}\right) = 120^\circ$$

$$\therefore \omega(t_2 - t_1) = \frac{2\pi}{T}(t_2 - t_1) = 120^\circ - 60^\circ = 60^\circ$$

$$\therefore \text{Time taken } t_2 - t_1 = \frac{60^\circ}{2\pi}T = \frac{60^\circ}{360^\circ} \times 12s = \boxed{2s}$$

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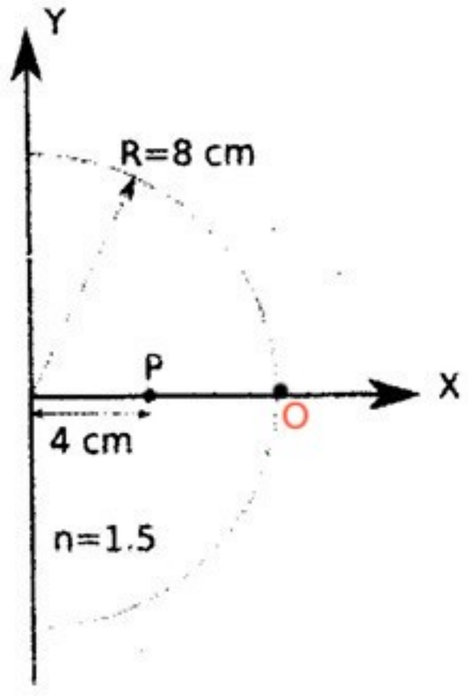
2. *[Q74]* A glass (with refractive index 1.5) hemisphere in air with radius of curvature 8 cm has a spot P at x=4 cm. Viewed along x from the right, the spot will be imaged by the spherical surface at x = ?



The equation for a spherical refracting surface is:

$$\boxed{\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}} \tag{9.16}$$

Equation (9.16) gives us a relation between object and image distance in terms of refractive index of the medium and the radius of curvature of the curved spherical surface. It holds for any curved spherical surface.

$$\therefore \frac{n_2}{v} = \frac{n_2 - n_1}{R} + \frac{n_1}{u} \ \{ \text{R \& u are -ve, as they are measured to the left}$$

$$\Rightarrow \frac{1}{v} = \frac{1 - 1.5}{-8} + \frac{1.5}{-4} = \frac{0.5}{8} - \frac{1.5}{4} = -\frac{2.5}{8}$$

$$\Rightarrow v = -3.2\ cm. \text{ As v is measured from } O\ (X = 8cm),$$

$$x = v + 8 = -3.2 + 8 = \boxed{4.8\ cm}$$

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3. *[Q75]* A ray of light is incident at an angle 60° on an interface between glass with refractive index 1.5 and a liquid with refractive index n, the ray will be totally internally reflected only if n < ____.

Critical angle condition, $\sin i_c = \frac{n_l}{n_g}$. For TIR, i should be $> i_c$.

$$\therefore \text{ Here, } 60^\circ > i_c \quad \text{or} \quad i_c < 60^\circ$$

$$\therefore \sin i_c < \sin 60^\circ \Rightarrow \frac{n_l}{n_g} < \frac{\sqrt{3}}{2} \Rightarrow n_l < \frac{\sqrt{3}}{2} n_g$$

$$\text{Given } n_g = 1.5 = \frac{3}{2} \Rightarrow n_l < \frac{\sqrt{3}}{2} \cdot \frac{3}{2} \Rightarrow \boxed{n_l < \frac{3\sqrt{3}}{4}}$$

HCU 2011

1. *[Q63]* A glass plane-convex lens has its flat side toward the object. The convex side has a radius of curvature of 30.0 cm. The index of refraction of the glass for violet light (400nm) is 1.537 and that for red light (700nm) is 1.517. The colour purple is a mixture of red and violet. If a purple object is placed 80.0 cm from this lens, where are the red (S'_r) and violet (S'_v) images formed?

