```
    {
        BYTE color=getpixel(h+(60-1)*8,v+(1-1)*16);
        if(color!=0)
        {
            setfillstyle(1,12);
            floodfill(h*20+10,v*20+10,14);
        }
        else
        {
            setfillstyle(1,0);
            floodfill(h*20+10,v*20+10,14);
        }
    }
    gotoxy(60,1);
    printf("%c",32);
    showmouse();
    showval();
}
RESULT: The output of the above-listed program is shown in Fig. 3.30.
```

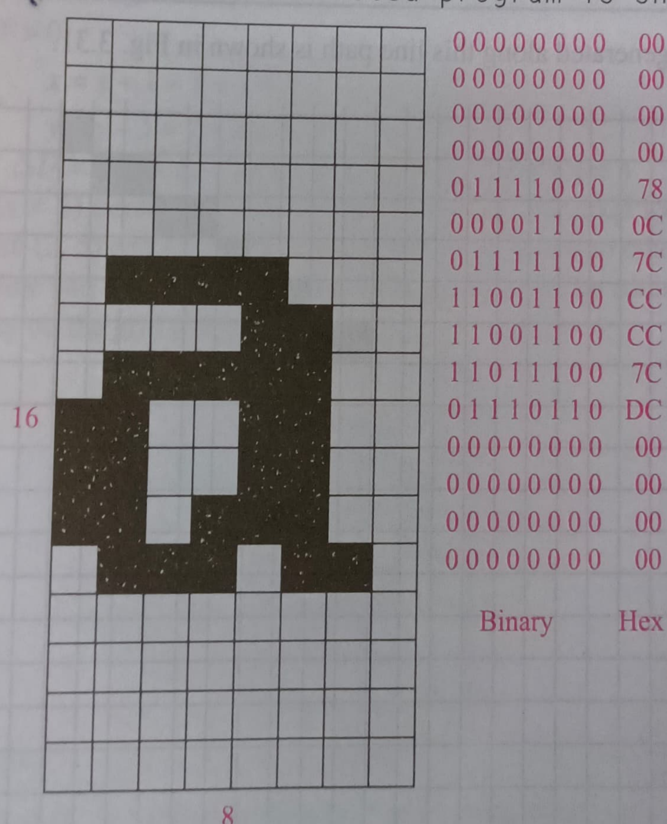


Fig. 3.30 *Output of the above-listed program*

Solved Exercises

3.1 Digitize a line from (10, 12) to (20, 18) on a raster screen using Bresenham's straight line algorithm. The result may be shown on a Cartesian graph.

Solution

These steps can be followed to digitize the sample end points say (10, 12) and (20, 18).
Slope of the sample line is 0.6, which is less than 1, as $\Delta x = 10$ and $\Delta y = 6$.
The initial decision parameter $r_o = 2\,\Delta y - \Delta x = 2 \times 6 - 10 = 2.$
The increments for calculating successive decision parameter are $2\,\Delta y = 12$ and $2\Delta y - 2\,\Delta x = -8$
Plot the initial point $(x_o, y_o) = (10, 12)$, the successive points can be tabulated as follows:

P	r_p	(x_{p+1}, y_{p+1})
0	2	(11, 13)
1	–6	(12, 13)
2	6	(13, 14)
3	–2	(4, 14)
4	10	(15, 15)
5	2	(16, 16)
6	–6	(17, 16)
7	6	(18, 17)
8	–2	(19, 17)
9	10	(20, 18)

A plot of the pixels generated along this line path is shown in Fig. 3.31.

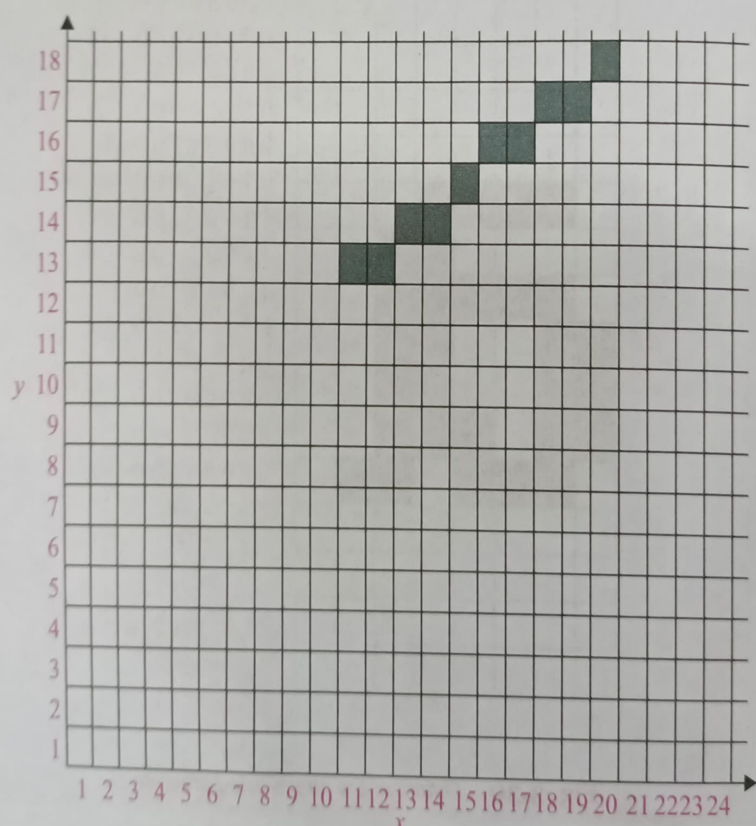


Fig. 3.31 *The plot of the above-calculated line*

3.2 Find the pixel location approximating the first octant of a circle having a centre (2, 3) and a radius of 2 units, using Bresenham circle algorithm. Use this to plot the complete circle on a Cartesian graph representing pixel grids.

Solution

$$x_c = 2, y_c = 3 \text{ and } R = 2$$
$$x = 0, y = R = 2$$
$$\Delta D = 2(1 - R) = 2(1 - 2) = -2$$

Iteration 1: Since $(y = 2) > (x = 0)$

putpixel $[(x_c + x), (y_c + y)]$? putpixel (2, 5)

Since $\Delta D = -2 < 0$

$$\delta = \delta_{HD} = 2\Delta D + 2y - 1 = 2(-2) + 2 \times 2 - 1 = -1$$

Since $\delta < 0$

$$x = x + 1 = 0 + 1 = 1$$
$$y = 2 \text{ (unchanged)}$$
$$\Delta D = \Delta D + 2x + 1 = -2 + 2 \times 1 + 1 = 1$$

Iteration 2: Since $(y = 2) > (x = 1)$

putpixel $[(x_c + x), (y_c + y)]$? putpixel (3, 5)

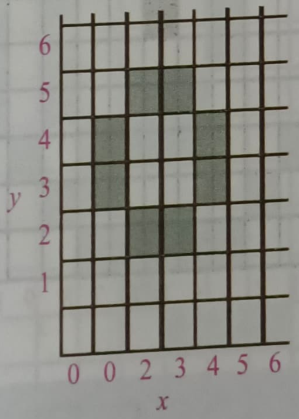
Since $\Delta D > 0$

$$\delta = \delta_{VD} = 2x - 2\,\Delta D + 1 = 2 \times 1 - 2 \times 1 + 1 = 1$$

Since $\delta > 0$

$$x = x + 1 = 1 + 1 = 2$$
$$y = y - 1 = 2 - 1 = 1$$
$$\Delta D = \Delta D + 2x - 2y + 1 = 1 + 2 \times 2 - 2 \times 1 + 1 = 4$$

Since $(y = 1) < (x = 2)$ iteration 3 will not run. Therefore, the pixel locations in the first octant of the given circle are (2, 5) and (3, 5). This octant will be mirrored to other octants about $x = 0$, $y = 0$, $x = y$ and $x = -y$ line and will then be located to the given centre by adding x_c and y_c to the calculated pixels. The display on the grid will be as follows:



3·3 Plot a circle centred at (5, 5) having a radius of 5 units using midpoint circle algorithm and Cartesian graph.

Solution

Calculate the circle in the first octant $x = 0$ to $x = y$ and then plot to the other octants based on symmetry.

The first point to plot is $(x_o + x_c, y_o + y_c) = (0 + 5, R + 5) = (5, 10)$

Initial decision parameter $r_o = 1 - R = 1 - 5 = -4$

The increment terms are $2x_o = 2\ (0) = 0$ and $2y_o = 2\ (5) = 10$

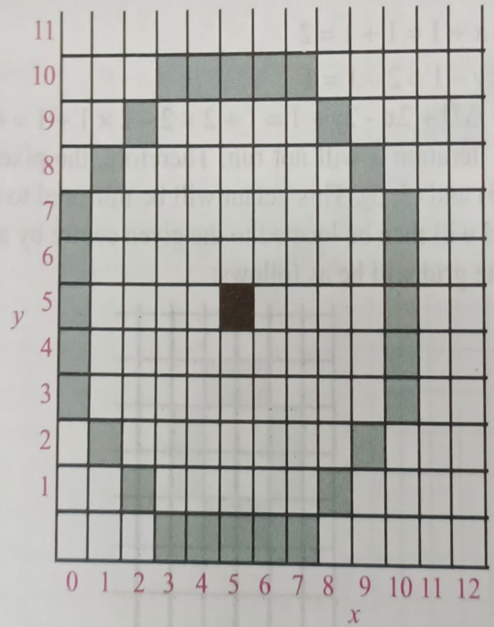
For each position thereafter starting at $p = 0$, if $r_p < 0$, then the next point will be $(x_p + 1, y_p)$ or else $(x_p + 1, y_p - 1)$.

The next decision parameter $r_{p+1} = r_p + 2x_{p+1} + 1$ if $r_p < 0$ or else $r_{p+1} = r_p + 2x_{p+1} - 2y_{p+1} + 1$, where $x_{p+1} = x_p + 1$ and $y_{p+1} = y_p - 1$.

