FOURIER SERIES

Tips & Tricks

8 • General Formula byheart ആക്കണം :-

9 Let $f(x)$ be defined over any interval (c, d)

10 Point to remember : $\boxed{L = \text{Half-the-period}}$

11 Then
$$f(x) = \underbrace{\frac{a_0}{2}}_{\substack{\text{vertical shift}\\ \text{(see below)}}} + \sum_{n=1}^{\infty} \underbrace{a_n \cos\frac{n\pi x}{L}}_{\substack{\text{horizontally}\\ \text{symmetric}\\ \text{(even) term}}} + \underbrace{b_n \sin\frac{n\pi x}{L}}_{\substack{\text{horizontally}\\ \text{antisymmetric}\\ \text{(odd) term}}}$$

12

period $2L$

1
$$a_n = \frac{1}{L}\int_c^d f(x)\cos\frac{n\pi x}{L}\,dx$$
{ including a_0 (put $n=0$) ഈ formula മതി
otherwise $n = 1, 2, 3 \ldots$

2

3
$$b_n = \frac{1}{L}\int_c^d f(x)\sin\frac{n\pi x}{L}\,dx$$
{ $n = 1, 2, 3 \ldots$

4 e.g. if period $= 2\pi$, $L = \pi$. If period is π, $L = \frac{\pi}{2}$

if period $= L$, $L \to L/2$. Then $\frac{1}{L} \to \frac{2}{L}$ //

5 (half-range expansion-ൽ ~~[illegible]~~ വരുന്നത് ഈ 2 ആണ്
see next page)

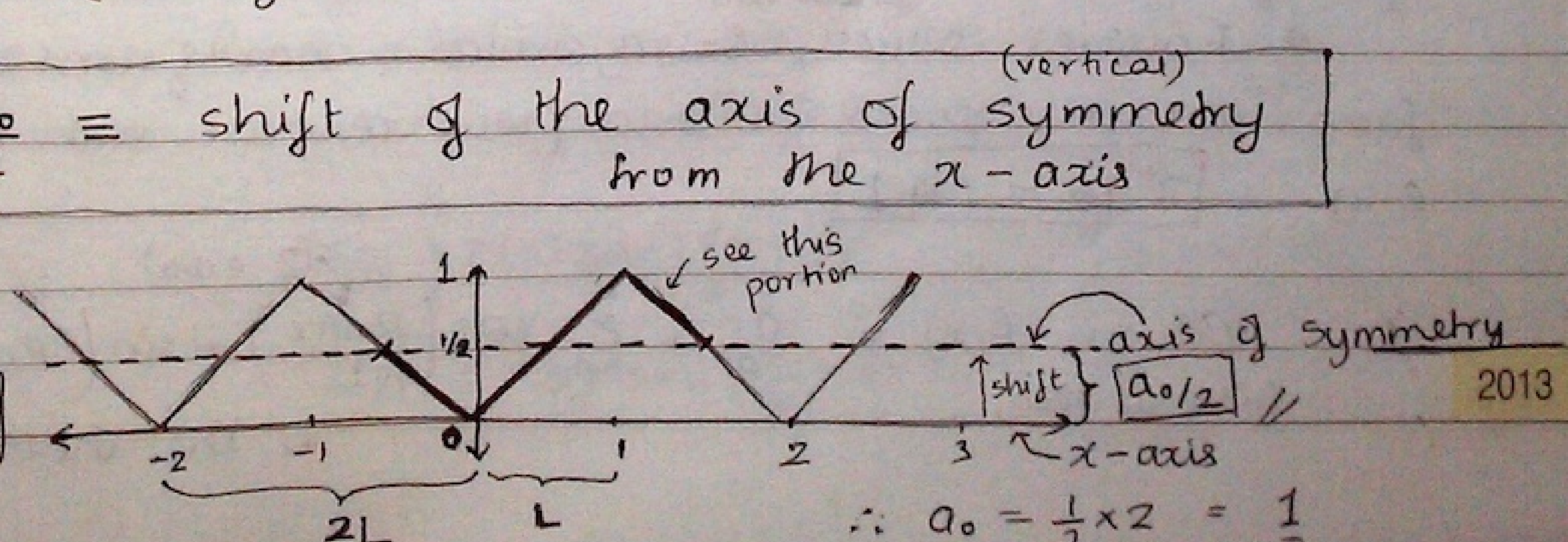
• SYMMETRY :-

→ Vertical symmetry :- if $f(x)$ (f(x) ന്റെ ഗ്രാഫ്) is vertically symmetric (antisymmetric) i.e. about x-axis, then :

$$\boxed{\frac{a_0}{2} \equiv \text{shift of the axis of (vertical) symmetry from the } x\text{-axis}}$$

e.g.

