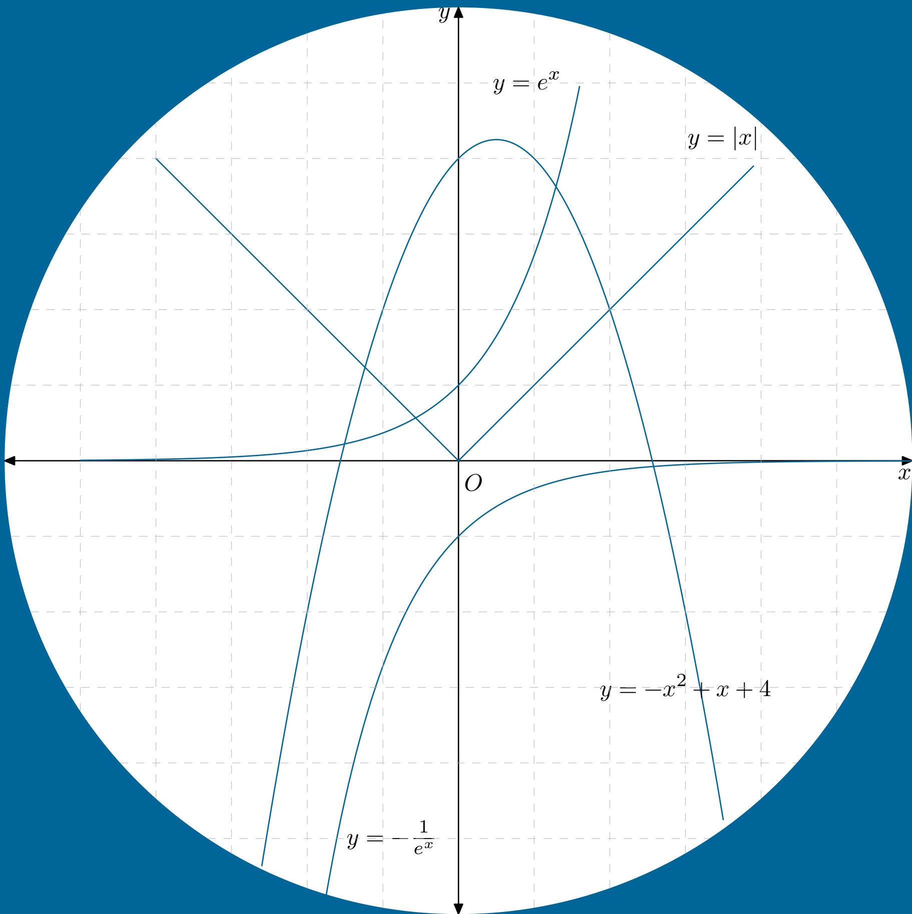
Shiv Shankar Dayal



A Variable in Algebra

A problem-oriented approach

A Variable in Algebra
Early Draft [May 24, 2025]

Dedicated to my family
and Free Software Community

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Preface

This is a book on algebra which, covers basics of algebra till high school level. It covers the most essential topics to take up a bachelor's course where knowledge of algebra is required. There is no specific purpose for writing this book. This is a book for self study and is not recommended for courses in schools and universities. I will try to cover as much as I can and will keep adding new material over a long period. I have no interest in writing a book in a fixed way which serves a university or college course as I have always loved freedom. Life, freedom and honor in that order are important.

Algebra is probably one of the most fundamental subjects in Mathematics as further study of subjects like trigonometry, coordinate geometry and rest all depend on it. That is the primary reason I have chosen it to be the first subject in mathematics to be dealt with. It is very important to understand algebra for the readers if they want to advance further in mathematics.

How to Read This Book?

Every chapter has theory. Read that first. Make sure you understand that. Of course, you have to meet the prerequisites for the book. Then, go on and try to solve the problems. In this book, there are no pure problems. Almost all have answers except those which are of similar kind and repetitive in nature for the sake of practice. If you can solve the problem then all good else look at the answer and try to understand that. Then, few days later take on the problem again. If you fail to understand the answer you can always email me with your work and I will try to answer to the best of my ability. However, if you have a local expert seek his/her advice first. Just that email is bad for mathematics.

Note that mathematics is not only about solving problems. If you understand the theory well, then you will be able to solve problems easily. However, problems do help with the enforcement of theory in your mind.

I am a big fan of old MIR publisher's problem books, so I emphasize less on theory and more on problems. I hope that you find this style much more fun as a lot of theory is boring. Mathematics is about problem solving as that is the only way to enforce theory and find innovtive techniques for problem solving.

Some of the problems in certain chapters rely on other chapters which you should look ahead or you can skip those problems and come back to it later. Since this books is meant for self study answers of most of the problems have been given which you can make use of. However, do not use for just copying but rather to develop understanding.

Who Should Read This Book?

Since this book is written for self study anyone with interest in algebra can read it. That does not mean that school or college students cannot read it. You need to be selective as to what you need for your particular requirements. This is mostly high school course with a little bit of lower classes' course thrown in with a bit of detail here and there.

Prerequisite

You should have knowledge till grade 10th course. Attempt has been made to keep it simple and give as much as background to the topic which is reasonable and required. However, not everything will be covered below grade 10.

Goals for Readers

The goal of for reading this book is becoming proficient in solving simple and basic problems of algebra. Another goal would be to be able to study other subjects which require this knowledge like trigonometry or calculus or physics or chemistry or other subjects. If you can solve 95% problems after 2 years of reading this book then you have achieved this goal.

All of us possess a certain level of intelligence. At average any person can read this book. But what is most important is you have to have interest in the subject. Your interest gets multiplied with your intelligence and thus you will be more capable than you think you can be. One more point is focus and effort. It is not something new which I am telling but I am saying it again just to emphasize the point. Trust me if you are reading this book for just scoring a nice grade in your course then I have failed in my purpose of explaining my ideas.

A lot of problems are given in the book for practice and you should try to solve all of these. Solutions are given to assist you for understading. However, use them as a last resort. Slowly more and more problems will be added. There are very easy problems which should be practiced to progress towards more difficcult problems.

Also, if you find this book useful feel free to share it with others without hesitation as it is free as in freedom.

What Makes This Book Different?

The license. Most books are copyrighted by publishers, while some by the authors. This is released under GNU FDL. This means it is free as in freedom. You can modify, use it anyway you like, sell, resell, and so on. The only condition is that if you modify then modifications have to sent back to me as well so that all of us can benefit from that modification. You can even print, and sell it to make profit out of it. Check license in the appendix for full details.

Make It Better

You can help by checking for spellings, grammar, adding problems, and solutions. Perhaps writing new chapters or extending existing ones. The book's sources are hosted at https://gitlab.com/shivshankar.dayal/algebra-context. Send me a pull request, and I will approve it after review. You have my thanks in advance for improving it.

Confession

I feel like an absolute thief while writing this book for nothing given in this book is mine. All of it belongs to others who did the original work and I have just copied shamelessly. I have nothing new to put in the book. This book is just the result of the pain I feel when I see young children wasting their life for they are poor. And therefore, this book is licensed under GNU FDL. Even if I manage to create few new problems it is still based on knowledge of other pioneers of the subject but perhaps that is how we are supposed to progress bit-by-bit.

Acknowledgements

I am in great debt of my family and free software community because both of these groups have been integral part of my life. Family has provided direct support while free software community has provided the freedom and freed me from the slavery which comes as a package with commercial software. I am especially grateful to my wife, son and parents because it is their time which I have borrowed to put in the book. To pay my thanks from free software community I will take one name and that is Richard Stallman who started all this and is still fighting this never-ending war. When I was doing the Algebra book then I realized how difficult it is to put Math on web in HTML format and why Donald Knuth wrote TeX. Also, TeX was one of the first softwares to be released as a free software.

Now as this book is being written using ConTeXt so obviously Hans hagen and all the people involved with it have my thanks along with Donald Knuth. I use `Emacs` with `Auctex` and hope that someday I will use it in a much more productive way someday. `Emacs` macros have helped me a lot with typing this book, and I do not see any other editor, which makes macros so easy to use.

I have used METAPOST for drawing all the diagrams. It is a wonderful program and works very nicely.

I would like to thank my parents, wife and son for taking out their fair share of time and the support which they have extended to me during my bad times. After that I would like to pay my most sincere gratitude to my teachers particularly H. N. Singh, Yogendra Yadav, Satyanand Satyarthi, Kumar Shailesh and Prof. T. K. Basu. Now is the turn of people from software community. I must thank the entire free software community for all the resources they have developed to make computing better. However, few names I know and here they go. Richard Stallman is the first, Donald Knuth, Edger Dijkstra, John von Neumann after that. I am not a native English speaker and this book has just gone through one pair of eyes therefore chances are high that it will have lots of errors(particularly with commas and spelling mistakes). At the same time it may contain lots of technical errors. With time and revisiosn those errors will be removed.

Shiv Shankar Dayal
Nalanda, 2023

I
Theory and Problems

Chapter 1
Logarithm

Definition: A number x is called the logarithm of a number y to the base b if $b^x = y$, where $b > 0, b \neq 1, y > 0$.

Mathematically, it is represented by the equation $\log_b y = x$ or $b^x = y$.

Notes:

1. The conditions $b > 0, b \neq 1$ and $y > 0$ are necessary in the definition of logarithm.

2. When $b = 1$ suppose logarithm is defined, and we have to find the value of $\log_1 y$. Let $\log_1 y = x \Rightarrow 1^x = y \Rightarrow 1 = y$.

   If $\log_1 2$ is defined then $1 = 2$. So we see that $b = 1$ leads to meaningless results. Similarly, it is true for $b \neq 1$.

3. Similarly if $y < 0$, then $b^x = y$, which is meaningless as L.H.S. is positive while R.H.S. is negative.

4. Let the condition to be true when $b = 0$. Thus, $0^x = y \Rightarrow 0 = y$. Thus, if $\log_0 2$ is defined then $0 = 2$. Hence, our assumption leads to failure.

5. No number can have two different logarithms to a given base. Assume that a number N has two different logarithms x and y with base b. Then, $\log_b N = x$ and $\log_b N = y$

   $\Rightarrow N = b^x$ and $N = b^y$

   $\Rightarrow b^x = b^y \Rightarrow x = y$

6. When the number or base is negative the value of logarithm comes out to be a complex number with non-zero imaginary part.

   Let $\log_e(-5) = x \Rightarrow \log_e(5.e^{i\pi}) = x$ (In complex numbers $e^{i\pi} = -1$)

   $x = \log_e 5 + i\pi$

1.1 Important Results

1. $\log_b 1 = 0$

   Proof: Let $\log_b 1 = x \Rightarrow b^x = 1 \Rightarrow x = 0$

2. $\log_b b = 1$

   Proof: Let $\log_b b = x \Rightarrow b^x = b \Rightarrow x = 1$

3. $b^{\log_b N} = N$

Proof: Let $\log_b N = x \Rightarrow b^x = N \Rightarrow b^{\log_b N} = N$

1.2 Important Formulas

1. $\log_b(x.y) = \log_b x + \log_b y, (x > 0, y > 0)$

   Proof: Let $\log_b x = m \Rightarrow b^m = x$. Similarly, $b^n = y$

   $xy = b^{m+n} = b^o$ (say)

   $m + n = o \Rightarrow \log_b(x.y) = \log_b x + \log_b y$

   Corollary: $\log_b(xyz) = \log_b x + \log_b y + \log_b z$

   If $x, y < 0$, then $\log_b(x.y) = \log_b |x| + \log_b |y|$

2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y, (x, y > 0)$

   Proof: Let $\log_b x = m \Rightarrow b^m = x$ and $\log_b y = n \Rightarrow b^n = y$

   $\frac{x}{y} = b^{m-n}$ and $\log_b\left(\frac{x}{y}\right) = o \Rightarrow b^o = \frac{x}{y}$

   $\Rightarrow m - n = o \Rightarrow \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

   $\log_b\left(\frac{x}{y}\right) = \log_b |x| - \log_b |y|, (x, y < 0)$

3. $\log_b N^k = k \log_b N$

   Proof: Let $\log_b N = x \Rightarrow b^x = N$

   Let $\log_b N^k = y \Rightarrow b^y = N^k \Rightarrow b^y = b^{kx} \Rightarrow y = kx$

   $\Rightarrow \log_b N^k = k \log_b N$

4. $\log_b a = \log_c a \log_b c$

   Proof: Let $\log_b a = x \Rightarrow b^x = a$

   $\log_c a = y \Rightarrow c^y = a$

   $\log_b c = z \Rightarrow b^z = c$

   $b^x = a = c^y = b^{yz} \Rightarrow x = yz \Rightarrow \log_b a = \log_c a \log_b c$

   Alternatively, we can also write it as $\log_b a = \frac{\log_c a}{\log_c b}$

5. $\log_{b^k} N = \frac{1}{k} \log_b N[b > 0]$

   Proof: From previous item we can infer that $\log_{b^k} N = \frac{\log N}{\log b^k} = \frac{1}{k} \log_b N$

$\log_{b^k} N = \frac{1}{k} \log_{|b|} N [b < 0, k = 2m, m \in N]$

6. $\log_b a = \frac{1}{\log_a b}$

   Proof: Let $\log_b a = x \Rightarrow b^x = a$

   Also let $\log_a b = y \Rightarrow a^y = b = a^{xy} \Rightarrow xy = 1$

   $\Rightarrow \log_b a = \frac{1}{\log_a b}$

1.3 Bases of Logarthims

There are two popular bases for logarithms. Common base is 10 and another is e. When base is 10, logarithm is known as *common logarithm* and when base is e, logarithm is known as *natural* or *Napierian logarithm.*

$\log_{10} x$ is also written as $lg\, x$ and $\log_e x$ as $ln\, x$.

1.4 Characteristics and Mantissa

Typically a logarithm will have an integral part and a fractional part. The integral part is called *characteristics* and fractional part is called *mantissa.*

For example, if $\log x = 4.7$ then 4 is characteristics and .7 is mantissa of logarithm. If characteristics is less that zero then at times it is written with a bar above it. For example, $\log x = -5.3 = \bar{5}.3$

As you can easily figure out the number of possitive integers having base b and characteristics n is $b^{n+1} - b^n$.

1.5 Inequality of Logarithms

If $b > 1$ and $\log_b x_1 > \log_b x_2$ then $x_1 > x_2$. If $b < 1$ and $\log_b x_1 > \log_b x_2$ then $x_1 < x_2$.

1.6 Expansion of Logarithm and Its Graph

The logarithm series is given below:

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

So we can see that rate of increment of logarithm function decreases. Rate of increment of logarithm function is given by $\frac{1}{x}$ at any point x, as we will learn when we study Calculus and derivatives.

1.7 Problems

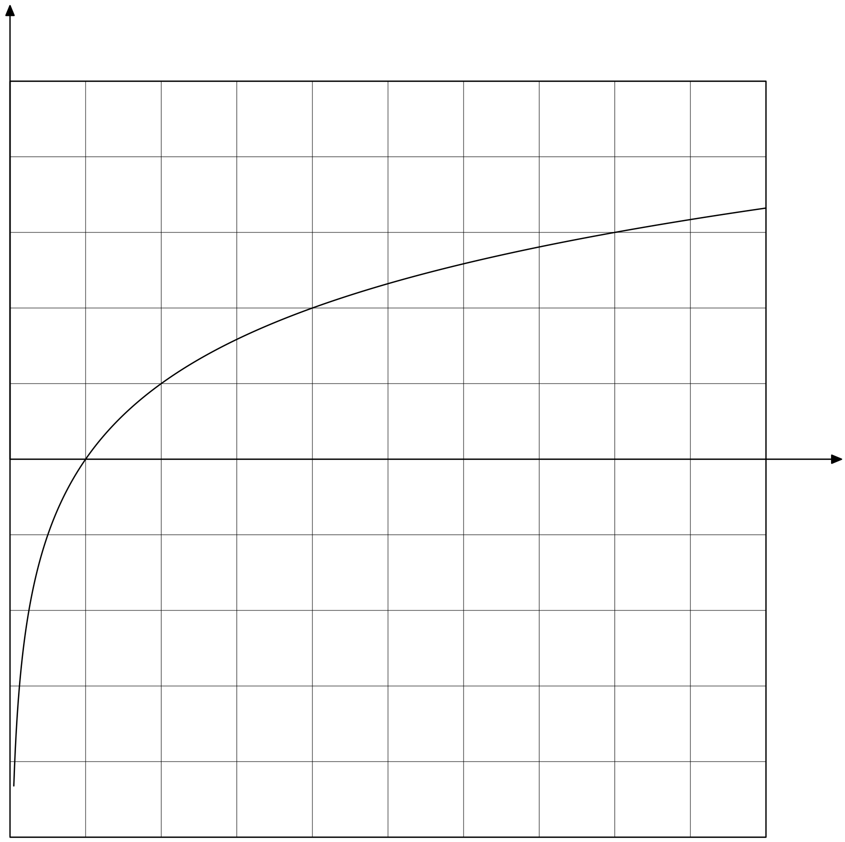


Figure 1.1 Graph of $\log 2$.

1. Find the value of x, where $\log_{\sqrt{8}} x = \frac{10}{3}$.
2. Prove that $\log_b a . \log_c b . \log_a c = 1$.
3. Prove that $\log_3 \log_2 \log_{\sqrt{5}} 625 = 1$.
4. If $a^2 + b^2 = 23ab$, then prove that $\log \frac{a+b}{5} = \frac{1}{2}(\log a + \log b)$.
5. Prove that $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$.
6. Find the value of $\log \tan 1^\circ + \log \tan 2^\circ + ... + \log \tan 89^\circ$.
7. Evaluate $\log_9 \tan \frac{\pi}{6}$.
8. Evaluate $\frac{\log_{a^2} b}{\log_{\sqrt{a}} b^2}$.

9. Evaluate $\log_{\sqrt{5}} .008$.

10. Evaluate $\log_{2\sqrt{3}} 144$.

11. Prove that $\log_3 \log_2 \log_{\sqrt{3}} 81 = 1$.

12. Prove that $\log_a x \log_b y = \log_b x \log_a y$.

13. Prove that $\log_2 \log_2 \log_2 16 = 1$.

14. Prove that $\log_a x = \log_b x \log_c b \ldots \log_n m \log_a n$.

15. Prove that $a^x = 10^x \log_{10} a$.

16. If $a^2 + b^2 = 7ab$, prove that $\log\{\frac{1}{3}(a+b)\} = \frac{1}{2}(\log a + \log b)$.

17. Prove that $\frac{\log a \log_a b}{\log b \log_a b} = -\log_a b$.

18. Prove that $\log(1+2+3) = \log 1 + \log 2 + \log 3$.

19. Prove that $2\log(1+2+4+7+14) = \log 1 + \log 2 + \log 4 + \log 7 + \log 14$.

20. Prove that $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1$.

21. Simplify $\frac{\log_9 11}{\log_5 13} \div \frac{\log_3 11}{\log_{\sqrt{5}} 13}$.

22. Simplify $3^{\sqrt{\log_3 2}} - 2^{\sqrt{\log_2 3}}$.

23. Find the least integer n such that $7^n > 10^5$, given that $\log_{10} 343 = 2.5353$.

24. If a, b, c are in G.P., prove that $\log_a x, \log_b x, \log_c x$ are in H.P.

25. Prove that $\log \sin 8x = 3 \log 2 + \log \sin x + \log \cos x + \log \cos 2x + \log \cos 4x$.

26. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$ then prove that $xyz + 1 = 2yz$.

27. If a and b are the lengths of the sides and c be the length of the hypotenuse of a right-angle triangle and $c - b \neq 1$ and $c + b \neq 1$, prove that $\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a$.

28. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, then prove that $x^x y^y z^z = 1$.

29. If $\frac{yz \log(yz)}{y+z} = \frac{zx \log(zx)}{z+x} = \frac{xy \log(xy)}{x+y}$, prove that $x^2 = y^y = z^2$.

30. Prove that $(yz)^{\log y - \log z}(zx)^{\log z - \log x}(xy)^{\log x - \log y} = 1$.

31. Prove that $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \ldots + \frac{1}{\log_{1988} N} = \frac{1}{\log_{1988!} N}$.

32. If $0 < x < 1$, prove that $\log(1+x) + \log(1+x^2) + \log(1+x^4) + \ldots$ to $\infty = -\log(1-x)$.

33. Find the sum of the series $\frac{1}{\log_2 a} + \frac{1}{\log_4 a} + \dots$ up to n terms.

34. If $\log_4 10 = x, \log_2 20 = y$ and $\log_5 8 = z$, prove that $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$.

35. If $x = \log_a bc, y = \log_b ca, z = \log_c ab$, prove that $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$.

36. Prove that $\frac{1}{1+\log_b a+\log_b c} + \frac{1}{1+\log_c a+\log_c b} + \frac{1}{1+\log_a b+\log_a c} = 1$.

37. Prove that $x^{\log y - \log z} y^{\log z - \log x} z^{\log x - \log y} = 1$.

38. If $\frac{\log a}{y-z} = \frac{\log b}{z-x} = \frac{\log c}{x-y}$, prove that $a^x b^y c^z = 1$.

39. If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{x-y}$, prove that $y^z z^y = z^x x^z = x^y y^x$.

40. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, prove that $a^{b+c} b^{c+a} c^{a+b} = 1$.

41. If $\frac{\log x}{q-r} = \frac{\log y}{r-p} = \frac{\log z}{p-q}$, prove that $x^{q+r} y^{r+p} z^{p+q} = x^p y^q z^r$.

42. If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, prove that $x = a^{\frac{1}{1-\log_a z}}$.

43. Let $f(x) = \frac{1}{1-\log_e x}$, $f(y) = e^{f(z)}$ and $z = e^{f(x)}$, prove that $x = e^{f(y)}$.

44. Show that $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \dots + \frac{1}{\log_{43} n} = \frac{1}{\log_{43!} n}$.

45. Show that $2(\log a + \log a^2 + \log a^3 + \dots + \log a^n) = n(n+1)\log a$.

46. Find the number of digits in 12^{12}, without actual computation. [Given $\log 2 = 0.301$ and $\log 3 = 0.477$]

47. How many positive integers have a characteristics of 2 when base is 3.

48. Prove that $\log_a x \log_b y = \log_b x \log_a y$.

49. If a, b, c are in G.P., prove that $\log_a x, \log_b x, \log_c x$ are in H.P.

50. How many zeros are there between the decimal point and first significant digit in 0.0504^{10}? Given $\log 2 = 0.301, \log 3 = 0.477, \log 7 = 0.845$.

51. Find the number of digits in 72^{15} without actual computation. Given $\log 2 = 0.301$ and $\log 3 = 0.477$.

52. How many positive integers have characteristics 2 when base is 5?

53. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of digits in $3^{15} \times 2^{10}$.

54. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of digits in 6^{20}.

55. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of digits in 5^{25}.

56. Solve $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p x)\}] = 0$.

57. Solve $\log_7 \log_5(\sqrt{x+5} + \sqrt{x}) = 0$.

Solve the following equations:

58. $\log_2 x + \log_4(x + 2) = 2$.

59. $\log_{x+2} x + \log_x(x + 2) = \frac{5}{2}$.

60. $\log(x + 1) = 2 \log x$.

61. $2 \log_x a + \log_{ax} a + 3 \log_{a^2x} a = 0$. Given $a > 0$.

62. $x + \log_{10}(1 + 2^x) = x \log_{10} 5 + \log_{10} 6$.

63. $x^{\frac{3}{4}(\log_2 x)^2 + \log_x 2 - \frac{5}{4}} = \sqrt{2}$.

64. $(x^2 + 6)^{\log_3 x} = (5x)^{\log_3 x}$.

65. $(3 + 2\sqrt{2})^{x^2-6x+9} + (3 - 2\sqrt{2})^{x^2-6x+9} = 6$.

66. $\log_8\left(\frac{8}{x^2}\right) \div (\log_8 x)^2 = 3$.

67. $\sqrt{\log_2(x)^4} + 4 \log_4 \sqrt{\frac{2}{x}} = 2$.

68. $2 \log_{10} x - \log_x 0.01 = 5$.

69. $\log_{\sin x} 2 \log_{\cos x} 2 + \log_{\sin x} 2 + \log_{\cos x} 2 = 0$.

70. $2^{x+3} + 2^{x+2} + 2^{x+1} = 7^x + 7^{x-1}$.

71. $\log_{\sqrt{2} \sin x}(1 + \cos x) = 2$.

72. $\log_{10}[198 + \sqrt{x^3 - x^2 - 12x + 36}] = 2$.

73. If $\log 2 = 0.30103$ and $\log 3 = 0.47712$, solve the equation $2^x 3^{2x} - 100 = 0$.

74. $\log_x 3 \log_{\frac{x}{3}} 3 + \log_{\frac{x}{81}} 3 = 0$.

75. $\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$.

76. $\log_2(x^2 - 1) = \log_{\frac{1}{2}}(x - 1)$.

77. $\log_5\left(5^{\frac{1}{x}+125}\right) = \log_5 6 + 1 + \frac{1}{2x}$.

78. $\log_{100} |x + y| = \frac{2}{1}$ and $\log_{10} y - \log_{10} |x| = \log_{100} 4$.

79. $2\log_2 \log_2 x + \log_{\frac{1}{2}} \log_2(2\sqrt{2}x) = 1$.

80. $\log_{\frac{3}{4}} \log_8(x^2+7) + \log_{\frac{1}{2}} \log_{\frac{1}{4}}(x^2+7)^{-1} = 2$.

81. $\log_{10} x + \log_{10} x^{\frac{2}{1}} + \log_{10} x^{\frac{1}{4}} + \ldots$ to $\infty = y$ and $\frac{1+3+5+\ldots+(2y-1)}{4+7+10+\ldots+(3y+1)} = \frac{20}{7\log_{10} x}$.

82. $18^{4x-3} = (54\sqrt{2})^{3x-4}$.

83. $4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$.

84. $3^{4\log_9(x+1)} = 2^{2\log_2(x+3)}$.

85. $\frac{6}{5}a^{\log_a x \log_{10} a \log_a 5} - 3^{\log_{10}\frac{x}{10}} = 9^{\log_{100} x + \log_4 2}$.

86. $2^{3x+\frac{1}{2}} + 2^{x+\frac{1}{2}} = 2^{\log_2 6}$.

87. $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$.

88. For $x > 1$, show that $2\log_{10x} x - \log_x .01 \geq 4$.

89. Show that $|\log_b a + \log_a b| > 2$.

90. Solve $\log_{0.3}(x^2+8) > log_{0.3} 9x$.

91. Solve $\log_{x-2}(2x-3) > \log_{x-2}(24-6x)$.

92. Find the interval in which x will lie if $\log_{0.3}(x-1) < \log_{0.09}(x-1)$.

93. Solve $\log_{\frac{1}{2}} x \geq \log_{\frac{1}{3}} x$.

94. Solve $\log_{\frac{1}{3}} \log_4(x^2-5) > 0$.

95. Solve $\log(x^2-2x-2) \leq 0$.

96. Solve $\log_2^2(x-1)^2 - \log_{0.5}(x-1) > 5$.

97. Prove that $\log_2 17 \log_{\frac{1}{5}} 2 \log_3 \frac{1}{5} > 2$.

98. Show that $\log_{20} 3$ lies between $\frac{1}{2}$ and $\frac{1}{3}$.

99. Show that $\log_{10} 2$ lies between $\frac{1}{4}$ and $\frac{1}{3}$.

100. Solve $\log_{0.1}(4x^2-1) > \log_{0.1} 3x$.

101. Solve $\log_2(x^2-24) > \log_2 5x$.

102. Show that $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > 2$.

103. Without actual computation find greater among $(0.01)^{\frac{1}{3}}$ and $(0.001)^{\frac{1}{5}}$.

104. Without actual computation find greater among $\log_2 3$ and $\log_3 11$.

105. Solve $\log_3(x^2 + 10) > \log_3 7x$.

106. Solve $x^{\log_{10} x} > 10$.

107. Solve $\log_2 x \log_{2x} 2log_2 4x > 1$.

108. Solve $\log_2 x \log_3 2x + \log_3 x \log_2 4x > 0$.

109. Find the value of $\log_{12} 60$ if $\log_6 30 = a$ and $\log_{15} 24 = b$.

110. If $\log_a x, \log_b x$ and $\log_c x$ are in A.P. and $x \neq 1$, prove that $c^2 = (ac)^{\log_a b}$.

111. If $a = \log_{\frac{1}{2}} \sqrt{0.125}$ and $b = \log_3\left(\frac{1}{\sqrt{24}-\sqrt{17}}\right)$ then find whether $a > 0, b > 0$.

112. Which one is greater among $\cos(\log_e \theta)$ and $\log_e(\cos\theta)$ if $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$.

113. If $\log_2 x + \log_2 y \geq 6$, prove that $x + y \geq 16$.

114. If a,b,c be three distinct positive numbers, each different from 1 such that $\log_b a \log_c a - \log_a a + \log_a b \log_c b - \log_b b + \log_a c \log_b c - \log_c c = 0$ then find abc.

115. If $y = 10^{\frac{1}{1-\log x}}$ and $z = 10^{\frac{1}{1-\log y}}$, prove that $x = 10^{\frac{1}{1-\log z}}$.

116. If n is a natural number such that $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$ and $p_1, p_2, p_3, \dots, p_k$ are distinct primes, then show that $\log n \geq k \log 2$.

117. The numbers $3, 3\log_y x, 3\log_z y, 7\log_x z$ form and A.P. then prove that $x^{18} = y^{21} = z^{28}$.

118. Prove that $\log_4 18$ is an irrational number.

119. If $x, y, z > 1$ are in G.P. then prove that $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in H.P.

120. Find the value of $\log_{30} 8$, if $\log_{30} 3 = a$ and $\log_{30} 5 = b$.

121. Find the value of $\log_{54} 168$, if $\log_7 12 = a$ and $\log_{12} 24 = b$.

122. If $a \neq 0$ and $\log_x(a^2 + 1) < 0$ then find the interval in which x lies.

123. If $\log_{12} 18 = a$ and $\log_{24} 54 = b$, prove that $ab + 5(a - b) = 1$.

124. If a,b,c are in G.P., show that $\log_a x, \log_b x, \log_c x$ are in H.P.

125. If $a, a_1, a_2, \dots, a_n$ are in G.P. and $b, b_1, b_2, \dots, b_n$ in A.P. with positive terms and also the common difference of A.P. and common rations of G.P. are positive, show that there exists a system of logarithm for which $\log a_n - b_n = \log a - b$ for any n. Find the base of this system.

126. If $\log_3 2, \log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in A.P., find the value of x.

127. Prove that $\log_2 7$ is an irraational number.

128. If $\log_{0.5}(x-2) < \log_{0.25}(x-2)$, then find the interval in which x lies.

Chapter 2
Progressions

There are three different progressions: arithmetic progression, geometric progression and harmonic progression. We start this chapter with arithmetic progression or A.P.

2.1 Arithmetic Progressions

Consider sequences like $1, 2, 3, 4, \ldots$ or $-1, -2, -3, -4, \ldots$ or $1, 3, 5, 7, \ldots$ or $a, a+d, a+2d, \ldots$

These sequences increase or decrease with a common difference. When quantities increase or decrease with a common difference they are said to be in *Arithmetic Progression*. The *common difference* can be found by subtracting any term of the series that follows it. For example for the first series it is 1 and for the last it is d.

Consider the series $a, a+d, a+2d, a+3d, \ldots$

Simple observation tells us that 1st term is a, 2nd term is $a+d$, the 3rd term is $a+2d$ and hence the nth term will be $a+(n-1)d$. These terms are typically written as $t_1, t_2, t_3, \ldots, t_n$.

2.1.1 nth Term of Arithmetic Progression

Following above discussion, we can clearly say that the nth term of an arithmetic progression is given by $t_n = a+(n-1)d$, where a is called the first term and d the common difference.

$$t_n = a + (n-1)d \tag{2.1}$$

2.1.2 Sum of an Arithmetic Progression

Let S_n represent the sum of first n terms of an arithmetic progression, then we can write.

$$S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-2)d] + [a+(n-1)d]$$

Writing the terms in reverse order we have

$$S_n = [a+(n-1)s] + [a+(n-2)d] + \cdots + (a+d) + a$$

Adding term by term, we get

$$2S_n = [2a+(n-1)d] + [2a+(n-1)d] + \cdots \text{ to } n \text{ terms}$$

$$2S_n = n[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[2a+(n-1)d] \tag{2.2}$$

We also see that $S_n = \frac{n}{2}(t_1 + t_n)$

We also see that if a series is

$$1 + 2 + 3 + \cdots + n = \sum_{i=0}^{n} i = \frac{n(n+1)}{2}. \tag{2.3}$$

2.1.3 Arithmetic Mean

When three quantities are in arithmetic progression the quantity in the middle is known to be arithmetic mean of the other two. For example, if a, b, c are in A.P., then b is said to be arithmetic mean of a and c. In general, it is written $b = \frac{a+c}{2}$. This can be examined further. Let $b = a + d$, then $c = a + 2d$. Clearly, $b = \frac{a+c}{2}$.

It is also possible to insert n numbers between any two numbers such that all of them are in A.P. Consider two numbers a and b in between which we want to insert n numbers such that they are in A.P. Clearly, b will become $n+2$th term of A.P. Let common difference be d then we can write $b = a + (n+1)d \Rightarrow d = \frac{b-a}{n+1}$. Now all the n arithmetic means can be deduced. Let those be $m1, m2, \cdots, m_n$ then $m_1 = a + \frac{b-a}{n+1}, m_2 = a + \frac{2(b-a)}{n+1}, \cdots, m_n = a + \frac{n(b-a)}{n+1}$.

First A.M. $= a + d = \frac{an+b}{n+1}$

Second A.M. $= a + 2d = \frac{a(n-1)+b}{n+1}$

...

nth A.M. $= a + nd = \frac{a+nb}{n+1}$

$$A_n = \frac{a+nb}{n+1} \tag{2.4}$$

Suppose there are n terms of an A.P., then the arithmetic mean of those n terms is given by $\frac{t_1+t_2+\cdots+t_n}{n}$.

2.1.4 Deducing Number of Terms

We know that $S_n = \frac{n}{2}[2a + (n-1)d]$. Say S_n, a and d are known and we have to evaluate n. This being a quadratic equaion will have two roots for n. If the results are positive and integral then there is no problem in interpreting the results. In some cases for a negative root a suitable interpretation can be given.

Example: How many terms of the series $-8, -6, -4, \cdots$ must be added for the sum to be 36?

$\frac{n}{2}[-16 + (n-1)2] = 36 \Rightarrow n^2 - 9n - 36 = 0 \Rightarrow n = 12, -3$

If we take 12 terms of the series, we have $-8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, 14$. The sum of these terms is 36 and sum of last three terms is also 36 which is represented by $n = -3$.

2.1.5 Properties of an A.P.

1. If a fixed number is added to or subtracted from each item of a given A.P., then the resulting is also an A.P., and it has the same common difference as that of the given A.P.

2. If each term of an A.P. is multiplied or divided by a non-zero fixed constant then the resulting sequence is also an A.P. The common difference is multiplied or divided by the same factor.

3. If $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are two arithmetic progressions then $a_1+b_1, a_2+b_2, a_3+b_3, \cdots$ are also in A.P.

4. If we have to choose three unknown terms in an A.P. then it is best to choose them as $a-d, a, a+d$.

5. If we have to choose four unknown terms in an A.P. then it is best to choose them as $a-3d, a-d, a+d, a+3d$.

6. In an A.P., the sum of terms equidistant from the beginning and end is constant and is equal to the sum of first and last term.

7. Any term of an A.P., except the first, is equal to half the sum of terms which are equidistant from it:

$$a_n = \frac{1}{2}(a_{n-k} + a_{n+k}),\ k < n, \text{ and for } k = 1$$

$$a_n = \frac{1}{2}(a_{n-1} + a_{n+1})$$

8. $t_n = S_n - S_{n-1}, n \geq 2$

9. If $t_n = pn + q$ i.e. a linear expression in n then it will form an A.P. of common difference $p = t_n - t_{n-1}$ and first term $p + q$. For example, if $t_n = 3n + 4$, then it is an A.P. of common difference 3 anda the first term as 7.

10. If $S_n = an^2 + bn + c$ i.e. a quadratic function in n, then the series in an A.P. where $a = 2a$, twice the coefficient of n^2.

2.1.6 Sum of Squares and Cubes and More

We observe that

$$i^3 - (i-1)^3 = 3i^3 - 3i + 1 \Rightarrow \sum_{i=1}^{n}[i^3 - (i-1)^3] = 3\sum_{i=0}^{n} i^2 - \frac{3n(n+1)}{2} + n \tag{2.5}$$

$$n^3 = 3\sum_{i=0}^{n} i^2 - \frac{3n(n+1)}{2} + n \Rightarrow 3\sum_{i=0}^{n} i^2 = n^3 + \frac{3n(n+1)}{2} - n$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2.6}$$

Following in a similar fashion, we can show that

$$\sum_{i=0}^{n} = \left\{\frac{n(n+1)}{2}\right\}^2 \tag{2.7}$$

More powers can be evaluated in a similar fashion.

2.2 Geometric Progressions

A succession of numbers is said to be in geometric progressions or geometric sequence if the ratio of any term and the term preceeding it is constant throughout. This constant is called *common ratio* of the G.P.

Example: 1, 2, 4, 8, 16, ...

Here, $\frac{t_2}{t_1} = \frac{t_3}{t_2} = ... = 2$.

Also, $1, 3, 9, 27, ...$ are in geometric progression whose first term is 1 and common ratio is 3.

Also, $2, -4, 8, -16, ...$ are in geometric progression whose firts term is 2 and common ratio is -2.

2.2.1 Properties of a G.P.

1. If the each term of a G.P. be multiplied by a non-zero number, then the sequence obtained is also a G.P.

   Proof: Let the given G.P. be $a, ar, ar^2, ar^3, ...$

   Let k be a non-zero number, the sequence obtained by multiplying each term of the given G.P. by k is $ak, ark, ar^2k, ar^3k, ...$

   Clearly, the series is in G.P. with the same common ratio as previous ratio i.e. r.

   Again, dividing each term of G.P. $a, ar, ar^2, a^3, ...$ we obtain the sequence $\frac{a}{k}, \frac{ar}{k}, \frac{ar^2}{k}, ...$

   It is clear that this new sequence is also a G.P., whose common ratio is r.

2. The reciprocals of the terms of a G.P. are also in G.P.

   Proof: Let the G.P. be $a, ar, ar^2, ...$, the sequence whose terms are reciprocals of this G.P. is $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \cdots$

   It is clear that this sequence is in G.P., whose first term is $\frac{1}{a}$ and common ratio is $\frac{1}{r}$.

2.2.2 Sum of the First n Terms of a G.P.

Let a be the first term and r be the common ratio of a G.P. and S_n be the sum of its first n terms

Case I: When $r \neq 1$

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting, we get $(1-r)S_n = a - ar^n = a(1-r^n)$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Case II: When $r = 1$

$S_n = a + a + \cdots + a = na$ and this G.P. is also an A.P. whose common difference is 0.

2.2.3 Sum of Infinite Terms of a G.P.

If $|r| \geq 1$ then sum would be $\pm\infty$. However, if $|r| < 1$ then sum would be finite.

We have obtained that $S_n = \frac{a(1-r^n)}{1-r}$

We see that as n approaches ∞, r^n will approach 0. Thus, $S_\infty = \frac{a}{1-r}$

2.2.4 Recurring Decimals

Recurring decimals are a very interesting and nice example to demonstrate the infinite G. P. and the value can be obtained by the formula derived in previous section. Consider a recurring decimal $\dot{7}$.

$$.\dot{7} = .777777\text{...to } \infty$$

$$= .7 + .07 + .007 + .0007 + \cdots$$

$$= \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots$$

$$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \cdots$$

$$= 7\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right)$$

$$= \frac{7}{9}$$

2.2.5 Geometric Mean

Like arithmetic means; we also have geometric means. Say two numbers a and b are in G.P. and x is a geometric mean between them then by definition a, x, b will be in G.P. Then,

$$\frac{x}{a} = \frac{b}{x}$$

$$\Rightarrow x^2 = ab \Rightarrow x = \sqrt{ab}$$

If $G_1, G_2, \ldots, G_n$ are n geometric means between two numbers a and b, then $G_1G_2 \ldots G_n = (\sqrt{ab})^n$

Proof: b is the $n+2$nd term. Thus, $b = ar^{n+1}$ where common ratio is r.

Thus, $G_1 = ar, G_2 = ar^2, \cdots, G_n = ar^n$

$G_1G_2 \ldots G_n = a^n r^{1+2+\cdots+n} = a^n r^{\frac{n(n+1)}{2}}$

$= \sqrt{(ab)^n}$

If $a_1, a_2, \ldots, a_n$ are n positive numbers in G.P. then their geometric mean is given by $G = (a_1a_2 \ldots a_n)^{\frac{1}{n}}$

Thus, first G.M. $= ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$

Second G.M. $= ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$

...

nth G.M. $= ar^n = a\left(\frac{b}{a}\right)^{n/(n+1)}$

2.2.6 Notes

1. Odd number of terms in a G.P. should be taken as $\cdots \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, \cdots$

2. Even number of terms in a G.P. should be taken as $\cdots, \frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5, \cdots$

3. If $a_1, a_2, \ldots, a_n$ and $b_1, b-2, \ldots, b_n$ be two G.P. of common ratios r_1 and r_2 then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots$ also form G.P., where common ratios will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

4. Let $a_1, a_2, a_3, \ldots$ be a G.P. of positive terms, then $\log a_1, \log a_2, \log a_3, \ldots$ will be an A.P. and vice-versa.

   Let a be the first term and r be the common ratio of the G.P. then $a_i = ar^{i-1}$. Now $\log a_i = \log a + (i-1)\log r$ which represents ith term of an A.P. with first term as $\log a$ and common difference $\log r$.

   Conversely, let us assume that $\log a_1, \log a_2, \log a_3, \ldots$ are in A.P. then $a_i = x^{a+(i-1)d} = x^a x^{i-1d}$ where x is the base of the logarithm. This shows that $a_1, a_2, a_3, \ldots$ will be in G.P., whose first term is x^a and whos ecommon ratio is x^d.

5. Increasing and decreasing G.P.

   Case I: Let the first term a be positive. Then if $r > 1$, then it is an increasing G.P. but if $0 < r < 1$ then it is a decreasing G.P.

case II: Let the first term a be negative. Then if $r > 1$, then it is a decreasing G.P. but if $0 < r < 1$ then it is an increasing G.P.

2.2.7 Arithmetico Geometric Series

If the termms of an A.P. are multiplied y corresponding terms of a G.P., then the new series obtained is called an Arithmetico-Geometric series.

Exmaple: If the terms of the arithmetic series $2 + 5 + 8 + \cdots$ are multiplied with the corresponsing terms of the geometric series $x + x^2 + x^3 + \cdots$ then the resulting arithmetico-geometric series is $2x + 5x^2 + 8x^3 + \cdots$

2.2.8 Sum of n terms of an Arithmetico-Geometric Series

Let $a_1, a_2, \ldots, a_n$ be an A.P. and $b_1, b_2, \ldots, b_n$ be a G.P. Let d be the common difference of the A.P. and r be the common ratio of the G.P. Also, let $a = a_1$ and $b = b_1$, then

$$S_n = ab + (a+d)br + (a+2d)br^2 + \cdots + [a+(n-1)d]br^{n-1}$$

$$rS_n = abr + (a+d)br^2 + (a+2d)br^3 + \cdots + [a+(n-1)d]br^n$$

$$\Rightarrow (1-r)S_n = ab + dbr + dbr62 + \cdots + dbr^{n-1} - [a+(n-1)d]br^n$$

$$= ab + \frac{dbr(1-r^{n-1})}{(1-r) - [a+(n-1)d]br^n}$$

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r}(r \neq 1)$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, therefore , sum of an infinite number of terms of an arithmetico-geometric series is given by

$$S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

2.3 Harmonic Progressions

Consider an A.P. then an H.P. is formed by terms given by reciprocal of terms of the A.P. respectively. So if the terms of A.P. are $a_1, a_2, \ldots, a_n$ then terms of H.P. are given by $\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$.

When we study H.P. and its properties we do that by studying the properties of the corresponding A.P.

2.3.1 Harmonic Means

Numbers H_1, H_2, ... , H_n are said to be the n H.M. between two numbers a and b, if $a, H_1, H_2, \ldots, H_n, b$ are in H.P. For example, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ are the H.M. between 1 and $\frac{1}{5}$ because $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ are in H.P.

Let a and b be the two given quantities and H be the H.M. between them. Then a, H, b will be in H.P.

$\therefore \frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ will be in H.P.

$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \Rightarrow H = \frac{2ab}{a=b}$

Let $H_1, H_2, \ldots, H_n$ be the n H.M. between two given quantities a and b, and d be the c.d. of the corresponding A.P. Then $a, H_1, H_2, \ldots, H_n, b$ will be in H.P.

$\therefore \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \cdots, \frac{1}{H_n}, \frac{1}{b}$ will be in A.P.

$\frac{1}{b} = t_{n+2} = \frac{1}{a} + (n+1)d \Rightarrow d = \frac{a-b}{ab(n+1)}$

$\therefore \frac{1}{H_1} = \frac{1}{a} + d \Rightarrow H_1 = \frac{ab(n+1)}{a+nb}$

$H_2 = \frac{ab(n+1)}{2a+(n-1)b}$

...

$H_n = \frac{ab(n+1)}{an+b}$

2.4 Relation between A.M., G.M. and H.M.

Let a and b be two real, positive and unequal quantities and A, G and H be the single A.M., G.M. and H.M. between them respectively.

Then, $A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$

$AH = ab = G^2$ and thus A, G, H form a G.P.

Similarly it can be probve that $A > G > H$

For equal a and b, it can be easily verified that $A = G = H$

2.5 Problems

1. If nth term of a sequence is $2n^2 + 1$, find the sequence. Is this seuquence in A.P.?
2. Find the first five terms of the sequence for which $t_1 = 1, t_2 = 2$ and $t_{n+2} = t_n + t_{n+1}$.
3. Write the sequence whose nth term is $3n + 5$.
4. Write the sequence whose nth term is $2n^2 + 3$.
5. Write the sequence whose nth term is $\frac{3n}{2n+4}$.

6. Write the first three terms of sequence defined by $t_1 = 2, t_{n+1} = \frac{2t_n+1}{t_n+3}$.

7. If nth term of a sequence is $4n^2 + 1$, find the sequence. Is this sequence an A.P.?

8. If nth term of a sequence is $2an + b$, where a, b are constants, is this sequence an A.P.?

9. Find the 5th term of the sequence whose first three terms are 3, 3, 6 and each term after the second is the sum of two preceding terms.

10. Consider the sequence defined by $t_n = an^2 + bn + c$. If $t_1 = 1, t_2 = 5$ and $t_3 = 11$ then find the value of t_{10}.

11. Show that the seuquence 9, 12, 15, 18, ... is an A.P. Find its 16^{th} term and the general term.

12. Show that the sequence $\log a, \log(ab), \log(ab^2), \log(ab^3), \ldots$ is an A.P. Find its nth term.

13. Find the sum to n terms of the sequence $\langle t_n \rangle$, where $t_n = 5 - 6n, n \in N$.

14. How many terms are there in the A.P. $3, 7, 11, \ldots, 407$?

15. If a, b, c, d, e are in A.P. find the value of $a - 4b + 6c - 4d + e$.

16. In a certain A.P. 5 times the 5th term is equal to 8 times the 8th term, then prove that 13th term is zero.

17. Find the term of the series $25, 22\frac{3}{4}, 20\frac{1}{2}, 18\frac{1}{4}, \cdots$ which is numerically smallest positive number.

18. A person was appointed in the pay scale of Rs. $700 - 40 - 1500$. Find in how many years he will reach the maximum of the scale.

19. Find the A.P. whose 7th and 13th terms are respectively 34 and 64.

20. Is 55 a term of the sequence 1, 3, 5, 7, ...? If yes, find which term it is.

21. Find the first negative term of the sequence 2000, 1995, 1990, ...

22. How many terms are identical in two arithmetic progressions 2, 4, 6, 8, ... up to 100 terms and 3, 6, 9, ... up to 80 terms.

23. Find the number of all positive integers of 3 digits which are divisible by 5.

24. Is 105 a term of the arithmetic progression 4, 9, 14, ...?

25. Find the first negative term of the sequence 999, 995, 991, ...

26. Each of the series $3 + 5 + 7 + \cdots$ and $4 + 7 + 10 + \cdots$ is continued to 100 term. Find how many terms are identical?

27. If m times the mth term of an A.P. is equal to n times the nth term, find its $(m + n)$th term.

28. If a, b, c be the pth, qth and rth terms respectively of an A.P., prove that $a(q-r)+b(r-p)+c(p-q)=0$.

29. Find the number of integers between 100 and 1000 that are divisible by 7 and not divisible by 7.

30. If a, b, c be the pth, qth and rth terms respectively of an A.P., prove that $(a-b)r+(b-c)p+(c-a)q=0$.

31. The sum of three numbers in A.P. is 27 and the sum of their squares is 293. Find the numbers.

32. The sum of four integers in A.P. is 24 and their product is 945. Find the numbers.

33. If the pth term of an A.P. is q and the qth term is p, find the first term and common difference. Also, show that $(p+q)$th term is zero.

34. For an A.P. show that $t_m + t_{2n+m} = 2t_{m+n}$.

35. Divide 15 into three parts which are in A.P. and the sum of their squares is 83.

36. Three numbers are in A.P. Their sum is 27 and the sum of their squares is 275. Find the numbers.

37. The sum of three numbers in A.P. is 12 and the sum of their cubes is 408. Find the numbers.

38. Divide 20 into four parts which are in A.P. such that the product of first and fourth is to product of second and third is 2 : 3.

39. The sum of three numbers in A.P. is -3 and their product is 8. Find the numbers.

40. Divide 32 into four parts which are in A.P. such that the ratio of product of extremes to the product of means is 7 : 15.

41. If $(b+c-a)/a, (c+a-b)/b, (a+b-c)/c$ are in A.P. then prove that $1/a, 1/b, 1/c$ are also in A.P.

42. If $a, b, c \in R+$ form an A.P., then prove that $a+1/bc, b+1/ca, c+1/ab$ are also in A.P.

43. If a, b, c are in A. P., then prove that $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.

44. If a, b, c are in A.P., then prove that $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are also in A.P.

45. If a, b, c are in A.P., then prove that $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are also in A.P.

46. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P. then prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are also in A.P.

47. If a, b, c are in A.P. then prove that $b+c, c+a, a+b$ are also in A.P.

48. If a^2, b^2, c^2 are in A.P. then prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

49. If a, b, c are in A.P., show that $2(a-b) = a-c = 2(b-c)$.

50. If a, b, c are in A.P., then prove that $(a-c)^2 = 4(b^2-ac)$.

51. In an A.P. if $S_n = t_1 + t_2 + \cdots + t_n$ (n odd), $S_2 = t_2 + t_4 + \cdots + t_{n-1}$, then find the value of S_1/S_2 in terms of n.

52. Find the degree of the equation $(1+x)(1+x^6)(1+x^{11})\cdots(1+x^{101})$.

53. Prove that a sequence is an A.P. if the sum of its terms is of the form $An^2 + Bn$, where A, B are constants.

54. If the sequence $a_1, a_2, \ldots, a_n$ form an A.P., then prove that $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$.

55. Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots, a_{24}$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

56. If the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to next n terms. Then, find the ratio of sum of first $2n$ terms to the sum of next $2n$ terms.

57. If the sum of n terms of a series be $5n^2 + 3n$, find its nth term. Are the terms of this series in A.P.?

58. Find the sum of the series $(a+b)^2 + (a^2+b^2) + (a-b)^2 + \cdots$ to n terms.

59. Find $1 - 3 + 5 - 7 + 9 - 11 + \cdots$ to n terms.

60. The interior angles of a polygon are in A.P. The smallest angle is 120°and the commnon difference is 5°. Find the number of sides of the polygon.

61. 25 trees are planted in a straight line at intervals of 5 meters. To water them the gardener must bring water for each tree separately from a well 10 meters from the first tree. How far he will have to travel to water all the trees beginning with the first if he starts from the well.

62. If a be the first term of an A.P. and the sum of its first p terms is equal to zero, show that the sum of the next q terms is $-\frac{a(p+q)}{p-1}q$.

63. The sum of the first p terms of an A.P. is equal to the sum of its first q terms, prove that the sum of its first $(p+q)$ terms is zero.

64. Prove that the sum of latter half of $2n$ terms of a series in A.P. is equal to the one third of the sum of first $3n$ terms.

65. If $S_1, S_2, S_3, \ldots, S_p$ be the sum of n terms of arithmetic progressions whose first terms are respectively $1, 2, 3, \ldots$ and common differences are $1, 2, 3, \ldots$ prove that

$$S_1 + S_2 + S_3 + \cdots + S_p = \frac{np}{4}(n+1)(p+1)$$

66. If a,b and c be the sum of p, q and r terms rspectively of an A.P., prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

67. If the sum of m terms of an A.P. is equal to half the sum of $(m+n)$ terms and is also equal to half the sum of $(m+p)$ terms, prove that $(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$.

68. If there are $(2n+1)$ terms in an A.P., then prove that the ratio of sum of odd terms and the sum of even terms is $n+1:n$.

69. The sum of n terms of two series in A.P. are in the ration $(3n-13):(5n+21)$. Find the ratio of their 24th terms.

70. If the mth term of an A.P. is $\frac{1}{n}$ and nth term of an A.P. is $\frac{1}{m}$ then prove that the sum to mn terms is $\frac{mn+1}{2}$.

71. If the sum of m terms of an A.P.is n and the sum of its n terms is m, show that sum of $(m+n)$ terms is $-(m+n)$.

72. If S be the sum of $2n+1$ terms of an A.P., and S_1 that of alternate terms beginning with the first, then show that $\frac{S}{S_1}=\frac{2n+1}{n+1}$

73. If a, b, c be the 1st, 3rd, nth terms respectively of an A.P., prove that the sum of n terms is $\frac{c+a}{2}+\frac{c^2-a^2}{b-a}$.

74. The sum of n terms of two series in A.P. are in ratio $(3n+8):(7n+15)$. Find the ratio of their 12th terms.

75. If the ratio of the sum of m terms and n terms of an A.P. is $m^2:n^2$, prove that the ratio of its mth and nth term wil be $(2m-1):(2n-1)$.

76. How many terms are in the G.P. 5, 20, 80, ..., 5120?

77. How many terms are in the G.P. 0.03, 0.06, 0.12, ... , 3.84?

78. A boy agrees to work at the rate of one rupee the first day, two rupee the second day, four rupees the third day, eight rupees the fourth day and so on. How much would he get on $20th$ day?

79. The population of a city in January 1987 was 20,000. It increased at the rate of 2% per annum. Find the population of the city in January 1997.

80. The sum of n terms of a sequence is 2^n-1, find its nth term. Is the sequence in G.P.?

81. If the fifth term of a G.P. is 81 and second term is 24. Find the G.P.

82. The seventh term of a G.P. is 8 times the fourth term. Find the G.P. when its 5th term is 48.

83. If the 5th and 8th terms of a G.P. be 48 and 384 respectively, find the G.P

84. If the 6th and 10th terms of a G.P. are $\frac{1}{16}$ and $\frac{1}{256}$ respectively, find the G.P.

85. If the pth, qth and rth terms of a G.P. be $a, b, c(a, b, c > 0)$, then prove that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

86. If the $(p+q)$th term of a G.P. is a and the $(p-q)$th term is b, show that its pth term is $\sqrt{ab}$.

87. If the pth, qth and rth terms of a G.P. be x, y and z respectively, prove that $x^{q-r}.y^{r-p}.z^{p-q} = 1$.

88. The first term of a G.P. is 1. The sum of third and fifth terms is 90. Find the common ratio of G.P.

89. Fifth term of a G.P. is 2. Find the product of its first nine terms.

90. The fourth, seventh and last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and number of terms in the G.P.

91. Three numbers are in G.P. If we double the middle term they form an A.P. Find the common ratio of the G.P.

92. If p, q and r are in A.P. show that pth, qth and rth term of a G.P. are in G.P.

93. If a, b, c and d are in G.P., show that $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$.

94. Three non-zero numbers a, b and c are in A.P. Increasing a by 1 or increading c by 2,the numbers are in G.P. Then find b.

95. Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the mean by 5, they will be in A.P. Find the numbers.

96. If the product of three numbers in G.P. be 216 and their sum is 19, find the numbers.

97. A number consists of three digits in G.P. The sum of the right hand and left hand digits exceed twice the middle digit by 1 and the sum of left hand and middle digit is two-third of the sum of the middle and right hand digits. Find the number.

98. In a set of four numbers, the first three are in G.P. and the last three are in A.P. with a common difference of 6. If the first number is same as fourth, find the four numbers.

99. The sum of three numbers in G.P. is 21 and the sum of their squares is 189. Find the numbers.

100. The prodduct of three consecutive terms of a G.P. is -64 and the first term is four times the third. Find the terms.

101. Three numbers whose sum is 15 are in A.P. If $1, 4, 19$ be added to them respectively the resulting numbers are in G.P. Find the numbers.

102. From three numbers in G.P. other three numbers in G.P. are subtracted. Resulting numbers are found to be in G.P. again. Prove that the three sequences have the same common ratio.

103. If a, b, c, d are in G.P., show that $(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2$.

104. If a,b,c,d are in G. P., then show that $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ad+bc+cd)^2$.

105. If $a^x=b^y=c^z$ where x, y, z are in G.P., show that $\log_b a=\log_c b$.

106. If the continued product of three numbers in a G.P. is 216 and the sum of their products in pairs is 156, find the numbers.

107. If a, b, c, d are in G.P., show that $(a+b)^2, (b+c)^2, (c+d)^2$ are in G.P.

108. If a, b, c, d are in G.P., show that $(a-b)^2, (b-c)^2, (c-d)^2$ are in G.P.

109. If a, b, c, d are in G.P., show that $a^2+b^2+c^2, ab+bc+cd, b^2+c^2+d^2$ are in G.P.

110. If a, b, c, d are in G.P., show that $\frac{1}{(a+b)^2}, \frac{1}{(b+c)^2}, \frac{1}{(c+d)^2}$ are in G.P.

111. If a, b, c, d are in G.P., show that $a(b-c)^3=d(a-b)^3$.

112. If a, b, c, d are in G.P., show that $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$.

113. If a, b, c are in G.P., show that $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3$.

114. If a, b, c are in G.P., show that $(a^2-b^2)(b^2+c^2)=(b^2-c^2)(a^2+b^2)$.

115. If a, b, c are in G.P., show that $\log a, \log b, \log c$ are in A.P.

116. Find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ to n terms.

117. Find $1+2+4+8+\cdots$ to 12 terms.

118. Find $1-3+9-27+\cdots$ to 9 terms.

119. Find $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\cdots$ to n terms.

120. Find the sum of n terms of the series $(a+b)+(a^2+2b)+(a^3+3b)+\cdots$ to n terms.

121. A man agrees to work at the rate of one dollar the first day, two dollars the second day, four dollars the third day, eight dollars the fourth day and so on. How much would he get at the end of 120 days.

122. Find the sum to n terms of the series $8+88+888+\cdots$.

123. Find the sum to n terms of the series $6+66+666+\cdots$.

124. Find the sum to n terms of the series $4+44+444+\cdots$.

125. Find the sum to n terms of the series $.5+.55+.555+\cdots$.

126. Find $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$ to n terms.

127. If you had a choice of a salary of a salary of \$1000 a day for a month of 31days or \$1 for the first day, doubling every day which choice would you make?

128. How many terms of the series $1 + 3 + 3^2 + 3^3 + \cdots$ must be taken to make 3280?

129. Find the least value of n for which $1 + 3 + 3^2 + \cdots + 3^{n-1} > 1000$.

130. Find $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ to ∞.

131. A person starts collecting \$1 first day, \$3 second day, \$9 third day and so on. What will be his collection in 20 days.

132. Find the sum of $\left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 5\right) + \left(x^6 + \frac{1}{x^6} + 8\right) + \cdots$ to n terms.

133. How many terms of the series $1 + 2 + 2^2 + \cdots$ must be taken to make 511?

134. Find the least value of n such that $1 + 2 + 2^2 + \cdots + 2^{n-1} \geq 300$.

135. Determine the no. of terms of a G.P. if $a_1 = 3, a_n = 96$ and $S_n = 189$.

136. Express $0.4\dot{2}\dot{3}$ as a rational number.

137. Find $\frac{1}{5} + \frac{1}{7} + \frac{1}{5^2} + \frac{1}{7^2}$ to ∞.

138. Prove that the sum of n terms of the series $11 + 103 + 1005 + \cdots$ is $\frac{10}{9}(10^n - 1) + n^2$.

139. Find the sum to n terms of the series $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \cdots$.

140. If S be the sum, P be the product and R the sum of reciprocals of n terms in G.P., prove that $P^2 = \left(\frac{S}{R}\right)^n$.

141. Find $1 + \frac{x}{1+x} + \frac{x^2}{(1+x)^2} + \cdots$ to ∞ if $x > 0$.

142. Prove that in an infinite G.P. whose common ratio is r is numerically less than one, the ratio of any term to the sum of all the succeediing terms is $\frac{1-r}{r}$.

143. If $S_1, S_2, S_3, \ldots, S_p$ are the sum of infinite geometric series whose first terms are $1, 2, 3, \ldots, p$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{p+1}$ respectively, prove that $S_1 + S_2 + S_3 + \cdots + S_p = p(p+3)/2$.

144. If $x = 1 + a + a^2 + a^3 + \cdots$ to ∞ and $y = 1 + b + b^2 + b^3 + \cdots$ to ∞, show that $1 + ab + a^2b^2 + a^3b^3 + \cdots$ to $\infty = \frac{xy}{x+y-1}$, where $0 < a < 1$ and $0 < b < 1$.

145. Find the sum to infinity for the series $1 + (1 + a)r + (1 + a + a^2)r^2 + \cdots$, where $0 < a < 1$ and $0 < r < 1$.

146. After striking the floor a certain ball rebound to $\frac{4}{5}$th of the height from which it has fallen. Find the total distance it travels before coming to rest if it is gently dropped from a height of 120 meters.

147. If a be the first term and b be the nth term and p be the product of n terms of a G.P., show that $p^2 = (ab)^n$.

148. Show that the ratio of sum of n terms of two G.P.'s having the same common ratio is equal to the ratio of their nth terms.

149. If S_1, S_2, S_3 be the sum of $n, 2n, 3n$ terms respectively of a G.P. show that $(S_2 - S_1)^2 = S_1(S_3 - S_2)$.

150. If S_n denotes the sum of n terms of a G.P.,whose first term is a and common ratio is r, find $S_1 + S_2 + \cdots + S_{2n-1}$.

151. The sum of n terms of a series is $a.2^n - b$, find its nth term. Are the terms of this series in G.P.

152. Find $\frac{1}{1+x^2}\left[1 + \frac{2x}{1+x^2} + \left(\frac{2x}{1+x^2}\right)^2 + \cdots \text{ to } \infty\right]$ where $x \geq 0$.

153. The sum of an infinite G.P. whose common ratio is numerically less than 1 is 32 and the sum of their first two terms is 24. Find the terms of the G.P.

154. The sum of infinite number of terms of a decreasing G.P. is 4 and the sum of the squares of its terms to infinity is $\frac{16}{3}$, find the G.P.

155. If $p(x) = (1 + x^2 + x^4 + \cdots + x^{2n-2})/(1 + x + x^2 + \cdots + x^{n-1})$ is a polynomial in x, then find the possible values of n.

156. If $x = a + \frac{a}{r} + \frac{a}{r^2} + \cdots \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - \cdots \infty$ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + \cdots \infty$, then prove that $\frac{xy}{z} = \frac{ab}{c}$.

157. A G.P. consists of an even number of terms. If the sum of all terms is 5 times the sum of the terms occupying odd places, then find the common ratio.

158. If sum of n terms of a G.P. is $3 - \frac{3^{n+1}}{4^{2n}}$, then find the common ratio.

159. In an infinite G.P. whose terms are all positive, the common ratio being less than unity, prove that any term $>, =, <$ the sum of all the succeeding terms according as the common ratio $<, =, > \frac{1}{2}$.

160. Prove that $(666 \ldots n \text{ digits})^2 + 888 \ldots n \text{ digits} = 444 \ldots 2n \text{ digits}$.

161. Find the sum $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \cdots$ to n terms.

162. If the sum of the series $\sum_{n=0}^{\infty} r^n$, $|r| < 1$ is S, then find the sum of the series $\sum_{n=0}^{\infty} r^{2n}$.

163. If for a G.P. $t_m = \frac{1}{n^2}$ and $t_n = \frac{1}{m^2}$ then find the term $t_{\frac{m+n}{2}}$.

164. If a, b, c be three successive terms of a G.P. with common ratio r and $a < 0$ satisfying the condition $c > 4b - 3a$, then prove that $r > 3$ or $r < 1$.

165. If $(1-k)(1+2x+4x^2+8x^3+16x^4+32x^5) = 1-k^6$, where $k \neq 1$, then find $\frac{k}{x}$.

166. If $(a^2+b^2+c^2)(b^2+c^2+d^2) \le (ab+bc+cd)^2$, where a, b, c, d are non-zero real numbers, then show that they are in G.P.

167. If $a_1, a_2, \ldots, a_n$ are n non-zero numbers such that $(a_1^2 + a_2^2 + \cdots + a_{n-1}^2)(a_2^2 + a_3^2 + \cdots + a_n^2) \le (a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n)^2$, then show that $a_1, a_2, \ldots, a_n$ are in G.P.

168. α, β be the roots of $x^2 - 3x + a = 0$ and γ, δ be the roots of $x^2 - 12x + b = 0$ and the numbers $\alpha, \beta, \gamma, \delta$ form an increasing G.P., then find the values of a and b.

169. There are $4n+1$ terms in a certain sequence of which the first $2n+1$ terms are in A.P. of common difference 2 and the last $2n+1$ terms are in G.P. of common ratio $\frac{1}{2}$. If the middle terms of both the A.P. and G.P. are same then find the mid term of the sequence.

170. If $f(x) = 2x+1$ and three unequal numbers $f(x), f(2x), f(4x)$ are in G.P, then find the number of values for x.

171. Three distinct real numbers, a, b, c are in G.P. such that $a+b+c = xb$, then show that $x < -1$ or $x > 3$.

172. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in A.P., such that $|a| < 1, |b| < 1, |c| < 1$, then show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P. as well.

173. Given that $0 < x < \frac{\pi}{4}, \frac{\pi}{4} < y < \frac{\pi}{2}$ and $\sum_{k=0}^{\infty}(-1)^k \tan^{2k} x = p$, $\sum_{k=0}^{\infty}(-1)^k \cot^{2k} y = q$ then prove that $\sum_{k=0}^{\infty} \tan^{2k} x \cot^{2k} y$ is $\frac{1}{\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}}$

174. An equilateral triangle is drawn by joining the mid-points of a given equilateral triangle. A third equilateral triangle is drawn inside the second in the same manner and the process is continued indefinitely. If the side of first equilateral triangle is $3^{1/4}$ inch, then find the sum of areas of all these triangles.

175. If $S = exp\{(1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x \cdots \text{to} \infty) \log_e 4\}$ satisfies the roots of the equation $t^2 - 20t + 64 = 0$ for $0 < x < \pi$ then find the values of x.

176. If $S \subset (-\pi, \pi)$, denote the set of values of x satisfying the equation

$8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{to} \infty} = 4^3$ then find the value of S.

177. If $0 < x < \frac{\pi}{2}$ and $2^{\sin^2 x + \sin^4 x + \cdots \text{ to } \infty}$ satisfies the roots of the equation $x^2 - 9x + 8 = 0$, then find the value of $\cos x/(\cos x + \sin x)$.

178. If $S_\lambda = \sum_{r=0}^{\infty} \frac{1}{\lambda^r}$, then find $\sum_{\lambda=1}^{n} (\lambda - 1) S_\lambda$.

179. If a, b, c are in A.P. then prove that $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ are in G.P. $\forall x \neq 0$.

180. If $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} = \frac{c+de^x}{c-de^x}$ then prove that a, b, c, d are in G.P.

181. If x, y, z are in G.P. and $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in A.P. then prove that $x = y = z$ but their common values are not necessarily zero.

182. If a, b, c are three unequal numbers such that a, b, c are in A.P. and $b - a, c - b, a$ are in G.P. then prove that $a : b : c = 1 : 2 : 3$.

183. The sides a,b,c of a triangle are in G.P. such that $\log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a$ are in A.P., then prove that $\triangle ABC$ is an obtuse angled triangle.

184. If the roots of the equation $ax^3 + bx^2 + cx + d = 0$ be in G.P. then prove that $c^3 a = b^3 d$.

185. Find the 100th term of the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots$.

186. If pth term of an H.P. is qr, and qth term is rp, prove that rth term is pq.

187. If the pth, qth and rth terms of an H.P. be respectively a, b and c, then prove that $(q - r)bc + (r - p)ca + (p - q)ab = 0$.

188. If a, b, c are in H.P., prove that $\frac{a-b}{b-c} = \frac{a}{c}$.

189. If a, b, c, d are in H.P., then, prove that $ab + bc + cd = 3ad$.

190. If $x_1, x_2, x_3, \ldots, x_n$ are in H.P., prove that $x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n = (n - 1)x_1x_n$.

191. If a, b, c are in H.P., show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

192. If a^2, b^2, c^2 are in A.P. show that $b + c, c + a, a + b$ are in H.P.

193. Find the sequence whose nth term is $\frac{1}{3n-2}$. Is this sequence an H.P.?

194. If mth term of an H.P. be n and nth term be m, prove that $(m + n)$th term $= \frac{mn}{m+n}$ and (mn)th term $= 1$.

195. The sum of three rational numbers in H.P. is 37 and the sum of their reciprocals is $\frac{1}{4}$, find the numbers.

196. If a, b, c are in H.P., prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$.

197. If a, b, c are in H.P., prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$.

198. If x_1, x_2, x_3, x_4, x_5 are in H.P., prove that $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = 4x_1x_5$.

199. If x_1, x_2, x_3, x_4 are in H.P., prove that $(x_1 - x_3)(x_2 - x_4) = 4(x_1 - x_2)(x_3 - x_4)$.

200. If $b + c, c + a, a + b$ are in H.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

201. If $b + c, c + a, a + b$ are in H.P., prove that a^2, b^2, c^2 are in A.P.

202. If a, b, c are in A.P., prove that $\frac{bc}{ab+ac}, \frac{ca}{bc+ab}, \frac{ab}{ca+cb}$ are in H.P.

203. If a, b, c are in H.P., prove that $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

204. If a, b, c are in H.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

205. If a, b, c are in A.P., and x, y, z are in G.P.; show that $x^{b-c}.y^{c-a}.z^{a-b} = 1$.

206. If pth, qth, rth and sth term of an A.P. be in G.P., prove that $p - q, q - r, r - s$ are in G.P.

207. If pth, qth and rth terms of an A.P. and G.P. both be a, b and c, show that $a^{b-c}b^{c-a}c^{a-b} = 1$.

208. If a, b, c be in A.P. and b, c, d be in H.P., prove that $ad = bc$.

209. If $a^x = b^y = c^z$ and a, b, c are in G.P., show that x, y, z are in H.P.

210. If $\frac{x+y}{2}, y, \frac{y+z}{2}$ be in H.P., show that x, y, z are in G.P.

211. If x, y, z be in G.P., and $x + a, y + a, z + a$ be in H.P., prove that $a = y$.

212. If three positive numbers a, b, c are in A.P., G.P. and H.P. as well, then find their values.

213. If a, b, c be in A.P., b, c, d be in G.P. and c, d, e be in H.P., prove that a, c, e are in G.P.

214. If a, b, c be in A.P. and a^2, b^2, c^2 be in H.P., prove that $-\frac{a}{2}, b, c$ are in G.P. or else $a = b = c$.

215. If a, b, c are the pth, qth and rth terms of boht an A.P. and a G.P., prove that $a^b b^c c^a = a^c b^a c^b$.

216. An A.P. and a G.P. of positive terms have the same first term. The sum of their first, second and third terms are respectively $1, \frac{1}{2}$ and 2. Show that the sum of their fourth terms is $\frac{19}{2}$.

217. If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ and p, q, r be in A.P., show that x, y, z are in H.P.

218. An A.P. and a H.P. have the same first term a, the same last term b and the same number of terms n. Prove that the product of the rth term of A.P. and the $(n-r+1)$th terrm of term of H.P. is ab.

219. Prove that if from each term of the three consecutive terms of an H.P. half the second term be subtracted the resulting terms are in G.P.

220. If $y-x, 2(y-a), y-z$ are in H.P., prove that $x-a, y-a, z-a$ are in G.P.

221. If a, b, c be in A.P., p, q, r be in H.P, and ap, bq, cr be in G.P., show that $\frac{p}{r}+\frac{r}{p}=\frac{a}{c}+\frac{c}{a}$.

222. If a, b, x be in A.P., a, b, y be in G.P. and a, b, z be in H.P., prove that $4z(x-y)(y-z)=y(x-z)^2$.

223. If $x, 1, z$ be in A.P., $x, 2, z$ be in G.P., show that $x, 4, z$ are in H.P.

224. Find the sum of n terms of the series whose nth term is $12n^2-6n+5$.

225. Find the sum to n terms of the series $1^2+3^2+5^2+7^2+\cdots$.

226. Find the sum to n terms of the series $1.2.3+2.3.4+3.4.5+\cdots$.

227. Find the sum of the series $1.n+2.(n-1)+3.(n-2)+\cdots+n.1$.

228. Find the sum to n terms of the series $1+(1+2)+(1+2+3)+\cdots$.

229. Find the sum to n terms of the series $1+(2+3)+(4+5+6)+\cdots$.

230. Find the sum of series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$ to 16 terms.

231. Find $(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\cdots$ to 10 terms.

232. Find $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots$ to n terms.

233. Find the sum of $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots$ to infinity.

234. Find the sum of n terms of the series $1+5+11+19+\cdots$.

235. A sum is distributed among certain number of persons. Second person gets one rupee more than the first, third person gets two rupees more than the second, fourth person gets three rupees more than the third and so on. If the first person gets one rupee and the last person get 67 rupees, find the number of persons.

236. Natural numbers have been grouped in the following way $1,(2,3),(4,5,6),(7,8,9,10),\cdots$ Show that the sum of the numbers in the nth group is $\frac{n(n^2+1)}{2}$.

237. Find $1+3+7+15+\cdots$ to n terms.

238. Find $1+2x+3x^2+4x^3+\cdots$ to n terms.

239. Find $1 + 2.2 + 3.2^2 + 4.3^3 + \cdots + 100.2^{99}$.

240. Find $1 + 2^2x + 3^2x^2 + 4^2x^4 + \cdots$ to ∞, $|x| < 1$

241. If the sum of n terms of a sequence be $2n^2 + 4$, find its nth term. Is this sequence in A.P.?

242. Find the sum of n terms of the series whose nth term is $n(n-1)(n+1)$.

243. Find the sum of the series $1^3 + 3^3 + 5^3 + \cdots$ to n terms.

244. Find the sum of the series $1^2 + 4^2 + 7^2 + 10^2 + \cdots$ to n terms.

245. Find the sum of the series $1^2 + 2 + 3^2 + 4 + 5^2 + 6 + \cdots$ to $2n$ terms.

246. Find the sum of the series $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to n terms.

247. Find the sum of the series $1.3 + 3.5 + 5.7 + \cdots$ to n terms.

248. Find the sum of the series $1.2 + 2.3 + 3.4 + \cdots$ to n terms.

249. Find the sum of the series $1.2^2 + 2.3^2 + 3.4^2 + \cdots$ to n terms.

250. Find the sum of the series $2.1^2 + 3.2^2 + 4.3^2 + \cdots$ to n terms.

251. Find the sum of the series $1 + (1+3) + (1+3+5) + \cdots$ to n terms.

252. Find the sum of the series $1^2 + (1^2+2^2) + (1^2+2^2+3^2) + \cdots$ to n terms.

253. Find the sum of the series $1.2.3 + 2.3.5 + 3.4.7 + \cdots$ to n terms.

254. Find the sum of the series $1.2.3 + 2.3.4 + 3.4.5 + \cdots$ to n terms.

255. Find the sum of the series $1.3^2 + 2.5^2 + 3.7^2 + \cdots$ to 20 terms.

256. Find the sum of the series $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \cdots$ to n terms.

257. Find the sum of the series $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \cdots$ to 10 terms.

258. Find the sum of the series $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots$ to n terms.

259. Find the sum to infinity of the series $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} + \cdots$.

260. Find the sum of the series $2 + 6 + 12 + 20 + \cdots$ to n terms.

261. Find the sum of the series $3 + 6 + 11 + 18 + \cdots$ to n terms.

262. Find the sum of the series $1 + 9 + 24 + 46 + 75 + \cdots$ to n terms.

263. Find the nth term of the series $2 + 4 + 7 + 11 + 16 + \cdots$.

264. Find the sum to 10 terms of the series $1 + 3 + 6 + 10 + \cdots$.

265. The odd natural numbers have been divided in groups as $(1,3)$, $(5,7,9,11)$,

$(13,15,17,19,21,23)$, ... Show that the sum of numbers in the nth group is $4n^3$.

266. Show that the sum of numbers in each of the following groups is an square of an odd positive integer (1), $(2,3,4)$, $(3,4,5,6,7)$, ...

267. Find the sum to n terms of the series $2+5+14+41+\cdots$.

268. Find the sum to n terms of the series $1.1+2.3+4.5+8.7+\cdots$.

269. If $a_1, a_2, a_3, \cdots, a_{2n}$ are in A.P., show that $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$.

270. If $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are in A.P., whose common difference is d show that $\sin d$

$[\sec\alpha_1\sec\alpha_2 + \sec\alpha_2\sec\alpha_3 + \cdots + \sec\alpha_{n-1}\sec\alpha_n] = \tan\alpha_n - \tan\alpha_1$.

271. If $a_1, a_2, a_3, \ldots, a_n$ be in A.P., prove that $\frac{1}{a_1a_n} + \frac{1}{a_2a_{n-1}} + \cdots + \frac{1}{a_na_1} = \frac{2}{a_1+a_n}$

$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$.

272. If $a_1, a_2, a_3, \ldots$ be in A.P. such that $a_i \neq 0$, show that $S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_na_{n+1}} = \frac{n}{a_1a_{n+1}}$.

273. If $a_1, a_2, a_3, \ldots, a_n$ be in A.P. and $a_1 = 0$, show that $\frac{a_3}{a_2} + \frac{a_4}{a_3} + \cdots + \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$.

274. If $a_1, a_2, \ldots, a_n$ are in A.P., whose common difference is d, show that $\sum_{k=1}^{n} \frac{a_ka_{k+1}a_{k+2}}{a_k+a_{k+2}}$

$= \frac{n}{2}\left[a_1^2 + (n+1)a_1d + \frac{(n-1)(2n+5)}{6}d^2\right]$.

275. If x, y and z are positive real numbers different from 1, and $x^{18} = y^{21} = z^{28}$, show that $3, 3\log_y x, 3\log_z y, 7\log_x z$ are in A.P.

276. If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x}\,dx$, then $I_1, I_2, I_3, \ldots$ are in A.P.

277. Can there be an A.P. whose terms are distinct prime numbers?

278. Four distinct no. are in A.P. If one of these integers is sum of the squares of remaining three, then 0 must be one of the numbers in A.P.

279. In an A.P. of $2n$ terms the middle pair of terms are $p+q$ and $p-q$. Show that the sum of cubes of the terms in A.P. are $2np[p^2 + (4n^2-1)q^2]$.

280. Find the sum S_n of the cubes of the first n terms of an A.P. and show that the sum of the first n terms of the A.P. is a factor of S_n.

281. Show that any positive integral power (greater than 1) of a positive integer m, is the sum of m consecutive odd positive integers. Find the first odd integer for $m^r (r > 1)$.

282. If a be the sum of n terms and b^2 the sum of the square of n terms of an A.P., find the first term and common difference of the A.P.

283. If $a_1, a_2, \dots, a_n$ are in A.P., whose common diference is d, then find the sum of the series $\sin d[\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \cdots + \csc a_{n-1} \csc a_n]$.

284. If $a_1, a_2, \dots, a_n$ are in A.P. where $a_i > 0 \,\forall i$, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

285. If $a_1,\ a_2,\ \cdots,\ a_n$ are in A.P., whose common differemce is d show that $\sum_2^n \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \tan^{-1} \frac{a_n - a_1}{1+a_n a_1}$.

286. If $a_1, a_2, \dots, a_n$ are the first n items of an A.P. with first term a and common difference d such that $ad > 0$. Let $S_n = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} - \cdots + \frac{1}{a_{n-1} a_n}$ Prove that the product $a_1 a_n S_n$ does not depend on a or d.

287. If $a_1, a_2, \dots, a_n, a_{n+1}, \dots$ be in A.P., whose common difference is d and $S_1 = a_1 + a_2 + \cdots + a_n$, $S_2 = a_{n+1} + \cdots + a_{2n}$, $S_3 = a_{2n+1} + \cdots + a_{3n}$ Show that $S_1, S_2, S_3, \dots$ are in A.P. whose common difference is $n^2 d$.

288. If a, b, c are three terms of an A.P. such that $a \neq b$, show that $(b-c)/(a-b)$ is a rational number.

289. Prove that $\tan 70°, \tan 50° + \tan 20°, \tan 20°$ are in A.P.

290. If $\log_l x, \log_m x, \log_n x$ are in A.P. and $x \neq 1$, prove that $n^2 = (nl)^{\log_l m}$.

291. The length of sides of a right angled triangle are in A.P., show that their ratio is $3:4:5$

292. Find the values of a for which $5^{1+x} + 5^{1-x}, \frac{a}{2}, 25^x + 25^{-x}$ are in A.P.

293. If $\log 2, \log(2^x - 1)$ and $\log(2^x + 3)$ are in A.P., then find x.

294. If $1, \log_y x, \log_z y, -15 \log_x z$ are in A.P., then prove that $x = z^3$ and $y = z^{-3}$.

295. Show that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannot be terms of a single A.P.

296. A circle of one centimeter radius is drawn on a piece of paper and with the same center $3n - 1$ other circles are drawn of radii 2 cm, 3 cm, 4 cm and so on. The inner circle is painted blue, the ring between that and next circle is painted red, the next ring yellow then other rings blue, red, yellow and so on in this order. Show that the successive areas of each color are in A.P.

297. If $x, y, z(x, y, z \neq 0)$ are in A.P. and $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are also in A.P., then prove that $x = y = z$.

298. If θ and α are two real numbers such that $\frac{\cos^4 \theta}{\cos^2 \alpha}, \frac{1}{2}, \frac{\sin^4 \theta}{\sin^2 \alpha}$ are in A.P., prove that $\frac{\cos^{2n+2} \theta}{\cos^{2n} \alpha}, \frac{1}{2}, \frac{\sin^{2n+2} \theta}{\sin^{2n} \alpha}$ are also in A.P..

299. If $a_n = \int_0^{\pi} (\sin 2nx / \sin x)\, dx$, show that $a_1, a_2, a_3, \ldots$ are in A.P.

300. If $l_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$, show that $\frac{1}{l_2+l_4}, \frac{1}{l_3+l_5}, \frac{1}{l_4+l_6}, \cdots$ are in A.P. Find the common difference of A.P.

301. If $I_n = \int_0^{\pi} \frac{1-\cos 2nx}{1-\cos 2x} dx$, then show that $I_1, I_2, I_3, \ldots$ are in A.P.

302. If α, β, γ are in A.P.and $\alpha = \sin(\beta + \gamma), \beta = \sin(\gamma + \alpha)$ and $\gamma = \sin(\alpha + \beta)$. Prove that $\tan \alpha = \tan \beta = \tan \gamma$.

303. Suppose a, b, c are three positive real numbers in A.P., such that $abc = 4$. Prove that the minimum value of b is $4^{\frac{1}{3}}$.

304. Find the sum of n terms of the series: $\log a + \log \frac{a^3}{b} + \log \frac{a^5}{b^2} + \log \frac{a^7}{b^3} + \cdots$.

305. The first, second and the last terms of an A.P. are a,b, c respectively. Prove that the sum of al the terms is $\frac{(b+c-2a)(a+c)}{2(b-a)}$.

306. If S_n denotes the sum of n terms of an A.P., show that $S_{n+3} = 3(S_{n+2} - S_{n+1}) + S_n$.

307. If $a_1, a_2, \ldots, a_n$ are in arithmetic progression with common difference d, prove that $\sum_{r<s} a_r a_s = \frac{1}{2} n(n-1) [a_1^2 + (n-1) a_1 d + \frac{1}{12} (3n^2 - 7n + 2) d^2]$.

308. Balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second of two balls and so on. If 669 more balls are added, then all balls can be arranged in the shape of a square and each of the sides contained 8 balls less than each side of the triangle did. Determine the initial no. of balls.

309. Find the sum of the product of the first n natural numbers takes two at a time.

310. A postman delivered daily for 42 days 4 more letters each day than on the previous day. The total delivery made for the first 24 days of the period was the same as that for the last 18 days. How many letters did he deliver during the whole period?

311. If S_n denotes the sum to n terms of an A.P. and $S_n = n^2 p, S_m = m^2 p, m \neq n$, prove that $S_p = p^3$.

312. There are n A.P.'s whose common difference are $1, 2, 3, \ldots, n$ respectively the first term of each being unity. Prove that the sum of their nth terms is $\frac{n}{2}(n^2 + 1)$.

313. If $S_1, S_2, \ldots, S_m$ are the sum of n terms of m A.P.s whose first terms are $1, 2, \ldots, m$ and whose common differences are $1, 3, 5, \ldots, 2m-1$ respectively, show that $S_1 + S_2 + \cdots + S_m = \frac{1}{2}mn(mn+1)$

314. A straight line is drawn through the center of a square $ABCD$ intersecting side AB at point N so that $AN : NB = 1 : 2$. On this line take an arbitrary point M lying inside the square. Prove that the distances from M to the sides AB, AD, BC, CD of the square taken in that order, form an A.P.

315. If the sides of a right-angled triangle are in G.P., find the cosine of the greater acute angle.

316. Does there exist a geometric progression containing $27, 8$ and 12 as three of its terms? If it exists, how many such progressions are possible?

317. Show that $10, 11, 12$ cannot be terms of a G.P.

318. If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x \cos(nx)\, dx$, then prove that $I_1, I_2, I_3, \ldots$ are in G.P.

319. Let $I_n = \displaystyle\int_0^{\pi} \frac{\sin(2n-1)x}{\sin x} dx$. Show that $I_1, I_2, I_3, \ldots$ are in A.P. as well as in G.P.

320. Prove that the three successive terms of a G.P. will form sides of a triangle if the common ratio r satisfied the inequality $\frac{1}{2}(\sqrt{5}-1) < r < \frac{1}{2}(\sqrt{5}+1)$.

321. Find out whether $111\ldots1(91 \text{ digits })$ is a prime number.

322. Find the natural number a for which $\displaystyle\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfied the relation $f(x+y) = f(x)f(y)$ for all natural numbers x, y and further $f(1) = 2$.

323. In a certain test, there are n questions. In this test 2^{n-i} students give wrong answers to at least i questions $(1 \leq i \leq n.)$ If total no. of wrong answers given is 2047, find the value of n.

324. If $S_1, S_2, S_3, \ldots, S_{2n}$ are the sums of infinite geometric series whose first terms are respectively $1, 2, 3, \ldots, 2n$ and common ratio are respectively $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{2n+1}$, find the value of $S_1^2 + S_2^2 + \cdots + S_{2n-1}^2$.

325. A sqaure is given, a second square is made by joining the middle points of the first square and then a third square is made by joining the middle points of the sides of second square and so on till infinity. Show that the area of first square is equal to sum of the areas of all the succeeding squares.

326. If a is the value of x for which the function $7 + 2x\log 25 - 5^{x-1} - 5^{2-x}$ has the greatest value and $r = \displaystyle\lim_{x\to 0} \int_0^x \frac{t^2}{x^2\tan(\pi + x)} dt$, find $\displaystyle\lim_{n\to\infty} \sum_{n=1}^{n} ar^{n-1}$.

327. If pth, qth, rth terms of a G.P. are positive numbers a, b, c respectively, show that the vectors $(\log a).\vec{\imath} + (\log b)\vec{\jmath} + (\log c)\vec{k}$ and $(q-r)\vec{\imath} + (r-p)\vec{\jmath} + (p-q)\vec{k}$ are perpendicular.

328. The pollution in a normal atmosphere is less that $0.01\,\%$. Due to leakage of gas from a factory the pollution increased to $20\,\%$. If everyday 80% of the pollution us neutralised, in how many days the atmosphere will be normal?

329. The sides of a triangle are in G.P. and its largest angle is twice the smallest one. Prove that the common ratio of the G.P. lies in the interval $(1, \sqrt{2})$.

330. If a, b, c, d are in G.P., then prove that $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$.

331. If a, b, c, d, p are real and $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$. Show that a, b, c, d are in G.P. whose common ratio is p.

332. If $2x^4 = y^4 + z^4, xyz = 8$ and $\log_y x, \log_z y, \log_x z$ are in G.P., show that $x = y = z = 2$.

333. If a, b, c, d are in both A.P. and G.P. and $b = 2$, then find the number of such sequences.

334. If $\log_x a, a^{x/2}, \log_b x$ are in G.P., then find x.

335. The $(m+n)$th and $(m-n)$th terms of a G.P. are p and q respectively. Show that mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$ respectively.

336. If the pth, qth and rth terms of an A.P. are in G.P., then find the common ratio of the G.P.

337. A G.P. consists of $2n$ terms. If the sum of the terms occupying the odd places is S_1, and that of the terms in even places is S_2, show that the common ratio of the progression is S_2/S_1.

338. If S_n denotes the sum of n terms of a G.P. whose first term and common ratio are a and r respectively, show that

$$rS_n + (1-r)\sum_{n=1}^{n} S_n = na$$

339. Find the sum of $2n$ terms of the series where every even term is x times the term just before it and every odd term is y times the term just before it, the first term being 1.

340. Prove that in the sequence of numbers $49, 4489, 444889, \ldots$ in which every number is made by inserting 48 in the middle of previous number as indicated, each number is the square of an integer.

341. If there be m quantities in a G.P., whose common ratio is r and S_m denotes the sum of the first m terms then prove that the sum of their products taken two and two together is $\frac{r}{r+1}S_mS_{m-1}$.

342. Solve the following equations for x and y

$$\log_{10} x + \log_{10} x^{1/2} + \log_{10} x^{1/4} + \cdots = y$$

$$\frac{1 + 3 + 5 + (2y - 1)}{4 + 7 + 10 + \cdots + 3y + 1} = \frac{20}{7\log_{10} x}$$

343. If $a_1, a_2, \ldots, a_n$ are in G.P. and $S = a_1 + a_2 + \cdots + a_n$, $T = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$ and $P = a_1.a_2.\ldots.a_n$ show that $P^2 = \left(\frac{S}{T}\right)^n$.

344. Let a, b, c be respectively the sums of the first n terms, the next n terms and the next n terms of a G.P. show that a, b, c are in G.P.

345. If S_n denotes the sum to n terms of a G.P. whose first term and common ratio are a and r respectively, then prove that $S_1 + S_2 + \cdots + S_n = \frac{na}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

346. If S_n denotes the sum to n terms of a G.P. whose first term and common ratio are a and r respectively, then prove that $S_1 + S_3 + S_5 + \cdots + S_{2n-1} = \frac{na}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}$

347. Let s denote the sum of terms of an infinite geometric progression and σ^2 the sum of squares of the terms. Show that the sum of first n terms of this geometric progression is given by $s\left[1 - \left(\frac{s^2-\sigma^2}{s^2+\sigma^2}\right)^n\right]$, where $|r| < 1$.

348. Let $a_1, a_2, a_3, \ldots, a_n$ be a geometric progression with first term a and common ratio r, then the sum of the products $a_1, a_2, \ldots, a_n$ taken two at a time i.e. $\sum\limits_{i<j} a_i a_j = \frac{a^2 r(1-r^{n-1})(1-r^n)}{(1-r)^2(1+r)}$.

349. If $a_1, a_2, a_3, \ldots$ is a G.P. with first term a and common ratio r, show that $\frac{1}{a_1^2 - a_2^2} + \frac{1}{a_2^2 - a_3^2} + \cdots + \frac{1}{a_{n-1}^2 - a_n^2} = \frac{r^2(1-r^{2n-2})}{a^2 r^{2n-2}(1-r^2)^2}$.

350. If $a_1, a_2, a_3, \ldots$ is a G.P. with first term a and common ratio r, show that $\frac{1}{a_1^m + a_2^m} + \frac{1}{a_2^m + a_3^m} + \cdots + \frac{1}{a_{n-1}^m + a_n^m} = \frac{r^{mn-m} - 1}{a^m(1+r^m)(r^{mn-m} - r^{mn-2m})}$.

351. If $a_1, a_2, \ldots, a_{2n}$ are $2n$ positive real numbers which are in G.P. show that $\sqrt{a_1 a_2} + \sqrt{a_3 a_4} + \sqrt{a_5 a_6} + \cdots + \sqrt{a_{2n-1} a_{2n}} = \sqrt{a_1 + a_3 + \cdots + a_{2n-1}}\sqrt{a_2 + a_4 + \cdots + a_{2n}}$.

352. Find the solution of the system of equations $1 + x + x^2 + \cdots + x^{23} = 0$ and $1 + x + x^2 + \cdots + x^{19} = 0$.

353. A man invests \$$a$ at the end of the first year, \$$2a$ at the end of the second year, \$$3a$ at the end of the third year, and so on up to the end of nth year. If the rate of interest is \$$r$ per rupee and the interest is compounded annually, find the amount the man will receive at the end of $(n + 1)$th year.

354. Find the value of $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\infty\right)}$

355. If $A = 1 + r^a + r^{2a} + \cdots$ to ∞ and $B = 1 + r^b + r^{2b} + \cdots$ to ∞, prove that $r = \left(\frac{A-1}{A}\right)^{\frac{1}{a}} = \left(\frac{B-1}{B}\right)^{\frac{1}{b}}$.

356. If $s_1, s_2, \dots, s_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \dots, n$ and common ratios are $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n+1}$ respectively, then prove that $s_1 + s_2 + \cdots + s_n = \frac{1}{2}n(n+3)$.

357. If S_n be the sum of infinite G.P.'s whose first term is n and the common ratio is $\frac{1}{n+1}$, find $\lim\limits_{n\to\infty} \frac{S_1S_n+S_2S_{n-1}+\cdots+S_nS_1}{S_1^2+S_2^2+\cdots+S_n^2}$.

358. The sum of the terms of an infinitely decreasing G.P. is equal to the greatest value of the function $f(x) = x^3 + 3x - 9$ on the interval $[-5, 3]$, and the difference between the first and second terms is $f'(0)$. Prove that the common ratio of the progression is $\frac{2}{3}$.

359. Find the sum of the series $\frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \cdots\infty$.

360. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the sum to infinite number of terms of series $\cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + \cdots$ is finite, then show that x lies in the set $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

361. An A.P. and a G.P. with positive terms have the same number of terms and their first terms as well as the last terms are equal. Show that the sum of A.P. is greater than or equal to the sum of the G.P.

362. Given a G.P. and A.P. of positive terms $a, a_1, a_2, \dots, a_n, \dots$ and $b, b_1, b_2, \dots, b_n, \dots$ respectively, with the common ratio of the G.P. being different from 1, prove that there exists $x \in R, x > 0$ such that $\log_x a_n - b_n = \log_x a - b, \forall n \in N$.

363. If the $(m+1)$th, $(n+1)$th and $(r+1)$th terms of an A.P. are in G.P., and m, n, r are in H.P., show that the ratio of the first term to the common difference of the A.P. is $-n/2$.

364. If a, b, c are in G.P. and $a-b, c-a, b-c$ are in H.P., then show that $a + 4b + c = 0$.

365. If S_1, S_2 and S_3 denote the sum to $n(>1)$ terms of three sequences in A.P., whose first terms are unity and common differences are in H.P., prove that $n = \frac{2S_3S_1-S_1S_2-S_2S_3}{S_1-2S_2+S_3}$

366. Find a three-digit number such that its digits are in G.P. and the digits of the number obtained from it by subtracting 400 form an A.P.

367. If a, b, c be distinct positive numbers in G.P. and $\log_c a, \log_b c, \log_a b$ be in A.P., prove that the common difference of the progression is $3/2$.

368. If p be the first of the n arithmetic means between two numbers a and b and q the first of the n harmonic means between the same two numbers, prove that the value of q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

369. An A.P. and a G.P. each has p as first term and q as second term where $0 < q < p$. Find the sum to infinity, s of the G.P., and prove that the sum of first n terms of the A.P. may be written as $np - \frac{n(n-1)}{2} \cdot \frac{p^2}{s}$.

370. If $\log_x y, \log_z x, \log_y z$ are in G.P., $xyz = 64$ and x^3, y^3, z^3 are in A.P., then find x, y and z.

371. Find all complex numbers x and y such that $x, x + 2y, 2x + y$ are in A.P. and $(y+1)^2, xy+5, (x+1)^2$ are in G.P.

372. Find A.P. of distinct terms whose first term is 3 and second, tenth and thirty fourth terms form a G.P.

373. Let a, b, c, d be four positive real numbers such that the geometric mean of a and b is equal to the gerometric mean of c and d and the arithmetic mean of a^2 and b^2 is equal to the arithmetic mean of c^2 and d^2. Show that the arithmetic mean of a^n and b^n is equal to the arithmetic mean of c^n and d^n for every integral value of n.

374. The sum of first ten terms of an A.P. is equal to 155, and the sum of first two terms of a G.P. is 9. Find these progressions if the first term of the A.P. euqals the common ratio of the G.P. and the first term of G.P. equals the common difference of A.P.

375. If a, b, c be in H.P., prove that $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right) = \frac{4}{ac} - \frac{3}{b^2}$.

376. If a, b, c are positive real numbers which are in H.P. show that $\frac{a+b}{2a-b} + \frac{b+c}{2c-b} \geq 4$.

377. If $(a+b)/(1-ab), b, (b+c)/(1-bc)$ are in A.P., then prove that a, b^{-1}, c are in H.P.

378. Suppose a, b, c are in A.P. and $|a|, |b|, |c| < 1$ if $x = 1 + a + a^2 + \cdots \infty, y = 1 + b + b^2 + \cdots \infty$, $z = 1 + c + c^2 + \cdots \infty$ then prove that x, y, z are in H.P.

379. If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and a, b, c are in G.P. prove that x, y, z are in A.P.

380. If a, b, c be in A.P., l, m, n be in H.P. and al, bm, cn be in G.P. with common ratio not equal to 1 and a, b, c, l, m, n are positive show that $a : b : c = \frac{1}{n} : \frac{1}{m} : \frac{1}{l}$.

381. An A.P., a G.P. and an H.P. have the same first term a abd same second term b, show that $n + 2$th terms will be in G.P. is $\frac{b^{2n+2}-a^{2n+2}}{ab(b^{2n}-a^{2n})} = \frac{n+1}{n}$.

382. If an A.P. and a G.P. have the same 1st and 2nd terms then show that every other term of the A.P. will be less than the corresponding term of G.P. all the terms being positive.

383. If A, G, H are the arithmetic, geometric and harmonic means of two positive real numbers a and b, and if $A = kh$, prove that $A^2 = kG^2$. Find the ratio of a to b. For what value of k does the ratio exist.

