1.11.14

AI24BTECH11031 - Shivram S

Question:
If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$

Solution:
Let the two unit vectors be $\mathbf{a}$ and $\mathbf{b}$

$$\|\mathbf{a}+\mathbf{b}\|^2 = 1 \tag{0.1}$$
$$(\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = 1 \tag{0.2}$$
$$\mathbf{a}^\top\mathbf{a}+\mathbf{b}^\top\mathbf{b}+2\mathbf{a}^\top\mathbf{b} = 1 \tag{0.3}$$
$$\mathbf{a}^\top\mathbf{b} = \frac{-1}{2} \tag{0.4}$$

Hence,

$$\|\mathbf{a}+\mathbf{b}\| = \sqrt{(\mathbf{a}-\mathbf{b})^\top(\mathbf{a}-\mathbf{b})} \tag{0.5}$$
$$= \sqrt{\mathbf{a}^\top\mathbf{a}+\mathbf{b}^\top\mathbf{b}-2\mathbf{a}^\top\mathbf{b}} \tag{0.6}$$
$$= \sqrt{1+1-2\cdot\frac{-1}{2}} \tag{0.7}$$
$$= \sqrt{3} \tag{0.8}$$

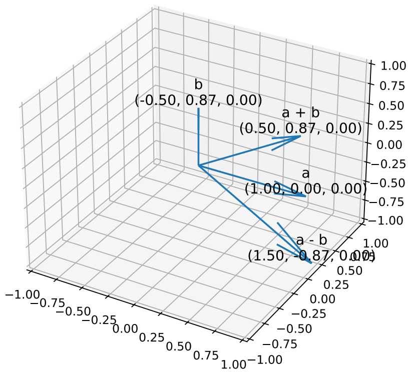

Fig. 0.1: Two unit vectors a and b such that $\|a + b\| = 1$ and $\|a - b\| = \sqrt{3}$.