The piece-wise exponential

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Question: Given that

$$f(t) = e^{-at}u(t) + e^{bt}u(-t) \tag{1.1}$$

$$u(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t > 0 \end{cases} \tag{1.2}$$

$$\int_{-\infty}^{\infty} f(t) = 1 \tag{1.3}$$

Find the possible values of (a, b) if these are the end points of the latus recta of the associated conic. Plot $f(t)$ for these values of (a, b)

Solution: We expand the integral as

$$\int_{-\infty}^{\infty} f(t) = \int_{-\infty}^{0} f(t) + \int_{0}^{\infty} f(t) \tag{1.4}$$

$$= \int_{-\infty}^{0} e^{bt} + \int_{0}^{\infty} e^{-at} \tag{1.5}$$

$$= \frac{1}{b} + \frac{1}{a} \tag{1.6}$$

Subsituting (1.3) in (1.6)

$$\frac{1}{a} + \frac{1}{b} = 1 \tag{1.7}$$

$$ab - a - b = 0 \tag{1.8}$$

Which is the equation of a conic. If we take a to be x and b to be y and then express this as a conic in standard form we get

$$\mathrm{g}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathrm{T}}\mathbf{x} + f \tag{1.9}$$

By comparison we obtain:

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{1.10}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{1.11}$$

$$f = 0 \tag{1.12}$$

We eigen decompose $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathrm{T}} \tag{1.13}$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.14}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{1.15}$$

And convert the conic into a standard conic to make it simpler to solve using affine transformations.

$$\mathbf{y}^{\mathrm{T}} \left(\frac{\mathbf{D}}{f_0} \right) \mathbf{y} = 1 \tag{1.16}$$

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{1.17}$$

$$\tag{1.18}$$

Where

$$f_0 = \mathbf{u}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{1.19}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.20}$$

Since eigenvalues of $\mathbf{D}$ are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$ We use another transformation to shift the negative eigenvalue of $\mathbf{D}$ to get the hyperbola in standard form, finally giving us

$$\mathbf{z}^{\mathrm{T}} \left(\frac{\mathbf{D_0}}{f_0} \right) \mathbf{z} = 1 \tag{1.21}$$

$$\mathbf{z}^{\mathrm{T}}\mathbf{D_0}\mathbf{z} - f_0 = 0 \tag{1.22}$$

$$\mathbf{y} = \mathbf{P}_0\mathbf{z} \tag{1.23}$$

$$\tag{1.24}$$

Here $\mathbf{P}_0$ is the reflection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{D}_0$ is defined as

$$\mathbf{D} = \mathbf{P}_0\mathbf{D}_0\mathbf{P}_0^{\mathrm{T}} \tag{1.25}$$
$$\mathbf{P}_0^{\mathrm{T}}\mathbf{D}\mathbf{P}_0 = \mathbf{D}_0 \tag{1.26}$$
$$\mathbf{D}_0 = \begin{pmatrix} \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{1.27}$$

Now we solve to find the endpoints of the latus recta of this standard conic

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.28}$$
$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{2} \tag{1.29}$$
$$c = \frac{e\mathbf{u}^{\mathrm{T}}\mathbf{n} \pm \sqrt{e^2(\mathbf{u}^T\mathbf{n})^2 - \lambda_2(e^2-1)(\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e(e^2-1)} \tag{1.30}$$
$$= \pm\frac{1}{\sqrt{2}} \tag{1.31}$$
$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} = \pm 2\mathbf{e}_2 \tag{1.32}$$
$$\tag{1.33}$$

Equation of latus recta is

$$\mathbf{n}^{\mathrm{T}}\mathbf{x} = \mathbf{n}^{\mathrm{T}}\mathbf{F} \tag{1.34}$$
$$\equiv \mathbf{n}^{\mathrm{T}}\mathbf{x} = \sqrt{2} \tag{1.35}$$

Let $\hat{\mathbf{z}}$ be the endpoints of the latus recta.
From (1.35) we can say $\hat{\mathbf{z}}$ is of the form $\begin{pmatrix} k_i \\ \pm 2 \end{pmatrix}$. Subsituting back in (1.22)

$$\mathbf{z}^{\mathrm{T}}\mathbf{D_0}\mathbf{z} - f_0 = 0 \tag{1.36}$$
$$-\frac{k_i^2}{2} + 2 - 1 = 0 \tag{1.37}$$
$$k_i^2 = 2 \tag{1.38}$$
$$k_i = \pm\sqrt{2} \tag{1.39}$$
$$\therefore \hat{\mathbf{z}} = \begin{pmatrix} \pm\sqrt{2} \\ \pm 2 \end{pmatrix} \tag{1.40}$$