384. If p be the rth term when n A.M.'s are inserted between a and b and q be the rth term when n H.M.'s are inserted between a and b, then show that $\frac{p}{a} + \frac{b}{q}$ is independent of n and r.

385. Two trains A and B start from the same station P at the same time. A covers half the distance between first station P and second station Q with speed x and other half distance with speed y. Train B covers the whole distance with speed $\frac{x+y}{2}$. Which train will reach Q earlier.

386. If n is a root of equation $x^2(1-ac) - x(a^2+c^2) - (1+ac) = 0$ and if n H.M.'s are inserted between a and c, show that the difference between the first and last mean is equal to $ac(a-c)$.

387. If $A_1, A_2, \dots, A_n$ are the n A.M.'s and $H_1, H_2, \dots, H_n$ the n H.M.'s between a and b, show that $A_r H_{n-r+1} = ab$ for $1 \le r \le n$.

388. Find the coefficient of x^{99} and x^{98} in the polynomial $(x-1)(x-2)(x-3)\dots(x-100)$.

389. Find the nth term and sum to n terms of the series $12, 40, 90, 168, 280, 432, \dots$

390. Find the nth term and the sum to n terms of the series $10, 23, 60, 169, 494, \dots$.

391. Find the sum of the series $3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + \cdots \infty$.

392. If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $H'_n = \frac{n+1}{2} - \left\{\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + \cdots + \frac{n-2}{2.3}\right\}$, show that $H_n = H'_n$.

393. Show that $\tan^{-1}\left(\frac{x}{1+1.2x^2}\right) + \tan^{-1}\left(\frac{x}{1+2.3x^2}\right) + \cdots + \tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right) = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$.

394. Find the sum to n terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots$.

395. Find $\displaystyle\sum_{k=n}^{n} \tan^{-1}\frac{2k}{2+k^2+k^4}$

396. Show that $\frac{1^4}{1.3} + \frac{2^4}{3.5} + \frac{3^4}{5.7} + \cdots + \frac{n^4}{(2n-1)(2n+1)} = \frac{n(4n^2+6n+5)}{48} + \frac{n}{16(2n+1)}$

397. If $a_1, a_2, \dots, a_n, \dots$ are in A.P. with first term a and common difference d, find the sum for $r > 1$ of $a_1a_2\dots a_r + a_2a_3\dots a_{r+1} + \cdots$ to n terms.

398. If $a_1, a_2, \dots, a_n, \dots$ are in A.P. and none of them is zero. Then prove that $\frac{1}{a_1a_2\dots a_r} + \frac{1}{a_2a_3\dots a_{r+1}} + \cdots + \frac{1}{a_na_{n+1}\dots a_{n+r-1}} = \frac{1}{(r-1)(a_2-a_1)}$
$\left[\frac{1}{a_1a_2\dots a_{r-1}} - \frac{1}{a_{n+1}a_{n+2}\dots a_{n+r-1}}\right]$

399. Find the sum to n terms of the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\cdots$.

400. Find the sum to n terms of the series $\frac{3}{2.4.6}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\cdots$.

401. Find $\frac{1}{1.3}+\frac{2}{1.3.5}+\frac{3}{1.3.5.7}+\cdots$ to n terms.

402. Find $\frac{2}{1.3}\cdot\frac{1}{3}+\frac{3}{3.5}\cdot\frac{1}{3^2}+\frac{4}{5.7}\cdot\frac{1}{3^3}+\cdots$ to n terms.

403. Find the sum of n terms of the series $\frac{1}{3}+\frac{3}{3.7}+\frac{5}{3.7.11}+\frac{7}{3.7.11.15}+\cdots$.

404. Find the sum of the series: $1+2(1-a)+3(1-a)(1-2a)+4(1-a)(1-2a)(1-3a)+\cdots$ to m terms.

405. Find the sum of the series $1+\frac{x}{b_1}+\frac{x(x+b_1)}{b_1b_2}+\frac{x(x+b_1)(x+b_2)}{b_1b_2b_3}+\cdots+\frac{x(x+b_1)\cdots(x+b_{n-1})}{b_1b_2\cdots b_n}$.

406. Let $S_k(n)=1^k+2^k+\cdots+n^k$, show that $nS_k(n)=S_{k+1}(n)+S_k(n-1)+S_k(n-2)+\cdots+S_k(2)+S_k(1)$.

407. Find the sum of all the numbers of the form n^3 which lie between 100 and 10000.

408. If S be the sum of the n consecutive integers beginning with a and t the sum of their squares, show that $nt-S^2$ is independent of a.

409. If $\sum_{x=5}^{n+5} 4(x-3)=Pn^2+Qn+R$, find the value of $P+Q$.

410. Find the sum to $2n$ terms of the series $5^3+4.6^3+7^3+4.8^3+9^3+4.10^3+\cdots$.

411. Find the sum to n terms of the series $\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\cdots$.

412. Find the sum to n terms of the series $1+5\left(\frac{4n+1}{4n-3}\right)+9\left(\frac{4n+1}{4n-3}\right)^2+13\left(\frac{4n+1}{4n-3}\right)^3+\cdots$.

413. Prove that the numbers of the sequence 121, 12321, 1234321, $\cdots$ are each a perfect square of an odd integer.

414. Prove that the sum to n terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+\cdots$ is $6n/(n+1)$.

415. Find the sum to n terms of the series $\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\cdots$.

416. Find the sum to n terms of the series $\frac{1}{(1+x)(1+ax)}+\frac{a}{(1+ax)(1+a^2x)}+\frac{a^2}{(1+a^2x)(1+a^3x)}+\cdots$.

417. Find the sum to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots$.

418. If $a_1, a_2, \ldots, a_n, \ldots$ are in A.P. with first term a and common difference d, then prove that $a_1a_2 + a_2a_3 + \cdots + a_na_{n+1} = \frac{[a+(n-1)d](a+nd)-(a-d)a(a+d)}{3d} = \frac{n}{3}[3a^2 + 2and + (n^2-1)d^2]$.

419. If $a_1, a_2, \ldots, a_n, \ldots$ are in A.P. with first term a and common difference d, then prove that $a_1a_2a_3 + a_2a_3a_4 + \cdots + a_na_{n+1}a_{n+2} =$

$\frac{[a+(n-1)d](a+nd)[a+(n+1)d][a+(n+2)d]-(a-d)a(a+d)(a+2d)}{4d} =$

$\frac{n}{4}[4a^3 + 6(n+1)a^2d + 2(2n^2+3n-1)ad^2 + (n^3-2n^2-n-2)d^3]$.

420. Find the sum to n terms of the series $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + \cdots$.

421. Let S_n denote the sum to n terms of the series $1.2 + 2.3 + 3.4 + \cdots$ and σ_{n-1} that to $n-1$ terms of the series $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \cdots$ Then prove that $18S_n\sigma_{n-1} - S_n = -2$.

422. Find $\frac{5}{1.2}\cdot\frac{1}{3} + \frac{7}{2.3}\cdot\frac{1}{3^2} + \frac{9}{3.4}\cdot\frac{1}{3^3} + \cdots$ to n terms.

423. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\infty = \frac{\pi^2}{6}$ then find $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty$.

424. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\infty = \frac{\pi^2}{6}$, then find $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\infty$.

425. If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then prove that $H_n = n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}\right)$.

426. Show that $\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^n}{x^{2^n}+1} = \frac{1}{x-1} - \frac{2^{n+1}}{x^{2^{n+1}}-1}$.

427. Show that $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right) = \frac{3}{2}\left(1-\frac{1}{3^{2^{n+1}}}\right)$.

428. If $x+y+z=1$ and x, y, z are positive numbers show that $(1-x)(1-y)(1-z) \geq 8xyz$.

429. If $a > 0, b > 0$ and $c > 0$, prove that $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$.

430. If $a+b+c=3$ and $a > 0, b > 0, c > 0$, find the greatest value of $a^2b^3c^2$.

431. Let $a_i + b_i = 1 (i = 1, 2, \ldots, n)$ and $a = \frac{1}{n}(a_1 + a_2 + \cdots + a_n), b = \frac{1}{n}(b_1 + b_2 + \cdots + b_n)$, show that $a_1b_1 + a_2b_2 + \cdots + a_nb_n = nab - (a_1-a)^2 - (a_2-a)^2 - \cdots - (a_n-a)^2$.

432. A sequence $a_1, a_2, a_3, \ldots, a_n$ of real numbers is such that $a_1 = 0, |a_2| = |a_1+1|, |a_3| = |a_2+1|, \ldots, |a_n| = |a_{n-1}+1|$. Prove that the arithmetic mean $(a_1 + a_2 + \cdots + a_n)/n$ of these numbers cannot be less than $-1/2$.

433. If a,b,c > 0, show that $(a+b)(b+c)(a+c) \geq 8abc$.

434. If $x+y+z=a$, show that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{a}$.

435. If n is a positive integer, show that $n^n \geq 1.3.5 \dots (2n-1)$.

436. Find the greatest value of $(7-x)^4(2+x)^5$ if $-2 < x < 7$.

437. If $a, b, c > 0$, show that $\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leq \frac{a+b+c}{2}$.

438. If $a, b, c > 0$, show that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$.

439. If $x_i > 0, i = 1, 2, 3, \dots, n$ show that $(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$.

440. If x,y are positive real numbers and m, n are positive integers, then show that $\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})} \leq \frac{1}{4}$.

441. If the arithmetic mean of $(b-c)^2$, $(c-a)^2$ and $(a-b)^2$ is the same as that of $(b+c-2a)^2$, $(c+a-2b)^2$ and $(a+b-2c)^2$, show that $a = b = c$.

Chapter 3
Complex Numbers

By definition a complex number has two parts: a real part and an imaginary part. You already know about real numbers and know about them. However, imaginary numbers is something different.

3.1 Imaginary Numbers

Imaginary numbers are called so because there cannot be physical representation of these quantities. Like we use real numbers for counting physical objects we cannot do that with imaginary numbers. In real world, they do not exist. Square root of negative numbers are called imaginary numbers. For example, $\sqrt{-1}$, $\sqrt{-2}.\sqrt{-3}$, ... and so on.

We denote $\sqrt{-1}$ with the Greek symbol ι, which stands for *iota*. We also use English letters i or j to represent this imaginary number. Clearly, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. If you examine carefully, you will find that following holds true:

$$i^{4m} = 1, i^{4m+1} = i, i^{4m+2} = -1 \text{ and } i^{4m+3} = -1, \forall m \in P$$

Gotcha:

Consider the following:

$$1 = \sqrt{1} = \sqrt{-1 * -1} = \sqrt{-1} * \sqrt{-1} = i * i = -1$$

However, the above result is wrong. The reason being is that for any two real numbers a and b, $\sqrt{a} * \sqrt{b} = \sqrt{ab}$ holds good if and only if two numbers are either zero or positive. Also, $\sqrt{1} \neq \sqrt{-1 * -1}$ because power of $-$ is $\frac{1}{2}$ which results in -1.

3.2 Definitions Related to Complex Numbers

A complex number is written as $a + ib$ or $x + iy$ or $a + jb$ or $x + jb$. Here, a, b, x, y are all real numbers. The complex numbers itself is denoted by z. Therefore, we have $z = x + iy$. Here, x is called the real part and is also denoted by $\mathfrak{R}(z)$ and y is called the imaginary part and is also denoted by $\mathfrak{I}(z)$.

A complex number is purely real if its imaginary part or y or $\mathfrak{I}(z)$ is zero. Similarly, a complex number is purely imaginary if its real part or x or $\mathfrak{R}(z)$ is zero. Clearly, as you can imagine that there can exist only one number which has both the parts as zero and certainly that is 0. That is, $0 = 0 + i0$.

The set of all complex number is typically denoted by C. Two complex numbers z_1 and z_2 are said to be true if there real parts are equal and imaginary parts are equal. That is if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then x_1 must be equal to x_2 and similarly for imaginary part for two complex numbers to be equal.

3.3 Simple Arithmetic Operations

3.3.1 Addition

$(a+ib)+(c+id)=(a+c)+i(b+d)$

3.3.2 Subtraction

$(a+ib)-(c+id)=(a-c)+i(b-d)$

3.3.3 Multiplication

$(a+ib)*(c+id)=ac+ibc+iad+bdi^2=(ac-bd)+i(bc+ad)$

3.3.4 Division

The complex number in denominator must not have both parts as zero. At least one part must be non-zero.

$$\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

3.4 Conjugate of a Complex Number

Let $z=x+iy$ be a complex number then its complex conjugate is a number with imaginary part made negative. It is written as $\overline{z}=x-iy.\overline{z}$ is the typical representation for conjugate of a complex number z.

3.4.1 Properties of Conjugates

1. $z_1=z_2\Leftrightarrow\overline{z_1}=\overline{z_2}$

   Clearly as we know for two complex numbers to be equal, both parts must be equal. So this is very easy to understand that if $x_1=x_2$ and $y_1=y_2$ then this bidirectional condition is always satisfied.

2. $\overline{(\overline{z})}=z$

   $z=x+iy$, hence, $\overline{z}=x-iy$. Hence, $\overline{(\overline{z})}=x-(-iy)=x+iy=z$

3. $z+\overline{z}=2\Re(z)$

   $z+\overline{z}=x+iy+x-iy=2x=2\Re(z)$.

4. $z-\overline{z}=2i\Im(z)$

   $z-\overline{z}=x+iy-(x-iy)=2iy=2i\Im(z)$

5. $z = \overline{z} \Leftrightarrow z$ is purely real.

   Clearly, $x + iy = x - iy \Rightarrow 2iy = 0 \Rightarrow y = 0$. Therefore, z is purely real. Conversely, if z is purely real then $z = x$, and thus $z = \overline{z}$.

6. $z + \overline{z} = 0 \Leftrightarrow z$ is purely imaginary.

   Clearly, $x + iy + x - iy = 0 \Rightarrow 2x = 0$. Therefore, z is purely imaginary. Conversely, if z is purely imaginary then $z = iy$, and thus $z + \overline{z} = 0$.

7. $z\overline{z} = [\Re(z)]^2 + [\Im(z)]^2$

   Clearly, $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = [\Re(z)]^2 + [\Im(z)]^2$

8. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

   $\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \overline{z_1} + \overline{z_2}$

9. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

   This can be proven like previous item.

10. $\overline{z_1 z_2} = \overline{z_1 z_2}$

    This can be proven like previous item.

11. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$

    It can be proven by multiplying and dividing by conjugate of denominator and then applying division formula given above.

12. If $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, where $a_0, a_1, \ldots, a_n$ and z are complex numbers, then

$$\overline{P(z)} = \overline{a_0} + \overline{a_1}(\overline{z}) + \overline{a_2}(\overline{z})^2 + \overline{a_n}(\overline{z})^n = \overline{P}(\overline{z})$$

    where

$$\overline{P}(z) = \overline{a_0} + \overline{a_1}z + \overline{a_2}z^2 + \ldots + \overline{a_n}z^n$$

13. If $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in z, and$Q(z) \neq 0$, then

$$\overline{R(z)} = \frac{\overline{P}(\overline{z})}{\overline{Q}(\overline{z})}$$

14. If

$$z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then } \overline{z} = \begin{vmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{vmatrix},$$

where $a_i, b_i, c_i (i = 1,2,3)$ are complex numbers.

3.5 Modulus of a Complex Number

Modulus of a complex number z is denoted by $|z|$ and is equal to the real number $\sqrt{x^2 + y^2}$. Note that $|z| \geq 0 \,\forall z \in C$.

3.5.1 Properties of Modulus

1. $|z| = 0 \Leftrightarrow z = 0$

   Clearly, this means $x^2 + y^2 = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow z = 0$.

2. $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

   Clearly, all result in $\sqrt{x^2 + y^2}$.

3. $-|z| \leq \Re(z) \leq |z|$.

   Clearly, $-\sqrt{x^2 + y^2} \leq x \leq \sqrt{x^2 + y^2}$.

4. $-|z| \leq \Im(z) \leq |z|$.

   Clearly, $-\sqrt{x^2 + y^2} \leq y \leq \sqrt{x^2 + y^2}$.

5. $z\overline{z} = |z|^2$

   Clearly, $(x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

Following relations are very easy and can be proved by the student. If z_1 and z_2 are two complex numbers then,

6. $|z_1 z_2| = |z_1||z_2|$

   $|z_1 z_2| = |x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} = \sqrt{(x_1 + y_1)^2 (x_2 + y_2)^2} = |z_1||z_2|$

7. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$

8. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \overline{z_1} z_2 + z_2 \overline{z_1} = |z_1|^2 + |z_2|^2 + 2\Re(z_1 \overline{z_2})$.

9. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \overline{z_1} z_2 - z_1 \overline{z_2} = |z_1|^2 + |z_2|^2 + 2\Re(z_1 \overline{z_2})$.

10. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\big(|z_1|^2 + |z_2|^2\big)$.

11. If a and b are real numbers, and z_1 and z_2 are complex numbers, then

    $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)\big(|z_1|^2 + |z_2|^2\big)$.

12. If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

13. If z_1 and z_2 are complex numbers then $|z_1 + z_2| \leq |z_1| + |z_2|$. This inequality can be generalized to more terms as well.

14. $|z_1 - z_2| \leq |z_1| + |z_2|$, $||z_1| - |z_2|| \leq |z_1| + |z_2|$ and $|z_1 - z_2| \geq ||z_1| - |z_2||$. These are trivial to prove.

3.6 Geometrical Representation

A complex number z which we have considered to be equal to $x + iy$ in our previous representations can be represented by a point P whose Cartesian coordinates are (x,y) referred to rectangular axes Ox and Oy where O is origin i.e. $(0, 0)$ and are called *real* and *imaginary* axes respectively. The xy two-dimensional plane is also called *Argand plane, complex plane* or *Gaussian plane.* The point P is also called the image of the complex number and z is also called the *affix* or *complex coordinate* of point P.

Now as you can easily figure out that all real numbers will lie on real axis and all imaginary numbers will lie on imaginary axis as their counterparts will be zero.

The modulus is given by the length of segment OP which is equal to $OP = \sqrt{x^2 + y^2} = |z|$. This, $|z|$ si the length of the OP. Given below is the graphical representation of the complex number.

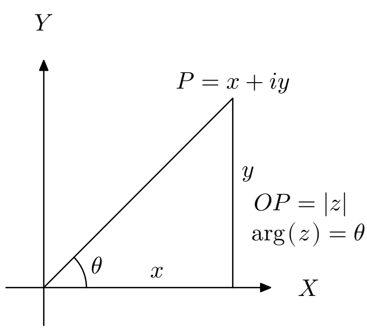


Figure 3.1 Complex number in argand plane or complex plane.

In the diagram, θ is known as the *argument* of z. This is nothing but angle made with positive direction (i.e. counter-clockwise) of real axis. Now, this argument is not unique. If θ is an argument of a complex number z then, $2n\pi + \theta$, where $n \in I$, where I is the set of integers. The value of argument for which $-\pi < \theta \leq \pi$ is called the *principal value* of *argument* or *principal argument.*

3.6.1 Different Arguments of a Complex Number

In the diagram, the argument is given as $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$, this value is for when z in first quadrant. When z will lie in second, third and fourth quadrants the arguments will be

$$arg(z) = \pi - \tan^{-1}\left(\frac{y}{|x|}\right), \arg(z) = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right) \text{ and } \arg(z) = -\tan^{-1}\left(\frac{|y|}{x}\right)$$

respecticely.

3.6.2 Polar Form of a Complex Number

If z is a non-zero complex number, then we can write $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$.

In this case, z is also given by $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$, where $n \in I$.

A Euler's Formula

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$ or $\arg(c)$ is θ, where c is the complex number.

3.6.3 Important Results Involving Arguments

If z, z_1 and z_2 are complex numbers, then

1. $\arg(\overline{(z)}) = -\arg(z)$. This can be easily proven as if $z = x + iy$, then $\overline{z} = x - iy$ i.e. sign of argument will get a -ve sign as y gets one.
2. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2n\pi$, where
$$n = \begin{cases} 0 \text{ if} & -\pi < \arg(z_1) + \arg(z_2) \leq \pi \\ 1 \text{ if} & -2\pi < \arg(z_1) + \arg(z_2) \leq -\pi \\ -1 \text{ if} & \pi < \arg(z_1) + \arg(z_2) \leq 2\pi \end{cases}$$
3. Similarly, $\arg(z_1\overline{z_2}) = \arg(z_1) - \arg(z_2)$.
4. $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi/2$.
5. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$.
6. $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$.
7. $|z_1 - z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 + \theta_2)$.

3.7 Vector Representation

Complex numbers can also be represented as vectors. Length of the vector is nothing but modulus of complex number and argument is the angle which the vector makes with read axis. It is denoted as $\overrightarrow{OP}$, where OP represents the vector of the complex number z.

3.8 Algebraic Operation's Representation

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, which are represented by two point P_1 and P_2 in the following diagrams.

3.8.1 Addition

Now, as we know that $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. Let us see how it looks using geometrically:

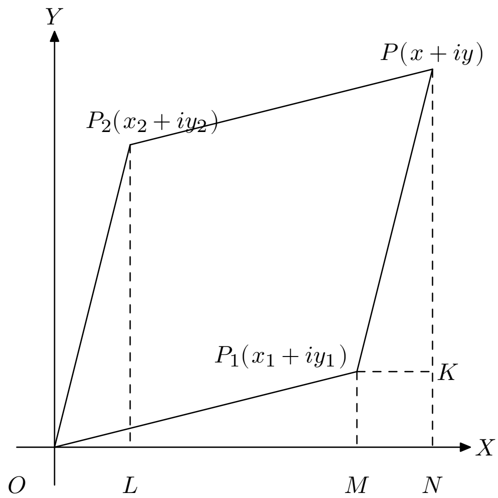


Figure 3.2 Complex numbers addition

Clearly, $z = z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$. Let P_1M, P_2L and PN be parallel to the y-axis; P_1K be parallet to the x-axis. This implied that triangle OP_2L and PP_1K are congruent.

We have $P_1K = OL = x_1$ and $P_2L = PK = y_1$

Thus, $ON = OM + MN = OL + P_1K = x_1 + x_2$ and $PN = PK + KN = P_2L + P_1M = y_2 + y_1$

So we can say that coordinates of P are $(x_1 + x_2, y_1 + y_2)$ which represents the complex number z.

We also see that this obeys vector addition i.e. $OP_1 + OP_2 = OP_1 + P_1P = OP$

3.8.2 Subtraction

In Figure 3.3, we first represent $-z_2$ by P_2' so that P_2P_2' is bisected at O. Complete the parallelogram OP_1PP_2'. Then it can be easily seen that P representd the difference $z_1 - z_2$.

As OP_1PP_2' is a parallelogram so $P_1P = OP_2'$. Using vetor notation, we have, $z_1 - z_2 = OP_1 - OP_2 = OP_1 + OP_2' = OP_1 + P_1P = P_2P1$

It follows that the complex number $z_1 - z_2$ is represented by the vector P_1P_2, where points P_1 and P_2 represent the complex numbers z_1 and z_2 respectively.

It should be noted that $\arg(z_1 - z_2)$ is the angle through which OX must be rotated in the anticlockwise direction to make it parallel with P_1P_2.

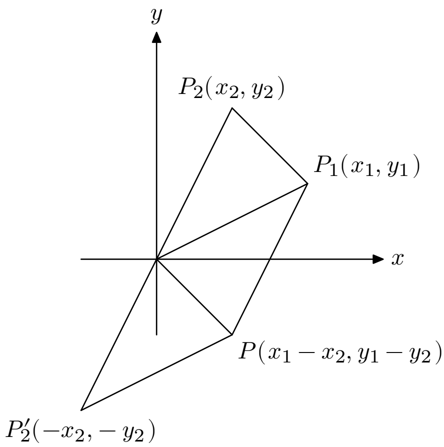


Figure 3.3 Complex numbers subtraction

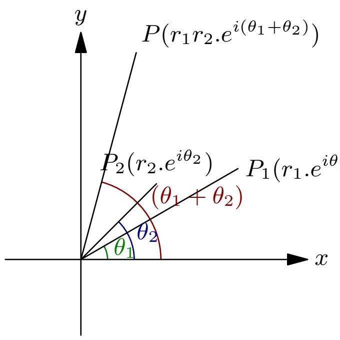


Figure 3.4 Complex numbers subtraction

3.8.3 Multiplication

For multiplication it is convenient to use Euler's formula of complex numbers.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then clealry, $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

3.8.4 Division

For division also it is convenient to use Euler's formula of complex numbers.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then clealry, $z_1/z_2 = r_1/r_2 e^{i(\theta_1-\theta_2)}$

3.9 Three Important Results

$z_1 - z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP}$

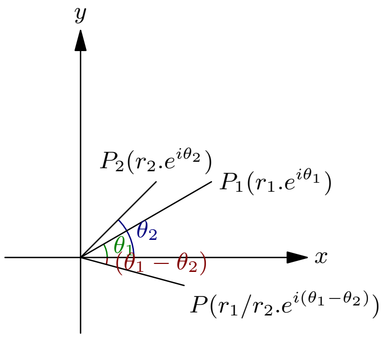


Figure 3.5 Complex numbers division

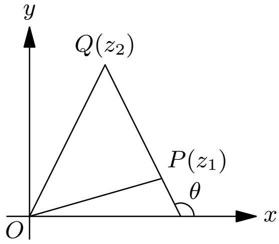


Figure 3.6 External angle

$\therefore |z_1 - z_2| = |\overrightarrow{QP}| = QP$ which is nothing but distance between P and Q.

$\arg(z_1 - z_2)$ is the angle made by $\overrightarrow{QP}$ with x-axis which is nothing but θ.

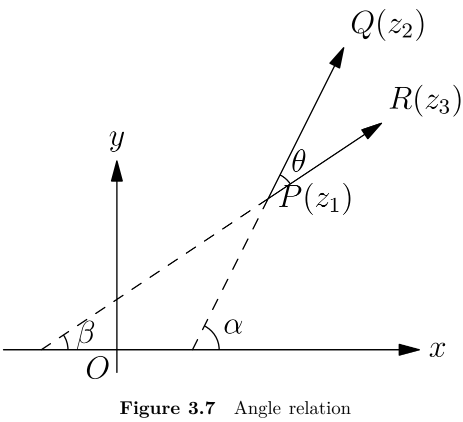


Figure 3.7 Angle relation between three complex numbers

In Figure 3.7, $\theta = \alpha - \beta = \arg(z_3 - z_1) - \arg(z_2 - z_1) \Rightarrow \theta = arg\frac{z_3 - z_1}{z_2 - z_1}$

Similarly if three complex numbers are vertices of a triangle then angles of those vertices can also be computed using previous results.

Similarly, for four points to be concyclic where those points are represented by z_1, z_2, z_3 and z_4 if

$$arg\left(\frac{z_2 - z_4}{z_1 - z_4} \cdot \frac{z_1 - z_3}{z_2 - z_4}\right) = 0$$

3.10 More Roots

3.10.1 Any Root of an Complex Number is a Complex Number

Let $x + iy$ be a complex number, where $y \neq 0$.

Let $(x + iy)^n = a \therefore x + iy = a^n$

Now, if a is real, a^n will also be real but from above a complex number $x + iy$ is equal to a real number, a^n, which is not possible. Hence, it must be complex.

3.10.2 Square Root of a Complex Number

Consider a complex number $z = x + iy$. Let $a + ib$ be its square root. Then

$$\sqrt{x + iy} = a + ib \Rightarrow x + iy = (a^2 - b^2) + 2abi$$

Equating real and imaginary parts

$$x = a^2 - b^2, y = 2ab \Rightarrow (a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$$

From these two equations, we have

$$a = \pm\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}}, b = \pm\sqrt{\frac{\sqrt{x^2 - y^2} - x}{2}}$$

3.10.3 Cube Roots of Unity

Let $x = \sqrt[3]{1} \Rightarrow x^3 - 1 = 0$

$\Rightarrow (x - 1)(x^2 + x + 1) = 0$

So the three roots are $x = 1, \frac{-1 \pm \sqrt{3}}{2}$ i.e. $1, \frac{-1 \pm \sqrt{3}i}{2}$.

It can be easily verified that if $\omega = \frac{-1 - \sqrt{3}i}{2}$, then $\omega^2 = \frac{-1 + \sqrt{3}i}{2}$, thus, three cube roots are represented as $1, \omega$ and ω^2. ω is the symbol used for representing cube root of unity.

A Important Identities

Following identities can be proved easily. The proof is left as an exercise to the reader.

1. $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
2. $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

3. $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
4. $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
5. $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
6. $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
7. $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$ or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$ or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$
8. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

3.10.4 nth Root of Unity

$1 = \cos 0 + i\sin 0 \Rightarrow \sqrt[n]{1} = \sqrt[n]{\cos 0 + i\sin 0}$

$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},$ where $k = 0, 1, 2, 3, 4, \ldots (n-1)$

$= e^{\frac{2k\pi}{n}} = 1, e^{\frac{i2\pi}{n}}, e^{\frac{i4\pi}{n}}, \ldots, e^{\frac{i2(n-1)\pi}{n}} = 1, \alpha, \alpha^2, \ldots, \alpha^{n-1},$ where $\alpha = e^{\frac{i2\pi}{n}}$

Similar to cube roots of unity it can be proven that $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$ and $1.\alpha.\alpha^2.\ldots\alpha^{n-1} = (-1)^{n-1}$

3.11 De Moivre's Theoremm

This theorem's proof uses mathematical induction, so read the chapter on it.

Statement: If n is any integer then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Proof: Case I. When n is 0. Clearly, $(\cos\theta + i\sin\theta)^0 = 1$

Case II. When n is a positive integer. Clearly is it true for $n = 1$

Let it is true for $n = m$. Then $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

For $n = m + 1$, $(\cos\theta + i\sin\theta)^{m+1} = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta) = \cos(m+1)\theta + i\sin(m+1)\theta$ [this result comes from trigonometry]

Thus, by mathematical induction we have proven the theorem for positive integers.

Case III. When n is negative number. For $n = -1$, $(\cos\theta + i\sin\theta)^{-1} = \frac{1}{\cos\theta + i\sin\theta}$

$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$

Let it be true for $n = -m$, $(\cos\theta + i\sin\theta)^{-m} = \cos m\theta - i\sin m\theta$

For $n = -(m+1)$, $(\cos\theta + i\sin\theta)^{-(m+1)} = \frac{\cos m\theta - i\sin m\theta}{\cos\theta + i\sin\theta}$

$= (\cos m\theta - i\sin m\theta)(\cos\theta - i\sin\theta) = \cos(m+1)\theta + i\sin(m+1)\theta$

Thus, it is proven for negative numbers as well. Proof for fractional powers is left as an exercise.

3.12 Some Important Geometrical Results

3.12.1 Section Formula

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ then $z = x + iy$, which divides the previous two points in the ratio $m : n$ can be given by using the results from coordinate geometry as below:

$$x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n} \text{ and } z = \frac{mz_2 + nz_1}{m+n}$$

Extending this section formula, we can say that if there is a point which is mid-point i.e. divides a line in two equal parts, then $m = 1$ and $n = 1$ then z is given by $\frac{1}{2}(z_1 + z_2)$.

3.12.2 Distance Formula

Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_1 - z_2|$.

3.12.3 Equation of a Line

The equation between two points z_1 and z_2 is given by the determinant

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$$

or,

$$\frac{z - z_1}{\overline{z} - \overline{z_1}} = \frac{z_1 - z_2}{\overline{z_1} - \overline{z_2}}$$

The parametric form is given by $z = iz_1 + (1-t)z_2$

3.12.4 Collinear Points

Three points z_1, z_2 and z_3 are collinear if and only if

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$$

3.12.5 Parallelogram

Four complex numbers $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ represent the vertices of a parallelogram if $z_1 + z_3 = z_2 + z_4$.This result comes from the fact that diagonals of a parallelogram bisect each other.

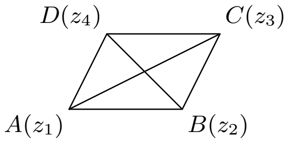


Figure 3.8 Parallelogram

3.12.6 Rhombus

Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a rhombus if $z_1 + z_3 = z_2 + z_4$ and $|z_4 - z_1| = |z_2 - z_1|$.

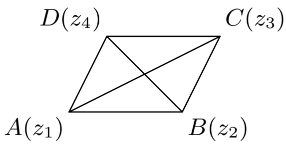


Figure 3.9 Rhombus

The diagonals must bisect each other. Thus, $z_1 + z_3 = z_2 + z_4$. Also, four sides of a rhombus are equal i.e. $AD = AB \Rightarrow |z_4 - z_1| = |z_2 - z_1|$.

3.12.7 Square

Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a square if $z_1 + z_3 = z_2 + z_4$, $|z_4 - z_1| = |z_2 - z_1|$ and $|z_3 - z_1| = |z_4 - z_2|$.

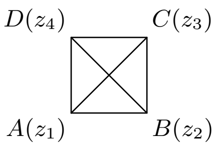


Figure 3.10 Square

The diagonals must bisect each other. Thus, $z_1 + z_3 = z_2 + z_4$. Also, four sides of a square are equal i.e. $AD = AB \Rightarrow |z_4 - z_1| = |z_2 - z_1|$.

Also the digonals are equal in length so $|z_3 - z_1| = |z_4 - z_2|$.

3.12.8 Rectangle

Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a square if $z_1 + z_3 = z_2 + z_4$ and $|z_3 - z_1| = |z_4 - z_2|$.

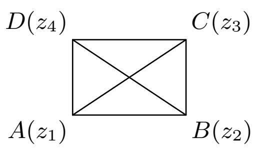


Figure 3.11 Rectangle

The diagonals must bisect each other. Thus, $z_1 + z_3 = z_2 + z_4$. Also, the digonals are equal in length so $|z_3 - z_1| = |z_4 - z_2|$.

3.12.9 Centroid of a Triangle

Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be the vertices of a $\triangle ABC$. Centroid $G(z)$ of the $\triangle ABC$ is the point of concurrence of the medians of all three sides and is given by

$$z = \frac{z_1 + z_2 + z_3}{3}$$

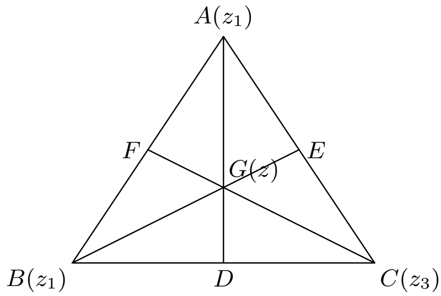


Figure 3.12 Centroid of a triangle.

3.12.10 Incenter of a Triangle

Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be the vertices of a $\triangle ABC$. Incenter $I(z)$ of the $\triangle ABC$ is the point of concurrence of the internal bisectors of and is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c}$$

where a, b, c are the lengths of the sides.

3.12.11 Circumcenter of a Triangle

Circumcenter $S(z)$ of a $\triangle ABC$ is the point of concurrence of perpendicular bisectors of sides of the triangle. It is given by

$$z = \frac{(z_2 - z_3)|z_1|^2 + (z_3 - z_1)|z_2|^2 + (z_1 - z_2)|z_3|^2}{\overline{z_1}(z_2 - z_3) + \overline{z_2}(z_3 - z_1) + \overline{z_3}(z_1 - z_2)}$$

$$= \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z_1} & z_1 & 1 \\ \overline{z_2} & z_2 & 1 \\ \overline{z_3} & z_3 & 1 \end{vmatrix}}$$

Also,

$$z = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

3.12.12 Orthocenter of a Triangle

The orthocenter $H(z)$ of the $\triangle ABC$ is the point of concurrence of altitudes of the side. It is given by

$$z = \frac{\begin{vmatrix} z_1^2 & \overline{z_1} & 1 \\ z_2^2 & \overline{z_2} & 1 \\ z_3^2 & \overline{z_3} & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z_1} & z_1 & 1 \\ \overline{z_2} & z_2 & 1 \\ \overline{z_3} & z_3 & 1 \end{vmatrix}}$$

$$= \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

$$= \frac{z_1 a \sec A + b z_2 \sec B + c z_3 \sec C}{a \sec A + b \sec B + c \sec C}$$

3.12.13 Euler's Line

The centroid G of a triangle lies on the segment joining the orthocenter H and the circumcenter S of the triangle. G divides the line H and S in the ratio $2 : 1$.

3.12.14 Length of Perpendicular from a Point to a Line

Length of a perpendicular of point $A(\omega)$ from the line $\overline{a}z + a\overline{z} + b = 0$, $(a \in C, b \in R)$ is given by

$$p = \frac{|\overline{a}\omega + a\overline{\omega} + b|}{2|a|}$$

3.12.15 Equation of a Circle

The equation of a circle with center z_0 and radius r is $|z - z_0| = r$ or $z = z_0 + re^{i\theta}, 0 \leq \theta \leq 2\pi$ or $z\overline{z} - z_0\overline{z} - \overline{z_0}z + z_0\overline{z_0} - r^2 = 0$

General equation of a circle is $z\overline{z} - a\overline{z} + \overline{a}z + b = 0$, $(a \in C, b \in R)$ such that $\sqrt{a\overline{a} - b} \geq 0$.

Center of this circle is $-a$ and radius is $a\overline{a} - b$.

An equation of the circle, one of whose diameter is the line segment joining z_1 and z_2 is $(z - z_1)(\overline{z} - \overline{z_2}) + (\overline{z} - \overline{z_1})(z - z_2) = 0$

An equation of the the circle passing through two points z_1 and z_2 is

$$(z - z_1)(\overline{z} - \overline{z_2}) + (\overline{z} - \overline{z_1})(z - z_2) + k\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$$

where k is a parameter.

3.12.16 Equation of a Circle Passing through Three Points

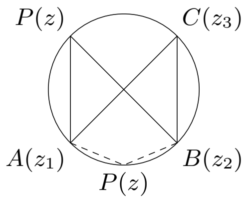


Figure 3.13 Circle through three points

We choose any point $P(z)$ on the circle. Two such points are shown in the figure above one is in same segment with C and the other one in different segement. So we have

$$\angle ACB = \angle APB \text{ or } \angle ACB + \angle APB = \pi$$

$$\arg \frac{z_3 - z_2}{z_3 - z_1} - arg \frac{z - z_2}{z - z_1} = 0 \text{ or } \arg \frac{z_3 - z_2}{z_3 - z_1} + arg \frac{z - z_2}{z - z_1} = \pi$$

Clearly, in both cases the fraction must be purely real. Thus we can apply the property of conjugates i.e. $z = \overline{z}$ which also gives us the condition for four concyclic points.

$$\Rightarrow \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \overline{\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}}$$

From this we can also deduce the condition for four points to be concyclic. Treating $P(z)$ as just another point $D(z_4)$, we can rewrite the abobe result as

$$\frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} = \overline{\frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}}$$

3.12.17 Finding Loci by Examination

1. $arg(z - z_0) = \alpha$

   If α is a real number and z_0 is a fixed point, then $arg(z - z_0) = \alpha$ represents a vector starting at z_0(exlcluding the point z_0) and making an angle α with real x-axis.

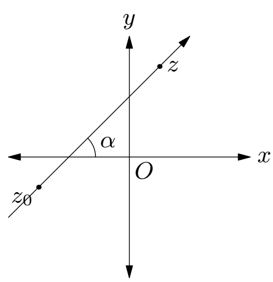


Figure 3.14

Now suppose z_0 is origin O, then the above equation becomes $arg(z) = \alpha$, which is a vector starting at origin and making an angle α, which is a vector starting at origin and making an angle α with x-axis.

2. If z_1 and z_2 are two fixed points such that $|z - z_1| = |z - z_2|$ then z represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$. And z, z_1, z_2 will form an isoscles triangle.

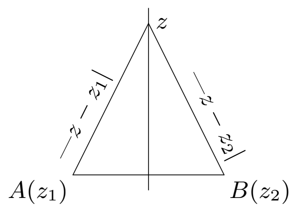


Figure 3.15

If z_1 and z_2 are two fixed points and $k > 0, k \neq 1$ is a real number then $\frac{|z-z_1|}{|z-z_2|} = k$ represents a circle.

3. $|z - z_1| + |z - z_2| = k$. Let z_1 and z_2 be two fixed points and k be a positive real number.

   i. Refer Figure 3.16, if $k > |z - z_2|$, then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of major axis $= k$.

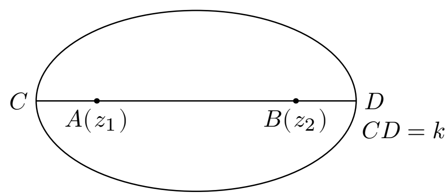


Figure 3.16 Locus of an Ellipse

   ii. If $k = |z - z_2|$, then it represents the line segment joining z_1 and z_2.

   iii. If $k < |z - z_2|$, thne it does not represent any curve/line in Argand plane.

4. If $|z - z_1| - |z - z_2| = k$. Let z_1 and z_2 be two fixed points and k be a positive real number.

   i. Refer Figure 3.17, if $k \neq |z - z_2|$, then it represnts a parabola with foci at $A(z_1)$ and $B(z_2)$.

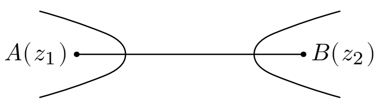


Figure 3.17 Locus of a Parabola

   ii. If $k = |z_1 - z_2|$, then it represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB

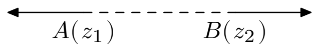


Figure 3.18

5. $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$. If z_1 and z_2 are two fixed points then it represents a circle with z_1 and z_2 as the endpoints of one of the diameters.

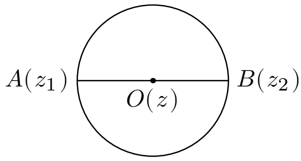


Figure 3.19

6. $\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha$. Let z_1 and z_2 be any two fixed points and α be a real number such that $0 \le \alpha \le \pi$.

   i. If $0 < \alpha < \pi$ and $\alpha \ne \pi/2$, then it represents a segment of a circle passing through $A(z_1)$ and $B(z_2)$.

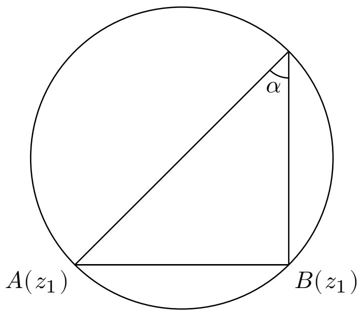


Figure 3.20

   ii. If $\alpha = \pi/2$, then it represents a circle with diameter as the line segment joining $A(z_1)$ and $B(z_2)$.

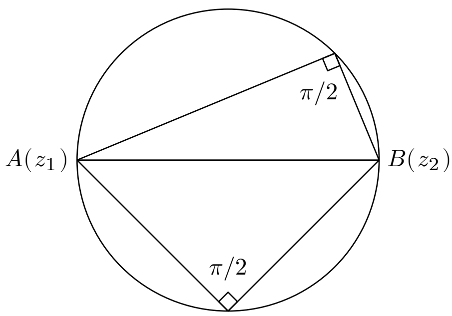


Figure 3.21

   iii. If $\alpha = \pi$, then it represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the line segment AB.

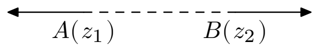


Figure 3.22

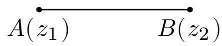


Figure 3.23

iv. If $\alpha = 0$, then it represents the straight line joining $A(z_1)$ and $B(z_2)$.

3.13 Problems

Find the square root of the following complex numbers:

1. $7 + 8i$

2. $a^2 - b^2 + 2abi$

3. $\sqrt[4]{-81}$

4. Find the square root of

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{31}{16}$$

Simplify the following in the form of $A + iB$

5. $i^{n+80} + i^{n+50}$

6. $\left(i^{17} + \frac{1}{i^{15}}\right)^3$

7. $\frac{(1+i)^2}{2+3i}$

8. $\left(\frac{1}{1+i} + \frac{1}{1-i}\right)\frac{7+8i}{7-8i}$

9. $\frac{(1+i)^{4n+7}}{(1-i)^{4n-1}}$

10. $\frac{1}{1-\cos\theta+2i\sin\theta}$

11. $\frac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cot u+i)(i+\tan v)}$ Evaluate:

12. i^5

13. i^{67}

14. i^{-59}

15. i^{2014}

16. If $a<0, b>0$, then prove that $\sqrt{ab}$ is equal to $\sqrt{|a|b}\,i$.

17. Prove that $i^n+i^{n+1}+i^{n+2}+i^{n+3}=0$.

18. Find the value of the sum $\sum_{n=1}^{13}(i^n+i^{n+1})$.

19. Simplify and find the value of $\frac{2^n}{(1+i)^{2n}}+\frac{(1+i)^{2n}}{2^n}$

20. Find different values of i^n+i^{-n}, $\forall\, n\in I$.

21. If $4x+(3x-y)\,i=3-6i$, then find the value of x and y.

22. Find the value of $\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)$.

23. Find the real values of x and y, if $\frac{(1+i)\,x-2i}{3+i}+\frac{(2-3i)\,y+i}{3-i}=i$.

24. Find the multiplicative inverse of $4-3i$.

25. If $z_1=2+3i$ and $z_2=1+2i$, then find the value of z_1/z_2.

26. If $z_1=9y^2-4-i10x$ and $z_2=8y^2-20i$ such that $z_1=\overline{z_2}$, then find $z=x+iy$.

27. Find z if $|z+1|=z+2(1+i)$, where $z\in C$.

28. Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$

29. If $\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$, where $x,y\in R$, then find x and y.

30. What is the real part of $(1+i)^{50}$.

31. If a complex number is z, such that $z+|z|=2+8i$, then find z.

32. Find the sum of sequence $S=i+2i^2+3i^3+...$ up to 100 terms.

33. Find the value of the sum $\frac{1}{1+i}+\frac{1}{1-i}+\frac{1}{-1+i}+\frac{1}{-1-i}+\frac{2}{1+i}+\frac{2}{1-i}+\frac{2}{-1+i}+\frac{2}{-1-i}+...+\frac{n}{1+i}+\frac{n}{1-i}+\frac{n}{-1+i}+\frac{n}{-1-i}$

34. Find the product of the real parts of the root $z^2-z-5+5i=0$.

35. Find the number of complex numbers satisfying $z^3+\overline{z}=0$.

36. Find the number of real roots of the equation $z^3+iz-1=0$.

37. In the following diagram, if given circle is unit circle then find the reciprocal of point A.

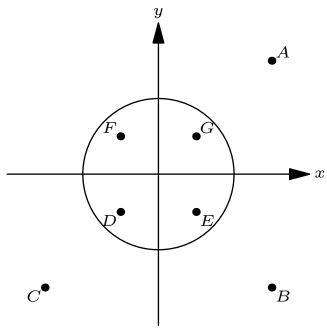


Figure 3.24

38. If $z = (3+7i)(p+iq)$, where $p, q \in I$, is purely imaginary, then find the minimum value of $|z|^2$.

39. If $\alpha = \left(\frac{a-ib}{a+ib}\right)^2 + \left(\frac{a+ib}{a-ib}\right)^2, \forall\, a, b \in R$, then prove that α is real.

40. If $\beta = \frac{z-1}{z+1}$ such that $|z| = 1$, then prove that β is imaginary.

41. If $|z-3i| = 3$ such the $\arg(z) \in \left(0, \frac{\pi}{2}\right)$, then find the value of $\cos(\arg(z)) - \frac{6}{z}$.

42. Find the polar form of the complex number $\frac{-16}{1+i\sqrt{3}}$

43. Let z and w be the two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$, then prove that $z = -\overline{w}$.

44. If $x - iy = \sqrt{\frac{a-ib}{c-id}}$, then prove that $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

45. Find the minimum value of $|z| + |z-2|$.

46. If $|z_1 - 1| < 1, |z_2 - 2| < 2$ and $|z_3 - 3| < 3$, then prove that the maximum value of $|z_1 + z_2 + z_3|$ is 12.

47. If α, β are two complex numbers, then prove that $|\alpha|^2 + |\beta|^2 = \frac{1}{2}(|\alpha+\beta|^2 + |\alpha-\beta|^2)$.

48. Show that for $z \in C$, $|z| = 0$, if and only if $z = 0$.

49. If z_1 and z_2 are $1-i$ and $2+7i$, then find $Im\left(\frac{z_1 z_2}{\overline{z_1}}\right)$.

50. If $|z-i| < 1$, then prove that $|z+12-6i| < 14$.

51. If $|z+6| = |2z+3|$, then prove that $|z| = 3$.

52. If $\sqrt{a-ib} = x - iy$, then prove that $\sqrt{a+ib} = x + iy$.

53. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^2}$, then find the value of $x_1 x_2 x_3 \ldots$ to ∞.

54. Find the value of $\frac{(\cos\theta+i\sin\theta)^4}{(\sin\theta+i\cos\theta)^2}$.

55. If $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then find $\Im(z)$.

56. Find the product of all values of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$.

57. If z_1 and z_2 are two non-zero complex numbers such that$|z_1+z_2|=|z_1|+|z_2|$, then find $\arg(z_1)-\arg(z_2)$.

58. If $z=1-\sin\alpha+i\cos\alpha$, where $\alpha\in\left(0,\frac{\pi}{2}\right)$, then find the modulus and principal value of the argument.

59. Find the value of expression $\left(\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right)^8$.

60. If $z_r=\cos\frac{2r\pi}{5}+i\sin\frac{2r\pi}{5}$, $r=0,1,2,3,4$, then find $z_1z_2z_3z_4z_5$.

61. If $z_n=\cos\frac{\pi}{(2n+1)(2n+3)}+i\sin\frac{\pi}{(2n+1)(2n+3)}$, then find $z_1z_2z_3\ldots\infty$.

62. If z_1, z_2 be two complex numbers and a, b are two real numbers, then prove that $|az_1-bz_2|^2+|bz_1+az_2|^2=(a^2+b^2)(|z_1|^2+|z_2|^2)$.

63. Show that the equation $\frac{A^2}{x-a}+\frac{B^2}{x-b}+\ldots+\frac{H^2}{x-h}=x+l$, where $A, B, \ldots H, a, b, \ldots, h$ and l are real, cannot have imaginary roots.

64. Find all real number x, such that $|1+4i-2^{-x}|\leq 5$.

65. Show that a unimodular complex number, not purely real can be expressed as $\frac{c+i}{c-i}$ for some real c.

66. If $(z^2+3)^2=-16$, then find $|z|$.

67. If $\frac{\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x}{1+2i\sin\frac{x}{2}}$ is real, then find the set of all possible values of x.

68. Prove that $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$.

69. If $x^2-x+1=0$, then find the value of $\sum_{n=1}^{5}\left(x^n+\frac{1}{x^n}\right)^5$.

70. If $3^{49}(x+iy)=\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)^{100}$, then find x and y.

71. For any two complex numbers z_1 and z_2, prove that $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\Re(z_1\overline{z_2})=|z_1|^2+|z_2|^2+2\Re(\overline{z_1}z_2)$.

72. If $|z_1|=|z_2|=1$, then prove that $|z_1+z_2|=\left|\frac{1}{z_1}+\frac{1}{z_2}\right|$.

73. If $|z-2| = 2|z-1|$, then prove that $|z|^2 = \frac{4}{3}\Re(z)$.

74. If $\sqrt[3]{a+ib} = x+iy$, then prove that $\frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2)$.

75. If $x+iy = \sqrt{\frac{a+ib}{c+id}}$, then prove that $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

76. If $z_1, z_2, \dots, z_n$ are cube roots of unity, then prove that $|z_k| = |z_{k+1}| \, \forall \, k \in [1, n-1]$.

77. If n is a positive integer greater than unity and z is a complex number satisfying the equation $z^n = (1+z)^2$, then prove that $\Re(z) < 0$.

78. Prove that $x^{3m} + x^{3n-1} + x^{3r-2} \; \forall \; m,n,r \in N$, is divisible by $1+x+x^2$.

79. If $(\sqrt{3}+i)^n = (\sqrt{3}-i)^n \;\; \forall \; n \in N$, then prove that minimum value of n is 6.

80. If $(\sqrt{3}-i)^n = 2^n, n \in I$, the set of integers, then prove that n is multiple of 12.

81. If $z^4 + z^3 + 2z^2 + z + 1 = 0$, then prove that $|z| = 1$.

82. If $z = \sqrt[7]{-1}$, then find the value of $z^{86} + z^{175} + z^{289}$.

83. If $z^3 + 2z^2 + 3z + 2 = 0$, then find all the non-real, complex roots of the equation.

84. If z is a non-real root of $z = \sqrt[5]{1}$, then find the value of $2^{|1+z+z^2+z^{-2}+z^{-1}|}$.

85. If z is a non-real root of unity, then find the value of $1+3z+5z^2+\dots+(2n-1)z^{n-1}$.

86. Find the value of $\sqrt{-1-\sqrt{-1-\sqrt{-1-\text{ to }\infty}}}$.

87. If $z = e^{\frac{i2\pi}{n}}$, then find the value of $(11-z)(11-z^2)\dots(11-z^{n-1})$.

88. If $\frac{3}{2+\cos\theta+i\sin\theta} = a+ib$, then prove that $a^2+b^2 = 4a-3$.

89. If $|2z-1| = |z-2|$, then prove that $|z| = 1$.

90. If x is real and $\frac{1-ix}{1+ix} = m+in$, then prove that $m^2+n^2 = 1$.

91. Find the general equation of the straigt line joining the points $z_1 = 1+i$ and $z_2 = 1-i$.

92. If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2 + 8z_3 = 0$, then prove that

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$$

93. Find the length of perpendicualr from $P(2-3i)$ to the line $(3+4i)z+(3-4i)\overline{z}+9 = 0$.

94. If a point z_1 is a reflection of a point z_2 through the line $z\overline{z} + \overline{b}z = c, b \neq 0$ in the argand plane, thne prove that $\overline{b}z_2 + b\overline{z_1} = c$.

95. The point represented by the complex number $2 - i$ is rotated by origin by an angle $\pi/2$ in the anti-clockwise direction. Find the new coordinates.

96. A particle P starts from the point $z_0 = 1 + 2i$. It first moves horizontally, away from origin by 5 units and then vertically, away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of vector $\hat{\imath} + \hat{\jmath}$ and it then rotaes about origin in anti-clockwise direction for an angle $\pi/2$ to reach z_2. Find the coordinates of z_2.

97. A man walks a distance of 3 units from the origin in North-East direction. Then he walks 4 units in North-West direction. Find the final coordinates.

98. If three complex numbers satisfty the relationship $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$, then prove that z_1, z_2 and z_3 form an equilateral triangle.

99. If z_1, z_2 and z_3 form an equilateral triangle then prove that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$, and hence $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

100. If z_1, z_2 and z_3 are vertices of an equilateral triangle and z_0 is the circumcenter then prove that $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

101. If z_1, z_2 and z_3 form a right-angled, isosceles triangle with right angle at z_3, then prove that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

102. Find the equation of the circle whose center is z_0 and radius is r.

103. If $z = 1 - t + i\sqrt{t^2 + t + 2}$, where t is a real parameter. Prove that locus of z in argand plane is a hyperbola.

104. Find the locus of z if $\overline{z} = \overline{a} + \frac{r^2}{z - a}$.

105. If the equation $|z - z_1|^2 + |z - z_2|^2 = k$ represents the equation of a cirlce, where $z_1 = 2 + 3i$, $z_2 = 4 + 3i$ are the ends of a diameter, then find the value of k.

106. If $|z + 1| = \sqrt{2}|z - 1|$, then show that locus of z is a circle.

107. Prove that the locus of z given by $\left|\frac{z-1}{z-i}\right| = 1$ is a straight line.

108. Find the condition for four complex numbers z_1, z_2, z_3 and z_4 to lie on a cyclic quadrilateral.

109. If z_1, z_2 and z_3 are complex numbers, such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, then show that these points lie on a circle passing through origin.

110. If $|z - \omega|^2 + |z - \omega^2|^2 = r^2$, where r is radius and ω, ω^2 are cube roots of unity and ends of diameter of the circle then find radius.

111. Find the region represented by $|z - 4| < |z - 2|$.

112. If $2z_1 - 3z_2 + z_3 = 0$, then find the geometrical relationship between them.

113. If $z = x + iy$, such that $|z+1| = |z-1|$ and $\arg \frac{z-1}{z+1} = \frac{\pi}{4}$, find x and y.

114. If $|z|^8 = |z-1|^8$, then prove that roots of this equation are collinear.

115. Prove that $z\overline{z} + a\overline{z} + \overline{a}z + b = 0$, represents a circle if $|a|^2 > b$.

116. If $z = (\lambda + 3) + i\sqrt{3 - \lambda^2}$, where $|\lambda| < \sqrt{3}$, then prove that it represents a circle.

117. If z is a complex number such that $|\Re(z)| + |\Im(z)| = k$, $\forall\ k \in R$, then find the locus of z.

118. Consider a sequence of complex numbers such that $z_{n+1} = z_n^2 + i$, $\forall\ n \geq 1$, where $z_1 = 0$. Find z_{111}.

119. The complex numbers whose real and imaginary parts are integers and satisfy the relation $z\overline{z}^3 + z^3\overline{z} = 350$, forms a rectangle in the argand plane. Find length of its diagonals.

120. If z_1, z_2 are two complex numbers and $\arg \frac{z_1+z_2}{z_1-z_2}$ but $|z_1 + z_2| \neq |z_1 - z_2|$ then find the figure formed by 0, z_1, z_2 and $z_1 + z_2$.

121. If z_1 and z_2 are complex numbers such that $a|z_1| = b|z_2|$, $a, b \in R$, then prove that $\frac{az_1}{bz_2} + \frac{bz_2}{az_1}$ lies on the segment $[-2, 2]$ of the real axis.

122. If z_1, z_2, z_3 are roots of the equation $z^3 + 3\alpha z^2 + 3\beta z + \gamma = 0$, such that they form an equilateral triangle then prove that $\alpha^2 = \beta$.

123. If $z_1^2 + z_2^2 + 2z_1z_2\cos\theta = 0$, then prove that z_1, z_1 and the origin form an isosceles triangle.

124. A, B and C represent z_1, z_2 and z_3 on argnad plane. The circumcenter of this triangle lies on the origin. If the altitude AD meets circumcircle again at P, then find the complex number representing P.

125. If z_1 and z_2 are the roots of the equation $z^2 + pz + q = 0$, where p,q can be complex numbers. Let A, B represent z_1, z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin then find p^2.

126. If $\Re\left(\frac{z+4}{2x-1}\right) = \frac{1}{2}$ then prove that locus of z is a straight line.

127. If z_1, z_2 and z_3 are vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If z_1, z_2, z_3 are in clockwise sense then find z_2 and z_3.

128. If $z_1 = \frac{a}{1-i}$, $z_2 = \frac{b}{2+i}$, $z_3 = a - bi$ for $a, b \in R$ and $z_1 - z_2 = 1$. Then find the centroid of the triangle formed by z_1, z_2 and z_3.

129. Let $\lambda \in R$. If the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$ form three vertices of an equilateral triangle in the argand plane, then find λ.

130. If a, b, c and u, v, w are complex numbers such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number then prove that the triangles are similar.

131. Find the intercept made by the circle $z\overline{z} + \overline{\alpha}z + \alpha\overline{z} + r = 0$ on real axis on the complex plane.

132. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then find the value of $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha)$.

133. Find the locus of the center of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally.

134. Prove that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$.

135. $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\Re(z_1\overline{z_2}) = 0$. Also, $w_1 = a + ic$, $w_2 = b + id$ then prove that $|w_1| = |w_2| = 1$ and $\Re(w_1\overline{w_2}) = 0$.

136. If $\left|\frac{z_1}{z_2}\right| = 1$ and $\arg(z_1z_2) = 0$, then prove that $|z_2|^2 = z_1z_2$.

137. Find the value of the expression $2\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right) + 3\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right) + 4\left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right) + \ldots + (n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$.

138. If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1+iz_2}{z_1-iz_2}\right| = 1$, then prove that $\frac{z_1}{z_2}$ is purely real.

139. If $z = -2 + 2\sqrt{3}i$, then find values of $z^{2n} + 2^{2n}z^n + 2^{4n}$.

140. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then find the values of $\frac{x}{y} + \frac{y}{x}$, $xy + \frac{1}{xy}$.

141. The complex numbers z_1 and z_2 such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, prove that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

142. If $A(z_1)$, $B(z_1)$ and $C(z_3)$ are the vertices of a $\triangle ABC$ in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then prove that the value of $z_2 = z_3 + i(z_1 - z_3)$.

143. If $z_1z_2 \in C$, $z_1^2 + z_2^2 \in R$, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then find the value of $z_1^2 + z_2^2$.

144. If $\sqrt{1-c^2} = nc - 1$ and $z = e^{i\theta}$, then find the value of $\frac{c}{2n}(1+nz)\left(1+\frac{n}{z}\right)$.

145. Consider an ellipse having its foci at $A(z_1)$ and $B(z_2)$ in the argand plane. If the eccentricity of the ellipse is e and it is known that origin is an interior point of the ellipse, then prove that $e \in \left(0, \frac{|z_1-z_2|}{|z_1|+|z_2|}\right)$

146. If $|z-2-i|=|z|\left|\sin\left(\frac{\pi}{4}-\arg(z)\right)\right|$, then find the locus of z.

147. Find the maximum area of the triangle formed by the complex coordinates zz_1 and z_2, which satisfy the relation $|z-z_1|=|z-z_2|$ and $\left|z-\frac{z_1+z_2}{2}\right|\leq r$, where $r>|z_1-z_2|$.

148. If $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ are complex numbers such that $|z_1|=1, |z_1|=2$ and $\Re(z_1z_2)=0$, and $\omega_1=a_1+\frac{ia_2}{2}$ and $\omega_2=2b_1+ib_2$, then prove that $|\omega_1|=1, |\omega_2|=2$ and $\Re(\omega_1\omega_2)=0$.

149. Let z be a complex number and a be a be a real number such that $z^2+az+a^2=0$, then prove that i) locus of z is a pair of straight lines ii) $\arg(z)=\pm\frac{2\pi}{3}$ iii) $|z|=|a|$

150. If $x+\frac{1}{x}=1$ and $p=x^{4000}+\frac{1}{x^{4000}}$ and q is the the digit at units place in $2^{2^n}+1, n\in N$ and $n>1$, then find $p+q$.

151. Consider an equilateral triangle $A\left(\frac{2}{\sqrt{3}}e^{i\pi/2}\right)$, $B\left(\frac{2}{\sqrt{3}}e^{-i\pi/6}\right)$ and $C\left(\frac{2}{\sqrt{3}}e^{-i5\pi/6}\right)$. If $P(z)$ is any point on the incircle then find the value of $AP^2+BP^2+CP^2$.

152. If $A_1, A_2, \ldots, A_n$ be the vertices of a regular polygon of n sides in a circle of unit radius and $a=|A_1A_2|^2+|A_1A_3|^2+\ldots+|A_1A_n|^2$, $b=|A_1A_2||A_1A_3|\ldots|A_1A_n|$, then find $\frac{a}{b}$.

153. If $\left(1+i\frac{x}{a}\right)\left(1+i\frac{x}{b}\right)\left(1+i\frac{x}{c}\right)\ldots=A+iB$, then prove that $\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)\ldots=A^2+B^2$.

154. Find the range of real number α for which the equations $z+\alpha|z-1|+2i=0; z=x+iy$ has a solution. Also, find the solution.

155. For every real number $a\geq 0$, find all the complex numbers satisfying the equation $2|z|-4az+1+ia=0$.

156. Show that $(x^2+y^2)^5=(x^5-10x^3y^2+5xy^4)+(5x^4y-10x^2y^3+y^5)^2$.

157. Express $(x^2+a^2)(x^2+b^2)(x^2+c^2)$ as sum of two squares.

158. If $(1+x)^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n$, then prove that $2^n=(a_0-a_2+a_4-\ldots)^2+(a_1-a_3+a_5-\ldots)^2$.

159. Dividing $f(z)$ by $z-i$, we get i as remainder and if we divide by $z+i$, we get $1+i$ as remainder. Find the remainder upon division of $f(z)$ by z^2+1.

160. If $|z|\leq 1, |w|\leq 1$, show that $|z-w|^2\leq(|z|-|w|)^2+[\arg(z)-\arg(w)]^2$.

161. If z is any complex number, then show that $\left|\frac{z}{|z|}-1\right|\leq|arg(z)|$.

162. If z is any complex number, then show that $|z-1|\leq||z|-1|+|z||argz|$.

163. If $\left|z+\frac{1}{z}\right|=a$, where z is a complex number and $a>0$, find the greatest and least values of $|z|$.

164. If z_1, z_2 be complex numebrs and c is a positive number, prove that $|z_1+z_2|^2<(1+c)|z_1|^2+\left(1+\frac{1}{c}\right)|z_2|^2$.

165. If z_1 and z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$, prove that $\frac{iz_1}{z_2}=x$ where x is a real number. Find the angle between the lines from origin to the points z_1+z_2 and z_1-z_2 in terms of x.

166. Let z_1, z_2 be any two complex numbers and a, b be two real numbers such that $a^2+b^2\neq 0$. Prove that $|z_1|^2+|z_2|^2-|z_1^2+z_2^2|\leq 2\frac{|az_1+bz_2|^2}{a^2+b^2}\leq |z_1|^2+|z_2|^2+|z_1^2+z_2^2|$.

167. If $b+ic=(1+a)z$ and $a^2+b^2+c^2=1$, prove that $\frac{a+ib}{1+c}=\frac{1+iz}{1-iz}$, where a, b,c are real numbers and z is a complex number.

168. If a, b, c, ..., k are all n real roots of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0$, where $p_1, p_2, \ldots, p_n$ are real, show that $(1+a^2)(1+b^2)\ldots(1+k^2)=(1-p_2+p_4+\ldots)^2+(p_1-p_3+\ldots)^2$.

169. If $f(x)=x^4-8x^3+4x^2+4x+39$ and $f(3+2i)=a+ib$, find $a:b$.

170. Let A and B be two complex numbers such that $\frac{A}{B}+\frac{B}{A}=1$, prove that the triangle formed by origin and these two points is equilateral.

171. If $n>1$, show that the roots of the equation $z^n=(1+z)^n$ are collinear.

172. If A,B,C and D are four complex number then show that $AD.BC\leq BD.CA+CD.AB$.

173. If a, $b\in R$ and a, $b\neq 0$, then show that the equation of line joining a and ib is $\left(\frac{1}{2a}-\frac{i}{2b}\right)z+\left(\frac{1}{2a}+\frac{i}{2b}\right)\overline{z}=1$.

174. If z_1 and z_2 are two compelx numbers such that $|z_1|-|z_2|=|z_1-z_2|$, then show that $\arg(z_1)-\arg(z_2)=2n\pi$ where $n\in I$.

175. Let A, B, C, D, E be points in the complex plane representing complex numbers z_1, z_2, z_3, z_4, z_5 respectvely. If $(z_3-z_2)z_4=(z_1-z_2)z_5$, prove that $\triangle ABC$ and $\triangle DOE$ are similar.

176. Let z and z_0 are two complex numbers and z, z_0, $z\overline{z_0}$, 1 are represented by points P, P_0, Q, A respectively. If $|z|=1$, show that the triangle POP_0 and AOQ are congruent and hence $|z-z_0|=|z\overline{z_0}-1|$, where O represents the origin.

177. If the line segment joining z_1 and z_2 is divided by P and Q in the ratio $a:b$ internally and externally, then find OP^2+OQ^2 where O is origin.

178. Let z_1, z_2, z_3 be three complex numbers and a, b, c be real numbers not all zero such that $a+b+c=0$ and $az_1+bz_2+cz_3=0$, then show that z_1, z_2, z_3 are collinear.

179. If $z_1 + z_2 + ... + z_n = 0$, prove that if a line passes through origin then all these do not lie of the same side of the line provided they do not lie on the line.

180. The points $z_1 = 9 + 12i$ and $z_2 = 6 - 8i$ are given on a complex plane. Find the equation of the angle formed by the vector representing z_1 and z_2.

181. If the vertices of a $\triangle ABC$ are represented by z_1, z_2, z_3 respectively, then show that the orthocenter of $\triangle ABC$ is $\frac{z_1 a \sec A + z_2 b \sec B + z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$ or

$\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$.

182. If the vertices of a $\triangle ABC$ are represented by z_1, z_2 and z_3 respectively, show that its circumcenter is $\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$.

183. Show that the circumcenter of the triangle whose vertices are given by the complex numbers z_1, z_2, z_3 is given by $z = \frac{\sum z_1 \overline{z_1}(z_2 - z_3)}{\sum \overline{z_1}(z_2 - z_3)}$.

184. Find the orthocenter of the triangle with vertices z_1, z_2, z_3.

185. $ABCD$ is a rhombus described in clockwise direction. Suppose that the vertices A, B, C, D are given by z_1, z_2, z_3, z_4 respectively and $\angle CBA = 2\pi/3$. Show that $2\sqrt{3} z_2 = (\sqrt{3} - i) z_1 + (\sqrt{3} + i) z_3$ and $2\sqrt{3} z_4 = (\sqrt{3} + i) z_1 + (\sqrt{3} - i) z_3$.

186. The points P, Q and R represent the numbers z_1, z_2 and z_3 respectively and the angles of the $\triangle PQR$ at Q and R are both $\frac{1}{2}(\pi - \alpha)$. Prove that $(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{\alpha}{2}$.

187. Points z_1 and z_2 are adjacent vertices of a regular polygon of n sides. Find the vertex z_3 adjacent to $z_2 (z_1 \neq z_3)$.

188. Let $A_1, A_2, \ldots, A_n$ be the vertices of an n sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, find the value of n.

189. If $|z| = 2$, then show that the points representing the complex numbers $-1 + 5z$ lie on a circle.

190. If $|z - 4 + 3i| \leq 2$, find the least and tghe greatest values of $|z|$ and hence find the limits between which $|z|$ lies.

191. If $z - 6 - 8i \leq 4$, then find the least and greatest value of z.

192. If $z - 25i \leq 15$ then find the least positive value of $\arg(z)$.

193. Show that the equation $|z - z_1|^2 + |z - z_2|^2 = k$ where $k \in R$ will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$.

194. If $|z - 1| = 1$, prove that $\frac{z-2}{z} = i \tan(\arg z)$.

195. Find the locus of z if $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$

196. If α is real and z is a complex number and u and v be the real and imaginary parts of $(z-1)(\cos\alpha - i\sin\alpha) + (z-1)^{-1}(\cos\alpha + i\sin\alpha)$, prove that the locus of points representing the complex number such that $v = 0$ is a circle of unit radius with center at point $(1, 0)$ and a straight line through the center of the circle.

197. If $|a_n| < 2$ for $n = 1, 2, 3, ...$ and $1 + a_1 z + a_2 z^2 + ... + a_n z^n = 0$, show that z does not lie in the interior of the circle $|z| = \frac{1}{3}$.

198. Show that the roots of the equation $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + ... + \cos\theta_n = 2$, where $\theta_1 + \theta_2 + ... + \theta_n \in R$ lies outside the circle $|z| = \frac{1}{2}$.

199. z_1, z_2, z_3 are non-zero, non-collinear complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, show that z_1, z_2, z_3 lie on a circle passing through origin.

200. A, B, C are the points representing the complex numbers z_1,z_2,z_3 respectively on the complex plane and the circumcenter of the $\triangle ABC$ lies on the origin. If the altitude of the triangle through the vertex A meets the circle again at P, prove that P represents the complex number $\frac{z_2 z_3}{z_1}$.

201. Two different non-parallel lines cut the circle $|z| = r$ at points a, b, c, d respectively. Prove that these two lines meet at a point given by $\frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$.

202. Let z_1, z_2, z_3 be three non-zero complex numbners such that $z_2 \neq 1$, $a = |z_1|$, $b = |z_2|$ and $c = |z_3|$. If $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ then show that $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3-z_1}{z_2-z_1}\right)^2$.

203. P is a point on a circle with OP as diameter. Two points Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q and R are represented by the complex numbers z_1, z_2 and z_3 respectively, show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$.

204. Find the equation in complex variables of all circles which are orthogonal to $|z| = 1$ and $|z-1| = 4$.

205. Find the real values of the parameter t for which there is at least one complex number $z = x + iy$ satisfying the condition $|z+3| = t^2 - 2i + 6$ and the ineuqality $z - 3\sqrt{3}i < t^2$.

206. If a, b, c and d are real and $ad > bc$, show that the imaginary parts of the complex number z and $\frac{az+b}{ca+d}$ have the same sign.

207. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_1 = \frac{i(z_2+1)}{z_2-1}$, prove that $x_1^2 + y_1^2 - x_1 = \frac{x_2^2-y_2^2+2x_2-2y_2+1}{(x_2-1)^2+y_2^2}$.

208. Simplify $\frac{(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3}{(\cos 2\theta + i\sin 2\theta)^5}$.

209. Find all complex numbers such that $z^2 + |z| = 0$.

210. Solve the equation $z^2 + z|z| + |z^2| = 0$.

211. If $a > 0$ and $z|z| + az + 1 = 0$, show that z is a negative real number.

212. For every real number $a > 0$, find all complex numbers z such that $|z|^2 - 2iz + 2a(1+i) = 0$.

213. Find the integral solution of the following equations: i. $(3+4i)^x = 5^{x/2}$ ii. $(1-x)^x = 2^x$ iii. $(1-i)^x = (1+i)^x$.

214. Find the common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$.

215. If $z_1 + z_2 + z_3 = \alpha$, $z_1 + z_2\omega + z_3\omega^2 = \beta$ and $z_1 + z_2\omega^2 + z_3\omega = \gamma$, express z_1, z_2, z_3 in terms of α, β, γ. Hence prove that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$.

216. If n is an odd integer greater than 3, but not a multiple of 3, prove that $x^3 + x^2 + x$ is a factor of $(x+1)^n - x^n - 1$.

217. If n is an odd integer greater than 3, but not a multiple of 3, prove that $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y)(x^2 + xy + y^2)$.

218. If $|z_1| = |z_1| = \cdots = |z_n| = 1$, prove that $|z_1 + z_2 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$.

219. If $\alpha, \beta \in \mathbb{C}$, show that $|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$.

220. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\Re(z_1\overline{z_2}) = 0$, then show that the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfy i. $|\omega_1| = 1$ ii. $|\omega_2| = 1$ iii. $\Re(\omega_1\overline{\omega_2}) = 0$.

221. Prove that $\left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| < 1$ if $|z_1| < 1, |z_2| < 1$.

222. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\frac{z - z_1}{z - z_2}$ is $\frac{\pi}{2}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

223. Find all complex numbers z for which $\arg\left(\frac{3z - 6 - 3i}{2z - 8 - 6i}\right) = \frac{\pi}{4}$ and $|z - 3 + i| = 3$.

224. If $|z| \le 1, |w| \le 1$, show that $|z - w|^2 \le (|z| - |w|)^2 + (\arg(z) - \arg(w))^2$.

225. If z is any non-zero complex number, show that $\left|\frac{z}{|z|} - 1\right| \le |\arg(z)|$ and $|z - 1| \le ||z| - 1| + |z||\arg(z)|$.

226. If $\left|z + \frac{1}{z}\right| = a$, where z is a complex number and $a > 0$, find the greatest value of $|z|$.

227. If z_1, z_2 are complex numbers and c is a positive number, prove that $|z_1 + z_2|^2 < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$.

228. If z_1 and z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1$, prove that $\frac{iz_1}{z_2} = x$, where x is a real number. Find the angle between the lines from the origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of x.

229. Let z_1, z_2 be any two complex numbers and a, b be two real numbers such that $a^2 + b^2 \neq 0$. Prove that $|z_1|^2 + |z_2|^2 - |z_1^2 + z_2^2| \leq 2\frac{|az_1+bz_2|^2}{a^2+b^2} \leq |z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2|$.

230. If $b + ic = (1 + a)z$ and $a^2 + b^2 + c^2 = 1$, prove that $\frac{a+ib}{1+c} = \frac{1+iz}{1-iz}$, where a, b, c are real numbers and z is a complex number.

231. For any two complex numbers z_1 and z_2 and any real numbers a and b, show that $|az_1 - bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$.

232. If α and β are any two complex numbers, show that $|\alpha + \beta|^2 = |\alpha|^2 + |\beta|^2 + \mathfrak{R}(\alpha\overline{\beta}) + \mathfrak{R}(\overline{\alpha}\beta)$.

233. Prove that $|1 - \overline{z_1}z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$.

234. If $a_i, b_i \in R$, $i = 1, 2, \ldots, n$, show that $\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq \left(\sum_{n=1}^{n} \sqrt{a_i^2 + b_i^2}\right)^2$.

235. Let $\left|\frac{\overline{z_1}-2\overline{z_2}}{2-z_1\overline{z_2}}\right| = 1$ and $|z_2| \neq 1$, where z_1 and z_2 are complex nubers, show that $|z_1| = 2$.

236. If z_1 and z_2 are complex numbers and $u = \sqrt{z_1z_2}$, prove that $|z_1| + |z_2| = \left|\frac{z_1+z_2}{2} + u\right| + \left|\frac{z_1+z_2}{2} - u\right|$

237. If z_1 and z_2 are roots of the equation $\alpha z^2 + 2\beta z + \gamma = 0$, then prove that $|\alpha|(|z_1| + |z_2|) = |\beta + \sqrt{\alpha\gamma}| + |\beta - \sqrt{\alpha\gamma}|$

238. If a, b, c are complex numbers such that $a + b + c = 0$ and $|a| = |b| = |c| = 1$, find the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

239. If $|z + 4| \leq 3$, find the least and greatest value of $|z + 1|$.

240. Show that for any two non-zero complex numbers z_1 and z_2, $(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right| \leq 2|z_1 + z_2|$

241. Show that the necessary and sufficient condition for both the roots of the equation $z^2 + az + b = 0$ to be unimodular are $|a| \leq 2$, $|b| = 1$ and $\arg(b) = 2\arg(a)$.

242. If z is a complex number, show that $|z| \leq |\mathfrak{R}(z)| + |\mathfrak{I}(z)| \leq \sqrt{2}|z|$.

243. If $\left|z - \frac{4}{z}\right| = 2$, show that the greatest value of $|z|$ is $\sqrt{5} + 1$.

244. If $\alpha, \beta, \gamma, \delta$ be the real roots of the equation $ax^4 + bx^3 + cs^2 + dx + e = 0$, show that $a^2(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2) = (a - c + e)^2 + (b - d)^2$.

245. If $a_i \in R, i = 1, 2, \ldots, n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, show that $\prod_{i=1}^{n}(1 + \alpha_i^2) = (1 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + \ldots)^2$

246. If the complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, prove that $z_1 + z_2 + z_3 = 0$.

247. If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then prove that the complex numbers z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in a unit circle.

248. If z_1, z_2, z_3 be the vertices of an equilateral triangle whose circumcenter is z_0, then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

249. Prove that the complex numbers z_1 and z_2 and the origin form an equilateral triangle if $z_1^2 + z_2^2 - z_1z_2 = 0$.

250. If z_1 and z_2 be the roots of the equation $z^2 + az + b = 0$, then prove that the origin, z_1 and z_2 form an equilateral triangle if $a^2 = 3b$.

251. Let z_1, z_2 and z_3 be the roots of the equation $z^3 + 3\alpha z^2 + 3\beta z + \gamma = 0$, where α, β and γ are complex numbers and that these represent the vertices of A, B and C of a triangle. Find the centroid of $\triangle ABC$. Show that the triangle will be equilateral, if $\alpha^2 = \beta$.

252. If z_1, z_2, z_3 are in A.P., prove that they are collinear.

253. If z_1, z_2 and z_3 are collinear points in argand plane then show that one of the following holds: $-z_1|z_2 - z_3| + z_2|z_3 - z_1| + z_3|z_1 - z_2| = 0, z_1|z_2 - z_3| - z_2|z_3 - z_1| + z_3|z_1 - z_2| = 0, z_1|z_2 - z_3| + z_2|z_3 - z_1| - z_3|z_1 - z_2| = 0$.

254. What region in the argand plane is represented by the inequality $1 < |z - 3 - 4i| < 2$.

255. Find the locus of point z if $|z - 1| + |z + 1| \leq 4$.

256. If $z = t + 5 + i\sqrt{4 - t^2}$ and t is real, find the locus of z.

257. If $\frac{z^2}{z-1}$ is real, show that locus of z is a circle with center $(1, 0)$ and radius unity.

258. If $|z^2 - 1| = |z|^2 + 1$, show that locus of z is a straight line.

259. Find the locus of the point z if $\frac{\pi}{3} \leq \arg(z) \leq \frac{3\pi}{2}$.

260. Find the locus of the point z if $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$.

261. Show that the locus of the point z satisfying the condition $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ is the semicircle above x-axis, whose diameter is the joints of the points $(-1, 0)$ and $(1, 0)$ excluding these points.

262. Find the locus of the point z if $\log_{\sqrt{3}} \frac{|z|^2-|z|+1}{2+|z|} < 2$.

263. If O be the center of the circle circumscribing the equilateral $\triangle ABC$ and its radius be unity and A lies on the x-axis. Find the complex numbers represented by B and C.

264. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at a point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1+i$ and $2-i$ respectively, then find the complex number represented by A.

265. If z_1, z_2, z_3 and z_4 are the vertices of a square taken in anticlockwise order, prove that $z_3 = -iz_1 + (1+i)z_2$ and $z_4 = (1-i)z_1 + iz_2$.

266. Let z_1, z_2 and z_3 are vertices of an equilateral triangle in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3.

267. If a and b are real numbers between 0 and 1 such that points $z_1 = a + i$, $z_2 = 1 + bi$, and $z_3 = 0$ form an equilateral triangle, then find a and b.

268. Let $ABCD$ be a square described in the anticlockwise sense in the argand plane. If A represents $3 + 5i$ and the center of the square represents $\frac{7}{2} + \frac{5}{2}i$. Find the numbers represented by B, C and D.

269. Find the vertices of a regular polygon of n sides, if its center is located at origin and one of its vertices is z_1.

270. Prove that the points $a(\cos\alpha + i\sin\alpha)$, $b(\cos\beta + i\sin\beta)$ and $c(\cos\gamma + i\sin\gamma)$ in the argand plane are collinear, if $bc\sin(\beta-\gamma) + ca\sin(\gamma-\alpha) + ab\sin(\alpha-\beta) = 0$.

271. A represents the number $6i$, B the number 3 and P the complex number z. If P moves such that $PA : PB = 2 : 1$, show that $z\overline{z} = (4+2i)z + (4-2i)\overline{z}$. Also, show that the locus of P is a circle, find its radius and center.

272. Show that if the points z_1, z_2, z_3 and z_4 taken in order are concyclic, then the expression $\frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}$ is purely real.

273. Let z_1, z_2, z_3 and z_4 be the vertices of a quadrilateral. Prove that the quadrilateral is cyclic if $z_1z_2 + z_3z_4 = 0$ and $z_1 + z_2 = 0$.

274. Show that the triangles whose vertices are z_1, z_2, z_3 and z_1', z_2', z_3' are similar if

$$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$$

275. If a, b, c and u, v, w are the complex numbers representing two triangles such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number, prove that the two triangles are similar.

276. Find the equation of perpendicular bisector of the line segment joining points z_1 and z_2.

277. Find the equation of a circle having the line segment joining z_1 and z_2 as diameter.

278. If $\left|\frac{z-z_1}{z-z_2}\right| = c, c \neq 0$, then show that locus of z is a circle.

279. If $|z| = 1$, find the locus of the point $\frac{2}{z}$.

280. If for any two complex numbers z_1 and z_2, $|z_1 + z_2| = |z_1| + |z_2|$, prove that $\arg(z_1) - \arg(z_2) = 2n\pi$.

281. Find the complex number z, the least in absolute value, which satisfies the condition $|z - 2 + 2i| = 1$.

282. Find the point in the first quadrant, on the curve $|z - 5i| = 3$, whose argument is minimum.

283. Find the set of points of the cooradinate plane, which satisfy the inequality

$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1$

284. Find the set of all points on the xy-plane whose coordinates satisfy the following condition: the number $z^2 + z + 1$ is real and positive.

285. Find the real values of the parameter a for which at least one complex number z satisfies the equality $|z - az| = a + 4$ and the inequality $|z - 1| < 1$.

286. Find the real values of the parameter t for whihc at least one complex number z satisfied the equality $|z + \sqrt{2}| = t^2 - 3t + 2$ and the inequality $|z + i\sqrt{2}| < t^2$.

287. Find the real value of a for which there is at least one complex number satisfying $|z + 4i| = \sqrt{a^2 - 12a + 28}$ and $|z - 4\sqrt{3}| < 1$.

288. Find the set of points belonging to the coordinate plane xy, for which the real part of the complex number $(1 + i)z^2$ is positive.

289. Solve the equation $2z = |z| + 2i$ in complex numbers.

290. Three points represented by the complex numbers a, b, c lie on a circle with center O and radius r. The tangent at c cuts the chord joining the points a, b at z. Show that $z = \frac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}$.

291. Show that all roots of the equation $a_1z^3 + a_2z^2 + a_3z + a_4 = 3$, where $|a_i| \leq 1, i = 1, 2, 3, 4$ lie outside the circle with center as origin and radius $\frac{2}{3}$.

292. Given that $\sum_{i=1}^{n} b_i = 0$ and $\sum_{i=1}^{n} b_i z_i = 0$, where b_is are non-zero real numbers, no three of z_i's form a straight line. Prove that z_i's are concyclic if $b_1b_2|z_1 - z_2|^2 = b_3b_4|z_3 - z_4|^2$.

293. A cubic equation $f(x) = 0$ has one real root α and two complex roots $\beta \pm i\gamma$. Points A, B and C represent these roots. Show that the roots of the derived equation $f'(x) = 0$ are complex if A falls inside one of the two equilateral triangles described on base BC.

294. Prove that the reflection of $\overline{a}z + a\overline{z} = 0$ in the real axis is $\overline{az} + az = 0$.

295. If $\alpha, \beta, \gamma, \delta$ are four complex numbers such that $\frac{\gamma}{\delta}$ is real and $\alpha\delta - \beta\gamma \neq 0$, then prove that $z = \frac{\alpha+\beta t}{\gamma+\delta t}, t \in R$ represents a straight line.

296. If ω, ω^2 are cube roots of unity, then prove that

   i. $(3 + 3\omega + 5\omega^2)^6 - (2 + 6\omega + 2\omega^2)^3 = 0$.

   ii. $(2 - \omega)(2 - \omega^2)(2 - \omega^{10})(2 - \omega^{11}) = 49$.

   iii. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$.

   iv. $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$.

   v. $1 + \omega^n + \omega^{2n} = 3$, where $n > 0, n \in I$ and is a multiple of 3.

   vi. $1 + \omega^n + \omega^{2n} = 0$, where $n > 0, n \in I$ and is not a multiple of 3.

297. Resolve into linear factors $a^2 + b^2 + c^2 - ab - bc - ca$.

298. If $x = a + b, y = a\omega + b\omega^2, z = a\omega^2 + b\omega$, prove that $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ and $xyz = a^3 + b^3$.

299. Resolve into linear factors:

   i. $a^2 - ab + b^2$

   ii. $a^2 + ab + b^2$

   iii. $a^3 + b^3$

   iv. $a^3 - b^3$

   v. $a^3 + b^3 + c^3 - 3abc$

300. Show that $x^{3p} + x^{3q+1} + x^{3r+2}$, where p, q, r are positive integers is divisible by $x^2 + x + 1$.

301. Show that $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$, where p, q, r, s are positive integers is divisible by $x^3 + x^2 + x + 1$.

302. If $p = a + b + c, q = a + b\omega + c\omega^2, r = a + b\omega^2 + c\omega$, where ω is a cube root of unity, prove that $p^3 + q^3 + r^3 - 3pqr = 27abc$.

303. If ω is a cube root of unity, prove that $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a - b - c)(2b - a - c)(2c - a - b)$.

304. If $ax + cy + bz = X, cx + by + az = Y, bc + ay + cz = Z$, show that

i. $(a^2+b^2+c^2-ab-bc-ca)(x^2+y^2+z^2-xy-yz-zx)=X^2+Y^2+Z^2-XY-YZ-ZX$

ii. $(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)=X^3+Y^3+Z^3-3XYZ$

305. Prove that $\left(\frac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4=\cos 8\theta+i\sin 8\theta$.

306. If $z^2-2z\cos\theta+1=0$, show that $z^2+z^{-2}=2\cos 2\theta$.

307. Prove that $(1+i)^n+(1-i)^n=2^{n/2+1}\cos\frac{n\pi}{4}$.

308. Show that the value of $\displaystyle\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$ is i.

309. Show that $e^{2mi\cot^{-1}p}\left(\frac{pi+1}{pi-1}\right)^m=1$.

310. Prove that $\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right)$.

311. If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, show that $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$ and $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$.

312. If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, show that $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$.

313. If α,β are the roots of the equation $t^2-2t+2=0$, show that a value of x, satisfying $\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{\sin\theta}{\sin^n\theta}$ is $x=\cot\theta-1$.

314. If $(1+x)^n=p_0+p_1x+p_2x^2+\ldots+p_nx^n$, show that $p_0-p_2+p_4-\ldots=2^{n/2}\cos\frac{n\pi}{4}$ and $p_1-p_3+p_5-\ldots=2^{n/2}\sin\frac{n\pi}{4}$.

315. If $(1-x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, show that $a_0+a_3+a_6+\ldots=\frac{1}{3}\left(1+2^{n+1}\cos\frac{n\pi}{3}\right)$.

316. If n is a positive integer and $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$, show that $c_0+c_4+c_8+\ldots=2^{n-2}+2^{n/2-1}\cos\frac{n\pi}{4}$.

317. Solve the equation $z^8+1=0$ and deduce that $\cos 4\theta=8\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)$.

318. Prove that the roots of the equation $8x^3-4x^2-4x+1=0$ are $\cos\frac{\pi}{7},\cos\frac{3\pi}{7},\cos\frac{5\pi}{7}$.

319. Solve the equation $z^{10}-1=0$ and deduce that $\sin 5\theta=5\sin\theta\left(1-\frac{\sin\theta}{\sin^2\frac{\pi}{5}}\right)\left(1-\frac{\sin\theta}{\sin^2\frac{2\pi}{5}}\right)$.

320. Solve the equation $x^7+1=0$ and deduce that $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\frac{1}{8}$.

321. Form the equation whose roots are $\cot^2 \frac{\pi}{2n+1}, \cot^2 \frac{2\pi}{2n+1}, \ldots, \cot^2 \frac{n\pi}{2n+1}$, and hence find the value of $\cot^2 \frac{\pi}{2n+1} + \cot^2 \frac{2\pi}{2n+1} + \ldots + \cot^2 \frac{n\pi}{2n+1}$.

322. If $\theta \neq k\pi$, show that $\cos\theta \sin\theta + \cos^2\theta \sin 2\theta + \ldots + \cos^n\theta \sin n\theta = \cot\theta(1 - \cos^n\theta \cos n\theta)$.

323. Show that $-3 - 4i = 5e^{i(\pi + \tan^{-1} 4/3)}$.

324. Solve the equation $2\sqrt{2}x^4 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$.

325. If $z_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, prove that $z_1 z_2 z_3 \ldots$ to $\infty = i$.

326. If $\cos\theta + i\sin\theta$ is a solution of the equation $p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$, prove that $p_1 \sin\theta + p_2 \sin 2\theta + \cdots + p_n = 0$ and $p_0 + p_2 \cos\theta + \cdots + p_n \cos n\theta = 0$, $p_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots, n$.

327. Show that $\left(\frac{1+\cos\phi + i\sin\phi}{1+\cos\phi - i\sin\phi}\right)^n = \cos n\phi + i\sin\phi$.

328. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then prove that

   i. $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \phi)$,

   ii. $xy + \frac{1}{xy} = 2\cos(\theta + \phi)$,

   iii. $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$, and

   iv. $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$.

329. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}$.

330. Find the equation whose roots are nth powers of the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$.

331. Find the values of A and B, where $Ae^{2i\theta} + Be^{-2i\theta} = 5\cos 2\theta - 7\sin 2\theta$.

332. If $x = \cos\theta + i\sin\theta$ and $\sqrt{1 - c^2} = nc - 1$, prove that $1 + c\cos\theta = \frac{c}{2n}(1 + nx)(1 + \frac{n}{x})$.

333. Show that the roots of equation $(1+z)^n = (1-z)^n$ are $i\tan\frac{r\pi}{n}$, $r = 0, 1, 2, \ldots, (n-1)$ excluding the value when n is even and $r = \frac{n}{2}$.

334. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, show that $\frac{(x+y)(xy-1)}{(x-y)(xy+1)} = \frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta}$

335. Show that $C_0^n + C_3^n + C_6^n + \cdots = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$.

336. Show that $C_1^n + C_4^n + C_7^n + \cdots = \frac{1}{3}\left[2^{n-2} + 2\cos\frac{(n-2)\pi}{3}\right]$.

337. Show that $C_2^n + C_5^n + C_8^n + \cdots = \frac{1}{3}\left[2^{n+2} + 2\cos\frac{(n+2)\pi}{3}\right]$.

338. If C_r stands for C_r^{4n}, prove that $C_0 + C_4 + C_8 + \cdots = 2^{4n-2} + (-1)^n 2^{2n-1}$.

339. If $(1-x+x^2)^{6n} = a_0 + a_1x + a_2x^2 + \cdots$, show that $a_0 + a_3 + a_6 + \ldots = \frac{1}{3}(2^{6n+1} + 1)$.

340. If $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots$, show that $a_0 + a_3 + a_6 + \ldots = \frac{1}{3}\left(1 + (-1)^n 2^{n+1}\cos\frac{n\pi}{3}\right)$.

341. Let $A = x+y+z$, $A' = x'+y'+z'$, $AA' = x''+y''+z''$, $B = x+y\omega+z\omega^2$, $B' = x'+y'\omega+z'\omega^2$, $BB' = x''+y''\omega+z''\omega^2$, $C = x+y\omega^2+z\omega$, $C' = x'y'\omega^2+z'\omega$, $CC' = x''+y''\omega^2+z''\omega$, then find x'', y'' and z'' in terms of x, y, z and $x', y'z'$.

342. Prove the equaity $(ax-by-cz-dt)^2 + (bx+ay-dz+ct)^2 + (cx+dy+az-bt)^2 + (dx-cy+bz+at)^2 = (a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2)$.

343. Prove the equality: $\frac{\cos n\theta}{\cos^n\theta} = 1 - C_2^n\tan^2\theta + C_4^n\tan^4\theta - \ldots + A$, where $A = (-1)^{n/2}\tan^n\theta$ if n is even, $A = (-1)^{(n-1)/2}.C_{n-1}^n\tan^n\theta$ if n is odd.

344. Prove the equality: $\frac{\sin n\theta}{\cos^n\theta} =^n C_1\tan\theta -^n C_3\tan^3\theta +^n C_5\tan^5\theta - \ldots + A$, where $A = (-1)^{(n-2)/2}.^nC_{n-1}\tan^{n-1}\theta$ if n is odd, $A = (-1)^{n/2}.\tan^n\theta$ if n is odd.

345. Prove the following equality:

$$2^{2m}\cos^{2m}x = \sum_{k=0}^{m-1} 2\binom{2m}{k}\cos 2(m-k)x + \binom{2m}{m}$$

346. Prove the following equality:

$$2^{2m}\sin^{2m}x = \sum_{k=0}^{m-1}(-1)^{m+k}2\binom{2m}{k}\cos 2(m-k)x + \binom{2m}{m}$$

347. Prove the following equality:

$$2^{2m}\cos^{2m+1}x = \sum_{k=0}^{m} 2\binom{2m+1}{k}\cos(2m-2k+1)x$$

348. Prove the following equality:

$$2^{2m}\sin^{2m+1}x = \sum_{k=0}^{m}(-1)^{m+k}2\binom{2m+1}{k}\cos(2m-2k+1)x$$

349. Let $u_n = \cos\alpha + r\cos(\alpha+\theta) + r^2\cos(\alpha+2\theta) + \ldots + r^n\cos(\alpha+n\theta)$, $v_n = \sin\alpha + r\sin(\alpha+\theta) + r^2\sin(\alpha+2\theta) + \ldots + r^n\sin(\alpha+n\theta)$, then show that

$$u_n = \frac{\cos\alpha - r\cos(\alpha-\theta) - r^{n+1}\cos[\alpha+(n+1)\theta] + r^{n+2}\cos(\alpha+n\theta)}{1-2r\cos\theta+r^2},$$

$$v_n = \frac{\sin\alpha - r\sin(\alpha-\theta) - r^{n+1}\sin[\alpha+(n+1)\theta] + r^{n+2}\sin(\alpha+n\theta)}{1-2r\cos\theta+r^2}$$

350. Simplify the following sums:

$$S = 1 + n\cos\theta + \frac{n(n-1)}{1.2}\cos 2\theta + \dots = \sum_{k=0}^{n} {}^nC_k\cos k\theta,\ [{}^nC_0 = 1]$$

$$S' = 1 + n\sin\theta + \frac{n(n-1)}{1.2}\sin 2\theta + \dots = \sum_{k=0}^{n} {}^nC_k\sin k\theta,\ [{}^nC_0 = 1]$$

351. If $\alpha = \frac{\pi}{2n}$ and $p < 2n$(p a popsitive integer), then prove that

$$\sin^{2p}\alpha + \sin^{2p}2\alpha + \dots + \sin^{2p}n\alpha = \frac{1}{2} + n\frac{1.3.5.\dots(2p-1)}{2.4.\dots 2p}$$

352. Prove that $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y)(x^2+xy+y^2)$ if n is an odd number and not divisible by 3.

353. Prove that $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y)(x^2+xy+y^2)^2$ if n, when divided by 6 has a remainder of 1.

354. Prove that the polynomial $(\cos\theta + x\sin\theta)^n - \cos n\theta - x\sin n\theta$ is divisible by x^2+1.

355. Prove that the polynomial $x^n\sin\theta - p^{n-1}x\sin n\theta + p^n\sin(n-1)\theta$ is divisible by $x^2 - 2px\cos\theta + p^2$.

356. Find out for what values of p and q the binomial x^4+1 is divisible by x^2+px+q.

357. Find the sum of the pth($p \in \mathbb{P}$) power of the roots of the equation $x^n = 1$.

358. Let $\epsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, $\forall\ n \in P$, and let $A_k = x + y\epsilon^k + z\epsilon^{2k} + \cdots + w\epsilon^{(n-1)k}$, $(k = 0, 1, 2, \dots, n-1)$ where $x, y, z, \dots, w$ are n arbitrary complex numbers. Prove that

$$\sum_{k=0}^{n-1}|A_k|^2 = n(|x|^2 + |y|^2 + \dots + |w|^2)$$

359. Prove the identity $x^{2n} - 1 = (x^2-1)\prod_{k=1}^{n-1}\left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right)$.

360. Prove the identity $x^{2n+1} - 1 = (x-1)\prod_{k=1}^{n}\left(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1\right)$.

361. Prove the identity $x^{2n+1} + 1 = (x+1)\prod_{k=1}^{n}\left(x^2 + 2x\cos\frac{2k\pi}{2n+1} + 1\right)$.

362. Prove the identity $x^{2n} + 1 = \prod_{k=0}^{n-1}\left(x^2 - 2x\cos\frac{(2k+1)\pi}{2n} + 1\right)$.

363. If n is even, then prove the identity $\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}}$.

364. If n is even, then prove the identity $\cos\frac{2\pi}{2n+1}\cos\frac{4\pi}{2n+1}\ldots\cos\frac{2n\pi}{2n+1}=\frac{(-1)^{n/2}}{2^n}$.

365. Prove that if $\cos\alpha+i\sin\alpha$ is the solution of the equation $x^n+p_1x^{n-1}+\cdots+p_n=0$, then $p_1\sin\alpha+p_2\sin 2\alpha+\cdots+p_n\sin n\alpha=0(p1, p2, \ldots, p_n$ are real).