J15 - A29



$\therefore a_0 = \frac{1}{2} \times 2 = 1$

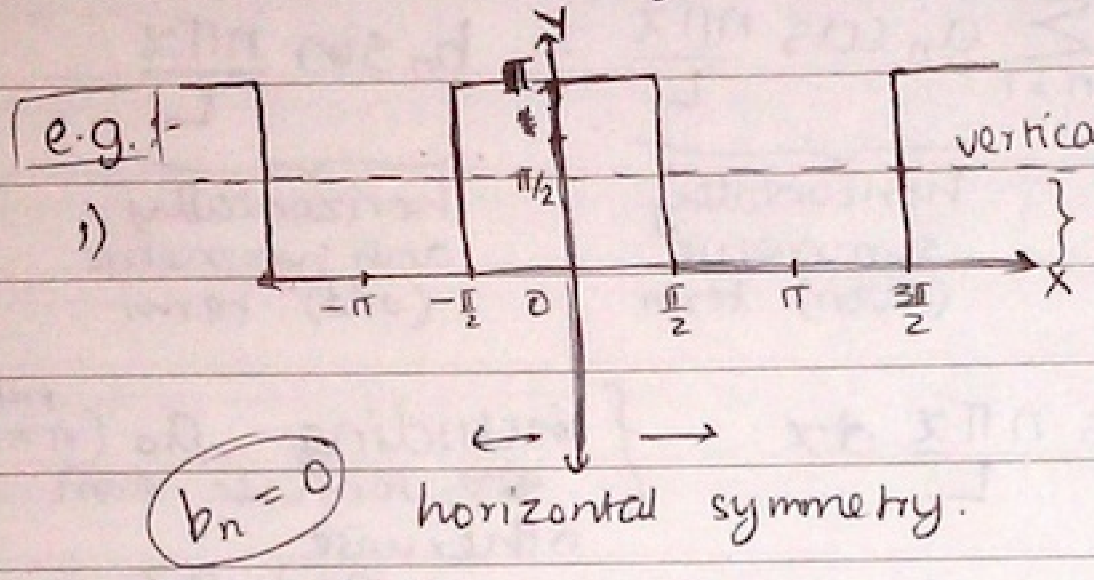
→ Horizontal symmetry (even or odd function) :-

(even function) 1) If $f(x)$ is symmetric about y-axis, there will be only cosine terms ; i.e, all $b_n = 0$ (a_0 ഉണ്ടാവും)

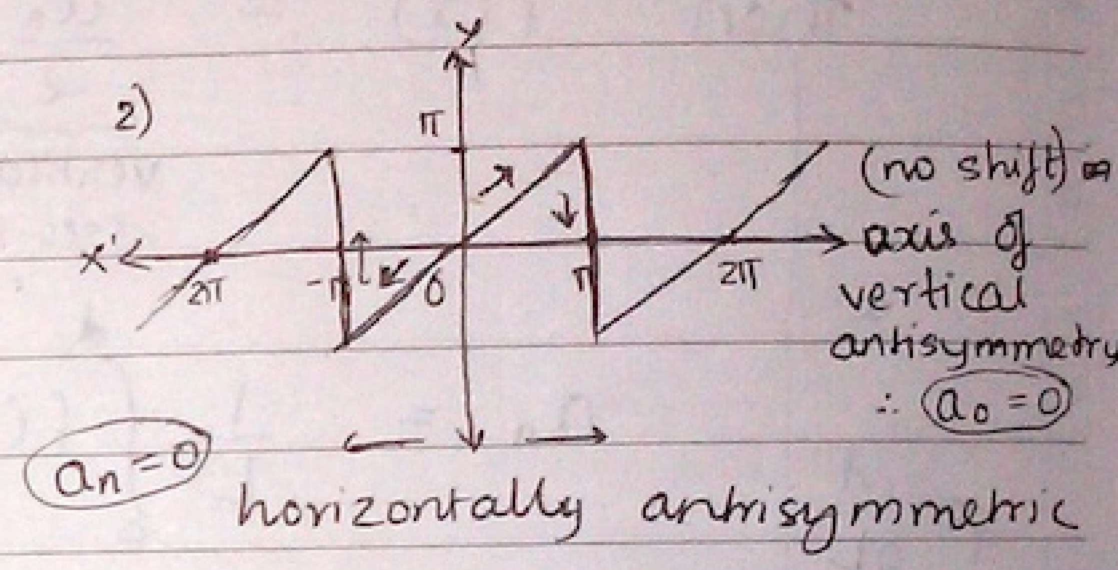
(odd function) 2) If $f(x)$ is antisymmetric about y-axis, there will be only sine terms ; i.e. all $a_n = 0$ (a_0 ഉണ്ടാവും)

