Assignment_Unit_3

Programme : B.Tech   Brach : CSE   Semester : 3rd

Subject Code : CSE 214   Subject Name : Discrete mathematics

Student ID : SUE019441   Student Name : SHIVAM KUMAR

PART-A

Q.1 - What do you mean by partial order set ?

Ans : A partially order set (Normally Poset) is a set A, together with a relation, $\leq$, that obeys below properties

(i) For all $a, b, c \in A$ (Reflexive), $(a,a) \in A$ is aRa

(ii) For all $a, b \in A$ If aRb and bRa then $a = B$ (Anti-Symmetric Property)

(iii) For $a, b, c \in A$ If aRb and bRc then aRc (Transitive property)

Q.2 Define Hasse Diagram.

Ans: A Hasse diagram is a graphical representation of a finite partially order set (POset) that shows the ordering of element without displaying all the relation explicitly.

Q.3 What are Lattices ?

Ans A Lattices is a special kind of poset, in which every pair of elements has both:

(i) A least upper bound (LUB) - Called the join (denoted $a \cup b$)

(2) A greatest lower bound (GLB) - Called the meet (denoted $a \cap b$)

Q-4 - Define Karnaugh maps.

Ans - A Karnaugh map (K-map) is a graphical tool used in digital logic design to simplify boolean algebra expressions.

It helps to reduce logic functions visually.

Q-5 - What is complemented Lattices?

Ans - A complemented lattices is a type of bounded Lattice in which every element has a complement.

Q-6 - Define Modular and Complete Lattices.

Ans - <u>Modular Lattices</u> - A Lattices $(A, \vee, \wedge)$ is called a modular lattices if it satisfies the modular law:

For all $a, b, c \in A$ if, $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$

<u>Complete Lattices</u> - A Lattices $(A, \vee, \wedge)$ is called a complete lattices if every subset of A has:

(1) A last upper bound $(\overset{(\text{Join})}{LUB})$: $\vee S$ or operation

(2) A greatest lower bound $(\underset{(\text{meet})}{GLB})$: $\wedge S$ AND operation

Q-7 - What do you mean by the Morphisms of Lattices?

Ans - Morphisms are structure - preserving functions between lattices. They help to compare or map one lattice to another, while maintaining lattice operations.

Q-8 - Define posets, lattices and Boolean algebra.

Ans - Poset - A poset is a set 'A' equipped with a binary relation '$\leq$' that satisfies:

(i) Reflexive - $a \leq a$

(ii) Anti-Symmetric - if $a \leq b$ and $b \leq a$ then $a = b$

(iii) Transitive - if $a \leq b$ and $b \leq c$ then $a \leq c$

Lattice - A lattice is a poset in which every pair of elements has:

(i) A least upper bound (join), detnoted $a \vee b$.

(ii) A greatest lower bound (meet), denoted $a \wedge b$.

Boolean Algebra - A boolean algebra is a complemented distributive lattice with:

(i) A least element (0) and greastest element (1)

(ii) complements for every element a: a' such that:

(a) $a \vee a' = 1$

(b) $a \wedge a' = 0$

Q-9 - If the set $P = \{1, 2, 3, 4, 6, 12\}$ with the relation $\leq$ defined as: $a \leq b \Leftrightarrow$ a divides b. Is P poset under the given operation?

Ans - Yes, $P = \{1, 2, 3, 4, 6, 12\}$ with relation $\leq$ defined as:

$a \leq b \Leftrightarrow$ a devides b

(i) Reflexivity - Every element devides itself

$\forall a \in P$, $a/a \Rightarrow a \leq a$

(ii) Anti - Symmentry - if a/b and b/a, then $a=b$

This is true because the only way two positive integers devide each other is if they are equal.

(iii) Transitivity - If a/b and b/c then a/c

This is known as property of divisibility.

Q-10 - Let $A = \{1, 2, 3, 4, 5, 6\}$ with relation $\leq$ defined as "a is less than or equal to b". draw Hesse diagram of $(A, \leq)$.