366. Prove the identity $\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{1}{2}(5-3\sqrt[3]{7})}$.

367. Prove the identity $\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{1}{2}(3\sqrt[3]{9}-6)}$.

368. Let $A=x_1+x_2\omega+x_3\omega^2$, $B=x_1+x_2\omega^2+x_3\omega$, where ω,ω^2 are complex roots of unity and x_1,x_2,x_3 are roots of the cubic equation $x^3+px+q=0$. Prove that A^3 and B^3 are the roots of the quadratic equation $x^2+27qx-27p^3=0$.

369. Solve the equation $\frac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+1)(a^4+10a^2+5)}=ax$.

370. Find the magnitude of the sum $S=C_1^n-3C_3^n+3^2C_5^n-3^3C_7^n+\cdots$.

371. Find the magnitude of the follwing sums:

$$\sigma=1-C_2^n+C_4^n-C_6^n+\cdots$$

$$\sigma'=C_1^n-C_3^n+C_5^n-C_7^n+\cdots$$

Chapter 4

Polynomials and Theory of Equations

4.1 Polynomial Functions

A function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is callled a polynomial function where $a_i \in \mathbb{C}$, where $i = 0, 1, 2, \ldots, n$ i.e. $i \geq 0$ and $i \in \mathbb{I}$. Since $a_i \in \mathbb{C}$, it is evident that $a_i \in \mathbb{R}$ because $\mathbb{R} \subset \mathbb{C}$. This equation will be called an equation of degree n if and only if $a_n \neq 0$. a_n is called *leading* coefficient of the polynomial. If the leading coefficient is 1 then the polynomial is also callled *monic* polynomial. A polynomial with one term is called *monomial*, with two terms, a *binomial* and with three terms it is called a *trinomial*. The most useful trinomials are quadratic equations, which we will study further in this chapter. If $f(x) = a_0$, then it is called a *constant* polynomial. If $n = 0$ implies $f(x) = a_0$, which will be a polynomial of degree 0. If $f(x) = 0$, then it is callled *zero* polynomial, in this case the degree is defined as $-\infty$ to satisfy the first two properties given below. We take domain and range of these polynimoials or functions as set of complex numbers, $\mathbb{C}$. A real number r or a complex number z, for which $f(r) = 0$ or $f(z) = 0$, then r and z are called *zeros, roots* or *solutions* of the polynomial.

If $f(x)$ is a polynomial of degree p, and $g(x)$ is a polynomial of degree q, then

1. $f(x) \pm g(x)$ is a polymnoial of degree $\max(p, q)$,
2. $f(x).g(x)$ is a polynomial of degree $p + q$, and
3. $f(g(x))$ is a polynomial of degree $p.q$, where $g(x)$ is not a constant polynomial.

The $f(x)$ shown at the beginning is a polynomial in one variable, and similarly, we can have polynomials in $2, 3, \ldots, m$ variables. The domain of such a polynomial of m variables is set of ordered m tuple of complex numbers and range is $\mathbb{C}$.

4.2 Division of Polynomials

If $P(x)$ and $D(x)$ are any two polynomials such that $D(x) \not\equiv 0$, then two unique polynomilas $Q(x)$ and $R(x)$ can be found such that $P(x) = D(x).Q(x) + R(x)$. Here, the degree of $R(x)$ would be less than the degree of $D(x)$ or $R(x) \equiv 0$. Like numbers $Q(x)$ denotes the quotient, and is called so, while $R(x)$ is called the remainder.

Particulalrly, if $P(x)$ is a polynomial with complex coefficients and z is a complex number, then a polynomial $Q(x)$ of degree 1 less than $P(x)$ will exist such that $P(x) = (x - z)Q(x) + R$, where R is a complex number.

4.3 Remainder Theorem

Theorem 1

If $f(x)$, a polynomial, is divided by $(x - \alpha)$, then the remainder is $f(\alpha)$.

Proof

$f(x) = (x-\alpha)Q(x) + R \Rightarrow f(\alpha) = (\alpha-\alpha)Q(x) + R \Rightarrow R = f(\alpha)$. □

4.4 Factor Theorem

Theorem 2

$f(x)$ has a factor $(x-\alpha)$, if and only if, $f(\alpha) = 0$

Proof

Following from remainder theorem, described above, if $R = f(\alpha) = 0$, then $f(\alpha) = (x-\alpha)Q(x)$, and thus, $f(x)$ has a factor $(x-\alpha)$. □

4.5 Fundamental Theorem of Algebra

Every polynomial of degree greater or equal than one has at least one root/solution/zero in the complex numbers. We can also say that for $f(x)$ introduced in the beginning with $n \geq 1$, then there exists a $z \in \mathbb{C}$, such that

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Now it is trivial to deduce that an nth degree polynomial will have exactly n roots i.e. $f(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1})(x-\alpha_n)$.

Notes:

1. Some of the roots of the polynomial may have repetition.
2. If a root α repeats m times, then m is called *multiplicity* of the root α or α is called m fold root.
3. Quadratic surds of the form $\sqrt{a} + \sqrt{b}$, where $\sqrt{a}$ and $\sqrt{b}$ are irrational numbers, then it will have its conjugate as a root. Similarly, if a complex root occurs, then it always occurs in pair with its complex conjugate as another root of the polynomial. However, if the coefficients are complex numbers then it is not mandatory for complex roots to appear in conjugate pairs.

4.6 Identity Theorem

Theorem 3

If $f(x)$, a polynomial of degree n, vanishes for at least $n+1$ distinct values of x, then it is identically 0.

Proof

We have $f(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1})(x-\alpha_n)$, and we let that it vanishes for α_{n+1}, then

$$f(x) = a(\alpha_{n+1} - \alpha_1)(\alpha_{n+1} - \alpha_2) \cdots (\alpha_{n+1} - \alpha_{n-1})(\alpha_{n+1} - \alpha_n) = 0$$

Because α_{n+1} is different from $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ none of the terms will vanish, which implies that $a = 0 \Rightarrow f(x) = 0$. $\square$

Corollary 1

Consider two poynomials $f(x)$ and $g(x)$ having degrees p and q respectively, such that $p \leq q$. If both of them have equal value for $q + 1$ distinct values of x, then they must be equal.

Proof

Let $h(x) = f(x) - g(x)$. This implies that the degree of $h(x)$ is at most q and it vanishes for $q + 1$ distinct values of x. $\Rightarrow h(x) = f(x) - g(x) = 0 \Rightarrow f(x) = g(x)$. $\square$

Corollary 2

If $f(x)$ is a periodic polynomial with some constant period T i.e. $f(x) = f(x + T)\ \forall\ x \in \mathbb{R}$, then $f(x) = c$.

Proof

Let $f(0) = x$, then $f(0) = f(T) = f(2T) = \cdots = c$. Thus, polynomials $f(x)$ and $g(x) = c$ take same values for infinite number of points. Hence, they must be identical. $\square$

4.7 Rational Root Theorem

Theorem 4

If $p, q \in \mathbb{Z}, q \neq 0$ such that they are relatively prime i.e. $gcd(p, q) = 1$, then if $\frac{p}{q}$ is a root of the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$, where $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{I}$ and $a_n = 0$, then p is a divisor of a_0 and q that of a_n.

Proof

Since $\frac{p}{q}$ is a root, we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \cdots + a_1 \frac{p}{q} + a_0 = 0$$

$\Rightarrow a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 q^{n-1} p + a_0 q^n = 0$

$\Rightarrow a_{n-1} p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_1 p q^{n-2} + a_0 q^{n-1} = -a_n \frac{p^n}{q}$

Everything on L.H.S. is integer and p, q are relatively prime therefore q must divide a_n. Similalry, it can be proven that a_0 is divisible by q. $\square$

Corollary 3 (Integer Root Theorem)

If roots of $x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$, where $0 \leq i \leq n - 1$ are integers and coefficients are also integer, are integer then all the roots divide a_0.

Proof

This corollary is a direct result from previous corollary. □

4.8 Vieta's Relations

If α_1, α_2, ..., α_n are n roots of the equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, then $\sum_{i=0}^{n} \alpha_i = -\frac{a_{n-1}}{a_n}$, $\sum_{1\le i\le j\le n} \alpha_i\alpha_j = \frac{a_{n-2}}{a_n}$, $\sum_{1\le i\le j\le k\le n} \alpha_i\alpha_j\alpha_n = -\frac{a_{n-3}}{a_n}$, $\cdots$, $\alpha_1\alpha_2 \ldots \alpha_n = (-1)^n\frac{a_0}{a_n}$.

These relations are denoted as $\sigma_1, \sigma_2, \ldots, \sigma_n$ as well. These relations are known as Vieta's relations.

4.9 Symmetric Functions

Consider functions $a + b + c$, $a^2 + b^2 + c^2$, $(a - b)^2 + (b - c)^2 + (c - a)^2$, and $(a + b)(b + c)(c + a)$ in which the terms can be interchanged without changing the overall function. Functions demonstrating such behavior are known as *symmetric* functions.

In general, if a function is of n variables then this definition warrants that any two variable can be interchanged without changing the function. Thus, we see that Vieta's relations are symmetric functions.

4.10 Common Roots of Polynomial Equations

If α is a common root of the polynomial equations $f(x) = 0$ and $g(x) = 0$, if and only if, it is a root of the HCF of the polynomilas $f(x)$ and $g(x)$. The HCF of two polynomials can be found exactly like HCF of two integers using Euclid's method.

4.11 Irreducabilty of Polynomials

When we talk of irreducability we talk in terms of set to which the coefficients of the polynomial belong. The set could be $\mathbb{Q}, \mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$.

An irreducible polynomial is, a non-constant polynomial which cannot have non-constant factors in the same set as coefficients of the polynomial itself.

Consider following example:

1. $x^2 - 5x + 6 = (x - 2)(x - 3)$
2. $x^2 - \frac{4}{9} = \left(x - \frac{2}{3}\right)\left(x + \frac{2}{3}\right)$

3. $x^2 - 5 = (x - \sqrt{5})(x + \sqrt{5})$

4. $x^2 + 9 = (x + 3i)(x - 3i)$

Over $\mathbb{I}$, first is reducible while other are irreducible, over $\mathbb{Q}$ first two are reducible bit last two are not, over $\mathbb{R}$, first three are reducible but last one is not and over $\mathbb{C}$ all are reducible.

4.11.1 Gausss' Lemma

If a polynomial with integer coefficients is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.

4.12 Eisenstein's Irreducibility Criterion Theorem

Theorem 5

Consider the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients. If there exists a prime p such that the following three conditions apply

1. *p divides each a_i for $0 \le i < n$,*

2. *p does not divide a_n, and*

3. *p^2 does not divide a_0,*

then $f(x)$ is irreducible over rational numbers and integers.

Proof

If possible, let us assume that $f(x) = g(x).h(x)$ such that $g(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ and $h(x) = c_l x^l + c_{l-1}x^{l-1} + \cdots + c_1 + c_0$, where $b_i, c_i \in \mathbb{Z}\ \forall\ i = 0, 1, 2, \ldots; b_k \neq, c_l \neq 0; 1 \le k, l \le n-1$.

Comparing leading coefficient on both sides, we have $a_n = b_k c_l$. As $p \nmid a_n \Rightarrow p \nmid b_k c_l \Rightarrow p \nmid b_k$ and $p \nmid c_l$.

Similarly, $a_0 = b_0 c_0$. As $p|a_0$ and $p^2 \nmid a_0 \Rightarrow p|b_0 c_0$, but both b_0 and c_0 cannot be divided by p. Without loss of generality, we suppose $p|b_0$ and $p \nmid c_0$. Suppose i be the smallest index such that b_i is not divisible by p. There is such an index i since $p \nmid b_k$, where $1 \le i \le k$. Depending on i and k, for $i \le k, a_i = b_i c_0 + b_{i-1}c_1 + \cdots + b_0 c_i$ and for $i > k, a_i = b_i c_0 + b_{i-1}c_1 + \cdots + b_{i-k}c_k$.

We have $p|a_i$ and by supposition p divides each one of $b_0, b_1, \ldots, b_{i-1} \Rightarrow p|b_i c_0$. But $p \nmid c_0 \Rightarrow p|b_i$, which is a contradiction, and therefore, $f(x)$ is irreducible. $\square$

4.13 Extended Eisenstein's Irreducibility Criterion Theorem

Theorem 6

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficient. If there exists a prime number p and an integer $k \in \{0, 1, 2, \ldots, n-1\}$ such that $p \mid a_0, a_1, \ldots, a_k; p \nmid a_{k+1}$

and $p^2 \nmid a_0$, then $f(x)$ has an irreducible factor of degree at least $k+1$. In particular if p can be taken so that $k = n-1$, then $f(x)$ is irreducible.

Proof

Suppose that $f(x) = g(x)h(x)$ such that

$$g(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$$

$$h(x) = c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0$$

where $b_i, c_i \in \mathbb{Z}\ \forall\ i = 0, 1, 2, \ldots; b_k \neq 0, c_r \neq 0; 1 \leq m, r \leq n-1$.

Since $a_0 = b_0 c_0$ is divisible by p and not by p^2, exactly one of b_0, c_0 is a multiple of p. Without loss of generality assume that $p \mid b_0$ and $p \nmid c_0$.

Now $p \mid a_1 = b_0 c_1 + b_1 c_0 \Rightarrow p \mid b_1 c_0 \Rightarrow p \mid b_1$.

Simmilarly, $p \mid a_2 = b_0 c_2 + b_1 c_2 + b_2 c_0 \Rightarrow p \mid b_2 c_0 \Rightarrow p \mid b_2$ and so on.

We conclude that all coefficients $b_0, b_1, \ldots, b_k$ are divisible by p. Now, $a_{k+1} = b_k c_1 + b_{k-1} c_2 + b_{k-2} c_2 + \ldots \Rightarrow p \nmid a_{k+1}$. It follows that degree of $g \geq k+1$. □

4.14 Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{C}$, the set of complex numbers, is called a *quadratic equation*. The numbers a, b, c are called *coefficients* of the equation. The quantity $b^2 - 4ac$ is called the *discriminant* of the equation. It is represented by D or Δ. A quadratic equation represents a parabola geometrically.

Examples:

1. $4x^2 + 4x + 1 = 0, a = 4, b = 4, c = 1$.
2. $7x^3 + 10 = 0$ is not a quadratic equation because the power of x is greater than 2.
3. $3x^2 - 2x^{1/2} + 7 = 0$ is not a quadratic equation is not a quaadratic equation because the power of x in the second term is not 1.
4. $2x^2 - 4 = 0, a = 2, b = 0, c = -4$.

A quadratic equation is called *incomplete* if one of b or c is zero. Thus, the last example above represents an incomplete quadratic equation.

An expression of the form $ax^2 + bx + c$ is called a *quadratic expression* while other elements are same as a quadratic equation.

If two expression in x are equal for all values of x then this statement of equality between the two expression is called an *identity*.

$f(x) = 0$ is said to be an identity in x if it is satisfied by all values of x in the domain of $f(x)$. Thus, an identity in x is satisfied by all values of x while an equation is satisfied for particular values of x.

Example: $(x+1)^2 = x^2 + 2x + 1$ is an identity in x.

Two equations are called *identical equations* if they have same roots.

Example: $x^2 - 5x + 4 = 0$ and $2x^2 - 10x + 8 = 0$ are identical equations because both have same roots 1 and 4.

Note:

1. Two equations in x are identical if and only if the coefficients of similar power of x in the two equations are proportional. Thus, if $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ are identical equations, then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$
2. An equation remains unchanged if it is multiplied or divided by non-zero number.

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_0$, where $a_0, a_1, a_2, \dots, a_n$ are constants $(a_0 \neq 0)$ and n is a positive integer is called a polynomial in x of degree n.

As a special case a constant is also called a polynomial of degree zero.

4.15 Rational Expression and Rational Function

An expression of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, is called a rational expression.

In the particular case, when $Q(x)$ is a non-zero constant, $\frac{P(x)}{Q(x)}$ reduces to a polynomial.Thus, every polynomial is a rational expression but the converse is not true.

Examples:

1. $\frac{x^2-5x+4}{x-2}$
2. $\frac{1}{x-7}$

4.16 Roots of a Quadratic Equation

The values x for which the equation $ax^2 + bx + c = 0$ are satisfied are called roots of the equation. They are also called roots of the quadratic expression $ax^2 + bx + c$

Every quadratic equation has at most two roots. Let $ax^2 + bx + c = 0$, where $a \neq 0$

Multiplying both sides of the equation with a

$a^2x^2 + abx + ac = 0 \Rightarrow (ax)^2 + 2.ax.\frac{b}{2} + \frac{b^2}{4} + ac - \frac{b^2}{4} = 0$

$\left(ax+\frac{b}{2}\right)^2=\frac{b^2-4ac}{4}\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

These are two roots of the quadratic equation. Let us suppose the above quadratic equation has three roots α, β and γ. These roots will satisfy the above equation. Thus,

$$a\alpha^2+b\alpha+c=0, a\beta^2+b\beta+c=0, a\gamma^2+b\gamma+c=0$$

Subtracting the first two, we get $(\alpha-\beta)[a(\alpha+\beta)+b]=0$

$\because \alpha\neq\beta \therefore a(\alpha+\beta)+b=0$

Similarly, $a(\alpha+\gamma)+b=0$

Subtracting these two, we get $a(\alpha-\gamma)=0$

$\because a\neq 0 \therefore \alpha=\gamma$

Thus, a quadratic equation has at most two roots.

4.17 Sum and Product of the Roots

From the two obtained we observe that $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

4.18 Nature of Roots

For equation $ax^2+bx+c=0$ when a,b,c are real.

1. When $D<0$

   In this case, both roots will be either imaginary or complex numbers depending on whether b is zero or not. These roots are conjugate of each other.

2. When $D=0$

   In this case, both roots will be equal.

3. When $D>0$

   In this case, both roots will be equal and unqual. If D is not a perfect square then roots are irrational and come as a pair of conjugate irrational numbers.

4. When D is a perfect square and a, b, c are rationals.

   In this case, both roots are real and unequal.

4.18.1 Conjugate Roots

Imaginary/complex roots of a quadratic equation with real coefficients always occur in conjugate pair.

Let $\alpha + i\beta$ be a root of the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real numbers. Thus,

$a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = 0$

$\Rightarrow (a\alpha^2 - a\beta^2 + b\alpha + c) + (2a\alpha\beta + b\beta)i = 0$

Equating real and imaginary parts

$a\alpha^2 - a\beta^2 + b\alpha + c = 0, 2a\alpha\beta + b\beta = 0$

Using $\alpha - i\beta$ as the second root of the equation

$a(\alpha - i\beta)^2 + b(\alpha - i\beta) + c = (a\alpha^2 - a\beta 62 + b\alpha + c) + (2a\alpha\beta + b\beta)i$

$= 0 + i.0$

Thus, we see that $\alpha - i\beta$ also satisfied the equation and is second root of the equation. Similarly, if the roots are irrational they also appear as conjugate pair.

4.19 Quadratic Expression and its Graph

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \tag{4.1}$$

4.19.1 When a Quadratic Equation is Always Positive/Negative

It follows from Eq. 4.1, that $f(x) > 0(< 0)\ \forall\ x \in \mathbb{R}$ if and only if $a > 0(< 0)$ and $D = b^2 - 4ac < 0$. See Figure 4.1(Figure 4.2). Also, it follows from eq:1 that $f(x) \geq 0(\leq 0)\ \forall\ x \in \mathbb{R}$ if and only if $a > 0(< 0)$ and $D = b^2 - 4ac = 0$. in this case $f(x) < 0(< 0)$ for each $x \in R, x \neq -b/2a$, and the graph of $y = f(x)$ touches the x-axis at $x = -b/2a$.

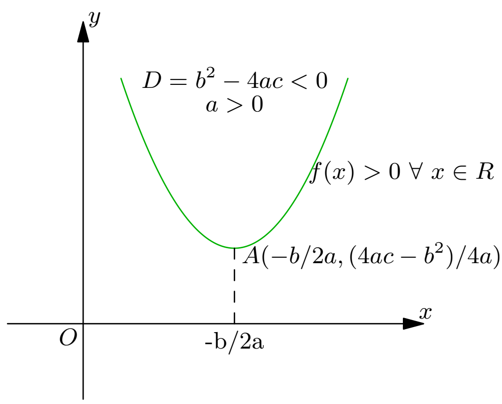


Figure 4.1 When quadratic equation is always positive

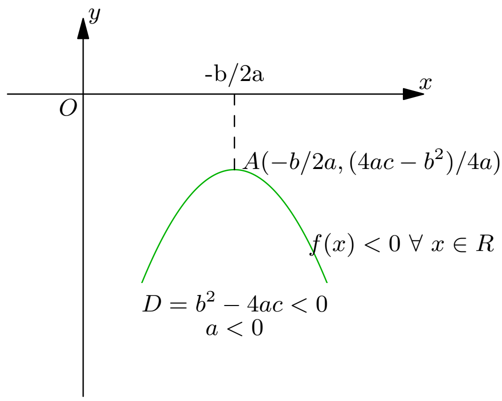


Figure 4.2 When quadratic equation is always negative

4.20 Sign of a Quadratic Equation

If $D = b^2 - 4ac > 0$, then eq. Equation 4.1 can be written as

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)\right]$$

$$= a(x - \alpha)(x - \beta)$$

If $D = b^2 - 4ac > 0$ and $a > 0$, then (See Figure 4.3)

$$f(x) = \begin{cases} > 0 \text{ for } x < \alpha \text{ or } x > \beta \\ > 0 \text{ for } \alpha < x < \beta = 0 \text{ for } x = \alpha,\beta \end{cases}$$

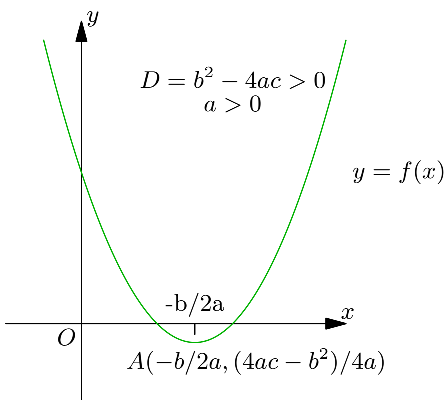


Figure 4.3 When $D > 0$ and $a > 0$

If $D = b^2 - 4ac > 0$ and $a < 0$, then (See Figure 4.4)

$$f(x) = \{ < 0 \text{ for } x < \alpha \text{ or } x > \beta > 0 \text{ for } \alpha < x < \beta = 0 \text{ for } x = \alpha,\beta$$

Note that if $a > 0$, then $f(x)$ attains the least value at $x = -b/2a$, a value which is achieved by differentiating the function once and at this point the tangent to parabola has slope 0. The least value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

If $a < 0$, then $f(x)$ is maximum at value $x = -\frac{b}{2a}$ and value of function has the same formula which is for least value shown above.

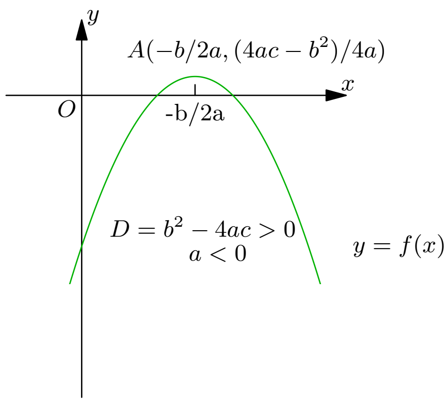


Figure 4.4 When $D > 0$ and $a < 0$

4.21 Position of Roots

Conditions for both roots to be more than a real number k

Form the Fig. Figure 4.5, we note that both the roots are more than k if and only if $D > 0$, $k < -\frac{b}{2a}$ and $f(k) > 0$.

In case $a < 0$, from Fig. Figure 4.6, both the roots are more than k if and only if $D > 0$, $k < -\frac{b}{2a}$ and $f(k) < 0$.

Combining the above two equations, we get the condition for the roots to be more than a real number k if and only if $D > 0$, $k < -\frac{b}{2a}$ and $af(k) > 0$. Similarly, condition for the roots to be more than a real number k if and only if $D > 0$, $k > -\frac{b}{2a}$ and $af(k) > 0$.

Conditions for a real number k to lie between two roots

Similarly, the real number k lies between the roots of the quadratic equation if and only if a and $f(k)$ are of opposite signs, i.e. if and only if $a > 0$, $D > 0$, $f(k) < 0$ or $a < 0$, $D > 0$, $f(k) > 0$.

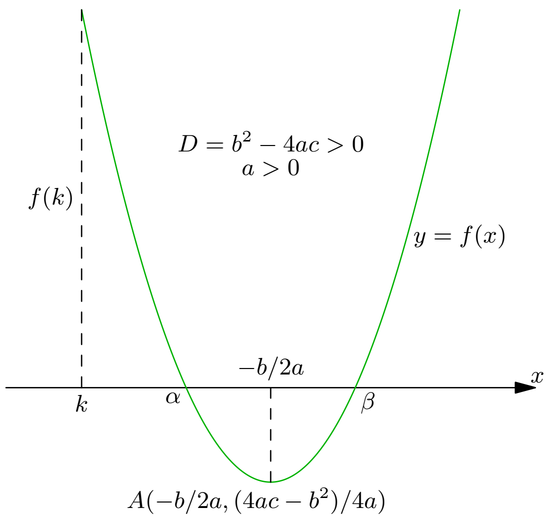


Figure 4.5 When $D > 0$ and $a > 0$

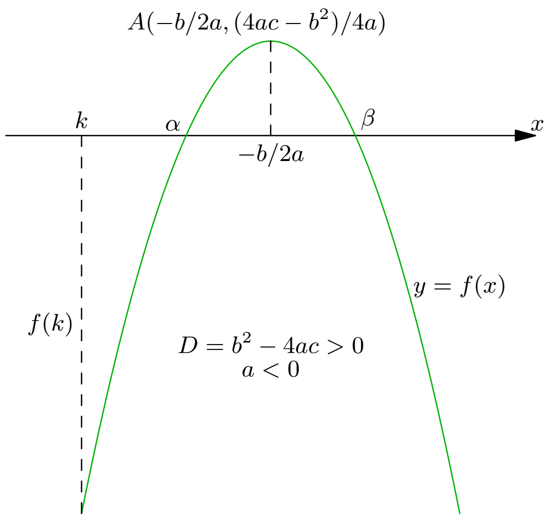


Figure 4.6 When $D > 0$ and $a < 0$

Combining these two, we get $D > 0$, $af(k) < 0$ as the condition for k to lie between two roots.

Conditions for exactly one root to lie in between (k_1, k_2) where $k_1 < k_2$

If $a > 0$, then exactly one root lies in the interval (k_1, k_2) if and only if $f(k_1) > 0$ and $f(k_2) < 0$. Also, same is true if anad only if $f(k_1) < 0$ and $f(k_2) > 0$. Combining these two we get $f(k_1)f(k_2) < 0$. This condition is also true if $a < 0$.

Conditions for both roots to lie in between (k_1, k_2) where $k_1 < k_2$

If $a > 0$, both the roots will lies in the interval (k_1, k_2) if and only if $D > 0$, $k_1 < -\frac{b}{2a} < k_2$, $f(k_1) > 0$ and $f(k_1) > 0$. In case $a < 0$, the conditions are $D > 0$, $k_1 < -\frac{b}{2a} < k_2$, $f(k_1) < 0$ and $f(k_1) < 0$.

Conditions for the quadratic equation to have repeated roots

The quadratic equation $f(x) = ax^2 + bx + c = 0, a \neq 0$ has a repeated root if and only if $f(\alpha) = f'(\alpha) = 0$, where α is the repeated root. In this case, $f(x) = a(x - \alpha)^2$, In fact, $\alpha = -b/2a$. Geometrically, the x-axis will be a tangent to the parabola at $x = -b/2a$. See Figure 4.7 and Fig. Figure 4.8.

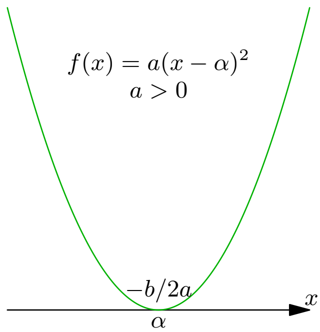


Figure 4.7 $f(\alpha) = 0, f'(\alpha) = 0$

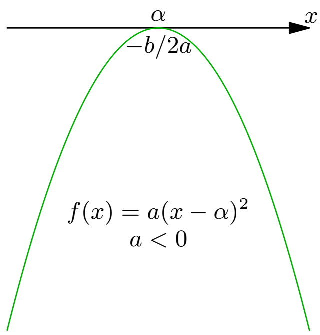


Figure 4.8 $f(\alpha) = 0, f'(\alpha) = 0$

bf Conditions for two quadratic equations to have one common root

Consider two quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x^2 + c' = 0$ having a common root α. Clearly, this common root will satisfy both the equations, i.e. $a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b'\alpha + c' = 0$.

Solving these two equations, we get

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \alpha^2 = \frac{bc' - b'c}{ab' - a'b}, \alpha = \frac{a'c - ac'}{ab' - a'b}$$

Eliminating α, we get

$$(a'c - ac')^2 = (bc' - b'c)(ab' - a'b)$$

This is the required condition for two quadratic equations to have one common root.

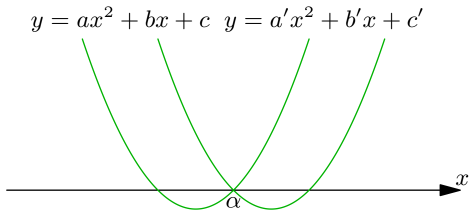


Figure 4.9 Common roots

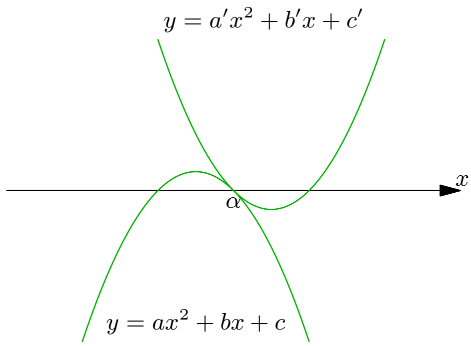


Figure 4.10 Common roots

To obtain the common root make coefficients of x^2 in both the equations same and subtract one equation from the other to obtain a linear equation in x, which you can solve to obtain the common root.

For having both roots common the two equations must be identical i.e. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

4.22 General Quadratic Equation in x and y

The general quadratic equation in x and y is given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\therefore x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2 + 2fy + c)}}{2a}$$

$$\Rightarrow x + hy + g = \pm\sqrt{(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac}$$

It can be resolved into two linear factors if $(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac$ is a perfect square and $h^2 - ab > 0$.

The condition for $(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac$ to be a perfect square is that its discriminant is 0, i.e.

$$4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

4.23 Equations of Higher Degree

The equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$, where $a_0, a_1 \dots, a_n \in \mathbb{C}$, the set of complex numbers and $a_0 \neq 0$, is said to be an equation of degree n. An equation of degree n has exactly n roots. Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ be the n roots. Then

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

$$\sum \alpha_i = -\frac{a_1}{a_0}, \sum \alpha_i\alpha_j = \frac{a_2}{a_0}, \dots, \prod \alpha_i = (-1)^n \frac{a_n}{a_0}$$

4.24 Cubic and Biquadratic Equation

If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}$$

Also, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + d + e = 0$, then

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a}, \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}, \alpha\beta\gamma\delta = \frac{e}{a}$$

4.25 Transformation of Equations

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0 \tag{4.2}$$

1. To form an equation whose roots are $k(\neq 0)$ times roots of the Equation 4.2, replace x by x/k.

2. To form an equation whose roots are the negatives of the roots of Equation 4.2, replace x by $-x$. Alternatively, change the sign of the coefficients of x^{n-1}, x^{n-3}, x^{n-5}, ... etc. in Equation 4.2.

3. To form an equation whose roots are k more than the roots of Equation 4.2, replace x by $x - k$ in .

4. to form an equation whose roots are reciprocals of roots in Equation 4.2, replace x by $1/x$ in Equation 4.2 and then multiply both sides by x^n.

5. To form an equation whose roots are squares of roots in Equation 4.2, replace x by $\sqrt{x}$. Then you can collect all terms involving $\sqrt{x}$ on one side and square both sides followed by simplification.

6. To form an equation whose roots are cubes of roots in Equation 4.2, replace x by $\sqrt[3]{x}$. Then you can collect all terms involving $\sqrt[3]{x}$ and $\sqrt[3]{x^2}$ on one side and cube both sides followed by simplification.

4.26 Descartes Rule

1. The maximum no. of positive real roots of Equation 4.2 is the number of changes of sign of coefficients from positive to negative and negative to positive.

2. The maximum no. of negtive real roots of Equation 4.2 is the number of changes of sign of coefficients from positive to negative and negative to positive in the equation $f(-x) = 0$.

4.27 Hints for Solving Polynomial Equations

1. To solve the equation of the form $(x-a)^{2n} + (x-b)^{2n} = A$, where $n \in \mathbb{P}$, put $y = x - \frac{a+b}{2}$.

2. To solve the equation of the form $a_0(f(x))^{2n} + a_1(f(x))^n + a_2 = 0$, put $(f(x))^n = y$ then we obtain two roots y_1, y_2 to solve again for $f(x) = y_1$, $f(x) = y_2$.

3. An equation of the form $(ax^2+bx+c_1)(ax^2+bx+c_2)\ldots(ax^2+bx+c_n) = A$ can be solved by putting $ax^2 + bx = y$.

4. An equation of the form $(x-a)(x-b)(x-c)(x-d) = Ax^2$, where $ab = cd$, can be reduced to a product of two quadratic polynomials by putting $y = x + \frac{ab}{x}$.

5. An equation of the form $(x-a)(x-b)(x-c)(x-d) = A$, where $a < b < c < d, b - a = d - c$ can be solved by putting $y = x - \frac{a+b+c+d}{4}$.

6. A polynomial $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x)\ \forall\ x, y$. All symmetric polynomials can be represented as a function of $x + y$ and xy.

4.28 Problems

1. What is the remainder when $x + x^9 + x^{25} + x^{49} + x^{81}$ is divided by $x^3 - x$?

2. Prove that the polynomial $x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$ is divisible by $x^9 + x^8 + x^7 + \cdots + x + 1$.

3. If $f(x)$ is a polynomial with integral coefficients and suppose that $f(1)$ and $f(2)$ are both odd, then prove that there exists no integer n for which $f(n) = 0$.

4. If f is a polynomial with integer coefficients such that there exists four distinct integers a_1, a_2, a_3, and a_4 such that $f(a_1) = f(a_2) = f(a_3) = f(a_4) = 1991$, show that there exists no integer b, such that $f(b) = 1993$.

5. Find a polynomial function of lowest degree with integral coefficients with $\sqrt{5}$ as one of its roots.

6. Find a polynomial of the lowest degree with integer coefficients whose one of the zeroes is $\sqrt{5} + \sqrt{2}$.

7. If $f(x)$ is a polynomial such that $x.f(x-1) = (x-4)f(x) \forall\ x\ \in \mathbb{R}$. Find all such $f(x)$.

8. Let $f(x)$ be a monic cubic equation such that $f(1) = 1, f(2) = 2, f(3) = 3$ then find $f(4)$.

9. Find a fourth degree equation with rational coefficients, one of whose roots is, $\sqrt{3} + \sqrt{7}$.

10. Form the equation of the lowest degree with rational coefficients which has $2 + \sqrt{3}$ and $3 + \sqrt{2}$ as two of its roots.

11. Find a polynomial equation of the lowest degree with rational coefficients whose one root is $\sqrt[3]{2} + 3\sqrt[3]{4}$.

12. Show that $(x-1)^2$ is a factor of $x^n - nx + n - 1$.

13. If a, b, c, d, e are all zeroes of the polynomial $6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$, find the value of $(1+a)(1+b)(1+c)(1+d)(1+e)$.

14. If $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the roots of the equation $x^n - 1, n \in \mathbb{N}, n \geq 2$, show that $n = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1})$.

15. If $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is a poynomial such that $f(1) = 10, f(2) = 20, f(3) = 30$, find the value of $\frac{f(12)+f(-8)}{10}$.

16. If the polynomial $x^{2k} + 1 + (x+1)^{2k}$ is not divisible by $x^2 + x + 1$, then find the value of $k \in \mathbb{N}$.

17. Find all polynomials $P(x)$ with real coefficients such that $(x-8)P(2x) = 8(x-1)P(x)$.

18. If $(x-1)^3$ divides $f(x) + 1$ and $(x+1)^3$ divides $f(x) - 1$, then find the polynomial $f(x)$ of degree 5.

19. Find the polynomial equation of lowest degree with rational coefficients, two of whose roots are $3+2i$ and $2+3i$.

20. Find the roots of the equation $x^4+x^3-19x^2-49x-30$, if all roots are rational numbers.

21. Find the rational roots of $2x^3-3x^2-11x+6=0$.

22. Solve $x^3-3x^3+5x-15=0$.

23. Show that $f(x)=x^{1000}-x^{500}+x+1=0$ has no rational roots.

24. If $x^2+ax+b+1=0$, where $a,b\in\mathbb{Z}$ and $b\neq -1$, has a root in integers then prove that a^2+b^2 is composite.

25. For what values of p, will the sum of squares of the roots $x^2-px+p-1=0$ be minimum?

26. Let α,β be two real numbers not equal to -1, such that α,β and $\alpha\beta$ are the roots of a cubic polynomial with rational coefficients. Prove or disprove that $\alpha\beta$ is rational.

27. Find the roots of the cubic equation $9x^3-27x^2+26x-8=0$, given that one of the roots of the equation is double the other.

28. If the product of two roots of the equation $4x^4-24x^3+31x^2+6x-8=0$ is 1, find all the roots.

29. One root of the equation $x^4-5x^3+ax^2+bc+c=0$ is $3+\sqrt{2}$. If all the roots of the equation are real, find extremum values of a,b,c; given that a,b and c are rational.

30. Find the rational roots of the equation $x^4-4x^3+6x^2-4x+1=0$.

31. Solve the equation $x^4+10x^3+35x^2+50x+24=0$, if some of two of its roots is equal to the sum of the other two roots.

32. Find the rational roots of $6x^4+x^3-3x^2-9x-4=0$.

33. Find the rational roots of $6x^4+35x^3+62x^2+35x+2=0$.

34. Given that the sum of two of the roots of $4x^3+ax^2-x+b=0$ is zero, where $a,b\in\mathbb{Q}$. Solve the equation for all values of a and b.

35. Find all a,b such that $x^3+ax^2+bx-8=0$ are real and in G.P.

36. Show that $2x^6+12x^5+30x^4+60x^3+80x^2+30x+45=0$ has no real roots.

37. Construct a polynomial equation, of the least degree with rational coefficients one of whose roots is $\sin 10^\circ$.

38. Construct a polynomial equation, of the least degree with rational coefficients one of whose roots is $\sin 20^\circ$.

39. Construct a polynomial equation, of the least degree with rational coefficients one of whose roots is $\cos 10^\circ$.

40. Construct a polynomial equation, of the least degree with rational coefficients one of whose roots is $\cos 20^\circ$.

41. Construct a polynomial equation, of the least degree with rational coefficients one of whose roots is $\tan 10^\circ$.

42. Construct a polynomial equation, of the least degree with rational coefficients one of whose roots is $\tan 20^\circ$.

43. Construct a polynomial equation, of the least degree with rational coefficients two of whose roots are $\sin 10^\circ$ and $\cos 20^\circ$.

44. If p, q, r are the real roots of $x^3 - 6x^2 + 3x + 1 = 0$, determine the possible values of $p^2q + q^2r + r^2p$.

45. The product of two of the four roots of the equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is 32. Determine the value of k.

46. If $x + y = 1$ and $x^4 + y^4 = c$, find $x^3 + y^3$ and $x^2 + y^2$ in terms of c.

47. Find all x and y that satisfy $x^3 + y^3 = 7$ and $x^2 + y^2 + x + y + xy = 4$.

48. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then prove that $\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} \times \frac{\alpha^2+\beta^2+\gamma^2}{2}$.

49. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then prove that $\frac{\alpha^7+\beta^7+\gamma^7}{7} = \frac{\alpha^5+\beta^5+\gamma^5}{5} \times \frac{\alpha^2+\beta^2+\gamma^2}{2}$.

50. If $\alpha + \beta + \gamma = 0$, then show that $3(\alpha^2 + \beta^2 + \gamma^2)(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^4 + \beta^4 + \gamma^4)$.

51. Show that there does not exist any distinct natural numbers a, b, c and d such that $a^3 + b^3 = c^3 + d^3$ and $a + b = c + d$.

52. Determine all the roots of the system of simultaneous equations $x + y + z = 3$, $x^2 + y^2 + z^2 = 3$, and $x^3 + y^3 + z^3 = 3$.

53. Given real numbers x, y, z, such that $x + y + z = 3$, $x^2 + y^2 + z^2 = 5$, $x^3 + y^3 + z^3 = 7$, find $x^4 + y^4 + z^4$.

54. If α, β are the roots of the equation $x^2 - (a + d)x + ad - bc = 0$, show that α^3 and β^3 are the roots of the equation $x^2 - (a^3 + d^3 + 3abc + 3bcd)x + (ad - bc)^3 = 0$.

55. If $a^3 + b^3 + c^3 = (a + b + c)^3$, prove that $a^5 + b^5 + c^5 = (a + b + c)^5$. Generalize your result.

56. If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, find the value of $p^3 + q^3 + r^3$.

57. Find the sum of the 5th powers of the roots of the equation $x^3 + 3x + 9 = 0$.

58. Find the sum of the 5th powers of the roots of the equation $x^3 - 7x^2 + 4x - 3 = 0$.

59. α, β, γ are the roots of the equation $x^3 - 9x + 9 = 0$. Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$ and $\alpha^{-5} + \beta^{-5} + \gamma^{-5}$.

60. Find the cubic equation whose roots are α, β and γ, such that $\alpha + \beta + \gamma = 9$, $\alpha^2 + \beta^2 + \gamma^2 = 29$, $\alpha^3 + \beta^3 + \gamma^3 = 99$. Also, find the value of $\alpha^4 + \beta^4 + \gamma^4$.

61. If $\alpha + \beta + \gamma = 4, \alpha^2 + \beta^2 + \gamma^2 = 7, \alpha^3 + \beta^3 + \gamma^3 = 28$, find $\alpha^4 + \beta^4 + \gamma^4$ and $\alpha^5 + \beta^5 + \gamma^5$.

62. Solve: $x^3 + y^3 + z^3 = a^3$, $x^2 + y^2 + z^2 = a^2$, $x + y + z = a$ in terms of a.

63. If α, β, γ be the roots of $2x^3 + x^2 + x + 1 = 0$, show that $\left(\frac{1}{\beta^3} + \frac{1}{\gamma^3} - \frac{1}{\alpha^3}\right)\left(\frac{1}{\gamma^3} + \frac{1}{\alpha^3} - \frac{1}{\beta^3}\right)\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} - \frac{1}{\gamma^3}\right) = 16$.

64. Find $x, y \in \mathbb{C}$ such that $x^5 + y^5 = 275$, $x + y = 5$.

65. Find real x such that $\sqrt[4]{97 - x} + \sqrt[4]{x} = 5$.

66. Find the common roots of the polynomials $x^3 + x^2 - 2x - 2$ and $x^3 - x^2 - 2x + 2$.

67. Find the common roots of $x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$, and solve the equations.

68. Show that the set of polynomials $P = \{p_k(x) : p_k(x) = x^{5k+4} + x^3 + x^2 + x + 1\}, k \in \mathbb{N}$ has a common non-trivial polynomial divisor.

69. Find the common roots of the equations $x^3 - 3x - 4x + 12 = 0$ and $x^3 + 9x^2 + 26x + 24 = 0$.

70. Find the common roots of the equations $x^4 - 5x^3 + 2x^2 + 20x - 24 = 0$ and $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$.

71. If d, e, f are in G.P. and the two quadratic equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in H.P.

72. If n is even and α, β are the roots of the equation $x^2 + px + q = 0$ and also of the equation $x^{2n} + p^n x^n + q^n = 0$ and $f(x) = \frac{(1+x)^n}{1+x^n}$ where $\alpha^n + \beta^n \neq 0, p \neq 0$, find the value of $f\left(\frac{\alpha}{\beta}\right)$.

73. Factorize $x^4 + 4$ as a product of irreducible polynomials over the sets $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.

74. Check if $x^4 + x^3 - x - 1$ is irreducible over $\mathbb{Z}$.

75. Check if $x^3 + x^2 + x + 3$ is irreducible over $\mathbb{Z}$.

76. Show that $x^4 + x^3 - x + 1$ is irreducible over $\mathbb{Z}$.

77. Prove that if the integer 'a' is not divisible by 5, then $f(x) = x^5 - x + a$ cannot be factored as the product of two non-constant polynomials with integer coefficients.

78. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients such that $|a_0|$ is a prime and $|a_0| > |a_1| + |a_2| + \cdots + |a_n|$. Prove that $f(x)$ is irreduicible over $\mathbb{Z}$.

79. Prove that $16x^3 - 35x^2 + 105x + 175$ is irreducible over $\mathbb{Z}$.

80. Prove that $x^3 - 3x^2 + 3x + 22$ is irreducible over $\mathbb{Z}$.

81. Let p be a prime number. Show that $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible.

82. Let $f(x) = x^n + 5x^{n-1} + 3$, $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as a product of two polynomials, each of which has all its coefficient integers and degree at least 1.

83. Prove that for any prime p, polynomial, $x^n - p$ is irreducible over $\mathbb{Z}$.

84. Prove that $x^7 + 48x - 24$ is irreducible over $\mathbb{Z}$.

85. Prove that $x^4 + 2x^2 + 2x + 2$ is not product of two polynomials $x^2 + ax + b$ and $x^2 + cx + d$, where a, b, c, d are integers.

86. Prove that $x^5 - 36x^4 + 6x^3 + 30x^2 + 24$ is irreducible over $\mathbb{Z}$.

87. Prove that $x^3 + 3x^2 + 3x + 5$ is irreducible over $\mathbb{Z}$.

88. Prove that $x^p + px + p - 1$ is reducible for some prime p then it must be '2'.

89. Let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial over $\mathbb{Z}$ and irreducible over it. Prove that $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_0$ is also irreducible over $\mathbb{Z}$ and use this to show that $21x^5 - 49x^3 + 14x^2 - 4$ is irreducible over $\mathbb{Z}$.

90. If $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ are distinct, then prove that $(x - a_1)(x - a_2)\cdots(x - a_n) - 1$ is irreducible over z.

91. Prove that $1 + x^p + x^{2p} + \cdots + x^{p(p-1)}$ is irreducible over $\mathbb{Z}$.

92. Solve for $x : 2p(p-2)x = p - 2$.

93. If x_1 and x_2 are non-zero roots of the equations $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$ respectively, prove that $\frac{a}{2}x^2 + bx + c$ has a root between x_1 and x_2, where $a \neq 0$.

94. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Show that there exists an integer M such that $P(n).P(n+1) = P(M)$ for any integer n.

95. Prove that, if the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then the roots of the equation cannot be rational numbers.

96. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, then prove that $\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}$ for all odd n.

97. Show that $\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}=a+b+c$.

98. Let $a_1,a_2,\dots,a_n$ be non-negative real numbers not all zero. Prove that $x^n-a_1x^{n-1}-\dots-a_{n-1}x-a_n=0$ has exactly one positive real root.

99. Let $P(x)$ be a real polynomial function, and $P(x)=ax^3+bx^3+cx+d$. Prove that if $P(x)\le 1$ for all x such that $|x|\le 1$, then $|a|+|b|+|c|+|d|\le 7$.

100. A person who left home between 4 p.m. and 5 p.m. returned between 5 p.m. and 6 p.m. and found that the hands of his watch has exactly changed places. When did he go out?

101. If $\alpha^{13}=1$ and $\alpha\ne 1$, find the quadratic equation whose roots are $\alpha+\alpha^3+\alpha^4+\alpha^{-4}+\alpha^{-3}+\alpha^{-1}$ and $\alpha^2+\alpha^5+\alpha^6+\alpha^{-6}+\alpha^{-5}+\alpha^{-2}$.

102. Determine all pairs of positive integers (m,n), such that $(1+x^n+x^{2n}+\dots+x^{mn})$ is divisible by $(1+x+x^2+\dots+x^m)$.

103. Show that $(a-b)^2+(a-c)^2=(b-c)^2$ is not solvable when a,b,c are all distinct.

104. If $P(x)$ is a polynomial of degree n such that $P(x)=2^x$ for $x=1,2,3,\dots,n+1$, find $P(x+2)$.

105. If a, b, c, d are all real and $a^2+b^2+c^2+d^2=ab+bc+cd+da$, then show that $a=b=c=d$.

106. Determine $x,y,z\in\mathbb{R}$, such that $2x^2+y^2+2x^2-8x+2y-2xy+2xz-16z=35=0$.

107. Find all real numbers satisfying $x^8+y^8=8xy-6$.

108. Solve the system of equations for real x and y, $5x\left(1+\frac{1}{x^2+y^2}\right)=12, 5y\left(1-\frac{1}{x^2+y^2}\right)=4$.

109. Solve the system $(x+y)(x+y+z)=18, (y+z)(x+y+z)=30, (z+x)(x+y+z)=2L$ in terms of L, where $x,y,z,L\in\mathbb{R}^+$.

110. Solve $x+y-z=4, x^2-y^2+z^2=-4, xyz=6$, where $x,y,z\in\mathbb{R}$.

111. Solve $3x(x+y-2)=2y\cdots(1), y(x+y-1)=9x\cdots(2)$.

112. Solve $xy+x+y=23\cdots(1), yz+y+z=31\cdots(2), zx+z+x=47\cdots(3)$.

113. Find all the solutions of the system of equations $y=4x^3-3x$, $z=4y^3-3y$ and $x=4z^3-3z$.

114. Let $x=p$, $y=q$, $z=r$ and $w=s$ be the unique solutions of the system of linear equations $x+a_iy+a_i^2z+a_i^3w=a_i^4, i=1,2,,$3,4. Express the solution of the following system in the terms of p, q, ,r and $s.x+a_i^2y+a_i^4z+a_i^6w=a_i^8, i=1,2,3,4$. Assume the uniqueness of the solution.

115. Find out all values of a and b, for which $xyz + z = a \cdots (1)$, $xyz^2 + z = b \cdots (2)$ and $x^2 + y^2 + z^2 = 4$ has only one solution.

116. Given a, b and c are positive real numbers, such that $a^2 + ab + \frac{b^2}{3} = 25$, $\frac{b^2}{3} + c^2 = 9$, $c^2 + ca + a^2 = 16$. Find out the value of $ab + 2bc + 3ca$.

117. Solve $\log_3(\log_2 x) + \log_{1/3}(\log_{1/2} y) = 1$, $xy^2 = 4$.

118. Solve $\log_2 x + \log_4 y + \log_4 z = 2, \log_3 y + \log_9 z + \log_9 x = 2, \log_4 z + \log_{16} x + \log_{16} y = 2$.

119. Find all real numbers x and y satisfying $\log_3 x + \log_2 y = 2$, $3^x - 2^y = 23$.

120. Let α, β, γ be the roots of $x^3 - x^2 - 1 = 0$, then find the value of $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1+\gamma}$.

121. Show that $(x-1)^2$ is a factor of $x^{m+1} - x^m - x + 1$.

122. Find all real solution x of the equation $x^{10} - x^8 + 8x^6 - 24x^4 + 32x^2 - 48 = 0$.

123. Solve $2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + \cdots + 2x + 3 = 0$ in $\mathbb{R}$.

124. Prove that $1 + x^{111} + x^{222} + x^{333} + x^{444}$ divides $1 + x^{111} + x^{222} + x^{333} + \cdots + x^{999}$.

125. If x, y, z are rational and strictly positive and if $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ show that $\sqrt{x^2 + y^2 + z^2}$ is rational.

126. If $a^2x^3 + b^2y^3 + c^2z^3 = p^5$, $ax^2 = by^2 = cz^2$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{p}$, find $\sqrt{a} + \sqrt{b} + \sqrt{c}$ only in terms of p.

127. If $ax^3 = by^3 = cz^3$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$; prove that $\sqrt[3]{ax^2 + by^2 + cz^2} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$.

128. Prove that, if (x, y, z) is a solution of the system of equations $x + y + z = a, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$. Then, at least one of the numbers x, y, z is 'a'.

129. If one root of the equation $2x^2 - 6x + k = 0$ is $\frac{1}{2}(a + 5i)$, where $i^2 = -1$; $k, a \in \mathbb{R}$, find the values of 'a' and 'k'.

130. If $x^3 + px^2 + q = 0$, where $q \neq 0$ has a root of multiplicity 2, prove that $4p^3 + 27q = 0$.

131. If $f(x)$ is a quadratic polynomial with $f(0) = 6$, $f(1) = 1$ and $f(2) = 0$, find $f(3)$.

132. Show that, if a, b, c are real number and $ac = 2(b + d)$, then, at least one of the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ has real roots.

133. Given any four positive, distinct, real numbers, show that one can choose three numbers A, B, C among them, such that all the quadratic equations have only real roots and all of them have only imaginary roots $Bx^2 + x + C = 0; Cx^2 + x + A = 0; Ax^2 + x + B = 0$.

134. Show that the equation $x^4 - x^3 - 6x^2 - 2x + 9 = 0$ has no negative roots.

135. If $a, b, c, d \in \mathbb{R}$ such that $a < b < c < d$, then show that, the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct.

136. Find the maximum no. of positive and negative real roots of the equation $x^3 + x^3 + x^2 - x - 1 = 0$.

137. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \neq 0$, show that the equation $P(x).Q(x)$ has at least two real roots.

138. Let $f(x)$ be the cubic polynomial $x^3 + x + 1$; suppose $g(x)$ is a cubic polynomial such that $g(0) = -1$ and the roots of $g(x) = 0$ are square roots of $f(x) = 0$. Determine $g(9)$.

139. If $p, q, r, s \in \mathbb{R}$, show that the equation $(x^2 + px + 3q)(x^2 + rx + q)(-x^2 + sx + 2q) = 0$ has at least two real roots.

140. If t_n denotes the nth term of an A.P., and $t_p = \frac{1}{q}$, $t_q = \frac{1}{p}$, then show that t_{pq} is a root of the equation $(p + 2q - 3r)x^2 + (q + 2r - 3p)x + (r + 2p - 3q) = 0$.

141. If p and q are odd integers, show that the equation $x^2 + 2px + 2q = 0$ has no rational roots.

142. Show that there cannot exist an integer n, such that $n^3 - n + 3$ divides $n^3 + n^2 + n + 2$.

143. If $s_n = 1 + q + q^2 + \cdots + q^n$ and $S_n = 1 + \frac{1+q}{2} + \left(\frac{1+q}{2}\right)^2 + \cdots + \left(\frac{1+q}{2}\right)^n$, then prove that $\binom{n+1}{1} + \binom{n+1}{2}s + \binom{n+1}{3}s^2 + \cdots + \binom{n+1}{n+1}s^n = 2^n S_n$.

144. Solve for x, y, z the equations $a = \frac{xy}{x+y}$, $b = \frac{yz}{y+z}$, $c = \frac{zx}{z+x}$ $\quad (a, b, c \neq 0)$.

145. Solve and find the non-trivial solutions: $x^2 + xy + zx = 0, y^2 + yz + zx = 0, z^2 + zx + zy = 0$

146. Solve: $x^2 + xy + y^2 = 7$, $y^2 + yz + z^2 = 19$, $z^2 + zx + x^2 = 3$.

147. Determine all solutions of the equation in $\mathbb{R}$, $(x^2 + 3x - 4)^3 + (2x^2 - 5x + 3)^3 = (3x^2 - 2x - 1)^3$.

148. Show that there is no positive integer, satisfying the condition that $n^4 + 2n^3 + 2n^2 + 2n + 1$ is a perfect square.

149. Find the possible solutions of the system of equations: $a^x = (x + y + z)^y$, $a^y = (x + y + z)^z$, $z^z = (x + y + z)^x$.

150. Show that $2x^3 - 4x^2 + x - 5$ cannot be factored into polynomials with integer coefficients.

151. The product of two of the four roots of the equation $x^4 + 7x^3 - 240x^2 + kx + 2000 = 0$ is -200, find k.

152. The product of the two of the four roots of $x^4 - 20x^3 + kx^2 + 590x - 1992 = 0$ is 24, find k.

153. Let a, b, c, d be any four real numbers not all equal to zero. Prove that the roots of the polynomial $f(x) = x^6 + ax^3 + bx^2 + cx + d$ can all not be real.

154. If a, b, c and p, q, r are real numbers, such that for every real number $x, ax^2 + 2bx + c \geq 0$ and $px^2 + 2qx + r \geq 0$, then prove that $apx^2 + bqx + cr \geq 0$ for all real x.

155. Find a necessary and sufficient condition on the natural number n, for the equation $x^n + (2 + x)^n + (2 - x)^n = 0$ to have an integral root.

156. Given that α, β and γ are the angles of a right angled triangle. Prove that $\sin\alpha \sin\beta \sin(\alpha-\beta) + \sin\beta \sin\gamma \sin(\beta-\gamma) + \sin\gamma \sin\alpha \sin(\gamma-\alpha) + \sin(\alpha-\beta) \sin(\beta-\gamma) \sin(\gamma-\alpha) = 0$.

157. Suppose a, b and c are three real numbers, such that the quadratic equation $x^2 - (a + b + c)x + (ab + bc + ca) = 0$ has roots of the form $\alpha \pm i\beta$, where $\alpha > 0$ and $\beta \neq 0$ are real numbers. Show that the numbers a, b, c are all positive and the numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}$ form the sides of a triangle.

158. Find the number of quadratic polynomials $ax^2 + bx + c$, where, a, b, c are distinct, $a, b, c \in \{1, 2, 3, \dots, 999\}$ and $(x + 1)$ divides $ax^2 + bx + c$.

159. Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic equations $x^2 + ax + b = 0$ and $x^2 + 2ax + b = 0$ have integer roots.

160. If the magnitude of the quadratic function $f(x) = ax^2 + bx + c$ never exceeds 1 for $0 \leq x \leq 1$, prove that the sum of the magnitudes of the coefficients cannot exceed 17.

161. Suppose that $-1 \leq ax^2 + bx + c \leq 1$ for $-1 \leq x \leq 1$, where a, b, c are real numbers, prove that $-4 \leq 2ax + b \leq 4$ for $-1 \leq x \leq 1$.

162. Find the polynomial $p(x) = x^2 + px + q$ for which $\max_{x\in[-1,1]}|p(x)|$ is minimal.

163. Find real numbers a, b, c for which $|ax^2 + bx + c| \leq 1\ \forall\ \ x < 1$ and $\frac{8}{3}a^2 + 2b^2$ is maximal.

164. Let $a, b, c \in \mathbb{R}$ and $a < 3$ and all roots of $x^3 + ax^2 + bx + c = 0$ are negative real numbers. Prove that $b + c < 4$.

165. $xp(x - 1) = (x - 30)p(x)\ \forall\ x \in \mathbb{R}$, find all such polynomial $p(x)$.

166. Find a polynomial $p(x)$ if it exist such that $xp(x - 1) = (x + 1)p(x)$.

167. Let $f(x)$ be a quadratic function. Suppose $f(x) = x$ has no real roots, then prove that $f(f(x)) = x$ has also no real roots.

168. The polynnomial $ax^4 + bx^3 + cx^2 + dx + e$ with integral coefficients is divisible by 7 for every integer x. Show that $7|a, 7|b, 7|c, 7|d, 7|e$.

169. Prove that $a^2 + ab + b^2 \geq 3(a + b - 1)\ \forall\ a,b \in \mathbb{R}$.

170. Let $p(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder on dividing $p(x^5)$ by $p(x)$.

171. Find the remainder when x^{2025} is divided by $(x^2+1)(x^2+x+1)$.

172. Prove that there does not exist a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, such that $p(0), p(1), p(2), \ldots$ are all prime numbers.

173. Solve: $x + \sqrt{a + \sqrt{x}} = a$ for real x and a.

174. Solve: $x^2 - \sqrt{a - x} = a$ for real x and a.

175. Solve: $\sqrt{a - \sqrt{a + x}} = x$ for real x and a.

176. The polynomial $ax^3 + bx^2 + cx + d = 0$ has integral coefficinets with ad odd and bc even. Prove that all the roots cannot rational.

177. If roots of $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has real roots, then find the minimum value of $a^2 + b^2$.

178. If the coefficient of x^k upon the expansion and collecting of terms in the expansion $\underbrace{\left(\cdots\left(\left((x-2)^2-2\right)^2-2\right)^2\cdots-2\right)^2}_{n\text{ times}}$ is a_k, then find a_0, a_1, a_2 and a_{2^k}.

179. Prove that the equations $x^2 - 3xy + 2y^2 + x - y = 0$ and $x^2 - 2xy + y^2 - 5x + 7y = 0$ imply the equation $xy - 12x + 15y = 0$.

180. If a and b are integers, and the solution of the equation $y - 2x - a = 0$ and $y^2 - xy + x^2 - b = 0$ are rational, then prove that the solutions are integers.

181. Solve the following system of equations for real numbers $a, b, c, d, e : 3a = (b + c + d)^3, 3b = (c + d + e)^3, c = (d + e + d)^3, d = (e + a + b)^3, 3e = (a + b + c)^3$.

182. Solve for real numbers x and y, simultaneously the equations $xy^2 = 15x^2 + 17xy + 15y^2$ and $x^2y = 20x^2 + 3y^2$.

183. Solve the system of equations in integers: $3x^2 - 3xy + y^2 = 7, 2x^2 - 3xy + 2y^2 = 14$.

184. In the sequence $a_1, a_2, a_3, \ldots, a_n$, the sum of any three consecutive terms is 40; if the third term is 10, and the eighth term is 8; find the 2013th term.

185. A sequence has first term 2007, after which every term is the sum of the squares of the digits of the preceding term. Find the sum of this sequence up to 2013 terms.

186. Find a finite sequence of 16 numbers, such that it reads same from left to right as from right to left, the sum of any 7 consecutive terms is -1, and the sum of any 11 consecutive terms is $+1$.

187. A two-pan balance is inaccurate since its balance arms are of different lengths and its pans are of different weights. Three objects of different weights A, B and C are

each weighed separately. When they are placed on left pan, they are balanced by weights A_1, B_1 and C_1 respectively. When A and B are placed in the right pan, they are balanced by A_2 and B_2 respectively. Determine the true weights of C in terms of A_1, B_1, C_1, A_2 and B_2.

188. If a and b are two of the roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

189. If $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are all polynomials, such that $P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$, prove that $(x - 1)$ is a factor of $P(x)$.

190. If $x^5 - x^3 + a$, prove that $x^6 \geq 2a - 1$.

191. The roots x_1, x_2 and x_3 of the equation $x^3 + ax + a = 0$, where a is real and $a \neq 0$, satisfy $\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \frac{x_3^2}{x_1} = -8$, find x_1, x_2 and x_3.

192. Let $p(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that $p(0) = 2007, p(1) = 2006, p(2) = 2005, \ldots, p(2007) = 0$; determine $p(2008)$.

193. If $P(x)$ denotes a polynomial of degree n, such that $P(k) = \frac{1}{k}$, for $k = 1, 2, 3, \ldots, n + 1$, find $P(n + 1)$.

194. If $P(x)$ denotes a polynomial of degree n, such that $P(k) = \frac{k}{k+1}$, for $k = 1, 2, 3, \ldots, n$, find $P(n + 1)$.

195. Let a, b and c denote three integers, and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$ and $P(c) = a$.

196. In the polynomial $P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + 1$, the coefficients $a_1, a_2, \ldots, a_{n-1}$ are non-negative, and it has n real roots. Prove that $P(2) \geq 3^n$.

197. Determine all the polynomials of degree n with each of its $n + 1$ coefficients equal to ± 1, which have only real roots.

198. Let $p(x)$ be a polynomial over $\mathbb{Z}$, and at three distinct integers it takes ± 1 value, prove that it has no integral roots.

199. Let α, β be the roots of $x^2 - 6x + 1 = 0$. Prove that $\alpha^n + \beta^n \in \mathbb{Z}\ \forall\ n \in \mathbb{N}$, also prove that $5 \nmid (\alpha^n + \beta^n)\ \forall\ n \in \mathbb{N}$.

200. Let $P(x)$ be a polynomial with real coefficients such that $P(x) \geq 0$ for every real x. Prove that $P(x) = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2$.

201. Is it possible to find three quadratic polynomial $f(x), g(x), h(x)$ such that the equation $f(g(h(x)))$ has eight roots $1, 2, 3, 4, 5, 6, 7, 8$.

202. Let $P(z) = az^3 + bz^2 + cz + d$, where a, b, c, d are complex numbers with $|a| = |b| = |c| = |d| = 1$. Show that $|P(z)| \geq \sqrt{6}$ for at least one complex number z satisfying $|z| = 1$.

203. Consider two monic polynomials $f(x)$ and $g(x)$ of degree 4 and 2 respectively over real numbers. Let there be an interval (a, b) of length more than 2 such that both $f(x)$ and $g(x)$ are negative for $x \in (a, b)$ and both are positive for $x < a$ and $x > b$. Prove that there is a real number α such that $f(\alpha) < g(\alpha)$.

204. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_i(x))\ \forall\ i = 1, 2, 3, ...$ Show that for any positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct.

205. Find all polynomials f satisfying $f(x^2) + f(x)f(x+1) = 0\ \forall x \in \mathbb{C}$.

206. Find all polynomials $P(x)$, for which $P(x)P(2x^2) = P(2x^3 + x)\ \ \forall x \in \mathbb{R}$.

207. Find all polynomials $f(x)$ such that $f(x)f(x+1) = f(x^2 + x + 1)$.

208. Find all polynomials $f(x)$ such that $f(x)f(-x) = f(x^2)$.

209. Prove that if a polynomial of degree 7 over $\mathbb{Z}$ is equal to ± 1 for 7 different integers then it is irreducible over $\mathbb{Z}$.

210. Prove that $(x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1$ is irreducible over $\mathbb{Z}$.

211. For what values of $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots?

212. If $a + b + c = 0$ and a,b,c are rational. Prove that the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are rational.

213. Show that if the roots of the equation $(a^2 + b^2)x^2 + 2(ac + bd)x + c^2 + d^2 = 0$ are real, they will be equal.

214. If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ be equal, prove that a, b, c are in H.P.

215. If $a + b + c = 0$ and a,b,c are real, prove that equation $(b - x)^2 - 4(a - x)(c - x) = 0$ has real roots and roots will not be equal unless $a = b = c$.

216. Show that if p,q,r,s are real numbers and $pr = 2(q + s)$ then at least one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.

217. If the equation $x^2 - 2px + q = 0$ has two equal roots, then the equation $(1 + y)x^2 - 2(p + y)x + (q + y) = 0$ will have its roots real and distinct only when y is negative and p is not unity.

218. If the equation $ax^2 + 2bx + c = 0$ has real roots. a,b,c being real numbers and if m and n are real numbers such that $m^2 > n^2 > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.'

219. If theq equations $ax + by = 1$ and $cx^2 + dy^2 = 1$ have only one solution, prove that $\frac{a^2}{c} + \frac{b^2}{d} = 1$ and $x = \frac{a}{c}, y = \frac{b}{d}$.

220. If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that $\frac{(r+1)^2}{r} = \frac{b^2}{ac}$.

221. If one root of the eq. $(l-m)x^2+lx+1=0$ be double of the other and if l be real, show that $m \leq \frac{9}{7}$.

222. If one root of the quadratic equation $ax^2+bx+c=0$ is equal to the nth power of the other, then show that

$$(ac^n)^{1/(n+1)}+(a^nc)^{1/(n+1)}+b=0$$

223. If the roots of the equation $ax^2+bx+c=0$ be in the ratio $p:q$, show that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{c}{a}}=0$$

224. If α and β be the roots of the equation $x^2+px+q=0$. Find the value of the following in the terms of p and q.

   i. $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

   ii. $(\omega\alpha+\omega^2\beta)(\omega^2\alpha+\omega\beta)$, where ω an imaginary cube root fo unity.

225. If α and β be the roots of the equation $A(x^2+m^2)+Amx+cm^2x^2=0$, prove that $A(\alpha^2+\beta^2)+A\alpha\beta+c\alpha^2\beta^2=0$.

226. If α and β be the roots of the euqation $ax^2+bx+c=0$, prove that $a\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)+b\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)=b$.

227. If a and b are the roots of the equation $x^2+px+1=0$ and c and d are the roots of the equation $x^2+qx+1=0$, show that $q^2-p^2=(a-c)(b-c)(a+d)(b+d)$.

228. If the roots of the equation $x^2+px+q=0$ differ from the roots of the equation $x^2+qx+p=0$ by the same quantity, show that $p+q+4=0$.

229. If α, β are the roots of the equation $ax^2+bx+c=0$ and $S_n=\alpha^n+\beta^n$, show that $aS_{n+1}+bS_n+cS_{n-1}=0$ and hence find S_5.

230. If the sum of roots of the equation $ax^2+bc+c=0$ is equal to the sum of the squares of their reciprocals, show that bc^2, ca^2, ab^2 are in A.P.

231. If α and β be the values of x obtained from the equation $m^2(x^2-x)+2mx+3=0$ and if m_1 and m_2 be the two values of m for which α and β are connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{3}$, find the value of $\frac{m_1^2}{m_2}+\frac{m_2^2}{m_1}$.

232. If the ratio of the roots of the equation $ax^2+bx+c=0$ be equal to the roots of equation $a_1x^2+b_1x+c_1=0$, prove that $\left(\frac{b}{b_1}\right)^2=\frac{ca}{c_1a_1}$.

233. Find the quantity equation with the rational coefficients one of whose roots is $\frac{1}{2+\sqrt{5}}$.

234. If α and β be the roots of the equation $ax^2+bx+c=0$, find the quantity equation whose roots are $\frac{1}{a\alpha+b}$ and $\frac{1}{a\beta+b}$.

235. If c,d are the roots of the equation $(x-a)(x-b)=k$, show that a,b are the roots of the equation $(x-c)(x-d)+k=0$

236. The coefficients of x in the equation $x^2+px+q=0$ was wrongly written as 17 in place of 13 and roots were found to be -2 and -15. Find the roots of the correct equation.

237. If α and β be the roots of the equation $x^2+px+q=0$, show that $\frac{\alpha}{\beta}$ is a root of the equation $qx^2-(p^2-2q)x+q=0$.

238. If $x^2-ax+b=0$ and $x^2-px+q=0$ have a common root and the second equation has equal roots then show that $b+q=\frac{ap}{2}$.

239. If $ax^2+2bx+c=0$ and $a_1x^2+2b_1x+c_1=0$ have a common root and $\frac{a}{a_1},\frac{b}{b_1},\frac{c}{c_1}$ are in A.P., show that a_1,b_1,c_1 are in G.P.

240. If each pair of the following three equations $x^2+p_1x+q_1=0$, $x^2+p_2x+q_2=0$, $x^2+p_3x+q_3=0$ have exactly one root in common, then show that $(p_1+p_2+p_3)^2=4(p_1p_2+p_2p_3+p_3p_1-q_1-q_2-q_3)$.

241. If the equations $x^2+cx+bc=0$ and $x^2+bx+ca=0$ have a common root, show that $a+b+c=0$; show that other roots are given by the equation $x^2+ax+bc=0$.

242. If $a,b,c\in\mathbb{R}$ and equations $ax^2+bx+c=0$ and $x^2+2x+9=0$ have a common root, show that $a:b:c=1:2:9$.

243. Find the value of p if the equation $3x^2-2x+p=0$ and $6x^2-17x+12=0$ have a common root.

244. Show that $|x|^2-|x|-2=0$ is an equation.

245. Show that $\frac{(x+b)(x+c)}{(b-a)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1$ is an indenity.

246. If a,b,c,a_1,b_1,c_1 are rational and equations $ax^2+2bx+c=0$ and $a_1x^2+2b_1x+c_1=0$ have one and only one root in common, prove that b^2-ac and $b_1^2-a_1c_1$ must be perfect squares.

247. If $(a^2-1)x^2+(a-1)x+a^2-4a+3=0$ be an identity in x, then find the value of a.

248. Solve $\left(x+\frac{1}{x}\right)^2=4+\frac{3}{2}\left(x+\frac{1}{x}\right)$.

249. Solve $(x+4)(x+7)(x+8)(x+11)+20=0$.

250. Solve $3^{2x+1}+3^2=3^{x+3}+3^x$.

251. Solve $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$.

252. A car travels 25 km per hour faster than a bus for a jouney of 500 km. The bus takes 10 hours more than the car. Find the speed of the bus and the car.

253. Show that the roots of the equation $(a+b)^2x^2-2(a^2-b^2)x+(a-b)^2=0$ are equal.

254. Show that the equation $3x^2+7z+8=0$ cannot be satisfied by any real values of x.

255. For what values of a will the roots of the equation $3x^2+(7+a)x+8-a=0$ be equal.

256. If the roots of the equation $(a^2+b^2)x^2+2(ac+bd)x+(c^2+d^2)=0$ are equal then show that $a:b=c:d$.

257. Prove that the roots of the equation $(b-c)x^2+2(c-a)x+(a-b)=0$ are always real.

258. Show that the roots of the equation $\frac{1}{x-a}+\frac{1}{a}+\frac{1}{x-1}=0$ are real for all real values of a.

259. Show that if $a+b+c=0$, the roots of the equation $ax^2+bx+c=0$ are rational.

260. Prove that the roots of the equation $(b+c-2a)x^2+(c+a-2b)x+(a+b-2c)=0$ are rational.

261. Show that the roots of the equation $x^2+rx+s=0$ will be rational if $r=k+\frac{s}{k}$, where r, s and k are rational.

262. Prove that roots of the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are always real and cannot be equal unless $a=b=c$.

263. If a,b,c are rational, show that the roots of the equation $a^2(b^2-c^2)x^2+b^2(c^2-a^2)x+c^2(a^2-b^2)=0$ are rational.

264. Show that the roots of the equation $(a^4+b^4)x^2+4abcdx+c^4+d^4=0$ cannot be different, if real.

265. If p,q,r are in H.P. and p and r are of the same sign, prove that the roots of the equation $px^2+2qx+r=0$ will be complex.

266. Prove that the roots of the equation $bx^2+(b-c)x+(b-c-a)=0$ are real if those of equation $ax^2+2bx+b=0$ are imaginary and vice-versa.

267. Prove that the values of x obtained from the equations $ax^2+by^2=1$ and $ax+by=1$ will be equal if $a+b=1$.

268. Prove that the values of x obtained from the equations $x^2+y^2=a^2$ and $y=mx+c$ will be equal if $c^2=a^2(1+m^2)$.

269. The roots of the equation $4x^2-(5a+1)x+5a=0$ are α and β. If $\beta=1+\alpha$, calculate the possible values of a, α and β.

270. If one root of the equation $5x^2+13x+k=0$ be reciprocal of another, find k.

271. Find the values of m, for which the equation $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$ has (a) equal roots, (b)the products of root is 2, and (c) the sum of roots is 6.

272. Find the relation between the coefficients of the quadratic equal $ax^2 + bx + c = 0$ if one root is n times the another.

273. If the roots of the equation $ax^2+bx+c = 0$ are in the ratio $3 : 4$, prove that $12b^2 = 49ac$.

274. If the roots of the equation $4x^2 + ax + 3 = 0$ are in the ratio $1 : 2$, show that the roots of the equation $ax^2 + 3x + a = 2$ are imaginary.

275. If one root of the equation $x^2 - px + q = 0$ be m times their difference, prove that $p^2(m^2 - 1) = 4m^2q$.

276. If the difference of the roots $x^2 - px + q = 0$ is unity, then prove that $p^2 - 4q = 1$ and $p^2 + 4q = (1 + 2q)^2$.

277. Find the condition that the equation $\frac{a}{x-a}+\frac{b}{x-b} = m$ may have roots equal in magnitude but opposite in sign.

278. Find the relation between coefficients of the euqation $ax^2 + bx + c = 0$ if one root exceeds other by k.

279. If one root of the equation $ax^2 + bx + c = 0$ be square of the other, show that $b^3 + a^2c + ac^2 = 3abc$.

280. Determine the value p for which one root of the equation $x^2 + px + 1 = 0$ is the square of the other.

281. If one root of the equation $x^2 + px + q = 0$ be the square of the other then show that $p^3 - q(3p - 1) + q^2 = 0$.

282. If α, β be the roots of the equation $2x^2 + 3x + 4 = 0$. Find the values of

   i. $\alpha^2 + \beta^2$

   ii. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

283. If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the values of $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ in terms of a, b, c.

284. If α, β are the roots of the equation $ax^2 + bx + c = 0$, prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}} = 0$.

285. Show that the two equations $x^2 - 2ax + b^2 = 0$ and $x^2 - 2bx + b^2 = 0$ are such that the G.M. of the roots of one is equal to the A.M. of the roots of the another.

286. If sum of the roots of the equation $px^2 + qx + r = 0$ be equal to the sum of their squares, show that $2pr = pq + q^2$.

287. If α, β be the roots of the equation $x^2 - px + q = 0$, prove that $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - \frac{4p^2}{q} + 2$.

288. If α, β be the roots of the equation $ax^2 + bx + c = 0$, find the value of $\frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$.

289. If α, β be the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$ and if λ_1 and λ_2 are the two values for which the roots α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then prove that

   i. $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = 254$

   ii. $\frac{\lambda_1^2}{\lambda_2} + \frac{\lambda_2^2}{\lambda_1} = 4048$

290. If α, β be the roots of the equation $x^2 + px + q = 0$ and γ, δ be the roots of the equation $x^2 + rx + s = 0$, find the values of

   i. $(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)$

   ii. $(\alpha - \gamma)(\beta - \delta) + (\beta - \gamma)(\alpha - \delta)$

   iii. $(\alpha - \gamma)^2 + (\beta - \delta)^2 + (\beta - \gamma)^2 + (\alpha - \delta)^2$

291. If α, β be the roots of the equation $x^2 - px + q = 0$ and $\nu_n = \alpha^n + \beta^n$, prove that $\nu_{n+1} = p\nu_{n-1} - q\nu_{n-1}$.

292. If α, β be the roots of the equation $x^2 + px + q = 0$ and γ, δ those of equation $x^2 + px + r = 0$, prove that $(\alpha - \gamma)(\alpha - \delta) = (\beta - \gamma)(\beta - \delta) = -(q + r)$.

293. If α, β be the roots of the equation $x^2 - 2px + q = 0$ and γ, δ those of equation $x^2 - 2rx + s = 0$ and if

   i. $\alpha\delta = \beta\gamma$, prove that $p^2 s = r^2 q$.

   ii. $\alpha, \beta, \gamma, \delta$ be in G.P., prove that $p^2 s = r^2 q$

   iii. $\alpha, \beta, \gamma, \delta$ be in A.P., prove that $s - q = r^2 - p^2$.

294. If the roots of the equation $ax^2 + 2bx + c = 0$ be α and β, and those of the equation $Ax^2 + 2Bx + C = 0$ be $\alpha + k$ and $\beta + k$, prove that $\frac{b^2 - ac}{B^2 - AC} = \frac{a^2}{A^2}$.

295. If the roots of the equation $ax^2 + bx + c = 0$ be α and β, and those of the equation $Ax^2 + Bx + C = 0$ be $\alpha + k$ and $\beta + k$, prove that $\frac{b^2 - 4ac}{B^2 - 4AC} = \frac{a^2}{A^2}$.

296. If the roots of the equation $x^2 + 2px + q = 0$ and $x^2 + 2qx + p = 0$ differ by a constant then show that $p + q + 1 = 0$.

297. If α, β be the roots of the equation $ax^2 + bx + c = 0$ then find the equations whose roots are

   i. $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

   ii. $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

iii. $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$

iv. $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$

v. $\frac{1}{(\alpha+\beta)^2}$ and $(\alpha-\beta)^2$

298. Find those equations whose roots are (a) reciprocal of the roots of (b) equal in magnitude but opposite in sign to the roots of the equation $ax^2+bx+c=0$.

299. If α, β be the roots of the equation $x^2+px+q=0$, find the value of (a) $\alpha^4+\beta^4$ (b) $\alpha^{-4}+\beta^{-4}$

300. If α, β be the roots of the equation $x^2-px+q=0$, find the equation whose roots are

i. $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$

ii. $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$

301. Find the values of p and q such that the equation $x^2+px+q=0$ has $5+3i$ as a root.

302. Form the quadratic equation whose one root is $3+4i$.

303. If one root of the equation $4x^2+2x-1=0$ be α then prove that its second root is $4\alpha^2-3\alpha$.

304. If $\alpha \neq \beta$ and $\alpha^2=5\alpha-3, \beta^2=5\beta-3$, form the quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

305. In copying a quadratic equation of the form $x^2+px+q=0$, the coefficient of x was wrongly written as -10 in place of -11 and the roots were found to be 4 and 6. Find the roots of the correct equation.

306. In writing a quadratic equation of the form $x^2+px+q=0$, the constant term was wrongly written as -6 in place of 2 and the roots were found to be 6 and -1. Find the correct equation.

307. Two candidaes attempt to solve a quadratic equation of the form $x^2+px+q=0$. One starts wiith a wrong values of p and finds the roots to be 2 and 6. The other starts with a wrong value of q and finds the roots too be 2 and -9. Find the correct roots.

308. If α, β be the roots of the quadratic equation $x^2+px+q=0$ and α_1, β_1 be the roots of the equation $x^2-px+q=0$. Form the quadratic equation whose roots are $\frac{1}{\alpha_1\beta}+\frac{1}{\alpha\beta_1}$ and $\frac{1}{\alpha\alpha_1}+\frac{1}{\beta\beta_1}$.

309. If $2+\sqrt{3}i$ is a root of the equation $x^2+px+q=0$, where p, q are real, then find them.

310. Find the equation whose one root is $\frac{1}{2+\sqrt{3}}$.

311. If α, β are the roots of equation $x^2 - px + q = 0$, show that $\alpha + \frac{1}{\beta}$ is a root of equation $qx^2 - p(1+q)x + (1+q)^2 = 0$.

312. Determine the value of m for which $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

313. Find the value of a if $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ have a common root.

314. If the equations $ax^2 + bx + x = 0$ and $bx^2 + cx + a = 0$ have a common root then show either $a + b + c = 0$ or $a = b = c$.

315. Find the value of m so that equations $x^2 + 10x + 21 = 0$ and $x^2 + 9x + m = 0$ may have a common root. Find also the equation formed by the other roots.

316. Show that the equations $x^2 - x - 12 = 0$ and $3x^2 + 10x + 3 = 0$ have a common root. Also, find the common root.

317. If the equations $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$ have a common root, show that $2p^2 + 3q^2 - 5pq + 1 = 0$.

318. Show that the equation $ax^2 + bx + c = 0$ and $x^2 + x + 1 = 0$ cannot have a common root unless $a = b = c$.

319. If the equations $x^2 + px + q = 0$ and $x^2 + p_1x + q_1 = 0$ have a common root, show that it must be either $\frac{pq_1 - p_1q}{q - q_1}$ or $\frac{q - q_1}{p_1 - p}$.

320. Prove that the two quadratic equations $ax^2 + bx + c = 0$ and $2x^2 - 3x + 4 = 0$ cannot have common root unless $6a = -4b = 3c$.

321. Prove that the equations $(q-r)x^2 + (r-p)x + p - q = 0$ and $(r-p)x^2 + (p-q)x + q - r = 0$ have a common root.

322. If the equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root, prove that their other roots satisfy the equation $x^2 - a(b+c)x + a^2bc = 0$.

323. If the equations $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the other root of the first, then prove that $(q-b)^2 = bq(p-a)^2$.

324. Show that $(x-2)(x-3) - 8(x-1)(x-3) + 9(x-1)(x-2) = 2x^2$ is an identity.

325. Show that $\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-a)(x-c)}{(b-a)(b-c)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2$ is an identity.

326. Show that $3x^{10} - 2x^5 + 8 = 0$ is an equation.

327. Solve the equation $\frac{x+2}{x-2} - \frac{x-2}{x+2} = \frac{5}{6}$.

328. Solve the equation $\frac{2\sqrt{x}+1}{3-\sqrt{x}} = \frac{11-3\sqrt{x}}{5\sqrt{x}-9}$.

329. Solve the equation $(x+1)(x+2)(x-3)(x-4) = 336$.

330. Solve the equation $\sqrt{x+1}+\sqrt{2x-5}=3$.

331. Solve the equation $2^{2x}+2^{x+2}-32=0$.

332. A pilot flies an aircraft with a cetain speed for a distance of 800 km. He could have saved 40 minutes by increasing the average speed of the aircraft by 40 km/hour. Find the average speed of the aircraft.

333. The length of a rectangle is 2 meters more than its width. If the length is increased by 6 meters and width is decreased by 2 meters, the area becomes 119 sq. mt. Find the dimensions of original rectangle.

334. Find the range of values of x for which $-x^2+3x+4>0$.

335. Find all integral values of x for which $5x-1<(x+1)^2<7x-3$.

336. Find all values of x for which the inequality $\frac{8x^2+16x-51}{(2x-3)(x+4)}>3$ holds.

337. Show that the expression $\frac{x^2-3x+4}{x^2+3x+4}$ lies between 7 and $\frac{1}{7}$ for real values of x.

338. If x be real, prove that the expression $\frac{x^2+34x-71}{x^2+2x-7}$ has no value between 5 and 9.

339. If x be real, show that the expression $\frac{4x^2+36x+9}{12x^2+8x+1}$ can have any real value.

340. Prove that if x is real, the expression $\frac{(x-a)(x-c)}{x-b}$ is capable of assuming all values if $a>b>c$ or $a<b<c$.

341. If $x+y$ is constant, prove that xy is maximum when $x=y$.

342. If x be real, find the maximum value of $3-6x-8x^2$ and the corresponding value of x.

343. Prove that $\left|\frac{12x}{4x^2+9}\right|\leq 1$ for all real values of x or the equality being satisfied only if $|x|=\frac{3}{2}$.

344. Prove that if the equation $x^2+9y^2-4x+3=0$ is satisfied for real values of x and y, x must lie between 1 and 3, and y must lies between $-\frac{1}{3}$ and $\frac{1}{3}$.

345. Find the value of a for which $x^2-ax+1-2a^2>0$ for all real values of x.

346. Determine a such that $x^2-11x+a$ and $x^2-14x+2a$ may have a common factor.

347. Find the condition that the expressions $ax^2+bxy+cy^2$ and $a_1x^2+b_1xy+c_1y^2$ may have factors $y-mx$ and $my-x$ respectively.

348. Find the values of m for which the expression $2x^2+mxy+3y^2-5y-2$ can be resolved into two linear factors.

349. If the expression $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ can be resolved into two rational factors prove that $a^3 + b^3 + c^3 = 3abc$.

350. Find the linear factors of $2x^2 - y^2 - x + xy + 2y - 1$.

351. Show that the expression $x^2 + 2(a + b + c)x + 3(ab + bc + ca)$ will be a perfect square if $a = b = c$.

352. If x is real, prove that $2x^2 - 6x + 9$ is always positive.

353. Prove that $8x - 15 - x^2 > 0$ for limited values of x and also find the limits.

354. Find the range of the values of x for which $-x^2 + 5x - 4 > 0$.

355. Find the range of the values of x for which $x^2 + 6x - 27 > 0$.

356. Find the solution set of inequation $\frac{4x}{x^2+3} \geq 1$, $x \in \mathbb{R}$.

357. Find the real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$.

358. If x be real and the roots of the equation $ax^2 + bx + c = 0$ are imaginary, prove that $a^2x^2 + abx + ac$ is always positve.

359. Prove that the expression $\frac{x^2-2x+4}{x^2+2x+4}$ lies between $\frac{1}{3}$ and 3 for real values of x.

360. If x be real, show that $\frac{2x^2-3x+2}{2x^2+3x+2}$ lies between 7 and $\frac{1}{7}$.

361. If$p > 1$ and x is real, show that $\frac{x^2-2x+p^2}{x^2+2x+p^2}$ lies between $\frac{p-1}{p+1}$ and $\frac{p+1}{p-1}$.

362. if x be real, prove that the expansion $\frac{(x-1)(x+3)}{(x-2)(x+4)}$ does not lie between $\frac{4}{9}$ and 1.

363. if $a^2 + c^2 > ab$ and $b^2 > 4c^2$ for real x, show that $\frac{x+a}{x^2+bx+c^2}$ cannot lie betwen two limits.

364. show that if x real, the expression $\frac{x^2-bc}{2x-b-c}$ has no real value between b and c.

365. show that no real values of x and y besides 4 can satisfy the equation $x^2 - xy + y^2 - 4x - 4y + 16 = 0$.

366. prove that if $x^2 + 12xy + 4y^2 + 4x + 8y + 20 = 0$ is satisfied by real values of x and y, x cannot lies between -2 and 1 whereas y cannot lie betweenn -1 and $\frac{1}{2}$.

367. a rectangular field, one of whose sides is a straight edge of a river is to be enclosed by 600 meters of fencing on the remaining three sides. what would be the length and breadth of the rectangle if the ecnlosed area is to be as large as possible.

368. find the condition that the expression $ax^2 + 2hxy + by^2$ may have two factors of the form $y - mx$ and $my + x$.

369. if $p(x) = ax^2 + bx + c$ and $q(x) = -ax^2 + bx + c$, where $ac \neq 0$, show that the equation $p(x).q(x) = 0$ has at least two real roots.

370. prove that the roots of the equation $bx^2 + (b-c)x + b - c - a = 0$ are real if those of equation $ax^2 + 2bx + b = 0$ are imaginary and vice-versa, where $a, b, c \in \mathbb{r}$.

371. if a, b, c are odd numbers, show that the roots of the equation $ax^2 + bx + c = 0$ cannot be rational.

372. if roots of the equation $ax^2 + 2bx + c = 0$ are real and distinct, then show that the roots of the equation $(a+c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$ are complex numbers and vice-versa.

373. if $n, r \in \mathbb{p}$ such that $0 < r < n$, then show that the roots of the quadratic equation $C^n_{r-1}x^2 + C^n_r x + C^n_{r+1} = 0$ are real and distinct.

374. show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solutions.

375. if a, b, c are non-zero, real numbers and the equation $az^2 + bz + c + i = 0$ have purely imaginary roots then prove that $a = b^2c$.

376. if a and b are integers and the roots of the equation $x^2 + ax + b = 0$ are rational, show that they will be integers.

377. show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0$, where q is an integer has no integral roots.

378. solve the equation $a^3(b-c)(x-b)(x-c) + b^3(c-a)(x-a)(x-c) + c^3(a-b)(x-a)(x-b) = 0$. also show that the roots are equal if $\frac{1}{\sqrt{a}} \pm \frac{1}{\sqrt{b}} \pm \frac{1}{c} = 0$.

379. if roots of the equation $ax^2 + bx + c = 0$ be $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, prove that $(a+b+c)^2 = b^2 - 4ac$.

380. if $f(x) = ax^2 + bx + c$, and α, β be the roots of the equation $px^2 + qx + r = 0$, show that $f(\alpha)f(\beta) = \frac{(cp-ar)^2 - (bp-aq)(cq-br)}{p^2}$. hence or otherewise, show that if $ax^2 + bx + c = 0$a nd $px^2 + qx + r = 0$ have a common root, then $bp - aq$, $cp - ar$ and $cq - br$ are in g.p.

381. if $a(p+q)^2 + 2pbq + c = 0$ and $a(p+r)^2 + 2bpr + c = 0$, then show that $qr = p^2 + \frac{c}{a}$.

382. If α, β are the roots of the equation $x^2 - p(x+1) - c = 0$, show that $(\alpha+1)(\beta+1) = 1 - c$. Hence, prove that $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = 1$.

383. If α, β be the roots of the equation $x^2 + px + q = 0$ and $x^{2n} + p^n x^n + q^n = 0$, where n is an even integer, prove that $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of the equation $x^n + 1 + (x+1)^n = 0$.

384. If the roots of the equation $x^2 - ax + b = 0$be real and differ by less than c, the show that b must lie between $\frac{a^2-c^2}{4}$ and $\frac{a^2}{4}$.

385. Let a, b and c be interger with $a > 1$, and let p be a prime number. Show that if $ax^2 + bx + c = p$ for two distinct integral values of x, then it cannot be equal to $2p$ for any integral value of x.

386. If α and β are the roots of equation $x^2 + px + q = 0$ and α^4, β^4 are the roots of the equation $x^2 - rx + s = 0$, show that the equation $x^2 - 4qx + 1q^2 - r = 0$ has real roots.

387. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $\alpha_1, -\beta$ are those of equation $a_1x^2 + b_1x + c_1 = 0$, show that α, α_1 are the roots of the equation

$$\frac{x^2}{\frac{b}{a} + \frac{b_1}{a_1}} + x + \frac{1}{\frac{b}{c} + \frac{b_1}{c_1}} = 0.$$

388. How many quadratic equations are possible which remains unchanged when its roots are squared?

389. If a, b, c are in G.P. then show that the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in H.P.

390. If the three equations $a^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common root, find a, b and the roots of the equaiton.

391. If $m(ax^2 + 2bx + c) + px^2 + 2qx + r$ cab be expressed in the form of $n(x + k)^2$, then show that $(ak - b)(qk - r) = (pk - q)(bk - c)$.

392. The real numbers x_2, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$are in A.P. Find the intervals in which β and γ must lie.

393. If equations $x^3 + 3px^2 + 3qx + r = 0$ and $x^2 + 2px + q = 0$ have a common root, show that $4(p^2 - q)(q^2 - pr) = (pq - r)^2$.

394. If $c \neq 0$ and the equations $x^3 + 2ax^2 + 3bx + c = 0$ and $x^3 + ax^2 + 2bx = 0$ have a common root, show that $(c - 2ab)^2 = (2b^2 - ac)(a^2 - b)$.

395. If equation $x^3 + ax + b = 0$ have only real roots, then prove that $4a^3 + 27b^2 \leq 0$.

396. Let α be a root of $ax^2 + bx + c = 0$ and β be a root of $-ax^2 + bx + c = 0$ show that there exists a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ that lie between α and β or β and α as the case may be$(\alpha, \beta \neq 0)$

397. If $a, b, c \in \mathbb{R}, a \neq 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real root then show that $(a + b + c)c > 0$.

398. If $a < b < c < d$, then show that the quadratic equation $(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real roots for all real values of λ.

399. If $ax + 3b + 6c = 0$, $(a, b, c \in \mathbb{R})$ then show that the equation $ax^2 + bx + c = 0$ has at least one root between 0 and 2.

400. If a, b, c be non-zero real numbers such that $\int_0^1 (1 + \cos^8 x(ax^2 + bx + c)\, dx = \int_0^2 (1 + \cos^8 x)\,(ax^2 + bx + c)\, dx$, show that the equation $ax^2 + bx + c = 0$ has at least one real root between 1 and 2.

401. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$. If $f(x) = x$ has non-real roots, show that the equation $f(f(x)) = x$ has all non-real roots.

402. Let $a, b, c \in \mathbb{P}$ and consider all quadratic equations of the form $ax^2 - bx + c = 0$, which have two distinct real roots in $]0, 1[$. Find the least positive integers a and b for which such a quadratic equation exist.

403. If equation $ax^2 - bx + c = 0$ have two distinct real roots in $(0, 1)$, $a, b, c \in \mathbb{N}$, then prove that $\log_5(abc) \geq 2$.

404. If equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then find the least value of $3a + b$.

405. If equation $2x^3 + ax^2 + bx + 4 = 0$ has three real roots, where $a, b > 0$, show that $a + b > -6$.

406. Show that equation $x^3 + 2x^2 + x + 5 = 0$ has only real root α such that $[\alpha] = -3$, where $[x]$ denotes the integral part of x.

407. Solve $(x^2 + 2)^2 + 8x^2 = 6x)(x^2 + 2)$.

408. Solve $3x^3 = (x^2 + \sqrt{18}x + \sqrt{32})(x^2 - \sqrt{18}x - \sqrt{32}) - 4x^2$.

409. Solve $(15 + 4\sqrt{14})^t + (15 - 4\sqrt{14})^t = 30$, where $t = x^2 - 2|x|$.

410. For $a \leq 0$, determine all the roots of the equation $x^2 - 2a|x - a| - 3a^2 = 0$.

411. Find all solution of equation $|x^2 - x - 6| = x + 2$, where x is a real number.

412. Solve the equation $2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$.

413. Solve $3^x + 4^x + 5^x = 6^x$.

414. Solve $(\sqrt{2 + \sqrt{3}})^x + \sqrt{2 - \sqrt{3}}^x = 2^x$.

415. Let $\{x\}$ and $[x]$ denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$.

416. For the same notation as previous problem, solve $[x]^2 = x(x - [x])$.

417. Solve $x^3 - y^3 = 127, x^2y - xy^2 = 42$.

418. Solve the system of equations $x - 2y + z = 0, 4x - y - 3z = 0, x^2 - 2xy + 3xz = 14$.

419. Solve $x^4 + y^4 = 82, x + y = 4$.

420. Solve $\sqrt{a(2^x-2)+1}=1-2^x$, $x\in\mathbb{R}$.

421. If $x\in\mathbb{I}$, find the integral values of m satisfying the equation $(x-5)(x+m)+2=0$.

422. Find ll the positive solutions of the system of equations $x^{x+y}=y^n$ and $y^{x+y}=x^{2n}y^n$, where $n>0$.

423. Solve the equation $(144^{|x|}-2(12)^{|x|}+a=0)$ for every value of the parameter a.

424. If m and n are odd integers, show that the equation $x^2+2mx+2n=0$ cannot have rational roots.

425. If $f(x)=ax^3+bx^2+cx+d$ has local extrema at two points of opposite sign, then prove that the roots of the equation $ax^2+bx+c=0$ are real and distinct.

426. If $a,b\in\mathbb{R}, b\neq 0$, prove that the roots of the quadratic equation $\frac{(x-a)(ax-1)}{x^2-1}=b$, can never be equal.

427. If $n,r\in\mathbb{P}$ such that $r<n$, then show that the roots of the quadratic equation $C^n_r x^2+2C^n_{r+1}x+C^n_{r+2}=0$ are real.

428. If a,b,c are rational, show that the roots of the equation $abc^2x^2+3a^2cx+b^2cx-6a^2-ab+2b^2=0$ are rational.

429. If the roots of the equation $ax^2+bx+c=0$ be in the ratio $m:n$, prove that $\sqrt{\frac{m}{n}}+\sqrt{\frac{n}{m}}+\frac{b}{\sqrt{ac}}=0$.

430. If one root of the equation $x^2+xf(a)+a=0$ is equal to the third power of the other, determin the function $f(x)$.

431. If α,β are the roots of the equation $x^2-px+q=0$, then find the quadratic equation the roots of which are $(a^2-\beta^2)(\alpha^3-\beta^3)$ and $\alpha^3\beta^2+\alpha^2\beta^3$.

432. If α,β are the roots of the equation $x^2-bx+c=0$, then find the quadratic equation the roots of which are $(\alpha^2+\beta^2)(\alpha^3+\beta^3)$ and $\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4$.

433. If the sum of the roots of the quadratic equation $ax^2+bx+c=0$ is equal to the sum of squares of their reciprocals, then show that $\frac{b^2}{ac}+\frac{bc}{a^2}=2$.

434. The time of oscillation of a rigid body about a horizontal axis at a distance h from the C.G. is given by $T=2\pi\sqrt{\frac{h^2+k^2}{gh}}$, where k is a constant. Show that there are two values of h for a given value of T. If h_1 and h_2 are two values of h, show that $h_1+h_2=\frac{gT^2}{4\pi^2}$ and $h_1h_2=k^2$.