Now transforming $\hat{\mathbf{z}}$ back using (1.17), (1.23) to get endpoints of original conic $\hat{\mathbf{x}}$.

$$\hat{\mathbf{x}} = \mathbf{P}(\mathbf{P}_0\hat{\mathbf{z}}) + \mathbf{c} \tag{1.41}$$

Which gives us values of $\hat{\mathbf{x}}$ to be

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 2+\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{2} \\ 2+\sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_3 = \begin{pmatrix} 2-\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$$

$$\hat{\mathbf{x}}_4 = \begin{pmatrix} -\sqrt{2} \\ 2-\sqrt{2} \end{pmatrix}$$

We can plot out the original conic in (1.9) to verify our solution

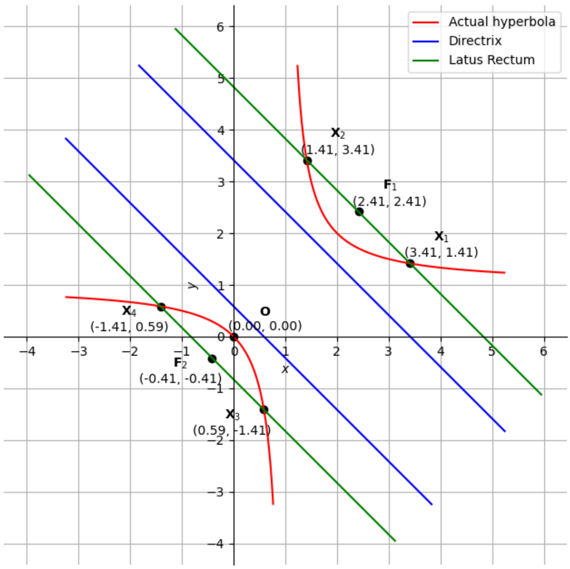

Fig. 1.1

Out of these, $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_4$ are unfit to be used as values for (a, b) as negative values of a or b will cause $f(t)$ to not be finite and thus the integral in (1.3) will not converge and condition will not be met.

Lastly, We plot the function $f(t)$ for $(a, b) = \hat{\mathbf{x}}_1$ and $(a, b) = \hat{\mathbf{x}}_2$

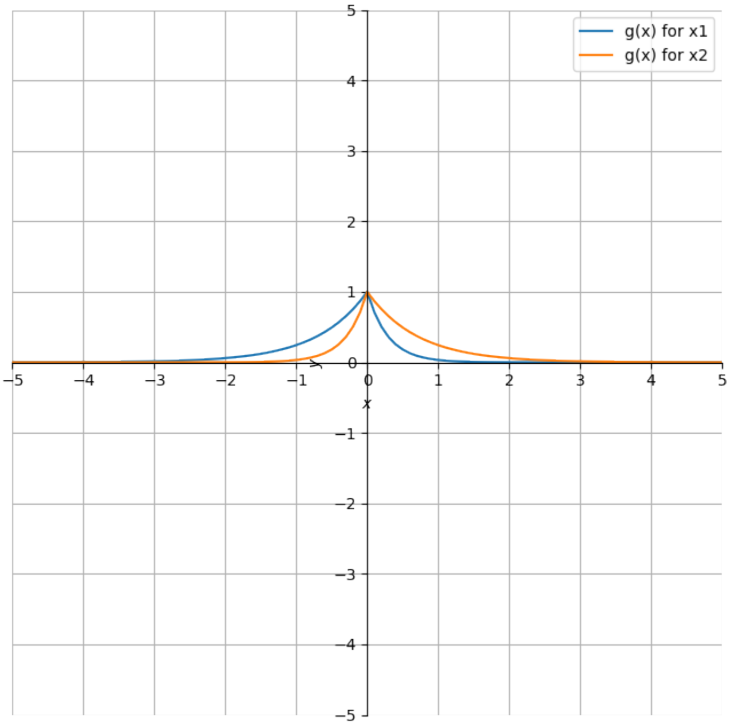

Fig. 1.2

Code for Figures 1.1 and 1.2 can be found at:

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Codes/solution.py
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