

The piece-wise exponential

Shiven Bajpai

Question: Given that

$$f(t) = e^{-at}u(t) + e^{bt}u(-t) \quad (1.1)$$

$$u(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t > 0 \end{cases} \quad (1.2)$$

$$\int_{-\infty}^{\infty} f(t) dt = 1 \quad (1.3)$$

Find the possible values of (a, b) if these are the end points of the latus recta of the associated conic. Plot $f(t)$ for these values of (a, b)

Solution: We expand the integral as

$$\int_{-\infty}^{\infty} f(t) dt = \int_{-\infty}^0 f(t) dt + \int_0^{\infty} f(t) dt \quad (1.4)$$

$$= \int_{-\infty}^0 e^{bt} dt + \int_0^{\infty} e^{-at} dt \quad (1.5)$$

$$= \frac{1}{b} + \frac{1}{a} \quad (1.6)$$

Substituting (1.3) in (1.6)

$$\frac{1}{a} + \frac{1}{b} = 1 \quad (1.7)$$

$$ab - a - b = 0 \quad (1.8)$$

Which is the equation of a conic. If we take a to be x and b to be y and then express this as a conic in standard form we get

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f \quad (1.9)$$

By comparison we obtain:

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad (1.10)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (1.11)$$

$$f = 0 \quad (1.12)$$

We eigen decompose \mathbf{V} as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (1.13)$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.14)$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \quad (1.15)$$

And convert the conic into a standard conic to make it simpler to solve using affine transformations.

$$\mathbf{y}^T \left(\frac{\mathbf{D}}{f_0} \right) \mathbf{y} = 1 \quad (1.16)$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (1.17)$$

$$(1.18)$$

Where

$$f_0 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 \quad (1.19)$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.20)$$

Since eigenvalues of \mathbf{D} are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$ We use another transformation to shift the negative eigenvalue of \mathbf{D} to get the hyperbola in standard form, finally giving us

$$\mathbf{z}^T \left(\frac{\mathbf{D}_0}{f_0} \right) \mathbf{z} = 1 \quad (1.21)$$

$$j(\mathbf{z}) = \mathbf{z}^T \mathbf{D}_0 \mathbf{z} - f_0 = 0 \quad (1.22)$$

$$\mathbf{y} = \mathbf{P}_0 \mathbf{z} \quad (1.23)$$

$$(1.24)$$

Here \mathbf{P}_0 is the reflection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and \mathbf{D}_0 is defined as

$$\mathbf{D} = \mathbf{P}_0 \mathbf{D}_0 \mathbf{P}_0^T \quad (1.25)$$

$$\mathbf{P}_0^T \mathbf{D} \mathbf{P}_0 = \mathbf{D}_0 \quad (1.26)$$

$$\mathbf{D}_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad (1.27)$$

Now we solve to find the endpoints of the latus recta of this standard conic

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (1.28)$$

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{2} \quad (1.29)$$

$$c = \frac{e \mathbf{u}^T \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^T \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)} \quad (1.30)$$

$$= \pm \frac{1}{\sqrt{2}} \quad (1.31)$$

$$\mathbf{F} = \frac{c e^2 \mathbf{n} - \mathbf{u}}{\lambda_2} = \pm 2 \mathbf{e}_2 \quad (1.32)$$

$$(1.33)$$

Equation of latus recta is

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{F} \quad (1.34)$$

$$\equiv \mathbf{n}^T \mathbf{x} = \sqrt{2} \quad (1.35)$$

$$\equiv \mathbf{x} = \mathbf{h} + k \mathbf{m} \quad (1.36)$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ \pm 2 \end{pmatrix} \quad (1.37)$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.38)$$

Let $\hat{\mathbf{z}}$ be the endpoints of the latus recta i.e the intersection of (1.36) with the conic (1.22)

$$k = \frac{1}{\mathbf{m}^T \mathbf{D}_0 \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{D}_0 \mathbf{h} + \mathbf{u}_0) \pm \sqrt{[\mathbf{m}^T (\mathbf{D}_0 \mathbf{h} + \mathbf{u}_0)]^2 - j(\mathbf{h})(\mathbf{m}^T \mathbf{D}_0 \mathbf{m})} \right) \quad (1.39)$$

$$= \pm \sqrt{2} \quad (1.40)$$

$$\therefore \hat{\mathbf{z}} = \begin{pmatrix} \pm \sqrt{2} \\ \pm 2 \end{pmatrix} \quad (1.41)$$