Solve - Hesse diagram

6
|
5
|
4
|
3
|
2
|
1

Each element is connected to its immediate successor, showing the ordering relation between elements.

Q-11 - If $a, b \in D_{18}$ and $a+b = \text{lcm}(a,b), a.b = \text{gcd}(a,b)$ and $a' = \frac{18}{a}$. show that $\{D_{18}, +, ., ', 1, 18\}$ is not a Boolean algebra.

$\because$ D_{18} represents the divisors of 18

Solve - $\therefore D_{18} = \{1, 2, 3, 6, 9, 18\}$

Need to show that $\{D_{18}, +, ., ', 1, 18\}$ is not a Boolean algebra.

Given, $a+b = \text{lcm}(a,b)$

$a.b = \text{gcd}(a,b)$

$a' = \frac{18}{a}$

$\because$ we know that A Boolean algebra must satisfy certain properties, including distributivity, existence of complement, and existence of identity elements for both operations.

Let $a = 6 \in D_{18}$; Then

$a' = \frac{18}{6} = 3 \in D_{18}$ (satisfied)

$a+a' = \text{lcm}(6,3) = 6 \neq 18 \Rightarrow a+a' \neq 1$ (Not satisfied)

$a.a' = \text{gcd}(6,3) = 3 \neq 1 \Rightarrow a.a' \neq 0$ (Not satisfied)

The complement operation does not satisfy the required properties:

$a+a' \neq 1$ and $a.a' \neq 0$

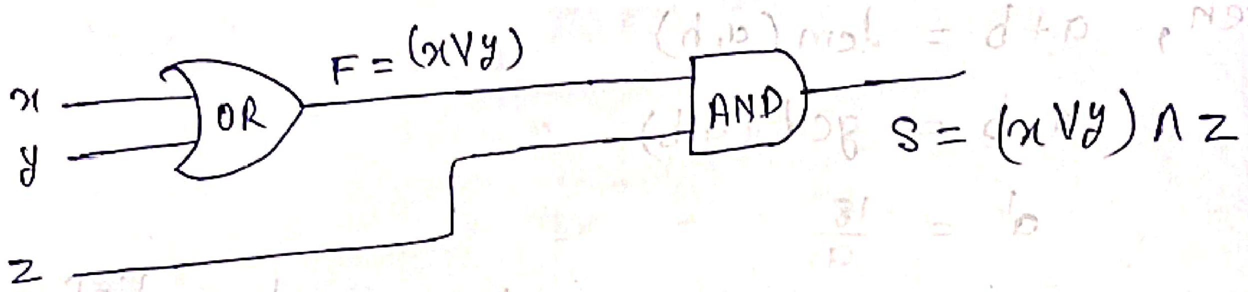
So $\{D_{18}, +, ., ', 1, 18\}$ is not a Boolean Algebra.

Q-12 - Draw the circuit represented by the Boolean function $f(x,y,z) = (x \vee y) \wedge z$.

Ans - The circuit for the boolean function $f(x,y,z) = (x \vee y) \wedge z$ can be draw using two logic gates

(1) OR gate for $(x \vee y)$ - x and y are input

(2) AND gate for $(x \vee y) \wedge z$ - $(x \vee y)$ and z are input



Q-13 - Draw the logic networks corresponding to the following Boolean expressions $xy + x\bar{y}$.