435. If α_1,α_2 be the roots of the equation $x^2+px+q=0$ and β_1,β_2 be the roots of $x^2+rx+s=0$ and the system of equations $\alpha_1y+\alpha_2z=0$ and $\beta_1y+\beta_2z=0$ has non-trivial solutions then show that $\frac{p^2}{r^2}=\frac{q}{s}$.

436. If a, b, c are in H.P. and α, β be the roots of $ax^2 + bx + c = 0$, show that $-(1+\alpha\beta)$ is the H.M. of α and β.

437. If α, β are roots of the equation $x + 1 = \lambda x(1 - \lambda x)$ and if λ_1, λ_2 are the two values of λ determined from the equation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = r - 2$, show that $\frac{\lambda_1^2}{\lambda_2^2} + \frac{\lambda_2^2}{\lambda_1^2} + 2 = 4\left(\frac{r+1}{r-1}\right)^2$.

438. If the roots of equation $ax^2 + bx + c = 0$ are reciprocals of those $lx^2 + mx + n = 0$, then prove that $a : b : c = n : m : l$, where a, b, c, l, m, n are all non-zero.

439. If x_1, x_2 be the roots of the equation $x^2 - 3x + A = 0$ and x_3, x_4 be those of equation $x^2 - 12x + B = 0$ and x_1, x_2, x_3, x_4 be an increasing G.P., find A and B.

440. Let p and q be roots fo the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in A.P., find the values of A and B.

441. Let α, β be the roots of the equation $x^2 + ax - \frac{1}{2a^2} = 0$, a being a real parameter, prove that $\alpha^4 + \beta^4 \geq 2 + \sqrt{2}$.

442. If α, β be the roots of the equation $x^2 - px + q = 0$ and $\alpha > 0, \beta > 0$, then find the value of $\alpha^{1/4} + \beta^{1/4}$.

443. If the difference between roots of the equation $ax^2 - bx + c = 0$ is same as the difference between the roots of equation $bx^2 - cx + a = 0$, then show that $b^4 - a^2c^2 = 4ab(bc - a^2)$.

444. If $f(x) = 0$ is a cubic equation with real roots α, β, γ on order of magnitudes, show that one root of the equation $f'(x) = 0$ lies between $\frac{1}{2}(\alpha + \beta)$ and $\frac{1}{2}(2\alpha + \beta)$ and the other root lies between $\frac{1}{2}(\beta + \gamma)$ and $\frac{1}{2}(2\beta + \gamma)$.

445. Show that the roots of the polynomial equation $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$ cannot be all real if $(n-1)a_1^2 - 2na_2 < 0$.

446. Let D_1 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$ and D_2 be the discriminant and γ, δ be the roots of the equation $px^2 + qx + r = 0$. If $\alpha, \beta, \gamma, \delta$ are in A.P., then prove that $D_1 : D_2 = a^2 : p^2$.

447. If α, β be the roots of the equatio $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ be those of equation $px^2 + qx + r = 0$, then show that $\frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$.

448. If α, β be the roots of the equation $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ be those of equation $px^2 + qx + r = 0$, then show that $2h = \frac{b}{a} - \frac{q}{p}$.

449. If α, β be the real and distinct roots of the equation $ax^2 + bx + c = 0$ and α^4, β^4 be those of equation $lx^2 + mx + n = 0$,prove that the roots of equation $a^2lx^2 - 4aclx + 2x^2l + a^2m = 0$ are real and opposite in sign.

450. If α, β be the roots of equation $ax^2 + bx + c = 0$ and γ, δ those of equation $lx^2 + mx + n = 0$, then find the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

451. If p, q be the roots of the equation $x^2 + bx + c = 0$, prove that b and c are the roots of the equation $x^2 + (p + q - pq)x - pq(p + q) = 0$.

452. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$, where $p \neq 1$, obtain the equation whose roots are $3p - 2q$ and $3q - 2p$.

453. If $\alpha \pm \sqrt{\beta}$ be the roots of the equation $x^2 + px + q = 0$, prove that $\frac{1}{\alpha} \pm \frac{1}{\sqrt{\beta}}$ will be the roots of the equation $(p^2 - 4q)(p^2x^2 + 4px) = 16q$.

454. If α, β be the roots of the equation $x^2 - px + q = 0$, form the equation whose roots are $\alpha^2\left(\frac{\alpha^2}{\beta} - \beta\right)$ and $\beta^2\left(\frac{\beta^2}{\alpha} - \alpha\right)$.

455. Let a, b, c, d be real numbers in G.P. If u, u, w satisfy the system of equations $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$, $6u + 9v = 4$, then show that the roots of the equation $\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b - c)^2 + (c - a)^2 + (d - b)^2]x + u + v + w = 0$ and $20x^2 + 10(a - d)^2x - 9 = 0$ are reciprocals of each other.

456. If $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of equation $(\beta_1 - x)(\beta_2 - x)\dots(\beta_n - x) + A = 0$, find the equation whose roots are $\beta_1, \beta_2, \dots, \beta_n$.

457. If $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of equation $x^n + nax - b = 0$, show that $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\dots(\alpha_1 - \alpha_n) = n(x^{n-1} + a)$.

458. If $\alpha, \beta, \gamma, \delta$ be the real roots of the equation $x^4 + qx^2 + rx + t = 0$, find the quadratic equation whose roots are $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2)$ and 1.

459. If α, β, γ be the roots of the equation $x^3 + px + q = 0$, find the cubic equation whose roots are $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}, \frac{\gamma+1}{\gamma}$.

460. Show that one of the roots of the equation $ax^2 + bx + c = 0$ may be reciprocal of of one of the roots of $a_1x^2 + b_1x + c_1 = 0$ if $(aa_1 - cc_1)^2 = (bc_1 - ab_1)(b_1c - a_1b)$.

461. If every pair of the equations $x^2 + px + qr = 0$, $x^2 + qa + pr = 0$ and $x^2 + rx + pq = 0$ have a common root, find the sum of the three common roots.

462. If equation $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ has equal roots and has common root with the equation $4x^2\sin^2\theta - 4x\sin\theta + 1 = 0$,find the value of θ.

463. If $a \neq 0$, find the value of a for which one of the roots of equation $x^2 - x + 3a = 0$ is double the roots of the equation $x^2 - x + a = 0$.

464. If by eliminating x between the equations $x^2 + ax + b = 0$ and $xy + l(x + y) + m = 0$, a quadratic equation in terms of y is formed whose roots are same as those of original quadratic equation in x, then prove that either $a = 2l$ or $b = m$ or $b + m = al$.

465. The roots of equation $10x^3 - cx^2 - 54x - 27 = 0$ are in H.P., then find c.

466. If a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$ such that $c^2 = -ab$, show that $(2q - p^2)^3.r = (pq - 4r)^3$.

467. Let $\alpha + i\beta, \alpha, \beta \in \mathbb{R}$ be roots of the equation $x^3 + qx + r = 0, q, r \in \mathbb{R}$. Find a real cubic equation independent of α and β, whose one root is 2α.

468. If α, β, γ be the roots of the equation $2x^3 + x^2 - 7 = 0$, show that $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = -3$.

469. The equations $x^3 + px^2 + qx + r = 0$ and $x^3 + p'x^2 + q'x + r' = 0$ have two common roots, find the quadratic equations whose roots are these common roots.

470. Find the condition that the roots of equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in G.P.

471. Find the condition that the roots of equation $x^3 - px^2 + qx - r = 0$ may be in H.P.

472. If $f(x) = x^3 + bx^2 + cx + d$ and $f(0), f(-1)$ are odd integers, prove that $f(x) = 0$ cannot have all integral roots.

473. If equation $2x^3 + ax^2 + bx + 4 = 0$ has three real roots $(a, b > 0)$, prove that $a + b \geq 6(2^{\frac{1}{3}} + 4^{\frac{1}{3}})$.

474. Find the condition that $a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ and $a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ have a common pair of repeated roots.

475. Let α be a non-zero real root of the equation $a_1x^2 + b_1x + c_1 = 0$. Find the condition for α to be repeated root of the equation $a_2x^3 + b_2x^2 + c_2x + d_2 = 0$.

476. If α, β, γ are real roots of the equation $x^3 - ax^2 + bx - c = 0$, prove that the area of the triangle whose sides are α, β, γ is $\frac{1}{4}\sqrt{a(4ab - a^3 - 8c)}$.

477. If $a < b < c < d$, then show that the quadratic equation $\mu(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real roots for all real μ and λ.

478. Show that equation $3x^5 - 5x^3 + 21x + 3\sin x + 4\cos x + 5 = 0$ can have at most one real root.

479. Find the integral part of the greatest root of equation $x^3 - 10x^2 - 11x - 100 = 0$.

480. If $n \in \mathbb{N}, a_0, a_1, a_2, \ldots, a_n \in \mathbb{I}$ and a_n and $a_0 + a_1 + \ldots + a_n$ are odd numbers, show that equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ cannot have integeral roots.

481. If the cubic equation $f(x) = 0$ has three real roots α, β, γ such that $\alpha < \beta < \gamma$, show that the equation $f(x) + 2f'(x) + f''(x) = 0$ has a root between α and γ.

482. Find the values of a for which all the roots of the equation $x^4 - 4x^3 - 8x^2 + a = 0$ are real.

483. If the equation $ax^2 - bx + c = 0$ has two distinct real roots between 1 and 2 where $a, b, c \in \mathbb{N}$, show that $a \geq 5$ and $b \geq 11$.

484. Show that the equation $(x - 1)^5 + (x + 2)^7 + (7x - 5)^9 = 10$ has exactly one real root.

485. Find the value of $\tan(\theta + \phi)$ and $\cot(\theta - \phi)$ where $\tan\theta$ and $\tan\phi$ are respectively actual and extraneous root of the equation $\sqrt{2x + 6} - \sqrt{x + 2} = 3$.

486. Solve $|x+1|-|x|+3|x-1|-2|x-2|=x+2$.

487. Solve $2^{|x+1|}-2^x=|2^x-1|+1$.

488. Solve $|x^2-2x|+y=1$, $x^2+|y|=1$.

489. Solve $|x^2+4x+3|+2x+5=0$.

490. Solve $x^2+\frac{9x^2}{(x+3)^2}=27$.

491. Solve $\frac{1}{[x]}+\frac{1}{[2x]}=\{x\}+\frac{1}{3}$, where $[x]$ denotes the integral part of x and $\{x\}=x-[x]$.

492. Solve $\frac{6}{5}a^{\log_a x\log_{10} a\log_a 5}-3^{\log_{10}\left(\frac{x}{10}\right)}=9^{\log_{100} x+\log_4 2}$.

493. Solve $\log_5\left(5^{\frac{1}{x}}+125\right)=\log_5 6+1+\frac{1}{2x}$.

494. Solve $x^{\frac{2}{3}\left[(\log_2 x)^2+\log_2 x-\frac{5}{4}\right]}=\sqrt{2}$.

495. Find all the real solutions of the equation $3x^2-8[x]+1=0$.

496. If $t>1$, solve the equation $(t+\sqrt{t^2-1})^{x^2-2x}+(t-\sqrt{t^2-1})^{x^2-2x}=2t$.

497. Obtain real solutions of the simultaneous equation

$$xy+3y^2-x+4y-7=0$$
$$2xy+y^2-2x-2y+1=0$$

498. Solve $2^{x-1}.27^{\frac{x}{x+2}}=3$.

499. Solve $4^x-3^{x-\frac{1}{2}}=3^{x+\frac{1}{2}}-2^{2x-1}$.

500. Solve $\log_{10}[98+\sqrt{x^3-x^2-12x+36}]=2$.

501. Solve $\log_{2x+3}(6x^2+23x+21)=4-\log_{3x+7}(4x^2+12x+9)$.

502. Prove that $2x^4+1402-y^4=0$ has no integral solution.

503. Solve for x, $|x-1|^{\log_3 x^2-2\log_x 9}=(x-1)^7$.

504. Solve $(\cos x)^{\sin^2 x-\frac{3}{2}\sin x+\frac{1}{2}}=1$.

505. Find the integral values of a for which the equation $(x+a)(x+1991)+1=0$ has integral roots.

506. Solve $2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$.

507. Solve $x + \log_{10}(1 + 2^x) = x \log_{10} 5 + \log_{10} 6$.

508. If $a > 0$, solve the equation $\log_a(ax) . \log_x(ax) + \log_{a^2}(a) = 0$.

509. Solve $\sqrt{11x - 6} + \sqrt{x - 1} = \sqrt{4x + 5}$.

510. Solve $\sqrt{3x^2 - 7x - 30} - \sqrt{2x^2 - 7x - 5} = x - 5$.

511. If x and y satisfy the equations $y = 2[x] + 3$ and $y = 3[x - 2]$ simultaneously, determine $[x + y]$.

512. If $x \in \mathbb{R}$ and $a_1, a_2, \dots, a_n \in \mathbb{R}$, then find the value for which $\sum_{i=1}^{n} (x - a_i)^2$ is least.

513. Let there be a quotient of two natural numbers in which the denominator is one less than the square of the numerator. If we add two to both nuerator and denominator, the quotient will exceed $\frac{1}{3}$, and if we subtract 3 from both numerator and denominator, the quotient will be between 0 and $\frac{1}{10}$. Determine the quotient.

514. Let $f(x)$ be a quadratic expression which is positive for all real x. If $g(x) = f(x) + f'(x) + f''(x)$, the for all real x, show that $g(x) > 0$.

515. By considering the quadratic equation $f(x) = (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \dots + (a_n x + b_n)^2$, prove the inequality $(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$.

516. Find the real values of m for which the equation $x(x + 1)(x + m)(x + m + 1) = m^2$ has four real roots.

517. Find all real values of a for which the equation $x^4 + (a - 1)x^3 + x^2 + (a - 1)x + 1 = 0$ possesses at least two distinct negative roots.

518. Find the real values of the parameter a for which the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ has at least two distinct negative roots.

519. If $a, b, c \in R$ and $a \neq 0$, solve the following system of equation in n unknowns $x_1, x_2, \dots, x_n$

$$ax_1^2 + bx_1 + c = x_2$$

$$ax_2^2 + bx_2 + c = x_3$$

$$\dots$$

$$ax_n^2 + bx_n + c = x_1$$

when (a) $(b - 1)^2 < 4ac$ (b) $(b - 1)^2 = 4ac$ (c) $(b - 1)^2 > 4ac$.

520. Solve the inequality $\log_x\left(x^2 - \frac{3}{16}\right) > 4$.

521. Find the values of m for which every solution of the inequality $\log_{\frac{1}{2}} x^2 \geq \log_{\frac{1}{2}}(x+2)$ is a solution of the ineuqality $49x^2 - 4m^4 \leq 0$.

522. Find all values of a for which the inequality, $1 + \log_5(x^2+1) \geq \log_5(ax^2+4x+a)$ is valid for all real x.

523. Find the values of the parameter a for which $1 + \log_2\left(2x^2+2x+\frac{7}{2}\right) \geq \log_2(ax^2+a)$ is satisfied by at least one real x.

524. Prove that the minimum value of $\frac{(a+x)(b+x)}{c+x}$, $x > -c$ is $\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$.

525. If x, a, b are real, prove that $4(a-x)(x-a+\sqrt{a^2+b^2}) \ngtr a^2+b^2$.

526. If β is such that $\sin 2\beta \neq 0$, show that for real x the expression $\frac{x^2+2x\cos 2\alpha+1}{x^2+2x\cos 2\beta+1}$ always lies between $\frac{\cos^2\alpha}{\cos^2\beta}$ and $\frac{\sin^2\alpha}{\sin^2\beta}$.

527. Show that for all real values of x, the expression $\frac{2a(x-1)\sin^2\alpha}{x^2-\sin^2\alpha}$ cannot lie between $2s\sin^2\frac{\alpha}{2}$ and $2a\cos^2\frac{\alpha}{2}$.

528. Show that the expression $\tan(x+\alpha)/\tan(x-\alpha)$ cannot lie between $\tan^2\left(\frac{\pi}{4}-\alpha\right)$ and $\tan^2\left(\frac{\pi}{4}+\alpha\right)$.

529. Prove that for real values of x the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ may have any value provided a lies between 1 and 7.

530. Prove that the expression $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will take all real values when x is real provided a^2-b^2 and c^2-d^2 have the same sign.

Chapter 5
Combinatorics

In this chapter we will study basic principles of counting, permutations and combinations. This study will enable you to further study the branch of mathematics called combinatorics. You would have certainly encountered a combinatorical problem in your life. It would be really surprising if you have not. Have you ever solved a Sudoku puzzle or Rubik's cube? Have you ever counted the number of poker hands that are full houses in order to determine the odds against a full house? Have you ever attempted to trace through a network without removing your pencil from paper and without tracing any part of network more than once? These are all combinatorical problems. As you can see that combinatorics has evolved from mathematical games.

With the invention of modern computers, we are enabled to solve more and more problems of combinatorics which were earlier not feasible due to calculations involved. The computer programs are often based on combinatorical algorithms which determine the speed and efficiency of the solution. Analysis of these programs and algorithms require sound knowledge of combinatorical mathematics and thinking. In computer science we write test cases for our programs, and those test cases can be enumerated by applying permutations and combinations on input data and states produced in the program. Combinatorics is a powerful tool for making sure that the tester does not miss any test case, which in mission-critical programs is of paramount importance.

The best way to learn combinatorics is to solve a lot of problems. This is in general true for all branches of mathematics but even more so for combinatorics because a problem which appears simple may be quite difficult to solve or require critical thinking. By solving problems of different kinds, and by repeating them the concepts will be enforced and discipline will develop.

We start with four basic counting principles and then we will progress into permutations and combinatins. To study the topic of permutations and combinations it is required to have basic knowledge in set theory which the reader is expected to know.

5.1 Four Basic Counting Principles

Let S be a set. A *partition* of S is a collection of $S_1, S_2, \ldots, S_m$ of subsets of S such that each element in S is in exactly one of these subsets:

$$S = S_1 \cup S_2 \cup \ldots \cup S_m$$

$$S_i \cap S_j = \phi (i \neq j)$$

Thus, the sets $S_1, S_2, \ldots, S_m$ are pairwise disjoint sets, and their union is S. The subsets $S_1, S_2, \ldots, S_m$ are called the *parts* of the partition. Note that by this definition a part of the partition may be empty, but usually there is no advantage in considering partitions with one or more empty sets. The number of objects of a set S is denoted by $|S|$, and is called the *size* of S.

5.1.1 Addition Principle

Suppose that a S is partitioned into pairwise disjoint partys $S_1, S_2, \dots, S_m$. The number of objects in S can be determined by finding the number of objects in each of the parts, and adding the numbers so obtained:

$$|S| = |S_1| + |S_2| + \dots |S_m|.$$

If the sets $S_1, S_2, \dots, S_m$ are allowed to overlap, then a more profound principle, the inclusion-exclusion principle can be used to count the number of objects in S.

We need to be careful when partitioning S into too many parts. For example, if we partition S into parts in such a way that each part contains only one element then addition princinple is becomes counting the number of parts, which is basically same as listing all objects of S. Thus the art of applying addition princinple is to partition the set S into not too many parts.

Example: In a university there are four mathematics courses, two economics courses, and three lietrature courses. A student is allowed to enroll into one course at most. Thus, we see that a student can take a course in $4 + 2 + 3 = 9$ ways.

Next principle is multiplication principle which will be stated for two sets, but it can be generalized to any finite number of sets.

5.1.2 Multiplication Principle

Let S be a set of ordered pairs (a, b), where the first object comes from a set of size p, and for each choice of object a there are q choices for object b. Then the size of S is $p \times q$:

$$|S| = p \times q$$

As in basic arithmetic multiplication is repeated addition, similarly multiplication principle is actuallly a consequence of the addition principle i.e. repeated addition. Let $a_1, a_2, \dots, a_p$ be p different choices for the object a. We partition S into parts $S_1, S_2, \dots, S_p$ where S_i is the set of ordered pairs in S with first object $a_i (i = 1, 2, \dots, p)$. The size of each S_i is q; hence, by the addition principle,

$$\begin{aligned} |S| &= |S_1| + |S_2| + \dots + |S_p| \\ &= q + q + \dots + q (pq'\text{s}) \\ &= p \times q \end{aligned}$$

The multiplication principle can be stated in another way as: If a first task has p outcomes, and no matter what the outcome of of the first task, a second task has q outcomes i.e. outcomes for two tasks are mutually exclusive, then the two tasks can be performed in $p \times q$ outcomes.

Example: Pencil comes in two different lengths, four different hardness, and three different thickness. How many different types of pencils are there?

The pencil has three different properties, which are exclusive of each other, and thus, we can apply multiplication principle. Hence, number of different types of pencils is $2 \times 4 \times 3 = 24$.

Example: The number of ways a man, woman, boy, and girl can be selected from three men, three women, five boys and four girls is $3 \times 3 \times 5 \times 4 = 180$.

Example: Determine the number of positive integers that are factors of the number

$$2^3 \times 3^4 \times 5^5 \times 7^7$$

The numbers $2, 3, 5$, and 7 are prime numbers. By the fundamental theorem of arithmetic, each factor is of the form

$$2^i \times 3^j \times 5^k \times 7^l$$

where $0 \leq i \leq 2$, $0 \leq j \leq 3$, $0 \leq k \leq 5$, and $0 \leq l \leq 7$. There are three choices for i, four for j, six for k, and eight for l. By multiplication principle, the number of factors is $3 \times 4 \times 6 \times 8 = 576$.

In the multiplication principle the q choices for object b may vary with the choices of a. The only requirement is that there be the same number q of choices, not necessarily the same choices.

Example: How many two-digit numbers have distinct, and nonzero digits?

A two-digit number ab can be regarded as an ordered pair (a, b), where a is the tens digit, and b is the units digit. Both are not allowed to be 0, and they must be different. Thus, we see that there are 9 ways to choose a, which are $1, 2, \ldots, 9$. Once a is chosen we cannot use the same digit for b, which means we are left with 8 choices for b. Here we see that choice of a makes a difference on what choices b has. However, for multiplication principle to be applicable what matters is that the number of choices remain constant which is 8 in this case. Applying multiplication principle, we arrive at the answer of the question as $9 \times 8 = 72$.

There is another way to arrive at the same result. Total number of two-digit number is $90, 10{,}11, 12, \ldots, 99$. Of these 90 numbers 9 have a zero in them$(10, 20, 30, \ldots, 90)$, and 9 have repeated digits$(11, 22, \ldots, 99)$. Thus, total number of required numbers equals $90 - 9 - 9 = 72$.

We can derive two important ideas from the previous example. First is that it is possible to solve a counting problem in many ways. The second idea is that to find the number of objects in a set A (in this csae the set of two-digit numbers with nonzero, and distinct digits) it may be easier to find the number of objects in a larger set U containing S (the set of all two-digit numbers), and then subtract the number of objects of U that do not belong to A (the two-digit numbers containing 0 or repeated digit). This leads us to subtraction principle.

5.1.3 Subtraction Principle:

Let A be a set, and let U be a larger set containing A. Let

$$\overline{A} = U \backslash A = \{x \in U : x \notin A\}$$

be the complement of A in U. Then the numebr $|A|$ of object in A is given by the rule

$$|A| = |U| - |\overline{A}|$$

The set U is usually some natural set containing all the objects under discussion (it is called *universal set*). Using the subtraction principle should be used only if it is easier to count the number of object in U nd $\overline{A}$ tha to count the number of objects in A.

Example: Most websites on internet have a lower limit of 8 characters as password length. Suppose if these passwordss are to made up of the digits $0, 1, 2, \ldots, 9$, and the lowercase letters $a, b, c, \ldots, z$ then how many passwords will have a repeated symbol?

There are a total of 10 digits, and 26 letters i.e. 36 symbols. So by two applications of multiplication principle, we get

$$|U| = 36^8 = 2{,}821{,}109{,}907{,}456$$

and

$$|\overline{A}| = 36.35.34.33.32.31.30.29 = 1{,}220{,}096{,}908{,}800$$

Therefore,

$$|A| = |U| - |\overline{A}| = 1{,}601{,}012{,}998{,}656.$$

Now we will formulate the last principle of counting principles.

5.1.4 Division Principle

Let S be a finite set that is partitioned into k parts in such a way that each part contains the same number of objects. Then the number of parts in the partition is given by the rule

$$k = \frac{|S|}{\text{number of objects in a part}}$$

Example; There are 240 rats in a collection of cages. If each cage contains 2 rats, the number of cages equals

$$\frac{240}{2} = 120.$$

Interesting problems of division principle will be found in the problems section.

Most counting problems can be classified as one of the following types:

1. Count the number of ordered arrangements or ordered selection of objects
   i. without repeating any object,
   ii. with repetion(perhaps limited) of objects permitted.
2. Count the number of unordered arrangements or unordered selection of objects
   i. without repeating any object,
   ii. with repetion(perhaps limited) of objects permitted.

We can represent repetition, and nonrepetition of objects as selection from a set, and a multiset. The latter might prove to be more useful in some cases. A *multiset* is like a set except that its members need not be distinct.[1] For example, a multiset M with three a's, two b's i.e. 5 elements of 2 different types. We usually indicate a multiset by specifying the number of times different types of elements occur in it. Thus, M is denoted by $\{3.a, 2.b\}$.[2]. The numbers 3, and 2 are the *repetition members* of the multiset M. Thus we can extrapolate that a set is a multiset with all repetition numbers equal to 1. Often there is no limit on number of repetitions i.e. infinite repetitions are allowed.[3]

5.2 Factorial of n

Factorial of n is denoted by $n!$. In the old style it is written as $\underline{|n}$. $n!$ is given by the first n natural numbers, i.e.

$$n! = 1.2.3.4 \ldots (n-1) .n$$

Also, $0! = 1$, which we will prove later.

Permutation means arrangement of objects along with selection. In the permutation of object order matter. If order of object changes then their permutation also changes. Combination of objects means selection of objects in such a way that order does not matter.

5.3 Permutation of Sets

Let $r \in \mathbb{P}$. By an *r-permutation* of a set S of n elements has a meaning of an ordered(by definition of permutation) arrangement of r of the n elements$(r \leq n)$. If $S = \{a, b, c\}$, then the three 1-permutations of S are

$$a \ b \ c,$$

the six 2-permutations of S are

$$ab \ ac \ ba \ bc \ ca \ cb,$$

and the six 3-permutations of S are

$$abc \ acb \ bac \ bca \ cab \ cba.$$

There are no 4-permutations of S because that will violate the assumption that $r \leq n$.

The r-permutations of an n-element set is denoted by $P(n, r)$ or ${}_nP_r$ or ${}^nP_r$ or P_r^n. If $r > n$ then ${}^nP_r = 0$. Clearly, ${}^nP_1 = n$ for each $n \in \mathbb{P}$.

For n and r positive integers with $r \leq n$,

1 Thus, a cardinal rule of sets is broken by multisets because a set is not supposed to have duplicates or repeated elements. The set $\{a, a, b\}$ is same as the set $\{a, b\}$ but not so for multisets

2 In standard set-theory's notation, we could denote the multiset M using ordered pairs as $\{(a, 3), (b, 2)\}$

3 In no circumstance, we need to consider different sizes of ∞.

$$^nP_r = n \times (n-1) \times ... \times (n-r+1).$$

Permutation of n objects taken r at a time is equivalent ot filling r different vacant spots from n different objects. We can fill first spot by n ways, second spot can be filled by remaining objects i.e. $n-1$ ways, and proceeding this way we find that rth spot can be filled in $n-r+1$ ways. Thus total number of ways is

$$n \times (n-1) \times ... (n-r+1).$$

We can rewrite the above as

$$\frac{n \times (n-1)...(n-r+1) \times (n-r) \times ... 2 \times 1}{(n-r) \times (n-r-1) \times ... 2 \times 1}$$

$$^nP_r = \frac{n!}{(n-r)!}$$

Alternatively, first place can be filled in n ways. Rest of $r-1$ spots from $n-1$ objects can be filled in $^{n-1}P_{r-1}$ ways. Thus, $^nP_r = n.^{n-1}P_{r-1}$. Similarly, $^{n-1}P_{r-1} = (n-1).^{n-2}P_{r-2}$. Proceeding this way we find that $^{n-r+1}P_1 = n-r+1$. Multiplying and cancelling common factors, we get $^nP_r = n \times (n-1) \times ... \times (n-r+1)$.

The number of permutations of n elements is $^nP_n = \frac{n!}{0!} = n!$. If we follow first result then it is evident that $0! = 1$.

5.3.1 Meaning of $\frac{1}{(-k)!}, k \in \mathbb{P}$

We have $^nP_r = \frac{n!}{(n-r)!}$. Putting $r = n+k$, we have $^nP_{n+k} = \frac{n!}{(-k)!}$. But the number of ways of arranging $n+1$ objects out of n different objects $= 0 \Rightarrow \frac{1}{(-k)!} = 0$.

Note: Although $(-k)!$ has no meaning by the definition of factorial but if we consider the above result then the formula for permutation becomes valid even for $r > n$.

5.3.2 Circular Permutation

Let us consider arranging objects along a circle. Let us consider that four persons A, B, C, and D are sitting around a table. We can have following arrangements:

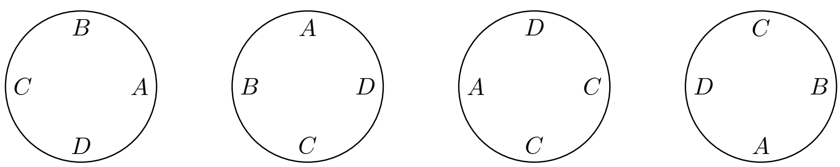


Figure 5.1

As shown four persons are sitting around a round table, and four anticlockwise rotations have lead to four arrangements. But if A,B,C,D are sitting in a row, and then are shiftedd such that last occupies the place of first, then the four arrangements will be different. Thus, if there are n objects then for each circular arrangement there are nn linear arrangements.

But for n different objects total number of linear arrengements are $n!$ so the total number of circular arrangements are

$$\frac{n!}{n} = (n-1)!\,.$$

Thus, we can say that number of circular r-permutations of a set of n elements is given by

$$\frac{{}^nP_r}{r} = \frac{n!}{r.(n-r)!}$$

5.3.3 Clockwise and Anti-Clockwise Arrangements

When clockwise and anticlockwise arranegemnts are same then total number of permutations will become half of what we computed in previous case i.e.

$$\frac{{}^nP_r}{2r} = \frac{n!}{2r.(n-r)!}$$

5.4 Combination of Sets

Consider a set S having n elements. A *combination* of a set S has a meaning of an unordered selection of the elements of S. The result of each selection is a *subset* A of the elements of $S : A \subset S$. Thus, the terms *combination* and *subset* are interchangeable.

Now let r be a non-negative integer. By an r-*combination* of a set S of n elements, we understand an unordered selection of r of the n objects of S. The result will be an r-subset of S.

If $S = \{a, b, c, d\}$, then

$$\{a, b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}$$

are the four 3-subsets of S. We denote the number of r-subsets or r-combinations of an n-element set by $\binom{n}{r}$ or ${}_nC_r$ or ${}^nC_r$ or C^n_r. Obviously,

$$\binom{n}{r} = 0 \quad \text{if } r > n.$$

Also,

$$\binom{0}{r} = 0 \quad \text{if } r > 0.$$

The following facts are easy to figure out for each non-negative integer n

$$\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1, \binom{n}{1} = n,$$

For $0 \leq r \leq n$,

$${}^nP_r = r!^n\,C_r.$$

Hence,

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Let S be an n-element set. Each r-permutation of S arises from following tasks

1. Choose r elements from S.
2. Arrange the chose r elements in some order.

The number of ways to carry out first task, by definition, is ^nC_r. The number of ways to carry out second task is $^nP_r = r!$. By the multiplication principle, we have $^nP_r = r!^n C_r$. Now applying the formula for permutations, we have

$$^nC_r = \frac{n!}{r!(n-r)!}.$$

5.5 Permutation of Multisets

Let S be a multiset with objects of k different types, where each object can be repeated infinitely. Then the number of r-permutations of S is k^r.

To prove this, we can choose the first item to be an object of any one of the k types. Since the number of repetitions are infinite the second item can be also chose in k ways. In fact, any item can be chosen in k ways due to infinite repetition. Following, multiplication principle, total number of such permutations is k^r.

Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \dots, n_k$ respectively. Let the size of S be $n = n_1 + n_2 + \dots + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_!! \, n_2! \dots n_k!}.$$

We can calculate this by thinking in terms of n places, and we want to put exactly one of the objects of S in each of the places. We have n_1 objects of one type in S, so we must choose a subset of n_1 places from the set of n places. We can do this in $^nC_{n_1}$ ways. After this we have $n - n_1$ places left, and we have n_2 objects of second type. So following similarly we can do this in $^{n-n_1}C_{n_2}$ ways. Following this way invoking multiplication principle, the number of permutations of S equals

$$^nC_{n_1}. {}^{n-n_1}C_{n_2}. {}^{n-n_1-n_2}C_{n_3} \dots {}^{n-n_1-n_2-\dots-n_{k-1}}C_{n_k}$$

which gives

$$\frac{n!}{n_1!(n-n_1)!}.\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}.\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\dots\frac{(n-n_1-n_2-\dots-n_{k-1})!}{n_k!(n-n_1-n_2-\dots-n_k)!}$$

which after cancellation, reduces to

$$\frac{n!}{n_1!\,n_2!\dots n_k!}$$

Let n be a positive integer, and let $n_1, n_2, \dots, n_k$ be positive integers with $n = n_1 + n_2 + \dots + n_k$. The number of ways to partition a set of n objects into k labeled boxes in which Box 1 contains n_1 objects, Box 2 contains n_2 objects, ..., Box k contains n_k objects equals

$$\frac{n!}{n_1!\,n_2!\dots n_k!}.$$

If the boxes are not labeled, and $n_1 = n_2 = \dots = n_k$, then the number of partitions equals

$$\frac{n!}{k!\,n_1!\,n_2!\dots n_k!}.$$

We can calculate this by direct application of the multiplication principle. So we first choose n_1 objects for the first box, then n_2 of the remaining $n - n_1$ objects for the second box and so on. By the multiplication principle, the number of ways is

$$^{n}C_{n_1}.^{n-n_1}C_{n_2}.^{n-n_1-n_2}C_{n_3}\dots^{n-n_1-n_2-\dots-n_{k-1}}C_{n_k}$$

which is same as the last result, i.e.

$$\frac{n!}{n_1!\,n_2!\dots n_k!}$$

If boxes are not labeled and $n_1 = n_2 = \dots = n_k$, then the result has to be divided by $k!$ because for each way of distributing the objects into the k unalbeled boxes there are $k!$ ways in which we can attach the labels to the boxes. Thus, using the division principle, we arrive at the result as

$$\frac{n!}{n_1!\,n_2!\dots n_k!}.$$

5.6 Combination of Multisets

If S is a multiset, then an r-combination of S is an unorndered selection of r of the objects of S. Thus, an r-combination of S is itslef a multiset, a *submultiset* of S of size r, or, for short, an r-submultiset. If S has n objects, then there is only one n-combination of S, namely, S itself. If S contains objects of k different types, then there are $k1$-combinations of S.

Let S be a multiset with objects of k types, each with an infinite repetitions, then the number of r-combinations of S equals

$$^{r+k-1}C_r =^{r+k-1} C_{k-1}.$$

Let k types of objects of S be $a_1, a_2, \dots, a_k$ so that

$$S = \{\infty.a_1, \infty.a_2, \dots, \infty.a_k\}$$

Any r-combination of S is of the form $\{x_1.a_1, x_2.a_2, \dots, x_k.a_k\}$, where $x_1, x_2, \dots, x_k$ are non-negative integers with $x_1 + x_2 + \dots + x_k = r$. The converse is also true. Thus, the number of r-combinations of S equals the number of solutions of the equation

$$x_1 + x_2 + \dots + x_k = r.$$

We will show that the number of solutions of this equation is given by number of permuations of the multiset

$$T = \{r.1, (k-1).*\}$$

of $r + k - 1$ objects of two different types. Given a permuation of T, the $k - 1*$'s divide the r1s into k groups. Let there be $x_1$1s to the left of the first $*$, $x_2$1s between the first and second $*, \dots$, and x_k1s to the right of last $*$. Clearly, $x_1 + x_2 + \dots + x_k = r$. The converse of this is also true. Thus, required combination is given by the formula

$${}^{r+k-1}C_r = {}^{r+k-1}C_{k-1}.$$

5.7 Some Important Indentities

1. ${}^nP_r = r.{}^{n-1}P_{r-1} + {}^{n-1}P_r$.
2. ${}^nC_r = {}^nC_{n-r}$.
3. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$.
4. ${}^nC_r = {}^nC_s \Rightarrow r = s$ or $r + s = n$.
5. ${}^nC_r = \frac{n-r+1}{r}.{}^nC_{r-1}$ $(1 \le r \le n)$.
6. If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_m$, where $m = n/2$. If n is odd, then the greatet value is ${}^nC_m$, where $m = (n-1)/2$ or $m = (n+1)/2$.
7. If $n = 2m + 1$, then ${}^nC_0 < {}^nC_1 < {}^nC_2 < \dots < {}^nC_m = {}^nC_{m+1}$. ${}^nC_{m+1} > {}^nC_{m+2} > \dots > {}^nC_n$.
8. If $n = 2m + 1$, then ${}^nC_0 < {}^nC_1 < {}^nC_2 < \dots < {}^nC_m > {}^nC_{m+1} > {}^nC_{m+1} > \dots > {}^nC_n$.
9. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$.
10. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots = {}^nC_1 + {}^nC_3 + \dots = 2^{n-1}$.
11. ${}^{2n+1}C_0 + {}^{2n+1}C_1 + \dots + {}^{2n+1}C_n = {}^{2n+1}C_{n+1} = {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \dots + {}^{2n+1}C_{2n+1} = 2^{2n}$.
12. $r.{}^nC_r = n.{}^{n-1}C_{r-1}$.

5.8 Some Useful Results

Number of selections of r objects out of n different objects:

1. When p paticular objects are always included $=^p C_p.^{n-p}C_{r-p} =^{n-p} C_{r-p}$.
2. When p paticular objects are excluded $=^{n-p} C_r$.
3. Number of selections of r objects out of n different objects such that p particular objects are not together in any selection $=^n C_r =^{n-p} C_{r-p}$.
4. Number of selection of r consecutive objects out of n objects in a row $= n - r + 1$.
5. Number of selection of r consecutive objects out of n objects along a circle $= n$ when $r < n$, 1 when $r = n$.
6. Number of selections of zero or more objects out of n different objects $=^n C_0 +^n C_1 +^n C_2 + ... +^n C_n = 2^n$.
7. Number of selections of one or more objects out of n different objects $=^n C_1 +^n C_2 + ... +^n C_n = 2^n - 1$.
8. Number of selections of zero or more objects out of n identical objects $= n + 1$.
9. Number of selections of one or more objects out of n identical objects $= n$.
10. Number of selection of one or more objects from $(p + q + r)$ objects, out of which r objects are identical and of one type, q objects are identical and of second type, r objects are identical and of third type $= (p + 1)(q + 1)(r + 1) - 1$.
11. Number of selection of one or more objects from $(p + q + r + n)$ objects, out of which r objects are identical and of one type, q objects are identical and of second type, r objects are identical and of third type and rest n are different $= (p + 1)(q + 1)(r + 1)(^n C_0 +^n C_1 +^n C_2 + ... +^n C_n) - 1 = (p + 1)(q + 1)(r + 1)2^n - 1$
12. Number of ways of distributing n different objects among 3 persons such that they gey x, y, z objects $=^n C_x.^{n-x}C_y.^{n-x-y}C_z.3! = \frac{n!}{x!y!z!}.3!$.
13. Number of ways of distributing n different objects in 5 sets having a, b, c, d, e objects$(a + b + c + d + d = n)$:
    i. When two sets have equal number of objects and three sets have equal number of objects $= \frac{n!}{a!b!c!d!e!2!3!}$
    ii. When all sets have equal number of objects $= \frac{n!}{a!b!c!d!e!5!}$
14. Number of ways of distributing n different objects among 5 persons
    i. When all person get different number of objects $= \frac{n!}{a!b!c!d!e!}.5!$.
    ii. When two persons get equal number of objects and three get equal number of objects $= \frac{n!}{a!b!c!d!e!2!3!}.5!$.
    iii. When all get equal number of objects $= \frac{n!}{a!b!c!d!e!5!}.5! = \frac{n!}{a!b!c!d!e!}$.

5.9 Permutations with Repetitions

The objective is to find permutation of r objects out of n objects of which p are of one type, q of second type and so on.

Let the different objects be denoted by $a, b, c, \ldots$

Consider the product

$$\left(1+\frac{ax}{1!}+\frac{a^2x^2}{2!}+\ldots+\frac{a^px^p}{p!}\right)\left(1+\frac{bx}{1!}+\frac{b^2x^2}{2!}+\ldots+\frac{b^qx^q}{q!}\right)\ldots$$

Required number of permutations = sum of all possible terms of the form $= \frac{r!}{p!q!\ldots}a^pb^q\ldots$ where $p+q+\ldots=r$

$= r!.$ coeff. of x^r in $\left[\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^p}{p!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^q}{q!}\right)\ldots\right]$

5.10 Combinations with Repetitions

The objective is to find combinations of r objects out of n objects under different cases of repetitions. To begin with we consider combinations of r objects taken out of n objects of which p are of one type, q of the second type and so on.

Let the different things be denoted by the letters $a, b, \ldots$

Consider the product $(1+ax+a^2x^2+\ldots+a^px^2)(1+bx+b^2x^2+\ldots+b^qx^2)\ldots$. All the terms in the product is of the same degree in the letters $a, b, \ldots$ as in x. The coefficient of x^r in the product is the number of ways of taking r of the letters $a, b, \ldots$ with the restriction that maximum number of a's is p, maximum number of b's is q and so on. Coeff. of x^r will not change if $a=b=\ldots=1$. Thus required number of combinaitons = Coeff. of x^r in $(1+x+x^2+\ldots+x^2)(1+x+x^2+\ldots+x^2)\ldots$

Similarly, number of combinaitons of r objects out of n objects of which p are of one type, q are of second type and $(n-p-q)$ things are all different = Coeff. of x^r in $[(1+x+x^2+\ldots+x^p)(1+x+x^2+\ldots+x^q)(1+x)(1+x)\ldots$ to $(n-p-q)$ factors $]$

$=$ Coeff. of x^r in $[(1+x+x^2+\ldots+x^p)(1+x+x^2+\ldots+x^q)(1+x)(1+x)^{n-p-q}]$

Similarly, number of combinations of r objects out of n objects of which p are of one type, q are of second type and so on, when each thing is taken at least once = Coeff. of x^r in $[(x+x^2+\ldots+x^p)(x+x^2+\ldots+x^q)\ldots]$

$=$ Coeff. of x^{r-3} in $[(1+x+x^2+\ldots+x^p)(1+x+x^2+\ldots+x^q)$

If n is a negative integer, then $(1+x)^n = 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\ldots$ to ∞ [this comes from binomial theorem]

So if n is a positive inetger then $(1+x)^{-n} = 1+\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2+\ldots$ to ∞

Coeff. of x^r in $(1-x)^{-n} = ^{n+r-1} C_r$ which is number of ways in which r identical objects can be distributed among n persons can get zero of more objects = Coeff. of x^r in $(1 + x + ... + x^r)^n = \left(\frac{1-x^{r+1}}{1-x}\right)^n = [(1-x^{r+1})(1-x)^{-n}]$.

= Coeff. of x^r in $(1-x)^{-n}$ (leaving powers higher than x^r) $= ^{n+r-1} C_r$.

5.11 Integral Solutions of Equations

As we have proved earlier, for equation $x_1 + x_2 + ... + x_r = n$ is equivalent of distributing r identical objects among n persons when each person getting zero or more things $= ^{n+r-1} C_r$

Similarly, number of non-negative integral solutions of equation $x + 2y + 3z + 4w = n$, equals coeff. of x^n in $[(1-x)^{-1}(1-x)^{-2}(1-x)^{-3}(1-x)^{-4}]$.

Similarly, number of positive integral solutions of equation $x + 2y + 3z + 4w = n$, equals coeff. of $x^{n-(1+2+3+4)}$ in $[(1-x)^{-1}(1-x)^{-2}(1-x)^{-3}(1-x)^{-4}]$.

5.12 Geometrical Applications of Combinations

Some basic geometrical results involving combinations are given below:

1. n non-concurrent and non-parallel straight lines, points of intersection are $^nC_2$.
2. The number of straight lines constructed out of n points, when no three points are collinear, are $^nC_2$.
3. Given n points, if m are collinear, then number of straight lines possible are $^nC_2 -^m C_2 + 1$.
4. In a polygon, total number of diagonals out of a n points, when no three points are collinear, are $\frac{n(n-3)}{2}$.
5. Number of triangles formed from n points, when no three points are collinear, are $^nC_3$.
6. Number of triangles formed out of n points in which m are collinear, $^nC_3 -^m C_3$.
7. Number of triangles constructed out of n points, when none of the side is common with the sides of polygon, are $^nC_3 -^n C_1 -^n C_1 .^{n-4}C_1$.
8. Number of parallelogram constructed by two system of parallel lines, when first set contains m parallel lines and second set contains n parallel lines, are $^nC_2 \times^m C_2$.
9. Number of squares formed by two system of parallel lines in which first set is perpendicular to second set of lines, when first set contains m parallel lines and second set contains n parallel lines is $\sum_{r=1}^{m-1}(m-r)(n-r);\ m < n$.

5.13 Number of Divisors and Sum of Divisors

Let $n = p_1^{n_1}.p_2^{n_2} \dots p_n^{n_k}$ where $p_1, p_2, \dots, p_k$ are distinct prime numbers and $n_1, n_2, \dots, n_k \in \mathbb{P}$. Obvously, any divisor of n is of the form $d = p_1^{m_1}.p_2^{m_2} \dots p_k^{m_k}$ where $m_1, m_2, \dots \in \mathbb{N}$ such that $0 \le m_i \le n_i, i = 1, 2, \dots, k$. Therefore, the total no. of divisors for n will be equal to the number of ways of selecting at least one from n_1 identical prime numbers p_1, n_2 primes p_2 and so on. The number of such ways is

$$(n_1 + 1)(n_2 + 1) \dots (n_k + 1).$$

These divisors will also include 1 and n, so obviously, number of divisors other than 1 and n is

$$(n_1 + 1)(n_2 + 1) \dots (n_k + 1) - 2$$

The sum of all divisors for n is given by

$$\sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \dots \sum_{r_k=0}^{n_k} p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$$

$$= \left(\frac{p_1^{n_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{n_2+1} - 1}{p_2 - 1}\right) \dots \left(\frac{p_k^{n_k+1} - 1}{p_k - 1}\right)$$

5.14 Exponent of Prime p in $n!$

Let $E_p(m)$ denote the exponent of the prime p in the positive integer m. We have

$$E_p(n!) = E_p[1.2.3.4 \dots (n-1).n]$$

The last integer amongst $1, 2, 3, \dots, (n-1), n$ which is divisible by p is $[n/p]p$, where $[x]$ denotes the greatest integer $\le x$. Therefore,

$$E_p(n!) = E_p\left(p.2p.3p \dots \left[\frac{n}{p}p\right]\right)$$

because the remaining integers from the set $(1, 2, 3, \dots, (n-1), n)$ are not divisible by p.

$$E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(1.2.3 \dots \left[\frac{n}{p}\right]\right)$$

The last integer amongst $1, 2, \dots, [n/p]$ which is divisible by p is

$$\left[\frac{[n/p]}{p}\right]p = \left[\frac{n}{p^2}\right]p$$

$$\Rightarrow E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1.2 \dots \left[\frac{n}{p^2}\right]\right)$$

Proceeding similarly,

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... \left[\frac{n}{p^s}\right]$$

where $p^s \leq n \leq p^{s+1}$

5.15 Inclusion-Exclusion Principle(PIE)

We have seen examples of subtraction principle. Inclusion exclusion principle is an extension of subtraction principle. In this type of problems, it is easier to make an indect coutnt of object in a set rather than to count the objects directly. Consder following examples:

Example: Count the permutations $i_1 i_2 \ldots i_n$ of $1, 2, \ldots, n$ in which 1 is not in the first position i.e $i_1 \neq 1$.

The number of permutations of $\{1, 2, \ldots, n\}$ with 1 in the first position is the same as the number $(n-1)!$ of permutations of $2, 3, \ldots, n$. Since the total umber permutations is $n!$, required number of permutations is $n! - (n-1)! = (n-1)\,.(n-1)!\,.$

Definition: The number of objects of the set S that have none of the properties $P_1, P_2, \ldots, P_m$ is given by the alternating expression

$$|\bar{A_1} \cap \bar{A_2} \cap \ldots \cap \bar{A_m}| =$$
$$|S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \ldots + (-1)^m |A_1 \cap A_2 \cap \ldots A_m|,$$

where the first sum is over all 1-subsets of $\{i\}$ of $\{1, 2, \ldots, m\}$, the second sum is over all 2-subsets $\{i, j\}$ of $\{1, 2, \ldots, m\}$ the third sum is all over 3-subsets $\{i, j, k\}$ of $\{1, 2, \ldots, m\}$, and so until the mth sum over all m-subsets of $\{1, 2, \ldots, 2\}$ of which the only one is itself.

The subtraction principle is the simplest instance of inclusion-exclusion principle. As a first generalization of the substraction principle, let S be a finite set of objects, and let P_1 and P_2 be two "properties" that each objects in S may or may not possess. We wish to count the number of object in S that have neither the properties of P_1 and P_2. Extending the subtracting principle, we can do this by first including of all objects of S in our count, then excluding all objects that have property $P1$ and excluding all objects that have property P_2, and then noting that we have excluded objects having both properties twice, readmitting all such objects once. Let A_1 be the subset of objects of S that have property A_1, and let A_2 be the subset that have property P_1. Then $\bar{A_1}$ consists of those which do not have property P_1, and similarly $\bar{A_2}$ consists of those which do not have property P_2. The objects of set $\bar{A_1} \cap \bar{A_2}$ are those that have neither property P_1 nor property P_1. Thus, we have

$$|\bar{A_1} \cap \bar{A_2}| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

To further prove this, we argue as follows. Consider an object x which has neither the property P_1, nor the property P_2. In this case the contribution towards the count by this object would be $1 - 0 - 0 + 0 = 1$. Next, we consider if the object x has property P_2, then its contribution is $1 - 1 - 0 + 0 = 0$. Similarly, if it has property P_1, then its contribution is $1 - 0 - 1 + 0 = 0$. For the last possibility when x has both the properties its contribution is $1 - 1 - 1 + 1 = 0$. As it is obvious any object will fall in either of these four possibilities and the total contribution is 1 only when it has neither of the properties. The inclusion-exclusion principle stated above is generalizatio of this two property example. We will now establish the validity of the general case.

First, we conisder an object x with none of the properties. Its contribution to the right side would be $1-0+0-0+...+(-1)^m 0=1$ since it is in S but in none of the other sets. Now consider an object y with exactly $n \geq 1$ of the properties. The contribution of y to $|S|=1={}^n C_0$. Its contribution to $\sum |A_i|$ is $n={}^n C_1$ since it has exactly n of the properties and so it is a member of exactly n of the sets out of $A_1, A_2, \ldots, A_m$. Similarly, the contribution of y to $\sum |A_i \cap A_j|$ is ${}^nC_2$ sinc ewe may select a pair of the properties y has in ${}^nC_2$ ways. Following similarly, the net contribution of y is

$$^nC_0 -^n C_1 +^n C_2 - ... + (-1)^{m n} C_m$$

which equal

$$^nC_0 -^n C_1 +^n C_2 - ... + (-1)^{n n} C_n$$

because

$$n \leq m$$

and ${}^nC_k = 0$ if $k > n$. The last expression is 0 from binomial theorem. Following similarly, we prove the inclusion-exclusion principle.

Definition: The number of objects of S which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by

$$|A_1 \cup A_2 \cup ... \cup A_m| =$$
$$\sum |Ai| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - ... + (-1)^{m+1} |A_1 \cap A_2 \cap ... \cap A_m|$$

The set $A_1 \cup A_2 \cup ... \cup A_m$ consiste of all those objects in S which possess at least one of the properties. Also,

$$|A_1 \cup A_2 \cup ... \cup A_m| = |S| - |\overline{A_1 \cup A_2 \cup ... \cup A_m}|.$$

From Demorgan's law

$$|\overline{A_1 \cup A_2 \cup ... \cup A_m}| = \bar{A}_1 \cap \bar{A}_2 \cap ... \cap \bar{A}_m$$

Following result from previous definition, we have the required equality.

5.15.1 A Special Case of PIE

For any set A having $n \geq 2$ element, $|A_1 + A_2 + \cdots + A_n = \sum_{i=0}^{n} |A_i| - \sum_{i<j} A_i A_j + \sum_{i,j<k} A_i A_j A_k| - \cdots + (-1)^n |A_1 A_2 \ldots A_n|$

In some problems we deal with properties $a_1, a_2, \ldots, a_n$ and numerical values associated with properties i.e. $n(a_1), n(a_2), \ldots, n(a_n), n(a_1 a_2), \ldots, n(a_{n-1} a_n) \ldots$ and so on.

We can have

1. $n(a_1) = n(a_2) = \cdots = n(a_n)$
2. $n(a_1a_2) = n(a_2a_3) = \cdots = n(a_1a_n) = n(a_2a_3) = \cdots = n(a_{n-1}a_n)$
3. $n(a_1a_2a_3) = n(a_1a_2a_4) = \cdots = n(a_ia_ja_k)$, where $i \neq j \neq k$

and so on.

Let $N(r)$ denote the common properties of $a_1, a_2, \ldots, a_n$ when taken r at a time. $N(0)$ is the value of $n(a_1'a_2' \ldots a_n')$, where a_i' is the complementary property of a_i, and N is the value of collection of zero property or at least one property.

Now we can rewrite the PIE in the form of

$$N(0) = N - C_1^n N(1) + C_2^n N(2) - C_3^n N(3) + \cdots + (-1)^n C_n^n N(n)$$

5.16 Derangements

Consider following problems. At a party 14 gentlemen check their overcoats. In how many ways can their overcoats be returned so that no gentleman get their own overcoat? In a cricket team there are 11 players who bat in a certain order. In how many ways those can bat so that no player bats at their pre-determined position? This type of problems fall in the category of following general problem.

Given an n-element set S in which each element has a specified position. We have to find the number of permutations of S in which no element is in its specified position. This can be exemplified by a set $S = \{1, 2, \ldots, n\}$ in which location of each integer is that specified by its position in the sequence $1, 2, \ldots, n$. A derangement $\{1, 2, \ldots, n\}$ is a permutation of $i_1i_2 \ldots i_n$ of $1, 2, \ldots, n$ sucb that $i_{\neq}1, i_2 \neq 2, \ldots, i_n \neq n$. Derangement of such an n-element set is denoted by D_n

For $n \geq 1$

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

Let T be the set of all $n!$ permutations of X. For $j = 1, 2, \ldots, n$ let P_j be the property that, in a permutation, j is in its proper position. Let A_j denote the set of permutations with property P_j. Thus,

$$D_n = |\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n|.$$

The permutations in A_1 are of the form $1i_2 \ldots i_n$, where $i_1 \ldots i_n$ is a permutation of $\{2, \ldots, n\}$. Thus, $|A_1| = (n-1)!$. We can write the general form as $|A_j| = (n-1)!$. For $A_j \cap A_k$, two elements have to be in the proper position. So, $|A_j \cap A_k| = (n-2)!$. For any integer k with $1 \leq k \leq n$, $|A_1 \cap A_2 \cap \ldots \cap A_k| = (n-k)!$. Since there are ${}^nC_k$ subsets of T, applying the inclusion and exclusion principle, we obtain

$$D_n = n! -^n C_1(n-1)! +^n C_2(n-2)! - \ldots + (-1)^{nn} C_n 0!$$

$$\Rightarrow D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$

5.17 The Bijection Principle

Let $P = \{a_1, a_2, \ldots, a_n\}$, and $Q = \{b_1, b_2, \ldots, b_n\}$. If $f : P \to Q$ is an injective function then $n \leq m$. If $f : P \to Q$ is a surjective function then $n \geq m$. If $f : P \to Q$ is injective and surjective then f is known to be a bijective function. For a bijective function $n = m$.

5.18 Occupancy Problems

The problems related to distribution of balls into boxes are called occupancy problems. We can have following cases:

5.18.1 Distinguishable Balls and Distinguishable Boxes

Number of ways to divide n non-identical balls in r different boxes such that each box gets 0 or more number of balls (empty boxes allowed) $= r^n$.

If no box is empty, then the number is found by the inclusion/exclusion principle or by recurrence relation or by generating function method. Using any one of them number of ways, to divide n non-identical balls in r different boxes such that each box gets at least one object, can be found.

$$= r^n - C_1^r(r-1)^n + C_2^r(r-2)^n - \cdots + (-1)^r C_{r-1}^r 1^n$$

5.18.2 Identical Balls and Distinguishable Boxes

Theorem 7

If an empty box is allowed, then the number of distributions is C_{r-1}^{n+r-1}.

Proof

Let $x_1, x_2, \ldots, x_r$ be the numnber of objects given to groups $1, 2, \ldots, r$ respectively.

Clearly, $x_1 + x_2 + \cdots + x_r = n$.

As each group can get zero or more, we have $0 \leq x_i \leq n \sim i = 1, 2, 3, \ldots, r$.

We observe that the number of integral solutions ofthe above equation is equal to number of ways of distributing n identical objects among r groups such that each gets zero or more

$= C_n^{n+r-1} = C_{r-1}^{n+r-1}$.

If no box is allowed to remain empty then the no. of ways become C_{r-1}^{n-1}. □

5.18.3 Distinguishable Balls and Identical Boxes

We mark the balls by n natural numbers $1, 2, \dots, n$. A partition of $\{1, 2, \dots, n\}$ in r part is a set if r non-empty subsets $A_1, A_2, \dots, A_r$ of $\{1, 2, \dots, n\}$ such that $A_1 \cup A_2 \cup \dots \cup A_r = \{1, 2, \dots, n\}$ and any two of the sets $A_1, A_2, \dots, A_r$ are disjoint.

We denote the number of partitions of $\{1, 2, \dots, n\}$ by $S(n, r)$, which is called a Stirling number of the second kind.

We can easily verify that $S(n, 1) = 1, S(n, n) = 1, S(n, r) = 0$, if $r > n$.

Now we will find $S(n, r)$ for $1 < r < n$. First possibility is that the number n is by itself a partition, which implies that the number $1, 2, \dots, n-1$ must form a $r-1$ partition. The number of such partitions is $S(n-1, r-1)$.

Second case is that, the number n is along with at least one of $1, 2, \dots, n-1$ in a partition, which implies that the numbers $1, 2, \dots, n-1$ must form a r partition, and n must be inserted in anyone of the r subsets. So n can be put in r ways. The number of such partitions is $rS(n-1, r)$.

Hence, $S(n, r) = S(n-1, r-1) + rS(n-1, r) \Rightarrow S(n, 2) = 2^{n-1} - 1$.

In general, we can easily find

$$S(n, r) = \frac{1}{r!}\left[r^n - C_1^r(r-1)^n + C_2^r(r-2)^n - \dots + (-1)^{r-1}C_{r-1}^r 1^n\right].$$

5.18.4 Identical Balls and Identical Boxes

We find distribution of n identical balls in r identical boxes so that no box remains empty.

The number of distributions = The number of ways of writing n as the sum $x_1 + x_2 + \dots + x_r$, where $x_i \in \mathbb{P}, i = 1, 2, \dots, r$ = Number of partitions of n in r parts.

This is same as finding number of integral solutions of $x_1 + x_2 + \dots + x_r = n$ with $1 \le x_1 \le x_2 \le \dots \le x_r$, which is equal to coeff. of x^n in $\frac{x^r}{(1-x)(1-x^2)\cdots(1-x^r)}$. Let us denote this number by $P_r(n)$.

We see that $P_1(n) = P_n(n) = 1, P_2(n)\left\lfloor \frac{n}{2} \right\rfloor, P_r(n) = 0, r > n$. Now we will find $P_r(n), 1 < r < n$. We divide all partition in two types. (i) At least one partition of size 1 (ii) Nopartition of size 1.

Number of partitions of in case (i) is $P_{r-1}(n-1)$. Number of partitions in case (ii) is $P_r(n-r)$. Now we add one ball in each part so that each part will have size of at least 2.

Hence, $P_r(n) = P_{r-1}(n-1) | P_r(n-r), 1 < r \le \left\lfloor \frac{n}{2} \right\rfloor$.

5.19 Dirichlet Drawer Principle or Pigeonhole Principle

The simplest form of the pigeonhole principle is the following quite obvious statement.

Theorem 8

If $n+1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.

Proof

We will prove this by contradiction. If each box contains at most one object, then the total number of objects is $1+1+...+1(n1s)=n$. Since we have $n+1$ objects for distribution, some box will contain at least two of the obect. $\square$

5.19.1 Pigeonhole Principle: Strong Form

Theorem 9

Let $x_1, x_2, \dots, x_n \in \mathbb{P}$. If $x_1+x_2+\cdots+x_n-n+1$ objects are distributed into n boxes, then at least one of the ith boxes contain at least x_i objects.

Proof

Suppose that we distribute $x_1+x_2+\cdots+x_n-n+1$ objects among n boxes. Let ith box contain x_i-1 objects, then total would be $\sum_{i=1}^{n}(xi-1)=x_1+x_2+\cdots+x_n-n$ objects. And hence, at least one of the ith box will contain at least x_i objects. $\square$

5.20 Problems

1. If ${}^nP_4 = 360$, find n.
2. If ${}^nP_3 = 9240$, find n.
3. If ${}^{10}P_r = 720$, find r.
4. If ${}^{2n+1}P_{n-1} :{}^{2n-1} P_n = 3 : 5$, find n.
5. If ${}^nP_4 = 12 \times^n P_2$, find n.
6. If ${}^nP_5 = 20 \times P_3^n$, find n.
7. If ${}^nP_4 : P_4^{n+1} = 3 : 4$, find n.
8. If ${}^{20}P_r = 6840$, find r.
9. If ${}^{k+5}P_{k+1} = \frac{11(k-1)}{2}.{}^{k+3}P_k$, find k.
10. If ${}^{22}P_{r+1} :{}^{20} P_{r+2} = 11 : 52$, find r.
11. If ${}^{m+n}P_2 = 90$ and ${}^{m-n}P_2 = 30$, find m and n.
12. If ${}^{12}P_r = 11880$, find r.
13. If ${}^{56}P_{r+6} :{}^{54} P_{r+3} = 30800 : 1$, find r.
14. Prove that ${}^1P_1 + 2.{}^2P_2 + 3.{}^3P_3 + \cdots + n.{}^nP_n =^{n+1} P_{n+1} - 1$.
15. If ${}^nC_{30} = C_4^n$, find n.
16. If ${}^nC_{12} = C_8^n$, find ${}^nC_{17}$ and ${}^{22}C_n$.
17. If ${}^{18}C_r = C_{r+2}^{18}$, find ${}^rC_6$.
18. If ${}^nC_{n-4} = 15$, find n.
19. If ${}^{15}C_r : C_{r-1}^{15} = 11 : 5$, find r.
20. If ${}^nP_r = 2520$ and $C_r^n = 21$, find r.
21. Prove that ${}^{20}C_{13} + C_{14}^{20} - C_6^{20} - C_7^{20} = 0$.
22. If ${}^nC_{r-1} = 36$, $C_r^n = 84$ and $C_{r+1}^n = 126$, find n and r.
23. How many numbers of four digits can be formed with digits $1, 2, 3, 4$ and 5 if repetition of digits is not allowed?
24. How many numbrs between 400 and 1000 can be made with the digits $2, 3, 4, 5, 6$ and 0, with no repetitions?

25. Find the number of numbers between 300 and 3000 that can be formed with the digits $0, 1, 2, 3, 4$ and 5 with no repetitions.

26. How many numbers of four digits greater than 2300 can be formed with digits $0, 1, 2, 3, 4, 5$ and 6 with no repetitions?

27. How many numbers can be formed by using any number of digits $0, 1, 2, 3$ and 4 with no repetitions?

28. How many numbers of four digits can be formed with the digits $1, 2, 3$ and 4? Find the sum of those numbers.

29. Find the sum of all four digit numbers that can be formed with the digits $0, 1, 2$ and 3.

30. Find the sum of all four digits that can be formed with $1, 2, 2$ and 3.

31. A person has to send invitation to 6 friends. In how many ways can he send invitations to them if he has 3 servants?

32. In how many ways 3 prizes can be given away to 7 boys when each is eligible for any number of prizes?

33. A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made?

34. A letter lock consists of three ring each marked with 10 different letters. In how many ways is it possible to make an unsuccessful attempts to open the lock?

35. How many numbers greater than 1000 but less than 4000 can be formed with the digits $0, 1, 2, 3$ and 4 with repetitions allowed?

36. In how many ways can 8 Indians, 4 Americans and 4 Englishmen be seated in a row so that persons of same nationality sit together?

37. There are 20 books of which 4 are single volume and the other are books of $8, 5$ and 3 volumes. In how many ways can all these books are arranged on a shelf so that volumes of the same book are not separated?

38. A library has two books each having three copies and three other books each having two copies each. In how many ways can all these books be arranged in a shelf so that copies of same books are not separated?

39. In how many ways 10 examination papers be arranged so that the best and worst papers never come together?

40. There are 5 boys and 3 girls. In how many ways can they be seated in a row so that not all girls sit together?

41. In how many ways can 7 I.A. and 5 I.Sc. students can be seated in a row so that no two of the I.Sc. students sit together?

42. In a class there are 7 boys and 3 girls. In how many different ways can they can be seated in a row so that no two of the three girls are consecutive?

43. In how many ways 4 boys and 4 girls can be seated in a row so that boys and girls alternate?

44. In how many ways 4 boys and 3 girls can be seated in a row so that boys and girls alternate?

45. In how many ways can the letters of the word "civilization" be rearranged?

46. How any different words can be formed from the word "university" so that all vowels are together?

47. In how many ways can the letters of the word "director" be arranged so that vowles are never together?

48. How many words can be formed by rearranging the letter of the word "welcome"? How many of them end with 'o'?

49. How many words can be formed with the letters of the word "California" in such a way that vowels occupy vowels' position and consonants occupy consonants' position?

50. How many different words can be formed with the letters of the word "pencil" when vowels occupy even place?

51. How many different words can be formed with five given letters of which three are vowel and two are consonants? How many will have no two vowels together?

52. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2 and 3?

53. In how many ways 5 Indians and 4 British can be seated at a round table if

    i. there is no restriction?

    ii. all British sit together?

    iii. all 4 British do not sit together?

    iv. no two British sit together?

54. In how many ways 5 Indians and 5 British can be seated along a circle so that they are alternated?

55. A round table conference is to be held between 20 delegates of 20 countries. In how many ways can they be seated if two particular delegates are always to sit together?

56. How many numbers of four digits can be formed with the digits 1, 2, 4, 5, 7 with no repetitions?

57. How many numbers of 5 digits can be formed with the digits 0, 1, 2, 3 and 4?

58. How many numbers between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5, 6 and 7; with no repetitions?

59. How many numbers between 100 and 1000 can be formed with the digits $0, 2, 3, 4, 8$ and 9; with no repetitions?

60. Find the total no. of nine digit numbers which have all different digits.

61. How many number between 1000 and 10000 can be formed with the digits $0, 1, 2, 3, 4$ and 5; with no repetitions?

62. How many different numbers greater than 5000 can be formed with the digits $0, 1, 5$ and 9; with no repetitions?

63. Find the number of numbers between 300 and 4000 that can be formed with the digits $0, 1, 2, 3, 4$ and 5; with no repetitions?

64. How many numbers of four digits divisible by 5 can be formed with the digits $0, 4, 5, 6$ and 7; with no repetitions?

65. How many even numbers of 5 digits can be formed with the digits $1, 2, 3, 4$ and 5?

66. How many numbers less than 1000 and divisible by 5 can be formed, in which no digit repeats?

67. How many numbers between 100 and 999 can be formed with the digits $0, 4, 5, 6, 7$ and 8? How many of them are odd?

68. Find the number of even numbers that can be formed with the digits $0, 1, 2, 3$ and 4; with no repetitions?

69. Find the number of numbers of six digits with the digit $1, 2, 3,$ 4,5 and 6, in which 5 alwyas occupied tens place; with no repetitions.

70. A number of four different digit is formed using the digits $1, 2, 3, 4, 5, 6$ and 7. How many such numbers can be formed? How many of them are greter than 3400?

71. Find the number of numbers of 4 digits formed with the digits $1, 2, 3, 4$ and 5, in which 3 occurs in the thousand's place and 5 occurs in the unit's place.

72. Find the number of numbers of 4 digits formed with the digits $0, 1, 2, 3, 4$ and 5; with no repetitions. How many of these are greter than 3000?

73. How many number of numbers can be formed by using any number of digits $0, 1, 2, 3, 5, 7$ and 9?

74. How many different numbers can be formed with the digits $1, 3, 5, 7$ and 9; when taken all at a time and what is their sum?

75. Find the sum of all four digit numbers that can be foemd with the digits $3, 2, 3, 4$.

76. Find the sum of all numbers greater than 10000 formed with the digits $0, 2, 4, 6$ and 8; with no repetitions.

77. Find the sum of all five digit numbers with the digits $3, 4, 5, 6$ and 7; with no repetitions.

78. Find the sum of all four digit numbers that can be formed with $0, 2, 3$ and 5.

79. A servant has to post 5 letters and there are 4 letter boxes. In how many ways he can post the letters?

80. In how many ways can 3 prizes be given to 5 students, when each student is eligible for any number of prizes?

81. In how many ways can n things be given to p persons? Each person can get any number of things$(n > p)$.

82. There are m men and n monkeys$(m < n)$. If a man can have any number of monkeys, in how many wasy every monkey have a master?

83. In how many ways the following 5 prizes be given to 10 students? First and second in mathematics; first and second in chemistry and first in physics?

84. There are stalls for 12 animals in a ship. In how many ways the shipload can be made if there are cows, calves and horses to transported with each being 12 in number?

85. In how many ways 5 delegates be put in 6 hotels of a city of there is no restriction?

86. Find the numbers of 5 digits that can be formed with the digits $0, 1, 2, 3$ and 4 if repetition is allowed.

87. In how many ways rings of 6 different types can be had in 4 fingers?

88. Find the number of 4 digit numbers greater than 3000 that can be formed with the digits $0, 1, 2, 3, 4$ and 5 if repetition is allowed.

89. In a town, the car plate numbers can be of three or four digits without digit 0. What is the maximum number of cars that can be numbered?

90. In how many ways can a ten question multiple choice examination with one correct answer can be answered if there are four choices to each question? If no two consecutive questions are answered the same way, how many ways are there?

91. There are two books each of three volumes and two books each of two volumes. In how may ways can the ten books be arranged on a table so that the volumes of the same book are not separated?

92. A library has 5 copies of 1 book, 4 copies of 2 books, 6 copies of 3 books and single copy of 8 books. In how many ways all the books can be arranged in so that copies of the same book stay together?

93. In a dinner part there are 10 Indians, 5 Americans and 5 Britishers. In how many ways they can be seated if all persons of the same nationality always sit together?

94. In a class there are 4 girls and 6 boys. In how many ways can they be seated in a rows so that no two girls are together?

95. Show that the number of ways in which n books can be arranged on a shelf so that two particular books shall not be together is $(n-2)(n-1)!$

96. You are given six balls of different colors (black, white, red, green, violet, yellow). In how many ways can you arrange them in a row so that black and white balls may never come together?

97. Six papers are set an examination, 2 of them in mathematics. In how many different orders can the papers be given if two mathematics papers are non successive?

98. In how many different ways can 15 I.Sc. and 12 B.Sc. students be arranged in a line so that no two B.Sc. students occupy consecutive positions?

99. In how many ways can 18 white and 19 black balls be arranged in a line so that no two white balls may be together. It is given that balls of same color are identical.

100. Show that the number of ways in which p positive and n negative signs mat be placed in a row so that no two negative signs may be together is C_n^{p+1}.

101. m men and n women are to be seated in a row so that no two women sit together. If $m > n$, then show that the number of ways in which they can be seated is $\frac{m!(m+1)!}{(m-n+1)!}$

102. 3 women and 5 men are to sit in a row. Find in how many ways they can be arranged so that no two women sit next to each other.

103. Find the number of ways of arranging 5 a's, 3 b's, 3 c's, 1 d, 2 e's and 1 f in a row, if letter c's are separated from one another.

104. Find the number of different permutations of the letters of the word “Banana”.

105. How many words can be formed from the letters of the word “circumference” taken all together?

106. There are three copies of each of four different books. In how many ways they can be arranged in a shelf?

107. Find the number of permutations of the letters of the word “Independence”.

108. How many different words can be formed can be formed with the letters of the word “Principal” so that the vowels are together?

109. How many words can be formed with the letters of the word “Mathematics”? In how many of them the vowels are togeter and consonants are together?x

110. In how many ways can the letters of the word “Director” be arranged so that the three vowels are together?

111. In how many ways can the letters of the word “Plantain” be arranged so that the three vowels are together?

112. Find the number of words that can be made by arranging the letters of the word “Intermediate” so that the relative order of vowels and consonants do not change.

113. In how many permutations of the word “Parallel” all the ls do not come together?

114. Find the number of words formed by the letters of the word “Delhi” which

   i. begin with D.

   ii. end with I.

   iii. the letter L being always in the middle.

   iv. begin with D and end with I

115. In how many ways can the letters of the word “Violent” be arraged so that vowels occupy only the odd places?

116. In how many ways can the letters of the word “Saloon” be arraged if consonants and vowels must occupy alternate places?

117. How many words can be formed out of the word “Article” so that vowels occupy the even places?

118. How many numbers greater than four million can be formed with the digits $2, 2, 3, 0, 3, 4$ and 5?

119. How many seven digits can be formed with the digits $1, 2, 2, 2, 3, 3$ and 5? How many of them are odd?

120. How many seven digits can be formed with the digits $1, 2, 3, 4, 3, 2$ and 1, so that odd digits always occupy the odd places?

121. How many numbers greater than $10,000$ can be formed with the digits $1, 1, 2, 3, 4$ and 0?

122. Find the number of numbers of four digits that can be made from the digits $0, 1, 2, 3, 4$ and 5 if the digits can be repeated in the same number. How many of these numbers have at least one digit repeated?

123. How many signals can be made by hoisting 2 blue, 2 red and 5 yellow flags on a flag at the same time?

124. How many signals can be made by hoisting 6 differently colored flags one above the other when any number of them can be hoisted at once?

125. Find the number of arrangements of the letter of the word “Delhi” if e always comes before i.

126. In how many ways can 5 men sit around a table?

127. In how many ways 5 boys and 5 girls can site around a table, if there is no restriction; if no two girls sit side-by-side?

128. In a class of students there are 6 boys and 4 girls. In how many ways can they be seated around a table so that all 4 girls sit together?

129. 5 boys and 5 girls from a line with the boys and girls alternating. Find the number of ways in which line can be made. In how many different ways could they form a circle so that boys and girls alternate?

130. In how many ways 6 boys and 5 girls can sit at a round table when no two girls sit next to each other?

131. In how many ways 50 pearls be arranged to form a necklace?

132. A round table conference is to be held between 20 delegates of 20 countries. In how many ways they and the host can be seated if two particular delegates are always to sit on the either side of the host?

133. Four gentlemen and four ladies are invited to a certain party. Find the number of ways of seating them around a table so that only ladies are seated on the two sides of each gentleman.

134. In how many ways can 7 Englishmen and 6 Indians sit around a table so that no two Indians are together?

135. If $C^{15}_{3r} = C^{15}_{r+3}$, find r.

136. If $C^{n}_{6} : C^{n-3}_{3} = 33 : 4$, find n.

137. Find the value of the expression $C^{47}_{4} + \sum_{j=1}^{5} C^{52-j}_{3}$.

138. Prove that the product of r consecutive integers is divisible by $r!$

139. Find the number of triangles, which can formed by joining the angular points of a polygon of m sides as vertices.

140. A man has 8 children to take them to a zoon. He takes three of them at a time to the zoo as often as he can without the same 3 children together more than once. How many times will he have to go to zoo? How many times a particular child will go?

141. On a new year day every student of a class sends a card to every other student. The postman delivers 600 cards. How many students are there in the class?

142. Show that a polygon of m sides has $\frac{m(m-3)}{2}$ diagonals.

143. Out of 6 gentelmen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady in each committee?

144. There are ten point on a plane. Of these ten points four points are in a straight line. With the exception of these four points, no other three points are in the same straight line. Find (a) the number of triangles formed, (b) the number of straight lines formed, and (c) the number of quadrilaterals formed, by joining these ten points.

145. There are 4 oranges, 5 apples and 6 mangoes in a fruit basket. In how many ways a person make a selection of fruits from the fruits basket.

146. Given 5 different green dyes, 4 different blue dyes and 3 different red dyes, how many combinations of dies can be chosen taking at least one green and one blue dye?

147. Find the number of divisors of $216,000$.

148. In an examination a minimum is to be secured in each of 5 subjects to pass. In how many ways can a student fail?

149. In how many ways 12 different things can be divided equally among 3 persons? Also find in how many ways can these 12 things be divided in three sets having 4 things.

150. How many different words of 4 letters can be formed with the letters of the word "Examination"?

151. How many quadrilaterals can be formed by joining vertices of a polygon of n sides?

152. A man has 7 friends and he wants to invite 3 of them at a party. Find out how many parties to each of his 3 friends he can give and how many times any particular friend will attend the parties.

153. Prove that the number of combinations of n things taken r at a time in which p particular things always occur is C^{n-p}_{r-p}.

154. A delegation of 6 members is to be sent abroad out of 12 members. In how many ways can the selection be made so that (a) a particular member is always included, and (b) a particular member is always exlcluded.

155. There are six students A, B, C, D, E and F. (a) In how many ways can they be seated in a line so that C and D do not sit together? (b) In how many ways can a committe of 4 be formed so as to always include C? (c) In how many ways can a committee of 4 be formed so as to always include C but exclude E?

156. There are n stations in a railway route. The number fo kinds of ticket printed (no return ticket) is 105. Find the number of stations.

157. There are 15 points in a plane of which 6 are collinear. How many different straight lines and triangles can be drawn by joining them?

158. There are 10 points in a plane out of which 5 are collinear. Find the number of quadrilaterals formed having vertices at points.

159. The three sides of a triangle have $3, 4$ and 5 interior points on them. Find the number of triangles that can be constructed using given interior points as vertices.

160. In how many ways can a team of 11 be chosen from 14 football players if two of them can be only goalkeepers?

161. A committee of 2 men and 2 women is to be chosen from 5 men and 6 women. In how many ways can this be done?

162. Find the number of ways in which 8 different articles can be distributed among 7 boys, if each boy is to receive at least one article.

163. Out of 7 men and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include at least 3 ladies?

164. A candidate is required to ansswer six out of ten questions which are divided into two groups, each containing five questions and he is not permitted to attempt more than 4 from any group. In how many ways can he make up his choices?

165. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students to be formed. Find in how many ways these committees can be formed if (a) a particular professor is included? (b) a particular professor is excluded.

166. From 6 boys and 7 girls, a committee of 5 is to be formed so as to include at least one girl. Find the number of ways in which this can be done.

167. From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done if (a) there is no restriction? (b) the committee is to include at least one lady?

168. From 8 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady.

169. In a group of 15 boys, there are 6 hockey players. In how many ways can 12 boys be selected so as to include at least 4 hockey players?

170. From 7 gentlemen and 4 ladies a boat party of 5 is to be formed. In how mny ways can this be done so as to include at least one lady?

171. A committee of 6 is to formed out of 4 boys and 6 girls. In how many ways can this be done if girls may not be outnumbered?

172. A person has 12 friends out of which 8 are relatives. In how many ways can he invite 7 friends such that at least 5 of them are relatives?

173. A student is required to answer 7 questions out of 12 questions which are divided into two groups of 6 questions each. He is not permitted to attempt more than 5 from either group. In how many ways can he choose the 7 questions?

174. Each of two parallel lines has a number of distinct points marked on them. On one line there are 2 points P and Q and on the other there are 8 points. Find the number of possible triangles out of these points. How many of these include P but exclude Q?

175. There are 7 men and 3 ladies contesting for 2 vacancies. An elector can vote for any no. of candidates not exceeding no. of vacancies. In how many ways can the elector vote?

176. A party of 6 is to be formed from 10 boys and 7 girls so as to include 3 boys and 3 girls. In how many ways can this party be formed if two particular girls cannot be together?

177. In an examination, the question paper consists of three different sections of $4, 5$ and 6 questions. In how many ways, can a student make a selection of 7 questions, selecting at least 2 questions from each section.

178. From 5 apples, 4 oranges and 3 mangoes, how many selections of fruits can be made?

179. Find the total no. of selections of at least one red ball from 4 red and 3 green balls if the balls of same color are different.

180. Find the number of different sums that can be formed with one dollar, one half dollar and one quarter dollar coin.

181. There are 5 questions in a question paper. In how many ways can a boy solve one or more questions?

182. In an election for 3 seats there are 6 candidates. A voter cannot vote for more than 3 candidates. In how many ways can he vote?

183. In an election the number of candidates is one more than the number of members to be elected. If a voter can vote in 30 different ways, find the number of candidates. (A voter has to vote for at least one candidate.)

184. In how many ways 12 different books can be distributed equally among 4 persons?

185. In how many ways 10 mangoes can be distributed among 4 person if any person can get any number of mangoes?

186. How many words can be formed out of 10 consonants and 4 vowels, such that each contains 3 consnants and 2 vowels?

187. A table has 7 seats, 4 being on one side facing the window and three being on the opposite side. In how many ways can 7 people beseated at the table if 3 people X, Y, Z must sit on the side facing the window?

188. A tea party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated.

189. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First two women choose chairs amongst the chair marked 1 to 4; and then men select the chairs from remaining. Find the number of possible arrangements.

190. Show that ${}^{2n}C_r(0 \leq r \leq 2n)$ is greatest when $r = n$.

191. How many different numbers of seven digits can be formed by using all the digits $1, 2, 3, 4, 3, 2, 1$, so that odd digits always occupy odd places?

192. Ten different letters of an alphabet are given. Words having five letters are formed from these given letters. Find the number of words which have at least one letter repeated.

193. How many ternary sequences of length 9 are there which either begin with 210 or end with 210?

194. Find the number of 7 digit numbers when the sum of those digits is even.

195. In how many ways 10 Indians, 5 Americans and 4 Britishers can be seated in a row so that all Indians are together?

196. In how many ways can the letters of the word ‘Arrange’ be arranged so that (a) the two r's are never together? (b) the two a's are together but not the two r's? (c) neither the two a's nor the two r's are together?

197. A man invites a party of $m + n$ friends to dinner and places m at around table and n at another. Find the number of arranging the guests.

198. Find the total no. of signals that can be made by five flags of different colors when any number of them may be used.

199. The letters of the word ‘Ought’ are written in all possible orders and these words are written out in a dictionary. Find the rank of ‘Tough’ in the dictionary.

200. The streets of a city are arranged like the lines of a chessboard. There are m streets running north and south and n east and west. Find the number of ways in which a man can travel from the N.W. to S.E. corner, going the shortest distance possible.

201. There are n letters and n corresponding envelops. In how many ways, can the letters be places in envelops (one letter in each envelop) so that no letter is put in the right envelop?

202. Find the number of non-congruent rectangles that can be formed on a chessboard.

203. Show that the no. of ways in which three numbers in A.P. can be selected from $1, 2, 3, \ldots, n$ in $\frac{1}{4}(n-1)^2$ or $\frac{1}{4}n(n-2)$; according as n is odd or even.

204. Two packs of52 playing cards are shuffled together. Find the number of ways in which a man can be dealt 26 cards so that he does not get two cards from the same suit and same denomination.

205. There is a polygon of n sides $(n > 5)$. Triangles are formed by joining the vertices of the polygon. How many triangles are there? Also, prove that number of these triangles which have no side in common with any of the sides of the polygon is $\frac{1}{6}n(n-4)(n-5)$.

206. n different objects are arranged in a row. In how many ways can 3 objects be selected so that (a) all three objects are consecutive, and (b) all three objects are not consecutive.

207. There are 12 intermeditate stations between two places, A and B. In how many ways can a train be made to stop at 4 of those 12 intermeditate stations so that no two of which are consecutive?

208. There are m points in a plane which are joined by straight lines in all possible ways and of these no two are coincident and no three of them are concurrent except at the points. Show that the number of points of intersection, other than the given points of the lines so formed is $\frac{m!}{8.(m-4)!}$.

209. Find the number of ways of choosing m coupon out of an unlimited number of coupons bearing the letters A, B and C so that they cannot be used to spell the word BAC.

210. A straight is a five-card hand containing consecutive values. How many different straights are tere? If the cards are not all from the same suit, then how many straights are there?

211. A is an n-element set. A subset P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1. Then a subset P_2 of A is chosen and again set A is reconstructed by replacing the elements of P_2. In this way m subsets are chosen, where $m > 1$. Find the number of ways of choosing $P_1, P_2, \ldots, P_m$ such that

   i. $P_1 \cup P_2 \cup \ldots \cup P_m$ contains exactly r elements of A.

   ii. $P_1 \cap P_2 \cap \ldots \cap P_m$ contains exactly r elements of A.

   iii. $P_i \cap P_j = \phi$ for $i \neq j$.

212. Find the number of ways in which m identical balls be distributed among $2m$ boxes so that no box contains more than one ball and show that it lies between $\frac{4^m}{\sqrt{2m+1}}$ and $\frac{4^m}{2\sqrt{m}}$.

213. If m parallel lines are intersected by n other parallel lines find the number of parallelograms thus formed.

214. From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done if the committee is to include at least one lady and if two particular ladies refuse to server on the same committee?

215. A man has 7 relatives, 4 of them are ladies and 3 are gentlemen. His wife also has 7 relatives, 3 of them are ladies and 4 are gentlemen. In how many ways can they invite to a dinner party of 3 ladies and 3 men so that there are 3 of the man's relatives and 3 of the wife's relatives?

216. Prove that if each of m points on one straight line be joined to each of the n points on the other straight line terminated by the points, then excluding the points given on the two lines, number of points of intersection of these lines is $\frac{1}{4}mn(m-1)(n-1)$.

217. John has x children with his first wife. Mary has $x+1$ children with her first husband. They marry and have children of their own. The whole family has 24 children. Assuming that two children of same parents do not fight, prove tha maximum possible no. of ways fight can take place is 191.

218. Find the number of divisors and sum of divisors of 2520.

219. Five balls of different colors are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many different ways can we place the balls so that no box remains empty.

220. Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$.

221. If a and b are positive integers, show that $\frac{(ab)!}{a!(b!)^a}$ is an integer.

222. A conference attended by 200 delegates is held in a hall. The hall has seven doors, marked A, B, ... , G. At each door, an entry book is kept and the delegates entering that door sign it in the order in which they enter. If each delegate is free to enter any time and through any door they like, how many different sets of seven lists would arise in all?

223. In how many ways 16 identical objects can be distributed among 4 persons if each person gets at least 3 objects?

224. Show that a selection of 10 balls can be made from an unlimited number of red, while, blue and green balls in 286 ways and that 84 of these contain balls of all four colors.

225. In how many ways 30 marks can be allotted to 8 questions if each question carries at least 2 marks?

226. In an examination, the maximum marks for each of the three papers is 50 each. Maximum marks for the fourth paper is 100. Find the number of ways in which a student can score 60% marks in aggregate.

227. Let n and k be positive integers, such that $n \geq \frac{k(k+1)}{2}$. Find the number of solutions $x_1, x_2, \ldots, x_k, x_1 \geq 1, x_2 \geq 2, \ldots, x_k \geq k$ all satisfyinng $x_1 + x_2 + \ldots + x_k = n$.

228. Find the number of integral solution of equation $x + y + z + w = 29, x > 0, y > 1, z > 2$ and $w \geq 0$.

229. Find the number of non-negative integral solutions of the equation $x + y + z + 4w = 20$.

230. Find the number of non-negative integral solutions to the system of equations $x + y + z + w + v = 20$ and $x + y + z = 5$.

231. Find the number of positive integral solutions of the inequality $3x + y + z \leq 30$.

232. Find the number of positive unique integral solution of the equation $a + b + c + d = 20$.

233. How many integers between 1 and 1,000,000 have the sum of digits 18?

234. Prove that the number of combinations of n letters together out of $3n$ letters of which n are a and n are b and the rest unlike is $(n + 2)\,2^{n-1}$.

235. An eight-oared boat is to be manned by a crew chose from 11 men of whom 3 can steer but cannot row and the rest cannot steer. In how many ways can the crew be arranged if two of them can only row the bow side?

236. Find the total number of ways of selecting five letters from the letters of the word ‘Independence’.

237. Find the number of combinations, and the number of permutations of the letters of the word ‘Parallel’, taken four at a time.

238. Prove that $\displaystyle\sum_{n=1}^{n} (n^2 + 1)\, n! = n.(n + 1)!$.

239. Find the value of n for which $\frac{^{n+4}P_4}{(n+2)!} - \frac{143}{4.n!} < 0$.

240. Find the value of n for which $\frac{195}{4.n!} - \frac{(n+3)(n+2)(n+1)}{(n+1)!} > 0$.

241. If $^{n-2}P_4 :^{n+2} C_8 = 16 : 57$, find the value of n.

242. If $^nP_r =^n P_{r+1}$ and $^nC_r =^n C_{r-1}$, find n and r.

243. If $^nP_{r-1} :^n P_r :^n P_{r+1} = a : b : c$, prove that $b^2 = a(b+c)$.

244. If $^{n+1}C_{r+1} :^n C_r :^{n-1} C_{r-1} = 11 : 6 : 3$, find n and r.

245. Show that $\sum_{k=m}^{n} {}^k C_r =^{n+1} C_{r+1} -^m C_{r+1}$.

246. Show that $C_r^n + 3.C_{r-1}^n + 3.C_{r-2}^n + C_{r-3}^n = C_r^{n+3}$.

247. Find r for which $^{18}C_{r-2} + 2.^{18}C_{r-1} +^{18} C_r \geq^{20} C_{13}$.

248. Prove that $^{4n}C_{2n} :^{2n} C_n = 1.3.5 \dots (4n-1) : [1.3.5 \dots (2n-1)]^2$.

249. Find the positive integral values of x such that $^{x-1}C_4 -^{x-1} C_3 - \frac{5}{4}(x-2)(x-3) < 0$.

250. Prove that $^{2n}P_n = 2^n.1.3.5 \dots (2n-1)$.

251. Show that there cannot exist two positive integers n and r for which $^nC_r,^n C_{r+1},^n C_{r+2}$ are in G.P.

252. Show that there cannot exist two positive integers n and r for which $^nC_r,^n C_{r+1},^n C_{r+2},^n C_{r+3}$ are in A.P.

253. For all positive integers show that $2.6.10 \dots (4n-6)(4n-2) = (n+1)(n+2) \dots (2n-2)2n$.

254. Show that $^{47}C_4 + \sum_{i=0}^{3} {}^{50-i} C_3 + \sum_{j=1}^{5} {}^{56-j} C_{53-j} =^{57} C_4$.

255. Show that $^nC_k + \sum_{j=0}^{m} {}^{n+j} C_{k-1} =^{n+m+1} C_k$.

256. Show that $^mC_1 +^{m+1} C_2 + \dots +^{m+n-1} C_n =^n C_1 +^{n+1} C_2 + \dots +^{n+m-1} C_m$.

257. How many numbers of 5 digits divisible by 25 can be made with the digits $0, 1, 2, 3, 4, 5, 6$ and 7?

258. How many numbers of 5 digits divisible by 4 can be made with the digits $1, 2, 3, 4$ and 5?

259. How many numbers of 4 digits divisible by 3 can be made with the digits $0, 1, 2, 3, 4$ and 5, digits being unrepeated in the same number? How many of these will be divisible by 6?

260. Find the sum of all the 4 digit numbers formed with the digits $1, 3, 3$ and 0?

261. Show that the number of permutation of n different objects taken not more than r at a time, when each object may be repeated any number of times is $\frac{n(n^r-1)}{n-1}$.

262. How many different 7 digit numbers are there sum of whose digits is even?

263. k numbers are chosne with replacement from the numbers $1, 2, 3, \ldots, n$. Find the number of ways of choosing the numbers so that the maximum number chosen is exactly $r(r \leq n)$.

264. Find the number of n digit numbers formed with the digits $1, 2, 3, \ldots, 9$ in which no two consecutive digits repeat.

265. A valid FORTRAN identifier consists of a string of one to six alphanumeric characters which are $A, B, \ldots, Z, 1, 2, \ldots, 9$ beginnning with a letter. How many valid FORTRAN identifiers are there.

266. Find the number of five digit number which can be made with at least one repeated digit.

267. Find the number of numbers between $20,000$ and $60,000$ having sum of digits even.

268. Find th enumber of ways in which the candidates $A_1, A_2, \ldots, A_{10}$ can be ranked, (a) if A_1 and A_2 are next to each other. (b) if A_1 is always above A_2.

269. $m+n$ chairs are placed in a line. You have to seat n men and m women on these chairs such that no man gets a seat between two women. In how many ways can these people be seated?

270. How many words can be made with the letters of the word ‘Intermediate’ if no vowel is between two consonants?

271. In how many ways can 5 identical black balls, 7 identical red balls and 6 identical green balls be arranged so that at least one ball is sperated from balls of the same color?

272. Ten guests are to be seated in a row of which three are ladies. The ladies insist on sitting together while two of gentlemen refuse to take consecutive seats. In how many ways can they be seated?

273. Show that the number of permutations of n different objects taken all at a time in which p particular objects are never together is $n! - (n-p+1)!\,p!$.

274. Find the number of ways in which six \`+' signs and four \`-' signs can be arranged so that no two \`-' signs occur together.

275. In how many ways can 3 ladies and 5 gentlemen arrange themselves about a round table so that every gentleman may have one lady by his side?

276. How many words of 7 letters can be formed by using the letters of the word ‘success’ so that (a) no two C's are together but not the two S, (b) neither the two C nor the two S are together?

277. A dictionary is made of the words that can be formed from the letters of the word 'Mother'. What is the position of the word 'Mother' in that dictionary if the words are printed in the same order as that of a dictionary.

278. A train going from Kolkata to Delhi stops at 7 intermediate stations. Five persons enter the train during the jouney with five different tickets of the same class. How many different set of tickets they could have had.

279. A train going from Cambridge to London stops at 9 intermediate stations. Six persons enter the train during the jouney with six different tickets of the same class. How many different set of tickets they could have had.

280. In how many ways can clear and cloudy days occur in a week? It is given that any day is entirely either clear or cloudy.

281. A student is allowed to select at most n books from a collection of $2n+1$ books. If the total no. of ways in which he can select at least one book is 63, find the value of n.

282. There are m bags which are numbered by m consecutive integers starting with the number k. Each bag contains as many different flowers as the number marked on the bag. A boy has to pick up k flowers from any of the bags. In how many different ways can he do it?

283. How many committes of 11 persons can be made out of 50 persons if three particular person are not to be included together?

284. There are m intermediate stations on a railway line between two place P and Q. In how many ways can the train stop at three of these intermediate stations, no two of which are consecutive?

285. A is an n-element set. A subset of P of A is chosen. The set A is reconstructed by replacing the elements of P. Then a subset Q of A is chosen. Find the number of ways of choosing P and Q such that (a) $P \cap Q$ contains exactly 2 elements, and (b) $P \cap Q = \phi$.

286. A is an n-element set. A subset P_1 is chosen. The set A is reconstructed by replacing the elements of P_1. Then a subset P_2 is chosen ad again the set is reconstructed by replacing elements of P_2. In this way m subsets $P_1, P_2, \ldots, P_m$ are chosen, where $m > 1$. Find the number of ways of choosing these subesets such that

    i. $P_1 \cup P_2 \cup \ldots \cup P_m$ contains all the elements of A except one.

    ii. $P_1 \cup P_2 \cup \ldots \cup P_m = A$.

    iii. $P_1 \cap P_2 \cap \ldots \cap P_m = \phi$.

287. There are three sections in a question paper, each containing 5 questions. A candidate has to solve any 5 questions, choosing at least one from each section. Find the number of ways in which the candidate can choose the questions.

288. Two numbers are selected at random from $1, 2, 3, \ldots, 100$ and are multiplied. Find the number of ways in which the two numbers can be selected so that the product thus obtained is divisible by 3.

289. In how many ways can a mixed doubles game in tennis be arranged from 5 married couples, if no husband and wife play in the same game?

290. There are n concurrent lines and another line parallel to one of them. How many different triangles will be formed by the $(n+1)$ lines?

291. In a plane there are n lines no two of which are parallel and no three are concurrent. How many different triangles can be formed with their points of intersection as vertices?

292. The England cricket team is to be selected out of fifteen players, five of them are bowlers. In how many ways can the team be selected so the team contains at least three bowler?

293. There are two bags each containing m balls. Find the number of ways in which equal no. of balls can be selected from both bags if at least one ball from each bag has to be selected.

294. A committee of 12 is to be formed from 9 women and 8 men. In how many ways can this be done if at least 5 wmen have to be included in a committee. In how many of these committees, the women are in majority and the men are in majority?

295. m equi-spaced horizontal lines are intersected by n equi-spaced vertical lines. If $m < n$ and the distance between two successive vertical lines, show that the number of squares formed by these lines $\frac{1}{6}m(m-1)(3n-m-1)$.

296. There are two sets of parallel lines, their equations being $x\cos\alpha + y\sin\alpha = p; p = 1, 2, 3, \dots, m$ and $y\cos\alpha - x\sin\alpha = q; q = 1, 2, 3, \dots, n(n > m)$, where α is a constant. Show that the lines form $\frac{1}{6}m(m-1)(3n-m-1)$ squares.

297. In how many different ways can a set A of $3n$ elements be partitioned in 3 equal number of elements?

298. In how many ways 50 different objects can be divided in 5 persons so that three of them get 12 objects each and two of them get 7 objects each?

299. If $a, b, c, \dots, k$ are positive integers such that $a + b + c + \dots + k \leq n$, show that $\frac{n!}{a!b!\dots k!}$ is a positive integer.

300. If $n \in N$, show that $\frac{(n^2)!}{(n!)^{n+1}}$ is an integer.

301. If $ab = n(a > 1, b > 1)$, then show that $(n-1)!$ is divisible by both a and b.

302. Show that $(kn)!$ is divisible by $(n!)^k$.

303. In how may ways 20 apples be distributed among 5 persons if each person can get any number of apples?

304. In how many ways r flags be displayed on n poles in a row, disregarding the limitation on the number of flags on a pole?

305. If $x + y + z = n$, where $x, y, z, n \in \mathbb{P}$, find the number of integral solution of this equation.

306. Find the number of integeral soolutions of $x + y + z = 0$, $x, y, z \geq -5$.

307. in an examination, the maximum marks for each of the three papers is n; for the fourth paper it is $2n$. Prove that the number of ways in which a student can get $3n$ marks is $\frac{1}{6}(n+1)(5n^2+10n+6)$.

308. Find the number of positive integral solutions of the equation $x_1 + x_2 + x_3 = 10$.

309. Find the number of non-negative integral solutions of equation $3x + y + z = 24$.

310. Find the number of non-negative integral solutions of equation $x + y + z + w = 29$, where $x \geq 1, y \geq 2, z \geq 3, w \geq 0$.

