E9 205 – Machine Learning for Signal Processing

Homework # 3
Due date: May. 16, 2021

Analytical in writing and report scanned and submitted.
Source code also need to be included.
Name of file should be "Assignment3_FullName.pdf" submitted to teams channel.
Assignment should be solved individually without consent.

May 3, 2021

1. **Kernel LDA** Deepak has learnt about linear discriminant analysis in his course. In a job interview, he is asked to find a way to perform dimensionality reduction in non-linear space. Specifically, he is given a set of N data points $\{\boldsymbol{x}_1,\ \boldsymbol{x}_2, .., \ \boldsymbol{x}_N\}$ and a non-linear transformation $\boldsymbol{\phi}(\boldsymbol{x})$ of the data. When he is asked is to define LDA in the non-linear space, he defines the within-class and between-class scatter matrices for a two-class problem as,

$$\begin{aligned}
\boldsymbol{S}_B &= (\boldsymbol{m}_2^{\phi} - \boldsymbol{m}_1^{\phi})(\boldsymbol{m}_2^{\phi} - \boldsymbol{m}_1^{\phi})^T \\
\boldsymbol{S}_W &= \sum_{k=1}^{2} \sum_{n \in C_k} \left[\boldsymbol{\phi}(\boldsymbol{x}_n) - \boldsymbol{m}_k^{\phi}\right]\left[\boldsymbol{\phi}(\boldsymbol{x}_n) - \boldsymbol{m}_k^{\phi}\right]^T
\end{aligned}$$

where $\boldsymbol{m}_k^{\phi} = \frac{1}{N_k} \sum_{n \in C_k} \boldsymbol{\phi}(\boldsymbol{x}_n)$ for $k = 1, 2$ and C_k denotes the set of data points belonging to class k. He also defines the Fisher discriminant as

$$J = \frac{\boldsymbol{w}^T \boldsymbol{S}_B \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{S}_W \boldsymbol{w}}$$

where $\boldsymbol{w}$ denotes the projection vector. He goes on to say that he can solve the generalized eigen value problem to find $\boldsymbol{w}$ which maximizes the Fisher discriminant. At this point, the interviewer suggests that $\boldsymbol{\phi}(\boldsymbol{x})$ can be infinite dimensional and therefore LDA suggested by Deepak cannot be performed. Deepak counters by saying that he could solve for the LDA using kernel function $k(\boldsymbol{x}_i, \boldsymbol{x}_j) = \boldsymbol{\phi}(\boldsymbol{x}_i)^T \boldsymbol{\phi}(\boldsymbol{x}_j)$. He goes on and shows that LDA can indeed be formulated in a kernel space and the projection of a new data point can be done using kernels (without computing $\boldsymbol{\phi}(\boldsymbol{x})$). How would you have found these two solutions if you were Deepak ? (**Points** 20)

2. By definiton, a kernel function $k(\boldsymbol{x}, \hat{\boldsymbol{x}}) = \boldsymbol{\phi}(\boldsymbol{x})^T \boldsymbol{\phi}(\hat{\boldsymbol{x}})$. A neccessary and sufficient condition for defining a kernel function is that the Gram matrix $\boldsymbol{K}$ is positive definite. Using

either of these definitions, prove the following kernel rules

$$
\begin{aligned}
k(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= ck_1(\boldsymbol{x}, \hat{\boldsymbol{x}}) \\
k(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= f(\boldsymbol{x})k_1(\boldsymbol{x}, \hat{\boldsymbol{x}})f(\hat{\boldsymbol{x}}) \\
k(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= \boldsymbol{x}^T\boldsymbol{A}\hat{\boldsymbol{x}} \\
k(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= k_1(\boldsymbol{x}, \hat{\boldsymbol{x}}) + k_2(\boldsymbol{x}, \hat{\boldsymbol{x}}) \\
k(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= k_1(\boldsymbol{x}, \hat{\boldsymbol{x}})k_2(\boldsymbol{x}, \hat{\boldsymbol{x}})
\end{aligned}
$$

where k_1,k_2 denote valid kernel functions, $c > 0$ is any scalar, $f(\boldsymbol{x})$ is any scalar function and $\boldsymbol{A}$ is symmetric positive definite matrix.

(**Points 10**)

3. **One-class SVM** Let $\boldsymbol{X} = \{\boldsymbol{x}_1,\ \boldsymbol{x}_2, .., \ \boldsymbol{x}_l\}$ be dataset defined in $\mathbb{R}^n$. An unsupervised outlier detection method consist of finding a center $\boldsymbol{a}$ and radius R of the smallest sphere enclosing the dataset in the high dimensional non-linear feature space $\boldsymbol{\phi}(\boldsymbol{x})$. In a soft margin setting, non-negative slack variables ζ_j (for $j = 1, .., l$) can be introduced such that, $||\boldsymbol{\phi}(\boldsymbol{x}_j) - \boldsymbol{a}||^2 \leq R^2 + \zeta_j$

   The objective function in this case is to minimize radius of the sphere with a weighted penalty for slack variables, i.e., $R^2 + C\sum_{j=1}^{l} \zeta_j$ where C is a penalty term for allowing a trade-off between training errors (distance of points outside the sphere) and the radius of the smallest sphere.

   (a) Give the primal form Lagrangian and the primal constraints for the one-class SVM. (**Points** 5)

   (b) Find the dual form in terms of kernel function and the KKT constraints for the one-class SVM. What are the support vectors ? Will support vectors change when $C > 1$ is chosen ? Give a numerically stable estimate of R (**Points** 15)

   (c) For a new data point $\boldsymbol{x}$, how will we identify whether it is an outlier or not (using kernel functions) ? (**Points** 5)

4. Use the following data source for the remaining two questions
   $leap.ee.iisc.ac.in/sriram/teaching/MLSP21/assignments/data/Data.tar.gz$
   Implementing Linear SVMs - 15 subject faces with happy/sad emotion are provided in the data. Each image is of 100×100 matrix. Perform PCA to reduce the dimension from 10000 to K. Implement a classifier on the training images with linear kernel based support vector machine. One potential source of SVM implementation is the LIBSVM package
   $http : //www.csie.ntu.edu.tw/\ cjlin/libsvm/$

   (a) Use the SVM to classify the test images. How does the performance change for various choice of kernels, parameter C and ϵ. How does the performance change as a function of K.

   (b) Compare the SVM classifier with LDA classifier and comment on the similarity and differences in terms of the problem formulation as well as the performance.

(**Points** 15)

5. **Supervised Sentiment Analysis** - Download the movie review data (each line is a individual review)
$http://www.leap.ee.iisc.ac.in/sriram/teaching/MLSP21/assignments/movieReviews1000.txt$

   a Split the data into two subsets. One for training (first 3000 reviews) and the other for testing (last 1000 reviews).

   b Use TF-IDF features and train PCA (using the training data) to reduce the data to 10 dimensions.

   c Train a SVM model. And check the performance on the test set in terms of review classification accuracy.

   d Compare different kernel choices - linear, polynomial and radial basis function. Report the number of support vectors used and the classification performance for different kernel choices.

(**Points** 30)