Ans - $F = xy + x\bar{y} = x(y + y') = x \cdot 1 = x$

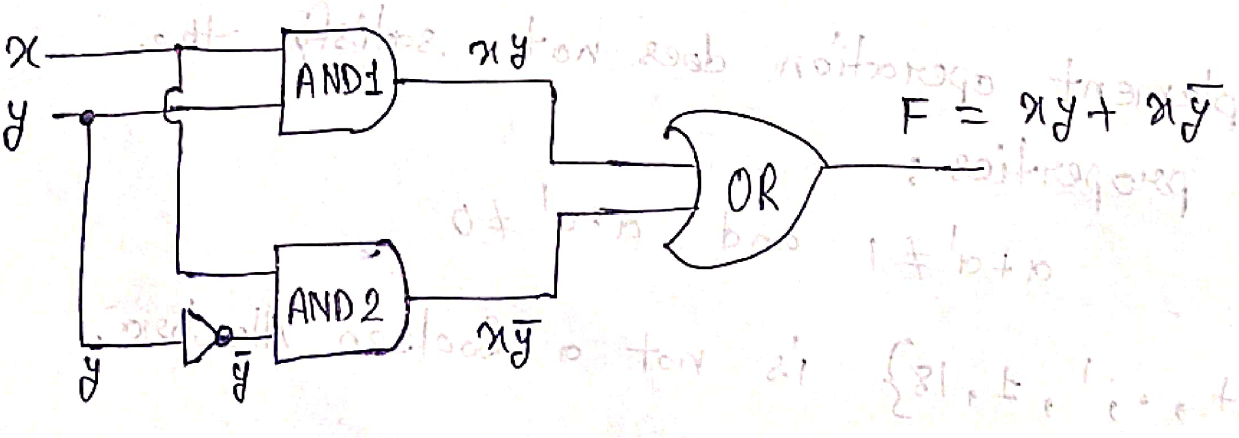
as per complement Law

(1) AND gate 1 for (xy) - x, y are input

(2) NOT gate for $(\bar{y})$ - y is input

(3) AND gate 2 for $(x\bar{y})$ - $x, \bar{y}$ are input

(4) OR gate for $(xy + x\bar{y})$ - xy, $x\bar{y}$ are input



Q-14 - Draw the logic networks corresponding to the following Boolean expressions $xy'z' + x'yz + xy'$

Ans - The circuit or logic networks for

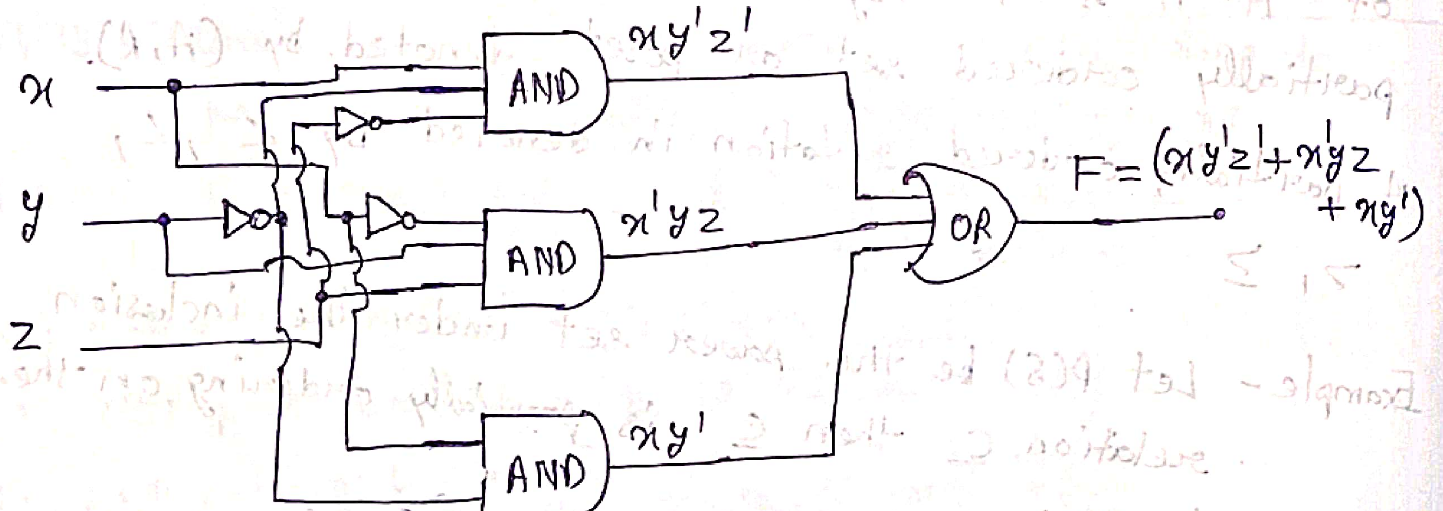
$$xy'z' + x'yz + xy'$$

can be draw using :