311. Find the number of non-negative integral solutions of the equation $a + b + c + d = 20$.

312. Find the number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots + x_k \leq n$.

313. Find the number of non-negative integral solutions of the equation $2x + 2y + z = 10$.

314. How many sets of 2 and 3 (different) numbers can be formed by using numbers between 0 and 180 (both inclusive) so that their average is 60.

315. If combinations of letters be formed by taking only 5 at a time out of the letters of the word 'Metaphysics', in how many of them will the letter T occur?

316. How many selections and arrangements of 4 letters can be made from the letters of the word 'Proportion'?

317. A five letter word is formed such that the letter in the odd numbered positions are taken from the letters which appear without repetitioni n the word 'Mathematics'. Further, the letters appearing in the even numbered positions are taken from the letter which appear with repetitions in the same word 'Mathematics'. In how many different ways can the five letter word be formed?

318. Box 1 contains six block lettered A, B, C, D, E and F. Box 2 contains four block lettered W, X, Y and Z. How many five letter codewords can be formed by using three blocks from box 1 and two blocks from box 2?

319. A tea party is arranged for $2m$ people along two sides of a long table with m chairs on each side. r men wish to sit on one particular side and s on the other. In how many ways can then be seated? $(r, s \leq m)$

320. A gentleman invites a party of 10 friends to a dinner and there are 6 places at round tale and the remaining 4 at another. Prove that the no. of ways in which he can arrange them among themselves is 151,200.

321. A family consists of a grandfather, m sons and daughters and $2n$ grandchildren. There are to be seated in a row for dinner. The grandchildren wish to occupy the n seats

at each end and grandfather refuses to have a grandchild on either side of him. In how many ways can the family be seated?

322. There are $2n$ guests at a dinner party. If the master and mistress of the house have fixed seats opposite one another and that there are two specified guests who must not be placed next to one another, find the number of ways the guests can be placed.

323. There are $4n$ objects of which n are alike and all the rest are different. Find the number of permutations of $4n$ objects taken $2n$ at a time, each permutation containing the n like objects.

324. A 7-digit number divisible by 9 is to be formed by using 7 digits out of digits $1, 2, 3, 4, 5, 6, 7, 8, 9$. Find the number of ways in which this can be done.

325. Find the number of 9-digit numbers divisible by nine using the digits from 0 to 9 if each digit is used atmost once.

326. Among 9! permutations of the digits $1, 2, 3, \dots, 9$. Consider those arrangements which have the property that if we take any five consecutive positions, the product of the digits in those positions is divisible by 7. Find the number of such arrangements.

327. Three distinct dice are rolled. Find the number of possible outcomes in which at least one die shows 5.

328. Find the number of possible outcomes in a throw of n distinct dice in which at least one of the dice shows an odd number.

329. Find the number of times the digit 5 will be written when listing integers from 1 to 1000.

330. If 33! is divisible by 2^n, then find the maximum value of n.

331. Let $E = \left\lfloor \frac{1}{3} + \frac{1}{50} \right\rfloor + \left\lfloor \frac{1}{3} + \frac{2}{50} \right\rfloor + \left\lfloor \frac{1}{3} + \frac{3}{50} \right\rfloor + \cdots$ up to 50 terms, then find the exponent of 2 in $E!$.

332. 3-digit numbers in which the middle one is a perfect square are formed using the digits 1 to 9. Find their sum.

333. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is $1, 16, 31$ etc.) This process is continued until a number is reached which has already been marked, then find the all unmarked numbers.

334. Let S be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the number of subsets A of S such that $x \in A$ and $2x \in S \Rightarrow 2x \in A$.

335. Prove that there are $2(2^{n-1} - 1)$ ways of dealing n distinct cards to two persons. The persons may receive unequal no. of cards, and each one receiving at least one card.

336. Find the number of ways in which one or more letters can be selected from the letters: A A A A B B B C D E.

337. Find the number of factors (excluding 1 and the expression itself) of the product of $a^7b^4c^3def$ where a, b, c, d, e, f are all prime numbers.

338. Find the no. of positive divisors of $b_1^{p_1}b_2^{p_2} \dots b_n^{p_n}$, where $b_1, b_2, \dots, b_n$ are prime numbers, and $p_1, p_2, \dots, p_n$ are positive integers.

339. In how many ways we can select two unit square on an ordinary chess board such that both square neither in same row nor in same column.

340. Find the number of pairings of a set of 2n elements [*e.g.*, $\{(1, 2), (3, 4), (5, 6)\}$ $\{(1, 3), (2, 4), (5, 6)\}$ are two pairings of the set $\{1, 2, 3, 4, 5, 6\}$].

341. There are 12 points in a plane, 5 of which are concyclic and out of remaining 7 points, no three are collinear and none concylic with previous 5 points. Find the number of circles passing through at least 3 points out of 12 given points.

342. In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.

343. There are two lines L_1 and L_2, and there are m and n points on these two lines. How many lines can be constructed using these points?

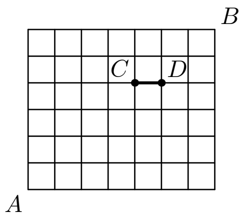
344. Let $P_i, i = 1, 2, \dots, 21$ be the vertices of a 21-sided regular polygon inscribed in a circle with center O. Triangle are formed by joining the vertices of the 21-sides polygon. How many of them are acute-angled triangles? How many of them are right-angled triangles? How many of them are obtuse-angled triangles? How many of them are equilateral triangles? How many of them are isosceles triangles?

345. Let U be a set containing n elements. A subset S of set U is chosen at random. The set U is reconstructed by replacing the elements of S, and another set T is chosen at random. Find the number of ways of choosing S and T such that $S \cup T$ contains exactly r elements.

346. Let U be a set containing n elements. A subset S of set U is chosen at random. The set U is then reconstructed by replacing the elements of S and another set T is chosen at random. Find the number of ways of selecting S and T such that $S = \overline{T}$.

347. What is the total no. of subsets of a set containing n elements?

348. Consider a network as shown in the figure.



Paths from A to B consists of the horizontal or vertical line segments. No diagonal movememnt is allowed. We can only move from left to right or down to up. How many paths are there from A to B? How many paths go via C? How many paths go via C to D?

349. Find the number of ways in which we can choose 3 squares on a chess board such that one of the squares has its two sides common to other two squares.

350. A person predicts the outcome of 20 cricket matches of his home team. Each match can result either in a win, loss or tie for the home team. Find the total number of ways in which he can make the predictions so that exactly 10 predictions are correct.

351. A forecast is to be made of the results of five cricket matches, each of which can be a win, a draw or a loss for Indian team. Find the number of different possible forecasts. Also find the number of forecasts containing 0, 1, 2, 3, 4 and 5 errors respectively.

352. In a club election the number of contestants is one more than the number of maximum candidates for which a voter can vote. If the total number of ways is which a voter can vote be 62, then find the number of candidates.

353. Every one of the 10 available lamps can be switched on to illuminate certain Hall. Find the total number of ways in which the hall can be illuminated.

354. In a unique hockey series between India and Pakistan, they decide to play on till a team wins 5 matches . Find the number of ways in which the series can be won by India, if no match ends in a draw.

355. There are n different books and p copies of each in a library. Find the number of ways in which one or more books can be selected.

356. A class has n students. We have to form a team of the students by including atleast two students and also by excluding atleast two students. Find the number of ways of forming the team.

357. If the $(n+1)$ numbers $a_1, a_2, a_3, \dots, a_{n+1}$, be all different and each of them is a prime number, then find the number of different factors (other than 1) of $a_1^m \cdot a^2 \cdot a^3 \dots a_{n+1}$.

358. In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70 then find the number of diagonals of the polygon.

359. In a plane there are two families of lines $y = x + r, y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. Find the number of squares of diagonals of the length 2 formed by the lines.

360. Find the number of triangles whose vertices are at the vertices of an octagon, but none of whose side happen to come from the sides of the octagon.

361. Let there be 9 fixed points on the circumference of a circle . Each of these points is joined to every one of the remaining 8 points by a straight line and the points are so positioned on the circumference that atmost 2 straight lines meet in any interior point of the circle. Find the number of such interior intersection points.

362. If a set A has m elements and another set B has n elements then find the number of functions from A to B.

363. Let $A = \{x : x \text{ is a prime number and } x < 30\}$. Find the number of different rational numbers whose numerator and denominator belongs to A.

364. Find the number of all three elements subsets of the set $\{a_1, a_2, a_3, \dots, a_n\}$ which contain a_3.

365. If the total number of m-element subsets of the set $A = \{a_1, a_2, a_3, \dots, a_n\}$ is k times the number of m-elements subsets containing a_4, then find n.

366. A set contains $(2n + 1)$ elements. Find the number of subsets of the set which contains at most n elements.

367. Find the number of subsets of the set $A = \{a_1, a_2, \dots, a_n\}$ which contain even number of elements.

368. Find the number of ways of choosing triplets (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \dots, n, n + 1\}$.

369. In the decimal number system, find the number of 6-digits numbers in which the digit in any place is greater than the digit to the left to it.

370. Find the number of 3-digit numbers of the form xyz such that $x < y$ and $z \leq y$.

371. Find the total number of 6-digit numbers $x_1x_2x_3x_4x_5x_6$ having the property $x_1 < x_2 \leq x_3 < x_4 < x_5 \leq x_6$.

372. If all the six digit numbers $x_1x_2x_3x_4x_5x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in the increasing order, then find the sum of the digits in the 72th number.

373. Let there be $n \geq 3$ circles in a plane. Find the value of n for which the number of radical centres, is equal to the number of radical axes. (Assume that all radical axes and radical centre exist and are different)

374. Find the number of functions f from the set $A = \{0, 1, 2\}$ into the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \leq f(j)$ for $i < j$ and, $i, j \in A$.

375. Show that the number of ways of selecting n objects out of $3n$ objects, n of which are alike and rest are different is $2^{2n-1} + \frac{(2n)!}{(n!)^2}$.

376. In a chess tournament, each participant was supposed to play exactly one game with each of the others. However, two participants withdraw after having played exactly 3 games each, but not with each other. The total number of games played in the tournament was 84. How many participants were there in all?

377. A positive integer n is called strictly ascending if its digits are in the increasing order. For example, 2368 and 147 are strictly ascending but $xml43679$ is not. Find the number of strictly ascending numbers $< 10^9$.

378. Consider the image given below:

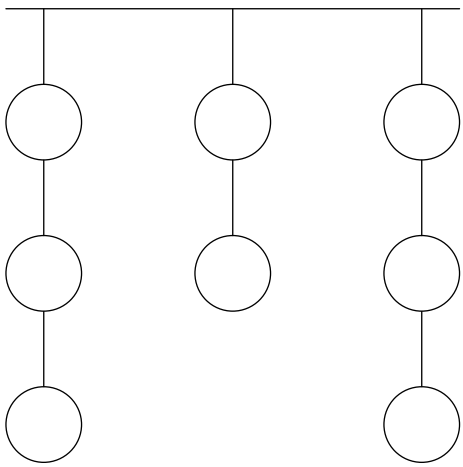


Figure 5.2

There are 8 clay targets, arranged in 3 columns, to be shot by 8 bullets. Find the number of ways in which they can be shot, such that no target is shot before all the targets below it, if any, are first shot.

379. How many hexagons can be constructed by joining the vertices of a quindecagon (15 sides) if none of the sides of the hexagon is also the side of the quindecagon?

380. Let A be a set of $n(\geq 3)$ distinct elements. Find the number of triplets (x, y, z) of the elements of A in which atleast two coordinates are equal.

381. Find the number of ways of arranging m numbers out of $1, 2, 3, \ldots, n$ so that maximum is $(n-2)$ and minimum is 2 (repetitions of numbers is allowed) such that maximum and minimum both occur exactly once, $(n > 5, m > 3)$.

382. Eight identical rooks are to be placed on an 8×8 chess-board. Find the number of ways of doing this, so that no two rooks are in attacking positions.

383. Define a good word as a sequence of letters that consists only of the letters A, B, and C - some of these letters may not appear in the sequence - and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?

384. Two n-digit integers (leading 0 allowed) are said to be equivalent if one is a permutation of the other. Thus 10,075 and 01,057 are equivalent. Find the number of 5-digit integers such that no two are equivalent.

385. If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these things so that no two of them are next to each other is $\frac{n}{6}(n-4)(n-5)$.

386. There are 20 persons including two brothers. In how many ways can they be arranged on a round table if there is exactly one person between the two brothers.

387. Find the number of different ways of painting a cube by using a different color for each face from six available colors. (Any two color schemes are called different if one cannot coincide with the other by a rotation of the cube.)

388. Find number of ways in which n things of which r alike and the rest distinct can be arranged in a circle distinguishing between clockwise and anti-clockwise arrangement.

389. In how many ways can we divide 52 playing cards among 4 players equally? In 4 parts equally?

390. 10 different toys are to be distributed among 10 children. Find the total number of ways of distributing these toys so that exactly 2 children do not get any toy.

391. In how many ways can 7 departments be divided among 3 ministers such that every minister gets at least one and atmost 4 departments to control?

392. Find the total number of ways of dividing 15 different things into groups of $8, 4$ and 3 respectively.

393. Find the number of ways of distributing 50 identical things among 8 persons in such a way that three of them get 8 things each, two of them get 7 things each, and remaining 3 get 4 things each.

394. If $3n$ different things can be equally distributed among 3 persons in k ways then find the number of ways to divide the $3n$ things in 3 equal groups.

395. Find the number of ways in which n different prizes can be distributed amongst m $(< n)$ persons if each is entitled to receive at most $n-1$ prizes.

396. n different toys have to be distributed among n children. Find the total number of ways in which these toys can be distributed so that exactly one child gets no toy.

397. Find the number of non-negative integral solutions of $x_1+x_2+x_3+x_4+x_5=8$.

398. Find the number of positive integral solutions of $x_1+x_2+x_3+x_4+x_5=8$.

399. Find the number of integral solutions of $x_1+x_2+x_3+x_4=14$, where $x_1 \geq -2, x_2 \geq 1, x_3 \geq 2$ and $x_4 \geq 0$.

400. How many integral solutions are there to $x+y+z+t=29$, when $x \geq 1, y \geq 2, z \geq 3$ and $t \geq 0$?

401. How many integral solutions are there of the system of equations $x_1+x_2+x_3+x_4+x_5=20$ and $x_1+x_2+x_3=5$, where $x_i \geq 0$?

402. In a box there are 10 balls, 4 red, 3 black, 2 white and 1 yellow. In how many ways can a child select 4 balls out of these 10 balls? (Assume that the balls of the same colour are identical)

403. There are three papers of 100 marks each in an examination. Find the number of ways in which a student can get 150 marks such that he gets atleast 60% in two papers.

404. Find the number of ways in which 30 marks can be allotted to 8 questions if each questions carries atleast 2 marks.

405. In an examination the maximum marks for each of three papers is n, and that for fourth paper is $2n$. Find the number of ways in which a candidate can get $3n$ marks.

406. In a shooting competition a man can score $5, 4, 3, 2$ or 0 points for each shot. Find the number of different ways in which he can score 30 in seven shots.

407. Find the number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 = 20$.

408. Find the number of ways to select 10 balls from an unlimited number of red, white, blue and green balls.

409. Find the number of ordered triples of positive integers which are solutions of the equation $x + y + z = 100$.

410. Find the number of integral solutions of $x_1 + x_2 + x_3 = 0$, with $x_i \geq -5$.

411. Find the number of integral solutions for the equation $x + y + z + t = 20$, where x, y, z, t are all ≥ -1.

412. Find the number of integral solutions of $a + b + c + d + e = 22$, subject to $a \geq -3, b \geq 1, c, d, e \geq 0$.

413. If a, b, c are three natural numbers in A.P. and $a + b + c = 21$ then find the possible number of values of the ordered triplet (a, b, c).

414. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$ then find the number of values of the ordered quadruplet (a, b, c, d).

415. Find the number of non-negative integral solution of the equation, $x + y + 3z = 33$.

416. Find the number of integral solutions of the equation $3x + y + z = 27$, where $x, y, z > 0$.

417. If a, b, c are positive integers such that $a + b + c \leq 8$ then find the number of possible values of the ordered triplet (a, b, c).

418. Find the number of non-negative integral solution of the inequation $x + y + z + w \leq 7$.

419. Find the number of non-negative even integral solutions of $x + y + z = 100$.

420. Find the number of non-negative integral solutions of $x + y + z + w \leq 23$.

421. Find the total number of positive integral solution of $15 < a + b + c \leq 20$.

422. Find the number of non-negative integer solutions of $(a+b+c)(p+q+r+s) = 21$.

423. Find the number of terms in a complete homogeneous expression of degree n in x, y and z.

424. In how many different ways can 3 persons A, B and C having 6 one rupee coins, 7 one rupee coins and 8 one rupee coins respectively donate 10 one rupee coins collectively if each one giving at least one coin. If each one can give 0 or more coin. Also answer the above questions for 15 rupees donation.

425. In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the fourth paper is 100. Find the number of ways in which a candidate can score 60% marks on the whole.

426. The minimum marks required for clearing a certain screening paper is 210 out of 300. The screening paper consists of 3 sections each of Physics, Chemistry, and Mathematics Each section has 100 as maximum marks. Assuming there is no negative marking and marks obtained in each section are integers, find the number of ways in which a student can qualify the examination (Assuming no subjectwise cut-off limit).

427. Find the number of ways in which the sum of upper faces of four distinct dices can be six.

428. How many integers > 100 and $< 10^6$ have the digital sum $= 5$?

429. In how many ways can 14 be scored by tossing a fair die thrice?

430. Find the number of positive integral solutions of $abc = 30$.

431. Find The number of positive integral solutions of the equation $abcde = 1050$.

432. Let y be an element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and x_1, x_2, x_3 be positive integers such that $x_1x_2x_3 = y$, then find the number of positive integral solutions of $x_1x_2x_3 = y$.

433. Let $x_i \in \mathbb{Z}$ such that $|x_1x_2 \ldots x_{10}| = 1080000$. Find the number of solutions.

434. Let $x_i \in \mathbb{Z}$, such that $|x_1| + |x_2| + \ldots + |x_{10}| = 100$. Find number of solutions.

435. Let there be n lines in a plane such that no two lines are parallel and no three are concurrent. Find the number of regions in which these lines divide the plane.

436. Determine the number of regions that are created by n mutually overlapping circles in a plane. Assume that no three circles passing through same points and every two circles intersect in two distinct points.

437. Determine number of ways to perfectly cover a $2 \times n$ board with dominoes (domino means a tile of size 2×1).

438. Tower of Brahma (or Tower of Hanoi) is a puzzle consisting of three pegs mounted on a board and n discs of different sizes. Initially all the n discs are stacked on the first peg so that any disc is always above a larger disc. The problem is to transfer

all these discs to peg 2, with minimum number of moves, each move consisting of transferring one disc from any peg to another so that on the new peg the transferred disc will be on top of a larger disc (i.e., keeping a disc on a smaller one is not allowed). Find the total (minimum) number of moves required to do this.

439. Five letters are written to five different persons and their addresses are written on five envelopes (one address on each envelope). In how many ways can the letters be placed in the envelopes so that no letter is placed in the correct envelope?

440. Find the number of positive integers from 1 to 1000, which are divisible by at least one of $2, 3$ or 5.

441. Find the number of ways in which two Americans, two Britishers, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.

442. In how many ways can 5 cards be drawn from a complete deck (of 52 cards) so that all the suites are present?

443. In how many ways can 6 distinguishable objects be distributed in four distinguishable boxes such that there is no empty box? If exactly one box is empty?

444. Find the number of ways to choose an ordered pair (a, b) of numbers from the set $\{1, 2, \dots, 10\}$ such that $|a–b| \leq 5$.

445. Suppose that in a poll made of 150 people, the following information was obtained: 70 of them read The Hindu, 80 read The Indian Express and 50 read Deccan Herald. 30 read both The Hindu and The Indian Express; 20 read both The Hindu and the Deccan Herald and 25 read both The Indian Express and Deccan Herald. Find at most how many of them read all the three.

446. Lewis Carroll, the famous author of Alice in Wonderland, Through the Looking Glass, The hunting of the Shark and other wonderful works, was a mathematician whose real name was Charles Lutwidge Dodgson (1832–1898). Here is a problem from his book 'A Tangled Tale'.

Let S be the set of pensioners, E the set of those who lost an eye, H those who lost an ear, A those who lost an arm and L those who lost a leg. Given that $n(E) = 70\,\%$, $n(H) = 75\,\%$, $n(A) = 80\%$ and $n(L) = 85\%$. Find what percentage at least must have lost all the four.

447. a, b, c, d be integers ≥ 0, $d \leq a$, $d \leq b$, and $a + b = c + d$. Prove that there exist sets A and B satisfying $n(A) = a$, $n(B) = b$, $n(A \cup B) = c$, $n(A \cap B) = d$.

448. How many positive integers of n digits exist such that each digit is $1, 2$ or 3? How many of these contain all three of the digits 1, 2 and 3 at least once?

449. A, B and C are the set of all the positive divisors of $10^{60}, 20^{50}$ and 30^{40} respectively. Find $n(A \cup B \cup C)$.

450. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 = 28$, where $3 \leq x_1 \leq 9$, $0 \leq x_2 \leq 8$ and $7 \leq x_3 \leq 17$.

451. I have six friends and during a certain vacation, I met them during several dinners. I found that I dined with all the six exactly on 1 day, with every five of them on 2 days, with every four of them on 3 days, with every three of them on 4 days and with every two of them on 5 days. Further every friend was present at 7 dinners, and every friend was absent at 7 dinners. How many dinners did I have alone?

452. A student on vacation for d days observed that it rained seven times morning or afternoon; when it rained in the afternoon, it was clear in the morning; there were five clear afternoon, and there were six clear mornings. Find d.

453. On a rainy day n people go to a party. Each of them leaves his raincoat at the counter of the gate. Find the number of ways in which the raincoats are handed over to the guests after the function is over so that no one receives his/her own raincoat.

454. Find the number of permutations of $1, 2, 3, 4, 5$ in which exactly one number occupies its natural position.

455. There are 5 boxes of 5 different colors. Also there are 5 balls of colors same as those of the boxes. In how many ways we can place 5 balls in 5 boxes such that all balls are placed in the boxes of colors not same as those of the ball. At least 2 balls are placed in boxes of the same color.

456. In how many ways 6 letters can be placed in 6 envelopes such that no letter is placed in its corresponding envelope. At least 4 letters are placed in correct envelopes. At most 3 letters are placed in wrong envelopes.

457. Find the numbers from 1 to 100 which are neither divisible by 2 nor by 3 nor by 7.

458. Find the number of numbers, from amongst $1, 2, 3, \dots, 500$, which are divisible by none of $2, 3, 5$.

459. Find the number of 3 element subsets of the set $\{1, 2, \dots, 10\}$, in which the least element is 3 or the greatest element is 7.

460. Find the number of n digit numbers, which contain the digits 2 and 7, but not the digits $0, 1, 8, 9$.

461. How many integers from 1 through 999 do not have any repeated digits?

462. Find the number of natural numbers less than or equal to 10^8 which are neither perfect squares, nor perfect cubes, nor perfect fifth powers.

463. In a certain state, license plates consist of from zero to three letters followed by from zero to four digits, with the provision, however, that a blank plate is not allowed. How many different license plates can the state produce? Suppose 85 letter combinations are not allowed because of their potential for giving offense. How many different license plates can the state produce?

464. If the number of ways of selecting K coupons one by one out of an unlimited number of coupons bearing the letters A, T, M so that they cannot be used to spell the word MAT is 93, then find K.

465. How many positive integers divide at least one of 10^{40} or 20^{30}?

466. Find the number of permutations of letters a, b, c, d, e, f, g taken all together if neither 'beg' nor 'cad' pattern appear.

467. Find the number of permutations of the letters of the word '$HINDUSTAN$' such that neither the pattern 'HIN' nor 'DUS' nor 'TAN' appears.

468. Find the number of permutations of the 8 letters $AABBCCDD$, taken all at a time, such that no two adjacent letters are alike.

469. Find the number of non-negative integer solutions of $x_1 + x_2 + x_3 = 15$, subject to $x_1 \leq 5, x_2 \leq 6$, and $x_3 \leq 7$.

470. According to the Gregorian calendar, a leap year is defined as a year n such that (i) n divides 4 but not 100; or (ii) n divides 400. Find the number of leap years from the year 1000 to the year 3000, inclusive.

471. Find the number of onto functions from a set containing 6 elements to a set containing 3 elements.

472. How many 6-digit numbers contain exactly three different digits?

473. Let D_n be the nth derangement number. Prove that $D_n = (n-1)(D_{n-1} + D_{n-2}), n > 2$.

474. Let D_n be the nth derangement number. Prove that $\lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e}$.

475. Five pairs of hand gloves of different colours are to be distributed to each of five people. Each person must get a left glove and a right glove. Find the number of distributions so that,exactly one person gets a proper pair.

476. Prove (combinatorially) that $\displaystyle\sum_{r=1}^{n} r!\,r = (n+1)! - 1$.

477. In maths paper there is a question on 'Match the column' in which column A contains 6 entries and each entry of column A corresponds to exactly one of the 6 entries given in column B written randomly. 2 marks are awarded for each correct matching and 1 mark is deducted from each incorrect matching. A student having no subjective knowledge decides to match all the 6 entries randomly. Find the number of ways in which he can answer, to get atleast 25% marks in this question.

478. Ten parabolas are drawn in a plane. Any two parabola intersect in four real, and distinct, points. No three parabola are concurrent. Find the total number of disjoint regions of the plane.

479. In how many ways can a 12 step staircase be climbed taking 1 step or 2 steps at a time?

480. A coin is tossed 10 times. Find the number of outcomes in which 2 heads are not successive.

481. Find the number of ways to pave a 1×7 rectangle by $1 \times 1, 1 \times 2, 1 \times 3$ tiles, if tiles of the same size are indistinguishable

482. Find the number of distributions of 5 distinguishable balls in 3 distinguishable cells, if empty cells are allowed, and if empty cells are not allowed.

483. How many terms are there in the expansion of $(a+b+c+d)^{24}$?

484. Find the number of ways of distributing five identical balls into three boxes so that no box is empty, and each box being large enough to accommodate all the balls.

485. Find the number of ways of distributing 10 identical balls in 3 boxes so that no box contains more than four balls and less than two balls.

486. Find the number of ways in which 14 identical toys can be distributed among three boys so that each one gets atleast one toy and no two boys get equal number of toys.

487. Find the number of distributions of 5 distinguishable balls in 3 identical boxes, empty boxes are allowed.

488. What is the number of necklaces that can be made from $6n$ identical blue beads and 3 identical red beads?

489. Find the number of ways in which n distinct objects can be put into two differentvboxes so that no box remains empty.

490. Find the number of ways in which n distinct objects can be kept into two identical boxes so that no box remains empty.

491. 10 identical balls are to be distributed in 5 different boxes kept in a row andvlabeled A, B, C, D and E. Find the number of ways in which the balls can be distributed in the boxes if no two adjacent boxes remain empty.

492. Find the number of ways in which 12 identical coins can be distributed in 6 different purses, if not more than 3 and not less than 1 coin goes in each purse.

493. Find the number of ways in which 30 coins of one rupee each be given to six persons so that none of them receive less than four rupees.

494. Find the number of ways of wearing 8 distinguishable rings on 5 fingers of right hand.

495. 15 identical balls have to be put in 5 different boxes. Each box can contain any number of balls. Find total number of ways of putting the balls into box so that each box contains at least 2 balls.

496. In how many ways can 3 blue, 4 red and 2 green balls be distributed in 4 distinct boxes? (Balls of the same colour are identical)

497. How many different ways can 15 candy bars be distributed to Tanya, Manya, Shashwat and Adwik, if Tanya cannot have more than 5 candy bars and Manya must have at least two. Assume all Candy bars to be alike.

498. Prove that the number of n digit quaternary sequences (whose digits are $0, 1, 2$, and 3), in which each of the digits 2 and 3 appear atleast once, is $4^n - 2.3^n + 2^n$.

499. Shivank has 15 ping-pong balls each uniquely numbered from 1 to 15. He also has a red box, a blue box, and a green box. How many ways can Shivank place the 15 distinct balls into the three boxes so that no box is empty? Suppose now that Shivank has placed 5 ping-pong balls in each box. How many ways can he choose 5 balls from the three boxes so that he chooses at least one from each box?

500. In how many ways we can place 9 different balls in 3 different boxes such that in every box at least 2 balls are placed?

501. In how many ways can we put 12 different balls in three different boxes such that first box contains exactly 5 balls.

502. A man has 3 daughters. He wants to bequeath his fortune of 101 identical gold coins to them such that no daughter gets more share than the combined share of the other two. Find the number of ways of accomplishing this task.

503. Divide the numbers $1, 2, 3, 4, 5$ into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.

504. Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance is at most $\sqrt{2}$.

505. Show that given a regular hexagon of side 2 cm and 25 points inside it, there are at least two points among them which are at most 1 cm distance apart.

506. Show that given a regular hexagon of side 2 cm and 25 points inside it, there are at least two points among them which are at most 1 cm distance apart.

507. If 7 points are chosen on the circumference or in the interior of a unit circle, such that their mutual distance apart is greater than or equal to 11, then one of them must be the centre.

508. $4^n + 1$ points lie within an equilateral triangle of side 1 cm. Show that it is possible to choose out of them, at least two, such that the distance between them is at most $\frac{1}{2^n}$ cm.

509. Let A be any set of 19 distinct integers chosen from the A.P. $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A, whose sum is 104.

510. Let $X \subset \{1, 2, 3, \ldots, 99\}$ and $n(x) = 10$. Show that it is possible to choose two disjoin non-empty proper subsets Y, Z of X such that $\sum\limits_{y \in Y} y = \sum\limits_{z \in Z} z$.

511. If repetition of digits is not allowed in any number (in base 10), show that among three four-digit numbers, two have a common digit occurring in them.

Also show that in base 7 system any two four-digit numbers without repetition of digits will have a common number occurring in their digits.

512. In base $2k$, $k \geq 1$ number system, any 3 non-zero, k-digit numbers are written without repetition of digits. Show that two of them have a common digit among them.

In base $2k+1$, $k \geq 1$ among any $3k+1$ digit non-zero numbers, there is a common digit occurring in any two numbers.

513. Let A denote the subset of the set $S = \{a, a+d, \cdots, a+2nd\}$ having the property that no two distinct elements of A add up to $2(a+nd)$. Prove that A cannot have more than $(n+1)$ elements. If in the set S, $2nd$ is changed to $a+(2n+1)d$, what is the maximum number of elements in A if in this case no two elements of A add up to $2a+(2n+1)d$?

514. Given any five distinct real numbers, prove that there are two of them, say x and y, such that $0 < \frac{x-y}{1+xy} < 1$.

515. Prove that, among any 52 integers, two can always be found, such that the difference of their squares, is divisible by 100.

516. There are 7 persons in a group, show that, some two of them, have the same number of acquaintances among them.

517. 51 points are scattered inside a square, with a side of one metre. Prove that some set of three of these points can be covered by a square, with side 20 cm.

518. Let $1 < a_1 < a_2 < a_3 < \cdots < a51 < 142$. Prove that, among the 50 consecutive differences $(a_i - a_{i-1})$ where $i = 1, 2, 3, \ldots, 51$, some value, must occur at least twelve times.

519. You are given 10 segments, such that, every segment is larger than 1 cm but shorter than 55 cm. Prove that, you can select three sides of a triangle, among these segments.

520. There are 9 cells in a 3×3 square. When these cells are filled by numbers $1, 2, 3$ only, prove that, of the eight sums obtained, at least, two sums are equal.

521. Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

522. Consider seven distinct positive integers, not exceeding 1706. Prove that, there are three of them, say a, b, c such that, $a < b + c < 4a$.

523. Consider a circle C with a radius of 16 and an annulus, or ring, A, with an outer radius of 3 and an inner radius of 2. Prove that wherever one might sprinkle a set S of 650 points inside C the annulus A can always be placed on the figure so that it covers at least 10 of the points.

524. On a rectangular table of dimensions 120" by 150", we set 14001 marbles of size $1''$ by $1''$. Prove that, no matter how these are arranged, one can place a cylindrical glass with diameter of $5''$ over atleast 8 marbles.

525. If a line is colored in 11 colors, show that, there exist two points, whose distance apart, is an integer, which have the same colour.

526. Show that, given 12 integers, there exists two of them whose difference is divisible by 11.

Chapter 6
Mathematical Induction

Any reasoning involving passage from particular assertions to general assertions, which derive their validity from the validity of particular assertions is called *induction*. *Mathematical induction* is a mathematical proof technique which enables us to draw conclusions about a general law on the basis of particular cases. It is used to prove a statement $P(n)$ holds for every natural number $n = 0, 1, 2, 3, ...$; that is, the overall statement is a seuqnece of infinitely many cases $P(0), P(1), P(2), P(3), ...$ The earliest rigorous use of induction was by Gersonides (1288-1344). The first explicit formulation of the principle was given by Pascal in his *Traité du triangle arithmétique* (1665).

In boolean algebra, a statement which is either true and false is called a *proposition*. $P(n)$ will denote a proposition whose truth value depends on natural numbers. For example, we recall the sum of first n natural numbers from arithmetic progression as $1+2+...+n = \frac{n(n+1)}{2}$ is denoted by $P(n)$, then we can write $P(n) = 1+2+...+n = \dfrac{n(n+1)}{2}$ Here $P(2)$ is true means the sum of first two natural numbers is equal to $1+2 = \frac{2.3}{2} = 3$.

Mathematical induction is used to prove propositions in many branches of algebra, geometry and analysis.

6.1 Principle of Finite Mathematical Induction

The proposition $P(n)$ is assumed to be true for all natural numbers if the following two conditions are satisfied:

1. The proposition $P(n)$ is true for $n = 1$ i.e. $P(1)$ is true.
2. $P(m)$ is true $\Rightarrow P(m+1)$ is true where m is an arbitrary natural number.

6.2 Extended Form of Mathematical Induction

1. If $P(n)$ is a proposition such that
    1. $P(1), P(2), \dots, P(k)$ are true.
    2. $P(m), P(m+1), \dots, P(m+k-1)$ are true implies $P(m+k)$ is true.
2. If $P(n)$ is a proposition such that
    1. $P(r)$ is true.
    2. $P(r), P(r+1), \dots, P(m)$ are true implies $P(m+1) istrue.$

6.3 Problems

1. Show that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

2. Show that $\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

3. Show that $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

4. Show that $\frac{1}{a+d} + \frac{a}{(a+d)(a+2d)} + \cdots + \frac{a}{[a+(n-1)d](a+nd)} = \frac{n}{a+nd}$.

5. Show that $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}\ \forall n \in \mathbb{N}$.

6. Show that $1.3 + 2.3^2 + 3.3^3 + \cdots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$.

7. Show that $1 + 4 + 7 + \cdots + 3n - 2 = \frac{n(3n-1)}{2}$.

8. Show that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

9. Show that $1 - 3^2 + 5^2 - 7^2 + \cdots + (4n-3)^2 - (4n-1)^2 = -8n^2$.

10. Show that $3.6 + 6.9 + 9.12 + \cdots + 3n(3n+3) = 3n(n+1)(n+2)$.

11. Prove the theorem of Nicomachus: $1^3 = 1, 2^3 = 3 + 5, 3^3 = 7 + 9 + 11, 4^3 = 13 + 15 + 17 + 19$ and so on.

12. Show that $\sum\limits_{r=1}^{n} r.C_r^n = n.2^{n-1}$.

13. Show that $\sum\limits_{r=1}^{n} r(2r+1) = \frac{n(n+1)(4n+5)}{6}$.

14. Show that $1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

15. Show that $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$.

16. $7 + 77 + 777 + \cdots + \underbrace{7\ldots77}_{n\text{ digits}} = \frac{7}{81}(10^{n+1} - 9n - 10)$.

17. Show that $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots+n} = \frac{2n}{n+1}$.

18. Show that $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}$.

19. Show that $1.3 + 2.3^2 + \cdots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$.

20. Show that $\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \sin\frac{n\alpha}{2}\csc\frac{\alpha}{2}\cos\frac{(n+1)\alpha}{2}$.

21. Show that $\tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \cdots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$.

22. Show that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\cdots+\tan^{-1}\frac{1}{n^2+n+1}=\tan^{-1}\frac{n}{n+2}$.

23. If $u_1=1, u_2=1$ and $u_{n+2}=u_{n+1}+u_n$, $n\geq 1. u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]\ \forall\, n\geq 1$.

24. If $p\in\mathbb{N}$, show that $p^{n+1}+(p+1)^{2n-1}$ is divisible by p^2+p+1 for every positive integer n.

25. Show that $2^n>2n+1\ \forall\, n>2$.

26. Show that $2^n>n^3$ if $n\geq 10$.

27. Show that $\tan n\alpha>n\tan\alpha$, where $0<\alpha<\frac{\pi}{4(n-1)}\ \forall n\in\mathbb{N}>1$.

28. Show that $n^4<10^n\ \forall\, n\geq 2$.

29. Show that $1^3+3^3+\cdots+(2n-1)^3=n^2(2n^2-1)$.

30. Show that $3.2^2+3^3.2^3+\cdots+3^n.2^{n+1}=\frac{12}{5}(6^n-1)$.

31. Show that $\frac{1}{1.4}+\frac{1}{4.7}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$.

32. Show that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

33. Show that $\cos\theta.\cos 2\theta\ldots\cos 2^{n-1}\theta=\frac{\sin 2^n\theta}{2^n\sin\theta}$.

34. Show that $\sin\alpha+\sin 2\alpha+\cdots+\sin n\alpha=\frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\frac{n+1}{2}\alpha$.

35. If $a_1=1$ and $a_{n+1}=\frac{a_n}{n+1}$, $n\geq 1$, show that $a_{n+1}=\frac{1}{(n+1)!}$.

36. If $a_1=1, a_2=5$ and $a_{n+2}=5a_{n+1}-6a_n$, $n\geq 1$, show that $a_n=3^n-2^n$.

37. If $u_0=2$, $u_1=3$ and $u_{n+1}=3u_n-2u_{n-1}$, show that $u_n=2^n+1$, $n\in\mathbb{N}$.

38. If $a_0=0, a_1=1$ and $a_{n+1}=3a_n-2a_{n-1}$, show that $a_n=2^n-1$.

39. If $A_1=\cos\theta$, $A_2=\cos 2\theta$ and for every natural number $m>2$, $A_m=2A_{m-1}\cos\theta-A_{m-2}$, prove that $A_n=\cos n\theta$.

40. For any positive number n, show that $(2\cos\theta-1)(2\cos 2\theta-1)\cdots(2\cos 2^{n-1}\theta-1)=\frac{2\cos 2^n\theta+1}{2\cos\theta+1}$.

41. Show that $\tan^{-1}\frac{x}{1.2+x^2}+\tan^{-1}\frac{x}{2.3+x^2}+\cdots+\tan^{-1}\frac{x}{n(n+1)+x^2}=\tan^{-1}x-\tan^{-1}\frac{x}{n+1}$, $x\in\mathbb{R}$.

42. Prove that $3+33+\cdots+\underbrace{33\ldots3}_{n\text{ digits}}=\frac{10^{n+1}-9n-10}{27}$.

43. Show that $\int_0^{\pi} \frac{\sin(2n+1)x}{\sin x} dx = \pi$.

44. Show that $\int_0^{\pi} \frac{\sin^2 nx}{\sin^2 x} dx = n\pi$.

45. Show that $\tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \cdots + \tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1}(n+1) - \frac{\pi}{4}$.

46. Show that if $n \in \mathbb{N}$, $n(n+1)(n+5)$ is divisible by 6.

47. Show that if $n \in \mathbb{N}$, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

48. Show that if $n \in \mathbb{P}$, and n is even then $n(n^2+20)$ is divisible by 48.

49. Show that if $n \in \mathbb{N}$, $4^n - 3n - 1$ is divisible by 9.

50. Show that if $n \in \mathbb{N}$, $3^{2n} - 1$ is divisible by 8.

51. Show that if $n \in \mathbb{N}$, $5.2^{3n-2} + 3^{3n-1}$ is divisible by 19.

52. Show that if $n \in \mathbb{N}$, $7^{2n} + 2^{3n-3}.3^{n-1}$ is divisible by 25.

53. Show that if $n \in \mathbb{N}$, $10^n + 3.4^{n+2} + 5$ is divisible by 9.

54. Show that if $n \in \mathbb{N}$, $3^{4n+2} + 5^{2n+1}$ is divisible by 14.

55. Show that if $n \in \mathbb{N}$, $3^{2n+2} - 8n - 9$ is divisible by 64.

56. Show that if $n \in \mathbb{N}$, $n^7 - n$ is divisible by 7.

57. Show that if $n \in \mathbb{N}$, $11^{n+2} + 12^{2n+1}$ is divisible by 133.

58. Show that if $n \in \mathbb{N}$, $10^{2n-1} + 1$ is divisible by 11.

59. Show that if $n \in \mathbb{N}$, $7^n - 3^n$ is divisible by 4.

60. Show that if $n \in \mathbb{N}$, $2.7^n + 3.5^n - 5$ is divisible by 24.

61. Show that if $n \in \mathbb{N}$, $3^{2n} - 1$ is divisible by 8.

62. Show that if $n \in \mathbb{N}$, $10^n + 3.4^{n+2} + 5$ is divisible by 9.

63. Show that if $n \in \mathbb{N}$, $5^{2n+1} + 2^{n+4} + 2^{n+1}$ is divisible by 23.

64. Show that if $n \in \mathbb{N}$, $7^{2n} - 1$ is divisible by 8.

65. Show that if $n \in \mathbb{N}$, $3^{2n+2} - 8n - 9$ is divisible by 8.

66. Show that if $n \in \mathbb{N}$, $41^n - 14^n$ is divisible by 27.

67. Show that if $n \in \mathbb{N}$, $15^{2n-1} + 1$ is divisible by 16.

68. Show that if $n \in \mathbb{N}$, $5^{2n+1} + 3^{n+2}.2^{n-1}$ is divisible by 19.

69. Show that if $n \in \mathbb{N}$, $10^n + 3.4^{n+2} + 5$ is divisible by 9.

70. Show that if $n \in \mathbb{N}$, $9^n - 8n - 1$ is divisible by 64.

71. Show that if $n \in \mathbb{N}$, $n^3 + 3n^2 + 5n + 3$ is divisible by 3.

72. Show that if $n \in \mathbb{N}$, $(n+1)(n+2)(n+3)(n+4)(n+5)$ is divisible by 120.

73. Show that if $n \in \mathbb{N}$, $n^5 - n$ is divisible by 5.

74. Show that if $n \in \mathbb{N}$, $(1+x)^n - nx - 1$ is divisible by x^2, where $x \neq 0$.

75. Show that $n(n^2 - 1)$ is divisible by 24, where $n \in$ odd positive integers.

76. Show that $n(n^2 + 20)$ is divisible by 48, where $n \in$ even positive integers.

77. Show that $2^{2n} + 1$ or $2^{2n} - 1$ is divisible by 5 according as n is odd or even positive integer.

78. Prove that $5^{2n} + 1$ is divisible by 13 if n is odd. Hence, deduce that 5^{99} leaves a remainder 8 when divided by 13.

79. Show that $4.6^n + 5^{n+1}$ leaves remainder 9 when divided by 20.

80. Show that if $n \in \mathbb{N}$, $3^n + 8^n$ is not divisible by 8.

81. Prove that $2^{2^n} + 1$ has last digit as 7 for $n > 1$.

82. Show that if $n \in \mathbb{N}$, $\frac{n^3}{3} + n^2 + \frac{5}{3}n + 1$ is a natural number.

83. Show that $x^n + y^n$ is divisible by $x + y$, where n is any odd integer.

84. Show that $x^n - y^n$ is divisible by $x - y$, where $n \in \mathbb{N}$.

85. Prove that $x(x^{n-1} - na^{n-1}) + a^n(n-1)$ is divisible by $(x-a)^2$ for all positive integers $n > 1$.

86. Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number.

87. Show that $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer.

88. Show that $2^n > n^2$, $n \geq 5$.

89. Show that $1 + 2 + \cdots + n \leq \frac{1}{8}(2n+1)^2$.

90. Show that $n^n < (n!)^2$, $n > 2$.

91. Show that $n! > 2^n$, $n > 3$.

92. Show that $n! < \left(\frac{n+1}{2}\right)^n$, $n > 1$.

93. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}$, $n > 1$.

94. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n+1} > 1 \forall n \in \mathbb{N}$.

95. Show that $1 + \frac{1}{4} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n} \forall n \in \mathbb{N}$ greater than 1.

96. Show that if $n \in \mathbb{N}$, $(2n + 7) < (n + 3)^2$.

97. Show that if $n \in \mathbb{N}$, $2^n > n$.

98. Show that if $n \in \mathbb{N}$, $1 + 2 + 3 + \dots + n < \frac{(2n+1)^2}{8}$.

99. Show that if $n \in \mathbb{N}$, $1^2 + 2^2 + \dots + n^2 > \frac{n^3}{3}$.

100. Show that if $n \in \mathbb{N}$, $2^n > n^2$ for $n \geq 5$.

101. Show that if $n \in \mathbb{N}$, $\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$ for $n > 1$.

102. Show that if $n \in \mathbb{N}$, $(1 + x)^n > 1 + nx$, $n > 1$ and $x > -1$, $x \neq 0$.

103. In a sequence $1, 4, 10, \dots, t_1 = 1, t_2 = 4$, and $t_n = 2t_{n-1} + 2t_{n-2}$ for $n \geq 3$. Show that $t_n = \frac{1}{2}\left[(1 + \sqrt{3})^n + (1 - \sqrt{3})^n\right] \forall n \in \mathbb{N}$.

104. If $x + y = a + b$, $x^2 + y^2 = a^2 + b^2$, prove that $x^n + y^n = a^n + b^n \forall n \in \mathbb{N}$.

105. Prove that $\frac{1}{2} . \frac{3}{4} . \cdots . \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$, where $n \in \mathbb{N}$.

106. Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} < \frac{25}{36}$, where $n \geq 2$, $n \in \mathbb{N}$.

107. Prove that $\sqrt{a + \sqrt{a + \sqrt{a + \dots n \sim \text{terms}}}} \leq \frac{1+\sqrt{4a+1}}{2}$, where $a \geq 0$.

108. Prove that $\sqrt{2\sqrt{3\sqrt{4 \dots \sqrt{n}}}} < 3$, where $n \geq 2$, $n \in \mathbb{N}$.

109. For $x^3 = x + 1$, $a_n = a_{n-1} + b_{n-1}$, $b_n = a_{n-1} + b_{n-1} + c_{n-1}$, $c_n = a_{n-1} + c_{n-1}$ prove that $x^{3n} = a_n x + b_n + c_n x^{-1} \forall n \in \mathbb{N}$, and $a = 0$, $b_0 = 1$, $c_0 = 0$.

110. Prove that, for all natural numbers n, $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is divisible by 2^n.

111. Prove that $x_1^2 + 3x_2^2 + 5x_3^2 + \dots + (2n - 1)x_n^2 \leq (x_1 + x_2 + \dots + x_n)^2$, where $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$.

112. Prove that $|\sin(x_1 + x_2 + \dots + x_n)| \leq |\sin x_1| + |\sin x_2| + \dots + |\sin x_n|$, where $x_1, x_2, \dots, x_n \in [0, \pi]$.

113. Prove that $\tan x_1 - \tan x_2 + \cdots + (-1)^n \tan x_n \geq \tan(x_1 - x_2 + \cdots + (-1)^n x_n)$, where $\frac{\pi}{2} > x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$.

114. Prove that $a_1^r - a_2^r + \cdots + (-1)^n a_n^r \geq (a_1 - a_2 + \cdots + (-1)^n a_n)^r$, where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0, r \geq 1$.

Chapter 7

Binomials, Multinomials and Expansions

An algebraic expression containing one term is called *monomial*, two terms is callled *binomial* and more than two is called is called *multinomial*. Examples of a monomial expressions are $2x$, $4y$, examples of binomial expressions are $a+b, x^2+y^2, x^3+y^3, x+\frac{1}{y}$ and exaamples of multinomial expressions are $1+x+x^2$, a^2+2a+b^2, $a^3+3a^2b+3ab^2+b^3$.

7.1 Binomial Theorem

Newton gave binomial theorem, by which we can expand any opwer of a binomial expression as a series. First we consider only positive integral values of exponent. For positive integral exponent the formula has the following form:

$$(a+x)^n =^n C_0 a^n x^0 +^n C_1 a^{n-1} x^1 +^n C_2 a^{n-2} x^2 + \ldots +^n C_n a^0 x^n$$

7.1.1 Proof by Mathematical Induction

Let

$$P(n) = (a+x)^n =^n C_0 a^n x^0 +^n C_1 a^{n-1} x^1 +^n C_2 a^{n-2} x^2 + \ldots +^n C_n a^0 x^n$$

When $n=1$, $P(1) = a+x =^1 C_0 a +^1 C_1 x$. When $n=2$, $P(2) = a^2+2ax+x^2 =^2 C_0 a^2 +^2 C_1 ax +^2 C_2 x^2$. Thus we see that $P(n)$ holds good for $n=1$ and $n=2$. Let $P(n)$ is true for $n=k$ i.e.

$$P(k) = (a+x)^k =^k C_0 a^k x^0 +^k C_1 a^{k-1} x^1 +^k C_2 a^{k-2} x^2 + \ldots +^k C_k a^0 x^k$$

Multiplying both sides with $(a+x)$

$$P(k+1) = (a+x)^{k+1} =^k C_0 a^{k+1} x^0 +^k C_1 a^k x +^k C_2 a^{k-1} x^2 + \ldots +^k C_k a x^k +$$
$$^k C_0 a^k x +^k C_1 a^{k-1} x^2 +^k C_2 a^{k-2} x^3 + \ldots +^k C_k x^{k+1}$$

Combining terms with equal powers of a and x, using the formula $^nC_r +^n C_{r+1} =^{n+1} C_{r+1}$ and rewriting $^kC_0$ and ^kC_k as $^{k+1}C_0$ and $^{k+1}C_{k+1}$, we get

$$P(k+1) =^{k+1} C_0 a^{k+1} x^0 +^{k+1} C_1 a^k x^1 +^{k+1} C_2 a^{k-1} x^2 + \cdots +^{k+1} C_{k+1} a^0 x^{k+1}$$

Thus, we see that $P(n)$ holds good for $n=k+1$ and we have proven binomial theorem by mathemtical induction.

7.1.2 Proof by Combination

We know that $(a+x)^n = (a+x)(a+x)\cdots[n \text{ factors}]$. If see only a, then we see that a^n exists and hence, a^n is a term in the final product. This is the term a^n, which can be written as ${}^nC_0a^nx^0$. If we take the letter $a, n-1$ times and x once then we observe ttat x can be taken in ${}^nC_1$ ways. Thus, we can say that the term in final product is ${}^nC_1a^{n-1}x$. Similarly, if we choose $a, n-2$ times and x twice then the term will be ${}^nC_2a^{n-2}x^2$. Finally, like a^n, x^n will exist and can be written as ${}^nC_nx^n$ for consistency. Thus, we have proven binomial theorem by combination.

7.2 Special Forms of Binomial Expansion

We have

$$(a+x)^n =^n C_0a^nx^0 +^n C_1a^{n-1}x^1 +^n C_2a^{n-2}x^2 + ... +^n C_na^0x^n \tag{7.1}$$

1. Putting $-x$ instead of x

$$(a-x)^n =^n C_0a^nx^0 -^n C_1a^{n-1}x^1 +^n C_2a^{n-2}x^2 - ... + (-1)^{nn}C_na^0x^n$$

2. Putting $a = 1$ in Eq. 7.1

$$(1+x)^n =^n C_0 +^n C_1x +^n C_2x^2 + ... +^n C_nx^n$$

3. Putting $x = -x$ in above equation

$$(1-x)^n =^n C_0 -^n C_1x +^n C_2x^2 - ... + (-1)^{nn}C_nx^n$$

7.3 General Term of a Binomial Expansion

We see that first term is $t_1 =^n C_0a^nx^0$, second term is $t_2 =^n C_1a^{n-1}x^1$ so general term will be

$$t_r =^n C_{r-1}a^{n-r+1}x^{r-1}$$

7.4 Middle Term of a Binomial Expansion

When n is an even number, i.e. $n = 2m$, $m \in \mathbb{P}$. Middle term will be $m+1$th term i.e.

$$t_{m+1} =^n C_ma^mmx^m.$$

When n an odd number, i.e. $n = 2m+1$ $m \in \mathbb{N}$. There will be two middle terms i.e. $m+1$th and $m+2$th terms will be middle terms. So

$$t_{m+1} =^n C_ma^{m+1}x^m, t_{m+2} =^n C_{m+1}a^mx^{m+1}$$

The middle terms have the largest coefficient. In case of two middle terms the coefficients of both the middle terms are equal.

7.5 Equidistant Coefficients

Binomial coefficients equidistant from start and end are equal. Coefficients of first term from start and end are ${}^nC_0$ and ${}^nC_n$ which are equal. Coefficients of second term from start and end are ${}^nC_1$ and ${}^nC_{n-1}$ which are equal. Similarly, coefficient of rth term from start is ${}^nC_{r-1}$ and from end is ${}^nC_{n-r+1}$. From combinations we know that ${}^nC_{r-1} =^n C_{n-r+1}$. Thus, it is prove that coefficients of terms equidistant from start and end are equal.

7.6 Properties of Binomial Coefficients

We have proven earlier that

$$(1+x)^n =^n C_0 +^n C_1x +^n C_2x^2 + \cdots +^n C_nx^n.$$

Putting $x = 1$, we get

$$2^n =^n C_0 +^n C_1 +^n C_2 + \cdots +^n C_n.$$

Putting $x = -1$, we get

$$0 =^n C_0 -^n C_1 +^n C_2 - \cdots + (-1)^{n n} C_n.$$

Adding the last two, we have

$$2^n = 2[^n C_0 +^n C_2 +^n C_4 + \cdots]$$

$$2^{n-1}\ {}^nC_0 +^n C_1 +^n C_2 + \cdots$$

Subtracting, we get

$$2^{n-1} =^n C_1 +^n C_3 +^n C_5 + \cdots$$

7.7 Multinomial Theorem

Consider the multinomila $(x_1 + x_2 + \cdots + x_n)^p$, where n and p are positive integers. The general term of such a multinomial is givenby

$$\frac{p!}{p_1!\,p_2!\ldots p_n!}x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}$$

such that $p_1, p_2, \ldots, p_n$ are non-negative integers and $p_1 + p_2 + \cdots + p_n = p$.

We can find the general term using the binomial theorem itself. General term in the expansion $[x_1 + (x_2 + x_3 + \cdots + x_n)]^n$ is

$$\frac{n!}{p!(n-p_1)!}x_1^{p_1}(x_2 + x_3 + \cdots + x_n)^{n-p_1}.$$

General term in expansion of $(x_2 + x_3 + \cdots + x_n)^{n-p_1}$ is

$$\frac{(n-p_1)!}{p_2!(n-p_1-p_2)!}x_2^{p_2}(x_3 + x_4 + \cdots + x_n)^{n-p_1-p_2}.$$

Proceding in this manner we obtain the general term given above.

7.7.1 Som Results on Multinomial Expansions

1. No. of terms in the multinomial $(x_1 + x_2 + \cdots + x_n)^p$ is number of non-negative integral solution of the equation $p_1 + p_2 + \cdots + p_n = p$ i.e. ${}^{n+p-1}C_p$ or ${}^{n+p-1}C_{n-1}$.

2. Largest coeff. in $(x_1 + x_2 + \cdots + x_n)^p$ is $\frac{n!}{(q!)^{n-r}[(q+1)!]^r}$, where q is the quotient and r is the remainder of p/n.

3. Coefficient of x^r in $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^p$ is $\sum \frac{n!}{p_0!p_1!p_2!\ldots p_n!} a_0^{p_0} a_1^{p_1} a_n^{p_n}$ where $p_0, p_1, \cdots, p_n$ are non-negative integers satisfying the equation $p_0 + p_1 + \ldots + p_n = n$ and $p_1 + 2p_2 + \cdots + np_n = r$.

7.8 Binomial Theorem for Any Index

7.8.1 Fractional Index

Let $f(m) = (1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \cdots$, where $m \in R$ then, $f(n) = (1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \cdots$

$$f(m)f(n) = (1+x)^{m+n} = f(m+n)$$

$$f(m)f(n)\ldots \text{ to } k \text{ factos } = f(m+n+\ldots) \text{ to } k \text{ terms}$$

Let $m, n, \ldots$ each equal to $\frac{j}{k}$

$$\Rightarrow \left[f\left(\frac{j}{k}\right)\right]^k = f(j)$$

but j is a positive integer, $f(j) = (1+x)^j$

$$\therefore (1+x)^{\frac{j}{k}} = f\left(\frac{j}{k}\right)$$

$$\therefore (1+x)^{\frac{j}{k}} = 1 + \frac{j}{k}x + \frac{\frac{j}{k}\left(\frac{j}{k}-1\right)}{1.2}x^2 + \ldots$$

And thus, we have proven binomial theorem for fractional index.

7.8.2 Negative Index

We can write

$$f(n)f(-n) = f(0) = 1$$

$$\Rightarrow f(-n) = \frac{1}{f(n)} = (1+x)^{-n} = 1 - nx + \frac{n(n-1)}{1.2}x^2 - \ldots$$

7.9 General Term in Binomial Theorem for Any Index

General term is given by

$$\frac{n.(n-1)\ldots(n-r+1)}{r!}x^r$$

The above expansion does not hold true when $|x|>1$ which can be quickly proved by making r arbitrarily large. For example, $(1-x)^{-1}=1+x+x^2+x^3+\ldots$. However, if we put $x=2$, then we have $(-1)^{-1}=1+2+2^2+\ldots$ which shows that when $x>1$ the above formula does not hold true.

From G.P. we know that $1+x+x^2+\ldots$ for r terms is

$$\frac{1}{1-x}-\frac{x^r}{1-x}$$

Thus, if r is very large and $|x|<1$, we can ignore the second fraction but not when $|x|>1$.

7.10 General Term for Negative Index

The $r+1$th term is given by

$$\frac{-n(-n-1)\ldots(-n-r+1)}{r!}(-x)^r$$

$$=\frac{n(n+1)\ldots(n+r-1)}{r!}x^r$$

7.11 Exponential and Logrithmic Series Expansions

Following expansions are useful for solving problem related to exponential and logarithmic series:

1. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ to ∞, where x is any number. e lies between 2 and 3.
2. If $a>0, a^x=e^{x\log_e a}=1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\ldots$
3. $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ to ∞ where $-1<x\leq 1$.

7.12 Problems

1. Expand $\left(x+\frac{1}{x}\right)^5$.
2. Use the bonimial theorem to find the exact value of $(10.1)^5$.
3. Simplify $(x+\sqrt{x-1})^6+(x-\sqrt{x-1})^6$.
4. If A be the sum of odd terms and B be the sum of even terms in the expansion of $(x+a)^n$, prove that $A^2-B^2=(x^2-a^2)^n$.
5. If n is a positive integer, prove that the integral part of $(7+4\sqrt{3})^n$ is an odd number.
6. If $(7+4\sqrt{3})^n=\alpha+\beta$, where α is a positive integer and β is a proper fraction, then prove that $(1-\beta)(\alpha+\beta)=1$.
7. Find the coefficient of $\frac{1}{y^2}$ in $\left(y+\frac{c^3}{y^2}\right)^{10}$.
8. Find the coefficient of x^9 in $(1+3x+3x^2+x^3)^{15}$.
9. Find the term independent of x in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.
10. Find the term independent of x in $(1+x)^m\left(x+\frac{1}{x}\right)^n$.
11. Find the coefficient of x^{-1} in $(1+3x^2+x^4)\left(1+\frac{1}{x}\right)^8$.
12. If a_r denotes the coefficient of x^r in the expansion $(1-x)^{2n-1}$, then prove that $a_{r-1}+a_{2n-r}=0$.
13. Find the vallue of k so that the term independent of x in $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$ is 405.
14. Show that there will be no term containing x^{2r} in the expansion $(x+x^{-2})^{n-3}$, if $n-2r$ is positive but not a multiple of 3.
15. Show that there will be a term independent of x in the expansion $(x^a+x^{-b})^n$, only if an is a multiple of $a+b$.
16. Expand $\left(x+\frac{1}{x}\right)^7$ using binomial theorem.
17. Use binomial theorem to expand $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$.
18. If $(1+ax)^n=1+8x+24x^2+...$, find a and n.
19. Find the 7th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.
20. Find the value of $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$.

21. If A be the sum of the odd terms and B be the sum of the even terms in the expansion $(x+a)^n$, show that $4AB = (x+a)^{2n} - (x-a)^{2n}$.

22. If n be a positive integer, prove that the integral part of $(5+2\sqrt{6})^n$ is an odd integer.

23. If $(3+\sqrt{8})^n = \alpha + \beta$, where α, n are positive integers and β is a proper fraction, then prove that $(1-\beta)(\alpha+\beta) = 1$.

24. Find the coefficient of x in the expansion of $\left(2x - \frac{3}{x}\right)^9$.

25. Find the coefficient of x^7 in the expansion of $(3x^2 + (5x)^{-1})^{11}$.

26. Find the coefficient of x^9 in the expansion of $(2x^2 - x^{-1})^{20}$.

27. Find the coefficient of x^{24} in the expansion of $(x^2 + 3ax^{-1})^{15}$.

28. Find the coefficient of x^9 in the expansion of $(x^2 - (3x)^{-1})^9$.

29. Find the coefficient of x^{-7} in the expansion of $\left(2x - \frac{1}{3x^2}\right)^{11}$.

30. Find the coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx}\right)^{11}$. Also, find the relation between a and b so that the coefficients are equal.

31. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, show that its coefficient is $\frac{2n!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$.

32. Find the term independent of x in the following binomial expansions:

    i. $\left(x + \frac{1}{x}\right)^{2n}$,

    ii. $\left(2x^2 + \frac{1}{x}\right)^{15}$,

    iii. $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$,

    iv. $\left(2x^2 - \frac{1}{x}\right)^{12}$,

    v. $\left(2x^2 - \frac{3}{x^3}\right)^{25}$,

    vi. $\left(x^3 - \frac{3}{x^2}\right)^{15}$,

    vii. $\left(x^2 - \frac{3}{x^3}\right)^{10}$, and

    viii. $\left(\frac{1}{2}x^{1/3} + x^{-1/3}\right)^8$.

33. If there is a term independent of x in $\left(x+\frac{1}{x^2}\right)^n$, show that it is equal to $\frac{n!}{\left(\frac{n}{3}\right)!\left(\frac{2n}{3}\right)!}$

34. Prove that in the expansion of $(1+x)^{m+n}$, coefficients of x^m and x^n are equal, $\forall\ m{,}n > 0{,}m, n \in \mathbb{N}$.

35. Give that the 4th term in the expansion of $\left(px+\frac{1}{x}\right)^n$ is $\frac{5}{2}$. Find n and p.

36. Find the middle term in the expansion of $\left(x-\frac{1}{2x}\right)^{12}$.

37. Find the middle terms in the expansion of $\left(2x^2-\frac{1}{x}\right)^7$.

38. Prove that the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}2^n$.

39. Show that the coefficient of the middle term in $(1+x)^{2n}$ is equal to the sum of coefficients of the two middle terms in $(1+x)^{2n-1}$.

40. Find the middle term in the expansions of;

    i. $\left(\frac{2x}{3}-\frac{3y}{2}\right)^{20}$,

    ii. $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$,

    iii. $\left(\frac{x}{y}-\frac{y}{x}\right)^7$,

    iv. $(1+x)^{2n}$, and

    v. $(1-2x+x^2)^n$.

41. Find the general and middle term of the expansion $\left(\frac{x}{y}+\frac{y}{x}\right)^{2n+1}$; n being a positive integer show that there is no term free of x and y.

42. Show that the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}.(-2)^n$.

43. If in the expansion of $(1+x)^{43}$, the coefficient of $(2r+1)$th term is equal to the coefficient of $(r+2)$th term, find r.

44. If the rth term in the expansion of $(1+x)^{20}$ has coefficient equal to that of the $(r+4)$th term, find r.

45. If the coefficient of $(2r+4)$th term and $(r-2)$th term in the expansion of $(1+x)^{18}$ are equal, find r.

46. If the coefficient of $(2r+5)$th term and $(r-6)$th term in the expansion of $(1+x)^{39}$ are equal, fin C_{12}^r.

47. Given positive integers $r > 1, n > 2, n$ being even and the coefficient of $3r$th term and $(r+2)$th term in the expansion of $(1+x)^{2n}$ are equal, find r.

48. If the coefficient of $(p+1)$th term in the expansion of $(1+x)^{2n}$ be equal to that of the $(p+3)$th term, show that $p = n-1$.

49. Show that the coefficient of $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of the rth and $(r+1)$th term in the expansion of $(1+x)^n$.

50. Find the greatest term in the expansion of $\left(7 - \frac{10}{3}\right)^{11}$.

51. Show that if the greatest term in the expansion of $(1+x)^{2n}$ has also the greatest coefficient x lies between $\frac{n}{n+1}$ and $\frac{n+1}{n}$.

52. Find the greatest terms in the expansions of:

    i. $\left(2 + \frac{9}{5}\right)^{10}$,

    ii. $(4-2)^7$, and

    iii. $(5+2)^{13}$.

53. Find the limits between which x must lie in order that the greatest term in the expansion of $(1+x)^{30}$ may have the greatest coefficient.

54. If $n \in \mathbb{P}$, then prove that $6^{2n} - 35n - 1$ is divisible by 1225.

55. Show that $2^{4n} - 2^n(7n+1)$ is some multuple of the square of 14, where $n \in \mathbb{P}$.

56. Show that $3^{4n+1} - 16n - 3$ is divisible by 256, if $n \in \mathbb{P}$.

57. If $n \in \mathbb{P}$, show that

    i. $4^n - 3n - 1$ is divisible by 9,

    ii. $2^{5n} - 31n - 1$ is divisible by 961,

    iii. $3^{2n+2} - 8n - 9$ is divisible by 64,

    iv. $2^{5n+5} - 31n - 32$ is divisible by 961 if $n > 1$, and

    v. $3^{2n} - 32n^2 + 24n - 1$ is divisible by 512 if $n > 2$.

58. If three consecutive coefficients in the expansion of $(1+x)^n$ be $165, 330$ and 462, find n and r.

59. If a_1, a_2, a_3 and a_4 be any four consecutive coefficients in the expansion of $(1+x)^n$, prove that $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$.

60. If 2nd, 3rd and 4th terms in the expansion of $(x+y)^n$ be $240, 720$ and 1080 respectively, find x, y and n.

61. If a, b, c be thre three consecutive terms in the expansion of some power of $(1+x)$, prove that the exponent is $\frac{2ac+ab+bc}{b^2-ac}$.

62. If 14the, 15th and 16th term in the expansion of $(1+x)^n$ are in A.P., find n.

63. If three consecutive terms in the expansion of $(1+x)^n$ be $56, 70$ and 56, find n and the position of the coefficients.

64. If three successive coefficients in the expansion of $(1+x)^n$ be $220, 495$ and 792, find n.

65. If 3rd, 4th and 5th terms in the expansion of $(a+x)^n$ be $84, 280$ and 560, find a, x and n.

66. If 6th, 7th and 8th terms in the expansion of $(x+y)^n$ be $112, 7$ and $\frac{1}{4}$, find x, y and n.

67. If a, b, c and d be the 6th, 7th, 8th and 9th terms respectively in any binomial expansion, prove that $\frac{b^2-ac}{c^2-bd} = \frac{4a}{3c}$.

68. If the four consecutive coefficients in any binomial expansion be a, b, c, and d, then prove that (a) $\frac{a+b}{a}, \frac{b+c}{b}, \frac{c+d}{c}$ are in H.P., and (b) $(bc+ad)(b-c) = 2(ac^2 - b^2d)$.

69. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in A.P. Find the value of n.

70. If the coefficients of the 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in A.P., show that $2n^2 - 9n + 7 = 0$.

71. If the coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are in A.P. show that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

72. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio $182 : 84 : 30$, prove that $n = 18$.

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, prove that

73. $C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n = n.2^{n-1}$.

74. $C_0 + 2.C_1 + 3.C_2 + \cdots + (n+1).C_n = (n+2)2^{n-1}$.

75. $C_0 + 3.C_1 + 5.C_2 + \cdots + (2n+1).C_n = (n+1)2^n$.

76. $C_1 - 2.C_2 + 3.C_3 - 4.C_4 + \cdots + (-1)^{n-1}n.C_n = 0$.

77. $C_0 + \frac{C_1}{2} + \frac{C_3}{3} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$.

78. $C_0 - \frac{C_1}{2} + \frac{C_3}{3} - \cdots + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}$.

79. $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots = \frac{2^n-1}{n+1}$.

80. $2.C_0 + 2^2.\frac{C_1}{2} + 2^3.\frac{C_2}{3} + \cdots + 2^{n+1}.\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$.

81. $C_0.C_r + C_1.C_{r+1} + \cdots + C_{n-r}.C_n = \frac{(2n)!}{(n+r)!(n-r)!}$.

82. $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \frac{(2n)!}{n!n!}$.

83. $\frac{C_1}{C_0} + 2.\frac{C_2}{C_1} + 3.\frac{C_3}{C_2} + \cdots + n.\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$.

84. $(1 + C_1 + C_2 + \cdots + C_n)^2 = 1 + C_1^{2n} + C_2^{2n} + \cdots + C_{2n}^{2n}$.

85. $(1 + C_1 + C_2 + \cdots + C_n)^5 = 1 + C_1^{5n} + C_2^{5n} + \cdots + C_{5n}^{5n}$.

86. $C_0 + 5.C_1 + 9.C_2 + \cdots + (4n+1).C_n = (2n+1)2^n$.

87. $1 - (1+x)C_1 + (1+2x)C_2 - (1+3x)C_3 + \cdots = 0$.

88. $3.C_1 + 7.C_2 + 11.C_3 + \cdots + (4n-1)C_n = (2n-1)2^{n+1}$.

89. $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \cdots = \frac{2^n}{n+1}$.

90. $C_0^n C_1^{n+1} + C_1^n C_2^{n+1} + \cdots + C_n^n C_{n+1}^{n+1} = \frac{(2n+1)!}{(n+1)!n!}$.

91. $C_0 - 2.C_1 + 3.C_2 - \cdots + (-1)^n(n+1)C_n = 0$.

92. $C_0 - 3.C_1 + 5.C_2 - \cdots + (-1)^n(2n+1)C_n = 0$.

93. $a - (a-1)C_1 + (a-2)C_2 - (a-3)C_3 + \cdots + (-1)^n(a-n)C_n = 0$.

94. $1^2.C_1 + 2^2.C_2 + 3^2C_3 + \cdots + n^2.C_n = n(n+1)2^{n-2}$.

95. If $n > 3$ and $n \in \mathbb{N}$, prove that $C_0.abc - C_1(a-1)(b-1)(c-1) + C_2(a-2)(b-2)(c-2) - \cdots + (-1)^n.C_n(a-n)(b-n)(c-n) = 0$

96. $C_0 - 2^2.C_1 + 3^2.C_2 - \cdots + (-1)^n(n+1)^2C_n = 0, n > 2$.

97. Prove that $\sum_{r=0}^{n} r^2.C_r p^r q^{n-r} = npq + n^2p^2$ if $p + q = 1$.

98. $2.C_0 + \frac{2^2}{2}.C_1 + \frac{2^3}{3}.C_2 + \cdots + \frac{2^{11}}{11}.C_{11} = \frac{3^{11}-1}{11}$.

99. $\frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_2 + \frac{2^4}{3.4}C_2 + \cdots + \frac{2^{n+2}}{(n+1)(n+2)}C_n = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$.

100. $C_1 - \frac{1}{2}C_2 + \frac{1}{3}C_3 - \cdots + (-1)^n\frac{1}{n}C_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

101. $\frac{C_0}{1} - \frac{C_1}{5} + \frac{C_2}{9} - \cdots + (-1)^n\frac{C_n}{4n+1} = \frac{n.4^n}{1.5.9...(4n+1)}$.

102. $\frac{C_0}{n} - \frac{C_1}{n+1} + \frac{C_2}{n+2} - \cdots + (-1)^n\frac{C_n}{2n} = \frac{n!(n-1)!}{(2n)!}$.

103. $\frac{C_0}{n(n+1)} - \frac{C_1}{(n+1)(n+2)} + \frac{C_2}{(n+2)(n+3)} - \cdots + (-1)^n \frac{C_n}{2n(2n+1)} = \frac{1}{(2n+1)} \cdot \frac{1}{^{2n}C_{n-1}}$.

104. $\frac{C_0}{k} - \frac{C_1}{k+1} + \frac{C_2}{k+2} - \cdots + (-1)^n \frac{C_n}{k+n} = \frac{n!}{k(k+1)...(k+n)}$.

105. Show that $C_0^2 - C_1^2 + C_2^2 - \cdots + (-1)^n.C_n^2 = 0$ or $(-1)^{n/2}.\frac{n!}{\left(\frac{n!}{2}\right)^2}$ according as n is odd or even.

106. Show that $C_r^n C_0 +^m C_{r-1}.^n C_1 +^m C_{r-2}.^n C_2 + \cdots +^m C_0.^n C_r =^{m+n} C_r$, where m, n, r are positive integers and $r < m$, $r < n$.

107. $^{2n}C_0^2 -^{2n} C_1^2 +^{2n} C_2^2 - \cdots + (-1)^{2n}.^{2n}C_{2n}^2 = (-1)^n.^{2n}C_n$.

108. Show that $C_1^2 + 2.C_2^2 + 3.C_3^2 + \cdots + n.C_n^2 = \frac{(2n-1)!}{[(n-1)!]^2}$.

109. Show that $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \cdots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{[(n+1)!]^2}$.

110. $C_0 - 2^2C_1 + 3^2C_2 - \cdots + (-1)^n(n+1)^2C_n = 0$, $n > 2$.

111. $\frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n \frac{C_n}{(n+1)(n+2)} = \frac{1}{n+2}$

112. $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \cdots + (-1)^n \frac{C_n}{n+2} = \frac{1}{(n+1)(n+2)}$.

113. $\frac{C_0}{3} - \frac{C_1}{4} + \frac{C_2}{4} - \cdots + (-1)^n \frac{C_n}{n+3} = \frac{2}{(n+1)(n+2)(n+3)}$.

114. $3.C_0 + 3^2\frac{C_1}{2} + 3^3\frac{C_2}{3} + \cdots + 3^{n+1}\frac{C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$.

115. If n is a positive integer in $(1+x)^n$, show that $2.\frac{\left(\frac{n!}{2}\right)^2}{n!}[C_0^2 - 2.C_1^2 + 3.C_2^2 - \cdots + (-1)^n.(n+1)C_n^2] = (-1)^{n/2}(2+n)$.

116. Show that $\sum_{0 \le i \le n} \sum_{i < j \le n} C_iC_j = 2^{2n-1} - \frac{(2n)!}{2(n!)^2}$.

117. Show that $^n_{r=0}C_r^3$ is equal to the coefficient of x^ny^n in the expansion of $[(1+x)(1+y)(x+y)]^n$.

118. Prove that the sum of coefficients in the expansion $(1+x-3x^2)^{2163}$ is -1.

119. If $(1+x-2x^2)^6 = 1 + a_1x + a_2x^2 + \cdots + a_{12}x^{12}$ show that $1 + a_3 + a_6 + a_9 + \cdots + a_{12} = 31$.

120. Find the sum of the rational terms in the expansion of $(2+\sqrt[5]{3})^{10}$.

121. Find the fractional pert of $\frac{2^{4n}}{15}$.

122. Show that the integer just above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}, $\forall\, n \in \mathbb{N}$.

123. Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$.

124. Show that $(101)^{50} > (100)^{50} + (99)^{50}$.

125. Find the sum of the series $\sum_{r=0}^{n} (-1)^r . {}^nC_r \Big[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \cdots \text{ to } m \text{ terms}\Big]$.

126. Find the last digit of the number $(32)^{32}$.

127. Prove that $\sum_{r=0}^{k} (-3)^{r-1} . {}^{3n}C_{2n-1} = 0$, where $k = \frac{3n}{2}$ and n is a positive even number.

128. If $t_0, t_1, t_2, t_3, \ldots$ be ther terms of expansion $(a + x)^n$, prove that $(t_0 - t_2 + t_4 - \cdots)^2 + (t_1 - t_3 + t_5 - \cdots)^2 = (a^2 + x^2)^n$.

If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$, show that

129. $a_0 + a_1 + a_2 + \cdots + a_{2n} = 3^n$.

130. $a_0 - a_1 + a_2 - \cdots + a_{2n} = 1$.

131. $a_0 + a_3 + a_6 + \cdots = 3^{2n-1}$.

132. If $S_n = 1 + q + q^2 + \cdots + q^n$ and $S'_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \cdots + \left(\frac{q+1}{2}\right)^n$, $q \neq 1$, prove that $C_1^{n+1} + C_2^{n+1}.S_1 + C_3^{n+1}.S_2 + \cdots + C_{n+1}^{n+1}.S_n = 2^n S'_n$.

133. Find the number of rational terms in the expansion of $(\sqrt[4]{9} + \sqrt[6]{8})^{1000}$.

134. Find the sum of rational terms in the expansion of $(\sqrt[3]{2} + \sqrt[5]{3})^{15}$.

135. Determine the values of x in the expansion of$(x + x\log_{10} x)^5$ if the third term in that expansion is 1,000,000.

136. Expand $\left(x + 1 - \frac{1}{x}\right)^3$.

137. Find the value of x for which the sixth term of $\left(\sqrt{2^{\log(10-3^x)}} + \sqrt[5]{2^{(x-2)\log 3}}\right)^m$ is equal to 21 and coefficients of second, third and fourth terms are the first, third and fifth terms of an A.P., given base of log is 10.

138. Find the values of x for which the sixth term of the expansion $\left[2^{\log_2 \sqrt{9^{x-1}+7}} + \frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$ is equal to 84.

139. If $n \in \mathbb{N}$, prove that $\frac{1}{(81)^n} - \frac{10}{(81)^n}.C_1^{2n} + \frac{10^2}{(81)^n}.C_2^{2n} - \frac{10^3}{(81)^n}.C_3^{2n} + \cdots + \frac{10^{2n}}{(81)^n} = 1$.

140. Find the value of $\lim_{n\to\infty} S_n = C_n - \frac{2}{3}C_{n-1} + \left(\frac{2}{3}\right)^2 C_{n-2} - \cdots + (-1)^n \left(\frac{2}{3}\right)^n C_0$.

141. If $E = (6\sqrt{6}+14)^{2n+1}$ and F be fractional part of E, prove that $EF = 20^{2n+1}$.

142. Find the digits at units, tens and hundreds place in the number $(17)^{256}$.

143. Show that for $n \geq 3$, $n^{n+1} > (n+1)^n$, $forall\ n \in \mathbb{P}$.

144. Show that $2 < \left(1+\frac{1}{n}\right)^n < 3\ \forall\ n \in \mathbb{N}$.

145. Show that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.

146. Show that $53^{53} - 33^{33}$ is divisible by 10.

147. Let k and n be positive integers and $S_k = 1^k + 2^k + \cdots + n^k$, show that $C_1^{m+1}S_1 + C_2^{m+1}S_2 + C_m^{m+1}S_m = (n+1)^{m+1} - n - 1$.

148. Find $\sum_{i=1}^{k}\sum_{k=1}^{n} C_k^n C_i^k$, $i \leq k$.

149. Prove that $\sum_{r=0}^{n}(-1)^r \cdot {}^nC_r \frac{1+r\log_e 10}{(1+\log_e 10^n)^r} = 0$.

150. Find the remainder when $32^{32^{32}}$ is divided by 7.

151. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_r = 1\ \forall\ r \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$.

152. Find the coefficient of x^{50} in $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}$.

153. Show that $C_n^n + C_n^{n+1} + C_n^{n+2} + \cdots + C_n^{n+1} = C_{n+1}^{n+k+1}$.

154. Find the coefficient of x^n in $(1+x+2x^2+3x^3+\cdots+nx^n)^2$.

155. Find the coefficient of x^k, $0 \leq k \leq n$ in the expansion of $1+(1+x)+(1+x)^2+\cdots+(1+x)^n$.

156. Find the coefficient of x^3 in $(x+1)^n + (x+1)^{n-1}(x+2) + (x+1)^{n-2}(x+2)^2 + \cdots + (x+2)^n$.

157. Simplify $\left(\frac{a+1}{a^{2/3}-a^{1/3}+1} - \frac{a-1}{a-a^{1/2}}\right)^{10}$ into a binomial and determine the term independent of a.

158. Find the coefficient of x^2 in $\left(x+\frac{1}{x}\right)^{10}(1-x+2x^2)$.

159. Find the coefficient of x^4 in the expansion of $(1+x-2x^2)^6$.

160. Find the term independent of x in $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

161. Find the term independent of x in $\left(x^2+\frac{1}{x^3}\right)^7(2-x)^{10}$.

162. Find the term independent of x in $(1+x+x^{-2}+x^{-3})^{10}$.

163. Let $(1+x^2)^2(1+x)^n=\sum_{k=0}^{n+4} a_k x^k$. If a_1, a_2 and a_3 are in A.P., find n.

164. Show that $C_1^m+C_2^{m+1}+C_3^{m+2}+\cdots+C_n^{m+n-1}=C_1^n+C_2^{n+1}+C_3^{n+2}+\cdots+C_n^{m+n-1}$.

165. If $n\in\mathbb{N}$ and $(1+x+x^2)^n=\sum_{r=0}^{2n} a_r x^r$, prove that (a) $a_r=a_{2n-r}$, (b) $a_0+a_1+a_2+\cdots+a_{n-1}=\frac{1}{2}(3^n-a_n)$, and (c) $(r+1)a_{r+1}=(n-r)a_r+(2n-r+1)a_{r-1}$, where $0<r<2n$.

166. If $(1-x^3)^n=\sum_{r=0}^{n} a_r.x^r.(1-x)^{3n-2r}$, where $n\in\mathbb{N}$, then find a_r.

167. Show that the coefficient of middle term in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$.

168. Find the value of r for which C_r^{200} is greatest.

169. Committees of how many persons should be made out of 20 persons so that the number of committees is maximum.

170. Show that the number of permutations which can be formed from $2n$ letters which are either ‘a’ or ‘b’ is greatest when the number of a's is equal to the number of b's.

171. Find the consecutive terms in the expansion of $(3+2x)^7$ whose coefficients are equal.

172. Find the sum of coefficients in the expansion of $(1+5x^2-7x^3)^{2000}$.

173. If the sum of the binomial coefficients in the expansion of $\left(3^{-\frac{x}{4}}+3^{\frac{5x}{4}}\right)^n$ is 64 and the term with greatest coefficient exceeds the third term by $n-1$ and $[\alpha]=x$, where $[\alpha]$ denotes the integral part of α, find the value of α.

174. Find the sum of the coefficients in the expansion of $(5p-4q)^n$, where $n\in\mathbb{P}$.

175. Find the sum of the coefficients in the expansion of the polynomial $(1-3x+x^3)^{201}.(1+5x-5x^2)^{503}$.

176. If the sum of the coefficients in the expansion of $(tx^2-2x+1)^n$ is equal to the sum of coefficients in the expansion of $(x-ty)^n$, where $n\in\mathbb{N}$, then find the value of t.

177. If $a_0, a_1, a_2, \ldots, a_n$ be the successive coefficient of $(1+x)^n$, show that $(a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots)^2 = a_0 + a_1 + \cdots + a_n = 2^n$.

178. Find the greatest term in the expansion of $\sqrt{3}\left(1 + \frac{1}{\sqrt{3}}\right)^{20}$.

179. In the expansion of $(x+a)^{15}$, if the eleventh term is the G.M. of the eighth and twelfth terms, which term in the expression is the greatest?

180. if the greatest term in the expansion of $(1+x)^{2n}$ has the greatest coefficient if and only if $x \in \left(\frac{10}{11}, \frac{11}{10}\right)$ and the fourth term in the expansion of $\left(kx + \frac{1}{x}\right)^m$, is $\frac{m}{4}$, then find the value of mk.

181. Given that the 4th term in the expansion of $\left(2 + \frac{3}{8}x\right)^{10}$ has the maximum numerical value, find the range of values of x for which this would be true.

182. Show that the roots of the equation $ax^2 + 2bx + c = 0$ are real and unequal, where a, b, c are three consecutive binomial expansion with positive integral index.

183. If $n \in \mathbb{P}$, show that $9^n + 7$ is divisible by 8.

184. If $n \in \mathbb{P}$, show that $3^{2n+1} + 2^{n+2}$ is divisible by 7.

185. Show that no three consecutive binomial coefficients can be in G.P. or H.P.

186. Let n be a positive integer and $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n}$, show that $a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2 = a_n$.

187. Let n be a positive integer and $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n}$, show that $a_0^2 - a_1^2 + a_2^2 - \cdots + (-1)^n a_{n-1}^2 = \frac{1}{2}a_n[1 - (-1)^n a_n]$.

188. Show that $\sum_{0 \le i < j} \sum_{0 \le j \le n} (C_i + C_j)^2 = (n-1)^{2n} C_n + 2^{2n}$, $(0 \le i \le j \le n)$.

189. Show that $\sum_{0 \le i < j} \sum_{0 \le j \le n} (i+j) C_i C_j = n\left(2^{2n-1} - \frac{1}{2} C_n^{2n}\right)$.

190. Show that $\frac{1}{m!} C_0 + \frac{n}{(m+1)!} C_1 + \frac{n(n-1)}{(m+2)!} C_2 + \cdots + \frac{n(n-1)\cdots 3.2.1}{(m+n)!} C_n = \frac{(m+n+1)(m+n+2)\cdots(m+2n)}{(m+n)!}$.

191. Show that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \cdots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!} C_1.C_2.\ldots.C_n$.

192. If n be a positive integer, prove that $\frac{1}{1!(n-1)!} + \frac{1}{3!(n-1)!} + \frac{1}{5!(n-5)!} + \cdots + \frac{1}{(n-1)!1!} = \frac{2^{n-1}}{n!}$.

193. Prove that $\sum_{r=0}^{n} (-1)^r . \left(\frac{C_r^n}{C_r^{r+3}}\right) = \frac{3!}{2(n+3)}$.

194. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ show that for $m \ge 2, C_0 - C_1 + C_2 - \cdots + (-1)^{m-1} C_{m-1} = (-1)^{m-1} \frac{(n-1)(n-2)\ldots(n-m+1)}{(m-1)!}$.

195. Find the G.C.D. of $C_1^{2n}, C_3^{2n}, C_5^{2n}, \ldots, C_{2n-1}^{2n}$.

196. Show that $\sum_{r=0}^{n} C_r^n . \sin rx \cos(n-r)x = 2^{n-1} \sin nx$.

197. $a.C_0 + (a-b).C_1 + (a-2b).C_2 + \cdots + (a-nb).C_n = 2^{n-1}(2a-nb)$.

198. $a^2.C_0 - (a-1)^2.C_1 + (a-2)^2.C_2 - \cdots + (-1)^n(a-n)^2.C_n = 0,\, n > 3$.

199. If $a_0, a_1, a_2, \ldots, a_n$ be in an A.P., prove that $a_0 - a_1.C_1 + a_2C_2 - \cdots + (-1)^n a_nC_n = 0$.

200. Show that $n > 3$, $\sum_{r=0}^{n}(-1)^r(a-r)(b-r)C_r = 0$.

201. Show that $n > 3$, $\sum_{r=0}^{n}(-1)^r(a-r)(b-r)(c-r)C_r = 0$.

202. Find the value of n for which $\frac{C_0}{2^n} + \frac{2.C_1}{2^n} + \cdots + \frac{(n+1)C_n}{2^n} = 16$ is true.

203. If $a_1, a_2, \ldots, a_{n+1}$ be an A.P., prove that $\sum_{k=0}^{n} a_{k+1}C_k = 2^{n-1}(a_1 + a_{n+1})$.

204. If $s = \frac{n+1}{2}[2a+nd]$ and $S = a + (a+d)C_1 + (a+2d)C_2 + \cdots + (a+nd)C_n$, prove that $(n+1)S = 2^n.s$.

205. If $(1 + x + x^2 + \cdots + x^p)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{np}x^{np}$, show that $a_1 + 2a_2 + 3a_3 + \cdots + np.a_{np} = \frac{1}{2}np(p+1)^n$.

206. Show that $\sum_{k=0}^{15} \frac{C_k^{15}}{(k+1)(k+2)} = \frac{2^{17}-18}{16.17}$.

207. Show that $\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \cdots + (-1)^n \frac{C_n}{3n+1} = \frac{3^n.n!}{1.4.5\ldots(3n+1)}$.

208. Show that $\sum_{r=0}^{n} \frac{(-1)^r C_r}{(r+1)(r+2)} = \frac{1}{n+2}$.

209. Prove that $\sum_{r=0}^{n} \frac{C_r.3^{r+3}}{(r+1)(r+2)(r+3)} = \frac{4^{n+3} - 1 - \frac{3}{2}(n+3)(3n+8)}{(n+1)(n+2)(n+3)}$.

210. Prove that $\sum_{r=0}^{n} \frac{r+2}{r+1} C_r = \frac{2^n(n+3)-1}{n+1}$.

211. Show that $\sum_{r=0}^{n} \frac{3^{r+4}C_r}{(r+1)(r+2)(r+3)(r+4)} = \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[4^{n+4} - \sum_{k=0}^{n} {}^{n+4}C_k 3^k\right]$.

212. Show that $\sum_{r=0}^{n-3} C_r C_{r+3} = \frac{(2n)!}{(n+3)!(n-3)!}$.

213. Show that the sum of the product taken two at a time from $C_0, C_1, C_2, \ldots$ is $2^{2n-1} - \frac{(2n-1)!}{n!(n-1)!}$.

214. If $S_n = C_0C_1 + C_1C_2 + \cdots + C_{n-1}C_n$ and $\frac{S_{n+1}}{S_n} = \frac{15}{4}$, find n.

215. Show that $C_0^2 + 2.C_1^2 + 3.C_2^2 + \cdots + (n+1)C_n^2 = \frac{(n+2)(2n-1)!}{n!(n-1)!}$.

216. Show that $C_0.C_n^{2n} - C_1.C_n^{2n-2} + C_2.C_n^{2n-4} - \cdots = 2^n$.

217. Show that $\sum_{0\le i\le j}\sum_{0\le j\le n}(i+j)(C_i + C_j + C_iC_j) = n^2.2^n + n\left(2^{2n-1} - \frac{(2n)!}{2(n!)^2}\right)$ $[0 \le i \le j \le n]$.

218. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, show that $a_0a_{2r} - a_1a_{2r+1} + a_2a_{2r+2} - \cdots + a_{2n-2r}a_{2n} = a_{n+r}$.

219. If P_n denoted the product of all coefficients in the expansion of $(1+x)^n$, show that $\frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{n!}$.

220. Show that $\sum_{r=1}^{n} r^3\left(\frac{C_r}{C_{r-1}}\right)^2 = \frac{1}{12}n(n+1)^2(n+2)$.

221. Show that $C_3 + C_7 + C_{11} + \ldots = \frac{1}{2}\left[2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right]$.

222. If $(1+x+x^2)^{20} = a_0 + a_1x + a_2x^2 + \cdots = a_{40}x^{40}$, then find the value of $a_0 + a_2 + a_4 + \cdots + a_{38}$.

223. If $(1+x+x^2)^{20} = a_0 + a_1x + a_2x^2 + \cdots = a_{40}x^{40}$, then find the value of $a_1 + a_3 + a_5 + \cdots + a_{37}$.

224. Show that $C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \cdots + (-1)^n\frac{C_n}{n} + \frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + \cdots + \frac{n-2}{2.3} = \frac{n+1}{2}$.

225. Show that $\sum_{0\le i<j}\sum_{0\le j\le n}\frac{i}{C_i} + \frac{j}{C_j} = \frac{n^2}{2}\sum_{r=0}^{n}\frac{1}{C_r}$ $[0 \le i \le j \le n]$.

226. Show that $\sum_{0\le i<j}\sum_{0\le j\le n} i.j.C_i.C_j = n^2\left[2^{2n-3} - \frac{1}{2}{}^{2n-2}C_{n-1}\right]$ $[0 \le i \le j \le n]$.

227. Prove that $C_1 - \left(1+\frac{1}{2}\right)C_2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)C_3 - \cdots + (-1)^n\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)C_n = \frac{1}{n}$.

228. Find the coefficient of x^5 in the expansion of $(1+2x+3x^2)^4$.

229. Find the coefficient of $x^3y^4z^2$ in the expansion of $(2x-3y+4z)^9$.

230. Find the number of terms in $(2x - 3y + 4z)^{100}$.

231. Find the coefficient of x^4 in the expansion of $(1 + x + x^2)^3$.

232. Find the coefficient of x^{10} in $(1 + x + x^2 + x^3 + x^4 + x^5)^3$.

233. Find the coefficient of x^7 in $(1 + 3x - 2x^3)^{10}$.

234. Find the coefficient of $x^3y^4z^5$ in $(xy + yz + zx)^6$.

235. Find the greatest coefficient in $(w + x + y + z)^{15}$.

236. Find the number of terms in $(a + b + c + d + e)^{100}$.

237. If $|x| < 1$, show that $(1 + x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ to ∞.

238. Find a, b so that the coefficient of x^n in the expansion of $\frac{(a+bx)}{(1-x)^2}$ may be $2n + 1$ and hence find the sum of the series $1 + 3\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + \cdots$.

239. Sum the series $1 + \frac{1}{3} + \frac{1.3.5}{3.6.9} + \cdots$ to ∞.

240. If $|x| < 1$, show that $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ to ∞.

241. If $|x| < 1$, show that $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ to ∞.

242. If $|x| < 1$, show that $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$ to ∞.

243. If $|x| < 1$, show that $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots$ to ∞.

244. If $|x| < 1$, show that $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots$ to ∞.

245. If $|x| < 1$, show that $(1 + x)^{-1/5} = 1 - \frac{x}{5} + \frac{3x^2}{25} - \frac{11x^3}{125} + \ldots$ to ∞.

246. Find the first four terms of $\left(\frac{2x}{3} - \frac{3}{2x}\right)^{-3/2}$.

247. Find the first three terms of $\left(1 - \frac{x}{2}\right)^{-2}$.

248. Find the coefficient of x^6 in $(1 - 2x)^{-5/2}$.

249. Find the $(r + 1)$th term and the its coefficients in $(1 - 2x)^{-1/2}$.

250. Show that $(1 + 2x + 3x^2 + 4x^3 + \ldots \text{ to } \infty)^{3/2} = 1 + 3x + 6x^2 + 10x^3 + \ldots$ to ∞, $|x| < 1$.

251. Sum the series $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots$ to ∞.

252. Sum the series $1 + \frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \ldots$ to ∞.

253. If $y = x - x^2 + x^3 - x^4 + ...$ to ∞, show that $x = y + y^2 + y^3 + ...$ to ∞.

254. Show that the coefficent of x^n in $(1 + x + x^2)^{-1}$ is $1, 0, -1$ as n is of the form $3m, 3m - 1, 3m + 1$.

255. Show that $\frac{1}{e} = 2\left[\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + ... \text{ to } \infty\right]$.

256. Sum the series $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + ...$ to ∞.

257. Show that $\log 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + ...$ to ∞.

258. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ to ∞, show that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + ...$ to ∞.

259. If α, β be the roots of the equation $ax^2 + bx + c = 0$, show that $\log(a - bx + cx^2) = \log a + (\alpha + \beta)x - \frac{(\alpha^2+\beta^2)}{2}x^2 + ...$ to ∞.

260. Sum the series $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \cdots$ to ∞.

261. Sum the series $\frac{1}{2!} + \frac{3}{4!} + \frac{5}{6!} + \cdots$ to ∞.

262. Sum the series $\frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+3}{4!} + \cdots$ to ∞.

263. Sum the series $\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \cdots$ to ∞.

264. Prove that $1 - \log 2 = \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots$ to ∞.

265. Prove that $\log(1 + x) - \log(x - 1) = 2\left[\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots \text{ to } \infty\right]$.

266. Prove that $2\log x - \log(x + 1) - \log(x - 1) = \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^5} + \cdots$ to ∞.

267. Prove that $\log[(1 + x)^{1+x} \log(1 - x)^{1-x}] = 2\left[\frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \cdots \text{ to } \infty\right]$

Chapter 8
Determinants

Let a,b,c,d be any four numbers, real or complex, then the symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

denotes $ad - bc$ and is called a *determinant* of second order. a,b,c,d are called elements of the determinant and $ad - bc$ is called value of the determinant.

As you can see, the elements of a determinant are positioned in the form of a square in its designation. The diagonal on which elements aa and dd lie is called the principal or primary diagonal of the determinant and the diagonal which is formed on the line of bb and cc is called the secondary diagonal. A row is constituted by elements lying in the same horizontal line and a column is constituted by elements lying in the same vertical line. Clearly, determinant of second order has two rows and two columns and its value is equal to the products of elements along primary diagonal minus the product of elements along the secondary diagonal. Thus, by definition

$$\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 18 - 12 = 6$$

Let a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be any nine numbers, then the symbol

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is another way of saying

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 b_2 \\ c_1 c_2 \end{vmatrix}$$

i.e. $a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

Rule to put + or - before any element: Find the sum of number of rows and columns in which the considered element occus. If the sum is even put a + sign before the element and if the sum is odd, put a $-$ sign before the element. Since a_1 occurs in first row and first column whose sum is $1 + 1 = 2$ which is an even number, therefore + sign occurs for it. Since a_2 occurs in first row and second column whose sum is $1 + 2 = 3$ which is an odd number, therefore $-$ sign occurs before it.

We have expanded the determinant along first row in previous case. The value of determinant does not change no matter which row or column we expand it along. Expanding the determinant along second row, we get

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Thus, we see that value of determinant remains unchanged irrespective of the change of row and column against which it is expanded.

Usually, an element of a determinant is denoted by a letter with two suffices, first one indicating the row and second one indicating the column in which the element occcur. Thus, a_{ij} element indicates that it has occurred in ith row and jth column. We also denote the rows by R_1, R_2, R_3 and so on. R_i denotes the ith row of determinant while R_j denotes jth row. Columns are denoted by C_1, C_2, C_3 and so on. C_i and C_j denote ith and jth column of determinant. Δ is the usual symbol for a determinant. Another way of denoting the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is $(a_1b_2c_3)$. The expanded form of determinant has $n!$ terms where n is the number of rows or columns.

Ex 1. Find the value of the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 9 \\ 2 & 1 & 6 \end{vmatrix}$$

$$\Delta = 1\begin{vmatrix} 4 & 9 \\ 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}$$

Expanding the determinant along first row $= 1(24-9) - 2(18-18) + 4(3-8) = -5$

Ex 2. Find the value of the determinant

$$\Delta = \begin{vmatrix} 3 & 1 & 7 \\ 5 & 0 & 2 \\ 2 & 5 & 3 \end{vmatrix}$$

Expanding the determinant along second row,

$$\Delta = -5\begin{vmatrix} 1 & 7 \\ 5 & 3 \end{vmatrix} + 0\begin{vmatrix} 3 & 7 \\ 23 & \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix}$$

$= -5(3-35) - 2(15-2) = 134$

8.1 Minors

Consider the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix}$$

If we leave the elements belonging to row and column of a particular element a_{ij} then we will obtain a second order determinant. The determinant thus obtained is called minor of a_{ij} and it is denoted by M_{ij}, since there are 9 elements in the above determinant we will have 9 minors.