Now transforming $\hat{\mathbf{z}}$ back using (1.17), (1.23) to get endpoints of original conic $\hat{\mathbf{x}}$.

$$\hat{\mathbf{x}} = \mathbf{P}(\mathbf{P}_0\hat{\mathbf{z}}) + \mathbf{c} \quad (1.42)$$

Which gives us values of $\hat{\mathbf{x}}$ to be

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 2 + \sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{2} \\ 2 + \sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_3 = \begin{pmatrix} 2 - \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_4 = \begin{pmatrix} -\sqrt{2} \\ 2 - \sqrt{2} \end{pmatrix}$$

We can plot out the original conic in (1.9) to verify our solution

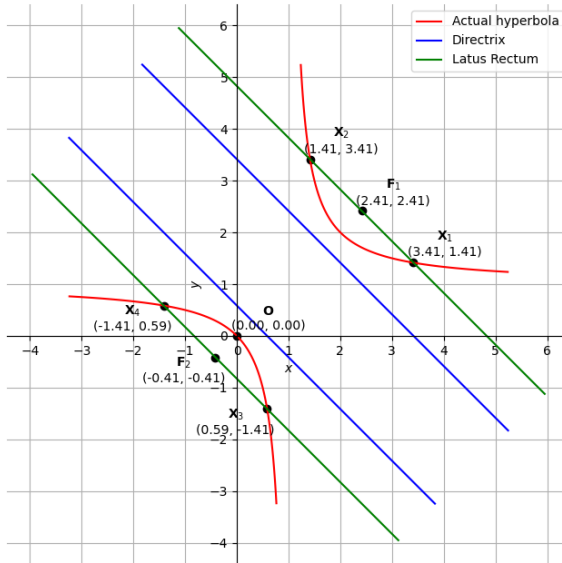


Fig. 1.1

Out of these, $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_4$ are unfit to be used as values for (a, b) as negative values of a or b will cause $f(t)$ to not be finite and thus the integral in (1.3) will not converge and

condition will not be met.

Lastly, We plot the function $f(t)$ for $(a, b) = \hat{\mathbf{x}}_1$ and $(a, b) = \hat{\mathbf{x}}_2$

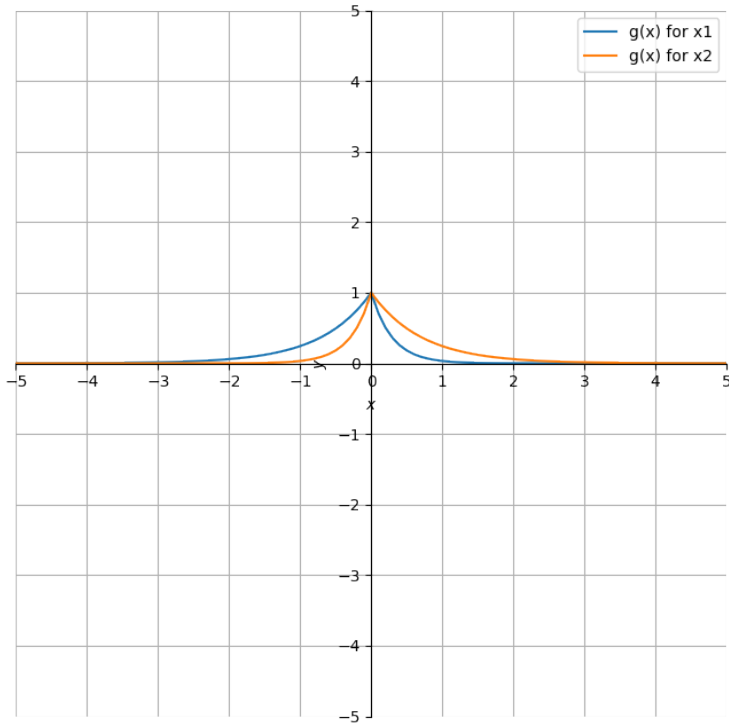


Fig. 1.2

Code for Figures 1.1 and 1.2 can be found at:

`Codes/solution.py`