e.g. :-

1)



2)



(not important) → Shifting horizontally (wave eqn-ൽ phase constant കൂടുന്നത് പോലെ)

$x \to x - \delta$, graph will shift δ to the right

$x \to x + \delta$, graph will shift δ to the left

• Calculation എളുപ്പമാക്കാൻ :-

→ ചോദിച്ചത് മാത്രം കണ്ടു പിടിക്കുക (General term വേണമെന്നില്ല)

→ $f(x) \to 0$ അല്ലാത്ത interval-ൽ integrate ചെയ്താൽ മതി

→ interval $\neq$ period (not always) ; period ചോദ്യത്തിൽ തന്നിട്ടില്ലെങ്കിൽ $n\pi x$-ന്റെ coefficient നോക്കുക. (denominator ആയിരിക്കും L)

→ Fourier series general form ചോദ്യത്തിൽ തന്നിട്ടുണ്ടെങ്കിൽ ആ form തന്നെ ഉപയോഗിക്കുക (or compare with our form)

e.g. $\boxed{\text{J16 - C41}}$

(2 extra)

Given $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{L}\right) + \sin\left(\frac{2n\pi x}{L}\right)$

general form,

in the interval $(0, L)$

$\frac{2n\pi x}{L} = \frac{n\pi x}{(L/2)}$; hence, Half-the-period $= \frac{L}{2}$ (period $= L$)

Then given $f(x) = x$, $x \in (0, \pi)$ {i.e. $L = \pi$

To find : b_2 ; directly calculate only b_2 :-

as $b_n = \frac{1}{L/2} \int_0^L f(x) \sin\left(\frac{2n\pi x}{L}\right) dx$, $L = \pi$,

$$b_2 = \frac{2}{\pi} \int_0^\pi x \sin\left(\frac{2\times 2\pi x}{\pi}\right) dx$$

(Integration by parts) $$= \frac{2}{\pi}\left\{-\left[x\frac{\cos 4x}{4}\right]_0^\pi + \int_0^\pi 1\cdot\frac{\cos 4x}{4}\, dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{4}(\pi \times 1 - 0) + \frac{1}{4}\cancel{\left[\frac{\sin 4x}{4}\right]_0^\pi}\right\}$$

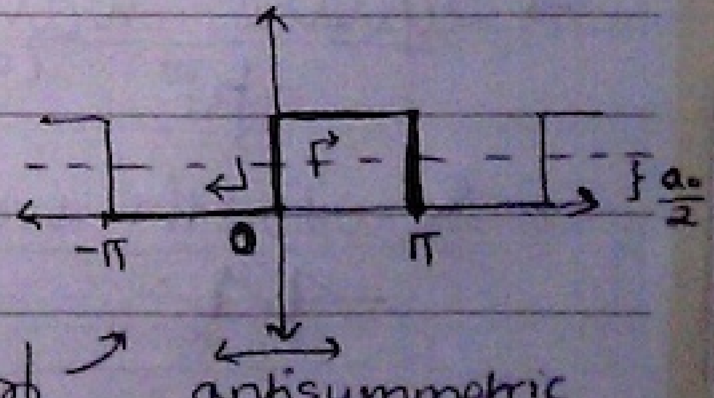
$$= \frac{2}{\pi} \times \frac{-\pi}{4} = \frac{-1}{2} = \boxed{-0.5}$$ //

e.g. $\boxed{\text{J17 - A28}}$:- period given : 2π, interval $(-\pi, \pi)$

→ $\boxed{\text{Harmonic}}$ എന്നാൽ $\sin\frac{n\pi x}{L}$, $\cos\frac{n\pi x}{L}$ terms ആണ്.
1st - harmonic : $n = 1$, 2nd harmonic; $n = 2$, etc.

