Shortest Distance Between Curves

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Question: Find the shortest distance between the curves $x^2+x+12 = y$ and $x^2+y^2 = 4$.

Solution:

Variable	Description	Value
C_1	Equation of first conic	$x^2 + x + 12 = y$
C_2	Equation of second conic	$x^2 + y^2 = 4$

TABLE 0: Variables Used

We can rewrite the equations of the curves as

$$\mathbf{x}^\top \mathbf{V}_1 \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f_1 = 0 \tag{1.1}$$

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, f_1 = 12 \tag{1.2}$$

and

$$\mathbf{x}^\top \mathbf{V}_2 \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f_2 = 0 \tag{1.3}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_2 = -4 \tag{1.4}$$

The shortest distance between the curves is along the common normal. Suppose the equation of the common normal is

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.5}$$

Suppose the line is normal to the second curve at the point $\mathbf{q}_2$. The equation of the normal can be written as

$$\left(\mathbf{V}_2 \mathbf{q}_2 + \mathbf{u}_2\right)^\top \mathbf{R} \left(\mathbf{x} - \mathbf{q}_2\right) = 0 \tag{1.6}$$

$$\tag{1.7}$$

where $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the rotation matrix. Suppose $\mathbf{q}_2 = r\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. We can rewrite the above equation as

$$\left(\mathbf{V}_2\mathbf{q}_2 + \mathbf{u}_2\right)^\top \mathbf{R}\mathbf{x} = \left(\mathbf{V}_2\mathbf{q}_2 + \mathbf{u}_2\right)^\top \mathbf{R}\mathbf{q}_2 \tag{1.8}$$

$$= \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}\right)\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix} \tag{1.9}$$

$$= r^2\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}^\top\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = 0 \tag{1.10}$$

Therefore $c = 0$ and the normal passes through the origin.

Suppose the line is normal to the first curve at the point $\mathbf{q}_1$. The equation of the normal can be written as

$$\left(\mathbf{V}_1\mathbf{q}_1 + \mathbf{u}_1\right)^\top \mathbf{R}\left(\mathbf{x} - \mathbf{q}_1\right) = 0 \tag{1.11}$$

Now, we can take $\mathbf{q}_1 = \begin{pmatrix}q_x\\ q_y\end{pmatrix}$ to get

$$\left(\mathbf{V}_1\mathbf{q}_1 + \mathbf{u}_1\right)^\top \mathbf{R}\mathbf{q}_1 = 0 \tag{1.12}$$

$$\begin{pmatrix}q_x + \frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}^\top\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}q_x\\ q_y\end{pmatrix} = 0 \tag{1.13}$$

$$q_xq_y + \frac{1}{2}q_y + \frac{1}{2}q_x = 0 \tag{1.14}$$

Now, since $\mathbf{q}_1$ lies on the first curve,

$$\begin{pmatrix}q_x\\ q_y\end{pmatrix}^\top\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}q_x\\ q_y\end{pmatrix} + 2\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}^\top\begin{pmatrix}q_x\\ q_y\end{pmatrix} + 12 = 0 \tag{1.15}$$

$$q_y = q_x^2 + q_x + 12 \tag{1.16}$$

Substituting the value of q_y in 1.14 gives us

$$2q_x^3 + 3q_x^2 + 26q_x + 12 = 0 \tag{1.17}$$

$$\tag{1.18}$$

Solving this equation gives us $q_x = -0.48$, which can be substituted in 1.14 to get $q_y = 11.75$. The other point of contact, q_2 is given by

$$\mathbf{q}_2 = \frac{2}{\|\mathbf{q}_1\|}\mathbf{q_1} = \begin{pmatrix}-0.0816\\ 1.998\end{pmatrix}$$

Thus, the minimum distance between the curves is given by

$$\|\mathbf{q}_2 - \mathbf{q}_1\| = 9.76 \tag{1.19}$$

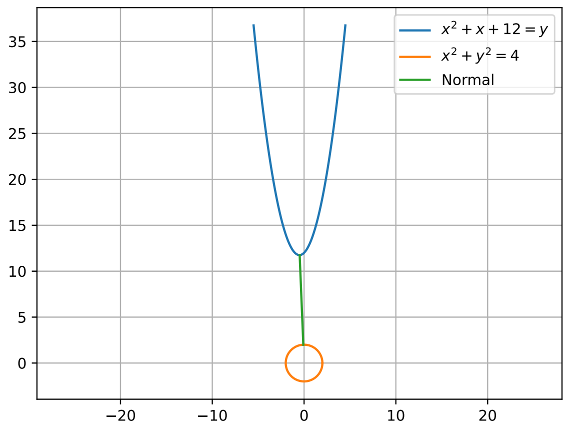

Fig. 1.1: The curves $x^2 + x + 12 = y$, $x^2 + y^2 = 4$, and their common normal.