Uncertainty

Chapter 13

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Outline

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

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Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time?

Problems:

1. Partial observability (road state, other drivers' plans, etc.)
2. Noisy sensors (traffic reports)
3. Uncertainty in action outcomes (flat tire, etc.)
4. Immense complexity of modeling and predicting traffic

Hence a purely logical approach either

1. Risks falsehood: "A_{25} will get me there on time", or
2. Leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport …)

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Methods for handling uncertainty

- Default or nonmonotonic logic:
  - Assume my car does not have a flat tire
  - Assume A_{25} works unless contradicted by evidence
- Issues: What assumptions are reasonable? How to handle contradiction?
- Rules with fudge factors:
  - $A_{25} \mid\!\rightarrow_{0.3}$ get there on time
  - $Sprinkler \mid\!\rightarrow_{0.99} WetGrass$
  - $WetGrass \mid\!\rightarrow_{0.7} Rain$
- Issues: Problems with combination, e.g., *Sprinkler* causes *Rain*??
- Probability
  - Model agent's degree of belief
  - Given the available evidence,
  - A_{25} will get me there on time with probability 0.04

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Probability

Probabilistic assertions summarize effects of

- laziness: failure to enumerate exceptions, qualifications, etc.
- ignorance: lack of relevant facts, initial conditions, etc.

Subjective probability:

- Probabilities relate propositions to agent's own state of knowledge

  e.g., $P(A_{25} \mid \text{no reported accidents}) = 0.06$

These are not assertions about the world

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

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Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time | …) = 0.04
$P(A_{90}$ gets me there on time | …) = 0.70
$P(A_{120}$ gets me there on time | …) = 0.95
$P(A_{1440}$ gets me there on time | …) = 0.9999

- Which action to choose?

  Depends on my preferences for missing flight vs. time spent waiting, etc.

  - Utility theory is used to represent and infer preferences
  - Decision theory = probability theory + utility theory

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Syntax

- Basic element: Random variable

- Similar to propositional logic: possible worlds defined by assignment of values to random variables.

- Boolean random variables
  e.g., *Cavity* (do I have a cavity?)
- Discrete random variables
  e.g., *Weather* is one of <*sunny,rainy,cloudy,snow*>
- Domain values must be exhaustive and mutually exclusive

- Elementary proposition constructed by assignment of a value to a random variable: e.g.,
  Weather = *sunny*, *Cavity* = *false* (abbreviated as $\neg$*cavity*)

- Complex propositions formed from elementary propositions and standard logical connectives e.g.,
  Weather = *sunny* $\vee$ *Cavity* = *false*

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Syntax

- Atomic event: A complete specification of the state of the world about which the agent is uncertain
  E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

  Cavity = *false* $\wedge$ *Toothache* = *false*
  Cavity = *false* $\wedge$ *Toothache* = *true*
  Cavity = *true* $\wedge$ *Toothache* = *false*
  Cavity = *true* $\wedge$ *Toothache* = *true*

- Atomic events are mutually exclusive and exhaustive

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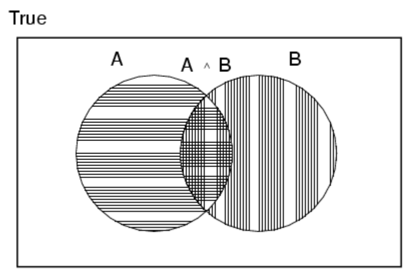
Axioms of probability

- For any propositions *A, B*
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$ and $P(false) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$
  -



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Prior probability

- Prior or unconditional probabilities of propositions
  e.g., P(*Cavity* = true) = 0.1 and P(*Weather* = sunny) = 0.72
  correspond to belief prior to arrival of any (new) evidence

- Probability distribution gives values for all possible assignments:
  P(*Weather*) = <0.72,0.1,0.08,0.1> (normalized, i.e., sums to 1)

- Joint probability distribution for a set of random variables gives the probability of every atomic event on those random variables

  P(*Weather,Cavity*) = a 4 × 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow
Cavity = true	0.144	0.02	0.016	0.02
Cavity = false	0.576	0.08	0.064	0.08

- Every question about a domain can be answered by the joint distribution

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Conditional probability

- Conditional or posterior probabilities
  e.g., P(*cavity* | *toothache*) = 0.8
  i.e., given that *toothache* is all I know

- (Notation for conditional distributions:
  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors)

- If we know more, e.g., *cavity* is also given, then we have
  P(*cavity* | *toothache,cavity*) = 1

- New evidence may be irrelevant, allowing simplification, e.g.,
  P(*cavity* | *toothache, sunny*) = P(*cavity* | *toothache*) = 0.8
- This kind of inference, sanctioned by domain knowledge, is crucial

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Conditional probability

- Definition of conditional probability:
  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$

- Product rule gives an alternative formulation:
  $P(a \wedge b) = P(a \mid b) P(b) = P(b \mid a) P(a)$

- A general version holds for whole distributions, e.g.,
  P(*Weather,Cavity*) = **P**(*Weather* | *Cavity*) **P**(*Cavity*)
- (View as a set of 4 × 2 equations, not matrix mult.)