For example, the minor of element

$$a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$$

The minor of element

$$a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$$

If we want to write the determinant in terms of minors then following is the expression obtained if we expand it along first row

$$\Delta = (-1)^{1+1} a_{11}M_{11} + (-1)^{1+2} a_{12}M_{12} + (-1)^{1+3} a_{13}M_{13}$$
$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

8.2 Cofactors

The minor M_{ij} multiplied with $(-1)^{i+j}$ is known as cofactor of the element a_{ij} and is denoted like A_{ij}. Thus, we can say that, $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

8.3 Theorems on Determinants

Theorem 10

The value of a determinant is not changed when rows are changed into corresponsing columns.

Proof

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding the determinant along first row,

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

If Δ' be the value of the determinant when rows of determinant Δ are changed into corresponding columns then

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Thus, we see that $\Delta = \Delta'$. □

Theorem 11

If any two rows or columns of a determinant are interchanged, the sign of determinant is changed, but its value remains the same.

Proof

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

Expanding the determinant along first row, $\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

Now $\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} [R_1 \leftrightarrow R_3]$

$= a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2)$

$= a_3b_2c_1 - a_3b_1c_2 - b_3a_2c_2 + b_3a_1c_2 + c_3a_2b_1 - c_3a_1b_2x$

$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$

$= -\Delta$ □

Theorem 12

The value of a determinant is zero if any two rows or columns are identical.

Proof

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\Delta[R_1 \leftrightarrow R_3]$$

Thus, $\Delta = -\Delta \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$. □

Theorem 13

A common factor of all elements of any row(or of any column) may be taken outside the sign of the determinant. In other owrds, if all the elements of the same row(or the same column) are multiplies by a constant, then the determinant becomes multiplied by that number.

Proof

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding the determinant along first row, $\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

and

$$\Delta' = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= ma_1(b_2c_3 - b_3c_2) - mb_1(a_2c_3 - a_3c_2) + mc_1(a_2b_3 - a_3b_2)$

$= m\Delta$ □

Theorem 14

If every element of some row or column is the the sum of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each term, the other only the second term. The remaining elements of both the determinants are the same as in the given determinant.

Proof

We have to prove that

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Let

$$\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$$

Then,

$$\Delta = (a_1+\alpha_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+\alpha_2)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+\alpha_3)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + \alpha_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \alpha_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + \alpha_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}.$$

Theorem 15

The value of a determinant does not change when any row or column is multiplied by a number or an expression and is then added to or subtracted from any other row or column.

Proof

We have to prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$$

Let

$$\Delta = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$$

then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m.0 = \Delta$$

8.4 Reciprocal Determinants

If

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

then

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \Delta^2$$

where capital letters denote the cofactors of corresponding small letters in Δ i.e. $A_i =$ cofactor of a_i, $B_i =$ cofactor of b_i and $C_i =$ cofactor of c_i in the determinant Δ. Here, the cofactors are sometimes called *inverse* elements and determinant made from them is called *reciprocal* determinant.

We know that,

$a_1A_1 + a_2A_2 + a_3A_3 = \Delta$, $b_1B_1 + b_2B_2 + b_3C_3 = \Delta$, $c_1C_1 + c_2C_2 + c_3C_3 = \Delta$, $a_1B_1 + a_2B_2 + a_3B_3 = 0$, $b_1A_1 + b_2A_2 + b_3A_3 = 0$, $a_1C_1 + a_2C_2 + a_3C_3 = 0$, $c_1A_1 + c_2A_2 + c_3A_3 = 0$, $b_1C_1 + b_2C_2 + b_3C_3 = 0$, $c_1B_1 + c_2B_2 + c_3B_3 = 0$. Let

$$\Delta_1 = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Now,

$$\Delta\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + a_2A_2 + a_3A_3 & a_1B_1 + a_2B_2 + a_3B_3 & a_1C_1 + a_2C_2 + a_3C_3 \\ b_1A_1 + b_2A_2 + b_3A_3 & b_1B_1 + b_2B_2 + b_3C_3 & b_1C_1 + b_2C_2 + b_3C_3 \\ c_1A_1 + c_2A_2 + c_3A_3 & c_1B_1 + c_2B_2 + c_3B_3 & c_1C_1 + c_2C_2 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}$$

$$\Delta\Delta_1 = \Delta^3$$

$$\Delta_1 = \Delta^2$$

Similarly, if Δ is a determinant of the n-th order and Δ' is the reciprocal determinant, then

$$\Delta' = \Delta^{n-1}$$

which can be proven by induction.

Any minor of Δ' of order r is equla to the complement of the corresponding minor of Δ multiplied with Δ^{r-1}, provided that $\Delta \neq 0$. The proof of this is straightforward and has been left as an exercise to the reader.

8.5 Two Methods of Expansions

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ and } D = \begin{vmatrix} a_1 & b_1 & c_1 & l \\ a_2 & b_2 & c_2 & m \\ a_3 & b_3 & c_3 & n \\ l' & m' & n' & r \end{vmatrix}$$

Let $A_1, B_1, ...$ be the cofactors of $a_1, b_1, ...$ in Δ.

In the expansion of D, the sum of the terms containing r is $r\Delta$: every other term contains one of the three l, m, m and one of the three l', m', n'.

Again, $\begin{vmatrix} a_1 & l \\ l' & r \end{vmatrix}$ and $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ are complementary minors of Δ;

hence, cofficients of ll' in $D = -$ cofficient of $a_1 r$ in $D = -$ coefficient of a_1 in $\Delta = -A_1$

and similarly, coefficient of mn' in $D = -$ coefficient of $c_2 r$ in $D = -$ coefficient of c_2 in $\Delta = -C_2$

Thus, we can show that

$$D = r\Delta - [A_1 ll' + B_2 mm' + C_2 nn' + C_2 mn' + B_2 m'n + A_2 nl' + C_1 n'l + B_1 lm' + A_2 l'm].$$

8.6 Symmetric Determinants

A determinant of nth order is often wirtten in the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = (a_{11} a_{22} \ldots, a_{nn})$$

Denoting any element by a_{ij}, the determinant is said to be *symmetric* if $a_{ij} = a_{ji}$. If $a_{ij} = -a_{ji}$, the determinant is *skew-symmetric*: it is implied that all the elements in the leading diagonal are zero. For example, if

$$\Delta_1 = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}, \Delta_2 = \begin{vmatrix} 0 & x & y \\ -x & 0 & y \\ -y & -x & 0 \end{vmatrix}$$

the determinant Δ_1 is symmetric and Δ_2 is skew-symmetric. We also say that Δ_1 is bordered by l, m, n.

If A_{ij}, A_{ji} are the cofactors of the elements a_{ij}, a_{ji} of a symmetric determinant Δ, then $A_{ij} = Aji$.

For A_{ij} is trandformed into A_{ji}, by changing rows into columns. Thus, if $\Delta = (a_{11}\ a_{22}\ a_{33})$

$$A_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{31} \\ a_{12} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = A_{32}.$$

Similarly, for the skew-symmetric determinants $A_{ij} = (-)1^{n-1}A_{ji}$, where n is the order of the determinant. Also, every skew-symmetric determinant of odd order is equal to zero (follows from the definition of skew-symmetric determinants).

8.7 System of Linear Equations

8.7.1 Consistent Linear Equations

A system of linear equations is said to be consistent if it has at least one solution.

Example: (i) System of equations $x + y = 2$ and $2x + 2y = 7$ is inconsistent because it has no solution i.e. no values of x and y exit which can satisfy the pair of equations. (ii) On the other hand equations $x + y = 2$ and $x - y = 0$ has a solution $x = 1, y = 1$ which satisfies the pair of equation making it a consistent system of linear equations.

8.8 Cramer's Rule

Cramer's rule is used to solve system of linear equations using determinants. Consider two equations $a_x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Solving this by cross multiplication, we have,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

8.8.1 System of Linear Equations in Three Variables

Let the given system of linear equations given in x, y and z be $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Let

$$\Delta \neq 0$$

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} [C_1 \to C_1 - yC_2 - zC_3]$$

$$= x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = x\Delta \Rightarrow x = \frac{\Delta_1}{\Delta}$$

Similalry,

$$y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

This rule which gives the values of x, y and z is known as Cramer's rule.

8.8.2 Nature of Solution of System of Linear Equations

From previous section we have arrived at the fact that $x\Delta = \Delta_1, y\Delta = \Delta_2, z\Delta = \Delta_3$

Case I. When $\Delta \neq 0$

In this case unique values of x, y, z will be obtained and the system of equations will have a unique solution.

Case II. When $\Delta = 0$

Sub Case I. When at least one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero.

Let $\Delta_1 \neq 0$ then $\Delta_1 = x\Delta$ will not be satisfied for any value of x because $\Delta = 0$ and hence no value is possible in this case. Same is the case for y and z.

Thus, no solution is feasible and system of equations become inconsistent.

Sub Case II. When $\Delta_1 = \Delta_2 = \Delta_3 = 0$

In this case infinite number of solutions are possible.

8.8.3 Condition for Consistency of Three Linear Equations in Two Unknonws

Consider a system of linear equations in x and ym $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ will be consistent if the values of x and y obtained from any two equations satisfy the third equations.

Solving first two equations by Cramer's rule, we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = k(\text{say})$$

Substituting these in third equation we get,

$$k[a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1)] = 0$$

$$a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition for consistency of three linear equations in two variables. If such a system of equations is consistent then number of solution is one i.e. a unique solution exists.

8.8.4 System of Homogeneous Linear Equations

A system of linear equations is said to be homogeneous if the sum of powers of the variables in each term is one. Let the three homogeneous equations in three unknowns x, y, z be $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$

Clearly, $x = 0, y = 0, z = 0$ is a solution of above system of equations. This solution is called trivial solution and any other solution is called non-triivial solution. Let the above system of equations has a non-trivial solution.

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

From first two we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = k(\text{say})$$

Substituting these in third equation we get

$$k[a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1)] = 0$$

$$a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the condition for system of equation to have non-trivial solutions.

8.9 Use of Determinants in Coordinate Geometry

8.9.1 Are of a Triangle

The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

8.9.2 Condition of Concurrency of Three Lines

Three lines are said to be concurrent if they pass through a common point i.e. they meet at a point.

Let $a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ be three lines.

These lines will be concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

8.9.3 Condition for General Equation in Second Degree to Represent a Pair of Straight Lines

The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

8.10 Product of Two Determinants

Let

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

then $\Delta_1\Delta_2$ is defined as

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1y_1 + a_2y_2 + a_3y_3 & a_1z_1 + a_2z_2 + a_3z_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_1y_1 + b_2y_2 + b_3y_3 & b_1z_1 + b_2z_2 + b_3z_3 \\ c_1x_1 + c_2x_2 + c_3x_3 & c_1y_1 + c_2y_2 + c_3y_3 & c_1z_1 + c_2z_2 + c_3z_3 \end{vmatrix}$$

8.11 Differential Coefficient of Determinant

Let

$$y = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x)] \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix},$$

where $f_i(x), g_i(x), h_i(x), i = 1, 2, 3$ are differentiable functions of x.

Now, $y = f_1(x)[g_2(x)h_3(x) - g_3(x)h_2(x)] - f_2(x)[g_1(x)h_3(x) - g_3(x)h_1(x)] + f_3(x)[g_1(x)h_2(x) - g_2(x)h_1(x)]$

$\therefore \frac{dy}{dx} = f_1'(x)[g_2(x)h_3(x) - g_3(x)h_2(x)] + f_1(x)[g_2'(x)h_3(x) - g_3'(x)h_2(x) + g_2(x)h_3'(x) - g_3(x)h_2'(x)] + -f_2'(x)[g_1(x)h_3(x) - g_3(x)h_1(x)] + -f_2(x)[g_1'(x)h_3(x) - g_1(x)h_3'(x) + g_1(x)h_3'(x) - g_3(x)h_3'(x)] + f_3'(x)[g_1(x)h_2(x) - g_2(x)h_1(x)] + f_3(x)[g_1'(x)h_2(x) - g_2'(x)h_1(x)x + g_1(x)h_2'(x) - g_2(x)h_1'(x)]$

$$= \begin{vmatrix} f_1'(x) & f_2'(x) & f_1'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

8.12 Problems

1. Evaluate $\begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix}$.

2. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

3. Evaluate $\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix}$ making use of relations between 2nd and 3rd column.

4. Evaluate $\begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix}$.

5. Evaluate $\begin{vmatrix} 18 & 1 & 17 \\ 22 & 3 & 19 \\ 26 & 5 & 21 \end{vmatrix}$.

6. Evaluate $\begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix}$.

7. Evaluate $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$.

8. Let a, b, c be positive and unequal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

9. Evaluate $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$.

10. Evaluate $\begin{vmatrix} 1+a_1 & a_2 & a_3 \\ a_1 & 1+a_2 & a_3 \\ a_1 & a_2 & 1+a_3 \end{vmatrix}$.

11. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$.

12. Show that $\begin{vmatrix} a-b+c & a+b-c & a-b-c \\ b-c+a & b+c-a & b-c-a \\ c-a+b & c+a-b & c-a-b \end{vmatrix} = 4(a^3+b^3+c^3-3abc)$.

13. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

14. Prove that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$.

15. Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$.

16. Prove that $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$.

17. If x,y,z are all different and if $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, prove that $xyz = -1$.

18. Evaluate $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$.

19. Show that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$.

20. Solve the equation $\begin{vmatrix} 15-x & 1 & 10 \\ 11-3x & 1 & 16 \\ 7-x & 1 & 13 \end{vmatrix} = 0$.

21. If $a+b+c=0$, solve the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$.

22. If $D_1 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$, $D_2 = \begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix}$ and $d = tx$, $e = hy$, $f = tz$, prove without expanding that $D_1 = -tD_2$

23. Show without expanding thet $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$.

24. If a, b, c are positive and are the pth, qth, rth terms of a G.P., respectively, then show without expanding thet $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$.

25. Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$.

26. Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$.

27. Evaluate $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$.

28. Evaluate $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$.

29. Evaluate $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$.

30. Prove that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & c+a \end{vmatrix} = 0$.

31. If a, b, c are the pth, qth, rth terms respectively of an H.P., show that $\begin{vmatrix} bc & p & 1 \\ ca & q & 1 \\ ab & r & 1 \end{vmatrix} = 0$.

32. If $\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & 1-2x & x-4 \\ x-2 & x+4 & 3x \end{vmatrix} = px^4 + qx^3 + rx^2 + sx + t$ be an indentity in x, where p, q, r, s and t are constants, find the value of t.

33. Prove that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$.

34. If a, b, c are in A.P., show that $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$.

35. If ω is a complex cube root of unity, prove that $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = 0$

36. Evaluate $\begin{vmatrix} k & k & k \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$.

37. Evaluate $\begin{vmatrix} a^2+x & b^2 & c^2 \\ a^2 & b^2+x & c^2 \\ a^2 & b^2 & c^2+x \end{vmatrix}$.

38. Evaluate $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

39. Evaluate $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$.

40. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = -2(a^3+b^3+c^3-3abc)$.

41. Show that $\begin{vmatrix} x+a & x+b & x+c \\ y+a & y+b & y+c \\ z+a & z+b & z+c \end{vmatrix} = 0$.

42. Show that $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$.

43. Show that $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$

44. Show that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$ and $(a+b+c)$ have the same sign.

45. Show that $\begin{vmatrix} bc & bc'+b'c & b'c' \\ ca & ca'+c'a & c'a' \\ ab & ab'+a'b & a'b' \end{vmatrix} = (ab'-a'b)(b'c-bc')(a'c-c'a)$.

46. Evaluate $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$.

47. Show that $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(ab+bc+ca)^3$.

48. Show that $\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3$.

49. Show that $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$.

Solve the following equations:

50. $\begin{vmatrix} a & a & x \\ a & a & a \\ b & x & b \end{vmatrix} = 0$.

51. $\begin{vmatrix} x & 2 & 3 \\ 6 & x+4 & 4 \\ 7 & 8 & x+8 \end{vmatrix} = 0$.

52. $\begin{vmatrix} x & 2 & 3 \\ 4 & x & 1 \\ x & 2 & 5 \end{vmatrix} = 0$.

53. $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$.

54. $\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0$.

Show without expanding at any stage that:

55. $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$

56. $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$

57. $\begin{vmatrix} 1 & \cos\alpha - \sin\alpha & \cos\alpha + \sin\alpha \\ 1 & \cos\beta - \sin\beta & \cos\beta + \sin\beta \\ 1 & \cos\gamma - \sin\gamma & \cos\gamma + \sin\gamma \end{vmatrix} = 2\begin{vmatrix} 1 & \cos\alpha & \sin\alpha \\ 1 & \cos\beta & \sin\beta \\ 1 & \cos\gamma & \sin\gamma \end{vmatrix}.$

58. $\begin{vmatrix} (a-1)^2 & a^2+1 & a \\ (b-1)^2 & b^2+1 & b \\ (c-1)^2 & c^2+1 & c \end{vmatrix} = 0$

59. $\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix} = 0.$

60. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

61. $\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}.$

62. $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}.$

63. find the value of the following determinant $\Delta = \begin{vmatrix} m! & (m+1)! & (m+2)! \\ (m+1)! & (m+2)! & (m+3)! \\ (m+2)! & (m+3)! & (m+4)! \end{vmatrix}$, then prove that $\frac{\Delta}{(m!)^3} - 4$ is divisible by m.

64. Solve the following system of equations using Cramer'r rule: $x+y=4$, $2x-3y=9$.

65. Solve the following system of equations using Cramer'r rule: $2x-y+3z=9$, $x+y+z=6$, $x-y+z=2$.

66. Determine the nature of solution for the equations: $2x+3y=6$, $4x+6y=10$.

67. Show that the following system of equations is consistent $x+y-z=1$, $2x+3x+z=4$, $4x+3y+z=16$.

68. Determine the nature of solution for the equations: $x+y=2$, $2x+2y=4$.

69. Determine whether the following system of equations is consistent: $2x + y = 13$, $6x + 3y = 18$, $x - y = -3$.

70. Show that the system of following euqations has non-trivial solutions: $x + y - 6z = 0$, $3x - y - 2x = 0$, $x - y + 2x = 0$.

71. For what value of k the following system of equations possess non-trivial solution. Also, find all the solutions of the system for that value of k, $x + y - kz = 0$, $3x - y - 2x = 0$, $x - y + 2x = 0$.

Solve the following equations by Cramer's rule:

72. $x - 2y = 0;\ 7x + 6y = 40.$

73. $x + y + z = 9;\ 3x + 2y - 3z = 0;\ z - x = 2.$

74. $x - y + z = 0;\ 2x + 3y - 5z = 7;\ 3x - 4y + 2z = -1.$

75. $2x + 3y - 3z = 0;\ 5x - 2y + 2z = 19;\ x + 7y - 5z = 5.$

76. $x + y + z = 1;\ ax + by + cz = k;\ a^2x + b^2y + c^2z = k^2$ where $a \neq b \neq c$.

Determine whether the following system of equations have no solution, unique solution or infinite number of solution:

77. $3x + 9y = 5;\ 9x + 27y = 10.$

78. $5x - 3y = 3;\ x + y = 7.$

79. $x + 2y = 5;\ 3x + 6y = 15.$

80. $2x + 3y + z = 5;\ 3x + y + 5z = 7;\ x + 4y - 2z = 3.$

81. $x + y - z = -2;\ 6x + 4y + 6z = 26;\ 2x + 7y + 4z = 31.$

82. Find the value of k such that following system of equations possess a non-trivial solution over the set of rationals Q. For that value of k find all the solutions of the system: $x + ky_3z = 0;\ x + ky - 2z = 0;\ 2x + 3y - 4z = 0.$

83. If a, b, c are different, show that the following system of equations has non-trivial solutions only when $a + b + c = 0$, $ax + by + cz = 0;\ bx + cy + az = 0;\ cz + ay + bz = 0.$

84. For what value of λ the following system of equations has non-trivial solutions: $3x - y + 4z = 0;\ x_2y - 3z = 0;\ 6x + 5y - \lambda z = 0.$

85. Let the three digit numbers $A28, 3B9, 62C$, where A, B, C are integers between 0 and 9, be divisible by a fixed integer k, show that the determinant $\begin{vmatrix} A & 2 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

86. Evaluate $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$

87. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the values of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$.

88. Show that $\begin{vmatrix} (x-a)^2 & b^2 & c^2 \\ a^2 & (x-b)^2 & c^2 \\ a^2 & b^2 & (x-c)^2 \end{vmatrix} = x^2(x-2a)(x-2b)(x-2c)$

$\left(x + \frac{a^2}{x-2a} + \frac{b^2}{x-2b} + \frac{c^2}{x-2c}\right)$.

89. If $a > 0, d > 0$, find the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}.$$

90. Show that $\begin{vmatrix} \frac{1}{a+x} & \frac{1}{a+y} & \frac{1}{a+z} \\ \frac{1}{a+y} & \frac{1}{b+y} & \frac{1}{b+z} \\ \frac{1}{c+x} & \frac{1}{c+y} & \frac{1}{c+z} \end{vmatrix} =$

$$\frac{(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)}{(a+x)(b+x)(c+x)(b+x)(b+y)(b+z)(c+x)(c+y)(c+z)}.$$

91. If $2s = a + b + c$, show that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & s^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & s^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c).$$

92. Show that

$$\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} =$$
$$(x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz).$$

93. Find the value of θ between 0 and $\pi/2$ and satisfying the equation:

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin\theta \end{vmatrix} = 0.$$

94. If $a^2 + b^2 + c^2 = 1$, then prove that

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ab(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & bc(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix} = \cos^2\phi.$$

95. If none of the a,b,c is zero, show that $\begin{vmatrix} -bc & b^2+ac & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$.

96. If u, v are functions of x, and $y = \frac{u}{v}$, show that $v^3 \frac{d^2x}{dy^2} = \begin{vmatrix} u & v & 0 \\ u' & b' & v \\ u'' & v'' & 2v' \end{vmatrix}$ where primes denote derivatives.

97. If $a \neq 0$ and $a \neq 1$, show that $\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = a^3\Big[1 + \frac{x(a^3-1)}{a^2(a-1)}\Big]$.

98. If $p+q+r=0$, prove that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

99. Show without expanding that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$.

100. Show without expanding that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = aA+B$, where A and B are determinants of 3rd order not involving x.

101. If $D_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & \frac{n(3n-1)}{2} \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$ show that $\displaystyle\sum_{r=1}^{n} D_r = 0$.

102. Without expanding the determinant, show that the value of $\begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$ is real.

103. Prove that $\begin{vmatrix} -2a & a+b & b+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$.

104. $f_r(x)$, $g_r(x)$, $h_r(x)$, where $r = 1,2,3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$ and

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

then find $F'(x)$.

105. Let α be a repeated root of a quadratic equation $f(x)=0$ and $A(x), B(x), C(x)$ be polynomials of degree 3,4,5 respectively. Show that $\Delta(x)=\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where prime denotes a derivative.

106. Prove that $\begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$ is independent of θ.

107. If f, g, h are differential functions of x and $\Delta=\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$ prove that $\Delta'=\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3y'')' & (x^3h'')' \end{vmatrix}$

108. If $f(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then show that $\frac{d^nf(x)}{dx^n}=0$, where $x=0$.

109. Prove that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(Q-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}=0.$

110. Prove that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2bc-b^2 & a^2 \\ b^2 & a^2 & 2bc-c^2 \end{vmatrix}=(a^3+b^3+c^3-3abc)^2.$

111. Prove that $\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}=0.$

112. For what value of m does the system of equation $3x+my=m$ and $2x-5y=20$ has a solution satisfying the conditions $x>0$, $y>0$.

113. Prove that the system of equation $3x-y+4z=0$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ has at least one solution for any real λ. Find the set of solutions when $\lambda=-5$.

114. For what value of p and q, the system of equations $2x+py+6z=8$, $x+2y+qz=5$, $x+y+3z=4$ has (a) no solution (b) a unique solution, and (c) infinite solutions.

115. Let λ and α be real. Find the set of all values of λ for which the system of equations: $\lambda x+y\sin\alpha-z\cos\alpha=0$, $x+y\cos\alpha+z\sin\alpha=0$, $-x+y\sin\alpha-z\cos\alpha=0$.

116. Evaluate $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

117. Evaluate $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$.

118. Evaluate $\begin{vmatrix} x & x(x^2+1) & x+1 \\ y & y(y^2+1) & y+1 \\ z & z(z^2+1) & z+1 \end{vmatrix}$.

119. If x, y, z are respectively lth, $2m$th, $3n$th terms of an H.P., then find the value of $\begin{vmatrix} yz & zx & xy \\ l & 2m & 3n \\ 1 & 1 & 1 \end{vmatrix}$.

120. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

121. Evaluate $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$.

122. Prove that $\begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$.

123. If $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$ are three 3 digit numbers such that each of them is divisible by k, then prove that the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is divisible by k.

124. If a_i, b_i, $c_i \in \mathbb{R}(i=1,\ 2,\ 3)$ and $x \in R$, show that $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

125. If a, b, c are the roots of the equation $px^3+qx^2+rx+s=0$, then find the value of $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$.

126. If $a<b<c$, prove that $\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} > 0$.

127. If a, b, c are distinct and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, show that $abc(ab+bc+ca) = a+b+c$.

128. Show that $x_1, x_2, x_3 \neq 0$, $\begin{vmatrix} x_1+a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2+a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3+a_3b_3 \end{vmatrix} = x_1x_2x_3$

$\left(1+\frac{a_1b_1}{x}+\frac{a_2b_2}{x}+\frac{a_3b_3}{x_3}\right)$.

129. Show that $\begin{vmatrix} \frac{1}{a+x} & \frac{1}{a+y} & 1 \\ \frac{1}{b+x} & \frac{1}{b+y} & 1 \\ \frac{1}{c+x} & \frac{1}{c+y} & 1 \end{vmatrix} = \frac{(a-b)(b-c)(c-a)(x-y)}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)}$.

130. Show that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

131. Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$.

132. If a, b, c are sides of a triangle, show that $\begin{vmatrix} a^2 & (s-a)^2 & (s-b)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = \frac{1}{2}P^2A^2$, where P denotes the perimeter of the triangle, A its area and $s = \frac{P}{2}$.

133. Show that $\begin{vmatrix} (x-a)^2 & ab & ac \\ ba & (x-b)^2 & bc \\ ca & cb & (x-c)^2 \end{vmatrix} = x^2(x-2a)(x-2b)(x-2c)$

$\left(x+\frac{a^2}{x-2a}+\frac{b^2}{x-2b}+\frac{c^2}{x-2c}\right)$.

134. If x, y, z are unequal and $\begin{vmatrix} x^3 & (x+a)^3 & (x-a)^3 \\ y^3 & (y+a)^3 & (y-a)^3 \\ z^3 & (z+a)^3 & (z-z)^3 \end{vmatrix} = 0$, prove that $a^2(x+y+z) = 3xyz$.

135. Show that $\begin{vmatrix} (1-x) & a & a^2 \\ a & a^2-x & a^3 \\ a^2 & a^3 & a^4-x \end{vmatrix} = x^2(1+a^2+a^3)-x^3$.

136. If $y = \sin px$ and $y_n = \frac{d^nx}{dy^n}$, find the value of $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$.

137. Evaluate $\begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$.

138. Evaluate $\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin\beta \\ 0 & -\sin\beta & \cos\beta \end{vmatrix}$.

139. Solve the equation $\begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix} = 0$.

140. Solve the equation for x, $\begin{vmatrix} C_r^x & C_r^{n-1} & C_{r-1}^{n-1} \\ C_r^{x+1} & C_r^n & C_{r-1}^n \\ C_r^{x+2} & C_r^{n+1} & C_{r-1}^{n+1} \end{vmatrix} = 0\ \forall n,r > 1$.

141. Solve the equation $\begin{vmatrix} u+a^2x & w'+abx & v'+acx \\ w'+abx & v+b^2x & u'+bcx \\ v'+acx & u'+bcx & w+c^2x \end{vmatrix} = 0$ expressing the result by means of determinants.

142. If $f(a,b) = \frac{f(b)-f(a)}{b-a}$ and $f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a}$, show that

$$f(a,b,c) = \begin{vmatrix} f(a) & f(b) & f(c) \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

143. If A, B, C are the angles of a $\triangle ABC$, then prove that $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$ is purely real.

144. If A, B, C are the angles of a $\triangle ABC$ such that $A \geq B \geq C$, find the minimum value of Δ, where $\Delta = \begin{vmatrix} \sin^2 A & \sin A\cos A & \cos^2 A \\ \sin^2 B & \sin B\cos B & \cos^2 B \\ \sin^2 C & \sin C\cos C & \cos^2 C \end{vmatrix}$. Also, show that $\Delta = \frac{1}{4}[\sin(2A-2B) + \sin(2B-2C) + \sin(2C-2A)]$.

145. Evaluate $\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$.

146. If $0 < x < \frac{\pi}{2}$, the find the values of x for which $\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ has maximum value.

147. If A, B, C are the angles of a triangle, show that $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0.$

148. If A, B, C are the angles of an isoscceles triangle, evaluate

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}.$$

149. For positive numbers $x, y, z \neq 1$, show that the numeric value of the determinant

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0.$$

150. If $a, b, c > 0$ and $x, y, z \in \mathbb{R}$, then show without expanding that

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix} = 0.$$

151. Without expanding the determinants, prove that $\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix} = 0.$

152. Evaluate $\sum_{n=1}^{N} U_n$ if $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}.$

153. If A, B,C are the angles of a triangle, then show without expanding that

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix} = 0.$$

154. Evaluate without expanding $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

155. Let $\Delta_i = \begin{vmatrix} i-1 & n & 6 \\ (i-1)^2 & 2n^2 & 4n-2 \\ (i-1)^3 & 3n^3 & 3n^2-2n \end{vmatrix}$. Show that $\sum_{n=1}^{n} \Delta_i = k$, a constant.

156. Let $m \in \mathbb{P}$ and $\Delta_r = \begin{vmatrix} 2r-1 & {}^mC_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2 m^2 & \sin^2 m & \sin^2(m+1) \end{vmatrix}$, then find the value of $\sum_{r=0}^{m} \Delta_r$.

157. Show that $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+1} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+1} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+1} \end{vmatrix}$

158. If $\Delta_r = \begin{vmatrix} r & n+1 & 1 \\ r^2 & 2n-1 & \frac{2n+1}{3} \\ r^3 & 3n+2 & \frac{n(n+1)}{2} \end{vmatrix}$, show that $\sum_{r=1}^{n} \Delta_r = 0$.

159. If $\Delta_r = \begin{vmatrix} 2^{r-1} & 2.3^{r-1} & 4.5^{r-1} \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$, show that $\sum_{r=1}^{n} \Delta_r = 0$.

160. Show without expanding that $\begin{vmatrix} x^2 & (x-1)^2 & (x-2)^2 \\ (x-1)^2 & (x-2)^2 & (x-3)^2 \\ (x-2)^2 & (x-3)^2 & (x-4)^2 \end{vmatrix}$ is independent of x.

161. Show without expanding that $\begin{vmatrix} 2 & 1+i & 3 \\ 1-i & 0 & 2+i \\ 3 & 2-i & 1 \end{vmatrix}$ is purely real.

162. Show without expanding that $\begin{vmatrix} x-3 & 2x+1 & 2 \\ 3x+2 & x+2 & 1 \\ 5x+1 & 5x+4 & 5 \end{vmatrix}$ is independent of x.

163. If a and x are real numbers and n is a positive integer, then show without expanding that $\begin{vmatrix} a^n-x & a^{n+1}-x & a^{n+2}-x \\ a^{n+3}-x & a^{n+4}-x & a^{n+5}-x \\ a^{n+6}-x & a^{n+7}-x & a^{n+8}-x \end{vmatrix} = 0$.

164. Find $\sum_{r=2}^{n} (-2)^r \begin{vmatrix} C_{r-2}^{n-2} & C_{r-1}^{n-2} & C_r^{n-2} \\ -3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$, $n > 2$.

165. If a, b, c are non-zero real numbers, show without expanding that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$.

166. Prove that $\begin{vmatrix} b+c-a-d & bc-ad & bc(a+d)-ad(b+d) \\ c+a-b-d & ca-bd & ca(b+d)-bd(c+a) \\ a+b-c-d & ab-cd & ab(c+d)-cd(a+b) \end{vmatrix} = -2(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$.

167. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca+ab-bc & bc+ab-ca & bc+ca-ab \\ (a+b)(a+c) & (b+c)(b+a) & (c+a)(c+b) \end{vmatrix} = 3(b-c)(c-a)(a-b)(a+b+c)(ab+bc+ca)$x.

168. Prove that $\begin{vmatrix} 1 & (m+n-l-p)^2 & (m+n-l-p)^4 \\ 1 & (n+l-m-p)^2 & (n+l-m-p)^4 \\ 1 & (l+m-n-p)^2 & (l+m-n-p)^4 \end{vmatrix} = 64(l-m)(l-n)(l-p)(m-n)(m-p)(n-p)$.

169. If u, v, w are differentiable functions of f and suffixes denote the derivatives w.r..t t, prove that $\frac{d}{dt}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$.

170. If $Y = sX$ and $Z = tX$, all the variables being differentiable functions of x, prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$, where suffixes denote the derivatives w.r.t. x.

171. If $f(x), g(x), h(x)$ are polynomials in x, find the condition that $\begin{vmatrix} f(x) & g(x) & h(x) \\ f(\alpha) & g(\alpha) & h(\alpha) \\ f(\beta) & g(\beta) & h(\beta) \end{vmatrix}$, which is a polynomial of degree 3, is expressible as $a(x-\alpha)^2(x-\beta)$.

172. Show that $\begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a + x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b + x\sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & c + x\sin\gamma \end{vmatrix}$ is independent of x.

173. If $f(x) = \begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$, show that $\int_0^{\frac{\pi}{2}} [f(x) + f'(x)]\,dx = \pi$.

174. Prove that $\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0$.

175. If $l_r\vec{i}$, $m_r\vec{j}$, $n_r\vec{k}$, $r = 1, 2, 3$ be three mutually perpendicular unit vectors, show that $\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \pm 1$.

176. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_i, B_i, C_i be the cofactors of a_i, b_i, c_i respectively and $\alpha_i, \beta_i, \gamma_i$ be the cofactors of A_i, B_i, C_i respectively, where $i = 1, 2, 3$, show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \Delta^6$

177. Using determinants, solve the equations: $x + 2y + 3z = 6, 2x + 4y + z = 17, 3x + 2y + 9z = 2$.

178. Solve the system of equations $ax + by + ca = d, a^2x + b^2y + c^2a = d^2, a^3x + b^3y + c^3a = d^3$. Will the solution always exist and be unique?

179. Determine the coefficients a, b, c of the quadratic function where $f(x) = ax^2 + bx + c$, if $f(1) = 0$, $f(2) = -2$ and $f(3) = -6$.

180. Determine the coefficients a, b, c of the quadratic function where $f(x) = ax^2 + bx + c$, if $f(0) = 6$, $f(2) = 11$, $f(-3) = 6$. Also, find $f(1)$.

181. Solve $(b+c)(y+z)-ax = b-c, (c+a)(z+x)-by = c-a, (a+b)(x+y)-cz = a-b$, where $a + b + c \neq 0$.

182. Examine the consistency of the system of equations $7x - 7y + 5z = 3, 3x + y + 5z = 7$ and $2x + 3y + 5z = 5$.

183. Find the value of k for which the following system of equations is consistent $x + y = 3, (1 + k)x + (2 + k)y = 8, x - (1 + k)y + (2 + k) = 0$.

184. Find the value of k for which the following system of equations is consistent $(k+1)^3x + (k + 2)^3y = (k + 1)^3, (k + 1)x + (k + 2)y = k + 3, x + y = 1$.

Chapter 9

Matrices

Matrices are an important concept which has numerous real life usage in various mathematical branches. Also, it has huge importance in modern computer science. It has its applications in computer graphics, artificial intelligence, data structures leading to various clever algorithms. Thus, it is of paramount importance that the reader understand this particular concept in a sound manner.

Definition: A *matrix* is a rectangular array of real or complex numbers. This rectangular array is made up of rows and columns much like determinants. Let us consider a matrix of $m \times n$ symbols, where m is number of rows and :n is the number of columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Such a matrix is called m by n matrix or a matrix of order $m \times n$. Sometimes a matrix is shown with parenthese instead of square brackets as shown in last example.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A compact way to write a matrix is $A = [a_{ij}], 1 \leq i \leq m; 1 \leq j \leq n$ or simply $[a_{ij}]_{m \times n}\, a_{ij}$ is an element located at i^{th} row and j^{th} column and is called $(i, j)^{th}$ element of the matrix. A matrix is just a rectangular array of numbers and unlike determinants it does not have a value.

9.1 Classification of Matrices

9.1.1 Equal Matrices

Two matrices are said to be *equal* if they have same order and each corresponding element is equal.

9.1.2 Row Matrix

A matrix having a single row is called a *row* matrix. For example, $[1,2, 3, 4]$.

9.1.3 Column Matrix

A matrix having a single column is called a *column* matrix. For example,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

9.1.4 Square Matrix

If $m = n$ i.e number of rows and columns are equal then the matrix is called a *square* matrix. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is a 3×3 matrix.

9.1.5 Diagonal Matrix

The diagonal from left-hand side upper corner to right-hand side lower corner is known as leading diagonal or principal diagonal. In the example of square matrix the elements of diagonal are $1, 5, 9$. When a matrix has all elements as zero except those belonging to its diagonal, then it is called a *diagonal* matrix. Equivalently, We can say that a matrix $[a_{ij}]_{m\times n}$ is a diagonal matrix if $a_{ij} = 0 \;\forall\; i \neq j$. For example, the square matrix example can be converted to a diagonal matrix like below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

For an $n \times n$ matrix the diagonal elements are represented as $[d_1, d_2 \ldots, d_n]$ This diagonal is also written with a *diag* prefix like $diag[d_1, d_2 \ldots, d_n]$.

9.1.6 Scalar Matrix

A diagonal matrix whose elements of the diagonal are equal is called *scalar* matrix. For example:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

For a square matrix $[a_{ij}]_{m\times n}$ to be a scalar matrix:

$$a_{ij} = \begin{cases} 0, & i \neq j \\ m, & i = j \end{cases} \;\forall\; m \neq 0$$

9.1.7 Unit Matrix or Identity Matrix

A diagonal matrix of order n, which has all elements of its diagonal as one, is called a *unit* or *identity* matrix. It is also denoted by I_n. We can rewrite it in concise way like we did for scalar matrix as

$$a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

9.1.8 Horizontal Matrix

An $m \times n$ matrix is called a *horizontal* matrix if $m < n$. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

9.1.9 Vertical Matrix

An $m \times n$ matrix is called a *vertical* matrix if $m > n$. For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

9.1.10 Triangular Matrix

A sqaure matrix in which all the elements below the diagonal are zero is called *upper triangular* matrix. Conversely, a sqaure matrix in which all the elements above the diagonal matrix is called *lower triangular* matrix. Thus, for a lower triangular matrix $a_{ij} = 0$ when $i < j$ and for an upper triangular matrix $a_{ij} = 0$ when $i > j$

Clearly, a diagonal matrix is both lower and upper triangular matrix. A triangular matrix is called strictly triangular if $a_{ii} = 0 \ \forall \ 1 \leq i \leq n$. Example of upper triangular matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

Example of lower triangular matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

9.1.11 Null or Zero Matrix

If all elements of a matrix is zero then it is a *null* or *zero* matrix.

9.1.12 Singular and Non-Singular Matrix

A matrix is said to be *non-singular* if $|A| \neq 0$ and s*ingular* if $|A| = 0$.

9.1.13 Trace of Matrix

If sum of the elements of a sqaure matrix A lying along the principal diagonal is called the *trace* of A, i.e. $tr(A)$. Thus, if $A = [a_{ij}]_{n \times n}$, then $tr(A) = \sum_{i=1}^{n} a_{ii}$

9.1.14 Properties of Trace of a Matrix

To prove the second and third properties of a trace of matrix we will have to use properties given further below on algebraic operations on a matrix. If $A = [a_{ii}]_{n\times n}$ and $B = [b_{ii}]_{n\times n}$ and λ is a scalar then

1. $tr(\lambda A) = \lambda tr(A)$
2. $tr(A + B) = tr(A) + tr(B)$
3. $tr(AB) = tr(BA)$

9.1.15 Determinant of a Matrix

Every square matrix A has a determinant associated with it. This is written as $det(A)$ or $|A|$ or Δ. We observe following for determinants of matrices:

1. If A_1, A_2, ..., A_n are square matrices of the same order then $|A_1A_2 \ldots A_n| = |A_1||A_2| \ldots |A_n|$.
2. If k is a scalar, then $|kA| = k^n|A|$, where n is the order of matrix.
3. If A and B are two matrices of equal order then $|AB| = |BA|$ even though $AB \neq BA$.

9.2 Algebra of Matrices

9.2.1 Addition of Matrices

If any two matrices are of same order then addition of those can be performed. The result is a matrix of same order with corresponding elements added. For example, consider two 3×3 matrices as given below:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}$$

then,

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ a_4 + b_4 & a_5 + b_5 & a_6 + b_6 \\ a_7 + b_7 & a_8 + b_8 & a_9 + b_9 \end{bmatrix}$$

9.2.2 Subtraction of Matrices

The conditions are same for subtraction to happen i.e. order of the matrices must be same. The result is like that of addition with resulting elements being the difference of original matrices. For example,

$$A - B = \begin{bmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ a_4 - b_4 & a_5 - b_5 & a_6 - b_6 \\ a_7 - b_7 & a_8 - b_8 & a_9 - b_9 \end{bmatrix}$$

where A and B are matrices from previous example. Following is observed for addition and subtraction:

1. Addition of matrices is commutative i.e. $A+B=B+A$ as well as associative i.e. $(A+B)+C=A+(B+C)$.
2. Cancellation laws are true in case of addition.
3. The equation $A+B=O$ has a unique solution in the set of all $m \times n$ matrices (where O is null matrix).

9.2.3 Scalar Multiplication

The scalar multiplication of a matrix A with a scalar λ is defined as $\lambda A = [\lambda a_{ij}]$.

9.2.4 Multiplication of two Matrices

The prerequisite for matrix multiplication is that number of columns of first matrix must be equal to number of rows of second matrix. The product is defined as

$$A_{m \times n} B_{n \times p} = \sum_{r=1}^{n} a_{mr} b_{rp}$$

It can be easily verified that the resulting matrix will have m rows and p columns.

A Properties of Matrix Multiplication

1. Commutative laws does not hold always for matrices.
2. If $AB = BA$, then they are called commutative matrices.
3. If $AB = -BA$, then they are called anti-commutative matrices.
4. Matrix multiplication is associative i.e. $(AB)C = A(BC)$. Proof of this has been left as an exercise.
5. Matrix mulplication is distributive wrt addition and subtraction i.e. $A(B \pm C) = AB \pm AC$.

9.2.5 Transpose of a Matrix

Let A be any matrix then its *transpose* can be obtained by ecxchanging rows and columns. It is denoted by A' or A^T and clearly, if order of A is $m \times n$ then A' will have order of $n \times m$.

A Properties of Transpose Matrices

1. $(A+B)' = A' + B'$.
2. $(A')' = A$.

3. $(kA)' = kA'$ where k is a constant.
4. $(AB)' = B'A'$.

Proofs of these properties are simple and have been left as an exercise.

9.2.6 Symmetric Matrix

A sqaure matrix $A = [a_{ij}]$ is called a *symmetric* matrix if $a_{ij} = a_{ji}\ \forall\ i,j$. We can also say thet a matrix is symmetric if and only if $A = A'$.

9.2.7 Skew Symmetric Matrix

A square matrix A is said to be a *skew symmetric* matrix if $a_{ij} = -a_{ji}\ \forall i, j$. Clearly, if a matrix is skew symmetric then elements of its diagonal are all zeros.

9.2.8 Orthogonal Matrix

A matrix is said to be orthogonal if $AA' = 1$.

Theorem 16

If A is a square matrix then $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Proof

$(A + A')' = A' + (A')' = A' + A$. Hence, $A + A'$ is a symmetric matrix. $(A - A')' = A' - A = -(A - A')$. Hence, $A - A'$ is a skew symmetric matrix. □

Theorem 17

Every square matrix can be shown as sum of a symetric matrix and a skew symmetric matrix.

Proof

Let A be any square matrix. $\frac{1}{2}(A + A') + \frac{1}{2}(A - A') = A$ hus, the matrix A is a sum of symmetrix matrix $A + A'$ and a skew symmetric matrix $A - A'$ □

9.2.9 Adjoint of a Matrix

Let $A = [a_{ij}]$ be a square matrix. Let $B = [A_{ij}]$ where A_{ij} is the cofactor of the element a_{ij} in the det. A. The transpose B' of the matrix B is called the adjoint of the matrix A and is written by $adj.A$. For example,

Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$

$$adj.A = B' = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & -4 & -1 \end{bmatrix}$$

$$A.adj(A) = adj(A).A = |A|I_n$$

9.2.10 Inverse of a Matrix

Following from above, inverse of a matrix is $\frac{adj(A)}{|A|}$. Inverse of a matrix A is denoted by A^{-1}.

9.2.11 Hermitian and Skew Hermitian Matrix

A sqaure matrix $A = [a_{ij}]$ is said to be a *Hermitian* matrix if $a_{ij} = \overline{a_{ij}}\ \forall\ i, j$ i.e. $A = A^{\theta}$. For example,

$$\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$$

is a Hermitian matrix.

Similarly, a sqaure matrix $A = [a_{ij}]$ is said to be a *skew Hermitian* matrix if $a_{ij} = \overline{a_{ji}}\ \forall\ i, j$ i.e. $A = -A^{\theta}$. For example,

$$\begin{bmatrix} 0 & -b+ic \\ b+ic & 0 \end{bmatrix}$$

is a skew Hermitian matrix. Following are observed for these types of matrices:

1. If A is a hermitian matrix, then $a_{ii} = \overline{a_{ii}} \Rightarrow a_{ii}$ is real, $\forall\ i$. Thus, members of diagonal of a Hermitian matrix are all real.

2. A Hermitian matrix over the set of real numbers is actually a real symmetric matrix.

3. If A is a skew Hermitian matrix, then $a_{ii} = -\overline{(a_{ii})} \Rightarrow a_{ii} = 0$ i.e. a_{ii} must be purely imaginary or zero.

4. A skew Hermitian matrix over the set of real numbers is acually a real skew-symmetric matrix.

9.2.12 Idempotent Matrix

A square matrix A is said to be *idempotent* if $A^2 = A$ i.e. multiplication of the matrix with itself yields itself.

9.2.13 Involuntary Matrix

A sqaure matrix A is said to be *involuntary* if $A^2 = I$ i.e. multiplication of the matrix with itself yields an indetity matrix.

9.2.14 Nilpotent Matrix

For a positive integer i if a square matrix satisfied the relationship $A^i = O$ then it is called a *nilpotent* matrix. Such smallest integer is called index of the nilpotent matrix.

9.3 Properties of adjoint and inverse matrices

1. If A is a sqaure matrix of order n, then $A(adj(A)) = |A|I_n = (adj(A))A$.

   Let $A = [a_{ij}]$, and let C_{ij} be a cofactor of a_{ij} in A. Then, $(adj(A)) = C_{ji}\ \forall\ 1 \le i, j \le n$. Now,

$$(A\ adj(A)) = \sum_{r=1}^{n}(A)_{ir}(adj(A))_{rj}$$

$$= \sum_{r=1}^{n} a_{ir}C_{rj} = \begin{cases} |A|, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$$\Rightarrow = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0\dots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}$$

$$= |A|I_n$$

   Similarly,

$$(adj(A)A)_{ij} = \sum_{r=1}^{n}(adj(A))_{ir}A_{rj}$$

$$= \sum_{r=1}^{n} C_{ri}a_{rj} = \begin{cases} |A|, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

2. Every invertible matrix possesses a unique matrix. Let A be a sqaure matrix of order $n \times n$. Let B and C be two inverses of A. Then, $AB = BA = I_n$ and $AC = CA = I_n$

$$AB = I_n \Rightarrow C(AB) = CI_n \Rightarrow (CA)B = CI_n \Rightarrow I_nB = CI_n$$

$$\Rightarrow B = C$$

3. Reversal law: If A and B are invertible matrices of same oreder, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. In general, if A,B, C, ... are invertible matrices then $(ABC\dots)^n = \dots C^{-1}B^{-1}A^{-1}$

   If the given matrices are invertible $|A| \neq 0$ and $|B| \neq 0 \Rightarrow |A||B| \neq 0$ Hence, AB is an invertible matrix. Now,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= A(I_n)A^{-1} = AA^{-1} = I_n$$

   Similarly,

$$(B^{-1}A^{-1})(AB) = I_n$$

4. If A is an invertible matrix, then A' is also invertible and $(A')^{-1} = (A^{-1})'$.

   A is an invertible matrix $\therefore |A| \neq 0 \Rightarrow |A'| \neq 0 [\because |A'| = |A|]$. Hence, A' is also invertible. Now,

$$AA^{-1} = I_n = A^{-1}A$$
$$(AA^{-1})' = (A^{-1}A)'$$
$$(A^{-1})'A' = I_n = A'(A^{-1})'$$
$$\Rightarrow (A')^{-1} = (A^{-1})'$$

5. If A is a non-singular square matrix of order n, then $|adjA| = |A|^{n-1}$.

   We have $A(adj(A)) = |A|I_n$

$$A(adj(A)) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \ldots & |A| \end{bmatrix}$$
$$|A(adj(A))| = |A|^n$$
$$|adj(A)| = |A|^{n-1}$$

6. Reversal law for adjoint: If A and B are non-singular sqaure matrices of the same order, then

   $adj(AB) = adj(B)\,adj(A)$ using $(AB)^{-1} = B^{-1}A^{-1}$

7. If A is an invertible square matrix, then $adj(A') = (adj(a))'$

8. If A is a sqaure non-singular matrix, then $adj(adj(A)) = A^{n-2}A$

   We know that $B(adj(B)) = |B|I_n$ for every sqaure matrix of order n. Replacing B by $adj(A)$, we get $(adj(A))[adj(adj(A))] = |adj(A)|I_n = |A|^{n-1}I_n$. Multiplying both sides by A

$$(A\ adj(A))[adj(adj(A))] = A\{|A|^{n-1}I_n\}$$
$$|A|I_n(adj(adj(A))) = |A^{n-1}|(AI_n)$$
$$adj(adj(A)) = |A^{n-2}|A$$

9. If A is a non-singular matrix then $|A^{-1}| == |A|^{-1}$ i.e. $|A^{-1}| = \frac{I}{A}$. Since $|A| \neq 0, \therefore AA^{-1} = I, |AA^{-1}| = |A| \Rightarrow |A||A^{-1}| = 1$

10. Inverse of k^{th} power of A is k^{th} power of the inverse of A.

9.4 Solution of Simultaneous Linear Equations

Consider the system of equations given below:

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n} = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n} = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn} = b_n
\end{cases}
$$

Let

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}
$$

The system of equations can be written as $AX = B \Rightarrow X = A^{-1}B$. If $|A| \neq 0$, the system of equations has only trivial solution and the number of solutions is finite. If $|A| = 0$, the system of equations has non-trivial solution and the number of solutions is infintite. If the number of equations is less than the number of unknonwns then it has non-trivial solutions.

9.5 Elementary Operations/Transformations of a Matrix

Following are elementary operations of a matrix:

1. The interchange of any two rows or columns.
2. The multiplication of any row or column with a non-zero number.
3. The addition to the elements of any row or columns the corresponding elements of any other row or column multiplied with any non-zero number.

Elementary operations are also called row or column operation.

9.5.1 Equivalent Matrices

If a matrix B can be obtained from a matrix A by elementary transformations, then they are called equivalent matrices and are written as $A\ B$.

Every elementary row or column transformation of $m \times n$ matrix (not identiry matrix) can be obtained by pre-multiplcation or post-multiplication with the corresponding elementary matrix obtained from the identity matrix $I_m(I_n)$ by subjecting it to the same elementary row or column transformation.

Let $C = AB$ be a product of two matrices. Any elementray row or column transformation of AB can be obained by subjecting the pre-factor A or post-factor B to the same elementary row or column transformation.

9.5.2 Method of Finding Inverse of a Matrix by Elementary Transformation

Let A be a non-singular matrix of order n. Then A can be reduced to the identity matrix I_n by a sequence of elementary transformations only. As we have discussed every elementary row transformation of a matrix is equivalent top pre-multiplication by the corresponding elementary matrix. Therefore, there exists elementary matrices $E_1, E_2, \dots, E_k$ such that $(E_1,E_2,\dots,E_k)A = I_n$ $(E_1,E_2,\dots,E_k)AA^{-1} = I_nA^{-1}$ $(E_1,E_2,\dots,E_k)I_n = A^{-1}$

9.6 Echelon Form of a Matrix

A matrix is said to be in echelon form if

1. Every row of A which has all its elements 0, occurs below row which has a non-zero element.
2. The first non-zero element in each non-zero row is 1.
3. The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

9.7 Rank of a Matrix

Let A be a matrix of order $m \times n$. If at least one of its minors of order r is different from zero and all minors of order $r+1$ are zero, then the number r is called the rank of the matrix A and is denoted by $\rho(A)$.

1. The rank of a zero matrix is zero and rank of an identity matrix of order n is n.
2. The rank of a non-singular matrix of order n is n.
3. The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

9.8 Application of Matrices to Geometry or Computer Graphics

As said earlier matrices are very useful to represent many operaion in computer graphics or geometry. It will require some knowledge of coordinate geometry.

9.8.1 Reflection Matrix

Consider a point $P(x, y)$ and its reflection $Q(x_1, y_1)$ along x-axis.

This may be written as $x_1 = x + 0; y_1 = 0 - y$. This system of equation can be written in matrix form as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is reflection matrix of a point along x-axis. Similarly, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is reflection matrix along y-axis.

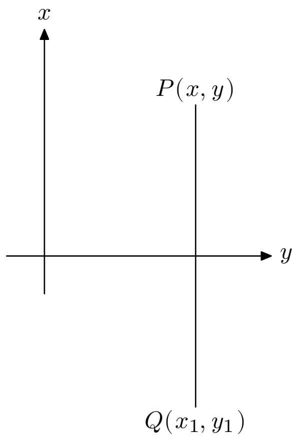


Figure 9.1

Similarly, the reflection matrix through origin is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Similarly, reflection along the line $y = x$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Similarly, reflection along the line $y = x \tan\theta$ is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

9.8.2 Rotation Through an Angle

The rotation matrix in such a form would be $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for anti-clockwise rotation.

9.9 Problems

1. Find the number of matrices having 12 elements.

2. Write down the matrix $A = [a_{ij}]_{2\times 3}$ where $a_{ij} = 2i - 3j$.

3. If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $B = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$, then find $A + B$.

4. If $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, find X.

5. If $\begin{bmatrix} x^2 - 4x & x^2 \\ x^2 & x^3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ x - +2 & 1 \end{bmatrix}$, then find x.

6. Find x, y, z and a for which $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, find $4A - 3B$.

8. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, find AB and BA. Also, show that $AB \neq BA$.

9. If A, B, C are three matrices such that $A = \begin{bmatrix} x & y & z \end{bmatrix}$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then find ABC.

10. Find the transpose and adjoint of the matrix A, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$.

11. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

12. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and verify that $AA^{-1} = 1$.

13. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$, hence obtain A^{-1}.

14. Solve the following equations by matrix method: $5x + 3y + z = 16$, $2x + y + 3z = 19$ and $x + 2y + 4z = 25$.

15. Find the product of two matrices A and B where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it for solving the equations $x + y + 2z = 1, 3x + 2y + z = 7$ and $2x + y + 3z = 2$.

16. If $\begin{bmatrix} x+y & 2 \\ 1 & x-y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$, then find x and y.

17. If $\begin{bmatrix} x-y & 2x+x_1 \\ 2x-y & 3x+y_1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ and co-ordinates of points P and Q be (x, y) and (x_1, y_1), then find PQ.

18. Find X and Y if $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

19. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, find the matrix C such that $A + C = B$.

20. If $A = \begin{bmatrix} 2 & 3 & 4 \\ -3 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & -4 & -5 \\ 1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -1 & 2 \\ 7 & 0 & 3 \end{bmatrix}$, find the matrix X such that $2A + 3B = X + C$.

21. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$, find $A - 2B + 3C$.

22. If $P(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, then show that $P(x).P(y) = P(x+y) = P(y).P(x)$

23. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, find A^2.

24. If $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$, then find A^2B^2.

25. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, find AB and BA and show that $AB \neq BA$.

26. Find the product of the following two matrices: $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$.

27. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, find $A^2 - 5A - 14I$, where I is a unit matrix.

28. Verify that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 4A^2 + A = O$.

29. If $A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$, find A^2.

30. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 5x + 7I$.

31. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$, show that $AB = BA$.

32. Let $f(x) = x^2 - 5x + 6$, find $f(A)$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

33. If the matrix $A = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$, then verify that $A^2 - 12A - I = 0$, where I is a unit matrix.

34. Show that $\left(\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

35. Let $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I, the identity matrix of order 2. Show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

36. Without using the concept of inverse of matrix, find the matrix $\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

37. Find x so that $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$.

38. Prove that the product of two matrices $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a zero matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

39. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then show that $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$, where n is a positive integer.

40. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $A^n = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix}$, where n is a positive integer.

41. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that $(aI + bA)^n = a^n I + na^{n-1}bA$, where I is a unit matrix of order 2 and n is a positive integer.

42. Under what condition is the marix equation $A^2 - B^2 = (A + B)(A - B)$ true?

43. A man buys 8 dozens of mangoes, 10 dozens of apples and 4 dozens of bananas.Mangoes cost USD 18 per dozen, apples 9 per dozen and bananas 6 per dozen. Represent the quntities by a row and a column matrix. Also, find the total cost.

44. A trust fund has USD 30,000 that is to be invested in two different types of bonds. The first bond pays 5% interest per year and second bond pays 7% interest per year. Using matrix multiplication determine how to divide USD 30,000 among the two types of bonds if the turst find must obtain an annual interest of USD 2000.

45. A store has in stock 20 dozen shirts, 15 dozen trousers and 25 dozen pair of socks. If the selling prices are USD 50 per shirt, 90 per trouser and 12 per pair of socks, then find the toal amount store owner will get after selling all the items in the stock.

46. Co-operative store of a particular school has 10 dozen physics books, 8 dozen chemisty books and 5 dozen mathematics books. Their selling prices are USD 8.3, 3.45, 4.5 each respectively. Find the total amount the store owner will receive after selling all the books.

47. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I_2 = A'A$.

48. Express the following matrix as a sum of a symmetric matrix and skew symmetric matrix $\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$.

49. Show that the following matrix is orthogonal $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

50. Show that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

51. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find $adj(A)$.

52. For the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ verify that $A(adj\ A) = |A|I$.

53. For the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$, show that $A(adj\ A) = 0$.

54. Find the inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

55. Find the inverse of $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$.

56. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc \neq 0$, then find the inverse of A.

57. If $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, verify thet $(AB)^{-1} = B^{-1}A^{-1}$.

58. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

59. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, find $(AB)^{-1}$.

60. Solve the following system of equations by matrix method: $3x - 2y = 7$ and $5x + 3y = 1$.

61. Solve the following system of equations by matrix method: $2x - 3y + 3z = 1$, $2x + 2y + 3z = 2$ and $3x - 2y + 2z = 3$.

62. Examine following system of equations for consistency: $2x + 3y = 5$ and $6x + 9y = 10$.

Chapter 10
Inequalities

Till now we have used formulas and methods to prove results. Now we will study inequalities which may require original thinking to (dis)prove them. This makes them fun and at times irritating. There is no set method to solve them. Certainly there are techniques and with time and practice you will learn these techniques to get better at solving these.

Ineuqalities come up in different branches of mathematics; for example in algebra, geometry and trigonometry. They are very useful in establishing many relations among various quantities. Certain inequalities are very useful in studying properties of many common expressions which lead to interesting observations. In this chapter we will only study algebraic inequalitites. The problems given are quite basic and simple. We start with some useful theorems for these inequalities.

There are some facts which are the very important for proving inequalities. Some of them are as follows:

1. If $x \geq y$ and $y \geq z$ then $x \geq z$, for any $x, y, z \in \mathbb{R}$.
2. If $x \geq a$ and $y \geq b$ then $x + a \geq y + b$, for any $x, y, a, b \in \mathbb{R}$.
3. If $x \geq y$ then $x + z \geq y + z$, for any $x, y, z \in \mathbb{R}$.
4. If $x \geq y$ and $a \geq b$ then $xa \geq yb$, for any $x, y \in \mathbb{R}^+$ or $a, b \in \mathbb{R}^+$.
5. If $x \in \mathbb{R}$ then $x^2 \geq 0$, with equality holding if and only if $x = 0$. More generally for $a_i \in \mathbb{R}^+$ and $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ holds $a_i x_i^2 + a_2 x_2^2 + \cdots + a_n x_n^2 \geq 0$, with equality holding if and only if $x_1 = x_2 = \cdots = x_n = 0$.

10.1 Strum's Method

Strum's method is given by the German mathematician *Friedrich Otto Rudolf Sturm*. Sturm's method helps prove a large number of different inequalities under certain conditions along with various other applications.

Theorem 18

Prove that if the product of positive numbers $x_1, x_2, \cdots, x_n (n \geq 2)$ is euqal to 1, then $x_1 + x_2 + \cdots + x_n \geq n$.

Proof

If $x_1 = \cdots = x_n$, then $x_1 + \cdots + x_n = n$. So we see that the statement is true if all the numbers are equal and are unity. Now we consider the case when at least two numbers are different such that one is greater than 1 and the other one is smaller. Let us assume that these are x_1 and x_2 which does not cause loss of generality, and that $x_1 < 1 < x_2$. Note that $x_1 + x_2 > 1 + x_1 x_2 [\because (1 - x_1)(x2 - 1) > 0]$. If given numbers are substitued by $1, x_1 x_2, x_3, \ldots, x_n$, then the product is equal to 1 and $1 + x_1x_2 + x_3 + \cdots + x_n < x_1 + x_2 + \cdots + x_n$. Repeating this we will find $n-1$ numbers equal to 1 and the nth number equal to $x_1x_2 \ldots x_n$. Thus, $x_1 + x_2 + \cdots + x_n < 1$. We see that equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$ □

Theorem 19

Prove that if the sum of the numbers $x_1, x_2, \ldots, x_n (n \geq 2)$ is equal to 1, then prove that $x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{1}{n}$.

Proof

If $x_1 = x_2 = \ldots = x_n = \frac{1}{n}$ then $x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{1}{n}$. Like previous theorem we consider two numbers x_1 and x_2 such that one of them is greater than $\frac{1}{n}$ while the other is smaller than $\frac{1}{n}$. Assume that these two numbers are x_1 and x_2, which does not cause loss of generality, and that $x_1 < \frac{1}{n}$ and $x_2 > \frac{1}{n}$. So we obtain a sequence of numbers $\frac{1}{n}, x_1 + x_2 - \frac{1}{n}, x_3, \ldots, x_n$ suhc that their sum remains equal to 1. We can easily prove that $x_1^2 + x_2^2 > \frac{1}{n^2} + \left(x_1 + x_2 - \frac{1}{n}\right)^2$, and hence

$$x_1^2 + x_2^2 + \cdots + x_n^2 > \frac{1}{n^2} + \left(x_1 + x_2 - \frac{1}{n}\right)^2 + x_3^2 + \cdots + x_n^2.$$

Repeating this we obtain a sequence in which all terms will be equal to $\frac{1}{n}$, and sum of their square is less than the sum of squares of numbers $x_1, x_2, \ldots, x_n$ i.e. $x_1^2 + x_2^2 + \cdots + x_n^2 > \frac{1}{n^2} +$ to n times. From this it follows that equality holds if and only if $x_1 = x_2 = \cdots = x_n$. □

10.2 A.M., G.M., H.M. and Q.M.

Theorem 20 ((A.M.– G.M. – H.M. – Q.M. Inequality))

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, then

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}. \tag{10.1}$$

Proof

Consider the numbers $\frac{x_1}{\sqrt[n]{x_1x_2\cdots x_n}}, \frac{x_2}{\sqrt[n]{x_1x_2\cdots x_n}}, \cdots, \frac{x_n}{\sqrt[n]{x_1x_2\cdots x_n}}$, we see that product is equal to 1. From theorem 18, we have that

$$\frac{x_1}{\sqrt[n]{x_1x_2\cdots x_n}} + \frac{x_2}{\sqrt[n]{x_1x_2\cdots x_n}} + \cdots + \frac{x_n}{\sqrt[n]{x_1x_2\cdots x_n}} \geq n \Rightarrow \frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n}.$$

The above inequality is also known as Cauchy's inequality.

In the above inequality, if we substitute $x_i = \frac{1}{x_i}$, then

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Consider the numbers $\frac{x_1}{x_1+x_2+\cdots+x_n}, \frac{x_2}{x_1+x_2+\cdots+x_n}, \cdots, \frac{x_n}{x_1+x_2+\cdots+x_n}$, and note that their sum is equal to 1. According to theorem 19, we have

$$\left(\frac{x_1}{x_1 + x_2 + \cdots + x_n}\right)^2 + \left(\frac{x_2}{x_1 + x_2 + \cdots + x_n}\right)^2 + \cdots + \left(\frac{x_n}{x_1 + x_2 + \cdots + x_n}\right)^2 \geq \frac{1}{n}$$

$$\Rightarrow \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2.$$

Hence, all the inequalities have been proven. □

10.3 Cauchy-Bunyakovsky-Schwarz Inequality

Theorem 21 ((Cauchy-Bunyakovsky-Schwarz Inequality))

Let $a_1, a_2, \ldots, a_n,\ b_1, b_2, \ldots, b_n \in R$. Then

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n\right)^2 \geq \left(a_1 b_1 + a_2 b_2 + \cdots a_n b_n\right)^2. \tag{10.2}$$

Proof

Let $x_k = \sqrt{\left(a_1^2 + a_2^2 + \cdots + a_k^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2\right)}$, where $k = 1, 2, \ldots, n$. In this case,

$$x_{k+1} = \sqrt{\left(a_1^2 + a_2^2 + \cdots + a_k^2 + a_{k+1}^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2 + b_{k+1}^2\right)}$$

$$\sqrt{\left[\left(\sqrt{a_1^2 + a_2^2 + \cdots + a_k^2}\right)^2 + a_{k+1}^2\right]\left[\left(\sqrt{b_1^2 + b_2^2 + \cdots + b_k}\right)^2 + b_{k+1}^2\right]}$$

$$\geq \sqrt{\left(\sqrt{a_1^2 + a_2^2 + \cdots + a_k^2}.\sqrt{b_1^2 + b_2^2 + \cdots + b_k^2} + a_{k+1}.b_{k+1}\right)^2} = x_k + a_{k+1} b_{k+1}$$

Alternative Proof.

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n\right)^2 - \left(a_1 b_1 + a_2 b_2 + \cdots a_n b_n\right)^2 =$$
$$\sum_{i,j=1 i \geq j}^{n} \left(a_i b_j - b_j a_i\right)^2 \geq 0.$$

10.3.1 Titu's Lemma

Lemma 1

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers then

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\ldots+\frac{a_n^2}{b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n} \tag{10.3}$$

Proof

This is a direct consequence of *Cauchy-Bunyakovsky-Schwarz Inequality.* It is obtained by substituting $a_i=\frac{x_i}{\sqrt{y_i}}$ and $b_i=\sqrt{y_i}$ into Cauchy-Bunyakovsky-Schwarz Inequality. Equality holds if and only if $a_i=kb_i$ for a non-zero real constant k. □

10.4 Chebyshev's Inequality

Theorem 22

Let $a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n$ be real numbers such that $a_1\leq_2\leq a_2\leq\cdots\leq a_n$ and $b_1\leq b_2\leq\cdots\leq b_n$ or $a_1\geq_2\geq a_2\geq\cdots\geq a_n$ and $b_1\geq b_2\geq\cdots\geq b_n$, then the inequality

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\left(\frac{b_1+b_2+\cdots+b_n}{n}\right)\leq\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \tag{10.4}$$

holds. The inequality is strict unless at least one of the sequences is a constant sequence.

Proof

We have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i-a_jb_j)=\sum_{i=1}^{n}\left(na_ib_i-a_i\sum_{j=1}^{n}b_j\right)=n\sum_{i=1}^{n}a_ib_i-\sum_{i=1}^{n}a_i\sum_{j=1}^{n}b_j$$

Simiarly

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_jb_j-a_jb_i)=n\sum_{j=1}^{n}a_jb_j-\sum_{j=1}^{n}a_j\sum_{i=1}^{n}b_i$$

From these two equations, we get

$$\begin{aligned}n\sum_{j=1}^{n}a_jb_j-\sum_{j=1}^{n}a_j\sum_{i=1}^{n}b_i&=\frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i-a_ib_j+a_jb_j-a_jb_i)\right]\\&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i-a_j)(b_i-b_j)\end{aligned}$$

Since both the sequences are either decreasing or increasing, we will have $(a_i-a_j)(b_i-b_j)\geq 0$. Thus, we have

$$n\sum_{j=1}^{n}a_jb_j-\sum_{j=1}^{n}a_j\sum_{i=1}^{n}b_i\geq 0.$$

Here equality holds if and only if for each of the indexes i,j either $a_i=a_j$ or $b_i=b_j$. □

Remark

If the the order of sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ in the orevious theorem are reverses then the inequlaity reverses as well.

The proof is similar to the proof of the theorem.

Remark

Chebyshev's inequality can be generalized to three or more sets of real numbers, with the constraint that sets are in increasing or decreasing order.

Remark

If the two sequeqnces are non-increasing or non-decreasing, and let $p_1, p_2, \dots, p_b$ be a sequence of non=negative real numbers such that $\sum_{i=1}^{n} p_i$ is positive. Then the following inequality holds

$$\left(\frac{\sum\limits_{i=1}^{n} p_i a_i b_i}{\sum\limits_{i=1}^{n} p_i} \right) \geq \left(\frac{\sum\limits_{i=1}^{n} p_i a_i}{\sum\limits_{i=1}^{n} p_i} \right) \left(\frac{\sum\limits_{i=1}^{n} p_i b_i}{\sum\limits_{i=1}^{n} p_i} \right).$$

The proof is similar to the theorem. This is called Chebyshev's inequality with weights.

10.5 Surányi's Inequality

Theorem 23

Let $a_1, a_2, \dots, a_n$ be non-negative real numbers, and let $n \in P$. Then

$$\begin{aligned} (n-1)\left(a_1^n + a_2^n + \cdots + a_n^n\right) + n a_1 a_2 \cdots a_n &\geq \\ \left(a_1 + a_2 + \cdots + a_n\right)\left(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}\right). \end{aligned} \tag{10.5}$$

Proof

We will prove this by mathematical induction. Due to symmetry and homegeneity of the inequality we may assume $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1 + a_2 + \cdots + a_n = 1$. For $n = 1$ equality occurs. Let us assume that for $n = 1$ the inequality holds i.e.

$$(k-1)\left(a_1^k + a_2^k + \cdots + a_k^k\right) + k a_1 a_2 \cdots a_k \geq a_1^{k-1} + a_2^{k-1} + \cdots + a_k^{k-1}.$$

We need to prove that:

$$k \sum_{i=1}^{k+1} a_i^{k+1} + (k+1) \prod_{i=1}^{k+1} a_i - \left(1 + a_{k+1}\right) \sum_{i=1}^{k+1} a_i^k \geq 0.$$

Hence

$$k a_{k+1} \prod_{i=1}^{k} a_i \geq a_{k+1} \sum_{i=1}^{k} a_i^{k-1} - (k-1) a_{k+1} \sum_{i=1}^{k} a_i^k.$$

Using this last inequality, it remains to prove that:

$$\left(k\sum_{i=1}^{k+1}a_i^{k+1}-\sum_{i=1}^{k}a_i^{k}\right)-a_{k+1}\left(k\sum_{i=1}^{k}a_i^{k}-\sum_{i=1}^{k}a_i^{k-1}\right)+$$
$$a_{k+1}\left(\prod_{i=1}^{k}a_i+(k-1)\,a_{k+1}^{k}-a_{k+1}^{k-1}\right)\geq 0.$$

We have

$$\prod_{i=1}^{k}a_i+(k-1)\,a_{k+1}^{k}-a_{k+1}^{k-1}=\prod_{i=1}^{k}(a_i-a_{k+1}+a_{k+1})+(k-1)\,a_{k+1}^{k}-a_{k+1}^{k-1}$$
$$\geq a_{k+1}^{k}+a_{k+1}^{k-1}\sum_{i=1}^{k}(a_i-a_{k+1})+(k-1)\,a_{k+1}^{k}-a_{k+1}^{k-1}=0.$$

Also

$$\left(k\sum_{i=1}^{k+1}a_i^{k+1}-\sum_{i=1}^{k}a_i^{k}\right)-a_{k+1}\left(k\sum_{i=1}^{k}a_i^{k}-\sum_{i=1}^{k}a_i^{k-1}\right)\geq 0$$
$$\Rightarrow k\sum_{i=1}^{k}a_i^{k+1}-\sum_{i=1}^{k}a_i^{k}\geq a_{k+1}\left(k\sum_{i=1}^{k}a_i^{k}-\sum_{i=1}^{k}a_i^{k-1}\right)$$

By Chebyshev's inequality, we have

$$k\sum_{i=1}^{k}a_i^{k}\geq\sum_{i=1}^{k}a_i\sum_{i=1}^{k}a_i^{k-1}=\sum_{i=1}^{k}a_i^{k-1}$$
$$\Rightarrow k\sum_{i=1}^{k}a_i^{k}-\sum_{i=1}^{k}a_i^{k-1}\geq 0.$$

and since $a_1+a_2+\cdots+a_{k+1}=1$, by the assumption $a_1\geq a_2\geq\cdots\geq a_{k+1}$, we deduce that

$$a_{k+1}\leq\frac{1}{k}$$

So it is enough to prove that

$$k\sum_{i=1}^{k}a_i^{k+1}-\sum_{i=1}^{k}a_i^{k}\geq\frac{1}{k}\left(k\sum_{i=1}^{k}a_i^{k}-\sum_{i=1}^{k}a_i^{k-1}\right).$$

which is equivalent to

$$k\sum_{i=1}^{k}a_i^{k+1}+\frac{1}{k}\sum_{i=1}^{k}a_i^{k-1}\geq 2\sum_{i=1}^{k}a_i^{k}$$

Since AM $\geq$ GM we have that

$$ka_i^{k+1} + \frac{1}{k}a_i^{k-1} \geq 2a_i^k \ \forall \ i$$

Adding this inequality for $i = 1, 2, \dots, k$ we obtain the required inequality. □

10.6 Rearrangement Inequality

Theorem 24

Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ (or $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$) be real numbers. If $a_1', a_2', \dots, a_n'$ is any permutation of $a_1, a_2, \dots, a_n$ then the equality

$$\sum_{i=1}^{n} a_i b_{n+1-i} \leq \sum_{i=1}^{n} a_i; b_i \leq \sum_{i=1}^{n} a_i b_i, \tag{10.6}$$

holds. Thus the sum $\sum_{i=1}^{n} a_i b_i$ is maximum when the two sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ are oredered similarly. And the sum is minimum when these are ordered in opposite manner.

Proof

We start by assuming that both a_i's and b_i's are non-decreasing. Suppose $\langle a_i' \rangle \neq \langle a_i \rangle$. Let r be the largest index such that $a_r' \neq a_r$ i.e. $a_r' \neq a_r$ and $a_i' = a_i$ for $r < i \leq n$. This implies that a_r' is from the set $\{a_1, a_2, \dots, a_{r-1}\}$ and $a_r' < a_r$. Further this also shows that $a_1', a_2', \dots, a_r'$ is a permutation of $a_1, a_2, \dots, a_r$. Thus we can find indices $k < r$ and $l < r$ such that $a_k' = a_r$ and $a_r' = a_l$. It follows that

$$a_k' - a_r' = a_r - a_l \geq 0, b_r - b_k \geq 0$$

We now interchange a_r' and a_k' to get a permutation of $a_1'', a_2'', \dots, a_n''$ of $a_1', a_2', \dots, a_n'$; thus

$$\begin{cases} a_i'' = a_i', & \text{if} i \neq r, k \\ a_r'' = a_k' = a_r, a_k'' = a_r' = a_l \end{cases}$$

Consider the sums

$$S'' = a_1''b_1 + a_2''b_2 + \cdots + a_n''b_n, \ \ S' = a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n,$$

and the difference $S'' - S'$:

$$\begin{aligned} S'' - S' &= \sum_{i=1}^{n} (a_i'' - a_i') b_i \\ &= (a_k'' - a_k') + (a_r'' - a_r') b_r \\ \& &= (a_r' - a_k') b_k + (a_k' - a_r') b_r \\ &= (a_k' - a_r')(b_r - b_k). \end{aligned}$$

$\because a_k' - a_r' \geq 0$ and $b_r - b_k \geq 0$, we can say that $S'' \geq S'$. We observe that the permutations $a_1'', a_2'', \dots, a_n''$ of $a_1', a_2', \dots, a_n'$ has th eproperty that $a_i'' = a_i = a_i$ for $r < i \leq n$ and $a_r'' = a_k' = a_r$. Hence the permutation $\langle a_i'' \rangle$ in place of $\langle a_i' \rangle$ may be considered and the steps can be continued like above. After at most $n-1$ such steps, we will arrive at the original permutation $\langle a_i \rangle$ from $\langle a_i' \rangle$. At each step the corresponding sum has the same order as a_i's i.e. non-decreasing. Thus,

$$a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n \le a_1b_1 + a_2b_2 + \cdots + a_nb_n \tag{10.7}$$

For the other part, let us put $c_i = a_{n+1-i}', d_i = -b_{n+1-i}$. Then $c_1, c_2, \dots, c_n$ is a permutation of $a_1, a_2, \dots, a_n$ and $d_1 \le d_2 \le \cdots \le d_n$. Using the inequality (Equation 10.7) for the sequences $\langle c_i \rangle$ and $\langle d_i \rangle$, we get

$$c_1d_1 + c_2d_2 + \cdots + c_nd_n \le a_1d_2 + a_2d_2 + \cdots + a_nd_n.$$

Thus,

$$-\sum_{i=1}^{n} a_{n+1-i}'b_{n+1-i} \le -\sum_{n=1}^{n} a_ib_{n+1-i}.$$

Thus,

$$a_1'b_1 + a_2'b_2 + \cdots + a_n'b_n \ge a_1b_1 + a_2b_{n-1} + \cdots + a_nb_1, \tag{10.8}$$

which is the other part of the inequality.

For the equality, we consider pairs k, l with $1 \le k < l \le n$, either $a + k' = a_l'$ or $a_k' > a_l'$ and $b_k = b_l$, then the equality holds for (Equation ??rearrangement:2). For (Equation 10.8), for each k, l with $1 \le k < l \le n$, either $a_{n+1-k}' \ge a_{n+1-l}'$ and $b_{n+1-k} = b_{n+1-l}$. □

Corollary 4

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be real numbers and $\beta_1, \beta_2, \dots, \beta_n$ be a permutation of $\alpha_1, \alpha_2, \dots, \alpha_n$. Then

$$\sum_{i=1}^{n} \alpha_i\beta_1 \le \sum_{i=1}^{n} \alpha_i^2.$$

The equality holds if and only if $\langle \alpha_i \rangle = \langle \beta_i \rangle$.

Proof

Let $\alpha_1', \alpha_2', \dots, \alpha_n'$ be a permutation of $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\alpha_1' \le \alpha_2' \le \dots \le \alpha_n'$. Then we can find a bijections σ of $\{1,2,\dots,n\}$ onto itself such that $\alpha_i' = \alpha_{\sigma(i)}, 1 \le j \le n$; i.e. σ is a permutation on the set $\{1,2,\dots,n\}$. Let $\beta_i' = \beta_{\sigma(i)}$. Then $\beta_1', \beta_2', \dots, \beta_n'$ is a permutation of $\alpha_1' \le \alpha_2' \le \dots \le \alpha_n'$. Applying the rearrangement inequality to $\alpha_1' \le \alpha_2' \le \dots \le \alpha_n'$ and $\beta_1', \beta_2', \dots, \beta_n'$, we get

$$\sum_{i=1}^{n} \alpha_i'\beta_i' \le \sum_{i=1}^{n} (\alpha_i')^2 = \sum_{i=1}^{n} \alpha_i^2.$$

We also have

$$\sum_{i=1}^{n} \alpha_i'\beta_i' = \sum_{i=1}^{n} \alpha_{\sigma(i)}\beta_{\sigma(i)} = \sum_{i=1}^{n} \alpha_i\beta_i,$$

because σ is a bijection on $\{1,2,\dots,n\}$. Thus,

$$\sum_{i=1}^{n} \alpha_i \beta_i \le \sum_{i=1}^{n} \alpha_i^2.$$

Say that equality holds and $\langle \alpha_i \rangle \neq \langle \beta_i \rangle$. Then $\langle \alpha_i' \rangle \neq \langle \beta_i' \rangle$. Let k be the largest index such that $\alpha_k' \neq \beta_k'$ for $k < i \neq n$. Let m be the least integer such that $\alpha_k' = \beta_m'$. If $m > k$, then $\beta_m' = \alpha_k'$ and hence $\alpha_k' = \alpha_m'$. This implies that $\alpha_k' = \alpha_{k+1}' = \dots = \alpha_m'$ and hence $\beta_{k+1}' = \dots = \beta_m'$. We now have an $m_1 > m$ such that $\alpha_k' = \beta_{m_1}'$. Using m_1 as pivot, we get $\alpha_k' = \alpha_{k+1}' = \dots = \alpha_m' = \dots = \alpha_m'$ and $\beta_{k+1}' = \dots = \beta_m' = \dots = \beta_{m_1}'$. It can be concluded that $\alpha_k' = \beta_l'$ for some $l < k$, thus forcing $m < k$.

Clearly $\beta_m' \neq \beta_k'$ by our choice of k. We know that equality holds if and only if for any two indexes $r \neq s$, either $\alpha_r' = \alpha_s'$ or $\beta_r' = \beta_s'$. Since $\beta_m' \neq \beta_k'$, we must have $\alpha_m' = \alpha_k'$. But then we have $\alpha_m' = \alpha_{m+1}' = \dots = \alpha_k'$. From the minimality of m, we see that $k - m + 1$ equal elements $\alpha_m', \alpha_{m+1}', \dots, \alpha_k'$ must be among $\beta_m', \beta_{m+1}', \dots, \beta_n'$ and since $\beta_k' \neq \alpha_k'$, we must have $\alpha_k' = \beta_l'$ for some $l > k$. But then using $\beta_l' = \alpha_l'$, we have

$$\alpha_m' = \alpha_{m+1}' = \dots = \alpha_k' = \dots = \alpha_l'.$$

Thus the number of equal elements gets enlarged to $l - m + 1 > k - m + 1$. Since this process cannot be continues indefinitely, we conclude that $\langle \alpha_i' \rangle = \langle \beta_i' \rangle$ which will be followed by $\langle \alpha_i \rangle \neq \langle \beta_i \rangle$. □

Corollary 5

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers and let $\beta_1, \beta_2, \dots, \beta_n$ be a permutation of $\alpha_1, \alpha_2, \dots, \alpha_n$. Then

$$\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} \ge n.$$

Equality holds if and only if $\langle \alpha_i \rangle \neq \langle \beta_i \rangle$.

Proof

Let $\alpha_1', \alpha_2', \dots, \alpha_n'$ be a permutation of $\alpha_1, \alpha_2, \dots, \alpha_n$ suhc that $\alpha_1' \le \alpha_2' \le \dots \alpha_n'$. Like in previous corollary, we can find a permutation σ of $\{1,2,\dots,n\}$ such that $\alpha_i' = \alpha_{\sigma(i)}$ for $1 \le i \le n$. We defien $\beta_i' = \beta_{\sigma(i)}$. Then $\langle \beta_i' \rangle$ is a permutation of $\langle \alpha_i' \rangle$. Using the rearrangement theorem, we get

$$\sum_{i=1}^{n} \beta_i' \left(-\frac{1}{\alpha_i'} \right) \le \sum_{i=1}^{n} \alpha_i' \left(-\frac{1}{\alpha_i'} \right) = -n.$$

Thus, we have the desired inequality. Like previous case we camn derive the equality. □

10.7 Young's Inequality

Theorem 25

If $p \in [1,\infty)$ and $q = p/(p-1)$. $q \in [1,\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $a, b > 0$, then

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab \tag{10.9}$$

Proof

Taking log of L.H.S. $\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right)$

Notice that, since $\dfrac{1}{p} + \frac{1}{q} = 1$, so the L.H.S. is just a convex combination of a^p and b^q. Since $\log x$ is a concave function, we have

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{\log a^p}{p} + \frac{b^q}{q} = \log a + \log b = \log(ab).$$

Hence, the inequality is proved(since $\log x$ is strictly increasing).

Alternative Proof.

Using generalized AM-GM inequality,

$$\frac{x^p}{p} + \frac{b^q}{q} \geq \left[(x^p)^{1/p}(y^q)^{1/q}\right] = xy.$$

Proof

Aliter: Since $\frac{1}{p} + \frac{1}{q} = 1$, we can write $p = \frac{m+n}{m}$, $q = \frac{m+n}{n}$, where $m, n \in \mathbb{P}$.

Let $a = x^{1/p}$ and $b = y^{1/q}$, then $\frac{a^p}{p} + \frac{b^q}{q} = \frac{mx+ny}{m+n}$.

However, from A.M.-G.M. inequality, $\frac{mx+ny}{m+n} \geq (x^m y^n)^{\frac{1}{m+n}} = ab$. □

10.8 Hölder's Inequality

Theorem 26

Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be real numbers and p, q be two positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. (Such a pair of indices is called a pair of conjugate indices.) Then the inequality holds

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q} \tag{10.10}$$

holds. Equality holds if and only if $|a_i|^p = c|b_i|^q$, $1 \leq i \leq n$, for some real constant c.

Proof

Following Young's inequality, consider

$$x = \frac{|a_k|}{\left(\sum\limits_{i=1}^{n} |a_i|^p\right)^{1/p}}, y = \frac{|b_k|}{\left(\sum\limits_{i=1}^{n} |b_i|^q\right)^{1/q}}$$

so we get

$$\frac{|a_k|^p}{p\left(\sum\limits_{i=1}^{n} |a_i|^p\right)} + \frac{|b_k|^q}{q\left(\sum\limits_{i=1}^{n} |b_i|^q\right)} \geq \frac{|a_k||b_k|}{\left(\sum\limits_{i=1}^{n} |a_i|^p\right)^{1/p}\left(\sum\limits_{i=1}^{n} |b_i|^q\right)^{1/q}}$$

Now summing over k, we obtain

$$\frac{1}{p} + \frac{1}{q} \geq \frac{\sum\limits_{i=1}^{n} |a_k b_k|}{\left(\sum\limits_{i=1}^{n} |a_i|^p\right)^{1/p}\left(\sum\limits_{i=1}^{n} |b_i|^q\right)^{1/q}}$$

Thus, we have

$$\sum_{i=1}^{n} |a_k b_k| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}.$$

It is now trivial to prove the condition for equality. □

Remark

If we take $p = q = 3$, Hölder's inequality reduces to the Cauchy-Schwarz inequality.

Remark

If either of p and q is negativem Hölder's inequality is reversed.

Remark

Hölder's inequality can have a version with weights. In addition to what we have, we also consider consider weights $w_1, w_2, \ldots, w_n$ then following equality holds

$$\sum_{i=1}^{n} w_i |a_i b_i| \leq \left(\sum_{i=1}^{n} w_i |a_i|^p\right)^{1/p}\left(\sum_{i=1}^{n} w_i |b_i|^q\right)^{1/q}$$

Given below is generalized Hölder's inequaltiy and the proof is similar like above.

Theorem 27

Let $a_{ij}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n,$ be positive humbers and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then

$$\sum_{i=1}^{m}\left(\prod_{j=1}^{n} a_{ij} a_{ij}^{\alpha_j}\right) \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)^{\alpha_j}. \tag{10.11}$$

10.9 Minkowski's Inequality

Theorem 28

Let $p \geq 1$ be a real number and $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be real numbers. Then

$$\left(\sum_{i=1}^{n} |ai + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p} \tag{10.12}$$

Here equality holds if and only if $a_i = \lambda b_i$ for some constant $\lambda, 1 \leq i \leq n$.

Proof

We assume that $p > 1$, because the result is clear for $p = 1$. Observe the following:

$$\sum_{i=1}^{n} |ai + b_i|^p = \sum_{i=1}^{n} |a_i + b_i|^{p-1}|a_i + b_i| \leq \sum_{i=1}^{n} |a_i + b_i|^{p-1}|a_i| + \sum_{i=1}^{n} |a_i + b_i|^{p-1}|b_i|.$$

Let q be the conjugate index of p. Using Hölder's inequaity to each sum on the right hand side, we have

$$\sum_{i=1}^{n} |a_i + b_i|^{p-1}|a_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |a_i + b_i|^{(p-1)\,q}\right)^{1/q}.$$

Since p, q are conjugate indexes, we get $(p-1)\,q = p$. It follows that

$$\sum_{i=1}^{n} |a_i + b_i|^{p-1}|a_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q}.$$

Similarly,

$$\sum_{i=1}^{n} |a_i + b_i|^{p-1}|b_i| \leq \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q}.$$

It now follows that

$$\sum_{i=1}^{n} |a_i + b_i|^p \leq \left[\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}\right] \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1/q}.$$

If we use $1 - (1/1) = 1/p$, we finally get the required inequaity.

Like Hölder's inequality the equality can be proven for this using the same conditions. □

Remark

For $0 < p < 1$, the inequality (Equation 10.12) gets reversed.

10.10 Convex and Concave Functions

Most of the inequalities discussed so far are consequencce of inequalities for a special class of functions, known as *convex* and *concave* functions. Consider the function $f(x) = x^n \ \forall\ n > 1$ defined on $\mathbb{R}$. Consider the case of $n = 2$, then on the graphs of this function, the chord joining any two points always lies above the graph. In fact taking $a < b$, and the point $ka + (1-k)b$ between a and b, we see that

$$^2 - ka^2 - (1-k)b^2 = -k(1-k)(a-b)^2 \leq 0.$$

Thus,

$$f(ka + (1-k)b) \leq kf(a) + (1-k)f(b).$$

This property is the defining property of a convex function. The family of convex functions obey a class of inequalities known as Jensen's inequality.

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if for all x, y in I and k in the interval $[0, 1]$, the following inequality holds:

$$f(kx + (1-k)y) \leq kf(x) + (1-k)f(y). \tag{10.13}$$

If the inequality is strict for all $x \neq y$, f is said to be strictly convex on I. If the inequality is reverse for same conditions then f is said to be concave and similarly for strictly concave f.

There are other equivalent properties of a convex function. Let x_1, x_2, x_3 are in I such that $x_1 < x_2 < x_3$ and we take $k = \frac{x_3 - x_2}{x_3 - x_1}$ which gives us

$$1 - k = \frac{x_2 - x_1}{x_3 - x_1}, \text{ and } x_2 = kx_1 + (1-k)x_3.$$

We have

$$\begin{aligned} f(x_2) &= f(kx_1 + (1-k)x_3) \\ &\leq kf(x_1) + (1-k)f(x_3) \\ &= \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3). \end{aligned}$$

We can write this as

$$f\,\frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq \frac{f(x_2) - f(x_3)}{x_2 - x_3},$$

for all $x_1 < x_2 < x_3$ in I. We can also write this as:

$$\frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0.$$

Consider $z_1 = (a, f(a))$ and $z_2 = (b, f(b))$ as two points on f. The equation of line joining these two points is given by

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - z}(x - a).$$

Any point between a and b is of the form $x = ka + (1-k)b$. Thus,

$$\begin{aligned} g(x) &= g(ka + (1-k)b) \\ &= f(a) + \frac{f(b) - f(a)}{b - a}(ka + (1-k)b - a) \\ &= f(a) + (1-k)[f(b) - f(a)] \\ &= kf(a) + (1-k)f(b) \\ &\geq f(ka + (1-k)b) = f(x) \end{aligned}$$

Thus, $(x, g(x))$ lies above $(x, f(x))$, a point on f.

We can look at this in another way. A subset E of the plane $\mathbb{R}^2$ is said to be convex if for every pair of points z_1 and z_2 in E, the line joining z_1 and z_2 lies entirely in E. With every function $f : I \to \mathbb{R}$, we associate a subset of $\mathbb{R}^2$ by

$$E(f) = \{(x, y) : a \leq x \leq b, f(x) \leq y\}.$$

Theorem 29

The function $f : I \to \mathbb{R}$ is convex if and only if $E(f)$ is a convex subset of $\mathbb{R}^2$.

Proof

Let f be convex. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be two points of $E(f)$. Consider any point on the line zoining z_1 and z_2. Then,

$$\begin{aligned} z &= kz_1 + (1-k)z_2 \\ &= (kx_1 + (1-k)x_2, ky_1 + (1-k)y_2) \end{aligned}$$

for some $k \in [0, 1]$. We see that $a \leq kx_1 + (1-k)x_2 \leq b$. Moreover,

$$\begin{aligned} f(kx_1 + (1-k)x_2) &\leq kf(x_1) + (1-k)f(x_2) \\ &\leq ky_1 + (1-k)y_2. \end{aligned}$$

Thus it follows that $z \in E(f)$, proving that $E(f)$ is convex.

Conversely let $E(f)$ be convex. Let x_1, x_2 be two points in I and let $z_1 = (x_1, f(x_1))$ and $z_2 = (x_2, f(x_2))$. Then z_1 and z_2 are in $E(f)$. By conexity of $E(f)$, the point $kz_1 + (1-k)z_2$ also lies in $E(f)$ for each $k \in [0, 1]$. Thus,

$$(kx_1 + (1-k)x_2, kf(x_1) + (1-k)f(x_1)) \in E(f)$$

The definition of $E(f)$ shows that

$$f(kx_1 + (1-k)x_2) \leq kf(x_1) + (1-k)f(x_2).$$

This shows that f is convex on the interval I. $\square$

Following theorem gives description about slope of a function's graph.

Theorem 30

Let $f : I \to \mathbb{R}$ be a convex function and $a \in I$ be a fixed point. Define a function $P : I \setminus \{a\} \to \mathbb{R}$ by

$$P(x) = \frac{f(x) - f(a)}{x - a}.$$

Then P is a non-decreasing function on $I \setminus \{a\}$.

Proof

Let f is convex on I and let x, y be two points in $I, x \neq a, x \neq b$ such that $x < y$. Then exactly one of the three possibilities will be possible:

$$a < x < y;\ x < a < y\ x < y < a.$$

Consider the case $a < x < y$; other cases can be handled similarly. We can write

$$x = \frac{x-a}{y-a} y + \frac{y-x}{y-a} a.$$

The convexity of f shows that

$$f\left(\frac{x-a}{y-a} y + \frac{y-x}{y-a} a\right) \leq \frac{x-a}{y-a} f(y) + \frac{y-x}{y-a} f(a).$$

This is equivalent to

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(y) - f(a)}{y - a}.$$

Thus $P(x) \leq P(y)$. This shows that $P(x)$ is a non-decreasing function for $x \neq a$. □

Interestingly, the converse is also true; if $P(x)$ is a non-decreasing function on $I \setminus \{a\}$ for every $a \in I$, then $f(x)$ is convex. We fix $x < y$ in I and let $a = kx + (1-k)y$ where $k \in (0{,}1)$. (The cases $k = 0$ or 1 are obvious.) In this case

$$P(x) = \frac{f(x) - f(a)}{x - a} = \frac{f(x) - f(a)}{(1-k)(x-y)}$$

$$P(y) = \frac{f(y) - f(a)}{y - a} = \frac{f(y) - f(a)}{k(y-x)}.$$

The condition $P(x) \leq P(y)$ implies that $f(a) \leq kf(x) + (1-k)f(y)$. Hence convexity of f is proven.

There is another easy way of deciding wherther a function is convex or concave for twice differentiable functions. If f is convex on an interval I and if its second derivative exists on I, then f is convex(strictly convex) on I if $f''(x) \geq 0(> 0)$ for all $x \in I$. Similarly f is concave(striclty concave) on I if $f''(x) \leq 0(< 0)$ for all $x \in I$.

When we defined conex function the inequality involved two points x, y; refer to (Equation 10.13). Jensen's inequaity extends this to any finite number of points.