(i) 3 NOT gates : for $\bar{x}, \bar{y}, \bar{z}$

(ii) 3 AND gates : one for each product term ($xy'z'$, $x'yz$, xy')

(iii) 1 OR gates : to combine all three terms ($xy'z' + x'yz + xy'$)



Part C- Long Questions and Answers

Q-15 - Which of the following relations on $A = \{0, 1, 2, 3\}$ are partial ordering ? Determine the properties of a partial ordering that the others lack ?

(a) $R_1 = \{(0,0), (1,1), (2,2), (3,3)\}$

(b) $R_2 = \{(0,0), (1,1), (2,0), (2,3), (2,2), (3,2), (3,3)\}$

(c) $R_3 = \{(0,0), (1,1), (1,2), (2,2), (3,3)\}$

Ans - To determine if a relation on $A = \{0,1,2,3\}$ is a partial ordering, it must satisfy three properties:

1. Reflexivity : For all a in A, aRa
2. Antisymmetry : For all a and b in A, if aRb and bRa, then $a = b$
3. Transitivity : For all a, b, and c in A, if aRb and bRc, then aRc.

(a) $R_1 = \{(0,0),(1,1),(2,2),(3,3)\}$

1. Reflexivity - Yes, (a,a) is in R_1 for all a in A
2. Antisymmetry - Yes, there are no pairs (a,b) and (b,a) in R_1 where $a \neq b$.
3. Transitivity - Yes, there are no pair (a,b) and (b,c) in R_1 where $a \neq b$ and $b \neq c$, so transitivity holds vacuously.

- Therefore, R_1 is a partial ordering.

(b) $R_2 = \{(0,0),(1,1),(2,0),(2,3),(2,2),(3,2),(3,3)\}$

1. Reflexivity :- Yes, (a,a) is in R_2 for all a in A.
2. Antisymmetry :- No, (2,3) and (3,2) are in R_2, but $2 \neq 3$
3. Transitivity :- Not applicable due to failure of antisymmetry.

- Therefore, R_2 is not a partial ordering due to lack of antisymmetry.

(c) $R_3 = \{(0,0),(1,1),(1,2),(2,2),(3,3)\}$

1. Reflexivity :- Yes, (a,a) is in R_3 for all a in A

2. Antisymmetric :- Yes, there are no pairs (a,b) and (b,a) in R_3 where $a \neq b$

3. Transitive :- Yes, for $(1,2)$ and no other pair starting with 2, transitivity holds

- Therefore, R_3 is a partial ordering.

Q-16- Let D_m denote the positive divisor of m ordered by divisibility. Draw the Hasse diagram of

