2021 ST 27-39

AI24BTECH11031 - Shivram S

1) If the marginal probability density function of the k^{th} order statistic of a random sample of size 8 from a uniform distribution on $[0, 2]$ is
$$f(x) = \begin{cases} \frac{7}{32}x^6(2-x), & 0 < x < 2, \\ 0, & \text{otherwise,} \end{cases}$$
then k equals ______.
2) For $\alpha > 0$, let $\left\{X_n^{(\alpha)}\right\}_{n\geq 1}$ be a sequence of independent random variables such that
$$P\left(X_n^{(\alpha)} = 1\right) = \frac{1}{n^{2\alpha}} = 1 - P\left(X_n^{(\alpha)} = 0\right).$$
Let $S = \left\{\alpha > 0 : X_n^{(\alpha)} \text{ converges to 0 almost surely as } n \to \infty\right\}$. Then the infimum of S equals ______ (round off to 2 decimal places).
3) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on $[0, 2]$. For $n \geq 1$, let
$$Z_n = -\log_e\left(\prod_{i=1}^{n}(2 - X_i)\right)^{\frac{1}{n}}$$
Then, as $n \to \infty$, the sequence $\{Z_n\}_{n\geq 1}$ converges almost surely to ______ (round off to two decimal places).
4) Let $\{X_n\}_{n\geq 0}$ be a time-homogeneous discrete time Markov chain with state space $\{0, 1\}$ and transition probability matrix
$$\begin{pmatrix} 0.25 & 0.75 \\ 0.75 & 0.25 \end{pmatrix}$$
If $P(X_0 = 0) = P(X_0 = 1) = 0.5$ then
$$\sum_{k=1}^{100} E\left[(X_{2k})^{2k}\right]$$
equals ______.
5) Let $\{0, 2\}$ be a realization of a random sample of size 2 from a binomial distribution with parameters 2 and p, where $p \in (0, 1)$. To test $H_0 : p = \frac{1}{2}$ against $H_1 : p \neq \frac{1}{2}$, the observed value of the likelihood ratio test statistic equals ______ (round off to 2 decimal places).
6) Let X be a random variable having the probability density function
$$f(x) = \begin{cases} \frac{3}{13}(1-x)(9-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\frac{4}{3}E\left[X\left(X^2-15X+27\right)\right]$ equals ______ (round off to two decimal places).

7) Let (Y, X_1, X_2) be a random vector with mean vector $\begin{pmatrix}5\\2\\0\end{pmatrix}$ and variance-covariance matrix $\begin{pmatrix}10 & 0.5 & -0.5\\ 0.5 & 7 & 1.5\\ -0.5 & 1.5 & 2\end{pmatrix}$. Then the value of the multiple correlation coefficient between Y and its best linear predictor on X_1 and X_2 equals ______ (round off to two decimal places).

8) Let $\underline{X_1}$, $\underline{X_2}$ and $\underline{X_3}$ be a random sample from a bivariate normal distribution with unknown mean vector $\underline{\mu}$ and unknown variance-covariance matrix Σ, which is a positive definite matrix. The p-value corresponding to the likelihood ratio for testing $H_0 : \underline{\mu} = \underline{0}$ against $H_1 : \underline{\mu} \neq \underline{0}$ based on the realization $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}4\\-2\end{pmatrix}, \begin{pmatrix}-5\\0\end{pmatrix}\right\}$ of the random sample equals ______ (round off to two decimal places).

9) Let $Y_i = \alpha + \beta x_i + \epsilon_i, i = 1, 2, 3$ where x_i's are fixed covariates, α and β are unknown parameters, and ϵ_i's are independent and identically distributed random variables with mean zero and finite variance. Let $\hat{\alpha}$ and $\hat{\beta}$ be the ordinary least squares estimators of α and β respectively. Given the following observations

y_i	0.62	26.86	54.02
x_i	3.29	21.53	48.69

the value of $\hat{\alpha} + \hat{\beta}$ equals ______ (round off to two decimal places).

10) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^3 \sin x & x = 0 \text{ or } x \text{ is irrational} \\ \frac{1}{q^3} & x = \frac{p}{q}, p \in \mathbb{Z} \setminus \{0\}, q \in \mathbb{N} \text{ and } \gcd(p, q) = 1 \end{cases}$$

where $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all positive integers and $\gcd(p, q)$ denotes the greatest common divisor of p and q. Then which one of the following statements is true?

a) f is not continuous at 0
b) f is not differentiable at 0
c) f is differentiable at 0 and the derivative of f at 0 equals 0
d) f is differentiable at 0 and the derivative of f at 0 equals 1

11) Let $f\,[0,\ \infty) \to \mathbb{R}$ be a function, where $\mathbb{R}$ denotes the set of all real numbers. Then which of the following statements is true?

a) If f is bounded and continuous, then f is uniformly continuous
b) If f is uniformly continuous, then $\lim_{x\to\infty} f(x)$ exists.
c) If f is uniformly continuous, then the function $g(x) = f(x)\sin x$ is also uniformly continuous
d) If f is continuous and $\lim_{x\to\infty} f(x)$ is finite, then f is uniformly continuous.

12) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f'(x) + 2f(x) > 0$ for all $x \in \mathbb{R}$ where f' denotes the derivative of f and $\mathbb{R}$ denotes the set of all real

numbers. Then which one of the following statements is true?

a) $f(x) > 0$ for all $x > 0$ and $f(x) < 0$ for all $x < 0$
b) $f(x) < 0$ for all $x \neq 0$
c) $f(x) > 0$ for all $x \neq 0$
d) $f(x) < 0$ for all $x > 0$ and $f(x) > 0$ for all $x < 0$

13) Let M be the collection of all 3×3 real symmetric positive definite matrices. Consider the set

$$S = \left\{ \mathbf{A} \in M : \mathbf{A}^{50} - \frac{1}{4}\mathbf{A}^{48} = \mathbf{0} \right\}$$

where $\mathbf{0}$ denotes the 3×3 zero matrix. Then the number of elements in S equals

a) 0　　b) 1　　c) 8　　d) ∞