→ Note :- Overtone എന്നാലും same ആണ്, BUT :-
1st overtone : $n = 2$, 2nd overtone, $n = 3$, etc.
($n = 1$ is called Fundamental

Here we have to find : ~~ratio of~~
$$\frac{\text{coefficient of 1st harmonic}}{\text{coefficient of 3rd harmonic}} = \frac{b_1}{b_3}$$



(b_1 & b_3 എടുത്താൽ മതി കാരണം $a_n = 0$; $f(x)$ odd-function ആണെന്ന് graph വരച്ച് നോക്കിയാൽ antisymmetric ആണല്ലോ.)

$$\therefore \frac{b_1}{b_3} = \frac{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{1\pi x}{\pi}\right)dx}{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{3\pi x}{\pi}\right)dx} = \frac{\int_0^\pi 1.\sin x\, dx}{\int_0^\pi 1.\sin 3x\, dx}$$ { $f(x) = 0$ for $-\pi <$

$$\Rightarrow \frac{b_1}{b_3} = \frac{\cancel{-}[\cos x]_0^{\pi}}{\cancel{-}\left[\frac{\cos 3x}{3}\right]_0^{\pi}} = \frac{\cancel{(-1-1)}}{\cancel{(-1-1)}} \times 3 = \boxed{3} \; //$$

• **Integration tips** :- (sem 4 Maths - Fourier series - even / odd functions)

→ If $f(x)$ is even (y-symmetric) $\displaystyle\int_{-L}^{L} f(x)\,dx = 2 \times \int_0^L f(x)\,dx$

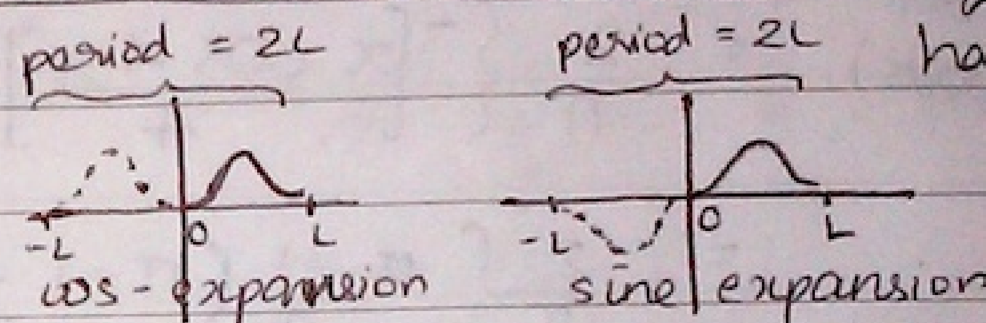
even × even, odd × odd ⇒ even

→ If $f(x)$ is odd (y-antisymmetric) $\displaystyle\int_{-L}^{L} f(x)\,dx = 0$

odd × even, even × odd ⇒ odd

ഇതാണ് half-range expansion ൽ 2

• **Half range expansion** :-



Periodic അല്ലാത്ത functions നെ periodic ആക്കി Fourier series കണ്ടുപിടിക്കാനുള്ള technique. Function SHOULD be defined in the interval $(0, L)$

cosine expansion

$$a_n = 2 \times \frac{1}{L} \int_0^L f(x) \cos\frac{n\pi x}{L}\,dx$$

$b_n = 0$

sine - expansion

$$b_n = 2 \times \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$a_n = 0$

(half-the-period is still L)

* **Some common Fourier series** :-

→ **Square wave** :- $(0, 2\pi)$

$$f(x) = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)}$$

i.e., odd multiples of x 1, 3, 5, . . .

→ **Sawtooth wave** $(-\pi, \pi)$

$$f(x) = \frac{A}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

peak-to-peak amplitude $= A$

→ $\sin^2 x$: We have formula,

$$\boxed{\sin^2 x = \underbrace{\frac{1}{2}}_{\frac{a_0}{2}} - \underbrace{\frac{1}{2}\cos 2x}_{a_2}}$$

→ $\cos^2 x$:- Similarly,

$$\boxed{\cos^2 x = \underbrace{\frac{1}{2}}_{\frac{a_0}{2}} + \underbrace{\frac{1}{2}\cos 2x}_{a_2}}$$

(both taking period 2π)