(a) D_{12} (b) D_{15} (c) D_{16} (d) D_{17}

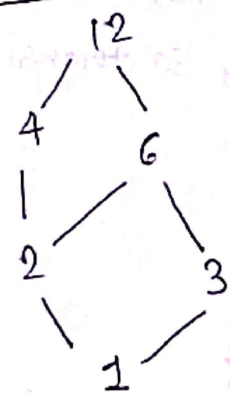
Ans -

(a) D_{12}

~~Hasse D~~

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

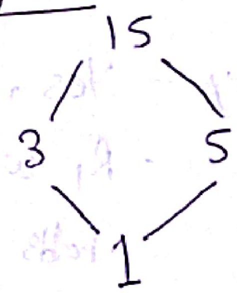
Hasse Diagram -



(b) D_{15}

$D_{15} = \{1, 3, 5, 15\}$

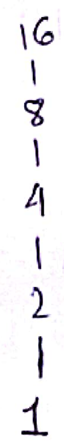
Hasse Diagram -



(c) D_{16}

$D_{16} = \{1, 2, 4, 8, 16\}$

Hasse Diagram -



(d) D_{17}

$D_{17} = \{1, 17\}$

Hasse Diagram -

Q-17 - Let $L_1 = \{a, b, c\}$ and $L_2 = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be the sets of two alphabets and multiplies of 30 respectively. Prove that the set $\{P(L_1), \cap, \cup\}$ and $\{L_2, gcd, lcm\}$ are isomorphic.

Solve - Given, $L_1 = \{a, b, c\}$, $L_2 = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Need to prove,

$$\{P(L_1), \cap, \cup\} \cong \{L_2, gcd, lcm\}$$

$P(L_1) = \{\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Each subset of L_1 can be mapped to a divisor of 30, L_2

$\{\phi\} \to 1$	$\{c\} \to 5$	$\{b, c\} \to 15$
$\{a\} \to 2$	$\{a, b\} \to 6$	$\{a, b, c\} \to 30$
$\{b\} \to 3$	$\{a, c\} \to 10$	

This mapping is bijective and preserves:

Intersection $\cap \leftrightarrow gcd$

Union $\cup \leftrightarrow lcm$

Example -

$\{a\} \cap \{b\} = \phi \to$ maps to $gcd(2, 3) = 1$

$\{a\} \cup \{b\} = \{a, b\} \to$ maps to $lcm(2, 3) = 6$

Since the structure is preserved and mapping is one to one

$$\{P(L_1), \cap, \cup\} \cong \{L_2, gcd, lcm\}$$

Hence, they are isomorphic. Proved

Q-18 - If $a, b \in D_{20}$ and $a+b = \text{lcm}(a,b)$, $a.b = \gcd(a,b)$ and $a' = \frac{20}{a}$ show that $\{D_{20}, +, \cdot, ', 1, 20\}$ is not a Boolean algebra.

Solve - $\because$ D_{20} represents the divisors of 20

$\therefore$ $D_{20} = \{1, 2, 4, 5, 10, 20\}$

Need to show that $\{D_{20}, +, \cdot, ', 1, 20\}$ is not a Boolean algebra.

Given, $a+b = \text{lcm}(a,b)$

$a.b = \gcd(a,b)$

$a' = \frac{20}{a}$

$\because$ we know that A Boolean algebra must satisfy certain properties including distributivity, existence of complement, and existence of identity elements for both operations.

Let $a = 2 \in D_{20}$; Then

$a' = \frac{20}{2} = 10 \in D_{20}$ (satisfied)

$a+a' = \text{lcm}(2, 10) = 10 \neq 20$ (not satisfied)

$a \cdot a' = \gcd(2, 10) = 2 \neq 1$ (not satisfied)

The complement operation does not satisfy boolean algebra required properties.

So, $\{D_{20}, +, \cdot, ', 1, 20\}$ is not a Boolean Algebra

Q-19 - Consider the Boolean algebra $\{D_{210}, +, \cdot, ', 1, 210\}$ where $a+b = \text{lcm}(a,b)$, $a \cdot b = \gcd(a,b)$, $a' = \frac{210}{a}$

Then find atoms of D_{210}?

Is $X = \{1, 2, 105, 210\}$ a subalgebra?

Is $Y = \{1, 2, 3, 6\}$ a subalgebra?

Find two sub algebra of 8 elements.

Solve ⇒ ∵ D_{210} represents the divisors of 210

∴ $D_{210} = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$

(i) Finding Atoms of D_{210} –

∵ we know that Atoms in a Boolean algebra are the minimal elements that cover the least element (1 in this case). In the context of divisors and the operations defined:

(1) Prime Divisors : The atoms of D_{210} are the prime divisors of 210

(2) Prime factorization of 210 ⇒ $210 = 2 \times 3 \times 5 \times 7$

∴ Atoms of D_{210}

The atoms are the prime factors: $\{2, 3, 5, 7\}$.

