Twenty-third International Olympiad,1982

1. . The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0\,(or)\,1$$

$$f(2) = 0, f(3) > 0, and f(9999) = 3333.$$

Determine $f(1982)$.

2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1, 2, 3, M_i$ is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection. of T_i in the interior bisector of angle A_i. Prove that the lines M_1, S_1,M_2S_2 and M_3S_3 are concurrent.

3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with following properties:
$x_0 = 1$, and for all $i \geq 0, x_{i+1} \leq x_i$.
(a) Prove that for every such sequence, there is $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ... + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + ... + \frac{x_{n-1}^2}{x_n} < 4.$$

4. Prove that if n is a positive integer such that the equation.

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions. Show that the equation has no solutions in integers when $n = 2891$.

5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by

the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear.

6. Let S be a square with sides of length 100, and let L be a path with in S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, ...A_{n-1}A_1$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in & such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.