The successive calculation can be tabulated as in the following:

P	r_p	(x_{p+1}, y_{p+1})	$2x_{p+1}$	$2y_{p+1}$	Actual points $[(x_c + x), (y_c + y)]$
0	–4	(1, 5)	2	10	(6, 10)
1	–1	(2, 5)	4	10	(7, 10)
2	4	(3, 4)	6	8	(8, 9)
3	3	(4, 3)	8	6	(9, 8)
4	6	(5, 2)	10	4	(10, 7)

As $x_5 > y_5$ the algorithm will not proceed. The plot will be as shown below after symmetrical plotting to other octants.



Review Questions

3.1 Define graphics primitives. Mention some typical graphics primitives.

3.2 Why it is preferred to take unit x increment or unit y increment corresponding to slope $m \leq 1$ or slope $m \geq 1$ in line-drawing algorithms.

3.3 Justify the approach of using integer arithmetic in a Bresenham line-drawing algorithm. Explain how rasterization accuracy is preserved despite using integer arithmetic.

3.4 Develop an algorithm to draw a thick line between two points.

3.5 Digitize a line from (1, 2) to (12, 18) on a raster screen using Bresenham's straight-line algorithm. Compare it with line generated using a DDA algorithm.

3.6 Find the pixel location approximating the first octant of a circle having a centre (10, 13) and a radius of 5 units using the Bresenham circle algorithm. Use this to plot the complete circle on pixel grids.

3.7 Plot a circle centred at (2, 5) having radius of 7 units using the midpoint circle algorithm.

3.8 Compare the advantages of Bresenham line-drawing algorithm over the DDA algorithm.

3.9 Modify Bresenham's line-drawing algorithm so that it will produce a dashed line. The dash length should be independent of slope. [University Question]

3.10 A line will be drawn from (x_1, y_1) and (x_2, y_2). Scan conversion starts from both (x_1, y_1) to (x_2, y_2) and (x_2, y_2) to (x_1, y_1) simultaneously, following Bresenham's algorithm.

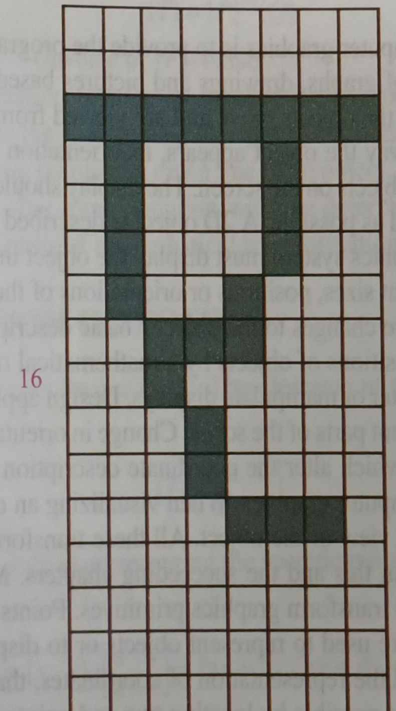
1. Write algorithm steps for such an implementation
2. What is the advantage of this technique? [University Question]

3.11 How can Bresenham's line-drawing algorithm be modified so that the antialiasing effects are produced while generating a straight line? [University Question]

3.12 When 8–way symmetry is used to obtain a full circle from pixel coordinates generated for the 0° to 45° octant, some pixels are set or plotted twice. This phenomenon is sometimes referred to as over-strike. Identify where overstrike occurs. [University Question]

3.13 Distinguish between seed filling and scan line-filling algorithm. Apply any of these algorithms to fill the polygon defined by (1, 1), (1, 5) and (5, 2). The seed pixel may be taken at any suitable location inside the polygon if required.

3.14 Derive all the 16 hexadecimal codes sequence to be inserted in the font table to create a user-defined font of size 16 pixel × 8 pixel, as shown in the figure below.



3.15 A pie chart needs to be drawn to show the percentage of total marks obtained by students. Suppose 8% of students got 90% and above; 52% got 75% to 85%; 30% got 60% to 75% and the remaining obtained below 60% marks. No one failed (<50% marks). Write a pseudo code to draw the pie chart and fill the segments of the pie chart with different colours.

3.16 Enumerate the advantages and disadvantages of flood fill and boundary filling algorithms.

```
translate2d(getmaxx()/2, getmaxy()/2 + 100);
setcolor(GREEN);
line(initialmatrix[0][0],initialmatrix[1][0],initialmatrix[0][1],
initialmatrix[1][1]);
line(initialmatrix[0][1],initialmatrix[1][1],initialmatrix[0][2],
initialmatrix[1][2]);
line(initialmatrix[0][2],initialmatrix[1][2],initialmatrix[0][0],
initialmatrix[1][0]);
getch();
}
```

Solved Exercises

4.1 Develop a general form of scaling matrix about a fixed point (x_f, y_f).

Solution

As a scaling matrix, described in Section 4.9, is always with respect to the origin so any object that needs to be scaled with respect to any arbitrary fixed point (x_f, y_f) is required to be translated so as to bring the scaling reference point to the origin, scaled and then translated to the same location again. The sequence of transformation will be thus $[T][S][T]^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_f & -y_f & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_f & y_f & x_1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ x_f(1-s_x) & y_f(1-s_y) & 1 \end{bmatrix}$$

4.2 Magnify the triangle $P(0, 0)$, $Q(2, 2)$ and $R(10, 4)$ to four times its size while keeping $R(10, 4)$ fixed.

Solution

The process is equivalent to scaling the triangle four times with respect to fixed point $R(10, 4)$. Hence identifying the parameters as shown in previous problem, the required parameters are

$$s_x = s_y = 4,\ x_f = 10 \text{ and } y_f = 4$$

The transformation matrix will be $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 10(1-4) & 4(1-4) & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -30 & -12 & 1 \end{bmatrix}$

which on applying to the set of three points simultaneously one gets

$$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 10 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -30 & -12 & 1 \end{bmatrix} = \begin{bmatrix} -30 & -12 & 1 \\ -22 & -4 & 1 \\ 10 & 4 & 1 \end{bmatrix}$$

Hence the resultant triangle is $P'(-30, -12)$, $Q(-22, -4)$ and $R(10, 4)$

4.3 Find the reflection of a point (p, q) about a line $y = mx + c$.

Solution

Referring to Section 4.13, the series of transformations are $[T] = [T_{trans}][R_\theta][R_{ref}][R_\theta]^{-1}[T_{trans}]^{-1}$

Hence, merging all the matrices given in the Section 4.13, the required matrix will be

$$[T] = \begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} & 0 \\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2} & 0 \\ \dfrac{-2cm}{1+m^2} & \dfrac{2c}{1+m^2} & 1 \end{bmatrix}$$

which on applying to the point (p, q), the matrix will be

$$[p \ q \ 1][T] = [p \ q \ 1]\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} & 0 \\ \dfrac{2m}{1+m^2} & -\dfrac{1-m^2}{1+m^2} & 0 \\ \dfrac{-2cm}{1+m^2} & \dfrac{2c}{1+m^2} & 1 \end{bmatrix}$$

Hence

$$P' = p\frac{1-m^2}{1+m^2} + q\frac{2m}{1+m^2} - \frac{2cm}{1+m^2} = \frac{p(1-m^2) + 2qm - 2cm}{1+m^2}$$

$$Q' = p\frac{2m}{1+m^2} - q\frac{1-m^2}{1+m^2} + \frac{2c}{1+m^2} = \frac{2pm + q(m^2-1) + 2c}{1+m^2}$$

are the required transformed points.

A group of points forming any object can be similarly transformed using its homogeneous matrix notation and multiplying it to the transformation matrix, as demonstrated in Exercise 4.2.

4.4 An object is defined with respect to a coordinate system whose units are measured in meters. If an observer's coordinate system uses centimeter as the basic unit, what is the coordinate transformation used to describe object coordinate in the observer's coordinate system?

Solution

Since there are 100 cm in a meter, the required transformation can be described by a coordinate system scaling transformation by $s_{x,} = s_{y,} = 1/100$. Thus, the transformation matrix using homogeneous coordinate system, the scaling transformation matrix will be

$$[T] = \begin{bmatrix} \dfrac{1}{100} & 0 & 0 \\ 0 & \dfrac{1}{100} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(refer Section 4.9).}$$

4.5 Transform the square $P(0, 0)$, $Q(10, 0)$, $R(10, 10)$ and $S(0, 10)$ into a master picture coordinate system with half of its size with centre at $(-1, -1)$.

Solution

The centre of the square *PQRS* is at $C(5, 5)$. It is required to first scale the square to half its size by keeping C fixed. Then apply the translation so as to move the centre C of the square to the master picture centre, that is, $(-1, -1)$.

Hence, the scaling factors will be $s_x = s_y = ½$ and reference point for scaling will be $(x_f, y_f) = (5, 5)$. For translation to the master picture, -6 units along x and -6 units along y, to locate the centre of the scaled square to the centre of the master picture.

(refer Exercise 4.1 for fixed point scaling)

The scaling matrix will be $[S] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 5\left(1-\frac{1}{2}\right) & 5\left(1-\frac{1}{2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 2.5 & 2.5 & 1 \end{bmatrix}$

and the translation matrix will be $[T_r] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -6 & 1 \end{bmatrix}$

Thus the combined transformation matrix will be $[S]\,[T_r]$

$$= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 2.5 & 2.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ -3.5 & -3.5 & 1 \end{bmatrix}$$

This can be verified by multiplying the transformation matrix with the given set of vertices of a square.

Transforming the diagonally opposite corners of the square,

$$\begin{bmatrix} 0 & 0 & 1 \\ 10 & 10 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ -3.5 & -3.5 & 1 \end{bmatrix} = \begin{bmatrix} -3.5 & -3.5 & 1 \\ 1.5 & 1.5 & 1 \end{bmatrix}$$

The resulting square can be written as $P(-3.5, -3.5)$, $Q(1.5, -3.5)$, $R(1.5, 1.5)$ and $S(-3.5, 1.5)$. The centre of the square is at $[(1.5 - 3.5)/2 = -1, (1.5 - 3.5)/2 = -1] = (-1, -1)$ and the edge of it is $1.5 + 3.5 = 5.0$ which is half the size of the initial square.

4.6 Use composite transformation to fix the triangle $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^T$ inside the square $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}^T$ so that its base coincides with the bottom edge of the square, and the top vertex touches the middle of the top edge of the square.