(ii) Check if $X = \{1, 2, 105, 210\}$ is a subalgebra.

For X to be a subalgebra, it must be closed under $+$, $\cdot$, and $'$ operations.

$2 + 105 = \text{lcm}(2, 105) = 210$ (in X)

$2 \cdot 105 = \gcd(2, 105) = 1$ (in X)

$2' = \frac{210}{2} = 105$ (in X)

$105' = \frac{210}{105} = 2$ (in X)

Therefore, X is a subalgebra.

(iii) check if $Y = \{1,2,3,6\}$ is a subalgebra.

For Y to be subalgebra, it must be closed under the operations:

$2+3 = \text{lcm}(2,3) = 6$ (in Y)

$2 \cdot 3 = \gcd(2,3) = 1$ (in Y)

$2' = \frac{210}{2} = 105$ (not in Y)

Therefore, Y is a subalgebra.

(iv) Find two sub algebra of 8 elements

To find subalgebra of 8 elements, we look for subsets of D_{210} that are closed under the operations and contain 8 elements, including 1 and 20. and also that are closed under lcm gcd, and the complement operation defined by $a' = \frac{210}{a}$

One potential subalgebra is $\{1,2,3,6,35,70,105,210\}$.

Another potential subalgebra is $\{1,2,5,10,21,42,105,210\}$.

Q-20 - Express $E(x,y,z)$ in its complete sum-of-products form.

(a) $E(x,y,z) = (x'+y)' + x'y$

(b) $E(x,y,z) = y(x+yz)'$

(c) $E(x,y,z) = x(xy + y' + x'y)$

Solve - To express each Boolean function $E(x,y,z)$ in its complete Sum of products (SOP) form, we follow these steps:

1. Simplify the expression using Boolean ~~gt~~ algebra.
2. Find the truth table.
3. Write the minterms (product term where output $=1$)
4. Express the function as a sum of those min terms.

(a) $E(x,y,z) = (x'+y)' + x'y$ --- (i)

apply De Morgan's Theorem

$(x'+y)' = x'' \cdot y' = x \cdot y'$ Put in Eq-(i)

$E(x,y,z) = x \cdot y' + x' \cdot y$

Truth Table

x	y	z	$E(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Minterms (rows with 1)

$(0,1,0) \rightarrow x'yz'$

$(0,1,1) \rightarrow x'yz$

$(1,0,0) \rightarrow xy'z'$

$(1,0,1) \rightarrow xy'z$

Complete SOP form –

$$E(x,y,z) = x'yz' + x'yz + xy'z' + xy'z$$

(b) $E(x,y,z) = y(x+yz)' \cdots (ii)$

apply De Morgan's Theorem

$(x+yz)' = x' \cdot (yz)' = x'(y'+z')$ Put in Eq-(ii)

$E(x,y,z) = y \cdot x' \cdot (y'+z') = yx'y' + yx'z' = 0 + yx'z'$

$E(x,y,z) = x'yz'$

Truth Table

x	y	z	$E(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Only 1 in row $(0,1,0) \rightarrow x'yz'$

Complete SOP form –

$$E(x,y,z) = x'yz'$$

(c) $E(x,y,z) = x(xy + y' + x'y)$

Simplify the inner part

$$(xy + y' + x'y) = y' + xy + x'y = y' + y(x+x')$$

$$= y' + y(1) = y' + y = 1$$

Now,

$$E(x,y,z) = x \cdot 1 = x$$

Truth Table

x	y	z	$E(x,y,z)$
0	X	X	0
1	X	X	1

whenever $x=1$, regardless of y,z the output is 1,
So there are 4 rows with output 1:

$(1,0,0) \rightarrow$ minterm: $xy'z'$

$(1,0,1) \rightarrow$ minterm: $xy'z$

$(1,1,0) \rightarrow$ minterm: xyz'

$(1,1,1) \rightarrow$ minterm: xyz

Even though the logic only depends on x, the complete SOP must include y and z for each case where the output is 1.