   Flat side towards the object (left) $\Rightarrow R_1 = \infty, \; R_2 = -30cm, u = -80cm$

   $$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \Rightarrow \frac{1}{f} = \frac{n-1}{-R_2} = \frac{n-1}{30}$$

   $$\therefore \frac{1}{v} - \frac{1}{u} = \frac{1}{f} = \frac{n-1}{30} \Rightarrow v = \left\{\frac{n-1}{30} + \frac{1}{u}\right\}^{-1}$$

   $$\Rightarrow S'_r = \left\{\frac{1.517-1}{30} - \frac{1}{80}\right\}^{-1} \Rightarrow \boxed{S'_r = +211.27\ cm}$$

   $$\Rightarrow S'_v = \left\{\frac{1.537-1}{30} - \frac{1}{80}\right\}^{-1} \Rightarrow \boxed{S'_v = +185.18\ cm}$$

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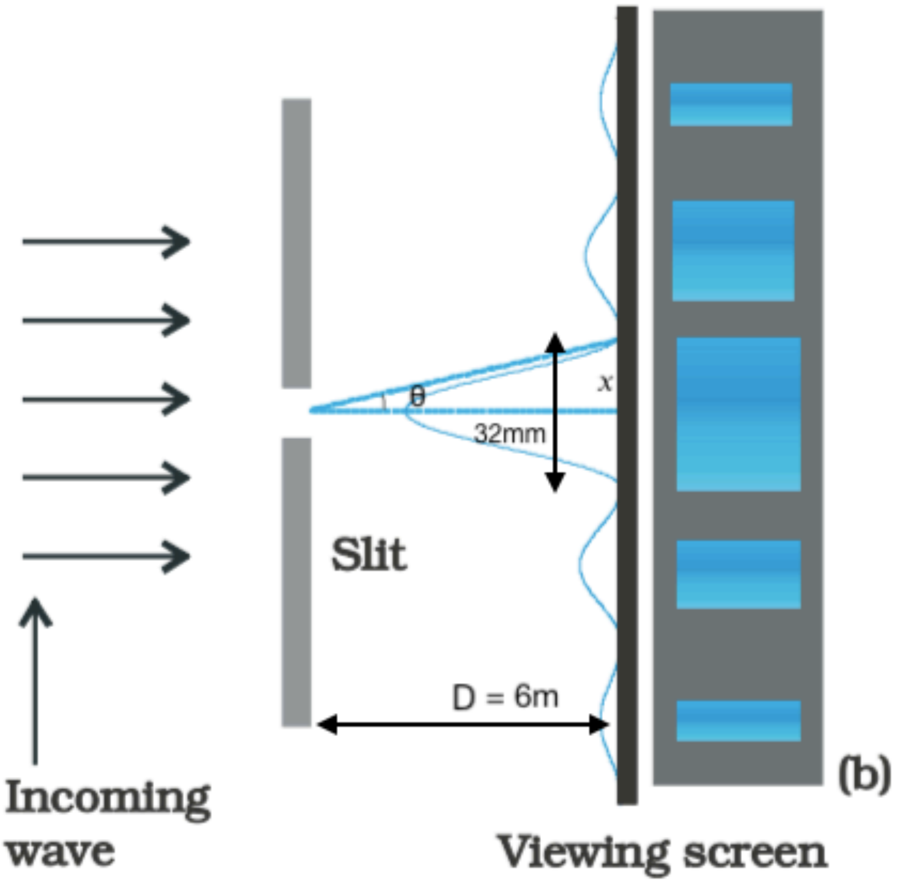
2. *[Q61]* A 633 nm He-Ne laser light passes through a narrow slit and the resulting diffraction pattern is observed on a screen kept 6 m away from the source. If the distance between the centers of the first minima on either side of the central maximum on the screen is 32 mm, how wide is the slot?