Solution

The size of the base of the triangle is $1 - (-1) = 2$, and the size of square edge is 1 unit. Hence, the triangle needs to be halved (scaled by ½) along x direction. The height of the triangle needs to be unchanged to make the top vertex of the triangle touch the top edge of the square, by keeping the base centre (0, 0), that is, origin fixed. Hence, it is required to be scaled with respect to the origin. The base along with the triangle can be translated to the square frame by translating the base of the triangle to the bottom edge of the square, which can be done by moving the centre of the base of the triangle to the centre of the bottom edge of the square, that is, the translation along is ½ units.

Hence, the scaling matrix will be $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and the translation matrix will be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

Thus the combined transformation will be $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

which when applied to the vertices of the triangle we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0.5 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The base of this triangle coincides with the bottom edge of the square, and the vertex touches the middle of the top edge of the square. The resulting triangle can be expressed normally as $\begin{bmatrix} 1 & 0 \\ 0.5 & 1 \\ 0 & 0 \end{bmatrix}$.

4.7 A triangle is defined by the vertices $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}^T$ and the 2×2 transformation matrix is $\begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix}$. Find the area of the triangle. Vertices of transformed triangle and its area thus prove that area of transformed triangle is equal to the product of area of original triangle and determinant of transformation matrix.

Solution

Area of original triangle is $= \dfrac{1}{2}\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ -2 & 0 & 1 \end{vmatrix} = ½ \times 8 = 4$ square unit.

Applying transformation matrix to the given triangle

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 4 & 8 \\ -12 & -8 \end{bmatrix}$, the elements of matrix are the vertices of transformed triangle.

Area of the transformed triangle $= \dfrac{1}{2}\begin{vmatrix} 12 & 8 & 1 \\ 4 & 8 & 1 \\ -12 & -8 & 1 \end{vmatrix} = 64$ square unit.

The determinant of transformation matrix $= \begin{vmatrix} 6 & 4 \\ 2 & 4 \end{vmatrix} = (24 - 8) = 16$

Thus,

Area of transformed triangle = Area of the original triangle × Determinant of transformation matrix

4.8 A unit square is given by $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}^T$ and the transformation matrix is $\begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix}$. Find the area of the square after being transformed. Find the relation between area of the square after being operated by the transformation matrix and the determinant of transformation matrix.

Solution

Area of the unit square = 1 × 1 = 1 square units.

Applying transformation to the original unit square $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 3 \\ 3 & 2 \\ 6 & 5 \end{bmatrix}$

Area of the square after being transformed $= \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 3 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 & 3 & 1 \\ 3 & 2 & 1 \\ 6 & 5 & 1 \end{bmatrix} = \frac{1}{2} \times 3 + \frac{1}{2} \times 3 = 3$

Determinant of the transformation matrix $= \begin{vmatrix} 3 & 2 \\ 3 & 3 \end{vmatrix} = (9 - 6) = 3$

Thus, Area of the square after being transformed = Determinant of the transformation matrix.

4.9 A mirror is placed such that it passes through (2, 0) and (0, 2). Find the reflected view of a triangle with vertices (3, 4), (5, 5) and (4, 7) in this mirror.

Solution

Equation of the mirror line is $y = -x + 2$. Hence the slope is $m = \tan^{-1}(-1) = 135°$.
To make the mirror line coincide with x axis, it is translated to pass through the origin and then rotated by 45° about origin.
The transformation matrix will be given by

$$[T_R]_{45°}\,[T_t]_{-2 0,0} = \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

These transformations are also applied to the given triangle. Now reflection of this triangle about the transformed mirror line, that is, x axis is given by

$$[T_M]_{y=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To get the normal image, apply the inverse transformations to the reflected triangle. Thus, the composite transformation matrix will be

$[T_R]_{45°}\,[T_t]_{-20,0}\,[T_M]_{y=0}\,[T_R]^{-1}_{45°}\,[T_t]^{-1}_{-20,0}$

$$= \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

The reflected traingle will thus be $\begin{bmatrix} 3 & 4 & 1 \\ 5 & 5 & 1 \\ 4 & 7 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -3 & 1 \\ -5 & -2 & 1 \end{bmatrix}$

4.10 Prove that a midpoint of a straight line $PQ[(0, 2), (3, 2)]$ after transformation $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ will be the same midpoint of the transformed straight line $P'Q'$ drawn after the transformation.

Solution

Applying transformation to the straight line PQ

$$\begin{bmatrix} P \\ Q \end{bmatrix} [T] = \begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

or
$$\begin{bmatrix} P' \\ Q' \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 9 & 8 \end{bmatrix}$$

Midpoint of the transformed straight line is $[x'_m, y'_m] = [(6 + 9)/2, (2 + 8)/2] = (7.5, 5)$

Midpoint of the original straight line $PQ = [x'_m, y'_m] = [(0 + 3)/2, (2 + 2)/2] = (1.5, 2)$

Applying transformation to the original midpoint

$$[1.5 \;\; 2] \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [7.5 \;\; 5]$$

This is same as midpoint of the transformed straight line.

4.11 Prove that a triangle $PQR \begin{bmatrix} 8 & 2 \\ 10 & 4 \\ 8 & 6 \end{bmatrix}$, after reflection about x–axis and then about $y = -x$ will be the same as the rotation about origin by an angle 270°.

Solution

Applying reflection about x axis to the triangle $\begin{bmatrix} 8 & 2 \\ 10 & 4 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 10 & -4 \\ 8 & -6 \end{bmatrix}$

Applying reflection about $y = -x$ to this (refer Fig. 4.12)

$$\begin{bmatrix} 8 & -2 \\ 10 & -4 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 4 & -10 \\ 6 & -8 \end{bmatrix} \qquad (4.6)$$

$S^x = S^x_o R$

Now, applying 270° rotation matrix to the original triangle

$$\begin{bmatrix} 8 & 2 \\ 10 & 4 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} \cos 270° & \sin 270° \\ -\sin 270° & \cos 270° \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 10 & 4 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 4 & -10 \\ 6 & -8 \end{bmatrix} \qquad (4.7)$$

Comparing 4.6 with 4.7, it can be seen that any object after the successive reflections about x–axis and then about $y = -x$ is the same as the rotation about origin by an angle 270°.

This proves that, after reflection about x–axis and then about $y = -x$ is the same as the rotation about origin by an angle 270°.

4.12 Prove that the reflection of a square $ABCD$ [(2, 2), (4, 2), (4, 4), (2, 4)] about x axis ($y = 0$) and then rotation of the resulting square about 60° will not be same if the order of transformation (first rotation and then reflection) is changed.

Solution

Reflecting the square about $y = 0$, i.e. x axis

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \\ 4 & -4 \\ 2 & -4 \end{bmatrix}$$

Now rotating the resultant points by 60°

$$\begin{bmatrix} 2 & -2 \\ 4 & -2 \\ 4 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} \cos 60° & \sin 60° \\ -\sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \\ 4 & -4 \\ 2 & -4 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{bmatrix} = \begin{bmatrix} 2.732 & 0.732 \\ 3.732 & 2.464 \\ 5.464 & 1.464 \\ 4.64 & 0.268 \end{bmatrix}$$

Now, proceeding with rotation first and then reflection

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -0.732 & -2.732 \\ 0.268 & -4.464 \\ -1.464 & -5.464 \\ -2.464 & -3.732 \end{bmatrix}$$

Hence, it can be seen that both the calculations arrived at with different sets of points prove the desired result.

4.13 Locate the new position of the triangle [(5, 4), (8, 3), (8, 8)] after its rotation by 90° clockwise about its centroid.

Solution

To achieve the required transformation, it is required to first bring the centre of the triangle to the origin then rotate it about the origin and finally translate back the centre to the same location.
Centroid of the triangle will be at [(5 + 8 + 8)/3, (4 + 3 + 8)/3] = (7, 5)
Translating the triangle centroid to the origin

$$\begin{bmatrix} 5 & 4 & 1 \\ 8 & 3 & 1 \\ 8 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & -5 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

Now, applying clockwise rotation through 90° to the resulting vertices of translated triangle with centre at origin

$$\begin{bmatrix} -2 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

This is to be translated back to the actual position, hence

$$\begin{bmatrix} -1 & 2 & 1 \\ -2 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 1 \\ 5 & 4 & 1 \\ 10 & 4 & 1 \end{bmatrix}$$

This is the rotated triangle, with centre located at [(6 + 5 + 10)/3, (7 + 4 + 4)/3] = (7, 5), which is same as before.

4.14 Find the position of a triangle $PQR \begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix}$ after its reflection about a line $x - 2y = -4$.

Solution

The given line can be rewritten as $y = ½x + 2$ which shows that the line will pass through the origin if it is translated by – 2 units along y axis. The line can be made coincident to x axis by rotating it by $-\tan^{-1}(½)$ degrees about origin. The simple reflection about x – axis can be applied and the transformed position is rotated and translated back to the original position.

Thus the composite transformation will be

$$[T] = [T_{trans}][R_\theta][R_{ref}][R_\theta]^{-1}[T_{trans}]^{-1}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ -0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$x-2y=-4$
$-2y=-4-x$
$2y=x+4$
$y=\frac{x}{2}+2$
$m=1/2, c=2$
$\theta=\tan^{-1}(1/2)$

CHAPTER 4

$$[T] = \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ -1.6 & 3.2 & 1 \end{bmatrix}$$

And the transformed vertices of the triangle can be obtained by

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix}[T] = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix}\begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ -1.6 & 3.2 & 1 \end{bmatrix} = \begin{bmatrix} 2.8 & 2.4 & 1 \\ 5.6 & 2.8 & 1 \\ 4/4 & 1.2 & 1 \end{bmatrix}$$

This gives the required triangle.

4.15 Prove that an intersection point of two straight lines $PQ[(-1,-1), (3, 5/3)]$ and $RS[(-1/2, 3/2), (3, -1/2)$ will be exactly the same as the intersection point of transformed intersecting lines $P'Q'$ and $R'S'$ drawn after the transformation. The transformation may be taken as $\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$.