Complete SOP form -

$$E(x,y,z) = xy'z' + xy'z + xyz' + xyz$$

Q-21 - Construct circuit that produce the following output $F(x,y,z) = (x+y+z)\left(\overline{\bar{x} y \bar{z}}\right)$

Ans - The circuit or logic networks for

$$F(x,y,z) = (x+y+z)\left(\overline{\bar{x} y \bar{z}}\right)$$

can be draw using:

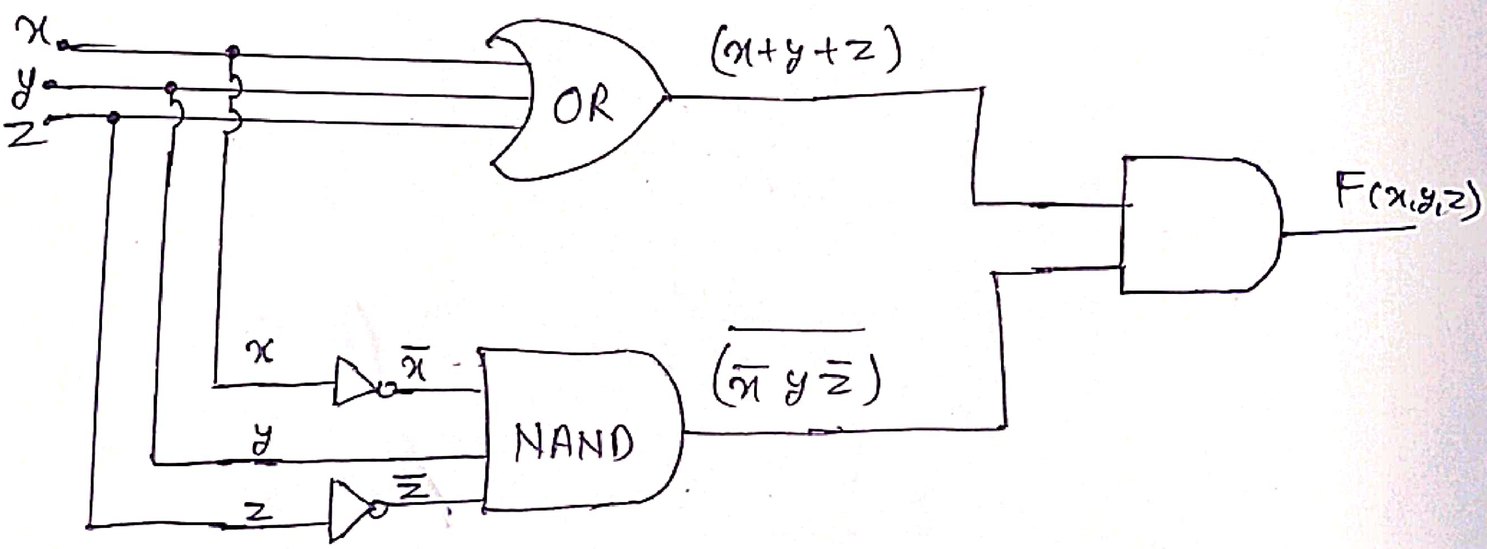
(i) 1 OR gate : for $x+y+z$

(ii) 2 NOT gate : for $\bar{x}$, $\bar{y}$

(iii) 1 NAND gate : for $\left(\overline{\bar{x} y \bar{z}}\right)$

(iv) 1 AND gate : for $(x+y+z)\left(\overline{\bar{x} y \bar{z}}\right)$

Circuit :-



Final Output $F(x,y,z) = (x+y+z)\left(\overline{\bar{x} y \bar{z}}\right)$