For first minima, $a \sin\theta = n\lambda, n = 1$

Given $2x = 32mm \Rightarrow x = 16mm.\ \therefore \sin\theta = \frac{x}{D} = \frac{16mm}{6m}$

$$\therefore a = \frac{\lambda}{\sin\theta} = \frac{633 \times 10^{-9}m \times 6m}{16 \times 10^{-3}m} = \boxed{0.24mm}$$

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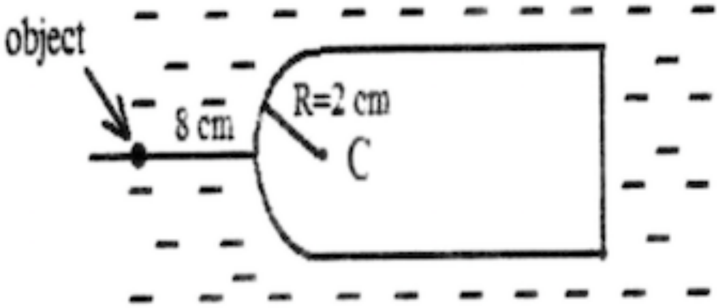
3. A cylindrical glass rod of refractive index n_g = 1.52 is immersed in water (refractive index n_w = 1.33). One end of the glass rod has a polished hemi-spherical surface with radius R=2cm. A small object is placed at a distance 8 cm to the left of the vertex of the hemi-spherical surface. The image distance (s) and the magnification (m) are given by?



$\rightarrow$ A cylindrical lens does not affect rays parallel to its length. Therefore here, refraction is only by the spherical end. Hence,

$$\text{We have } \frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R} \Rightarrow \frac{n_g}{v} = \frac{n_g - n_w}{R} + \frac{n_w}{u}$$

$$= \frac{1.52 - 1.33}{2} + \frac{1.33}{-8} = \frac{0.19}{2} - \frac{1.33}{8} = -\frac{1.14}{16}$$

$$\therefore s = v = \frac{n_g}{-1.14/16} = 1.52 \times \frac{16}{-1.14} = \boxed{-21.33\ cm}$$

$$\text{For a single spherical surface, magnification is } \textbf{NOT } \frac{v}{u}, \text{ but: } m = \frac{v/n_2}{u/n_1}$$

$$\therefore \text{Here,} \quad m = \frac{-21.33/1.52}{-8/1.33} = \boxed{+2.33}$$

HCU 2014

1. *[Q8]* A simple pendulum executes oscillations of period 10 s on the surface of Earth. This pendulum is taken to another planet having same mass but one fourth of Earth's size.The period of oscillations of the pendulum would now be?

$$T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow \frac{T'}{T} = \sqrt{\frac{g}{g'}} \;...(1)$$
$$g = \frac{GM}{R^2} \Rightarrow \frac{g}{g'} = \frac{R'^2}{R^2} \Rightarrow \sqrt{\frac{g}{g'}} = \frac{R'}{R}$$

$$\therefore eq.\ (1) \Rightarrow T' = T \cdot \frac{R'}{R} = 10s \times \frac{1}{4} = \boxed{2.5s}$$

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2. *[Q22]* In a harmonic oscillator, if the transition energy from 3rd state to 2nd state is 4.8 eV, what is the ground state energy?

$$\text{For a harmonic oscillator, } E = \left(n + \frac{1}{2}\right)h\nu$$

$$\therefore E_3 - E_2 = (3-2)h\nu = h\nu = 4.8eV \quad \left\{ \because \frac{1}{2}h\nu \ \text{ cancels out} \right.$$

$$\therefore \text{ Therefore ground state energy, } E_0 = \frac{1}{2}h\nu = \frac{4.8}{2}eV = \boxed{2.4\ eV}$$

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3. *[Q42]* The displacement of a body executing simple harmonic motion is given by

$$y = 5\sin\left(2\pi t + \frac{\pi}{4}\right).$$

Its initial displacement, assuming all quantities are in SI unit, is?

$$\rightarrow \text{ Initial displacement, } y(0) = 5\sin\left(2\pi \cdot 0 + \frac{\pi}{4}\right) = 5\sin\left(\frac{\pi}{4}\right) = \boxed{\frac{5}{\sqrt{2}}}$$

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4. *[Q65]* A metal rim of inner diameter 59.4 cm at 20°C is required to be fixed on a wooden wheel of diameter 59.6 cm. The temperature to which the rim must be heated to fit on to the wheel (given the coefficient of linear expansion of the metal is 1.2×10^{-5}/°C) is?