Solution

The equation of the lines PQ and RS may be written as $-(2/3)x + y = -(1/3)$ and $x + y = 1$, respectively. In matrix notation it can be represented as

$$[x\ y]\begin{bmatrix} -2/3 & 1 \\ 1 & 1 \end{bmatrix} = [-1/3\quad 1]$$

The intersection point may be obtained by multiplying both sides by the inverse of transformation matrix given.

$$[x_p\ y_p] = [-1/3\quad 1]\begin{bmatrix} -3/5 & -3/5 \\ 3/5 & 2/5 \end{bmatrix} = [4/5\quad 1/5]$$

Transforming the given lines simultaneously by multiplying with the given transformation, the resulting lines $P'Q'$ and $R'S'$ can be given in matrix notation as

$$[x'\ y'] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1\quad 1]$$

that is, $x' = 1$ and $y' = 1$.

The intersection point of these transformed lines is (1, 1).

Transforming the intersection point (x_p, y_p), $(x'_p, y'_p) = [4/5\quad 1/5]\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} = [1\quad 1]$ which is the same as obtained earlier. Hence, the desired result has been proved.

Review Questions

4.1 Prove that two successive reflections about any coordinate axes is equivalent to a single rotation about the origin.

4.2 Prove that a triangle ABC given by $[(-0.5, 1), (0, 2), (0.5, 1)]$ after being reflected about $y = x$ is the same as it is being reflected relative to x axis followed by counterclockwise rotation of 90°.

4.3 Show that uniform scaling, that is, same scaling factors taken along both directions, and a rotation form a commutative pair of operations.

4.4 Use triangle $PQR[(2, 3), (5, 8), (7, 2)]$ to show that two successive reflections about either of the coordinate axes is equivalent to a single rotation about the origin.

4.5 A circular disc of diameter "d" is rolling down an inclined plane starting from rest. Assume there is no slip, develop the set of transformations required to produce this animation. [*University Question*]

```
    printf("\n\t\t\tPress any key to Clip the drawn polygon");
    getch();
    //Draw Final Status
    cleardevice();
    setcolor(GREEN);
    rectangle(body.xmin, body.ymin, body.xmax, body.ymax);
    draw_polygon(outlen, polyout, YELLOW);
    //Wait for a keystroke to exit and restore previous screen mode
    getch();
    restorecrtmode();
}
Press any key to clip the drawn polygon
```

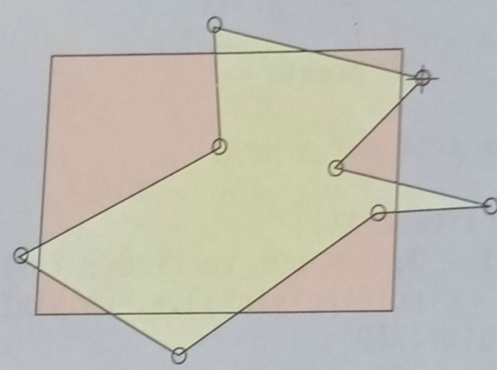


Fig. 5.21(a) *Output of the above-listed program (Screen 1)*

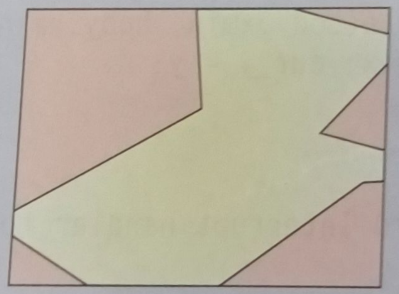


Fig. 5.21(b) *Output of the above-listed program (Screen 2)*

Solved Exercises

5.1 Determine the parametric representation of the line segment between position vector $P_1(2, 4)$ and $P_2(6, 4)$.

Solution

A line segment can be parametrically represented as $P(t) = P_1(t) + [P_2(t) - P_1(t)]t$ where $0 \leq t \leq 1$.

$$x(t) = x_1 + (x_2 - x_1)t$$

and $$y(t) = y_1 + (y_2 - y_1)t$$

Solving for x at $t = 0$, $x(0) = x_1 = 2$ and at $t = 1$, $x(1) = x_2 = 6$

Hence,
$$x(t) = 2 + (6 - 2)t = 2 + 4t.$$
Similarly,
$$y(t) = 4 + (4 - 4)t = 4$$
Thus the parametric representation for the line is given by
$$x(t) = 2 + 4t.$$
$$y(t) = 4, \qquad \text{where } 0 \le t \le 1.$$

5.2 Find the normalization transformation that maps a window whose corners are (2, 2), (10, 6), (8, 10) and (0, 6) onto a viewport which is the entire normalized device screen with lower left corner at A.

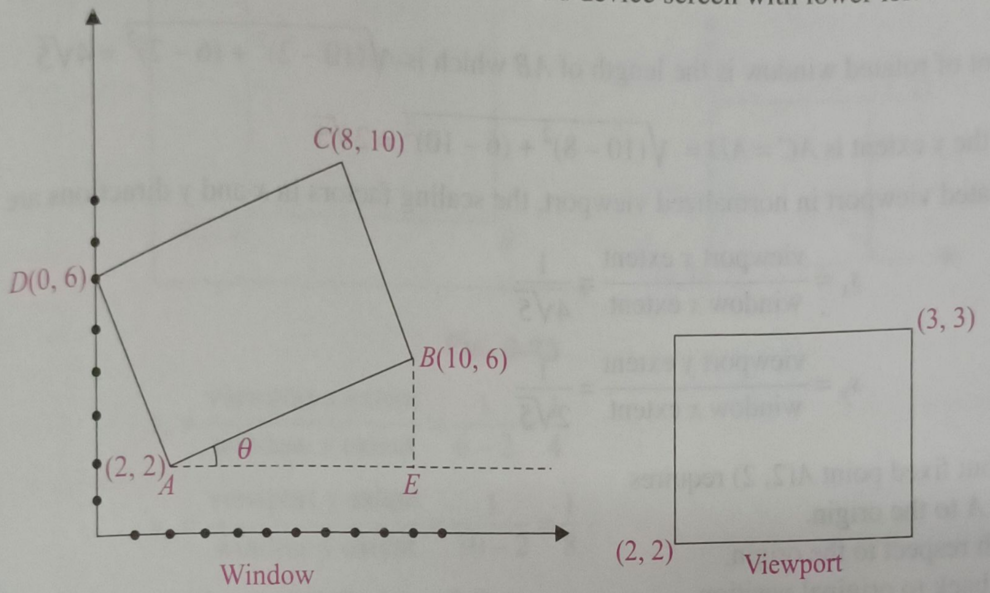


Fig. 5.22

Solution

Here as the window edges are not parallel to the coordinate axes, and so rotate the window about A so that it is aligned with the axes.

Now,
$$\sin\theta = \frac{BE}{AB} = \frac{6-2}{\sqrt{(10-2)^2 + (6-2)^2}}$$
$$= \frac{4}{\sqrt{64+16}} = \frac{4}{\sqrt{80}} = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

and
$$\cos\theta = \frac{AE}{AB} = \frac{10-2}{\sqrt{(10-2)^2 + (6-2)^2}}$$
$$= \frac{8}{\sqrt{64+16}} = \frac{2}{\sqrt{5}}$$

Rotate the rectangle in a clockwise direction. Hence, θ is negative.

The rotation matrix about $A(m, n) \equiv (2, 2)$, (refer Section 4.12)

$$[T_{R,\theta}]_A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ m(1-\cos\theta)+n\sin\theta & m(1-\cos\theta - m\sin\theta) & 1 \end{bmatrix}$$

Replacing θ by $-\theta$,

$$[T_{R,\theta}]_A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ m(1-\cos\theta)+n\sin\theta & m(1-\cos\theta + m\sin\theta) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 2\left(1-\frac{2}{\sqrt{5}}\right)+2\left(\frac{-1}{\sqrt{5}}\right) & 2\left(1-\frac{2}{\sqrt{5}}\right)+2\left(\frac{1}{\sqrt{5}}\right) & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 2\left(1-\frac{3}{\sqrt{5}}\right) & 2\left(1-\frac{1}{\sqrt{5}}\right) & 1 \end{bmatrix}$$

The x extent of rotated window is the length of AB which is $\sqrt{(10-2)^2+(6-2)^2} = 4\sqrt{5}$

Similarly, the y extent is $AC = AD = \sqrt{(10-8)^2+(6-10)^2} = 2\sqrt{5}$

For the rotated viewport in normalized viewport, the scaling factors in x and y directions are

$$s_x = \frac{\text{viewport } x \text{ extent}}{\text{window } x \text{ extent}} = \frac{1}{4\sqrt{5}}$$

$$s_y = \frac{\text{viewport } y \text{ extent}}{\text{window } x \text{ extent}} = \frac{1}{2\sqrt{5}}$$

scaling about fixed point $A(2, 2)$ requires

1. Translate A to the origin.
2. Scale with respect to the origin.
3. Return it back to original position.

i.e. the transformation procedure will be

$$[T_s]_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_f & -y_f & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_f & y_f & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ x_f(1-s_x) & y_f(1-s_y) & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{5}} & 0 \\ 2\left(1-\frac{1}{4\sqrt{5}}\right) & 2\left(1-\frac{1}{\sqrt{5}}\right) & 1 \end{bmatrix}$$

Now, the composite transformation matrix of given window to the normalized viewport will be given b

$$[T] = [T_{R,\theta}]_A [T_S]_A = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 2\left(1-\frac{3}{\sqrt{5}}\right) & 2\left(1-\frac{1}{\sqrt{5}}\right) & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{5}} & 0 \\ 2\left(1-\frac{1}{\sqrt{5}}\right) & 2\left(1-\frac{1}{2\sqrt{5}}\right) & 1 \end{bmatrix}$$

or, $$[T] = \begin{bmatrix} \frac{1}{10} & \frac{-1}{10} & 0 \\ \frac{1}{20} & \frac{1}{5} & 0 \\ \frac{17}{20} & \frac{9}{5} & 1 \end{bmatrix}$$

5.3 Find the normalization transformation for window to viewport which uses the rectangle whose lower left corner (2, 2) and upper right corner (6, 10) as a window and the viewport that has lower left corner at (0, 0) and upper right corner at (1, 1).