10.11 Jensen's Inequality

Theorem 31

Let $f : I \to \mathbb{R}$be a convex function. Let $x_1, x_2, \dots, x_n$ are points in I and $k_1, k_2, \dots, k_n$ are real numbers in the interval $[0, 1]$ such that $k_1 + k_2 + \cdots + k_n = 1$. Then

$$f\left(\sum_{i=1}^{n} k_i x_i\right) \leq \sum_{i=1}^{n} k_i f(x_i) \tag{10.14}$$

Proof

We will use induction to prove this. For $n = 2$, this is the definition of a convex function. Suppose the inequality (Equarion 10.14) is true for all $p < n$; i.e. for $p < n$ if $x_1, x-2, \dots, x_p$ are p points in I and $k_1, k_2, \dots, k_p$ are real numbers in [0,1] such that $\sum_{i=1}^{n} k-i = 1$, then

$$f\left(\sum_{i=1}^{p} k_i x_i\right) \leq \sum_{i=1}^{p} k + i f(x_i).$$

Now considering the conditions of the theorem,

$$y_1 = \frac{\sum\limits_{i=1}^{n-1} k_i x_i}{\sum\limits_{i=1}^{n-1} k_i},\ y_2 = x_n,\ \alpha_1 = \sum_{j=1}^{n-1} k_i,\ \alpha_2 = k_n.$$

We observe that $\alpha_2 = 1 - \alpha_1$, and y_1, y_2 are in I. Using the conexity of f, we get

$$\begin{aligned} f(\alpha_1 y_1 + \alpha_2 y_2) &= f(\alpha_1 y_1 + (1 - \alpha_1) y_2) \\ &\leq \alpha_1 f(y_1) + (1 - \alpha)1) f(y_2) \\ &= \alpha_1 f(y_1) + \alpha_2 f(y_2). \end{aligned}$$

However, we have

$$\alpha_1 y_1 + \alpha_2 y_2 = \sum_{i=1}^{n} k_i x_i.$$

Now we consider $f(y_1)$. If

$$\mu_i = \frac{k_i}{\sum\limits_{i=1}^{n-1} k_i},\ 1 \leq l \leq n-1$$

then it can be easily verifief that $\sum_{l=1}^{n-1} \mu_l = 1$. Using the induction hypothesis, we get

$$f\left(\sum_{l=1}^{n-1} \mu_l x_l\right) \leq \sum_{l=1}^{n-1} \mu_l f(x_l)$$

Since

$$\sum_{l=1}^{n-1} \mu_l x_l = y_1,$$

we get

$$f(y_1) \leq \frac{\sum\limits_{l=1}^{m-1} k_l f(x_l)}{\sum\limits_{i=1}^{n-1} k_i} = \frac{\sum\limits_{i=1}^{n-1} f(x_i)}{\alpha_1}$$

Thus we obtain

f

$$\begin{aligned}\left(\sum_{i=1}^{n} k_i f(x_i)\right) &\leq \alpha_1\left(\frac{\sum\limits_{i=1}^{n-1} k_i f(x_i)}{\sum\limits_{i=1}^{n-1} k_i}\right) + k_n f(x_n) \\ &= \sum_{i=1}^{n} k_i f(x_i).\end{aligned}$$

Thus, the theorem is proved by induction. □

Remark

If $f : I \to \mathbb{R}$ is concave, then the inequality (Equarion 10.14) gets reversed. If $x_1, x_2, \ldots, x_n$ are points in I and $k_1, k_2, \ldots, k_n$ are real numbers in the interval $[0, 1]$, such that $k_1 + k_2 + \cdots + k_n = 1$, then following inequality holds:

$$f\left(\sum_{i=1}^{n} k_i x_i\right) \geq \sum_{i=1}^{n} k_i f(x_i) \tag{10.15}$$

Remark

Using the concavity of $f(x) = \ln x$ on $(0, \infty)$, the AM-GM inequality can be proved. If $x_1, x_2, \ldots, x_n$ are points in $(0, \infty)$ and $k_1, k_2, \ldots, k_n$ are real numbers in the interval $[0,1]$ such that $k_1 + k_2 + \ldots + k_n = 1$, then we have

$$\ln\left(\sum_{i=1}^{n} k_i x_i\right) \geq \sum_{i=1}^{n} k_i \ln(x_i)$$

Proof

Taking $k_i = \frac{1}{n}$ for all i,

$$\ln\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n}\ln x_i = \sum_{i=1}^{n}\ln\left(x_i^{1/n}\right).$$

Using the fact that $g(x) = e^x = \exp(x)$ is strictly increasing on the interval $(-\infty, \infty)$, this leads to

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}x_i &\geq \exp\left(\sum_{i=1}^{n}\ln\left(x_i^{1/n}\right)\right)\\ &= \prod_{i=1}^{n}\exp\left(\ln\left(x_i^{1/n}\right)\right)\\ &= (x_1x_2\ldots x_n)^{1/n}.\end{aligned}$$

We can also prove generalized AM-GM inequality with this method.

$$\ln\left(\sum_{i=1}^{n}k_ix_i \geq \sum_{i=1}^{n}k_i\ln(x_i)\right) = \sum_{i=1}^{n}\ln x_i^{k_i},$$

Taking antilog

$$\sum_{i=1}^{n}k_ix_i \geq \prod_{i=1}^{n}x_i^{k_i}.$$

Nowfor any n positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, consider

$$k_i = \frac{\alpha_i}{\sum\limits_{j=1}^{n}\alpha_j}$$

Observe that k_i are in $[0, 1]$ and $\sum_{i=1}^{n}k_i = 1$. These choices of k_i give

$$\frac{\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \geq \left(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha)2}\right)^{1/(\alpha_1+\alpha_2+\cdots+\alpha_n)},$$

which is our generalized AM-GM inequality. □

Remark

Function $f(x) = x^p$ can be used to prove Hölder's inequality. We know that $f(x) = x^p$ is convex for $p \geq 1$ and concave for 0,$p < 1$ for $p \in (0, \infty)$. Let $x_1, x_2, \ldots, x_n$ be real numbers and $k_1, k_2, \ldots, k_n$ in $[0, 1]$, then we have

$$\left(\sum_{i=1}^{n}k_ix_i\right)^p \leq \sum_{i=1}^{n}k_ix_i^p \text{ for } p \geq 1$$

and

$$\left(\sum_{i=1}^{n}k_ix_i\right)^p \geq \sum_{i=1}^{n}k_ix_i^p \text{ for } 0 < p < 1.$$

Proof

Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be real numbers and $p > 1$ and q be conjugate numbers. Thus, $\frac{1}{p} + \frac{1}{q} = 1$. We need to assume that $b_i \neq 0$ for all i; else we may delete all those b_i which are zero without having an effect on the equality. Let

$$t = \sum_{i=1}^{n} |b_i|^q,\ k_j = \frac{|b_j|^q}{t},\ x_j = \frac{|a_j|}{|b_j|^{q-1}}$$

We have $k_j \in [0, 1]$ and $k_1 + k_2 + \dots + k_n = 1$. Using the conexity of x^p, we have

$$\left(\sum_{i=1}^{n} k_i x_i\right)^p \leq \sum_{i=1}^{n} k_i x_i^p,$$

which implies that

$$\left(\sum_{j=1}^{n} \frac{|b_j|^q}{t} \frac{|a_j|}{|b_j|^{q-1}}\right)^p \leq \sum_{j=1}^{n} \frac{|b_j|^1}{t} \frac{|a_j|^p}{|b_j|^{(q-1)p}} = \frac{1}{t} \sum_{j=1}^{n} |a_j|^p.$$

Futher simplification yields

$$\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} t^{1-(1/p)} = \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q\right)^{1/q}$$

For concave case the inequality is simply reversed. □

Theorem 32

Let $f : I \to \mathbb{R}$ be a convex function; $a_1 \leq a_2 \leq \dots \leq a_n, b_1, b_2, \dots, b_n$ are real numbers in I such that $a_1 + b_1 \in I$ and $a_n + b_n \in I$. Let $a'_1, a'_2, \dots, a'_n$ be a permutation of $a_1, a_2, \dots, a_n$. Then the follwoing inequality is true:

$$\sum_{i=1}^{n} f(a_i + b_{n+1-i}) \leq \sum_{i=1}^{n} f(a'_i + b_i) \leq \sum_{i=1}^{n} f(a_i + b_i).$$

Proof

We will use the proof of rearrangement inequality. Assume $\langle a'_i \rangle \neq \langle a_i \rangle$ and r be the largest index such that $a'_r \neq a_r$. Since $a_i = a'_i$ for $r < i \leq n$, we see that $a'_1, a'_2, \dots, a'_r$ is a permutation of $(a_1, a_2, \dots, a_r)$. Thus we can find $k < r, l < r$ such that $a'_k = a_r$ and $a'_r = a_l$. We deduce that $a'_k - a'_r = a_r - a_l \geq 0$ and $b_r - b_k \geq 0$. Interchanging a'_r and a'_k to get a permutation $(a''_1, a''_2, \dots, a''_n)$ of $(a'_1, a'_2, \dots, a'_n)$. Thus

$$a''_i = a'_i \text{ for } j \neq r,k,\ a''_r = a'_k = ar,\ a''_k = a'_r = a_l.$$

Let

$$S'' = \sum_{i=1}^{n} f(a''_i + b_i),\ S' = \sum_{i=1}^{n} f(a'_i + b_i).$$

Then,

$$\begin{aligned} S'' - S' &= f(a''_r + b_r) + f(a''_k + b_k) - f(a'_r + b_r) - f(a'_k + b_k) \\ &= f(a_r + b_r) + f(a_l + b_k) - f(a_l + b_r) - f(a_r + b_k). \end{aligned}$$

We notice that

$$a_l + b_k < a_r + b_k \text{ and } a_l + b_r < a_r + b_r.$$

These give

$$a_l + b_k < a_r + b_k \leq a_r + br, \, a_l + b_k \leq a_l + b_r < a_r + b_r.$$

If x_1, x_2, x_3 are in I, then the convexity of f implies that

$$(x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3).$$

Putting $x_1 = a_l + b_k$, $x_2 = a_r + b_k$ and $x_3 = a_r + b_r$, we get

$$(a_r + b_r - a_l - b_k) f(a_r + b_k) \leq (b_r - b_k) f(a_l + b_k) + (a_r - a_l) f(a_r + b_r).$$

Similarly putting $x_1 = a_l + b_k$, $x_2 = a_l + b_r$ and $x_3 = a_r + b_r$, we get

$$(a_r + b_r - a_l - b_k) f(a_l + b_r) \leq (a_r - a_l) f(a_l + b_k) + (b_r - b_k) f(a_r + b_r).$$

Adding, we get

$$\begin{gathered} (a_r + b_r - a_l - b_k) \{ f(a_r + b_k) + f(a_l + b_r) \} \leq \\ (a_r + b_r - a_l - b_k) \{ f(a_l + b_k) + f(a_r + b_r) \}. \end{gathered}$$

Since $a_l + b_k < a_r + b_r$, we arrive at

$$f(a_r + b_k) + f(a_l + b_r) \leq f(a_l + b_k) + f(a_r + b_r).$$

This proves that $S'' - S' \geq 0$.

Now we observe that the permutation $(a''_1, a''_2, \ldots, a''_n)$ has the property $a''_r = a_r$ and $a''_i = a_i$, for $r < j \leq n$. We may consider the $(a''_1, a''_2, \ldots, a''_n)$ in place $(a'_1, a'_2, \ldots, a'_n)$ and proceed as above. After at most $n-1$ steps we arrive at the original numbers $\langle a_i \rangle$ from $\langle a'_i \rangle$ and at each stage the corresponding sum in non-decreasing. Thus, finally we arrive at

$$\sum_{i=1}^{n} f(a'_i + n_i) \leq \sum_{i=1}^{n} f(a_i + b_i).$$

For the other inequality we define $c_i = a_{n+1-i}$ so that $c_1 \geq c_2 \geq \cdots \geq c_n$. We have to show that

$$\sum_{i=1}^{n} f(a_{n+1-i} + b_i) \leq \sum_{i=1}^{n} f(a'_i + b_i).$$

Setting $c'_i = a'_i$, we have

$$\sum_{i=1}^{n} f(c_i + b_i) \le \sum_{i=1}^{n} f(c_i' + b_i),$$

where $(c_1', c_2', \dots, c_n')$ is a permutation of $(c_1, c_2, \dots, c_n)$. We take $\langle c_i' \rangle \ne \langle c_i \rangle$ and let r be the smallest index such that $c_r' \ne c_r$. This forces that $c_r' \in \{c_{r+1}, c_{r+2}, \dots, c_n\}$ and $c_r' < c_r$. We see that $(c_r', c_{r+1}', \dots, c_n')$ is a permutation of $(c_r, c_{r+1}, \dots, c_n)$. We can find $k > r, l > r$ such that $c_k' = c_r$ and $c_r' = c_l$. This implies that $c_k' - c_r' = c_r - c_l \ge 0$ and $b_k - b_r \ge 0$. Now we can interchange c_r' and c_k' to get a permutation $(c_1'', c_2'', \dots, c_n'')$ of $(c_1', c_2', \dots, c_n')$; thus

$$c_i'' = c_i' \text{ for } i \ne r, k,\, c_r'' = c_k' = c_r,\, c_k'' = c_r' = c_l.$$

We compute the difference between

$$S'' = \sum_{i=1}^{n} f(c_i'' + b_i),\, S' = \sum_{i=1}^{n} f(c_i' + b_i),$$

and obtain

$$\begin{aligned} S'' - S' &= f(c_r'' + b_r) + f(c_k'' + b_k) - f(c_r' + b_r) - f(c_k' + b_k) \\ &= f(c_r + b_r) + f(c_l + b_k) - f(c_l + b_r) - f(c_r + b_k). \end{aligned}$$

We see that

$$c_l + b_r \le c_l + b_k < c_r + b_k,\, c_l + b_r \le c_r + b_r < c_r + b_k.$$

From the convexity of f

$$(c_r + b_k - c_l - b_r) f(c_l + b_k) \le (c_r - c_l) f(c_l + b_r) + (b_k - b_r) f(c_r + b_k),$$

and

$$(c_r + b_k - c_l - b_r) f(c_r + b_r) \le (b_k - b_r) f(c_l + b_r) + (c_r - c_l) f(c_r + b_k).$$

Adding, we get

$$\begin{gathered} (c_r + b_k - c_l - b_r)\{f(c_l + b_k) + f(c_r + b_r)\} \le \\ (c_r + b_k - c_l - b_r)\{f(c_l + b_r) + f(c_r + b_k)\}. \end{gathered}$$

We know that $c_r + b_k - c_l - n_r \ne 0$, so we have

$$f(c_l + b_k) + f(c_r + b_r) \le f(c_l + b_r) + f(c_r + b_k).$$

Thus, we see that $S'' \le S'$. We also see that the new sequence $\langle c_i'' \rangle$ has the property: $c_r'' = c_r$ and $C_i'' = c_i$ for $1 \le i < r$. Now we repeat the above argument by replacing $\langle c_i' \rangle$ with $\langle c_i'' \rangle$. At each step the sum will never increase. After at most $n - 1$ steps we arrive at the sequence $\langle c_i \rangle$. Thus, we find that the corresponding sum does not exceed to that of S'. Thus we get

$$\sum_{i=1}^{n} f(c_i + b_i) \le \sum_{i=1}^{n} f(c_i' + b_i),$$

which was to be proved. $\square$

10.12 Bernoulli's Inequality

Theorem 33

For every real number $r \geq 1$ and real number $x \geq -1$, we have

$$(1+x)^r \geq 1+rx$$

while for $0 \leq r \leq 1$ and real number $x \geq -1$ we have

$$(1+x)^r \leq 1+rx.$$

Proof

Using the convexity of $f(x) = \ln(x)$ on $(0, \infty)$. Since $x \geq -1$, we have $1+x \geq 0$. If $0 \leq r \leq 1$, we have

$$\ln(1+rx) = \ln(r(1+x)+1-r) \geq r\ln(1+x) + (1-r)\ln(1) = r\ln(1+x).$$

Taking antilog gives $(1+x)^r \leq 1+rx$. When $1 \leq r < \infty$,

$$\ln(1+x) = \ln\left(\frac{r-1}{r} + \frac{1}{r}(1+rx)\right) \geq \frac{r-1}{r}\ln(1) + \frac{1}{r}\ln(1+rx) = \frac{1}{r}\ln(1+rx).$$

This gives $(1+x)^r \geq 1+rx$. □

10.13 Popoviciu's Inequality

Theorem 34

Let $f : I \to \mathbb{R}$. If f is convex, then for any three p;oints x, y, z in I:

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \tag{10.16}$$

Proof

Without loss of generality, we can assume that $x \leq y \leq z$. If $x \leq y \leq \frac{x+y+z}{3}$, then

$$\frac{x+y+z}{3} \leq \frac{x+z}{2} \leq z \text{ and } \frac{x+y+z}{3} \leq \frac{y+z}{2} \leq z.$$

Therefore, there exists $s, t \in [0,1]$ such that

$$\frac{x+z}{2} = \left(\frac{x+y+z}{3}\right)s + z(1-s)$$

$$\frac{y+z}{2} = \left(\frac{x+y+z}{3}\right)t + z(1-t)$$

Adding, we get

$$\frac{x+y-2z}{2}=\frac{x+y-2z}{3}(s+t)\Rightarrow s+t=\frac{3}{2}.$$

As f is a convex function

$$f\left(\frac{x+z}{2}\right)\le s.f\left(\frac{x+y+z}{3}\right)+(1-s).f(z)$$

$$f\left(\frac{y+z}{2}\right)\le t.f\left(\frac{x+y+z}{3}\right)+(1-t).f(z)$$

and

$$f\left(\frac{x+y}{2}\right)\le\frac{1}{2}f(x)+\frac{1}{2}f(y).$$

Adding together last three inequalities we get the required inequality. The case when $\frac{x+y+z}{3}\le y$ is considered similarly, bearing in mind that $x\le\frac{x+z}{2}\le\frac{x+y+z}{3}$ and $x\le\frac{y+z}{2}\le\frac{x+y+z}{3}$.

When f is a concave function, the inequality gets reversed. □

10.14 Majorization

Definition: Given two seuquences $\langle a\rangle=(a_1,a_2,\ldots,a_n)$ and $\langle b\rangle=(b_1,b_2,\ldots,b_n)$ where $a_i,b_i\in\mathbb{R}\ \forall i\in\{1,2,\ldots,n\}$. We say that the sequence $\langle a\rangle$ majorizes the seuqnece $\langle b\rangle$, and write $\langle a\rangle\succ\langle b\rangle$, if the following conditions are fulfilled:

$$a_1\ge a_2\ge\cdots\ge a_n;$$

$$b_1\ge b_2\ge\cdots\ge b_n;$$

$$a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n;$$

$$a_1+a_2+\cdots+a_k\ge b_1+b_2+\cdots+b_k\ \forall k\in\{1,2,\ldots,n-1\}.$$

10.15 Karamata's Inequality

Theorem 35

Let $f:[a,b]\to\mathbb{R}$ be a convex function. Suppose that $(x_1,\ldots,x_n)\succ(y_1,\ldots,y_n)$ where $x_1,\ldots,x_n,y_1,\ldots,y_n\in[a,b]$. Then we have:

$$\sum_{i=1}^{n}f(x_i)\ge\sum_{i=1}^{n}f(y_i). \tag{10.17}$$

Proof

If $f(x)$ is a convex function over the interval (a,b), then $\forall a\le x_1\le x_2\le b$ and $g(x,y)=\frac{f(y)-f(x)}{y-x}$, $f(x_1,x)\le g(x_2,x)$. If $x<x_1$, then

$$g(x_1, x) = \frac{f(x_1) - f(x)}{x_1 - x} \leq \frac{f(x_1) - f(x)}{x_1 - x} = g(x_2 - x)\,.$$

We can argue similarly for other values of x.

We define a sequence $\langle C \rangle$ such that $c_i = g(a_i, b_i)$

We also define sequences $\langle A \rangle$ and $\langle B \rangle$ such that

$$A_i = \sum_{j=1}^{i} a_j, A_0 = 0 \text{ and } B_i = \sum_{j=1}^{i} b_j, B_0 = 0$$

If we assume that $a_i \geq a_{i+1}$ and similarly $b_i \geq b_{i+1}$, then we get that $c_i \geq c_{i+1}$. Now, we know that

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_I) = \sum_{i=1}^{n} c_i(a_i - b_i) = \sum_{i=1}^{n} c_i(A_i - A_{i-1} - B_i + B_{i+1})$$

$$= \sum_{i=1}^{n} c_i(A_i - B_i) - \sum_{i=0}^{n-1} c_{i+1}(A_i - B_i) = \sum_{i=1}^{n} (c_i - c_{i+1})(A_i - B_i) \geq 0$$

Therefore,

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i)\,.$$

10.16 Muirhead's Inequality

Theorem 36

If a sequence $\langle a \rangle$ majorises a sequence $\langle b \rangle$, and $x_1, x_2, \dots, x_n$ be a set of postiive real numbers then

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \dots x_n^{b_n} \tag{10.18}$$

Proof

We define a sequence $\langle c \rangle$ such that $\sum_{i=1}^{n} c_i = 0$, the we observe

$$\sum_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n} \geq n!$$

for real $x_1, x_2, \dots x_n$. By AM-GM we know that

$$\frac{\sum\limits_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}}{n!} \geq n! \sqrt{\prod_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}}$$

$$\Rightarrow n! \sqrt{\prod_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}} = n! \sqrt{\prod_{i=1}^{n} x_i^{(n-1)!(c_1 + c_2 + \dots c_n)}} = 1$$

$$\Rightarrow \sum_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n} \geq n!$$

We defined out sequence $\langle c \rangle$ such that $c_i = a_i - b_i$ which gives us $\sum c_i = \sum a_i - \sum b_i = 0$

Thus, $\sum_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n} - n! \geq 0$. Multiplying with $\sum_{sym} \prod_{i=1}^n x_i^{b_i}$, we get

$$\left(\sum_{sym} \prod_{i=1}^n x_i^{b_i} \right) \left(\sum_{sym} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n} - 1 \right)$$

$$= \sum_{sym} \prod_{i=1}^n x_i^{b_i + c_i} - \prod_{i=1}^n x_i^{b_1} \geq 0$$

$$\Rightarrow \sum_{sym} \prod_{i=1}^n x_i^{a_i} - \prod_{i=1}^n x_i^{b_1} \geq 0$$

Hence, it is proved. □

10.17 Schur's Inequality

Theorem 37

Let x, y, z be non-negative real numbers. For any $r > 0$, we have

$$\sum_{cyc} x^r (x - y)(x - z) \geq 0 \tag{10.19}$$

with equality if and only if $x = y = z$, or if two of x, y, z are equal and the third is 0.

Proof

When $r = 1$, the following case arises:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x).$$

Because L.H.S. is cyclic in x, y, z without loss of generality we can assume $x \geq y \geq z$. Rewriting L.H.S., we have

$$(x - y)\left[x^r(x - z) - y^r(y - z)\right] + z^r(z - x)(z - y).$$

We see that $x^r \geq y^r$ and $x - z \geq y - z$. Thus the expression inside brackets is non-negative. $(x - y)$ is also non-negative. z^r and $(z - x)(z - y)$ are also non-negtive. Thus entire expression is non-negative and hence the inequality is proven. □

Velentin Vornicu has given a general form of Schur's inequality. Consider $a, b, c, x, y, z \in \mathbb{R}$, where $a \geq b \geq c$, and either $z \geq y \geq z$ or $z \geq y \geq x$. Let $k \in \mathbb{Z}^+$, and let $f : \mathbb{R} \to \mathbb{R}_0^+$ be either convex or monotonic, then

$$f(x)(a - b)^k (a - c)^k + f(y)(b - a)^k (b - c)^k + f(z)(c - a)^k (c - b)^k \geq 0. \tag{10.20}$$

10.18 Symmetric Functions

Let $a_1, a_2, \dots, a_n$ be arbitrary real numbers. Considering the polynomial $P(x) = (x + a_1)(x + a_2) \cdots (x + a_n) = c_o x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n$. The the coefficients $c_0, c_1, \dots, c_n$ can be expressed as functions of a_1, a_2, a_n like $c_0 = 1, c_1 = a_1 + a_2 + \cdots + a_n$, $c_2 = a_1 a_2 + a_2 a_3 + \cdots, a_{n-1} a_n$, $c_3 = a_1 a_2 a_3 + a_2 a_3 a_4 + \cdots + a_{n-2} a_{n-1} a_n, \dots, c_n = a_1 a_2 \dots a_n$.

These are also called *elementary symmetric sum* and the first elementary symmetric sum of $f(x)$ is often written as $\sum_{sym} f(x)$ while the nth can be written as $\sum_{sym}^{n} f(x)$.

The *symmetric sum* $\sum_{sym} f(x_1, x_2, \dots, x_n)$ of a function $f(x_1, x_2, \dots, x_n)$ of n variables is defined to be $\sum_{\sigma} f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$, where σ ranges over over all permutations of $(1, 2, \dots, n)$. More generally symmetric sum of n variables is a sum that is unchanged by any permutatoin of its variables. Any symmetric sum can be written as a polynomial of elementary symmetric sums.

A *symmetric function* of n variables is a function that does not change by any permutation of its variables. Therefore,

$$\sum_{sym} f(x_1, x_2, \dots, x_n) = n!\, f(x_1, x_2, \dots, x_n)$$

We define *symmetric average* p_k as $\frac{c_k}{\binom{n}{k}}$.

10.19 Newton's Inequality

Theorem 38

For non-negative $x_1, x_2, \dots, x_n$ and $0 < k < nm$

$$d_k^2 \geq d_{k-1} d_{k+1}, \tag{10.21}$$

equality holds when all x_i's are equal.

Proof

We will prove this by mathematical induction. A proof by calculus is also possible but we will not prove by that method.

For $n = 2$, the inequality becomes AM-GM inequaltiy. Let the inequality hold for $n = m - 1$ for some positive integer $m \geq 3$.

Let d_k' be the symmetric averages of $x_1, x_2, \dots, x_{m-1}$. Note that $d_k = \frac{n-k}{n} d_k' + \frac{k}{n} d_{k-1}' x_m$.

$$\begin{aligned} d_{k-1} d_{k+1} &= \left(\frac{n-k+1}{n} d_{k-1}' + \frac{k-1}{n} d_{k-2}' x_m \right) \left(\frac{n-k-1}{n} d_{k+1}' + \frac{k+1}{n} d_k' x_m \right) \\ &= \frac{(n-k+1)(n-k-1)}{n^2} d_{k-1}' d_{k+1}' + \frac{(k-1)(n-k-1)}{n^2} d_{k-2}' d_{k+1}' x_m \end{aligned}$$

$$
\begin{aligned}
&+\frac{(n-k+1)(k+1)}{n^2}d'_{k-1}d'_k x_m+\frac{(k-1)(k+1)}{n^2}d'_{k-2}d'_k x_m^2\\
&\le \frac{(n-k+1)(n-k-1)}{n^2}d_k^{2}{}'+\frac{(k-1)(n-k-1)}{n^2}d'_{k-2}d'_{k+1}x_m\\
&+\frac{(n-k+1)(k+1)}{n^2}d'_{k-1}d'_k x_m+\frac{(k-1)(k+1)}{n^2}d_{k-1}^{2}{}' x_m^2\\
&\le \frac{(n-k+1)(n-k-1)}{n^2}d_k^{2}{}'+\frac{(k-1)(n-k-1)}{n^2}d'_{k-1}d'_k x_m\\
&+\frac{(n-k+1)(k+1)}{n^2}d'_{k-1}d'_k x_m+\frac{(k-1)(k+1)}{n^2}d_{k-1}^{2}{}' x_m^2\\
&=\frac{(n-k)^2}{n^2}d_k^{2}{}'+\frac{2(n-k)k}{n^2}d'_k d'_{k-1}x_m+\frac{k^2}{n^2}d_{k-1}^{2}{}' x_m^2-\left(\frac{d_k}{n}-\frac{d_{k-1}x_m}{n}\right)^2\\
&\le\left(\frac{n-k}{n}d'_k+\frac{k}{n}d'_{k-1}x_m\right)^2=d_k^2
\end{aligned}
$$

Hence, it is proven by induction. □

10.20 Maclaurin's Inequality

Theorem 39

For non-negative $x_1, x_2, \dots, x_n$ and $0 < k < nm$

$$d_1 \ge d_2^{1/2} \ge \cdots \ge d_n^{1/n}, \tag{10.22}$$

equality holds when all x_i's are equal.

Proof

Following Newton's inequality it is enough to show that $d_{n-1}^{1/(n-1)} \ge d_n^{1/n}$.

Since this is a homogeneous inequaqlity, it can be normalized. Thus, $d_n = \prod x_i = 1$ We then transform the inequality to(by exponentiating both sides by $n-1$)

$$\frac{\sum 1/x_i}{n} \ge 1^{(n-1)/n} = 1.$$

We know that the G.M. of $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}$ is 1 and hence the inequality is true by AM-GM. □

10.21 Aczel's Inequality

Theorem 40

If $a_1^2 > a_2^2 + \cdots + a_n^2$ or $b_1^2 > b_2^2 + \cdots + b_n^2$, then

$$(a_1b_1 - a_2b_2 - \cdots - a_nb_n)^2 \geq (a_1^2 - a_2^2 - \cdots - a_n^2)(b_1^2 - b_2^2 - \cdots - b_n^2) \tag{10.23}$$

Proof

Consider the function

$$f(x) = (a_1x - b_1)^2 - \sum_{i=2}^{n}(a_ix - b_i)^2$$
$$= (a_1^2 - a_2^2 - \cdots - a_n^2)x^2 - 2(a_1b_2 - a_2b_2 - \cdots - a_nb_n)x + (b_1^2 - b_2^2 - \cdots - b_n^2).$$

We have $f\left(\frac{b_1}{a_1}\right) = -\sum_{i=2}^{n}\left(a_i\frac{b_1}{a_1} - b_i\right)^2 \leq 0$, and from $a_1^2 > a_2^2 + \cdots + a_n^2$ we get $\lim_{x\to\infty} f(x) \to \infty$. Therefore, $f(x)$ must have at least one root, $\Leftrightarrow D = (a_1b_1 - a_2b_2 - \cdots - a_nb_n)^2 - (a_1^2 - a_2^2 - \cdots - a_n^2)(b_1^2 - b_2^2 - \cdots - b_n^2) \geq 0$. □

10.22 Carleman's Inequality

Theorem 41

Let $a_1, a_2, \ldots, a_n$ be n non-negative real numbers, where $n \geq 1$ then

$$\sum_{i=1}^{\infty}(a_1a_2\cdots a_i)^{1/i} < e\sum_{i=1}^{\infty} a_i, \tag{10.24}$$

unless all of a_i's are equal to zero.

Proof

Let us define $c_n = n\left(1+\frac{1}{n}\right)^n = \frac{(n+1)^n}{n^{n-1}}$. Then for all positive integers i,

$$(c_1 \ldots c_i)^{1/i} = i + 1$$

$$\Rightarrow \sum_{i=1}^{\infty}(a_1 \ldots a_i)^{1/i} = \sum_{i=1}^{\infty}\frac{(c_1a_1 \ldots c_ia_i)^{1/i}}{(c_1 \ldots c_i)^{1/i}} = \sum_{i=1}^{\infty}\frac{(c_1a_1 \ldots c_ia_i)^{1/i}}{i+1}.$$

Using AM-GM inequality, we get

$$\sum_{i=1}^{\infty}\frac{(c_1a_1 \ldots c_ia_i)^{1/i}}{i+1} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{c_ja_j}{i(i+1)} = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\frac{c_ja_j}{i(i+1)}.$$

Using the partial fraction for $\frac{1}{i(i+1)}$

$$\sum_{i=j}^{\infty}\frac{1}{i(i+1)} = \sum_{i=j}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{j}.$$

$$\Rightarrow \sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\frac{c_ja_j}{i(i+1)} = \sum_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^j a_i.$$

Since $\left(1+\frac{1}{j}\right)^j < e, \forall\ j \in I$ the inequality holds. $\square$

10.23 Sum of Squares(SOS Method)

Sum of sqaures or S.O.S. method revolves around the basic fact that sum of squares is a non-negative quantity. As you can see it requires knowledge only of very basic inequalitites which makes it highly desirable. By using SOS method we rewrite inequalitites as *sum of squares* to prove them as non-negative using only basic inequalities.

Proposition 1

Let $a, b, c \in \mathbb{R}$. Then $(a-c)^2 \leq 2(a-b)^2 + 2(b-c)^2$.

Proof

We have

$$(a-c)^2 \leq 2(a-b)^2 + 2(b-c)^2$$

$$\Leftrightarrow a^2 - 2ac + c^2 \leq 2(a^2 - 2ab + b^2) + 2(b^2 - 2bc + c^2)$$

$$\Leftrightarrow a^2 + 4b^2 + c^2 04ab - 4bc + 2ac \geq 0$$

$$\Leftrightarrow (a + c - 2b)^2 \geq 0,$$

which clearly holds. $\square$

Proposition 2

Let $a \geq b \geq c$. Then $(a-c)^2 \geq (a-b)^2 + (b-c)^2$.

Proof

We have

$$(a-c)^2 \geq (a-b)^2 + (b-c)^2$$

$$\Leftrightarrow a^2 - 2ac + c^2 \geq (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2)$$

$$\Leftrightarrow b^2 + ac - ab - b \leq 0$$

$$\Leftrightarrow (b-a)(b-c) \leq 0,$$

which is true for $a \geq b \geq c$. $\square$

Proposition 3

Let $a \geq b \geq c$. Then $\frac{a-c}{b-c} \geq \frac{a}{b}$.

Proof

Given $\frac{a-c}{b-c} \geq \frac{a}{b}$

$$\Leftrightarrow b(a-c) \geq a(b-c) \Leftrightarrow ac \geq bc \Leftrightarrow a \geq b.$$

Theorem 42

Consider the expression $S = S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2$, *where* S_a, S_b, S_c *are functions of* a, b, c.

1. *If* $S_a, S_b, S_c \geq 0$ *then* $S \geq 0$.
2. *If* $a \geq b \geq c$ *or* $a \leq b \leq c$ *and* $S_b, S_b + S_a, S_b + S_c \geq 0$ *then* $S \geq 0$.
3. *If* $a \geq b \geq c$ *or* $a \leq b \leq c$ *and* $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$ *then* $S \geq 0$.
4. *If* $a \geq b \geq c$ *and* $S_b, S_c, a^2S_b + b^2S_a \geq 0$ *then* $S \geq 0$.
5. *If* $S_a+S_b \geq 0$ *or* $S_b+S_c \geq 0$ *or* $S_c+S_a \geq 0$ *or* $S_a+S_b+S_c \geq 0$ *and* $S_aS_b+S_bS_c+S_cS_a \geq 0$ *then* $S \geq 0$.

Proof

1. If $S_a, S_b, S_c \geq 0$ then clearly $S \geq 0$.
2. Let us assume that $a \geq b \geq c$ or $a \leq b \leq c$ and $S_b, S_b + S_a, S_b + S_c \geq 0$.

   By Proposition (Preposition 2), it follows that $(a-c)^2 \geq (a-b)^2 + (b-c)^2$, so we have

$$\begin{aligned} S &= S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \\ &\geq S_a(b-c)^2 + S_b[(a-b)^2 + (b-c)^2] + S_c(a-b)^2 \\ &= (b-c)^2(S_a + S_b) + (a-b)^2(S_b + S_c). \end{aligned}$$

   Thus, $S \geq 0$ because $S_a + S_b, S_b + S_c \geq 0$.
3. Let us assume that $a \geq b \geq c$ or $a \leq b \leq c$ and $S_a, S_c, S_a + 2S_b, S_c + 2S_b \geq 0$.

   Then if $S_b \geq 0$ clearly $S \geq 0$.

   For case when $S_b \leq 0$, by Proposition (Preposition 1), we have $(a-c)^2 \leq 2(a-b)^2 + 2(b-c)^2$. Therefore

$$\begin{aligned} S &= S_b(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2 \\ &\geq S_a(b-c)^2 + S_b[2(a-b)^2 + 2(b-c)^2] + S_c(a-b)^2 \\ &= (b-c)^2(S_a + 2S_b) + (a-b)^2(S_c + 2S_b) \end{aligned}$$

   which is true for the given conditions.
4. Given $a \geq b \geq c$ and $S_b, S_c, a^2S_b + b^2S_a \geq 0$

   By Proposition (Preposition 3), we have $\frac{a-c}{b-c} \geq \frac{a}{b}$. Therefore

$$\begin{aligned}S &= S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2 \geq S_a(b-c)^2 + S_b(a-c)^2\\ &= (b-c)^2\left[S_a + S_b\left(\frac{a-c}{b-c}\right)^2\right] \geq (b-c)^2\left[S_a + S_b\left(\frac{a}{b}\right)^2\right]\\ &= (b-c)^2\left(\frac{b^2S_a + a^2S_b}{b^2}\right),\end{aligned}$$

which is true for given conditions.

5. We assume that $S_b + S_c \geq 0$. Then

$$\begin{aligned}S &= S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2\\ &= S_a(b-c)^2 + S_b[(c-b)+(b-a)]^2 + S_c(a-b)^2\\ &= (S_b+S_c)(a-b)^2 + 2S_b(c-b)(b-a) + (S_a+S_b)(b-c)^2\\ &= (S_b+S_c)\left(b-a+\frac{S_b}{S_b+S_c}(c-b)\right)^2 + \frac{S_aS_b+S_bS_c+S_cS_a}{S_b+S_c}(c-b)^2 \,\&\geq 0.\end{aligned}$$

Every difference $\sum_{cyc} x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n} - \sum_{cyc} x_1^{\beta_1}x_2^{\beta_2}\ldots x_n^{\beta_n}$ where $\alpha_1+\alpha_2+\cdots+\alpha_n = \beta_1+\beta_2+\cdots+\beta_n$ can be written in SOS form.

Some special cases are given below:

1. $a^2+b^2+c^2-ab-bc-ca = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$
2. $a^3+b^3+c^3-3abc = \frac{a+b+c}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$
3. $a^b+b^2c+c^2a-ab^2-bc^2-ca^2 = \frac{(a-b)^3+(b-c)^3+(c-a)^3}{3}$
4. $a^3+b^3+c^3-a^2b-b^2c-c^2a = \frac{(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2}{3}$
5. $a^4+b^4+c^4-a^3b-b^3c-c^3b = \frac{(3a^2+2ab+b^2)(a-b)^2+(3b^2+2bc+c^2)(b-c)^2+(3c^2+2ca+a^2)(c-a)^2}{4}$
6. $a^3b+b^3c+c^3a-ab^3-bc^3-ca^3 = \frac{a+b+c}{3}\left[(b-a^3)+(c-b)^3+(a-c)^3\right]$
7. $a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 = \frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{2}$

Theorem 43

Consider two polynomials having the same degree and same number of variables A and B. The difference of these two polynomilas can be written in SOS form:

$$\sum_{cyc} a_1^{\alpha_1}a_2^{\alpha_2}\ldots a_n^{\alpha_n} - \sum_{cyc} a_1^{\beta_1}a_2^{\beta_2}\ldots a_n^{\beta_n} = \sum P_{ij}(a)(a_i-a_j)^2,$$

where $\alpha_1+\alpha_2+\cdots+\alpha_n = \beta_1+\beta_2+\cdots+beta_n = m$ *and* $a = (a_1, a_2, \ldots, a_n)$.

Proof

We need to prove the following lemma first.

Lemma 2

If $a = (a_1, a_2, \dots, a_n)$ and $\alpha_1 + \alpha_2 + \alpha_n = m$, then:

$$\sum_{cyc} a_1^n - \sum_{cyc} a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n} = \sum P_{ij}(a)\,(a_i - a_j)^2$$

We prove this lemma by induction over k, which will be the number of elements except 0 belonging to the set $\alpha_1, \alpha_2, \dots, \alpha_n$.

If $k = 1$, the theorem is obviously true.

If $k = 2$, the expression becomes $\sum_{cyc} a_1^m - \sum_{a_1}^t a_2^{m-t} = \sum P_{ij}(a)\,(a_i - a_j)^2$

We observe that $ta^m + (m-t)\,b^m - ma^t b^{m-t} = P(a,b)\,(a-b)^2$. We also observe that $f(x) = tx^n + (m-t) - mx^t = 0$ has one repeated root which is 1 because $f(1) = f'(1) = 0$. Therefore $f(x)$ can be written like $Q(x)\,(x-1)^2$ where degree of Q will be $m-2$.

Let $x = \frac{a}{b}$, then we have: $b^m f\left(\frac{a}{b}\right) = ta^m + (m-t)\,b^m - ma^t b^{m-1} = b^{m-2} Q\left(\frac{a}{b}\right)(a-b)^2$.

However, b^{m-2} is a polynomial having 2 variables a, b because Q is a $m-2$ degree polynomial. If our proposition is already true with k, the number of elements except for 0 in the set of α, with $k+1$ we can transform this into the case of k as given below:

$a_1^{\alpha_1} a_2^{\alpha_2} \dots a_{k+1}^{\alpha_{k+1}} = \frac{\alpha_1 a_1^{\alpha_1+\alpha_2} + \alpha_2 a_2^{\alpha_1+\alpha_2} - (\alpha_1+\alpha_2)\,a_1^{\alpha_1} a_2^{\alpha_2}}{\alpha_1+\alpha_2} . a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}} \frac{\alpha_1}{\alpha_1+\alpha_2} a_1^{\alpha_1+\alpha_2}$

$a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}} + \frac{\alpha_2}{\alpha_1+\alpha_2} a_2^{\alpha_1+\alpha_3} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}}$

With $k = 2$: $\frac{\alpha_1 a_1^{\alpha_1+\alpha_2} + \alpha_2 a_2^{\alpha_1+\alpha_2} - (\alpha_1+\alpha_2)\,a_1^{\alpha_1} a_2^{\alpha_2}}{\alpha_1+\alpha_2} = H_{12}(a)\,(a_1 - a_2)^2$, we have:

$a_1^{\alpha_1} a_2^{\alpha_2} \dots a_{k+1}^{k+1} = Q_{12}(a)\,(a_1 - a_2)^2 + \frac{\alpha_1}{\alpha_1+\alpha_2} a_1^{\alpha_1+\alpha_2} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}} + \frac{\alpha_2}{\alpha_1+\alpha_2} a_2^{\alpha_1+\alpha_3} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}}$

$\therefore \sum_{cyc} a_1^m - \sum_{cyc} a_1^{\alpha_1} a_2^{\alpha_2} \dots a_{k+1}^{\alpha_{k+1}} = -\sum_{cyc} Q_{12}(a)\,(a_1 - a_2)^2 + \sum_{cyc} a_1^m - \frac{\alpha_1}{\alpha_1+\alpha_2}$

$\sum a_1^{\alpha_1+\alpha_2} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}} + \sum \frac{\alpha_2}{\alpha_1+\alpha_2} a_2^{\alpha_1+\alpha_3} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}} \sum_{a_1}^m - \sum_{cyc} a_1^{\alpha_1} \dots a_{k+1}^{\alpha_{k+1}}$

$= -\sum Q_{12}(a)\,(a_1 - a_2)^2 + \frac{\alpha_1}{\alpha_1+\alpha_2}\left(\sum_{cyc} a_1^m - \sum_{cyc} a_1^{\alpha_1+\alpha_2} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}}\right) +$

$\frac{\alpha_2}{\alpha_1+\alpha_2}\left(\sum_{cyc} a_1^m - \sum_{cyc} a_2^{\alpha_1+\alpha_2} a_3^{\alpha_3} \dots a_{k+1}^{\alpha_{k+1}}\right)$

So we see that these can be written in SOS form recursively. Hence proved. $\square$

10.24 Problems

Prove the following inequalities:

1. $a^2 + b^2 \geq 2ab$.
2. $\sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$, where $a > 0, b > 0$.
3. $\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$
4. $\frac{a+b}{2} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$, where $a > 0, b > 0$.
5. $a + b > 1 + ab$, where $b < 1 < a$.
6. $a^2 + b^2 > c^2 + (a + b - c)^2$, where $b < c < a$.
7. $2 \leq \frac{a}{b} + \frac{b}{a}$, where $ab > 0$.
8. $\frac{a}{b} + \frac{b}{a} \leq -2$, where $ab < 0$.
9. $x_1 \leq \frac{x_1+\cdots+x_n}{n} \leq x_n$, where $x_1 \leq \cdots \leq x_n$.
10. $\frac{x_1}{y_1} \leq \frac{x_1+\cdots+x_n}{y_1+\cdots+y_n} \leq x_n$, where $\frac{x_1}{y_1} \leq \cdots \leq \frac{x_n}{y_n}$ and $y_i > 0, i = 1, \ldots, n$.
11. $x_1 \leq (x_1 \ldots x_n)^{\frac{1}{n}} \leq x_n$, where $n \geq 2, 0 \leq x_1 \leq \ldots \leq x_n$.
12. $|a_1| + \cdots + |a_n| \geq |a_1 + a_2 + \cdots + a_n|$.
13. $\frac{a_1+\cdots+a_n}{n} \geq \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}$, where $n \geq 2, a_i > 0, i = 1, \ldots, n$.
14. $(a + b)\sqrt{\frac{a+b}{2}} \geq a\sqrt{b} + b\sqrt{a}$, where $a > 0, b > 0$.
15. $\frac{1}{2}(a + b) + \frac{1}{4} \geq \sqrt{\frac{a+b}{2}}$, where $a > 0, b > 0$.
16. $a(x + y - a) \geq xy$, where $x \leq a \leq y$.
17. $\frac{1}{x-1} + \frac{1}{x+1} > \frac{2}{x}$, where $x > 1$.
18. $\frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} > \frac{1}{2k+1} + \frac{1}{2k+2}$, where $k \in \mathbb{N}$.
19. $\frac{ab}{(a+b)^2} \leq \frac{(1-a)(1-b)}{[(1-a)+(1-b)]^2}$, where $0 < a \leq \frac{1}{2}, 0 < b \leq \frac{1}{2}$.
20. $\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$, where $k \in \mathbb{N}$.

21. $2^{n-1} \geq n$, where $n \in \mathbb{N}$.

22. $\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{5} + \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{1}{7} + \cdots + \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{100}{101} \cdot \frac{1}{103} < 1$.

23. $\frac{1-a}{1-b} + \frac{1-b}{1-a} \leq \frac{a}{b} + \frac{b}{a}$, where $0 < a, b \leq \frac{1}{2}$.

24. $\sum\limits_{i=1}^{n} \frac{1}{1-a_i} \sum\limits_{i=1}^{m} (1 - a_i) \leq \sum\limits_{i=1}^{n} \frac{1}{a_i} \sum\limits_{i=1}^{n} a_i$, where $0 < a_1, \ldots, a_n \leq \frac{1}{2}$.

25. $1 + \frac{1}{2^3} + \cdots + \frac{1}{n^3} < \frac{5}{4}$, where $n \in \mathbb{N}$.

26. $\frac{1}{1+a+b} \leq 1 - \frac{a+b}{2} + \frac{ab}{3}$, where $0 \leq a \leq 1, 0 \leq b \leq 1$.

27. $|x - y| < |1 - xy|$, where $|x| < 1, |y| < 1$.

28. $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \frac{2}{a} + \frac{2}{b} - \frac{2}{c}$, where $a > 0, b > 0, c > 0$.

29. $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} < \frac{1}{abc}$, where $a^2 + b^2 + c^2 = \frac{5}{3}$ and $a > 0, b > 0, c > 0$.

30. $3(1 + a^2 + a^4) \geq (1 + a + a^2)^2$.x

31. $(ac + bd)^2 + (ad - bc)^2 \geq 144$, where $a + b = 4, c + d = 6$.

32. $x_1^2 + x_2^2 + \cdots + x_{2n}^2 + na^2 \geq a\sqrt{2}(x_1 + x_2 + \cdots + x_{2n})$.

33. $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} \leq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2\sqrt{abc}}$, where $a > 0, b > 0, c > 0$.

34. $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2) < 0$, where $0 < a < b < c$.

35. $a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2$ where $a \geq b \geq c > 0$.

36. $\frac{y}{x} + \frac{y}{z} + \frac{x+z}{y} \leq \frac{(x+z)^2}{xz}$, where $0 < x \leq y \leq z$.

37. $\sqrt{1 + \sqrt{a}} + \sqrt{1 + \sqrt{a + \sqrt{a^2}}} + \cdots + \sqrt{1 + \sqrt{a + \cdots + \sqrt{a^n}}} < na$, where $n \geq 2$, $a \geq 2, n \in \mathbb{N}$.

38. $[5x] \geq [x] + \frac{[2x]}{2} + \frac{[3x]}{3} + \frac{[4x]}{4} + \frac{[5x]}{5}$, where $[x]$ is the integer part of the real number x.

39. $(n!)^2 \geq n^n$, where $n \in \mathbb{N}$.

40. $x^6 + x^5 + 4x^4 - 12x^3 + 4x^2 + x + 1 \geq 0$.

41. $\log^2 \alpha \geq \log \beta \log \gamma$, where $\alpha > 1, \beta > 1, \gamma > 1, \alpha^2 \geq \beta\gamma$.

42. $\log_4 5 + \log_5 6 + \log_6 7 + \log_7 8 > 4.4$.

43. $\frac{1}{3} + \frac{2}{3.5} + \cdots + \frac{n}{3.5 \ldots (2n+1)} < \frac{1}{2}$, where $n \in \mathbb{N}$.

44. $\frac{2^3+1}{2^3-1}\cdots\frac{n^3+1}{n^3-1}<\frac{3}{2}$, where $n\geq 2, n\in\mathbb{N}$.

45. $1.1!+2.2!+\cdots+n.n!<(n+1)!$, where $n\in\mathbb{N}$.

46. $\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots\left(1+\frac{1}{n^2}\right)<2$, where $n\geq 2, n\in\mathbb{B}$.

47. $\left(1-\frac{1}{p_1^2}\right)\left(1-\frac{1}{p_2^2}\right)\cdots\left(1-\frac{1}{p_n^2}\right)>\frac{1}{2}$, where $1<p_1<p_2<\cdots<p_n$, $p_i\in\mathbb{N}$, $i=1,2,\dots n$.

48. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots-\frac{1}{999}+\frac{1}{1000}<\frac{2}{5}$.

49. $\frac{a+b}{1+a+b}\leq\frac{a}{1+a}+\frac{b}{1+b}$, where $a\geq 0, b\geq 0$.

50. $\frac{a+b}{2+a+b}\geq\frac{1}{2}\left(\frac{a}{1+a}+\frac{b}{1+b}\right)$, where $a\geq 0, b\geq 0$.

51. $\sum_{i=1}^{n}\frac{a_1+2a_2+\cdots+ia_i}{i^2}\leq 2\sum_{i=1}^{n}a_i$, where $a_i\geq 0$, $i=1,2,\dots,n$.

52. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{41}{42}$, where $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$, $a,b,c\in\mathbb{N}$.

53. $\frac{4x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}>2$, where $x,y,z>0$.

54. $1<\frac{a}{a+b+d}+\frac{b}{a+b+c}+\frac{c}{b+c+d}+\frac{d}{a+c+d}<2$, where $a,b,c,d>0$.

55. $a+b>c+d$, where $a,b,c,d\geq\frac{1}{2}$ and $a^2+b>c^2+d$, $a+b^2>c+d^2$.

56. $(b-a)(9-a^2)+(c-a)(9-b^2)+(c-b)(9-c^2)\leq 24\sqrt{2}$, where $0\leq a\leq b\leq c\leq 3$.

57. If $0<a,b,c<1$, then one of the numbers $(1-a)b$, $(1-b)c$, $(1-c)a$ is not greater than $\frac{1}{4}$.

58. Let $a>0$, $b>0$, $c>0$, and $a+b+c=1$. Prove that $\sqrt{a+\frac{1}{4}(b-c)^2}+\sqrt{b+\frac{1}{4}(c-a)^2}+\sqrt{c+\frac{1}{4}(b-a)^2}\leq 2$.

59. Let $a>0, b>0, c>0$, and $a+b+c=1$. Prove that $\sqrt{a+\frac{1}{4}(b-c)^2}+\sqrt{b}+\sqrt{c}\leq\sqrt{3}$.

60. Find the smallest possible value of the expression: $\frac{a^4}{b^4}+\frac{b^4}{a^4}-\frac{a^2}{b^2}-\frac{b^2}{a^2}+\frac{a}{b}+\frac{b}{a}$, where $a,b>0$.

61. $\frac{(1-x_1)(1-x_2)\dots(1-x_n)}{x_1x_2\dots x_n}\geq(n-1)^n$, where $n\geq 2$, $x_i>0$, $i=1,2,\dots,n$ and $x_1+x_2+\cdots+x_n=1$.

62. $\frac{1}{1+x_1}+\frac{1}{1+x_2}+\cdots+\frac{1}{1+x_n}\geq\frac{n}{1+\sqrt[n]{x_1\dots x_n}}$, where $n\geq 2$, $x_1\geq 1, x_2\geq 1,\dots,x_n\geq 1$.

63. $abc+bcd+cda+dab\leq\frac{1}{27}+\frac{176}{27}abcd$, where $a,b,c,d\geq 0$, and $a+b+c+d=1$.

64. $0 \le xy + yz + zx - 2xyz \le \frac{7}{27}$, where $x, y, z \ge 0$, and $x + y + z = 1$.

65. Suppose that for numbers $x_1, x_2, \ldots, x_{1997}$, the following conditions holds: (a) $-\frac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$, $i = 1, 2, \ldots, 1997$, (b) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$. Find the greatest possible value of the expression $x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$.

66. Prove that $\cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_n(\tan\alpha_1 + \tan\alpha_2 + \cdots + \tan\alpha_n) \le \frac{(n-1)^{(n-1)/2}}{n^{(n-2)/2}}$, where $n \ge 2$ and $0 \le \alpha_i < \frac{\pi}{2}$, $i = 1, 2, \ldots, n$.

67. Prove that $\sum_{i=1}^{n} x_i^k(1 - x_i) \le a_k$, where $k \ge 2$, $k \in \mathbb{N}$, and $a_k = \max_{[0;1]}[x^k(1-x) + (1-x)^k x]$, $x_i \ge 0$, $i = 1, 2, \ldots, n$, $x_1 + x_2 + \cdots + x)n = 1$, $n \ge 2$.

68. $2(n-1)(x_2x_3 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_2x_n + \cdots + x_{n-1}x_n) - n^{n-1}x_1x_2 \ldots x_n \le n - 2$, where, $n \ge 2$, $x_1, x_2, \ldots x_n \ge 0$ and $x_1 + x_2 + \cdots + x_n = 1$.

69. $\frac{x_1+x_2+\cdots+x_n}{n} - \sqrt[n]{x_1x_2 \ldots x_n} \le$

$\frac{(\sqrt{x_1}-\sqrt{x_2})^2+(\sqrt{x_1}-\sqrt{x_3})^2+\cdots+(\sqrt{x_1}-\sqrt{x_n})^2+\cdots+(\sqrt{x_{n-1}}-\sqrt{x_n})^2}{n}$, where $n \ge 2, x_1, x_2, \ldots, x_n \ge 0$.

70. *Turkevici's Inequality*: $(n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) + \sqrt[n]{x_1^2x_2^2 \ldots x_n^2} \ge (x_1 + x_2 + \cdots + x_n)^2$, where $n \ge 2, x_2, x_2, \ldots, x_n \ge 0$.

71. Prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$, where $a > 0, b > 0, c > 0$.

72. Prove that $\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2$, where $a > 0, b > 0, c > 0$.

73. $(a + b)(b + c)(c + a) \ge 8abc$, where $a > 0, b > 0, c > 0$.

74. $(a+b+c-d)(b+c+d-a)(c+d+a-b)(d+a+b-c) \le (a+b)(b+c)(c+d)(d+a)$, where $a > 0, b > 0, c > 0, d > 0$.

75. (*Schur's Inequality*) $a^3 + b^3 + c^3 + 3abc \ge a^2b + ab^2 + b^2c + bc^2 + ca^2 + c^2a$, where $a > 0, b > 0, c > 0$.

76. $\left(1 + \frac{4a}{b+c}\right)\left(1 + \frac{4b}{c+a}\right)\left(1 + \frac{4c}{a+b}\right) > 25$, where $a > 0, b > 0, c > 0$.

77. $\frac{\log(a-1)}{\log a} < \frac{\log a}{\log(a+1)}$, where $a > 1$.

78. (*Schur's Inequality*) $abc \ge (a + b - c)(c + a - b)(b + c - a)$, where $a > 0, b > 0, c > 0$.

79. $x^8 + y^8 \ge \frac{1}{128}$, if $x + y = 1$.

80. $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \ge 12.5$, if $a > 0, b > 0$ and $a + b = 1$.

81. $\left(x_1+\frac{1}{x_1}\right)^2+\dots+\left(x_n+\frac{1}{x_2}\right)^2 \geq \frac{(n^2+1)^2}{n}$, if $n \geq 2, x_1>0,\dots,x_n>0$ and $x_1+\dots+x_n=1$.

82. $a^4+b^4+c^4 \geq abc(a+b+c)$.

83. $x^2+y^2 \geq 2\sqrt{2}(x-y)$, if $xy=1$.

84. $\sqrt{6a_1+1}+\sqrt{6a_2+1}+\sqrt{6a_3+1}+\sqrt{6a_4+1}+\sqrt{6a_5+1} \leq \sqrt{55}$, if $a_1>0,\dots,a_5>0$ and $a_1+\dots+a_5=1$.

85. $6a+4b+5c \geq 5\sqrt{ab}+3\sqrt{bc}+7\sqrt{ca}$, where $a \geq 0, b \geq 0, c \geq 0$.

86. $2(a^4+b^4)+17>16ab$.

87. $\left(\frac{1+nb}{n+1}\right)^{n+1} \geq b^n$, where $n \in \mathbb{N}, b>0$.

88. $\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$, where $n \in \mathbb{N}$.

89. $\left(1+\frac{1}{n}\right)^{n+1} < \left(1+\frac{1}{n+1}\right)^{n+2}$, where $n \in \mathbb{N}$.

90. $\left(1+\frac{m}{n-1}\right)^{(n-1)/m} < \left(1+\frac{m}{n}\right)^{n/m} < \left(1+\frac{m-1}{n}\right)^{n/(m-1)}$, where $m>1, n>1$ and $m,n \in \mathbb{N}$.

91. $n! < \left(\frac{n+1}{2}\right)^n$, where $n=2,3,4,\dots$.

92. $n(n+1)^{1/n} < n+S_n$, where $S_n=\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}$, $n=2,3,4,\dots$.

93. $n-S_n>(n-1)^{1/(1-n)}$, where $S_n=\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}$, $n=3,4,\dots$.

94. $(q^n-1)(q^{n+1}+1) \geq 2nq^n(q-1)$, where $q>1, n \in \mathbb{N}$.

95. $a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd \geq 10$, where $a,b,c,d>0$, and $abcd=1$.

96. $\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq \left(\frac{1+abc}{2\sqrt{abc}}\right)^3$, where $a,b,c>0$.

97. $\left(a+\frac{1}{b}-t\right)\left(b+\frac{1}{c}-t\right)\left(c+\frac{1}{a}-t\right) \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1-t)^2+4-3t$, where $a, b, c, t>0$ and $abc=1$.

98. $n\sqrt[n]{a_1a_2\dots a_n}-(n-1)\sqrt[n-1]{a_1a_2\dots a_{n-1}} \leq a_n$, where $a_i>0, i=1,2,\dots,n$, $n=3,4,\dots$.

99. $\sqrt[n]{a_1a_2\dots a_n}+\sqrt[n]{b_1b_2\dots b_n}+\dots+\sqrt[n]{k_1k_2\dots k_n}$

$\leq \sqrt[n]{(a_1+b_1+\dots+k_1)(a_2+b_2+\dots+k_2)\cdots(a_n+b_n+\dots+k_n)}$ where

$a_1,a_2,\dots,a_n,b_1,b_2,\dots,b_n,\dots,k_1,k_2,\dots,k_n>0$.

100. $na^k-ka^n \leq n-k$, where $n>k$, $n,k \in \mathbb{N}$, $a>0$.

101. $\frac{x_1^2}{x_2}+\frac{x_2^3}{x_3^2}+\cdots+\frac{x_n^{n+1}}{x_1^n}\geq x_1+x_2+\cdots+x_n$, where $n\geq 2, n\in\mathbb{N}, x_1=\min(x_1,x_2,\ldots,x_n)>0$.

102. $\frac{a^{x_1-x_2}}{x_1+x_2}+\frac{a^{x_2-x_3}}{x_2+x_3}+\cdots+\frac{a^{x_n-x_1}}{x_n+x_1}\geq\frac{n^2}{2\sum\limits_{i=1}^{n}x_i}$, where $a>0, x_i>0, i=1,2,\ldots,n$.

103. $\sqrt[p]{x_1+1}+\sqrt[p]{x_2+1}+\cdots+\sqrt[p]{x_n+1}\leq n+1$, where $n\geq 2$, $x_1, x_2, x_n>0$, $x_1+x_2+\cdots+x_n=p$, $p\in\mathbb{N}$, $p\geq 2$.

104. $x^k(1-x^m)\leq\frac{k^{k/m}.m}{(k+m)^{1+k/m}}$, where $0\leq x\leq 1$, $k,m\in\mathbb{N}$.

105. $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}\geq\frac{3\sqrt{3}}{2}$, where $x,y,z>0$ and $x^2+y^2+z^2=1$.

106. $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}\geq\frac{9+3\sqrt{3}}{2}$, where $x,y,z>0$ and $x^2+y^2+z^2=1$.

107. Find the minimum value of the funciton $f(x)=\frac{1}{\sqrt[n]{1+x}}+\frac{1}{\sqrt[n]{1-x}}$ in $[0,1)$, where $n\in\mathbb{N}, n>1$.

108. Find the minimum value of the funciton $f(x)=ax^m+\frac{b}{x^n}$ in $(0,\infty)$, where $a,b>0$, $m,n\in\mathbb{N}$.

109. Find in $[a,b]\,(0<a<b)$ a point x_0 such that the function $f(x)=(x-a)^2(b^2-x^2)$ attains its maximum value in $\lfloor a,b\rfloor$ at x_0.

110. Find the greatest possible value of the product xyz given $x,y,z>0$, and $2x+\sqrt{3}y+\pi z=1$.

111. Find the maximum and minimum values of the function $y=\frac{x}{ax^2+b}$, where $a,b>0$.

112. Find the maximum value of the function $y=\frac{5\sqrt{x^2+6x+8}+12}{x+3}$.

113. Find the maximum value of the function $y=\frac{\sqrt[3]{(x^2+1)^2(x^2+3)}}{3x^3+4}$.

114. Solve the system of equations: $x+y=2, xy-z^2=1$.

115. Solve the system of equations: $x+y+z=3, x^2+y^2+z^2=3$.

116. Given $a+b+c+d+e=8$, $a^2+b^2+c^2+d^2+e^2=16$, find the greatest possible value of e.

117. Find the minimum value of the expression $\frac{x_1}{x_2}+\frac{x_3}{x_4}+\frac{x_5}{x_6}$ if $1\leq x_1\leq x_2\leq x_3\leq x_4\leq x_5\leq x_6\leq 1000$.

118. Solve the equation $x^4+y^4+2=4xy$.

119. Find all integer solutions of the equation $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}=3$.

120. Prove that $x_1^{\alpha}+x_2^{\alpha}+\cdots+x_n^{\alpha} \geq x_1^{\beta}+x_2^{\beta}+\cdots+x_n^{\beta}$, where $n \geq 2, x_1>0, x_2>0, \cdots, x_n>0, \alpha>\beta \geq 0$, and $x_1 x_2 \ldots x_n=1$.

121. Prove that $x_1^{\alpha}+x_2^{\alpha}+\cdots+x_n^{\alpha} \geq x_1^{\beta}+x_2^{\beta}+\cdots+x_n^{\beta}$, where $n \geq 2, x_1>0, x_2>0, \cdots, x_n>0, \alpha \geq (n-1)|\beta|$, and $x_1 x_2 \ldots x_n=1$.

122. Prove that $x^2 y+y^2 z+z^2 x \leq \frac{4}{27}$, where $x, y, z \geq 0$ and $x+y+z=1$.

123. Prove that $\frac{1+a}{1+ab}+\frac{1+b}{1+bc}+\frac{1+c}{1+cd}+\frac{1+d}{1+da} \geq 4$, where $a, b, c, d>0$ and $abcd=1$.

124. Prove that $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d} \geq 4$, where $a, b, c, d>0$ and $abcd=1$.

125. Prove that $2ST>\sqrt{3(S+T)[S(bd+df+fb)+T(ac+ce+ea)]}$, where $0<a<b<c<d<e<f$ and $a+c+e=S, b+d+f=T$.

126. Prove that $\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{4} \leq \sqrt[4]{a . \frac{a+b}{2} . \frac{a+b+c}{3} . \frac{a+b+c+d}{4}}$, where $a>0, b>0, c>0, d>0$.

127. Prove that $a^{12}+(ab)^6+(abc)^4+(abcd)^3 \leq 1.43(a^{12}+b^{12}+c^{12}+d^{12})$.

128. $\left(1+\frac{1}{n}\right)^n>2$, where $n \in \mathbb{N}$.

129. $(1+a_1)(1+a_2) \cdots(1+a_n) \leq 1+\frac{S}{1!}+\cdots+\frac{S^n}{n!}$, where $n \geq 2, S=a_1+a_2+\cdots+a_n, a_i>0, i=1,2, \ldots, n$.

130. $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64$, where $a, b, c>0$ and $a+b+c=1$.

131. $\frac{a^n-1}{a^n(a-1)} \geq n+1-a^{\frac{n(n+1)}{2}}$, where $a>0, a \neq 1$.

132. $na^{n+1}+1 \geq (n+1) a^n$, where $a>0$.

133. $(\sqrt{k}+\sqrt{k+1})(\sqrt{k+1}+\sqrt{k+2}) \cdots(\sqrt{n}+\sqrt{n+1}) \geq(\sqrt{n}-\sqrt{k})(\sqrt{n}+\sqrt{k}-1)+2$, where $n>k, n, k \in \mathbb{N}$.

134. $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} \geq n$, where $a_i>0, i=1,2, \ldots, n$.

135. $a_{n+1}+\frac{1}{a_1(a_2-a_1)(a_3-a_2) \cdots(a_{n+1}-a_n)} \geq n+2$, where $0<a_k<a_{k+1}, k=1,2, \ldots, n$.

136. $1+\frac{x}{2} \leq \frac{1}{\sqrt{1-x}}$, where $0 \leq x<1$.

137. $\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{d^4}+\frac{d^4}{e^4}+\frac{e^4}{a^4} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}$, where $abcde \neq 0$.

138. $\left(\frac{a}{b}\right)^{1999}+\left(\frac{b}{c}\right)^{1999}+\left(\frac{c}{d}\right)^{1999}+\left(\frac{d}{a}\right)^{1999} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$, where $a, b, c, d>0$.

139. Prove that $\sqrt{\frac{a_1+a_2}{a_3}}+\sqrt{\frac{a_2+a_3}{a_4}}+\cdots+\sqrt{\frac{a_{n-1}+a_n}{a_1}}+\sqrt{\frac{a_n+a_1}{a_2}} \geq n \sqrt{2}$, where $n>2$ and $a_1>0, a_2>0, \ldots, a_n>0$.

140. Prove that $\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\le\frac{3\sqrt{3}}{4}$, where $x^2+y^2+z^2=1$.

141. Prove that $\left(\frac{1}{a_1^2}-1\right)\left(\frac{1}{a_2^2}-1\right)\cdots\left(\frac{1}{a_n^2}-1\right)\ge(n^2-1)^n$, where $n\ge 2$, $a_1>0, a_2>0,\dots,a_n>0$ and $a_1+a_2+\dots+a_n=1$.

142. Find the maximum and minimum value of the expression $(1+u)(1+v)(1+w)$ if $0<u\le\frac{7}{16}, 0<v\le\frac{7}{16}, 0<w\le\frac{7}{16}$, and $u+v+w=1$.

143. Find the maximum value of the expression x^py^q if $x+y=a, x>0, y>0$ and $p,q\in\mathbb{N}$.

144. Find the maximum value of the expression $a+2c$ if for all x, one has $ax^2+bx+c\le\frac{1}{\sqrt{1-x^2}}$, where $|x|<1$.

145. Prove that $\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$, where $a>0, b>0, c>0$.

146. Prove that $\frac{1+a_1}{1-a_1}.\frac{1+a_2}{1-a_2}\cdots\frac{1+a_{n+1}}{1-a_{n+1}}\ge n^{n+1}$, where $-1<a_1,a_2,\dots,a_{n+1}<1$ and $a_1+a_2+\dots+a_{n+1}\ge n-1$.

147. Prove that $(a+b)^3(b+c)^3(c+d)^3(d+a)^3\ge 16a^2b^2c^2d^2(a+b+c+d)^4$, where $a>0, b>0, c>0, d>0$.

148. Prove that $\left[\left(1+\frac{a}{b}\right)^2+\left(1+\frac{b}{c}\right)^2+\left(1+\frac{c}{a}\right)^2\right]\left[\left(1+\frac{b}{a}\right)^2+\left(1+\frac{c}{b}\right)^2+\left(1+\frac{a}{c}\right)^2\right]\ge$ $4\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)^2$, where $a>0, b>0, c>0$.

149. Prove that $(a^2+bc)^3(b^2+ac)^3(c^2+ab)^3\ge 64(a^3+b^3)(b^3+c^3)(c^3+a^3)a^3b^3c^3$, where $a>0, b>0, c>0$.

150. Prove that $a+\sqrt{ab}+\sqrt[3]{abc}\le\frac{4}{3}(a+b+c)$, where $a>0, b>0, c>0$.

151. Prove that $a+\sqrt{ab}+\sqrt[3]{abc}\le 3.\sqrt[3]{a.\frac{a+b}{2}.\frac{a+b+c}{3}}$, where $a>0, b>0, c>0$.

152. Prove that $(ab)^{\frac{5}{4}}+(bc)^{\frac{5}{4}}+(ca)^{\frac{5}{4}}\le\frac{\sqrt{3}}{9}$, where $a>0, b>0, c>0$ and $a+b+c=1$.

153. Prove that $a^2+b^2+c^2\ge 14$ if $a+2b+3c\ge 14$.

154. Prove that $ab+\sqrt{(1-a^2)(1-b^2)}\le 1$ if $|a|\le 1, |b|\le 1$.

155. Prove that $\sqrt{c(a-c)}+\sqrt{c(b-c)}\le\sqrt{ab}$ if $a>c, b>c, c>0$.

156. Prove that $a\sqrt{a^2+c^2}+b\sqrt{b^2+c^2}\le a^2+b^2+c^2$.

157. Prove that $\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, where $a>0, b>0, c>0$.

158. Prove that $\sqrt{a}(a+c-b)+\sqrt{b}(a+b-c)+\sqrt{c}(b+c-a) \leq \sqrt{(a^2+b^2+c^2)(a+b+c)}$, where a, b, c are lengths of sides of a triangle.

159. Prove that $(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right) \geq n^2$, where $a_1>0, a_2>0, \dots, a_n>0$.

160. Prove that $\frac{a_1^2+a_2^2+\dots+a_n^2}{n} \geq \left(\frac{a_1+a_2+\dots+a_n}{n}\right)^2$.

161. Prove that $a_1a_2+a_2a_3+\dots+a_9a_{10}+a_{10}a_1 \geq -1$ if $a_1^2+a_2^2+\dots+a_{10}^2=1$.

162. Prove that $x^4+y^4 \geq x^3y+xy^3$.

163. Prove that $\left(|a_1|^3+|a_2|^3+\dots+|a_n|^3\right)^2 \leq \left(a_1^2+a_2^2+\dots+a_n^2\right)^3$.

164. Prove that $3(a^2+b^2+c^2+x^2+y^2+z^2)+6\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} \geq (a+b+c+x+y+z)^2$.

165. Prove that $a^2+b^2+c^2 \geq ab+bc+ca$.

166. Prove that $(a_1+a_2+\dots+a_n)(a_1^7+a_2^7+\dots+a_n^7) \geq (a_1^3+a_2^3+\dots+a_n^3)(a_1^5+a_2^5+\dots+a_n^5)$, where $a_1>0, a_2>0, \dots, a_n>0$.

167. Prove that $\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a} \leq 12$, where $\frac{3}{2} \leq a \leq \frac{50}{3}$.

168. Prove that $a+b+c \leq abc+2$, where $a^2+b^2+c^2=2$.

169. Prove that $2(a+b+c)-abc \leq 10$, where $a^2+b^2+c^2=9$.

170. Prove that $1+abc \geq 3.min(a,b,c)$, where $a^2+b^2+c^2=9$.

171. Prove that $\left(\sum_{i=1}^{n} a_i^{k+1}\right)\left(\sum_{i=1}^{n} a_i^{-1}\right) \geq n\left(\sum_{i=1}^{n} a_i^{k}\right)$, where $k, n \in \mathbb{N}$ and $a_1>0, a_2>0, \dots, a_n>0$.

172. Prove that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$, where $a>0, b>0, c>0$.

173. Prove that $\frac{a_1^k+a_2^k+\dots+a_n^k}{n} \geq \left(\frac{a_1+a_2+\dots+a_n}{n}\right)^k$, where $k, n \in \mathbb{N}$ and $a_1>0, a_2>0, \dots, a_n>0$.

174. Prove that $\left(1+\frac{1}{\sin\alpha}\right)\left(1+\frac{1}{\cos\alpha}\right) > 5$, where $0<\alpha<\frac{\pi}{2}$.

175. Find the smallest possible value of the expression $(u-v)^2+\left(\sqrt{2-u^2}-\frac{9}{v}\right)^2$ if $0<u<\sqrt{2}, v>0$.

176. Prove that $x_1^2+\left(\frac{x_1+x_2}{2}\right)^2+\dots+\left(\frac{x_1+x_2+\dots+x_n}{n}\right)^2 \leq 4(x_1+x_2+\dots+x_n)^2$. This inequality is a particular case of *Hardy's inequality* $\sum_{k=1}^{n}\left(\frac{a_1+a_2+\dots+a_k}{k}\right)^p \leq \left(\frac{p}{p-1}\right)^p \cdot \sum_{k=1}^{n} a_k^p$, where $p>1, a_i \geq 0, i=1,2,\dots,n$.

177. Prove that $\frac{1}{a_1}+\frac{2}{a_1+a_2}+\dots+\frac{n}{a_1+a_2+\dots+a_n}<2\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$, where $a_1>0, a_2>0, \dots, a_n>0$.

178. Prove that $(\sin\alpha_1+\sin\alpha_2+\dots+\sin\alpha_n)^2+(\cos\alpha_1+\cos\alpha_2+\dots+\cos\alpha_n)^2\le n^2$.

179. Prove that $\frac{a_1+a_2+\dots+a_n}{n}\ge\sqrt[n]{a_1a_2\dots a_n}$, where $n\ge 2, a_1>0, a_2>0, \dots, a_n>0$.

180. Prove that $\sqrt{a_1b_1}+\sqrt{a_2b_2}+\dots+\sqrt{a_nb_n}\le\sqrt{a_1+a_2+\dots+a_n}\cdot\sqrt{b_1+b_2+\dots+b_n}$, where $a_i\ge 0, b_i\ge 0, i=1,2,\dots,n$.

181. Prove that $(x_1y_2-x_2y_1)^2+(x_2y_3-x_3y_2)^2+(x_1y_3-x_3y_1)^2\le(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)$.

182. Prove that $\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)^2\le\left(\sum_{i=1}^{n}a_ix_i\right)\left(\sum_{i=1}^{n}\frac{b_i}{x_i}\right)$, where $x_i>0, a_i>0, b_i>0, i=1,2,\dots n$.

183. Prove that $\left(\sum_{i=1}^{n}x_iy_i\right)\left(\sum_{i=1}^{n}\frac{x_i}{y_i}\right)\ge\left(\sum_{i=1}^{n}x_i\right)^2$, where $x_i>0, y_i>0, i=1,2,\dots,n$.

184. Prove that $ax+by+cz+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}\ge\frac{2}{3}(a+b+c)(x+y+z)$.

185. Prove that $(p_1q_1-p_2q_2-\dots-p_nq_n)^2\ge(p_1^2-p_2^2-\dots-p_n^2)(q_1^2-q_2^2-\dots-q_n^2)$, if $p_1^2\ge p_2^2+\dots+p_n^2, q_1^2\ge q_2^2+\dots+q_n^2$.

186. Prove that $\sqrt{x^2+xy+y^2}\sqrt{y^2+yz+z^2}+\sqrt{y^2+yz+z^2}\sqrt{z^2+zx+x^2}+\sqrt{z^2+zx+x^2}\sqrt{x^2+xy+y^2}\ge(x+y+z)^2$.

187. Prove that $a_1(b_1+a_2)+a_2(b_2+a_3)+\dots+a_n(b_n+a_1)<1$, where $n\ge 3, a_1, a_2, \dots, a_n>0$ and $a_1+a_2+\dots+a_n=1, b_1^2+b_2^2+\dots+b_n^2=1$.

188. Prove that $\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{z+x}{2}\right)^2}\ge\sqrt{6}$, where $x,y,z\ge 0, x^2+y^2+z^2=1$.

189. Prove that $\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\ge 2\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}$, where $a,b,c>0$.

190. Prove that $\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\ge\sqrt{3}$, where $a,b,c\ge 0$ and $a+b+c=1$.

191. Prove that $\sqrt{\frac{a+b}{2}-ab}+\sqrt{\frac{b+c}{2}-bc}+\sqrt{\frac{c+a}{2}-ca}\ge\sqrt{2}$, where $a,b,c\ge 0$ and $a+b+c=2$.

192. Prove that $\sqrt{1-xy}\sqrt{1-yz}+\sqrt{1-yz}\sqrt{1-zx}+\sqrt{1-zx}\sqrt{1-xy}\ge 2$, where $x,y,z\ge 0$ and $x^2+y^2+z^2=1$.

193. Prove that $x\sqrt{1-yz}+y\sqrt{1-zx}+z\sqrt{1-xy}\ge\frac{2\sqrt{2}}{3}$, where $x,y,z\ge 0$ and $x+y+z=1$.

194. Prove the following identity $(a_1c_1 + a_2c_2 + \cdots + a_nc_n) - (a_1d_1 + a_2d_2 + \cdots + a_nd_n)(b_1c_1 + b_2c_2 + \cdots + b_nc_n) = \sum_{1\leq i<k\leq n} (a_ib_k - a_kb_i)(c_id_k - c_kd_i)$.

195. Prove that $(a_1c_1 + a_2c_2 + \cdots + a_nc_n) - (a_1d_1 + a_2d_2 + \cdots + a_nd_n) \geq (a_1d_1 + a_2d_2 + \cdots + a_nd_n)(b_1c_1 + b_2c_2 + \cdots + b_nc_n)$, where $b_id_1 \geq 0 (i = 1, 2, \ldots, n)$ or $b_id_1 < 0 (i = 1, 2, \ldots, n)$ and $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \ldots \leq \frac{a_n}{b_n}, \frac{c_1}{d_1} \leq \frac{c_2}{d_2} \leq \ldots \leq \frac{c_n}{d_n}$.

196. Find the maximum and minimum value of the expression $\frac{\sqrt{x^2+y^2}+\sqrt{(x-2)^2+(y-1)^2}}{\sqrt{x^2+(y-1)^2}+\sqrt{(x-2)^2+y^2}}$.

197. Find the minimum value of the expression $\left(\frac{1}{x^n}+\frac{1}{a^n}-1\right)\left(\frac{1}{y^n}+\frac{1}{b^n}-1\right)$, where $x, y, a, b > 0, x + y = 1, a + b = 1$.

198. Prove that $4 \leq a^2 + b^2 + ab + \sqrt{4 - a^2}\,\sqrt{9 - b^2} \leq 19$, where $0 \leq a \leq 2$ and $0 \leq b \leq 3$.

199. Prove that $n\sqrt{m-1} + m\sqrt{n-1} \leq mn$, where $m \geq 1, n \geq 1$.

200. Prove that $\sqrt{m^2 - n^2} + \sqrt{2mn - n^2} \geq m$, where $m > n > 0$.

201. Prove that $x > \sqrt{x-1} + \sqrt{x(\sqrt{x} - 1)}$, where $x \geq 1$.

202. Prove that $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq n\sqrt{\frac{2}{n+1}}$, where $n \in \mathbb{N}$.

203. Prove that among seven arbitrary numbers one can find two numbers x and y such that $0 \leq \frac{x-y}{1+xy} < \frac{\sqrt{3}}{3}$.

204. Prove that $\frac{|a-b|}{\sqrt{1+a^2}\sqrt{1+b^2}} \leq \frac{|a-c|}{\sqrt{1+a^2}\sqrt{1+c^2}} \leq \frac{|b-c|}{\sqrt{1+b^2}\sqrt{1+c^2}}$.

205. *Huygen's inequality:* $\sqrt[n]{(a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n)} \geq \sqrt[n]{a_1a_2\ldots a_n}\,\sqrt[n]{b_1b_2\ldots b_n}$, where $a_i > 0, b_i > 0, i = 1, 2, \ldots, n$.

206. *Milne's inequality:* $\frac{a_1b_1}{a_1+b_1} + \frac{a_2b_2}{a_2+b_2} + \cdots + \frac{a_nb_n}{a_n+b_n} \leq$

$\frac{(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)}{(a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n)}$, where $a_i > 0, b_i > 0, i = 1, 2, \ldots, n$.

207. Prove that $\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1-z_1^2} + \frac{1}{x_2y_2-z_2^2}$, where $x_1 > 0, x_2 > 0$ and $x_1y_1 - z_1^2 > 0, x_2y_2 - z_2^2 > 0$.

208. Prove that $\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leq \frac{2}{3}\sqrt{abc}$, where $a \geq 1, b \geq 1, c \geq 1$.

209. Prove that $\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} + \sqrt{d-a} \leq \frac{3\sqrt{3}}{4}\sqrt{abcd}$, where $a \geq 1, b \geq 1, c \geq 1, d \geq 1$.

210. Prove that $\left(\frac{a^2-b^2}{2}\right)^2 \geq \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2}$, where $a, b \geq \frac{1}{2}$.

211. Prove that $x_1 + x_2 + \dots + x_n \le \frac{n}{3}$, where $x_1^3 + x_2^3 + \dots + x_n^3 = 0$ and $x_i \in [-1, 1]$, $i = 1, 2, \dots, n$.

212. Prove that $|x_1^3 + x_2^3 + \dots + x_n^3| \le 2n$, where $x_1 + x_2 + \dots + x_n = 0$ and $x_i \in [-2, 2]$, $i = 1, 2, \dots, n$.

213. Prove that $1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \le \frac{3\sqrt{2}}{4}$, where $a, b, c > 0$.

214. Prove that $\sqrt{1-a} + \sqrt{1-b} + \sqrt{1-c} + \sqrt{1-d} \ge \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}$, where $a, b, c, d > 0$, $a^2 + b^2 + c^2 + d^2 = 1$.

215. Prove that $\frac{a+b+c}{3} - \sqrt[3]{abc} \le \max\Big[\big(\sqrt{a} - \sqrt{b}\big)^2, \big(\sqrt{b} - \sqrt{c}\big)^2, \big(\sqrt{c} - \sqrt{a}\big)^2\Big]$, where $a > 0, b > 0, c > 0$.

216. Given that $a^2 + b^2 = 1$. Prove that (i) $|a + b| \le \sqrt{2}$, (ii) $|a - b| \le \sqrt{2}$, (iii) $|ab| \le \frac{1}{2}$, and (iv) $|ab^2 + a^2b| \le \frac{1}{\sqrt{2}}$.

217. Prove that $|xy - \sqrt{(1-x^2)(1-y^2)}| \le 1$, where $|x| \le 1, |y| \le 1$.

218. Prove that $\sqrt{1-x^2} + \sqrt{1-y^2} \le 2\sqrt{1 - \left(\frac{x+y}{2}\right)^2}$, where $|x| \le 1, |y| \le 1$.

219. Prove that $\frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \dots + \frac{a_n}{1-a_n} \ge \frac{n(a_1+a_2+\dots+a_n)}{n-(a_1+a_2+\dots+a_n)}$, where $0 \le a_1 < 1, 0 \le a_2 < 1, \dots, 0 \le a_n < 1$.

220. Prove that $\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \le \frac{3}{2}$, where $a, b, c > 0$ and $a + b + c = abc$.

221. Prove that $\frac{|x-y|}{1+a|x-y|} + \frac{|y-z|}{1+a|y-z|} \ge \frac{|x-z|}{1+a|x-z|}$, where $a > 0$.

222. Prove that $\frac{2x(1-x^2)}{(1+x^2)^2} + \frac{2y(1-y^2)}{(1+y^2)^2} + \frac{2z(1-z^2)}{(1+z^2)^2} \le \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}$, where $x > 0, y > 0, z > 0$ and $xy + yz + zx = 1$.

223. Prove that $\sqrt{a_1 + a_2 + \dots + a_n} \le \sqrt{1}(\sqrt{a_1} - \sqrt{a_2}) + \sqrt{2}(\sqrt{a_2} - \sqrt{a_3}) + \dots + \sqrt{n}(\sqrt{a_n} - \sqrt{a_{n+1}})$, where $a_1 \ge a_2 \ge \dots \ge a_{n+1} = 0$.

224. Prove that $\frac{1}{\frac{1}{1+a_1}+\frac{1}{1+a_2}+\dots+\frac{1}{1+a_n}} - \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}} \ge \frac{1}{n}$, where $a_1 > 0, a_2 > 0, \dots, a_n > 0$.

225. Prove that $a + b + c - 2\sqrt{abc} \ge ab + bc + ca - 2abc$, where $0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1$.

226. Prove that $\sqrt{a(1-b)(1-c)} + \sqrt{b(1-c)(1-a)} + \sqrt{c(1-a)(1-b)} \le 1 + \sqrt{abc}$, where $0 \le a \le 1, 0 \le b \le 1, 0 \le c \le 1$.

227. Prove that $[(x+y)(y+z)(z+x)]^2 \ge xyz(2x+y+z)(2y+z+x)(2z+x+y)$, where $x, y, z \ge 0$.

228. Prove that $\frac{ab(1-a)(1-b)}{(1-ab)^2} < \frac{1}{4}$, where $0 < a < 1, 0 < b < 1$.

229. Prove that $\max(a_1, a_2, \dots, a_n) \geq 2$, where $n > 3, a_1 + a_2 + \dots + a_n \geq n, a_1^2 + a_2^2 + \dots + a_n^2 \geq n^2$.

230. Prove that $\sqrt{a_1 + \frac{(a_n - a_{n-1})^2}{4(n-2)}} + \dots + \sqrt{a_{n-2} + \frac{(a_n - a_{n-1})^2}{4(n-2)}} + \sqrt{a_{n-1}} + \sqrt{a_n} \leq \sqrt{n}$, where $n \geq 3$, $a_1, a_2, \dots, a_n \geq 0$ and $a_1 + a_2 + \dots + a_n = 1$.

231. Prove that $2\sqrt{(x^2-1)(y^2-1)} \leq 2(x-1)(y-1) + 1$, where $0 \leq x, y \leq 1$.

232. Prove that $a^3 + b^3 + c^3 - 3abc \leq \sqrt{(a^2+b^2+c^2)^3}$.

233. Pprove that $\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \dots + \frac{1}{n-1+x_n} \leq 1$, where $x_1, x_2, \dots, x_n > 0$ and $x_1.x_2.\dots.x_n = 1$.

234. Prove that $\frac{x}{\sqrt{1-x}} + \frac{1}{\sqrt{1-y}} \geq \frac{x+y}{\sqrt{1-\frac{x+y}{2}}}$, where $0 \leq x, y < 1$.

235. Prove that $\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \dots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1}+\sqrt{x_2}+\dots+\sqrt{x_n}}{\sqrt{n-1}}$, where $n \geq 2$, $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n > 0$ and $x_1 + x_2 + \dots + x_n = 1$.

236. Prove that $\frac{x}{\sqrt{4y^2+1}} + \frac{y}{\sqrt{4x^2+1}} \leq \frac{1}{\sqrt{2}}$, where $0 \leq x, y \leq \frac{1}{2}$.

237. Prove that $0 \leq ab + bc + ca - abc \leq 2$, where $a, b, c > 0$ and $a^2 + b^2 + c^2 + abc = 4$.

238. Prove that $a + b + c \leq 3$, whhere $a, b, c > 0$ and $a^2 + b^2 + c^2 + abc = 4$.

239. Prove that $(x-1)(y-z)(z-1) \leq 6\sqrt{3} - 10$, where $x, y, z > 0$ and $x + y + z = xyz$.

240. Prove that $\sqrt[3]{\frac{x+y}{2z}} + \sqrt[3]{\frac{y+z}{2x}} + \sqrt[3]{\frac{z+x}{2y}} \leq \frac{5(x+y+z)+9}{8}$, where $x, y, z > 0$ and $xyz = 1$.

241. Prove that among four arbitrary numbers there are two numbers a and b such that $\frac{1+ab}{\sqrt{1+a^2}\sqrt{1+b^2}} > \frac{1}{2}$.

242. Given that $x + y + z = 0$ and $x^2 + y^2 + z^2 = 6$, find all possible values of the expression $x^2y + y^2z + z^2x$.

243. Let (h_n) be a sequence such that $h_1 = \frac{1}{2}$ and $h_{n+1} = \sqrt{\frac{1-\sqrt{1-h_n^2}}{2}}$, $n = 1, 2, \dots$. Prove that $h_1 + h_2 + \dots + h_n \leq 1.03$.

244. Prove that $abc \geq (a+b-c)(b+c-a)(c+a-b)$, where $a, b, c > 0$.

245. Prove that $(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$.

246. Prove that $(a+b)^2(a^2+b^2)^2 \cdots (a^n+b^n)^2 \geq (a^{n+1}+b^{n+1})^n$, where $a, b > 0$.

247. Prove that $(a_1^\alpha + a_2^\alpha + \dots + a_n^\alpha)^\beta \leq (a_1^\beta + a_2^\beta + \dots + a_n^\beta)^\alpha$, where $0 < \beta < \alpha, a_1 > 0, a_2 > 0, \dots, a_n > 0$.

248. *Nesbitt's inequality:* $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$, where $a,b,c>0$.

249. Prove that $\sqrt{\frac{a}{b+c+d}}+\sqrt{\frac{b}{a+c+d}}+\sqrt{\frac{c}{a+b+d}}+\sqrt{\frac{d}{a+b+c}}>2$, where $a,b,c,d>0$.

250. Prove that $\sqrt[3]{\frac{abc+abd+acd+bcd}{4}}\leq\sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}}$, where $a,b,c,d>0$.

251. Prove that $2\sqrt{ab+bc+ac}\leq 3\sqrt[3]{(b+c)(c+a)(a+b)}$, where $a,b,c>0$.

252. Prove that $8(x^3+y^3+z^3)^2\geq 9(x^2+yz)(y^2+xz)(z^2+xy)$, where $x,y,z>0$.

253. Prove that $4a^3+4b^3+4c^3+15abc\geq 1$, where $a,b,c\geq 0$ and $a+b+c=1$.

254. Prove that $a^3+b^3+c^3+abcd\geq min\left(\frac{1}{4},\frac{1}{9}+\frac{d}{27}\right)$, where $a,b,c\geq 0$ and $a+b+c=1$.

255. Prove that $\frac{a_1+a_2+\dots+a_n}{n}\geq\frac{1}{n}\sqrt{\frac{a_1^2+a_2^2+\dots+a_n^2}{n}}+\left(1-\frac{1}{n}\right)\sqrt[n]{a_1a_2\dots a_n}$, where $n\geq 2$, $a_i>0$, $i=1,2,\dots,n$.

256. *Turkevici's Inequality:* $a^4+b^4+c^4+d^4+2abcd\geq a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2$, where $a,b,c,d\geq 0$.

257. Prove that $\frac{a_1^3}{b_1}+\frac{a_2^3}{b_2}+\dots+\frac{a_n^3}{b_n}\geq 1$, where $a_i,b_i>0, i=1,2,\dots,n$, and $(a_1^2+a_2^2+\dots+a_n^2)^3=b_1^2+b_2^2+\dots+b_n^2$.

258. Prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+c}{b+c}+\frac{b+a}{c+a}+\frac{c+b}{a+b}$, where $a,b,c>0$.

259. Prove that $\sqrt{\frac{a_1^n}{a_1^n+\lambda a_1a_2\dots a_n}}+\sqrt{\frac{a_2^n}{a_2^n+\lambda a_1a_2\dots a_n}}+\dots+\sqrt{\frac{a_n^n}{a_n^n+\lambda a_1a_2\dots a_n}}\geq\frac{n}{\sqrt{1+\lambda}}$, where $n\geq 2$, $a_1,2_2,\dots,a_n>0$ and $\lambda\geq n^2-1$.

260. Prove that $\left(\sqrt[k]{2}-1\right)(a_1+a_2+\dots+a_n)<\sqrt[k]{2a_1^k+2^2a_2^k+\dots+2^na_n^k}$, where $k\in\mathbb{N}, k\geq 2$, $a_1,a_2,\dots,a_n>0$.

261. Prove that $3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)\geq xyz(x+y+z)^3$, where $x,y,z>0$.

262. Prove that $(x_1+x_2+\dots+x_n+y_1+y_2+\dots+y_n)^2\geq 4n(x_1y_1+x_2y_2+\dots+x_ny_n)$, where $x_1\leq x_2\leq\dots\leq x_n\leq y_1\leq y_2\leq\dots\leq y_n$.

263. Prove that $\frac{\ln z-\ln y}{z-y}<\frac{\ln z-\ln x}{z-x}<\frac{\ln y-\ln x}{y-x}$, where $0<x<y<z$.

264. Prove that $a^bb^cc^dd^a\geq b^ac^bd^ca^d$, where $0\leq a\leq b\leq c\leq d$.

265. Prove that $\frac{x_1}{S-x_1}+\frac{x_2}{S-x_2}+\dots+\frac{x_n}{S-x_n}\geq\frac{n}{n+1}$, where $n\geq 2$, $S=x_1+x_2+\dots+x_n$, $x_1,x_2,\dots,x_n>0$.

266. Prove that $a^3+b^3+c^3+6abc\geq\frac{1}{4}(a+b+c)^3$, where $a,b,c\geq 0$.

267. Prove that $a^2(2b+2c-a)+b^2(2c+2a-b)+c^2(2a+2b-c)\geq 9abc$, where a,b,c are side lengths of a triangle.

268. Prove that $\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{b_1 b_2 \dots b_n} \le \sqrt[n]{(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)}$, where $n \ge 2$, $a_i > 0$, $b_i > 0$, $i = 1, 2, \dots, n$.

269. Prove that $\sqrt[n]{(n+1)!} - \sqrt[n]{n!} \ge 1$, where $n \ge 2, n \in \mathbb{N}$.

270. Prove that $\sqrt[n]{F_{n+1}} > 1 + \frac{\grave{}}{\sqrt[n]{F_n}}$, where $n \ge 2$, $F_1 = 1$, $F_2 = 2$, $F_{k+2} = F_{k+1} + F_k$, $k = 1, 2, \dots$.

271. Prove that $\sqrt[n]{C_{n+1}^n} > 2\left(1 + \frac{1}{\sqrt[n]{n+1}}\right)$, where $n = 2, 3, \dots$.

272. Prove that $(1+a_1)(2+a_2)\cdots(n+a_n) \ge n^{\frac{n}{2}}$, where $n \ge 2, n \in \mathbb{N}$, $a_1, a_2, \dots, a_n > 0$ and $a_1 a_2 \dots a_n = 1$.

273. Prove that $\sqrt[n]{\frac{(a_1+b_1)(a_2+b_2)\dots(a_n+b_n)}{(a_1-c_1)(a_2-c_2)\dots(a_n-c_n)}} \ge \frac{\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{b_1 b_2 \dots b_n}}{\sqrt[n]{a_1 a_2 \dots a_n} - \sqrt[n]{c_1 c_2 \dots c_n}}$, where $n \ge 2, n \in \mathbb{N}, b_i > 0$, $a_i > c_i > 0$, $i = 1, 2, \dots, n$.

274. Prove that $\sqrt[3]{ab} + \sqrt[3]{cd} \le \sqrt[3]{(a+c+d)(a+b+c)}$, where $a, b, ,c, d \ge 0$.

275. Prove that $x^(x^2-1)^2 + y^2(y^2-1)^2 \ge (x^2-1)(y^2-1)(x^2+y^2-1)$.

276. Prove that $(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5) + (x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5) + \dots + (x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4) \ge 0$.

277. Prove that $0 \le ab + bc + ca - abc \le 2$, where $a, b, c \ge 0$ and $a^2 + b^2 + c^2 + abc = 4$.

278. Prove that $x^\lambda(x-y)(x-z) + y^\lambda(y-z)(y-x) + z^\lambda(z-y)(z-x) \ge 0$, where $x, y, z > 0$.

279. Prove that $\sqrt[3]{\left(\frac{a}{b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{a+c}\right)^2} + \sqrt[3]{\left(\frac{c}{a+b}\right)^2} \ge \frac{3}{\sqrt[3]{4}}$, where $a, b, c > 0$.

280. Prove that $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a+b+c)^3$, where $a, b, c > 0$.

281. Prove that $abc + abd + bcd + acd - abcd \le 3$, where $a, b, c, d > 0$ and $a^3 + b^3 + c^3 + d^3 + abcd = 5$.

282. Prove that $0 \le ab + bc + ca - abc \le 2$, where $a, b, c \ge 0$ and $a^2 + b^2 + c^2 + abc = 4$.

283. Prove that $a^2 + b^2 + c^2 + 2abc + 1 \ge (ab + bc + ca)$, where $a, b, c \ge 0$.

284. Prove that $\frac{x+y+z}{xy+yz+zx} \le 1 + \frac{1}{48}[(x-y)^2 + (y-z)^2 + (z-x)^2]$, where $x, y, z > 0$ and $xy + yz + zx + xyz = 4$.

285. Let $a_1, a_2, \dots, a_{n+1}$ be $n+1$ positive real numbers such that $a_1 + a_2 + \dots + a_n = a_{n+1}$. Prove that $\sum_{i=1}^{n} \sqrt{a_i(a_{n+1}) - a_i} \le \sqrt{\sum_{i=1}^{n} a_{n+1}(a_{n+1} - a_i)}$.

286. Prove that $\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \ge 1$, where $a, b, c > 0$ and $a, b, c \in \mathbb{R}$.

287. Prove that $a^2 + b^2 + c^2 \geq \sqrt{3}abc$, where $a, b, c > 0$ and $a, b, c \in \mathbb{R}$ such that $abc \leq a + b + c$.

288. For any positive real numbers a, b, c prove that $\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \geq \frac{27}{(a+b+c)^2}$.

289. Let a, b, c be three sides of a triangle such that $a + b + c = 2$. Prove that $1 \leq ab + bc + ca - abc \leq 1 + \frac{1}{27}$.

290. If a, b, c be positive real numbers such that $a + b + c = 1$. Prove that $\sqrt{ab + c} + \sqrt{bc + a} + \sqrt{ca + b} \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$.

291. If a, b, c, d are positive real numbers, prove that $\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} \geq \sqrt[4]{\frac{abc+bcd+cda+abd}{4}}$.

292. Let a, b, c be the sides of a triangle such that $a + b + c = 2$. Prove that $a^2 + b^2 + c^2 + 2abc < 2$.

293. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 1$, prove that $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + a + b + c \geq 4\sqrt{3}$.

294. Find all triples (a, b, c) of real numbers which satisfy the system of equations:

$$a + b + c = 6,$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2 - \frac{4}{abc}.$$

295. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that $\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \geq \frac{3}{5}$.

296. Let a, b, c and α, β, γ be positive real numbers such that $\alpha + \beta + \gamma = 1$. Prove that $b\alpha + b\beta + c\gamma + 2\sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)(ab + bc + ca)} \leq a + b + c$.

297. Prove that for all real numbers a and b, $a^2 + b^2 + 1 > a\sqrt{b^2 + 1} + b\sqrt{a^2 + 1}$.

298. For a fixed positive integer n, compute the minimum value of the sum $x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n}$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n$.

299. Let a, b, c, d be positive real numbers such that $a + b + c + d \leq 1$. Prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \leq \frac{1}{64abcd}$.

300. Let a, b, c be positive real numbers, all less than 1, such that $a + b + c = 2$. Prove that $abc \geq 8(1 - a)(1 - b)(1 - c)$.

301. Prove that $\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8$, where a, b, c are positive real numbers.

302. Prove that $abc \leq 1$, where a, b, c are real numbers such that $(1+a)(1+b)(1+c) = 8$.

303. Prove that $\sum_{i=1}^{n} \frac{a_i}{2-a_i}]geq \frac{n}{2n-1}$, where $a_1, a_2, \ldots, a_n \in \mathbb{R}, n \geq 2$ such that $\sum_{i=1}^{n} a_i = 1$.

304. Prove that $\sum_{i=1}^{n} \frac{a_i^2}{a_i+a_{i+1}} \geq \frac{1}{2}$, where $a_1, a_2, \ldots, a_n$ are positive numbers such that $\sum_{i=1}^{n} a_i = 1$ and $a_1 = a_{n+1}$.

305. Prove that $\frac{1}{a} + \frac{4}{b} + \frac{9}{c} + \frac{16}{d} \geq \frac{100}{a+b+c+d}$, where $a, b, c, d \in \mathbb{R}$.

306. Prove that $\sum_{i=1}^{n} \frac{a_i^2}{1-2a_i} \geq \frac{1}{n-2}$, where $n > 2, 0 < a_1, a_2, \ldots, a_n < \frac{1}{2}$ such that $\sum_{i=1}^{n} a_i = 1$.

307. Prove that $x_1 + x_2 + \cdots + x_n \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}$, where $n \geq 2$, $x_1 + x_2 + \cdots + x_n \geq x_1y_1 + x_2y_2 + \cdots + x_ny_n$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are positive real numbers.

308. If $x_1, x_2, \ldots, x_n$ are n positive real numbers, prove that $\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots +$

$\frac{x_n}{1+x_1^2+x_2^2+\cdots+x_n^2} < \sqrt{2}$.

309. If a, b, c are poositive real numbers, prove that $3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq abc(a+b+c)^3$.

310. Let $P(x) = ax^2 + bx + c$ be a quadratic polynomial with non-negative coefficients and let α be a positive real number. Prove that $P(\alpha)P(1/\alpha) \geq P(1)^2$.

311. If a, b, c, d, e are positive, real numbers, prove that $\sum \frac{a}{b+c} \geq \frac{5}{2}$ where sum is taken cyclically over a, b, c, d, e.

312. Let a, b, c be non-negative real numbers such that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2$. Prove that $ab + bc + ca \leq \frac{3}{2}$.