- Chain rule is derived by successive application of product rule:

$$\begin{aligned}\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1,\ldots,X_{n-1})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\ &= \mathbf{P}(X_1,\ldots,X_{n-2})\, \mathbf{P}(X_{n-1} \mid X_1,\ldots,X_{n-2})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\ &= \ldots \\ &= \pi_{i=1}\text{^}n\, \mathbf{P}(X_i \mid X_1, \ldots ,X_{i-1})\end{aligned}$$

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Inference by enumeration

- Start with the joint probability distribution:
-

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ cavity	**.016**	**.064**	**.144**	**.576**

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

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- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2

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Inference by enumeration

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	catch	*¬ catch*	*catch*	*¬ catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ cavity	**.016**	**.064**	**.144**	**.576**

- Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$
$$= \frac{0.016+0.064}{0.108 + 0.012 + 0.016 + 0.064}$$
$$= 0.4$$

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Normalization

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Denominator can be viewed as a normalization constant α

P(*Cavity* | *toothache*) = α, **P**(*Cavity,toothache*)
= α, [**P**(*Cavity,toothache,catch*) + **P**(*Cavity,toothache*,¬ *catch*)]
= α, [<0.108,0.016> + <0.012,0.064>]
= α, <0.12,0.08> = <0.6,0.4>

General idea: compute distribution on query variable by fixing evidence variables and summing over hidden variables

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Inference by enumeration, contd.

Typically, we are interested in
the posterior joint distribution of the query variables **Y**
given specific values **e** for the evidence variables **E**

Let the hidden variables be **H** = **X** - **Y** - **E**

Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha\mathbf{P}(\mathbf{Y},\mathbf{E} = \mathbf{e}) = \alpha\Sigma_h\mathbf{P}(\mathbf{Y},\mathbf{E}= \mathbf{e}, \mathbf{H} = \mathbf{h})$$

- The terms in the summation are joint entries because **Y**, **E** and **H** together exhaust the set of random variables

- Obvious problems:
  1. Worst-case time complexity $O(d^n)$ where d is the largest arity
  2. Space complexity $O(d^n)$ to store the joint distribution
  3. How to find the numbers for $O(d^n)$ entries?

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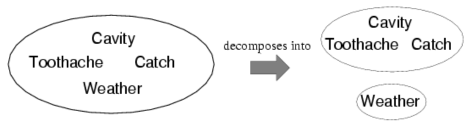
Independence

- *A* and *B* are independent iff
  P(*A*|*B*) = **P**(*A*) or **P**(*B*|*A*) = **P**(*B*) or **P**(A, B) = **P**(*A*) **P**(*B*)



  P(*Toothache, Catch, Cavity, Weather*)
  = **P**(*Toothache, Catch, Cavity*) **P**(*Weather*)

- 32 entries reduced to 12; for *n* independent biased coins, $O(2^n) \rightarrow O(n)$

- Absolute independence powerful but rare

- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

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Conditional independence

- **P**(*Toothache, Cavity, Catch*) has $2^3 - 1 = 7$ independent entries

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  (1) **P**(*catch* | *toothache, cavity*) = **P**(*catch* | *cavity*)

- The same independence holds if I haven't got a cavity:
  (2) **P**(*catch* | *toothache,*$\neg$*cavity*) = **P**(*catch* | $\neg$*cavity*)

- *Catch* is conditionally independent of *Toothache* given *Cavity*:
  P(*Catch* | *Toothache,Cavity*) = **P**(*Catch* | *Cavity*)

- Equivalent statements:
  P(*Toothache* | *Catch, Cavity*) = **P**(*Toothache* | *Cavity*)

  P(*Toothache, Catch* | *Cavity*) = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*)

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Conditional independence contd.

- Write out full joint distribution using chain rule:
  P(*Toothache, Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch, Cavity*)

  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch* | *Cavity*) **P**(*Cavity*)

  = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*) **P**(Cavity)

  I.e., 2 + 2 + 1 = 5 independent numbers

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.
- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

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Bayes' Rule

- Product rule $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$
  $\Rightarrow$ Bayes' rule: $P(a \mid b) = P(b \mid a)\, P(a) / P(b)$
- or in distribution form
  $\mathbf{P}(Y|X) = \mathbf{P}(X|Y)\, \mathbf{P}(Y) / \mathbf{P}(X) = \alpha \mathbf{P}(X|Y)\, \mathbf{P}(Y)$
- Useful for assessing diagnostic probability from causal probability:
  - P(Cause|Effect) = P(Effect|Cause) P(Cause) / P(Effect)
  - E.g., let *M* be meningitis, *S* be stiff neck:
    $P(m|s) = P(s|m)\, P(m) / P(s) = 0.8 \times 0.0001 / 0.1 = 0.0008$
  - Note: posterior probability of meningitis still very small!

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Bayes' Rule and conditional independence

P(*Cavity* | *toothache* $\wedge$ *catch*)

= α**P**(*toothache* $\wedge$ *catch* | *Cavity*) **P**(*Cavity*)

= α**P**(*toothache* | *Cavity*) **P**(*catch* | *Cavity*) **P**(*Cavity*)

- This is an example of a naïve Bayes model:

  P(Cause,Effect$_1$, … ,Effect$_n$) = **P**(Cause) π_i**P**(Effect$_i$|Cause)

- Total number of parameters is linear in *n*

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Summary

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- Independence and conditional independence provide the tools

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