Solution

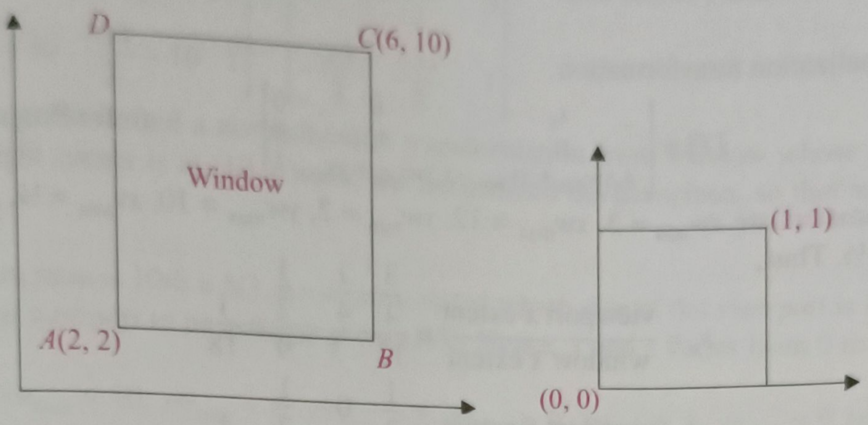


Fig. 5.23

$$s_x = \frac{\text{viewport } x \text{ extent}}{\text{window } x \text{ extent}} = \frac{1}{6-2} = \frac{1}{4}$$

$$s_y = \frac{\text{viewport } y \text{ extent}}{\text{window } y \text{ extent}} = \frac{1}{10-2} = \frac{1}{8}$$

Overall transformation will sequentially be as follows:

1. Translate the window to the origin.
2. Scale with respect to the origin to the required scaling factor.
3. Return it to viewport position.

This can be represented by transformation matrices as follows:

$$[T_S]_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -xw_{\min} & -yw_{\min} & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ xv_{\min} & yv_{\min} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ -s_x xw_{\min} + xv_{\min} & -s_y yw_{\min} + yv_{\min} & 1 \end{bmatrix}$$

These parameters can be well identified and substituted in the above equation as follows:

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ \frac{-1}{4}.2 + 0 & \frac{-1}{8}.2 + 0 & 1 \end{bmatrix}$$

Thus,

$$[T] = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix}$$

5-4 Find the viewing from a window in world coordinates with x extent 3 to 12 and y extent 2 to 10 onto a viewport with x extent ¼ to ¾ and y extent 0 to ½ in normalized device space and then map a workstation window with x extent ¼ to ½ and y extent ¼ to ½ in normalized device space into a workstation viewport with x and y extent both 1 to 12 on the physical display device.

Solution

For normalization transformation

$$[T] = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ -s_x xw_{\min} + xv_{\min} & -s_y yw_{\min} + yv_{\min} & 1 \end{bmatrix} \qquad \text{(refer Program 5.2)}$$

the parameters are $xw_{\min} = 3$, $xw_{\max} = 12$, $yw_{\min} = 2$, $yw_{\max} = 10$, $xv_{\min} = ¼$, $xv_{\max} = ¾$, $yv_{\min} = 0$ and $yv_{\max} = ½$. Thus,

$$s_x = \frac{\text{viewport } x \text{ extent}}{\text{window } x \text{ extent}} = \frac{\frac{3}{4} - \frac{1}{4}}{12 - 3} = \frac{\frac{1}{2}}{9} = \frac{1}{18}$$

$$s_y = \frac{\text{viewport } y \text{ extent}}{\text{window } y \text{ extent}} = \frac{\frac{1}{2} - 0}{10 - 2} = \frac{\frac{1}{2}}{8} = \frac{1}{16}$$

Substituting s_x and s_y

$$[T_1] = \begin{bmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ \frac{-1}{18}.3 + \frac{1}{4} & -\frac{1}{16}.2 + 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ \frac{1}{12} & -\frac{1}{8} & 1 \end{bmatrix}$$

The parameters for workstation transformation are $xw_{\min} = ¼$, $xw_{\max} = ½$, $yw_{\min} = ¼$, $yw_{\max} = ½$, $xv_{\min} = 1$, $xv_{\max} = 12$, $yv_{\min} = 1$ and $yv_{\max} = 12$.

$$s_x = \frac{12 - 1}{\frac{1}{2} - \frac{1}{4}} = \frac{11}{\frac{1}{4}} = 44$$

$$s_y = \frac{12 - 1}{\frac{1}{2} - \frac{1}{4}} = \frac{11}{\frac{1}{4}} = 44$$

And

$$[T_2] = \begin{bmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ -44 \cdot \frac{1}{4} + 1 & -44 \cdot \frac{1}{4} + 1 & 1 \end{bmatrix} \begin{bmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ -10 & -10 & 1 \end{bmatrix}$$

Thus, the complete viewing transformation will be $[T] = [T_1] \times [T_2]$

$$[T] = \begin{bmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ \frac{1}{12} & -\frac{1}{8} & 1 \end{bmatrix} \begin{bmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ -10 & -10 & 1 \end{bmatrix}$$

$$\text{Hence, } [T] = \begin{bmatrix} \frac{22}{9} & 0 & 0 \\ 0 & \frac{11}{4} & 0 \\ \frac{11}{3} - 10 & \frac{-11}{2} - 10 & 1 \end{bmatrix} = \begin{bmatrix} \frac{22}{9} & 0 & 0 \\ 0 & \frac{11}{4} & 0 \\ -\frac{19}{3} & \frac{-31}{2} & 1 \end{bmatrix}$$

5.5 Preserving the aspect ratio, find a normalization transformation from window whose lower left is at (0, 0) and upper right corner is at (10, 6) onto the normalized device screen, so that aspect ratios are preserved.

Solution

The window aspect ratio is 10/6 = 5/3. As it is not stated which side of the viewport is larger, we shall take the size of the viewport to be as large as possible. Hence, x and y varies from 0 to 1 and 0 to 5/3, respectively.

Thus $xw_{\min} = 0$, $xw_{\max} = 10$, $yw_{\min} = 0$, $yw_{\max} = 6$, $xv_{\min} = 0$, $xv_{\max} = 1$, $yv_{\min} = 0$ and $yv_{\max} = 3/5$.

Scaling factors are

$$s_x = \frac{\text{viewport } x \text{ extent}}{\text{window } x \text{ extent}} = \frac{1-0}{10-0} = \frac{1}{10} \qquad s_y = \frac{\text{viewport } y \text{ extent}}{\text{window } y \text{ extent}} = \frac{\frac{3}{5}-0}{6-0} = \frac{3}{30} = \frac{1}{10}$$

and the transformation matrix will be

$$[T] = \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ \frac{-1}{10} \cdot 0 + 0 & \frac{-1}{10} \cdot 0 + 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(refer Program 5.2)}$$

5.6 A clipping window $PQRS$ has left corner at (3, 4) and upper right corner at (10, 9). Find the section of the clipped line AB shown in the Fig. 5.20 using the Cohen–Sutherland line-clipping algorithm. Also find the region codes on which the end points of the lines CD and EF rest.

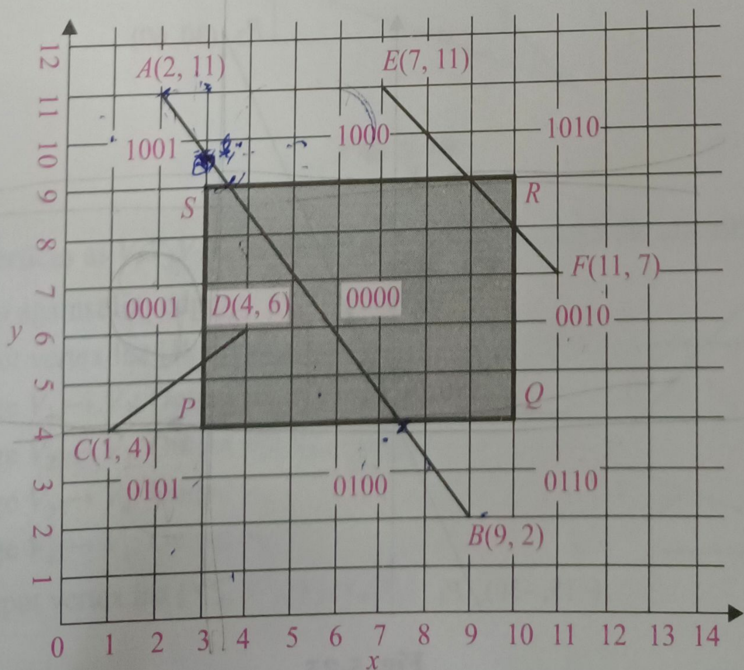


Fig. 5.24

CHAPTER 5

Solution

The region code for any point (x, y) will be set according to the scheme based on Cohen–Sutherland algorithm (refer Section 5.6).

Bit 1 = sign $(y - yw_{max})$ = sign $(y - 9)$ Bit 2 = sign $(yw_{min} - y)$ = sign $(4 - y)$

Bit 3 = sign $(x - xw_{max})$ = sign $(x - 10)$ Bit 4 = sign $(xw_{min} - x)$ = sign $(3 - x)$

$$\text{Here sign (a)} = \begin{cases} 1 \text{ if a} > 0 \\ 0 \text{ if a} < 0 \end{cases}$$

Hence the region code of $A(2, 11) \rightarrow 1001$ and $B(9, 2) \rightarrow 0100$

$C(1, 4) \rightarrow 0001$ and $D(4, 6) \rightarrow 0000$

$E(7, 11) \rightarrow 1000$ and $F(11, 7) \rightarrow 0010$

Considering clipping of line AB,

Slope of the line AB is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2 - 11}{9 - 2} = -\dfrac{9}{7}$

As the line has the slope less than 1, and the line has the code 1001, it intersects xw_{min} for which y can be calculated as $y = y_1 + m(x - x_1) = 11 + m\,(xw_{min} - 2)$ which gives $y = 9.71$ that lies outside the window and above yw_{max}. Thus, x is to be calculated as $x = x_1 + (y - y_1)/m = 2 + (yw_{max} - 11)/-9/7 = 3.55$

And, the intersecting point is (3.55, 9). Similarly considering the lower end point of the line AB, the region code for the end point B is 0100, hence it intersects the lower line $yw_{min} = 4$, and the x coordinate for the clipping point can be calculated as $x = 2 + (yw_{min} - 11)/-9/7 = 7.44$

Hence, the intersecting point is (7.44, 4).