> If the substance is in the form of a long rod, then for small change in temperature, ΔT, the fractional change in length, $\Delta l/l$, is directly proportional to ΔT.
>
> $$\frac{\Delta l}{l} = \alpha_1 \Delta T \qquad (11.4)$$
>
> where α_1 is known as the **coefficient of linear expansion** and is characteristic of the material

$$\therefore \Delta T = \frac{1}{\alpha}\frac{\Delta l}{l} \Rightarrow \int_{T_1}^{T_2} dT = \frac{1}{\alpha}\int_{l_1}^{l_2} \frac{dl}{l}$$

$$\Rightarrow T_2 - T_1 = \frac{1}{\alpha}\ln\left[\frac{l_2}{l_1}\right] \ ...(1)$$

$$\text{Here, } l = 2\pi r = \pi d \Rightarrow \frac{l_2}{l_1} = \frac{\pi d_2}{\pi d_1} = \frac{d_2}{d_1}$$

$$\therefore eq.\ (1) \Rightarrow T_2 = T_1 + \frac{1}{\alpha}\ln\left[\frac{d_2}{d_1}\right] = 20^\circ C + \frac{\ln[59.6/59.4]}{1.2 \times 10^{-5}} = \boxed{300.11^\circ C}$$

JAM 2016

[Q59] The maximum and minimum speeds of a comet that orbits the Sun are 80 and 10 km/s respectively. The ratio of the aphelion distance of the comet to the radius of the Earth's orbit is _____. (Assume that Earth moves in a circular orbit of radius 1.5×10^8 km with a speed of 30 km/s.)

$\rightarrow$ **aphelion/apogee**: far point; i.e. $r_{\max}$ & $v_{\min}$

$\rightarrow$ **perihelion/perigee**: near point; i.e. $r_{\min}$ & $v_{\max}$

To find: ratio of *aphelion distance* to radius of Earth's orbit $= \dfrac{r_{\max}}{R_E}$.

$$\text{For the comet, } h = r^2\frac{d\theta}{dt} = r \cdot r\omega = r_{\max}v_{\min} = r_{\min}v_{\max} \quad ...(1)$$

$$\therefore \frac{r_{\max}}{r_{\min}} = \frac{v_{\max}}{v_{\min}} = \frac{80}{10} = 8 \Rightarrow r_{\max} = 8r_{\min}$$

$$\therefore e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} = \frac{8r_{\min} - r_{\min}}{8r_{\min} + r_{\min}} = \frac{7}{9}$$

$$\therefore l = (1-e)r_{\max} = \left(1 - \frac{7}{9}\right)r_{\max} = \frac{2r_{\max}}{9} \quad ...(2)$$

$$\text{For Earth (circular orbit), } e = 0,\ \therefore l = r_{\max} = r_{\min} = R_E,\ h = R_E V_E \quad ...(3)$$

$$\text{For all planets, } \frac{h^2}{l} = \text{a constant.}$$

$$\therefore eq.\ (1), (2), (3) \Rightarrow \frac{r_{\max}^2 v_{\min}^2}{2r_{\max}/9} = \frac{R_E^2 V_E^2}{R_E}$$

$$\Rightarrow \frac{9r_{\max}v_{\min}^2}{2} = R_E V_E^2 \Rightarrow \frac{r_{\max}}{R_E} = \frac{2}{9}\frac{V_E^2}{v_{\min}^2} = \frac{2}{9}\cdot\frac{30^2}{10^2} = \boxed{2}$$

$\rightarrow$ *Refer Upadhyaya's Mechanics* ***p. 305-310***