Thus the clipped line is LM = [(3.55, 9), (7.44, 4)].

5.7 Use the Liang–Barsky line-clipping algorithm to clip the line $P_1(-15, -30) - P_2(30, 60)$ against the window having diagonally opposite corners as (0, 0) and (15, 15).

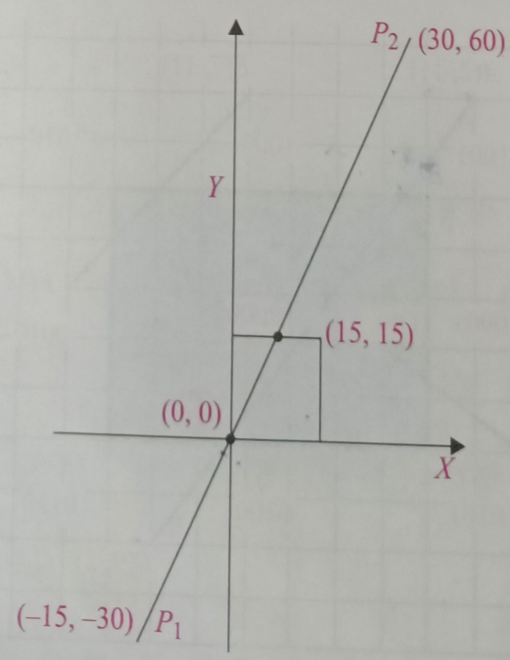


Fig. 5.25

Solution

Line coordinates $P_1 = (-15, -30), P_2 = (30, 60)$

Window coordinates $x_{min} = 0, x_{max} = 15, y_{min} = 0, y_{max} = 15$

$dx = 30 - (-15) = 45 \quad dy = 60 - (-30) = 90$

$d_1 = -dx = -45, \quad q_1 = x_1 - x_{min} = -15 - 0 = -15, \quad u_1 = q_1/d_1 = 1/3$

$d_2 = dx = 45, \quad q_2 = x_{max} - x_1 = 15 - (-15) = 30, \quad u_2 = q_2/d_2 = 2/3$

$d_3 = -dy = -90, \quad q_3 = y_1 - y_{min} = -30 - 0 = -30, \quad u_3 = q_3/d_3 = 1/3$

$d_4 = dy = 90, \quad q_4 = y_{max} - y_1 = 15 - (-30) = 45, \quad u_4 = q_4/d_4 = 1/2$

for $(d_i < 0)$ u_1 = MAXIMUM of (1/3, 1/3, 0) = 1/3

for $(d_i > 0)$ u_2 = MINIMUM of (2/3, 1/2, 1) = 1/2

Since $u_1 < u_2$ there is a visible section

Computing new end points

$$x'_1 = x_1 + dx \times u_1 = -15 + (45 \times 1/3) = 0$$
$$y'_1 = y_1 + dy \times u_1 = -30 + (90 \times 1/3) = 0$$
$$x'_2 = x_1 + dx \times u_2 = -15 + (45 \times 1/2) = 7\tfrac{1}{2}$$
$$y'_2 = y_1 + dy \times u_2 = -30 + (90 \times 1/2) = 15$$

Hence, the visible line will be $P_1'(0, 0) - P_2'(7\tfrac{1}{2}, 15)$.

5.8 Write the steps for clipping the polygon given in Fig. 5.21 using the Sutherland–Hodgman polygon-clipping algorithm.

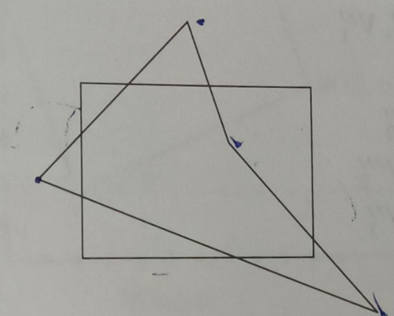


Fig. 5.26

Solution

Naming the vertices as $V_1V_2V_3V_4$ as shown and moving through the algorithmic steps as below

STEP 1: Clip against left edge

Input vertex list $[V_1V_2V_3V_4]$

Edge $V_1 \rightarrow V_2$: Output V_1'

Edge $V_2 \rightarrow V_3$: Output V_2', V_3

Edge $V_3 \rightarrow V_4$: Output V_4

Edge $V_4 \rightarrow V_1$: Output V_1

Output vertex list $[V'_1, V'_2, V_3, V_4\ V_1]$

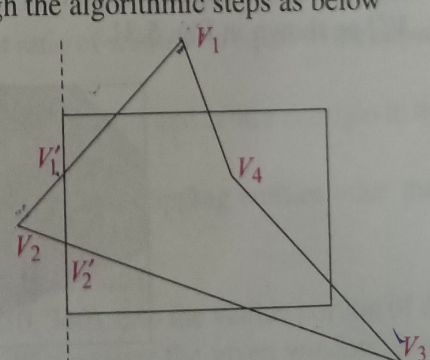


Fig. 5.27

CHAPTER 5

STEP 2: Clip against bottom edge

Input vertex list $[V_1', V_2', V_3, V_4\ V_1]$

Edge $V_1' \rightarrow V_2'$: Output V_2'

Edge $V_2' \rightarrow V_3$: Output V_2''

Edge $V_3 \rightarrow V_4$: Output V_3', V_4

Edge $V_4 \rightarrow V_1$: Output V_1

Edge $V_1 \rightarrow V_1'$: Output V_1'

Output vertex list $[V_2', V_2'', V_3', V_4, V_1, V_1']$

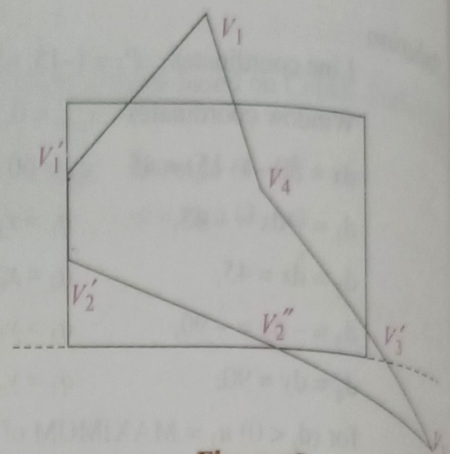


Fig. 5.28

STEP 3: Clip against right edge

Input vertex list $[V_2', V_2'', V_3', V_4, V_1, V_1']$

Edge $V_1 \rightarrow V_1'$: Output V_1'

Edge $V_1' \rightarrow V_2'$: Output V_2'

Edge $V_2' \rightarrow V_2''$: Output V_2''

Edge $V_2'' \rightarrow V_3'$: Output V_2'''

Edge $V_3' \rightarrow V_4$: Output V_3'', V_4

Edge $V_4 \rightarrow V_1$: Output V_1

Output vertex list $[V_1', V_2', V_2'', V_2''', V_3'', V_4, V_1]$

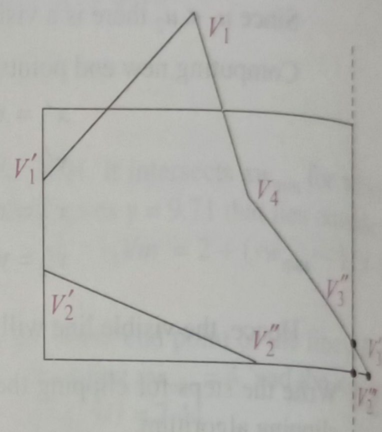


Fig. 5.29

STEP 4: Clip against top edge

Input vertex list $[V_1', V_2', V_2'', V_2''', V_3'', V_4, V_1]$

Edge $V_1 \rightarrow V_1'$: Output V_1'', V_1'

Edge $V_1' \rightarrow V_2'$: Output V_2'

Edge $V_2' \rightarrow V_2''$: Output V_2''

Edge $V_2'' \rightarrow V_2'''$: Output V_2'''

Edge $V_2''' \rightarrow V_3''$: Output V_3''

Edge $V_3'' \rightarrow V_4$: Output V_4

Edge $V_4 \rightarrow V_1$: Output V_4'

Output edge list $[V_1'', V_1', V_2', V_2'', V_2''', V_3'', V_4, V_4']$

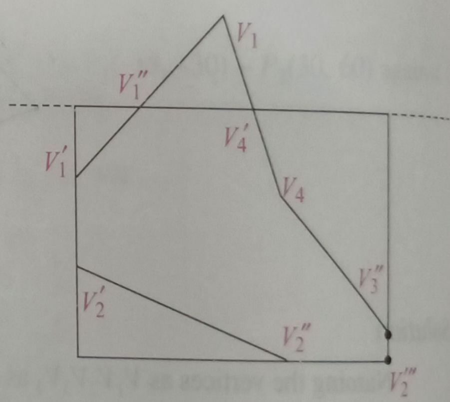


Fig. 5.30

Thus the final clipped polygon is $[V_1'', V_1', V_2' V_2''', V_2'', V_3'', V_4, V_4']$ as shown in Fig. 5.31.

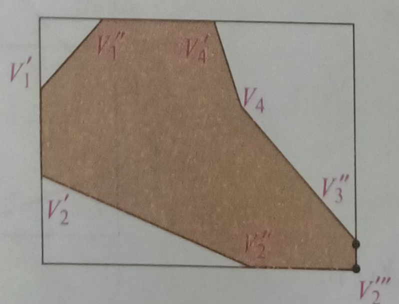


Fig. 5.31

Note: The intersection with a clipping window boundary can be calculated using slope intercept form of the line equation. A line (x_1, y_1) and (x_2, y_2) having end points, which is the edge of the given polygon in consideration.

The y coordinate of the intersection point with a vertical boundary can be obtained with the calculation $y = y_1 + m(x - x_1)$, where $m = (y_2 - y_1)/(x_2 - x_1)$ and x may be set either as xw_{min} or xw_{max}. Similarly, for the intersection with horizontal boundary the x coordinate can be calculated.

$x = x_1 + (y - y_1)/m$, where y may be set to yw_{min} or to yw_{max}. The vertices can be found by substituting $x = xw_{min}$ or xw_{max}.

Review Questions

5.1 Determine the parametric representation of the line segment between position vector $P_1(2.5, 9.4)$ and $P_2(6.0, 5.7)$.

5.2 Use Sutherland–Hodgman algorithm for line clipping to clip a line [(0, 0), (10, 10)] against rotated window shown in Fig. 5.32.

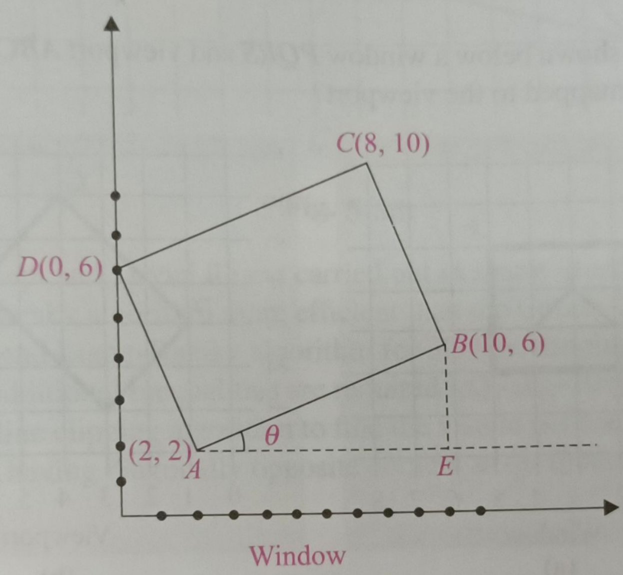


Fig. 5.32

5.3 Describe how clipping a line against an elliptical window (or viewport) situated at origin and semi-major axis 'a' and semi-minor axis 'b' might proceed.

5.4 Prove that for viewing transformation $s_x = s_y$ if and only if aspect ratio of window is equal to the aspect ratio of the viewport.

5.5 Find the normalization transformation matrix for a window of radius 4 units and centre at origin to the viewport of radius 1 unit and centre at (1, 1).

5.6 How can Cohen Sutherland algorithm be extended for 3D clipping, and clipping entities other than straight line?
[**Hint**: How many more bits are required in Cohen-Sutherland algorithm?]

5.7 Given a clipping window $P(0, 0)$, $Q(340, 0)$, $R(340, 340)$ and $S(0, 340)$, find the visible portion of the lines AB [(–170, 595), (170, 255)] and CD [(425, 85), (595, 595)] against the given window, using Cohen–Sutherland algorithm.

```
initgraph(&gd,&gm,"");
//defining origin centered 30 pixel initialmatrix a rectangle
initialmatrix[0][0] = 300; initialmatrix[0][1] = 340;
initialmatrix[0][2] = 340; initialmatrix[0][3] = 300;
initialmatrix[1][0] = 260; initialmatrix[1][1] = 260;
initialmatrix[1][2] = 220; initialmatrix[1][3] = 220;
initialmatrix[2][0] = 100; initialmatrix[2][1] = 100;
initialmatrix[2][2] = 100; initialmatrix[2][3] = 100;
initialmatrix[3][0] = 1;   initialmatrix[3][1] = 1;
initialmatrix[3][2] = 1;   initialmatrix[3][3] = 1;
setcolor(RED);
//Drawing initialmatrix (Rectangle)
line(initialmatrix[0][0],initialmatrix[1][0],initialmatrix[0][1],
initialmatrix[1][1]);
line(initialmatrix[0][1],initialmatrix[1][1],initialmatrix[0][2],
initialmatrix[1][2]);
line(initialmatrix[0][2],initialmatrix[1][2],initialmatrix[0][3],
initialmatrix[1][3]);
line(initialmatrix[0][3],initialmatrix[1][3],initialmatrix[0][0],
initialmatrix[1][0]);
translate3d(-320,-240,0);
float rotateangle = 45;
rotate3d(rotateangle, zaxis);
translate3d(320, 240, 0);
// scale3d(1.2, 1.5, 1.0);
//Drawing transformed initialmatrix (Rectangle)
setcolor(GREEN);
line(initialmatrix[0][0],initialmatrix[1][0],initialmatrix[0][1],
initialmatrix[1][1]);
line(initialmatrix[0][1],initialmatrix[1][1],initialmatrix[0][2],
initialmatrix[1][2]);
line(initialmatrix[0][2],initialmatrix[1][2],initialmatrix[0][3],
initialmatrix[1][3]);
line(initialmatrix[0][3],initialmatrix[1][3],initialmatrix[0][0],
initialmatrix[1][0]);
for(int i = 0; i<4; i++)
  printf("|%3.4f \t%3.4f \t%3.4f \t%3.4f \t|\n", initialmatrix[i][0],
initialmatrix[i][1], initialmatrix[i][2], initialmatrix[i][3]);
getch();
}
```

CHAPTER 7

Solved Exercises

7.1 Demonstrate local scaling taking scaling factors along the x, y and z axes as 2, 3 and 1 respectively, for a cube with homogeneous position vectors

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Use the same cube to demonstrate overall scaling by factor 2.

Solution

The transformation matrix for the local scaling will be

$$[T] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The resulting figure has homogeneous position vectors

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

The ordinary physical coordinate factor is the same as given above, as the homogeneous coordinate factor h is unity for each of the transformed position vectors.

Now applying overall scaling by a factor of 2 using transformation matrix, we have

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The resulting position vector for the vertices will now be

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Thus, the ordinary physical coordinates will be

$$[X^*] = \begin{bmatrix} 0 & 0 & 0.5 & 1 \\ 0.5 & 0 & 0.5 & 1 \\ 0.5 & 0.5 & 0.5 & 1 \\ 0 & 0.5 & 0.5 & 1 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ 0 & 0.5 & 0 & 1 \end{bmatrix}$$

and the size of the resulting cube reduces to half.

7.2 Apply 3D geometric transformations to make the given tetrahedron $ABCD$ rotate about the x axis, making it erect with its base ABC resting on the x–z plane. Next, magnify it four times about a fixed point $P[1, 1, 2]$.

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & \sqrt{5} & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution

The rotation matrix for the first transformation will be

$$[R] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90°) & \sin(-90°) & 0 \\ 0 & -\sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying this to the given matrix, we have

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} [R] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & \sqrt{5} & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -\sqrt{5} & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

For magnification about point $P(1, 1, 2)$, P should be brought to origin and the corresponding translation matrix would be

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}[T] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -\sqrt{5} & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & -\sqrt{5}-2 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

Then, for overall scaling by four times with reference to origin, the scaling factor should be 1/4 and the corresponding transformation matrix will be

$$[S] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

Applying scaling transformation, we get

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}[T][S] = \begin{bmatrix} -1 & -1 & -2 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & -\sqrt{5}-2 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 & 1/4 \\ 1 & -1 & -2 & 1/4 \\ 0 & -1 & -\sqrt{5}-2 & 1/4 \\ 0 & 0 & -3 & 1/4 \end{bmatrix}$$

Converting this to normal coordinates gives

$$\begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}[T][S] = \begin{bmatrix} -4 & -4 & -8 & 1 \\ 4 & -4 & -8 & 1 \\ 0 & -4 & -4\sqrt{5}-8 & 1 \\ 0 & 0 & -12 & 1 \end{bmatrix}$$

To bring back the reference point to its original location, the translation matrix will be

$$[T]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

The final position vector for the resulting tetrahedron will be

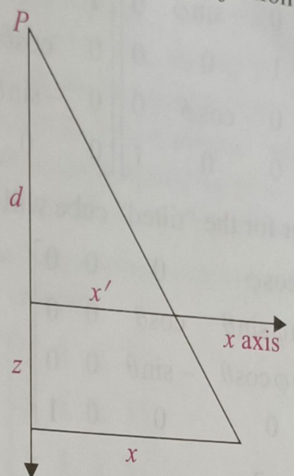
$$\begin{bmatrix} A'' \\ B'' \\ C'' \\ D'' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}[R][T][S][T]^{-1} = \begin{bmatrix} -4 & -4 & -8 & 1 \\ 4 & -4 & -8 & 1 \\ 0 & -4 & -4\sqrt{5}-8 & 1 \\ 0 & 0 & -12 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A'' \\ B'' \\ C'' \\ D'' \end{bmatrix} = \begin{bmatrix} -3 & -3 & -6 & 1 \\ 5 & -3 & -6 & 1 \\ 1 & -3 & -4\sqrt{5}-6 & 1 \\ 1 & 1 & -10 & 1 \end{bmatrix}$$

3 Determine the projected image on to the xy plane of a tetrahedron $ABCD = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, using standard single-point perspective transformation. The distance of the vanishing point P from the view plane may be taken as 5 units.

Solution

Here the view plane is the xy plane, and the centre of projection is $P = (0, 0, -5)$ on a negative z axis.



Here, $x' = \dfrac{d \cdot x}{z + d}$, $y' = \dfrac{d \cdot y}{z + d}$ and $z' = 0$

Thus, the transformation matrix may be written as

$$\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & d \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Applying this to the given set of position vectors for vertices of tetrahedron, we get

$$A'B'C'D' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 5 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 6 \\ 5 & 5 & 0 & 6 \end{bmatrix}$$

The normalized coordinates are $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 0)$ and $D = (5/6, 5/6, 0)$

7.4 Find the isometric projection for the computer display of a cube formed by a $\phi = 30°$ rotation about the y axis, followed by a $\theta = 45°$ rotation about the x axis and then parallelly projected on $z = 0$. The position vectors for the cube are:

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution

The matrix for the projection will be

$$[T] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vector for the "tilted" cube will be

$$[X][T] = [X] \begin{bmatrix} \cos\phi & 0 & 0 & 0 \\ \sin\phi\sin\theta & \cos\theta & 0 & 0 \\ \sin\phi\cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or,

$$[X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & 0 & 0 & 0 \\ \sin 30^\circ \sin 45^\circ & \cos 45^\circ & 0 & 0 \\ \sin 30^\circ \cos 45^\circ & -\sin 45^\circ & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or,

$$[X^*] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & 0 & 0 & 0 \\ 0.3535 & 0.7071 & 0 & 0 \\ 0.3535 & -0.7071 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.3535 & -0.7071 & 0 & 1 \\ 1.2195 & -0.7071 & 0 & 1 \\ 0.7071 & 0 & 0 & 1 \\ 1.5731 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.866 & 0 & 0 & 1 \\ 1.2195 & 0.7071 & 0 & 1 \\ 0.3535 & 0.7071 & 0 & 1 \end{bmatrix}$$

This gives the coordinates of the projected vertices of the isometric cube.

7.5 Extend the Cohen Sutherland line-clipping algorithm to determine the portion of line that falls within a well defined parallel view volume.

Solution

A 3D space is divided into six exterior overlapping regions—top, bottom, right, left, back and the front of the view volume—by planes defining its boundary. So the coding of six bits can be done on the basis of following scheme:

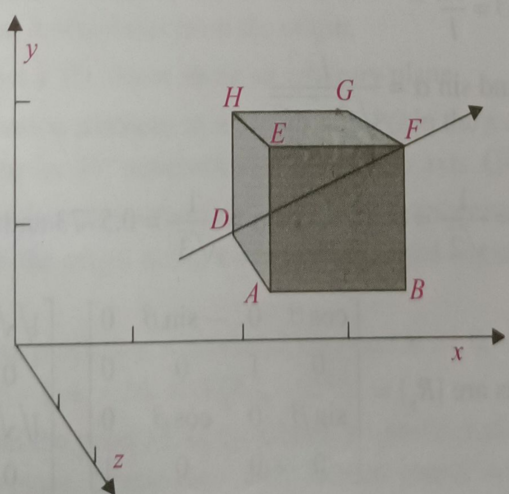
Each bit can be set to 1 if the following condition is true and 0 if false

Bit 1 ⇒ Point is to the left to the view volume ⇒ $(x - xv_{min})$ is negative
Bit 2 ⇒ Point is to the right of the view volume ⇒ $(xv_{max} - x)$ is negative
Bit 3 ⇒ Point is below the view volume ⇒ $(y - yv_{min})$ is negative
Bit 4 ⇒ Point is above the view volume ⇒ $(yv_{max} - y)$ is negative
Bit 5 ⇒ Point is behind the view volume ⇒ $(z - zv_{min})$ is negative
Bit 6 ⇒ Point is in front of the view volume ⇒ $(zv_{max} - z)$ is negative

If any line falls completely inside the view volume, the end points will have code '000000' for both its ends and the full line is accepted. If both the end points have the same bit 1, the line lies completely outside the view volume. In any other case the line is clipped against the plane of intersection. Knowing any three points on the plane, the equation of the bounding plane of the view volume can be determined. The point of intersection is determined by solving the equation for the plane and line.

Note: Point clipping can also be done based on the scheme given above. Only the point having code 000000 is considered inside the view volume.

7.6 Rotate a cube given by $[X] = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$ about its diagonal pointing away from the origin by 45°.

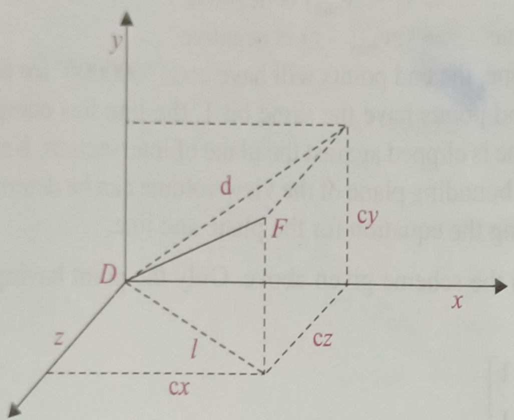


Solution

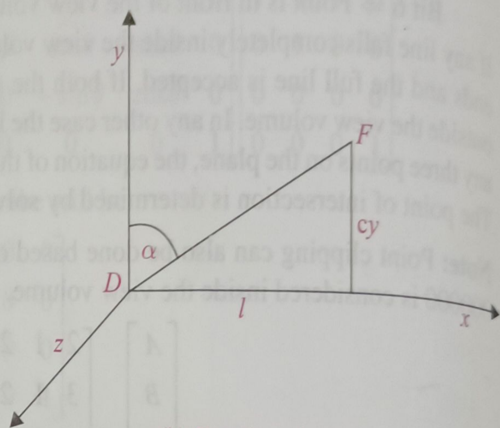
The diagonal passes through $D(2, 1, 1)$. Translate the point D to the origin so that the diagonal pa through the origin.

The required translation matrix will be $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & -1 & 1 \end{bmatrix}$

The rotation will now be performed in two steps to coincide the diagonal to the y axis (any one of axes can be chosen). First, rotate the cube about the y axis by an angle β to make F lie on the xy pla Thereafter, rotate it about the z axis by an angle α to align the diagonal DF to the y axis.



(a) Rotation about y



(b) Rotation about z

$$d^2 = cx^2 + cy^2 = (3-2)^2 + (2-1)^2 = 2$$

or $$d = \sqrt{2}$$

Similarly,

$$l^2 = cx^2 + cz^2 = (3-2)^2 + (2-1)^2 = 2$$

or $$l = \sqrt{2}$$

Now, $\cos\beta = \dfrac{cx}{l}$ and $\sin\beta = \dfrac{cz}{l}$

And $\cos\alpha = \dfrac{cy}{\sqrt{l^2 + cy^2}}$ and $\sin\alpha = \dfrac{l}{\sqrt{l^2 + cy^2}}$

Therefore, $\cos\alpha = \sin\alpha = \dfrac{1}{\sqrt{2}} = 0.7071$, $\cos\alpha = \dfrac{1}{\sqrt{3}} = 0.5773$ and $\sin\alpha = \sqrt{\dfrac{2}{3}} = 0.8165$

Thus, the rotation matrices are $[R_y] = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

And $[R_z] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & \sqrt{2/3} & 0 & 0 \\ -\sqrt{2/3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

After aligning the diagonal to the y axis, the cube is rotated by the required angle 45° about the y axis.

$$[R] = \begin{bmatrix} \cos 45^\circ & 0 & -\sin 45^\circ & 0 \\ 0 & 1 & 0 & 0 \\ \sin 45^\circ & 0 & \cos 45^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, the cube is to be brought back to the initial position in the reverse path of transformations i.e. by applying $[R_z]^{-1}[R_y]^{-1}[T]^{-1}$.

$$[X^*] = [X][T][R_y][R_z][R_{45^\circ}][R_z]^{-1}[R_y]^{-1}[T]^{-1}$$

Thus, $$[X^*] = \begin{bmatrix} 2.5059 & 1.3106 & 1.8047 & 1 \\ 3.3106 & 0.8047 & 1.4942 & 1 \\ 2.8047 & 0.4941 & 0.6894 & 1 \\ 2 & 1 & 1 & 1 \\ 2.8165 & 2.1154 & 1.2989 & 1 \\ 3.6212 & 1.6095 & 0.9883 & 1 \\ 3.1153 & 1.2988 & 0.1835 & 1 \\ 2.3106 & 1.8047 & 0.4941 & 1 \end{bmatrix}$$

Review Questions

7.1 Apply a suitable 3D transformation matrix to a line joining (1, 1, 1) and (2, 3, 4), to align it to the positive z axis and so that it originates from the origin.

7.2 Devise a method to reflect a 3D object about an arbitrary plane.

7.3 Determine 3D transformation matrices to scale the line PQ in the x direction by 3 by keeping point P fixed. Then rotate this line by 45° anticlockwise about the z axis. Given $P(1, 1.5, 2)$ and $Q(4.5, 6, 3)$.

7.4 Derive the necessary transformation matrix in 3D using a homogeneous coordinate system to scale by a factor S with respect to the origin along a line making equal angles with all three axes. [*University Question*]

7.5 For a standard perspective projection with vanishing point at (0, 0, – d) what is the projected image of a line segment joining $P(-1, 1, -2d)$ and $Q(2, -2, 0)$. [*University Question*]

7.6 An object is viewed from the point (5, 0, 0). Obtain a transformation matrix to get a projection of a point $P(x, y, z)$ on the yz plane. Obtain the transformation matrix in the projection plane which is now $x + 10 = 0$. [*University Question*]

7.7 Consider the xy plane to be the computer screen with the z axis pointing towards the viewer. A line $P(1, -1, 1)$ and $Q(1, -1, 0)$ is viewed from the point $V(0, 0, 20)$. Find where the points P and Q would be projected on the screen. [*University Question*]

7.8 Rotate a cube given by $[X] = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} 30 & 15 & 30 & 1 \\ 45 & 15 & 30 & 1 \\ 45 & 15 & 15 & 1 \\ 30 & 15 & 15 & 1 \\ 30 & 30 & 30 & 1 \\ 45 & 30 & 30 & 1 \\ 45 & 30 & 15 & 1 \\ 30 & 30 & 15 & 1 \end{bmatrix}$ about its diagonal pointing away from the orig by 30°.

7.9 Devise a method to clip a point outside the rectangular box (view volume) given the corner coordinat of the box. Edges of the box are parallel to the rectangular coordinate axes.

7.10 Determine the parallel projected image on to the xy plane of a tetrahedron $ABCD = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, af rotating it by 45° about a line which is equally inclined to all coordinate axes.

7.11 In a 3D coordinate system, a box is placed at the origin such that three edges are coincident with coordinate axis. Find the transformation matrix needed to show the top view of the box on xy scre with z projecting towards the viewer. [*University Questic*

7.12 Explain the terms projection plane, view plane and view volume with reference to 3D graphics. St and explain the anomalies of perspective projection.

7.13 A cube with length of side unity is placed so that the corner lies on the origin and the edges fo the three mutually perpendicular axes. Translate the cube along the xy plane so that the cube face centred at origin, then perform three-point perspective projection on the translated cube on the z = plane with centres of projections $x = -10$, $y = -10$ and $z = 10$ on the respective coordinate axes. [*University Questic*

7.14 A cube is given by $[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. The screen is the xy plane and z is projecting out of the scre towards the viewer. The cube is being viewed from the point (0, 2, 8). Work out the perspective vi and isometric view of the cube on the screen. [*University Questic*