313. Suppose a, b, c are positive real numbers. Prove that $3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$. When does equality hold?

314. Let $c_1, c_2, \ldots, c_n$ be n real numbers such that either $0 \leq c_i \leq 1$ for all i or $c_i \geq 1$ for all i. Prove that the inequality $\prod_{i=1}^{n}(1-p+pc_i) \leq 1-p+p\prod_{i=1}^{n} c_i$ holds, for any real p with $0 \leq p \leq 1$.

315. Let x_1, x_2, x_3, x_4 be real numbers in the interval $(0, 1/2]$. Prove that

$\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} \leq \frac{x_1^4+x_2^4+x_3^4+x_4^4}{(1-x_1)^4+(1-x_2)^4+(1-x_3)^4+(1-x_4)^4}$.

316. If $x_1, x_2, \dots, x_n$ be n real numbers such that $x_i \in (0, 1/2]$. Prove that $\frac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^n} \leq \frac{\prod_{i=1}^{n}(1-x_i)}{(\sum_{i=1}^{n}(1-x_i))^n}$.

317. Consider a sequence $\langle a_i \rangle$ of real numbers satisfying $a_{i+j} \leq a_i + a_j$. Prove that $a_1 + \frac{a_2}{2} + \dots + \frac{a_n}{n} \geq a_n, \forall n$.

318. For positive real numbers x, y, z, prove that $\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} \leq 1$, where the sum is taken cyclically over x, y, z.

319. Let x, y be non-negative real numbers such that $x + y = 2$. Prove that $x^3y^3(x^3+y^3) \leq 2$.

320. Let $\langle a_i \rangle$ and $\langle b_i \rangle$ be two sequences such that $0 < h \leq a_i \leq H$ and $0 < m \leq b_i \leq M$ for real h, H, m, M. Prove that $1 \leq \frac{(\sum a_i^2)(\sum b_i^2)}{(\sum a_ib_i)^2} \leq \frac{1}{4}\left(\sqrt{\frac{HM}{hm}} + \sqrt{\frac{hm}{HM}}\right)^2$.

321. Let $f : [0, a] \to \mathbb{R}$ be a convex function. Consider n points $x_1, x_2, \dots, x_n$ in $[0, a]$ such that $\sum_{i=1}^{n} x_i$ is also in $[0, a]$. Prove that $\sum_{i=1}^{n} f(x_i) \leq f\left(\sum_{i=1}^{n} x_i\right) + (n-1)f(0)$.

322. For any real number n, prove that $\binom{2n}{n}\sqrt{3n} < 4^n$.

323. Let a, b, c be positive real numbers and let x be a non-negative real number. Prove that $a^{x+2} + b^{x+2} + c^{x+2} \geq a^xbc + ab^xc + abc^x$.

324. Let $(a_1, a_2, \dots, a_n)$, $(b_1, b_2, \dots, b_n)$ and $(c_1, c_2, \dots, c_n)$ be three seuqnences of positive real numbers. Prove that $\sum_{i=1}^{n} a_ib_ic_i \leq \sqrt[3]{\sum_{i=1}^{n} a_i^3}\sqrt[3]{\sum_{i=1}^{n} b_i^3}\sqrt[3]{\sum_{i=1}^{n} c_i^3}$.

325. Prove for any three real numbers a, b, c, the inequality $3(a^2 - a - 1)(b^2 - b - 1)(c^2 - c - 1) \geq (abc)^2 - abc + 1$.

326. Consider a polynomial of the form $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + 1$, where $a_i \geq 0\ \forall 1 \leq i \leq n-1$. Suppose $P(x) = 0$ has n real roots. Prove that $P(2) \geq 3^n$.

327. Let $a_1 < a_2 < \dots < a_n$ be n positive integers. Prove that $(a_1 + a_2 + \dots + a_n)^2 \leq a_1^3 + a_2^3 + \dots + a_n^3$.

328. Consider a sequence $a_1, a_2, \dots, a_n$ of positive real numbers which add up to 1, where $n \geq 2$ is an integer. Prove that for any positive real numbers $x_1, x_2, \dots, x_n$ with $\sum_{i=1}^{n} x_i = 1$, the inequality $2\sum_{i<j} x_ix_j \leq \frac{n-2}{n+1} + \sum_{i=1}^{n} \frac{a_ix_i^2}{1-a_i}$, holds.

329. Let x_1, x_2, x_3, x_4 be four consecutive positive real numbers such that $x_1x_2x_3x_4 = 1$. Prove that $x_1^3 + x_2^3 + x_3^3 + x_4^3 \geq \min\left(x_1 + x_2 + x_3 + x_4, \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)$.

330. Let $\{x\}$ denote the fractional part of x i.e. $\{x\} = x - \lceil x \rceil$. Prove for any positive integer n, $\sum_{i=1}^{n} \{\sqrt{i}\} \leq \frac{n^2-1}{2}$.

331. If a, b, c are positive real numbers, prove that $\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$.

332. Let a, b, c be positive real numbers such that $abc > ab + bc + ca$. Prove that $abc \geq 3(a + b + c)$.

333. Let $a_1, a_2, \dots, a_n$ be n non-negative real numbers and let a denote the sum of these numbers. Prove that $\sum_{i=1}^{n-1} a_i a_{i+1} \leq \frac{a^2}{4}$.

334. Let a, b, c, d be complex numbers such that $ac \neq 0$. Prove that $\frac{\max(|ac|, |ad+bc|, |bd|)}{\max(|a|, |b|)(|c|, |d|)} \geq \frac{-1+\sqrt{5}}{2}$.

335. Let x_1, x_2, x_3, x_4 be non-negative real numbers such that $\sum_{i=1}^{n} \frac{1}{1+x_i} \leq 1$. Prove that $x_1 x_2 \cdots x_n \geq (n-1)^n$.

336. Prove that $\frac{1}{m+n-1} - \frac{1}{(m+1)(n+1)} \leq \frac{4}{45}$ for any two natural numbers m and n.

337. If a, b are two positive real numbers, prove that $a^b + b^a > 1$.

338. Let a, b be positive real numbers such that $a + b = 1$ and let p be a positive real. Prove that $\left(a + \frac{1}{a}\right)^p + \left(b + \frac{1}{b}\right)^p \geq \frac{5^p}{2^{p-1}}$.

339. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)$ $\left(c - 1 + \frac{1}{a}\right) \leq 1$.

340. Let x, y, z be real numbers in the interval $[-1, 2]$ such that $x + y + z = 0$. Prove that $\frac{(2-x)(2-y)}{(2+x)(2+y)} + \sqrt{\frac{(2-y)(2-z)}{(2+y)(2+z)}} + \sqrt{\frac{(2-z)(2-x)}{(2+z)(2+x)}} \geq 3$.

341. Let $\langle a_n \rangle$ be a sequence of distinct positive integers. Prove that $\sum_{i=1} \frac{a_i}{i^2} \geq \sum_{i=1}^{n} \frac{1}{i}$, for every positive integer n.

342. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.

343. Let $x_1, x_2, \dots, x_n$ be n positive real numbers. Prove that $\sum_{i=1}^{n} \frac{x_i^3}{x_i^2 + x_i x_{i+1} + x_{i+1}^2} \geq \frac{1}{3} \sum_{i=1}^{n} x_i$, where $x_1 = x_{n+1}$.

344. Suppose x, y, z are non-negative real numbers. Prove that $x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$.

345. Prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$, where a, b, c are positive real numbers.

346. If a, b are real numbers, prove that $a^2 + ab + b^2 \geq 3(a+b-1)$.

347. Define a sequence $\langle x_n \rangle$ by $x_1 = 2, x_{n+1} = \frac{x_n^4+9}{10x_n}$. Prove that $\frac{4}{5} < x_n \leq \frac{5}{4}\ \forall\ n > 1$.

348. Let a, b, c be positive real numbers such that $a^2 - ab + b^2 = c^2$. Prove that $(a-c)(b-c) \leq 0$.

349. Let a, b, c be positive real numbers. Prove that $\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$.

350. For all real numbers a, show that $(a^3 - a + 2)^2 \geq 4a^2(a^2+1)(a-2)$.

351. Let a, b, c be distinct real numbers. Prove that $\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right) + \left(\frac{2c-a}{c-a}\right)^2 \geq 5$.

352. Let $\alpha, \beta, x_1, x_2, \ldots, x_n$ be positive reals such that $\alpha + \beta = 1$, and $x_1 + x_2 + \cdots + x_n = 1$. Prove that $\sum_{i=1}^{n} \frac{x_i^{2m+1}}{\alpha x_i + \beta x_{i+1}} \geq \frac{1}{n^{2m-1}}$ for every positive integer m, where $x_{n+1} = x_1$.

353. Given positive reals a, b, c, d, prove that $\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2} \leq \sqrt{(a+c)^2 + (b+d)^2} + \frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}$.

354. With every natural number n, associate a real number a_n by $a_n = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$, where $\{p_1, p_2, \ldots, p_k\}$ is the set of all prime divisors of n. Show that for any natural number $N \geq 2$, $\sum_{i=2}^{N} a_1 a_2 \ldots a_n < 1$.

355. Let n be a fixed integer, with $n \geq 2$. Determine the least constant C such that the inequality $\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$ holds for all real numbers $x_1, x_2, \ldots, x_n$. Determine when the equality holds.

356. Let a, b, c, d be real numbers such that $(a^2 + b^2 - 1)(c^2 + d^2 - 1) > (ac + bd - 1)^2$. Prove that $a^2 + b^2 - 1 > 0$ and $c^2 - d^2 - 1 > 0$.

357. Let $x_1, x_2, \ldots, x_{100}$ be 100 positive integers such that $\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20$. Prove that at least two of the x_i's are equal.

358. Let $f(x)$ be a polynomial with integer coefficients and of degree $n > 1$. Suppose $f(x) = 0$ has n real roots in the interval $(0, 1)$, not all equal. If a is the leading coefficient of $f(x)$, prove that $|a| \geq 2^n + 1$.

359. Show that the equation $\frac{x}{y}+\frac{y}{z}+\frac{z}{w}+\frac{w}{x}=m$, has no solutions in positive reals for $m=2,3$.

360. Solve the system of equations: $x=\frac{4z^2}{1+4z^2}$, $y=z=\frac{4x^2}{1+4x^2}$, for real numbers x, y, z.

361. Suppose a, b are non-zero real numbers and that all the roots of the real polynomial $ax^n-ax^{n-1}+a_{n-1}x^{n-2}+\cdots+a_2x^2-n^2bx+b=0$ are real and positive. Prove that all the roots are in fact equal.

362. Find all triples (a, b, c) of positive integers such that product of any two leaves a remainder 1 when divided by the third number.

363. Find all positive solutions of the system: $x_1+\frac{1}{x_2}=4$, $x_2+\frac{1}{x_3}=1, \cdots, x_{1999}+\frac{1}{x_{2000}}=4$, $x_{2000}+\frac{1}{x_1}=1$.

364. Find all positive solutions of the system: $x+y+z=1$, $x^3+y^3+z^3+xyz=x^4+y^4+z^4+1$.

365. Let a, b be positive integers such that each equation $(a+b-x)^2=a-b$, $(ab+1-x)^2=ab-1$ has two distinct real roots. Suppose the bigger of these roots are the same. Show that the smaller roots are also the same.

366. Suppose the polynomial $P(x)=x^n+nx^{n-1}+a_2x^{n-2}+\cdots+a_n$ has real roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. If $\alpha_1^{16}+\alpha_2^{16}+\cdots+\alpha_n^{16}=n$. Find α_1,$\alpha_2, \ldots, \alpha_n$.

367. Find all the solutions of the following system of inequalities:

$$(x_1^2-x_3x_5)(x_2^2-x_3x_5)\le 0,$$
$$(x_2^2-x_4x_1)(x_3^2-x_4x_1)\le 0,$$
$$(x_3^2-x_5x_2)(x_4^2-x_5x_2)\le 0,$$
$$(x_4^2-x_1x_3)(x_5^2-x_1x_3)\le 0,$$
$$(x_5^2-x_2x_4)(x_1^2-x_2x_4)\le 0.$$

368. Solve the following system of equations, when a is a real number such that $|a|>1$:

$$x_1^2=ax_2+1,$$
$$x_2^2 \,\&= ax_3+1,$$
$$\vdots \quad \wr\vdots$$
$$x_{999}^2=ax_{1000}+1,$$
$$x_{1000}^2=ax_1+1.$$

369. Let $a_1, a_2, \ldots, a_n$ be n positive integers such that $\sum_{i=1}^n a_i=\prod_{i=1}^n a_i$. Let K_n denote this common value. Show that $K_n \ge n+s$, where s is the least positive integer such that $2^s-s\ge n$.

370. Let $z_1, z_2, z_3, \dots, z_n$ be n complex numbers such that $\,^n_{i=1}|z_i| = 1$. Prove that there exists a subset S of the set $\{z_1, z_2, \dots, z_n\}$ such that $\left|\sum_{z\in S} z\right| \geq \frac{1}{4}$.

371. Let $\langle a_n\rangle$ and $\langle b_n\rangle$ be two sequences of real numbers which are not proportional. Let $\langle x_n\rangle$ such that $\sum_{i=1}^{n} a_i x_i = 0, \sum_{i=1}^{n} b_i x_i = 1$. Prove that $\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)-(\sum_{i=1}^{n} a_i b_i)^2}$. When does equality hold?

372. Let $x_1, x_2, \dots, x_n$ be n positive real numbers. Prove that $\sum_{i=1}^{n} \frac{x_i}{2x_i+x_{i+1}+\dots+x_{i+n-2}} \leq n$, where $x_{n+i} = x_i$.

373. Let $x_1, x_2, \dots, x_n$ be $n \geq 2$ positive real numbers and k be a fixed integer such that $1 \leq k \leq n$. Show that $\sum_{\text{cyclic}} \frac{x_1+2x_2+\dots+2x_{k-1}+x_k}{x_k+x_{k+1}+\dots+x_n} \geq \frac{2n(k-1)}{n-k+1}$.

374. If z_1 and z_2 be two complex numbers such that $|z_1| \leq r, |z_2| \leq r$ and $z_1 \neq z_2$. Prove that for any natural number $n\left|\frac{z_1^n-z_2^n}{z_1-z_2}\right| \leq \frac{1}{2}n(n-1)r^{n-2}|z_1-z_2|$.

375. A sequence $\langle a_n\rangle$ is said to be convex if $a_n - 2a_{n+1} + a_{n+2} \geq 0$ for all $n \geq 1$. Let $a_1, a_2, \dots, a_{2n+1}$ be a convex sequence. Show that $\frac{a_1+a_3+\dots+a_{2n+1}}{n+1} \geq \frac{a_2+a_4+\dots+a_{2n}}{n}$, and equality holds if and only if $a_1, a_2, \dots, a_{2n+1}$ is an arithmetic progression.

376. Suppose $a_1, a_2, \dots, a_n$ are n positive real numbers. For each k, define $x_i = a_{i+1} + a_{i+2} + \dots + a_{i+n-1} - (n-2)a_i$, where $a_i = a_{i-n}$ for $i > n$. Suppose $x_k \geq 0$ for $1 \leq i \leq n$. Prove that $\prod_{i=1}^{n} a_i \geq \prod_{i=1}^{n} x_i$. Show that for $n = 3$ the inequality is still true without the non-negativity of x_i's, but for $n > 3$ these conditions are essential.

377. Let a, c be positive reals and b be a complex number such that $f(z) = a|z|^2 + 2Re(bz) + c \geq 0$, for all complex numbers z, where $Re(z)$ denoted the real part of z. Prove that $|b|^2 \leq ac$, and $f(z) \leq (a+c)(1+|z|^2)$. Show that $|b|^2 = ac$ only if $f(z) = 0$ for some $z \in \mathbb{C}$.

378. Suppose $x_1 \leq x_2 \leq \dots \leq x_n$ be n real numbers. Show that $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_k)^2$. Prove also that equality holds if and only if the sequence $\langle x_i\rangle$ is in A.P.

379. Suppose $\langle a_n\rangle$ is an infinite sequence of real numbers with the properties

1. there is some real constant c such that $0 \leq a_n \leq c$, for all $n \geq 1$, and
2. $|a_i - a_j| \geq \frac{1}{i+j}\ \forall i \neq j$.

Prove that $c \geq 1$.

380. Let a, b, c be positive reals such that $a + b + c = 1$. Prove that $a(1 + b - c)^{1/3} + b(1 + c - a)^{1/3} + c(1 + a - b)^{1/3} \leq 1$.

381. let $x_1, x_2, \dots, x_n$ be n positive reals which add up to 1. Find the minimum value of $\sum_{i=1}^{n} \frac{x_i}{1+\sum_{j \neq i} x_j}$.

382. If a, b, c, d are positive reals then find all possible values of $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$.

383. Let $\langle F_n \rangle$ be the Fibonacci sequence defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, for $n \geq 1$. Prove that $\sum_{i=1}^{n} \frac{F_i}{2^i} < 2$ for all $n \geq 1$.

384. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients such that $|P(0)| = P(1)$. Suppose all the roots of $P(x) = 0$ are real and lie in the interval $(0, 1)$. Prove that the product of the roots does not exceed $\frac{1}{2^n}$.

385. If x, y are real numbers such that $2x + y + \sqrt{8x^2 + 4xy + 32y^2} = 3 + 3\sqrt{2}$, prove that $x^2 y \leq 1$.

386. Determine the maximum value of $\sum_{i<j} x_i x_j (x_i + x_j)$, over all n-tuples $(x_1, x_2, \dots, x_n)$ of reals such that $x_i \geq 0$ for $1 \leq i \leq n$.

387. Let $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that $\sum_{i=1}^{n} (x_1 x_2 \cdots x_i)^{1/i} < 3\left(\sum_{i=1}^{n} x_i\right)$.

388. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be n real numbers with the property ${}_{i=1}^{n} a_i = 0$. Prove that $n a_1 a_n \sum_{i=1}^{n} a_i^2 \leq 0$.

389. Let a, b, c be positive real numbers. Prove that $\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}$.

390. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 2$. Prove that $x + y + z \leq 2 + xyz$. Find the conditions under which equality holds.

391. Let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ be such that $\sum_{i=1}^{n} x_i = 1$, where $n \geq 2$ is an integer. If $x_n \leq \frac{2}{3}$, prove that there exists a j such that $1 \leq j \leq n$ and $\frac{1}{3} \leq \sum_{i=1}^{j} x_i \leq \frac{2}{3}$.

392. Let x, y, z be non-negative real numbers such that $xy + yz + zx + xyz = 4$. Prove that $x + y + z \geq xy + yz + zx$.

393. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that $x^y + y^z + z^x \le \frac{4}{27}$.

394. Let x, y, z be real numbers and let p, q, r be real numbers in the interval $\left(0, \frac{1}{2}\right)$ such that $p + q + r = 1$. Prove that $pqr(x + y + z)^2 \ge xyr(1 - 2r) + yzp(1 - 2p) + zxq(1 - 2q)$. When does equality hold?

395. Let $x_1, x_2, \dots, x_n$ be n real numbers in the interval $[0, 1]$. Prove that $\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} x_i x_{i+1}\right) \le \left[\frac{n}{2}\right]$, where $x_{n+1} = x_1$.

396. Suppose x, y, z are positive real numbers such that $xyz \ge 1$. Prove that $\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \ge 0$.

397. Consider two sequences of positive real numbers, $a_1 \le a_2 \le \dots \le a_n$ and $b_1 \le b_2 \le \dots \le b_n$, such that $\sum_{i=1}^{n} a_i \ge \sum_{i=1}^{n} b_i$. Suppose there exists a j, $1 \le j \le n$, such that $b_i \le a_i$ for $1 \le i \le j$ and $b_i \ge a_i$ for $i > j$. Prove that $\prod_{i=1}^{n} a_i \ge \prod_{i=1}^{n} b_i$.

398. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \le \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}$.

399. Let $n \ge 4$ and let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \dots + a_n \ge n$, $a_1^2 + a_2^2 + \dots + a_n^2 \ge n^2$. Prove that $\max\{a_1, a_2, \dots, a_n\} \ge 2$.

400. Let $x_1 \le x_2 \le \dots \le x_{n+1}$ be $n + 1$ positive integers. Prove that $\sum_{i=1}^{n+1} \frac{\sqrt{x_{i+1} - x_i}}{x_{i+1}} < \sum_{i=1}^{n^2} \frac{1}{j}$.

401. Let a, b, c be three positive real numebrs which satisfy $abc = 1$ and $a^3 > 36$. Prove that $\frac{2}{3}a^2 < a^2 + b^2 + c^2 - ab - bc - ca$.

402. Let $z_1, z_2, \dots, z_n$ be n complex numbers and consider n positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ which have the property that $\sum 1/\lambda_i = 1$. Prove that $\left|\sum_{i=1}^{n} z_i\right|^2 \le \sum_{i=1}^{n} \lambda_1 |z_i|^2$.

403. Let a, b, c be three distinct real numbers. Prove that $2\min\{a, b, c\} < \sum a - \left(\sum a^2 - \sum ab\right)^{1/2} < \sum a + \left(\sum a^2 - \sum ab\right)^{1/2} < 3\max\{a, b, c\}$, where the sum is cyclic over a, b, c.

404. Show that for all complex numbers z with $\Re(z) > 1$, prove that $|z^{n+1} - 1| > |z^n||z - 1|, \forall\ n \geq 1$.

405. Suppose a, b, c are positive real numbers such that $x = a + b - c, y = b + c - a, z = c + a - b$. Prove that $abc(xy + yz + zx) \geq xyz(ab + bc + ca)$.

406. Let a, b, c be positive real numbers. Prove that $\sum \frac{a^3}{b^2-bc+c^2} \geq \frac{3\sum ab}{\sum a}$, where all sums are cyclic.

407. Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers satisfying $a = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{1}{\sqrt{3}}$. Prove that $\frac{a_1}{1-a_1^2} + \frac{a_2}{1-a_2^2} + \cdots + \frac{a_n}{1-a_n^2} \geq \frac{na}{1-a^2}$.

408. Suppose x, y, z are non-negative real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

1. $1 \leq \sum \frac{x}{1-yz} \leq \frac{3\sqrt{3}}{2}$, and
2. $1 \leq \sum \frac{x}{1+yz} \leq \sqrt{2}$.

The sums are cyclic over x, y and z.

409. Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove that $xy^2 + yz^2 + zx^2 \geq xy + yz + zx - \frac{2}{9}$.

410. Let a, b, c, d be positive real numbers such that $a + b + c + d = 2$. Prove that $\sum_{cyclic} \frac{a^2}{(a_1^2)^2} \leq \frac{16}{25}$.

411. Prove that $\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$ for all positive real numbers a, b and c.

412. If x, y are real numbers such that $x^3 + y^4 \leq x^2 + y^3$, prove that $x^3 + y^3 \leq 2$.

413. Let a, b, c be three positive real numbers. Prove that $\sum \frac{ab}{c(c+a)} \geq \sum \frac{a}{c+a}$, where the sum is cyclic over a, b and c.

414. Let x, y be two real numbers, where y is non-negative and $y(y + 1) \leq (x + 1)^2$. Prove that $y(y - 1) \leq x^2$.

415. Let x, y, z be positive real numbers. Prove that $\left(\frac{xy+yz+zx}{3}\right)^{1/2} \leq$

$\left(\frac{(x+y)(y+z)(z+x)}{8}\right)^{1/3}$.

416. Let a, b, c be positive real numbers such that $abc = 1$. Show that $\sum \frac{a^9+b^9}{a^6+a^3b^3+b^6} \geq 2$, where the sum is cyclical.

417. Let $a_1, a_2, \ldots, a_n (n > 2)$ be positive real numbers and let s be their sum. Let $0 < \beta \leq 1$ be a real number. Prove that $\displaystyle\sum_{i=1}^{n} \left(\frac{s-a_i}{a_i}\right)^{\beta} \geq (n-1)^{2\beta} \sum_{i=1}^{n} \left(\frac{a_i}{s-a_i}\right)^{\beta}$. When does equality hold?

418. For $n \geq 4$, let $a_1, a_2, \ldots, a_n$ be n positive real numbers such that ${}_{i=1}^{n} a_i^2 = 1$. Show that $\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n})^2$.

419. Does there exist an infinite sequence $\langle x_n \rangle$ of positive real numbers such that $x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n}$, $\forall\, n \geq 2$.

420. Let $a_1, a_2, \ldots, a_n$ be n positive real numbers and consider a permutation of $b_1, b_2, \ldots, b_n$ of it. Prove that $\displaystyle\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \sum_{i=1}^{n} a_i$.

421. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences of positive real numbers such that $\displaystyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$. Prove that $\displaystyle\sum_{i=1}^{n} \frac{a_i^2}{a_i+b_i} \geq \frac{1}{2}$.

422. Let x, y, z be positive real numbers. Prove that $\frac{y^2-x^2}{z+x} + \frac{z^2-y^2}{x+y} + \frac{x^2-z^2}{y+z} \geq 0$.

423. Find the greatest value of k such that for every triple (a, b, c) of positive real numbers, the inequality $(a^2 - bc)^2 > k(b^2 - ca)(c^2 - ab)$ holds.

424. Let a, b, c, d be positive real numbers. Prove that $\displaystyle\sum_{\text{cyclic}} \frac{a}{b+2c+d} \geq 1$.

425. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a) = 1$. Prove that $ab + bc + ca \leq \frac{3}{4}$.

426. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that $x^2 + y^2 + z^2 + 18xyz \leq 1$.

427. Let a, b, c be three positive real numbers such that $ab + bc + ca = 1$. Prove that $\left(\frac{1}{a} + 6b\right)^{1/3} + \left(\frac{1}{b} + 6c\right)^{1/3} + \left(\frac{1}{c} + 6a\right)^{1/3} \leq \frac{1}{abc}$.

428. Let $a_1, a_2, \ldots, a_n$ be $n > 1$ positive real numbers. For each k, $1 \leq k \leq n$, let $A_k = (a_1 + a_2 + \cdots + a_k)/k$. Let $g_n = (a_1 a_2 \cdots a_n)^{1/n}$ and $G_n = (A_1 A_2 \cdots A_n)^{1/n}$. Prove that $n\left(\frac{G_n}{A_n}\right)^{1/n} + \frac{g_n}{G_n} \leq n + 1$. Find the cases of equality.

429. Let x, y, z be real numbers in the interval $[0, 1]$. Prove that $3(x^2y^2 + y^2z^2 + z^2x^2) - 2xyz(x + y + z) \leq 3$.

430. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that $7(xy + yz + zx) \leq 2 + 9xyz$.

431. Let x, y, z be real numbers in the interval $[0, 1]$. Prove that $\frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1} \leq 2$.

432. Let a, b, c, d be positive real such that $a^3 + b^3 + 3ab = c + d = 1$. Prove that $\left(a + \frac{1}{a}\right)^3 + \left(b + \frac{1}{b}\right)^3 + \left(c + \frac{1}{c}\right)^3 + \left(d + \frac{1}{d}\right)^3 \geq 40$.

433. Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that $\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}$.

434. Let x, y, z be non-negative real numbers. Prove that $x^3 + y^3 + z^3 \geq x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy}$.

435. For all positive real numbers show that $4(ab + bc + ca) - 1 \geq a^2 + b^2 + c^2 \geq 3(a^3 + b^3 + c^3)$.

436. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}$.

437. Suppose a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3+b^3+c^3)}{abc}$.

438. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$.

439. Let a, b, c, d be non-negative real numbers such that $ab + bc + cd + da = 1$. Show that $\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$.

440. Find all real k for which the inequality $x_1^2 + x_2^2 + x_3^2 \geq k(x_1x_2 + x_2x_3)$ holds for all real numbers x_1, x_2, x_3.

441. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

442. Let a, b, c be non-negative reals such that $a + b \leq 1 + c, b + c \leq 1 + a, c + a \leq 1 + b$. Prove that $a^2 + b^2 + c^2 \leq 2abc + 1$.

443. If a, b, c are non-negative real numbers such that $a + b + c = 1$, then show that $\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{9}{10}$.

444. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab, b - bc, c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.

445. Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that (a) $y < x < 1$, and (b) $x^2 + y^2 < 1$.

446. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Let $k = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}$. Prove that the roots of the equation $x^2 + x + 4k = 0$ are real.

447. If a, b, c are three positive real numbers, prove that $\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq 3$.

448. If d is tghe largest among the positive numbers a, b, c, d, prove that $a(d-b) + b(d-c) + c(d-a) \leq d^2$.

449. If x, y, z are positive real numbers, prove that $(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2)$.

450. Suppose a, b, c are positive real bumbers. Prove that $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$.

451. Find all real p and q for which the equation $x^4 - \frac{8p^2}{q}x^3 + 4qx^3 - 3px + p^2 = 0$ has four positive roots.

452. Let a_1, a_2, a_3 be real numbers, each greater than 1. Let $S = a_1 + a_2 + a_3$ and suppose $S < \frac{a_i^2}{a_i - 1}$ for $i = 1, 2, 3$. Prove that $\frac{1}{a_1+a_2} + \frac{1}{a_2+a_3} + \frac{1}{a_3+a_1} > 1$.

453. Let a, b, c be positive real numbers such that $ab + bc + ca = \frac{1}{3}$. Prove that $\frac{a}{a^2-bc+1} + \frac{b}{b^2-ca+1} + \frac{c}{c^2-ab+1} \geq \frac{1}{a+b+c}$.

454. Suppose a, b, c are positive real numbers. Prove that $\frac{a^2b(b-c)}{a+b} + \frac{b^2c(c-a)}{b+c} + \frac{c^2a(a-b)}{c+a} \geq 0$.

455. Let $a_1, a_2, \ldots, a_n$ be $n > 2$ positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that $\sum_{i=1}^{n} \frac{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n}{a_i + n - 1} \leq \frac{1}{(n-1)^2}$.

456. Determine the largest value of k such that the inequality $\left(k + \frac{a}{b}\right)\left(k + \frac{b}{c}\right)\left(k + \frac{c}{ba}\right) \geq \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$ holds for positive real numbers a, b, c.

457. Let $x_1, x_2, \ldots, x_n$ be $n \geq 3$ positive real numbers. Prove that $\frac{x+_1 x_3}{x_1x_3+x_2x_4} + \frac{x_2x_4}{x_2x_4+x_3x_5} + \cdots + \frac{x_{n-1}x_1}{x_{n-1}x_1+x_nx_2} + \frac{x_nx_2}{x_nx_2+x_1x_3} \leq n-1$.

458. Let $a_1, a_2, \ldots, a_{2017}$ be positive real numbers. Prove that $\sum_{i=1}^{2017} \frac{a_i}{a_{i+1}+a_{i+2}+\cdots+a_{i+1008}} \geq \frac{2017}{1008}$, where indices are taken modulo 2017.

459. Let a, b, c be three positive real numbers such that $ab + bc + ca = 1$. Prove that $\sqrt{a + \frac{1}{a}} + \sqrt{b + \frac{1}{b}} + \sqrt{c + \frac{1}{c}} \geq 2(\sqrt{a} + \sqrt{b} + \sqrt{c})$.

460. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that $\frac{a^3+2}{b+2} + \frac{b^3+2}{c+2} + \frac{c^3+2}{a+3} \geq 3$.

461. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that $(2+a)(2+b) \geq cd$.

462. Find all real k such that $\frac{a+b}{2} \geq k\sqrt{ab} + (1-k)\sqrt{\frac{a^2+b^2}{2}}$ holds for all positive real numbers a, b.

463. Let a, b, c, d be real numbers having absolute value greater than 1 such that $abc + abd + acd + bcd + a + b + c + d = 0$. Probvve that $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} + \frac{1}{d-1} > 0$.

464. For all positive, real x, y show that $\frac{1}{x+y-1} - \frac{1}{(x+1)(y+1)} < \frac{1}{11}$.

465. Let a, b, c be three positive real numbers such that $abc = 1$. Prove that $\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{3}{2}$.

466. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that $\frac{a^2}{(b+c)^3} + \frac{b^2}{(c+a)^3 + \frac{c^2}{(a+b)^3}} \geq \frac{9}{8}$.

467. Suppose a, b, c are positive real numbers such that $ab + bc + ca \geq a + b + c$. Prove that $(a + b + c)(ab + bc + ca) + 3abc \geq 4(ab + bc + ca)$.

468. Let a, b, c, d be four real nubers such that $a + b + c + d = 0$. Prove that $(ab + ac + ad + bc + bd + cd)^2 + 12 \geq 6(abc + abd + acd + bcd)$.

469. Consider the expression $P = \frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} + \frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3} + \frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}$. Find the maximum value of P when x, y, z vary over the set of all positive real numbers.

470. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2 \ldots x_n = 1$. Let $S = x_1^3 + x_2^3 + \cdots + x_n^3$. Prove that $\frac{x_1}{S-x_1^3+x_1^2} + \frac{x_2}{S-x_2^3+x_2^2} + \cdots + \frac{x_n}{S-x_n^3+x_n^2} \leq 1$.

471. Let $a_1, a_2, \ldots, a_n$ be $n > 1$ positive real numbers whose sum is 1. Define $b_i = \frac{a_i^2}{\sum_{j=1}^n a_j^2}, 1 \leq i < 2$. Prove that $\displaystyle\sum_{i=1}^n \frac{a_i}{1-a_i} \leq \sum_{i=1}^n \frac{b_i}{1-b_i}$.

472. Suppose a, b, c, d are posotive real numbers. Prove that $\displaystyle\sum_{\text{cyclic}} \frac{a^4}{a^3+a^2b+ab^2+b^3} \geq \frac{a+b+c+d}{4}$.

473. Let a, b, c be non-negative real numberssatisfying $a^2 + b^2 + c^2 = 1$. Prove that $\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \geq 5abc + 2$.

474. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 \leq x + y + z$. Prove that $\frac{x^2+3}{x^3+1} + \frac{y^2+3}{y^3+1} + \frac{z^2+3}{z^3+1} \geq 6$.

475. For any three positive real numbers a, b, c prove that $\frac{a^2}{a+b} + \frac{b^2}{b+c} \geq \frac{3a+2b-c}{4}$.

476. Suppose a, b, c are non-negative real numbers such that $a^3 + b^3 + c^3 + abc = 4$. Prove that $a^3b + b^3c + c^3a \leq 3$.

477. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\left(a+\frac{1}{b}\right)^2 + \left(b+\frac{1}{c}\right)^2 + \left(c+\frac{1}{a}\right)^2 \geq 3(a+b+c+1)$.

478. Let a,b,c be positive reall numbers with $abc = 1$. Prove that $\frac{a}{c(a+1)} + \frac{b}{a(b+1)} + \frac{c}{b(c+1)} \geq \frac{3}{2}$.

479. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $\frac{1}{1+a^{2014}} + \frac{1}{1+b^{2014}} + \frac{1}{1+c^{2014}} > 1$.

480. For positive real numbers a, b, c, prove the inequality

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \geq \frac{9}{1+abc}.$$

481. Let x, y, z be positive real numbers such that $x+y+z = 3$. Prove that $\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx$.

482. Let a, b, c be positive real numbers. Prove that $\frac{9abc}{2(a+b+c)} \leq \frac{ab^2}{a+b} + \frac{bc^2}{b+c} + \frac{ca^2}{c+a} \leq \frac{a^2+b^2+c^2}{2}$.

483. For positive real numbers a, b, c, prove that $\frac{abc}{(1+a)(a+b)(b+c)(c+16)} \leq \frac{1}{81}$.

484. Let a, b, c, d be positive real numbers such that $a+b+c+d = 4$. Prove that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \geq 2$.

485. Let a, b, c be positive real numbers. Prove that $\frac{1+ab}{c} + \frac{1+bc}{a} + \frac{1+ca}{b} \geq \sqrt{a^2+2} + \sqrt{b^2+2} + \sqrt{c^2+2}$.

486. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that $\frac{a^2}{b^3+c^4+1} + \frac{b^2}{c^3+a^4+1} + \frac{c^2}{a^3+b^4+1} > \frac{1}{5}$.

487. *Janous Inequality*: Let a, b, c and x, y, z be two sets of positive real numbers. Prove that $\frac{x(b+c)}{y+z} + \frac{y(c+a)}{z+x} + \frac{z(a+b)}{x+y} \geq \sqrt{3(ab+bc+ca)}$.

488. Let x, y, z be positive real numbers such that $xy+yz+zx = 1$. Prove that $\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} \leq \frac{3\sqrt{3}}{4}$.

489. Let x, y, z be positive real numbers such that $x+y+z = 1$. Prove that $\frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{27}{8}$.

490. Let x, y, z be positive real numbers such that $x+y+z = 1$. Show that $\frac{z-xy}{x^2+xy+y^2} + \frac{x-yz}{y^2+yz+z^2} + \frac{y-zx}{z^2+zx+x^2} \geq 2$.

491. Let a, b, c be positive real numbers. Define $u = a+b+c, \frac{u^2-b^2}{3} = ab+bc+ca, w = abc$, where $v \geq 0$. Then $\frac{(u+v)^2(u-2v)}{27} \leq w \leq \frac{(u-v)^2(u+2v)}{27}$.

492. Let a, b, c be positive real numbers. Prove that $a^4+b^4+c^4 \geq abc(a+b+c)$.

493. Let a, b, c be real numbers such that $a^2+b^2+c^2 = 9$. Prove that $2(a+b+c)-abc \leq 10$.

494. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that $a^2+b^2+c^2+3abc \geq \frac{9}{4}$.

495. Determine the maximum value of k such that $a+b+c \geq k$ for all positive reals a, b, c with $a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \geq 1$.

496. If a, b, c are real numbers such that $a+b+c = 1$, prove that $10(a^3+b^3+c^3)-9(a^5+b^5+c^5) \geq 1$.

497. Let a, b, c be positive real numbers. Prove that $24abc \leq |a^3+b^3+c^3-(a+b+c)^3| \leq \frac{8}{9}(a+b+c)^3$. Also show that equality holds in both the inequalities if and only if $a = b = c$.

498. Find all $k > 0$ such that the inequality $\sqrt{a^2+kb^2}+\sqrt{b^2+ka^2} \geq a+b+(k-1)\sqrt{ab}$ holds positive real numbers a and b.

499. Let a, b, c be positive real numbers such that $abc = 1$. Prove that $a+b+c \geq \sqrt{\frac{1}{3}(a+2)(b+2)(c+2)}$.

500. Let $x_1, x_2, \ldots, x_n$ be $n \geq 3$ positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that $\sum_{i=1}^{n} \frac{x_i^8}{x_{i+1}(x_i^4+x_{i+1}^4)} \geq \frac{n}{2}$, where $x_1 = x_{n+1}$.

501. Let a, b, c be positive real numbers such that $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} = 1$. Prove that $\frac{a^2+b^2+c^2+ab+bc+ca-3}{5} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

502. For positive, real x, y, z show that $\frac{x(2x-y)}{y(2z+x)}+\frac{y(2y-z)}{z(2x+y)}+\frac{z(2z-x)}{x(2y+z)} \geq 1$.

503. Suppose $\frac{z(zx+yz+y)}{xy^2+z^2+1} \leq k$, for alll real numbers $x, y, z \in (-2, 2)$ with $x^2+y^2+z^2+xyz = 4$. Find the smallest value of k.

504. Suppose a, b, c are positive real numbers such that $a^3+b^3+c^3 = a^4+b^4+c^4$. Prove that $\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+c^3+a^3}+\frac{c}{c^2+c^3+a^3} \geq 1$.

505. Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that $\frac{a^4+5g^4}{a(a+2b)}+\frac{b^4+5c^4}{b(b+2c)}+\frac{c^4+5a^4}{c(c+2a)} \geq 1-(ab+bc+ca)$.

506. Let x, y, z be positive real numbers. Prove that $(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}$.

507. Suppose a, b, c are positive real numbers such that $abc = 1$. Prove that $\sum_{\text{cyclic}}\frac{a^2+bc}{a^2(b+c)}\geq ab+bc+ca$.

508. Let a, b, c be non-negative real numbers. Prove that $4(a^3+b^3+c^3)+15abc\geq(a+b+c)^3$.

509. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that $\frac{1}{a^4+b+c}+\frac{1}{b^4+c+a}+\frac{1}{c^4+a+b}\leq\frac{3}{a+b+c}$.

510. Let a, b, c be positive reals. Prove that $a^4(b+c)+b^4(c+a)+c^4(a+b)\leq\frac{1}{12}(a+b+c)^5$.

511. Let a, b, c be positive reals such that $ab+bc+ca=1$. Prove that $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c}\geq 2$.

512. Let a, b, c be positive reals such that $ab+bc+ca=1$. Prove that $\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2}\geq\frac{5}{2}$.

513. Let a, b, c be positive real numbers. Prove that $3+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\left[\frac{(a+1)(b+1)(c+1)}{1+abc}\right]$.

514. Let a, b, c be distinct positive real numbers such that $abc = 1$. Prove that $\sum_{\text{cyclic}}\frac{a^6}{(a-b)(a-c)}>15$.

515. Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that $a+b+c\leq 2abc+\sqrt{2}$.

516. Let a, b, c be positive real numbers. Prove that $\frac{(b+c-a)^2}{a^2+(b+c)^2}+\frac{(c+a-b)^2}{b^2+(c+a)^2}+\frac{(a+b-c)^2}{c^2+(a+b)^2}\geq\frac{3}{5}$.

517. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that $\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1}+\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1}+\sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1}\geq 6$.

II
Answers

Answers of Chapter 1 Logarithm

1. $\log_{\sqrt{8}} x = \frac{10}{3} \Rightarrow \log_{2^{\frac{3}{2}}} x = \frac{10}{3} \Rightarrow \frac{2}{3}\log_2 x = \frac{10}{3}$

   $\Rightarrow \log_2 x = 5 \Rightarrow x = 2^5 = 32.$

2. L.H.S. $= \log_b a . \log_c b \log_a c = \frac{\log a}{\log b} \cdot \frac{\log b}{\log c} \cdot \frac{\log c}{\log a} = 1 =$ R.H.S.

3. L.H.S. $= \log_3 \log_2 \log_{\sqrt{5}} (\sqrt{5})^8 = \log_3 \log_2 8 = \log_3 3 = 1 =$ R.H.S.

4. Given $a^2 + b^2 = 23ab \Rightarrow (a+b)^2 = 25ab \Rightarrow \frac{a+b}{5} = \sqrt{ab}$

   Taking log of both sides, we get

   $\log \frac{a+b}{5} = \frac{1}{2}(\log a + \log b)$.

5. L.H.S. $= 7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2$

   $= 7[\log 2^4 - \log 3.5] + 5[\log 5^2 - \log 2^3.3] + 3[\log 3^4 - \log 2^4.5]$

   $= 7[4\log 2 - \log 3 - \log 5] + 5[2\log 5 - 3\log 2 - \log 3] + 3[4\log 3 - 4\log 2 - \log 5]$

   $= \log 2 =$ R.H.S.

6. L.H.S. $= \log\tan 1^\circ + \log\tan 2^\circ + \ldots + \log\tan 89^\circ$

   $= (\log\tan 1^\circ + \log\tan 89^\circ) + (\log\tan 2^\circ + \log\tan 88^\circ) + \cdots + \log\tan 45^\circ$

   $= (\log\tan 1^\circ \cot 1^\circ) + (\log\tan 2^\circ \cot 2^\circ) + \cdots + \log\tan 45^\circ [\because \tan(90^\circ - \theta) = \cot\theta]$

   $= \log 1 + \log 1 + \cdots + \log 1 = 0 [\because \tan\theta\cot\theta = 1]$

7. Given $\log_9 \tan\frac{\pi}{6} = \log_9 \frac{1}{\sqrt{3}} = -\log_9 \sqrt{3} = -\log_9 9^{1/4} = -\frac{1}{4}$.

8. Given $\frac{\log_{a^2} b}{\log_{\sqrt{a}} b^2} = \frac{\frac{1}{2}\log_a b}{2.2\log_a b} = \frac{1}{8}$.

9. Given $\log_{\sqrt{5}} .008 = 2\log_5 \frac{8}{1000} = 2[\log_5 8 - \log_5 1000] = 2[\log_5 8 - \log 8.125]$

   $= 2[\log_5 8 - \log_6 8 - \log_5 125] = -2.\log_5 5^3 = -6.$

10. Given $\log_{2\sqrt{3}} 144 = \log_{2\sqrt{3}} (2\sqrt{3})^4 = 4.$

11. L.H.S. $= \log_3 \log_2 \log_{\sqrt{3}} 81 = \log_3 \log_2 \log_{\sqrt{3}} (\sqrt{3})^8 = \log_3 \log_2 8 = \log_3 3 = 1 =$ R.H.S.

12. L.H.S. $= \log_a x \log_b y = \frac{\log x}{\log a} \cdot \frac{\log y}{\log b} = \frac{\log x}{\log b} \cdot \frac{\log y}{\log a}$

$= \log_b x \log_a y =$ R.H.S.

13. L.H.S. $= \log_2 \log_2 \log_2 16 = \log_2 \log_2 \log_2 2^4 = \log_2 \log_2 4 = \log_2 2 = 1 =$ R.H.S.

14. R.H.S. $= \log_b x \log_c b \ldots \log_n m \log_a n = \frac{\log x}{\log b} \cdot \frac{\log b}{\log c} \cdots \frac{\log m}{\log n} \cdot \frac{\log n}{\log a}$

$= \frac{\log x}{\log a} = \log_a x =$ L.H.S.

15. Let $10^x \log_{10} a = z$.

Taking log of both sides, we get

$x \log_{10} a = \log z \Rightarrow \log_{10} a^x = \log z \Rightarrow z = a^x$.

16. Given $a^2 + b^2 = 7ab \Rightarrow a^2 + b^2 + 2ab = (a+b)^2 = 9ab$

$\Rightarrow \left(\frac{a+b}{3}\right)^2 = ab \Rightarrow \frac{a+b}{3} = \sqrt{ab} = (ab)^{1/2}$

Taking log of both sides,

$\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$.

17. L.H.S. $= \frac{\log_a \log_b a}{\log_b \log_a b}$

Let $\log_b a = z$, then L.H.S. $= \frac{\log_a z}{\log_b \frac{1}{z}} = -\frac{\log_a z}{\log_b z} = -\frac{\log z}{\log a} \cdot \frac{\log z}{\log b}$

$= -\frac{\log b}{\log a} = -\log_a b =$ R.H.S.

18. L.H.S. $= \log(1+2+3) = \log 6 = \log(1.2.3) = \log 1 + \log 2 + \log 3 =$ R.H.S.

19. L.H.S. $= 2\log(1+2+4+7+14) = 2\log 28 = \log 784$

$= \log(1.2.4.7.14) = \log 1 + \log 2 + \log 4 + \log 7 + \log 14 =$ R.H.S.

20. L.H.S. $= \log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80}$

$= \log 2 + 16[\log 2^4 - \log 3 - \log 5] + 12[\log 5^2 - \log 2^3 - \log 3] + 7[\log 3^4 - \log 2^4 - \log 5]$

$= \log 2 + 16[4\log 2 - \log 3 - \log 5] + 12[2\log 5 - 3\log 2 - \log 3] + 7[4\log 3 - 4\log 2 - \log 5]$

$= \log 2[1 + 64 - 36 - 28] + \log 3[28 - 16 - 112] + \log 5[24 - 7 - 15]$

$= \log 2 + \log 5 = \log 10 = 1$ [$\because$ default base of log is 10.]

21. Given $\frac{\log_9 11}{\log_5 13} \div \frac{\log_3 11}{\log_{\sqrt{5}} 13} = \frac{\log_{3^2} 11}{\log_5 13} \cdot \frac{\log_{5^{\frac{1}{2}}} 13}{\log_3 11}$

$= \frac{\frac{1}{2}\log_3 11}{\log_5 13} . \frac{2\log_5 13}{\log_3 11} = 1.$

22. Given, $3^{\sqrt{\log_3 2}} - 2^{\sqrt{\log_2 3}}$

Taking log with base 10,

$\sqrt{\log_3 2}\log 3 - \sqrt{\log_2 3}\log 2 = \sqrt{\frac{\log 2}{\log 2}(\log 3)^2} - \sqrt{\frac{\log 3}{\log 2}(\log 2)^2}$

$= \sqrt{\log 2 \log 3} - \sqrt{\log 3 \log 2} = 0.$

23. Given $\log_{10} 343 = 2.5353 \Rightarrow \log_{10} 7^3 = 2.5353 \Rightarrow \log_{10} 7 = o.8451$

For $7^n > 10^5 \Rightarrow n\log_{10} 7 > 5 \Rightarrow n > \frac{5}{0.8451}$

Thus, least such integer is 6.

24. Since a, b, c are in G.P., we can write $b^2 = ac$

Taking log of both sides, we get

$2\log b = \log a + \log c \Rightarrow \log a, \log b, \log c$ are in A.P.

i.e. $\frac{1}{\log a}, \frac{1}{\log b}, \frac{1}{\log c}$ are in H.P.

Multiplying each term with $\log x$,

$\frac{\log x}{\log a}, \frac{\log x}{\log b}, \frac{\log x}{\log c}$ are in H.P.

$\log_a x, \log_b x, \log_c x$ are in H.P.

25. R.H.S. $= 3\log 2 + \log\sin x + \log\cos x + \log\cos 2x + \log\cos 4x$

$= 2\log 2 + (\log 2. \sin x\cos x) + \log\cos 2x + \log\cos 4x$

$= 2\log 2 + \log\sin 2x + \log\cos 2x + \log\cos 4x = \log 2 + (\log 2. \sin 2x\cos 2x) + \log\cos 4x$

$= \log 2 + \log\sin 4x + \cos 4x = \log 2. \sin 4x\cos 4x$

$= \log\sin 8x =$ L.H.S.

26. We have to prove that $xyz + 1 = 2yz \Rightarrow x + \frac{1}{yz} = 2$

L.H.S. $= x + \frac{1}{yz}$, substituting the values of x, y and z,

$\log_{2a} a + \frac{1}{\log_{3a} 2a\log_{4a} 3a} = \frac{\log a}{\log 2a} + \frac{\log 3a. \log 4a}{\log 2a. \log 3a}$

$= \frac{\log a + \log 4a}{\log 2a} = \frac{\log(2a)^2}{\log 2a} = 2 =$ R.H.S.

27. We have to prove that $\log_{c+b} a + \log_{c-b} a = 2\log_{c+a} a \log_{c-b} a$

Dividing both sides by $\log_{c+b} a \log_{c-b} a$,

$\frac{1}{\log_{c-b} a} + \frac{1}{\log_{c+b} \log a} = 2$

$\Rightarrow \log_a(c-b) + \log_a(c+b) = 2$

$\Rightarrow \log_a(c^2 - b^2) = 2 \Rightarrow c^2 = a^2 + b^2$

which is true because c is hypotenuse and a and b are sides of a right-angle triangle.

28. Let $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$

$\log x = k(y-z), \log y = k(z-x), \log z = k(x-y)$

$\Rightarrow x\log x + y\log y + z\log z = k(xy - zx + yz - xy + zx - yz) = 0$

$\Rightarrow \log x^x + \log y^y + \log z^z = \log x^x y^y z^z = 0$

$\Rightarrow x^x y^y z^z = 1.$

29. Given $\frac{yz\log(yz)}{y+z} = \frac{zx\log(zx)}{z+x} = \frac{xy\log(xy)}{x+y}$

Dividing by xyz, $\frac{\log(yz)}{x(y+z)} = \frac{\log(zx)}{y(z+x)} = \frac{\log(xy)}{z(x+y)} = k$ (let)

$\log y + \log z = k(xy + yz), \log z + \log x = k(yz + xy), \log x + \log y = k(yz + zx)$

$\Rightarrow x\log x = kyz \Rightarrow x\log x = kxyz = y\log y = z\log z$

$\Rightarrow x^x = y^y = z^z.$

30. We have to prove that $(yz)^{\log y - \log z}(zx)^{\log z - \log x}(xy)^{\log x - \log y} = 1$

Taking log of both sides,

$\Rightarrow (\log y - \log z)(\log y + \log z) + (\log z - \log x)(\log z + \log x) + (\log x - \log y)(\log x + \log y) = 0$

$\Rightarrow (\log y)^2 - (\log z)^2 + (\log z)^2 - (\log x)^2 + (\log x)^2 - (\log y)^2 = 0$

$\Rightarrow 0 = 0.$

31. L.H.S $= \log_N 2 + \log_n 3 + \cdots + \log_n 1988$

$= \log_N(2.3.4.\ldots 1988) = \log_N 1988! = \frac{1}{\log_{1988!} N} =$ R.H.S.

32. L.H.S. $= \log(1+x) + \log(1+x^2) + \log(1+x^4) \ldots$ to ∞

$= \log(1 + x + x^2 + \ldots$ to $\infty)$

$= \log\frac{1}{1-x}[\because 0 < x < 1]$ (from the formula for the sum of an infinite G.P.)

$= -\log(1-x) =$ R.H.S.

33. Let $S_n = \frac{1}{\log_2 a} + \frac{1}{\log_4 a} + \cdots$ up to n terms

$S_n = \log_a 2 + \log_a 4 + \log_a 8 + \cdots$ up to n terms

$S_n = (1 + 2 + 3 + \cdots + n)\log_a 2 = \frac{n(n+1)}{2}\log_a 2.$

34. L.H.S. $= \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

$= \frac{1}{\log_4 10+\log_4 4} + \frac{1}{\log_2 20+\log_2 20} + \frac{1}{\log_5 8+\log_5 5}$

$= \frac{1}{\log_4 40} + \frac{1}{\log_2 40} + \frac{1}{\log_5 40}$

$= \log_{40} 4 + \log_{40} 2 + \log_{40} 5 = \log_{40}(4.2.5) = \log_{40} 40 = 1 =$ R.H.S.

35. L.H.S. $= \frac{1}{\log_a bc+1} + \frac{1}{\log_b ca+1} + \frac{1}{\log_c ab+1}$

$= \frac{1}{\log_a bc+\log_a a} + \frac{1}{\log_b ca+\log_b b} + \frac{1}{\log_c ab+\log_c c}$

$= \frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$

$= \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1 =$ R.H.S.

36. Given, $\frac{1}{1+\log_b a+\log_b c} + \frac{1}{1+\log_c a+\log_c b} + \frac{1}{1+\log_a b+\log_a c} = 1$

L.H.S. $= \frac{1}{\log_b a+\log_b a+\log_b c} + \frac{1}{log_c c+\log_c a+\log_c b} + \frac{1}{\log_a a+\log_a b+\log_a c}$

$= \frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc}$

Like previous problem the above expression will evaluate to 1.

37. We have to prove that $x^{\log y - \log z} y^{\log z - \log x} z^{\log x - \log y} = 1$

Taking log of both sides,

$(\log y - \log z)\log x + (\log z - \log x)\log y + (\log x - \log y)\log z = 0$

$\Rightarrow \log y \log z - \log z \log x + \log z \log y - \log x \log y + \log x \log z - \log y \log z = 0$

$\Rightarrow 0 = 0.$

38. Let $\frac{\log a}{y-z} = \frac{\log b}{z-x} = \frac{\log c}{x-y} = k$

$\Rightarrow x\log a = k(xy - zx), y\log b = k(yz - xy), z\log c = k(zx - yz)$

Adding all,

$x\log a + y\log b + z\log c = k(xy - zx + yz - xy + zx - yz) = 0$

$\log a^x b^y x^z = 0 \Rightarrow a^x b^y c^z = 1$

39. Let $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z} = \frac{1}{k}$

$\Rightarrow \log x = kx(y+z-x), \log y = ky(z+x-y), \log z = kz(x+y-z)$

Let $y^z z^y = z^x z^z = x^y y^x$

Taking log, we have

$z \log y + y \log z = x \log z + z \log x = y \log x + x \log y$

$\Rightarrow zky(z+x-y) + ykz(x+y-z) = xkz(x+y-z) + zkx(y+z-x) = ykx(y+z-x) + xky(x+z-y)$

$\Rightarrow yz^2 + xyz - y^2z + xyz + y^2 - z^2y = x^2z + xyz - xz^2 + xyz + xz^2 - x^2z = xy^2 + xyz - x^2y + x^2y + xyz - xy^2$

$\Rightarrow 2xyz = 2xyz = 2xyz.$

40. Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

$\Rightarrow \log a = k(b-c), \log b = k(c-a), \log c = k(a-b)$

$\Rightarrow (b+c)\log a = k(b^2-c^2), (c+a)\log b = k(c^2-a^2), (a+b)\log c = k(a^2-b^2)$

Adding all, $\log a^{b+c} + \log b^{c+a} + \log c^{a+b} = 0$

$\Rightarrow a^{b+c} b^{c+a} c^{a+b} = 1.$

41. Let $\frac{\log x}{q-r} = \frac{\log y}{r-p} = \frac{\log z}{p-q} = k$

$\Rightarrow \log x = k(q-r), \log y = k(r-p), \log z = k(p-q)$

$\Rightarrow (q+r)\log x = k(q^2-r^2), (r+p)\log y = k(r^2-p^2), (p+q)\log z = k(p^2-q^2)$

Adding all $\log x^{q+r} + \log y^{r+p} + \log z^{p+q} = 0$

$\Rightarrow x^{q+r} y^{r+p} z^{p+q} = 1.$

Similarly, $p \log x = kp(q-r), q \log y = kq(r-p), r \log z = kr(p-q)$

Adding all, $\log x^p + \log y^q + \log z^r = 0 \Rightarrow x^p y^q z^r = 1.$

42. Given $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$

$$\therefore z = a^{\frac{1}{1-\log_a a^{\left(\frac{1}{1-\log_a x}\right)}}} = a^{\frac{1}{1-\frac{1}{1-\log_a x}}}$$

Taking log of both sides with base a,

$$\log_a z = \frac{1}{1-\frac{1}{1-\log_a x}} = \frac{1-\log_a x}{-\log_a x} = 1 - \frac{1}{\log_a x}$$

$$\Rightarrow x = a^{\frac{1}{1-\log_a z}}.$$

43. Given $f(y) = e^{f(z)}$ and $z = e^{f(x)}$, where $f(x) = \frac{1}{1-\log_e x}$

$$f(y) = e^{\frac{1}{1-\log_e z}} = e^{\frac{1}{1-\log_e e^{\frac{1}{1-\log_e x}}}} = e^{\frac{1}{1-\frac{1}{1-\log_e x}}}$$

Following like above exercie $x = e^{f(y)}$.

44. L.H.S. $= \frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \cdots + \frac{1}{\log_{43} n}$

$= \log_n 2 + \log_n 3 + \log_n 4 + \cdots + \log_n 43 = \log_n(2.3.4 \ldots 43)$

$= \log_n 43! = \frac{1}{\log_{43!} n} =$ R.H.S.

45. L.H.S. $= (1 + 2 + 3 + \cdots + n).2\log a = \frac{n(n+1)}{2}.2\log a = n(n+1)\log a =$ R.H.S.

46. We will use of the fact that positive characteristics of n of a logarithmm means that there $n+1$ digits in the number.

Let $\log y = 12\log 12 = 12\log(2.2.3) = 12[2 \times 0.301 + 0.477] = 12.96$.

Thus, number of digits is 13.

47. We can use the fact that the number of positive integers having base b and characteristics n is $b^{n+1} - b^n$.

Thus, number of integer with base 3 and characteristics 2 is $3^3 - 3^2 = 18$.

48. L.H.S. $= \log_a x \log_b y = \frac{\log x}{\log a} \cdot \frac{\log y}{\log b} = \frac{\log x}{\log b} \cdot \frac{\log y}{\log x}$

$= \log_b x \log_a y =$ R.H.S.

49. Given a, b, c are in G.P. $\Rightarrow \frac{b}{a} = \frac{c}{b}$. Taking $\log_x$ of these

$\log_x b - \log_x a = \log_x c - \log_x b \Rightarrow 2\log_x b = \log_x a + \log_x a$. Thus, $\log_x a, \log_x b, \log_x c$ are in A.P., and hence,

$\log_a x, \log_b x, \log_c x$ are in H.P.

50. Let $y = (0.0504)^{10} \Rightarrow \log_{10} y = 10\log_{10}(0.0504) = 10\log_{10}(504 \times 10^{-4})$

$= -10\log_{10}[-4 + \log(2^3.3^2.7)] = -12.98$.

Thus, characteristics is -13. Therefore, number of zeros after decimal and first significant digit is 12.

51. Let $x = 72^{15} \therefore \log_{10} x = 15\log_{10} 72 = 15\log_{10}(2^3 \times 3^2) = 15[3\log_{10} 2 + 2\log_{10} 3]$

$= 15[3 \times 0.301 + 2 \times 0.477] = 15[0.903 + 0.954] = 15 \times 1.857 = 27.855$

So the characteristics is 27 and hence the number of digits will be 28.

52. Given $b = 5, n = 2$, therefore the number of integers will be $5^3 - 5^2 = 100$.

53. Let $x = 3^{15} \times 2^{10} \therefore \log_{10} x = 15 \log_{10} 3 + 10 \log_{10} 2$

$= 15 \times 0.477 + 10 \times 0.301 = 10.165$.

So no. of digits will be 11.

54. Let $x = 6^{20} \therefore \log_{10} x = 20 \log_{10}(2 \times 3) = 20[\log_{10} 2 + \log_{10} 3]$

$= 20[0.301 + 0.477] = 15.56$.

So no. of digits will be 16.

55. Let $x = 5^{25} \therefore \log_{10} x = 25 \log_{10} \frac{10}{2} = 25[1 - \log_{10} 2]$

$= 25 \times 0.699 = 17.475$

So no. of digits will be 18.

56. Given $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p x)\}] = 0$

$\Rightarrow 1 + \log_b\{1 + \log_c(1 + \log_p x)\} = 1$

$\Rightarrow \log_b\{1 + \log_c(1 + \log_p x)\} = 0$

$\Rightarrow 1 + \log_c(1 + \log_p x) = 1$

$\Rightarrow \log_c(1 + \log_p x) = 0$

$\Rightarrow 1 + \log_p x = 1$

$\Rightarrow \log_p x = 0 \Rightarrow x = 1$

57. Given $\log_7 \log_5(\sqrt{x+5} + \sqrt{x}) = 0 \Rightarrow \log_5(\sqrt{x+5} + \sqrt{x}) = 1$

$\Rightarrow \sqrt{x+5} + \sqrt{x} = 5 \Rightarrow \sqrt{x+5} = 5 - \sqrt{x}$

Squaring both sides,

$x + 5 = 25 + x - 10\sqrt{x} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$.

58. $\log_2 x + \log_4(x+2) = 2 \Rightarrow \log_2 x + \frac{1}{2}\log_2(x+2) = 2$

$\Rightarrow 2\log_2 x + \log_2(x+2) = 4 \Rightarrow \log_2 x^2(x+3) = 4$

$\Rightarrow x^2(x+2) = 16 \Rightarrow x = 2$

59. $\log_{(x+2)} x + \log_x(x+2) = \frac{5}{2} \Rightarrow \frac{1}{\log_x(x+2)} + \log_x(x+2) = \frac{5}{2}$

Let $z = \log_x(x+2) \Rightarrow \frac{1}{z} + z = \frac{5}{2}$

$2z^2 + 2 - 5z = 0 \Rightarrow z = 2, \frac{1}{2}$

$\Rightarrow \log_x(x+2) = 2, \frac{1}{2}$

$\Rightarrow x+2 = 2^2, x+2 = \sqrt{x}$

$x = 2, x^2 - 4x + 4 = 0 \Rightarrow x = \frac{3 \pm \sqrt{-7}}{2}$

However, x cannot be a complex number. $\therefore x = 2$.

60. $\frac{\log(x+1)}{\log x} = 2 \Rightarrow \log_x(x+1) = 2 \Rightarrow x+1 = x^2$

$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$

$\because x > 0, x = \frac{1+\sqrt{5}}{2}$.

61. $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0 \Rightarrow \frac{2}{\log_a x} + \frac{1}{\log_a ax} + \frac{1}{\log_a a^2x} = 0$

$\Rightarrow \frac{2}{\log_a x} + \frac{1}{\log_a a + \log_a x} + \frac{1}{\log_a a^2 + \log_a x} = 0$

$\Rightarrow \frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{1}{2+\log_a x} = 0$

Substituting $\log_a x = z, \frac{2}{z} + \frac{1}{1+z} + \frac{1}{2+z} = 0$

$\Rightarrow 6z^2 + 11z + 4 = 0 \Rightarrow z = -\frac{1}{2}, -\frac{4}{3}$

$\therefore x = a^{-\frac{1}{2}}, a^{-\frac{4}{3}}$.

62. $x + \log_{10}(1+2^2) = x\log_{10} 5 + \log_{10} 6$

$\Rightarrow \log_{10} 10^x + \log_{10}(1+2^x) = \log_{10} 5^x + \log_{10} 6$

$\Rightarrow \log_{10} 10^x(1+x^x) = \log_{10}(5^x.6)$

$\Rightarrow 2^x(1+2^x) = 2.3 \Rightarrow 2^x = 2, 1+2^x = 3 \Rightarrow x = 1$.

63. $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

Taking $\log_2$ of both sides,

$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \frac{1}{2}\log_2 2$

$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \frac{1}{2}$

Let $\log_2 x = z, \Rightarrow \left(\frac{3}{4}z^2 + z - \frac{5}{4}\right)z = \frac{1}{2}$

Solving this qubic equation yields $x = 2, \frac{1}{4}, \frac{1}{\sqrt[3]{2}}$.

64. Given $(x^2+6)^{\log_3 x} = (5x)^{\log_3 x}$

$\log_3 x$ has a possible value of 0, in that case $x = 1$

If $\log_3 x \neq 0, \Rightarrow x^2 + 6 = 5x \Rightarrow x = 2, 3.$

65. Given, $(3+2\sqrt{2})^{x^2-6x+9} + (3-2\sqrt{2})^{x^2-6x+9} = 6$

We observe that $3 + 2\sqrt{2} = \frac{1}{3-2\sqrt{2}}$, thus, given equation becomes

$(3+2\sqrt{2})^{x^2-6x+9} + (3+2\sqrt{2})^{-(x^2-6x+9)} = 6$

Let $z = (3+2\sqrt{2})^{x^2-6x+9} \Rightarrow z + \frac{1}{z} = 6 \Rightarrow z = 3 \pm 2\sqrt{2}$

Thus, $x^2 - 6x + 9 = \pm 1 \Rightarrow x = 2, 4$ because other roots are irrational.

66. Given, $\log_8\left(\frac{8}{x^2}\right) \div (\log_8 x)^2 = 3$

$\Rightarrow \log_8 8 - \log_8 x^2 = 3(\log_8 x)^2 \Rightarrow 1 - 2\log_8 x = 3(\log_8 x)^2$

Let $z = \log_8 x \Rightarrow 1 - 2z = 3z^2 \Rightarrow z = -1, \frac{1}{3} \Rightarrow x = 2, \frac{1}{8}.$

67. Given, $\sqrt{\log_2(x)^4} + 4\log_4\sqrt{\frac{2}{x}} = 2$

$\Rightarrow \sqrt{\log_2(x)^4} + 2\log_2\sqrt{\frac{2}{x}} = 2$

$\Rightarrow \sqrt{4\log_2 x} + \log_2\frac{2}{x} = 2$

$\Rightarrow \sqrt{4\log_2 x} + 1 - \log_2 x = 2 \Rightarrow \sqrt{4\log_2 x} = 1 + \log_2 x$

Squaring, $4\log_2 x = 1 + 2\log_2 x + (\log_2 x)^2 \Rightarrow (\log_2 x - 1)^2 = 0$

$\Rightarrow \log_2 x = 1 \Rightarrow x = 2.$

68. Given, $2\log_{10} x - \log_x 0.01 = 5 \Rightarrow 2\log_{10} x - \log_x(10)^{-2} = 5$

$\Rightarrow 2\log_{10} x - \log_x(10)^{-2} = 5 \Rightarrow 2\log_{10} x + 2\log_x 10 = 5$

$\Rightarrow 2\log_{10} x + \frac{2}{\log_{10} x} = 5$

Let $z = \log_{10} x \Rightarrow 2z + \frac{2}{z} = 5 \Rightarrow z = 2, \frac{1}{2}$

$\Rightarrow x = 100, \sqrt{10}.$

69. Given, $\log_{\sin x} 2\log_{\cos x} 2 + \log_{\sin x} 2 + \log_{\cos x} 2 = 0$

$\Rightarrow \log_{\sin x} 2(\log_{\cos x} 2 + 1) + \log_{\cos x} 2 = 0$

$\Rightarrow \frac{\ln 2}{\ln\sin x}\left(\frac{\ln 2}{\ln\cos x} + 1\right) + \frac{\ln 2}{\ln\cos x} = 0$

$\Rightarrow \frac{1}{\ln \sin x}\left(\frac{\ln 2}{\ln \cos x}+1\right)+\frac{1}{\ln \cos x}=0$

$\Rightarrow \frac{1}{\ln \sin x}\left(\frac{\ln 2}{\ln \cos x}+1\right)=-\frac{1}{\ln \cos x}$

$\Rightarrow \frac{1}{\ln \sin x}(\ln 2+\ln \cos x)=-1$

$\Rightarrow \ln(\sin 2x)=0 \Rightarrow x=2k\pi+\frac{\pi}{4}, k \in \mathbb{I}.$

70. Given, $2^{x+3}+2^{x+2}+2^{x+1}=7^x+7^{x-1}$

$\Rightarrow 2^{x+1}(2^2+2+1)=7^{x-1}(7+1) \Rightarrow 2^{x-2}=7^{x-2}$

Taking log of both sides

$(x-1)\log 2=(x-2)(\log 7), \because 2 \neq 7 \Rightarrow x=2.$

71. Given, $\log_{\sqrt{2}\sin x}(1+\cos x)=2$

$\Rightarrow 1+\cos x=(\sqrt{2}\sin x)^2=2\sin^2 x=2-2\cos^2 x$

$\Rightarrow 2\cos^2 x+\cos x-1=0 \Rightarrow \cos x=-1, \frac{1}{2}$

$\Rightarrow x=2n\pi, 2n\pi+\frac{\pi}{3}, n \in I$

72. Given, $\log_{10}[98+\sqrt{x^2-12x+36}]=2$

$\Rightarrow 98+\sqrt{x^2-12x+36}=10^2=100$

$\Rightarrow x^2-12x+36=4 \Rightarrow x^2-12x+32=0$

$\Rightarrow x=4, 8.$

73. Given, $2^x 3^{2x}-100=0 \Rightarrow x\log_{10}2+2x\log_{10}3=\log_{10}100=2$

Substituting values for $\log_{10}2$ and $\log_{10}3$, we get

$0.30103x+0.95424x=2 \Rightarrow x=1.593.$

74. Given, $\log_x 3 \log_{\frac{x}{3}} 3+\log_{\frac{x}{81}} 3=0$

$\Rightarrow \frac{1}{\log_3 x}\cdot\frac{1}{\log_x \frac{x}{3}}+\frac{1}{\log_3 \frac{x}{81}}=0$

$\Rightarrow \frac{1}{\log_3 x}\cdot\frac{1}{\log_3 x-\log_3 3}+\frac{1}{\log_3 x-\log_3 81}=0$

Let $z=\log_3 x, \Rightarrow \frac{1}{z}\cdot\frac{1}{z-1}+\frac{1}{z-4}=0$

$\Rightarrow z-4+z^2-z=0 \Rightarrow z^2-4=0 \Rightarrow z=\pm 2$

$\Rightarrow x=9, \frac{1}{9}.$

75. Given, $\log_{(2x+3)}(6x^2+23x+21) = 4 - \log_{(3x+7)}(4x^2+12x+9)$

$\Rightarrow \log_{(2x+3)}(2x+3)(3x+7) = 4 - \log_{(3x+7)}(2x+3)^2$

$\Rightarrow 1 + \log_{(2x+3)}(3x+7) = 4 - 2\log_{(3xx+7)}(2x+3)$

Let $z = \log_{(2x+3)}(3x+7)$,

$\Rightarrow 1 + z = 4 - \frac{2}{z} \Rightarrow z = 1, 2 \Rightarrow x = -4, -3, -\frac{1}{4}$.

For logarithm to be defined, $2x+3 > 0, 2x+3 \neq 1$ and $3x+7 > 0, 3x+7 \neq 1$.

Thus, $x = -\frac{1}{4}$ is the only valid solution.

76. Given, $\log_2(x^2-1) = \log_{\frac{1}{2}}(x-1)$

$\Rightarrow \log_2(x^2-1) = \log_{2^{-1}}(x-1) = -\log_2(x-1) = \log_2\frac{1}{x-1}$

$\Rightarrow x^2 - 1 = \frac{1}{x-1} \Rightarrow x = 0, x^2 - x - 1 = 0$

$\Rightarrow x = 0, \frac{1\pm\sqrt{5}}{2}$

For logarithm to be defined $x^2 - 1 > 0$ and $x - 1 > 0$

Thus, $x = \frac{1+\sqrt{5}}{2}$ is the only acceptable solution.

77. Given, $\log_5\left(5^{\frac{1}{x}+125}\right) = \log_5 6 + 1 + \frac{1}{2x}$

$\Rightarrow \log_5\left(5^{\frac{1}{x}+125}\right) - \log_5 6 = 1 + \frac{1}{2x}$

$\Rightarrow \log_5\left(\frac{5^{\frac{1}{x}+125}}{6}\right) = 1 + \frac{1}{2x}$

$\Rightarrow 5^{\frac{1}{x}+125} = 30.5^{\frac{1}{2x}}$

Let $z = 5^{\frac{1}{2x}}$

$\Rightarrow z^2 - 30z + 125 = 0 \Rightarrow z = 5, 25 \Rightarrow x = \frac{1}{2}, \frac{1}{4}$.

78. For $\log_{100}|x+y| = \frac{1}{2} \Rightarrow (x+y)^2 = 100$

And for $\log_{10} y - \log_{10}|x| = \log_{100} 4 \Rightarrow \log_{10}\frac{y}{|x|} = \log_{10} 2$

$\Rightarrow y = 2|x| \Rightarrow y^2 = 4x^2 \Rightarrow 5x^2 + 4x|x| = 100$

When $x > 0, x = \frac{10}{3}$ and when $x < 0, x = -10$

$\Rightarrow y = \frac{20}{3}, 20.$

79. Given, $2\log_2 \log_2 x + \log_{\frac{1}{2}} \log_2(2\sqrt{2}x) = 1$

$\Rightarrow \log_2(\log_2 x)^2 - \log_2 \log_2(2\sqrt{2}x) = 1$

$\Rightarrow \log_2\left(\frac{(\log_2)^2}{\log_2(2\sqrt{2}x)}\right) = 1$

$\Rightarrow \frac{(\log_2 x)^2}{\log_2(2\sqrt{2}x)} = 2$

$\Rightarrow (\log_2 x)^2 = \log_2(2\sqrt{2}x)^2$

$\Rightarrow (\log_2 x)^2 - 3 - 2\log_2 x = 0$

Let $z = \log_2 x$, then $z^2 - 2z - 3 = 0 \Rightarrow z = -1, 3$

$\Rightarrow x = \frac{1}{2}, 8$

For logarithm to be defined $x > 0, 2\sqrt{2}x > 0, \log_2 x > 0, \log_2(2\sqrt{2}x) > 0.$

Thus, $x = 8$ is only acceptable solution.

80. Given $\log_{\frac{3}{4}} \log_8(x^2 + 7) + \log_{\frac{1}{2}} \log_{\frac{1}{4}}(x^2 + 7)^{-1} = -2$

$\Rightarrow \log_{\frac{3}{4}} \log_{2^3}(x^2 + 7) + \log_{\frac{1}{2}} \log_{2^{-2}}(x^2 + 7)^{-}1 = -2$

$\Rightarrow \log_{\frac{3}{4}}\left[\frac{1}{3}\log_2(x^2 + 7)\right] + \log_{\frac{1}{2}}\left[\frac{1}{2}\log_2(x^2 + 7)\right] = -2$

Let $y = \log_2(x^2 + 7)$,

$\Rightarrow \log_{\frac{3}{4}}\left(\frac{y}{3}\right) + \log_{\frac{1}{2}} \frac{1}{2} + \log_{\frac{1}{2}} y = -2$

$\Rightarrow -\log_{\frac{3}{4}} 3 + \log_2 y. \log_{\frac{3}{4}} 2 - \log_2 y = -3$

$\Rightarrow \log_2 y\left(\log_{\frac{3}{4}} 2 - 1\right) = -3 + \log_{\frac{3}{4}} 3$

$\Rightarrow \log_2 y\left(\log_{\frac{3}{4}} 2 - \log_{\frac{3}{4}} \frac{3}{4}\right) = \log_{\frac{3}{4}}\left(\frac{3}{4}\right)^{-3} + \log_{\frac{3}{4}} 3$

$\Rightarrow \log_2 y. \log_{\frac{3}{4}} \frac{8}{3} = \log_{\frac{3}{4}} \frac{64}{9} = 2 \log_{\frac{3}{4}} \frac{8}{3}$

$\Rightarrow \log_2 y = 2 \Rightarrow y = 3 \Rightarrow x = \pm 3$, both of which are valid for the given equation.

81. Given, $\log_{10} x + \log_{10} x^{\frac{1}{2}} + \log_{10} x^{\frac{1}{4}} + \cdots$ to $\infty = y$

$\Rightarrow \left[1 + \frac{1}{2} + \frac{1}{4} + \cdots \text{ to } \infty\right] \log_{10} x = y$

$\Rightarrow \frac{1}{1-\frac{1}{2}} \log_{10} x = y \Rightarrow \log_{10} x = \frac{y}{2}$

Also given that $\frac{1+3+5+\cdots+(2y-1)}{4+7+10+\cdots+(3y+1)} = \frac{20}{7 \log_{10} x}$

$\Rightarrow \frac{\frac{y}{2}[2+(y-1)2]}{\frac{y}{2}[8+(y-1)3]} = \frac{20}{7 \log_{10} x}$

$\Rightarrow \frac{2y}{3y+5} = \frac{20}{7 \log_{10} x} = \frac{20 \times 2}{7y} \Rightarrow 7y^2 - 60y - 100 = 0$

$y = 10, -\frac{10}{7}$. Since number of terms cannot be fraction, therefore $y = 10$ and $x = 10^5$.

82. Given, $18^{4x-3} = (54\sqrt{2})^{3x-4}$

Taking log on both sides,

$\Rightarrow (4x-3) \log 18 = (3x-4) \log(18 \times 3\sqrt{2}) = \frac{3}{2}(3x-4) \log 18$

$\Rightarrow 4x - 3 = \frac{3}{2}(3x-4) \Rightarrow x = 6$

83. Given, $4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$

$\Rightarrow 4^{\log_{3^2} 3} + 9^{\log_2 2^2} = 10^{\log_x 83}$

$\Rightarrow 4^{\frac{1}{2}\log_3 3} + 9^{2 \log_2 2} = 10^{\log_x 83}$

$\Rightarrow 4^{\frac{1}{2}} + 9^2 = 83 = 10^{\log_x 83} \Rightarrow x = 10$

84. Given, $3^{4 \log_9(x+1)} = 2^{2 \log_2(x+3)}$

$\Rightarrow 3^{2 \log_3(x+1)} = x^2 + 3 [\because a^{\log_a N} = N]$

$\Rightarrow 3^{\log_3(x+1)^2} = x^2 + 2x + 1 = x^2 + 3 \Rightarrow x = 1$

85. $\frac{6}{5} a^{\log_a x \log_{10} a \log_a 5} - 3^{\log_{10} \frac{x}{10}} = 9^{\log_{100} x + \log_4 2}$

$\Rightarrow \frac{6}{5} a^{\log_{10} x \log_a 5} - 3^{\log_{10} x - 1} = 9^{\frac{1}{2}\log_{10} x + \frac{1}{2}\log_2 2}$

$\Rightarrow \frac{6}{5}\left(5^{\log_a 5}\right)^{\log_{10} x} - 3^{\log_{10} x - 1} = 3^{\log_{10} x + 1}$

$\Rightarrow \frac{6}{5}5^{\log_{10} x} = 6.5^{\log_{10} x - 1} = 3^{\log_{10} x - 1}(1 + 3^3)$

$\Rightarrow \left(\frac{5}{3}\right)^{\log_{10} x - 1} = \frac{10}{6}$

$\Rightarrow \log_{10} x - 1 = 1 \Rightarrow x = 100$

86. Given, $2^{3x+\frac{1}{2}} + 2^{x+\frac{1}{2}} = 2^{\log_2 6}$

$\Rightarrow 2^{3x}\sqrt{2} + 2^x\sqrt{2} = 6$

$\Rightarrow (2^x)^3 + 2^2 = 3\sqrt{2} \Rightarrow 2^x = \sqrt{2}, \frac{-\sqrt{2}\pm\sqrt{-10}}{2}$

Ignoring complex roots we have $x = \frac{1}{2}$.

87. $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$

$\Rightarrow (5 + 2\sqrt{6})^{x^2-3} + (5 + 2\sqrt{6})^{-(x^2-3)} = 10$

Let $z = (5 + 2\sqrt{6})^{x^2-3}$, then

$\Rightarrow z + \frac{1}{z} = 10 \Rightarrow z = 5 \pm 2\sqrt{6}$

$\therefore x = \pm 2, \pm\sqrt{2}$

88. $2\log_{10x} x - \log_x .01 \geq 4$

$\Rightarrow 2\log_{10} x - \log_x 10^{-2} \geq 4$

$\Rightarrow 2\log_{10} x + 2\log_x 10 \Rightarrow 2\log_{10} x + \frac{2}{\log_{10} x} \geq 4$

$= 2\left(\log_{10} x + \frac{1}{\log_{10} x}\right) \geq 4$

Let $z = \log_{10} x$, then $2\left(z + \frac{1}{z}\right) \geq 4$

$\Rightarrow 2\left[\left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^2 + 2\right] \geq 4$

which is true.

89. Let $E = \log_b a + \log_a b = \log_b a + \frac{1}{\log_b a}$

Let $z = \log_b a$, then $E = z + \frac{1}{z}$

Clearly, $z \neq 0$, or the problem will be undefined.

When $z > 0, E = z + \frac{1}{z} = \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^2 + 2 > 2$

When $z < 0, z = -y$ (let), then

$E = \left|-y - \frac{1}{y}\right| = y + \frac{1}{y} > 2.$

90. Given, $\log_{0.3}(x^2 + 8) > \log_{0.3} 9x$

$\Rightarrow x^2 + 8 < 9x \Rightarrow 1 < x < 8.$

91. $\log_{x-2}(2x - 3) > \log_{x-2}(24 - 6x)$

Case I: When $0 < x - 2 < 1 \Rightarrow 2 < x < 3$

Given inequality becomes $2x - 3 < 24 - 6x \Rightarrow x < \frac{27}{8}$

But $x < 3$ so 3 si still limiting value of x.

Case II: When $x - 2 > 1 \Rightarrow x > 3$

Given inequality becomes $2x - 3 > 24 - 6x \Rightarrow x > \frac{27}{3}$

However, for logarithm to be defined $2x - 3 > 0$ and $24 - 6x > 0$ and also $x - 2 > 0$.

Combining all these we get $2 < x < 3$.

92. Given, $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$

$\Rightarrow (x - 1)^2 > (x - 1) \Rightarrow x^2 - 3x + 2 > 0$

$\Rightarrow x < 1, x > 2$. For logarithm function to be defined $x > 1$, thus the interval for x will be $(2,\infty]$.

93. Given, $\log_{\frac{1}{2}} x \geq \log_{\frac{1}{3}} x$

$\Rightarrow \log_{\frac{1}{2}} x \geq \log_{\frac{1}{2}} x \log_{\frac{1}{3}} \frac{1}{2}$

$\Rightarrow \log_{\frac{1}{2}} x \left[1 - \log_{\frac{1}{3}} \frac{1}{2}\right] \geq 0$

$\Rightarrow \log_{\frac{1}{2}} x[1 - \log_3 2] \geq 0$

$\log_{\frac{1}{2}} x \geq 0 \Rightarrow x \leq 1$

For logarithm function to be defined $x > 0$, thus range of x will be $(0, 1]$.

94. Given, $\log_{\frac{1}{3}} \log_4(x^2 - 5) > 0$

$\Rightarrow \log_4(x^2 - 5) < 1 \Rightarrow x^2 - 5 < 4 \Rightarrow -3 < x < 3$

For logarithm to be defined $x^2 - 5 > 0$ and $\log_4(x^2 - 5) > 0$

$\Rightarrow x < -\sqrt{5}, x > \sqrt{5}$ and $x^2 - 5 > 1 \Rightarrow x < -\sqrt{6}, x > \sqrt{6}$

Combining all these conditions we get two ranges for x, $(-3, -\sqrt{6})$ and $(\sqrt{6}, 3)$.

95. Given, $\log(x^2 - 2x - 2) \leq 0 \Rightarrow x^2 - 2x - 2 \leq 1$

$\Rightarrow -1 \leq x \leq 3$

For logarithm to be defined $x^2 - 2x - 2 > 0$

$\Rightarrow x < 1 - \sqrt{3}, x > 1 + \sqrt{3}$

Combining all these ranges gives us the range as $[-1, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, 3]$.

96. Given, $\log_2^2(x - 1)^2 - \log_{0.5}(x - 1) > 5$

$\Rightarrow (2\log_2|x - 1|)^2 - \log_{0.5}(x - 1) > 5$

$\Rightarrow 4[\log_2(x - 1)]^2 + \log_2(x - 1) > 5$

Let $z = \log_2(x - 1)$, $\Rightarrow 4x^2 + z - 5 > 0$

$\Rightarrow 2 < -\frac{5}{4}, x > 1 \Rightarrow x < 1 + \frac{1}{2\sqrt[4]{2}}$

For log to be defined $x - 1 > 0 \Rightarrow x > 1$

When $z > 1, x > 3$

Thus, the range of x is $\left(1, 1 + \frac{1}{2\sqrt[4]{2}}\right) \cup (3, \infty)$.

97. We have to prove that $\log_2 17 \log_{\frac{1}{5}} 2 \log_3 \frac{1}{5} > 2$

$\Rightarrow \log_2 17 \log_3 2 > 2 \Rightarrow \log_3 17 > 2$

$\because 17 > 3^2 \therefore \log_3 17 > 2$

98. We have to prove that $\frac{1}{3} < \log_{20} 3 < \frac{1}{2}$

$\frac{1}{3} < \log_{20} 3 \Rightarrow 1 < \log_{20} 3^3 \Rightarrow 1 < \log_{20} 27$

which is true as the base is greater than 1 and the number is greater than the base.

$\log_{20} 3 < \frac{1}{2} \Rightarrow \log_{20} 3^2 < 1 \Rightarrow \log_{20} 9 < 1$

which is true as the base is greater than 1 and the number is less than the base.

99. We have to prove that $\frac{1}{4} < \log_{10} 2 < \frac{1}{2}$

$\frac{1}{4} < \log_{10} 2 \Rightarrow 1 < \log_{10} 2^4 = \log_{10} 16$

which is true because base is greater than 1 and the number is greater than the base.

$\log_{10} 2 < \frac{1}{2} \Rightarrow \log_{10} 2^2 < 1 \Rightarrow \log_{10} 4 < 1$

which is true as the base is greater than 1 and the number is less than the base.

100. Given $\log_{0.1}(4x^2 - 1) > \log_{0.1} 3x$

$\Rightarrow 4x^2 - 3x - 1 < 0 \Rightarrow (4x+1)(x-1) < 0$

Thus, $[-\infty, -\frac{1}{4}) \cup (1, \infty]$ is the initial solution.

Now, $x > 0$ is another restriction from R.H.S.

From L.H.S> $4x^2 - 1 > 0 \Rightarrow x < -\frac{1}{2}, x > \frac{1}{2}$

Combining all these we get, $\frac{1}{2} < x < 1$.

101. Given, $\log_2(x^2 - 24) > \log_2 5x$

$\Rightarrow x^2 - 24 > 5x \Rightarrow x < -3, x > 8$

But $x^2 - 24 > 0$ and also $x > 0$ for loarithm function to be defined.

$\therefore x > 8$.

102. We have to prove that $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > 2$

$\Rightarrow \log_\pi 3 + \log_\pi 4 > 2$

$\Rightarrow \log_\pi 12 > 2 \Rightarrow 12 > \pi^2$ which is true.

103. Given $(0.01)^{\frac{1}{3}}$ and $(0.001)^{\frac{1}{5}}$

Taking log of both with base 10,

$\frac{1}{3}\log_{01} 0.01$ and $\frac{1}{5}\log_{10} 0.001$

$-\frac{2}{3}$ and $-\frac{3}{5}$ out of which $-\frac{3}{5}$ is greater, therefore $(0.001)^{\frac{1}{5}}$ is greater.

104. $\log_3 11 > \log_3 9 = \log_3(3^2) = 2$ and $\log_2 3 < \log_2 4 = 2$.

Thus, $\log_3 11$ is geater.

105. Given, $\log_3(x^2+10) > \log_3 7x$

$\Rightarrow x^2+10 > 7x \Rightarrow x < 2, x > 5$

However, $x^2+10 > 0$ and $x > 0$ for logarithm to be defined.

Thus, intervals are $0 < x < 2$ and $x > 5$.

106. We have, $x^{\log_{10} x} > 10$

$\Rightarrow \log_{10} x \log_{10} x > 1 \Rightarrow \log_{10} x > \pm 1$

Thus range of values of x would be $(0, 0.1) \cup (10, \infty]$.

107. We have, $\log_2 x \log_{2x} 2 \log_2 4x > 1$

$\Rightarrow \frac{1}{\log_x 2}\left[\frac{1}{\log_2 2x}\log_2 2^2 x\right] > 1$

$\Rightarrow \frac{1}{\log_x 2}\left[\frac{1}{1+\log_2 x}\right]\left[2+\frac{1}{\log_x 2}\right] > 1$

Let $z = \log_x 2$, then

$\Rightarrow \frac{1}{z}\frac{z}{1+z}\left[2+\frac{1}{z}\right] > 1$

Solving this inequality and applying rules for definition of logarithm we have following range for x

$\left(2^{-\sqrt{2}}, \frac{1}{2}\right) \cup \left(1, 2^{\sqrt{2}}\right)$

108. Given, $\log_2 x \log_3 2x + \log_3 x \log_2 4x > 0$

Exchanging base, we have $\log_3 x \log_2 2x + \log_3 x \log_2 4x > 0$

$\Rightarrow \log_3 x(\log_2 2 + \log_2 x + \log_2 4 + \log_2 x) > 0$

$\Rightarrow \log_3 x(3 + 2\log_2 x) > 0$

For $log_3 x > 0, x > 1$ and for, $3 + 2\log_2 x^2 > 0 \Rightarrow \log_2 x^2 > -3$.

Also for $\log_3 x < 0, 0 < x < 1$ and for $3 + \log_2 x^2 < 0 \Rightarrow \log_2 x^2 < -3$

109. $\log_{12} 60 = \frac{\log_2 60}{\log_2 12} = \frac{\log_2(2^2\times 3\times 5)}{\log_2(2^2\times 3)}$

$= \frac{2+\log_2 3+\log_2 5}{2+\log_2 3}$

Let $\log_2 3 = x$ and $\log_2 5 = y$, then $\log_{12} 60 = \frac{2+x+y}{2+x}$

Given $a = \log_6 30 = \frac{\log_2 30}{\log_2 6} = \frac{\log_2(2\times 3\times 5)}{\log_2 2\times 3}$

$= \frac{1+\log_2 3+\log_2 5}{1+\log_2 3} = \frac{1+x+y}{1+x}$

Also given, $b = \log_{15} 24$, proceeding similarly $b = \frac{3+x}{x+y}$

From these two, we can write x and y in terms of x and y,

$x = \frac{b+3-ab}{ab-3}, y = \frac{2a-b-2+ab}{ab-1}$

Substituting these values for $\log_{12} 60$, we get

$\log_{12} 60 = \frac{2ab+2a-1}{ab+b+1}$

110. $\log_a x, \log_b x$ and $\log_c x$ are in A.P.

$\therefore 2\log_x b = \frac{1}{\log_x a} + \frac{1}{\log_x c}$

$\Rightarrow \frac{2}{\log_x b} = \frac{\log_x ac}{\log_x a \log_x c}$

$\Rightarrow 2\log_x c = \log_x ac \frac{\log_x b}{\log_x a} \Rightarrow \log_x c^2 = \log_x ac \log_a b$

$\Rightarrow c^2 = ac^{\log_a b}$.

111. $a = \log_{\frac{1}{2}} \sqrt{0.125} > 0$ because both base and number are less than 1.

$b = \log_3\left(\frac{1}{\sqrt{24}-\sqrt{17}}\right) = \log_3\left(\frac{\sqrt{24}+\sqrt{17}}{3}\right) > 0$

because both base and number are greater than 1.

112. Given $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$

Taking log natural of both sides

$\log_e e^{-\frac{\pi}{2}} < \log_e \theta < \log_e \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} < \log_e \theta < 1 < \frac{\pi}{2}\left[\because \log_e \frac{\pi}{2} < \log_e e\right]$

$\Rightarrow -\frac{\pi}{2} < \log_e \theta < \frac{\pi}{2}$

$\Rightarrow \cos(\log_e \theta) > 0$

Again, $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$

$\Rightarrow 0 < \theta < \frac{\pi}{2}[\because e^{-\frac{\pi}{2}} > 0]$

$\Rightarrow 0 < \cos\theta < 1 \Rightarrow \log_e \cos\theta < 0$

$\Rightarrow \cos(\log_e \theta) > \log_e(\cos\theta)$

113. Given, $\log_2 x + \log_2 y \geq 6 \Rightarrow \log_2 xy \geq 6 \Rightarrow xy \geq 64$

This means x and y are positive as negative values will not be valid for logarithm function.

A.M $\geq$ G.M $\Rightarrow \frac{x+y}{2} \geq xy \Rightarrow x + y \geq 16$.

114. Given, $\log_b a \log_c a - \log_a a + \log_a b \log_c b - \log_b b + \log_a c \log_b c - \log_c c = 0$

$\Rightarrow \frac{(\log a)^2}{\log b \log c} - 1 + \frac{(\log b)^2}{\log a \log c} - 1 + \frac{(\log c)^2}{\log a \log b} - 1 = 0$

Let $x = \log a$, $y = \log b$, $z = \log c$, then

$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} - 3 = 0$

$\Rightarrow \frac{x^3+y^3+z^3-3xyz}{xyz} = 0$

$\Rightarrow (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$

$\Rightarrow \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0$

$\because x, y, z$ are different the term inside brackets will be always positive. Thus.

$x + y + z = 0$, now substituting the original values,

$\log abc = 0 \Rightarrow abc = 1$.

115. Given, $y = 10^{\frac{1}{1-\log x}} \Rightarrow \log y = \frac{1}{1-\log x}$, and similarly, $\log z = \frac{1}{1-\log y}$

$\Rightarrow z = \frac{1}{1-\frac{1}{1-\log x}} = \frac{1-\log x}{-\log x} = -\frac{1}{\log x} + 1 \Rightarrow x = 10^{\frac{1}{1-\log z}}$.

116. Since n is a natural number and $p_1, p2, \ldots, p_k$ are distinct primes, therefore $a_1, a_2, \ldots, a_k$ are also natural numbers.

Now $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_K}$

$\Rightarrow \log n = a_1 \log p_1 + a_2 \log p_2 + \cdots + a_k \log p_k$

$\log n \geq \log 2 + \log 2 + \cdots + \log 2$ [since bases are primes so minimum value is is 2 and pwoers are natural numbers so they are greater than 1]

$\log n \geq k \log 2$

117. Let d be the common difference of the A.P., then

$3 \log_y x = 3 + d \Rightarrow \log_y x^3 = 3 + d \Rightarrow x^3 = y^{(3+d)}$

$3 \log_z y = 3 + 2d \Rightarrow y^3 = z^{(3+2d)}$

$7 \log_x z = 3 + 3d \Rightarrow z^7 = x^{(3+3d)}$

$y^3 = z^{(3+2d)} \Rightarrow y = z^{\frac{3+2d}{3}}$

$x^3 = y^{(3+d)} \Rightarrow x = y^{\frac{3+d}{3}} = z^{\frac{(3+d)(3+2d)}{9}}$

$z^7 = x^{(3+3d)} \Rightarrow x = z^{\frac{7}{3+3d}}$

$\therefore \frac{(3+d)(3+2d)}{9} = \frac{7}{3+3d} \Rightarrow d = \frac{1}{2}$

Thus, $x^{18} = y^{21} = z^{28}$.

118. We have, $\log_4 18 = \log_{2^2}(2 \times 3^2) = \frac{1}{2} + \log_2 3$

Thus, it will be enough to prove tha $\log_2 3$ is an irrational number.

Let $\log_2 3 = \frac{p}{q}$, where $p, q \in \mathbb{I}$

$\Rightarrow 2^{\frac{p}{q}} = 3 \Rightarrow 2^p = 3^q$

However, 2^p is an even number and 3^q is an odd number, and hence the equality will never be achieved. Therefore, $\log_2 3$ is an irrational number.

119. Given, x, y, z are in G.P. $\therefore \frac{y}{x} = \frac{z}{y}$

$\Rightarrow \ln\frac{y}{x} = \ln\frac{z}{y} \Rightarrow \ln y - \ln x = \ln z - \ln y$

$\Rightarrow \ln x, \ln y, \ln z$ are in A.P.

$\Rightarrow 1 + \ln x, 1 + \ln y, 1 + \ln z$ are in A.P.

$\Rightarrow \frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in H.P.

120. $\log_{30} 8 = \log_{30} 2^3 = 3\log_{30} 2 = 3\log_{30}\frac{30}{15}$

$= 3 - 3(\log_{30} 3 + \log_{30} 5) = 3(1 - a - b)$.

121. Given $\log_7 12 = a$ and $\log_{12} 24 = b$

Multiplying $ab = \log_7 24$

Adding 1 on both sides

$ab + 1 = \log_7 24 + \log_7 7 = \log_7 168$

Similarly, $8a = \log_7 12^8$ and $5ab = \log_7 168^5$

$\frac{ab+1}{8a-5ab}\frac{\log_7 168}{\log_7 12^8 - \log_7 168^5}$

Upon simplification we find that $\log_{54} 168 = \frac{ab+1}{8a-5ab}$

122. **Case I**: When $x > 1, x > a^2 + 1$. Also, $a^2 + 1 < 1 \therefore x > 1$

Case II: When $x < 1, x < a^2 + 1$. Also, $a^2 > 0 \therefore x < 1$.

In both the cases $x > 0$.

123. Given, $\log_{12} 18 = a$ and $\log_{24} 54 = b$

$\therefore ab + 5(a - b) = \frac{\log 18 \log 54}{\log 12 \log 24} + 5\left(\frac{\log 18}{\log 12} - \frac{\log 54}{\log 24}\right)$

$= \frac{\log 18 \log 54 + 5(\log 18 \log 24 - \log 54 \log 12)}{\log 12 \log 24}$

$\log 18 = \log 2 + 2\log 3, \log 12 = 2\log 2 + \log 3$

$\log 24 = 3\log 2 + \log 3, \log 54 = \log 2 + 3\log 3$

Now it is only a matter of substitution and simplification.

124. Given, a, b, c are in G.P. so we can write $b^2 = ac$

Taking *log* with base x,

$2\log_x b = \log_x a + \log_x c \Rightarrow \frac{2}{\log_b x} = \frac{1}{\log_a x} + \frac{1}{\log_b x}$

Thus, $\log_a x, \log_b x, \log_c x$ are in H.P.

125. Let r be the common ratio of the G.P. and d be the common difference of the A.P.

$\log a_n - b_n = \log a + n\log r - (b + nd) = \log a - b$

$\Rightarrow n\log r - nd = 0 \Rightarrow \log r = d \Rightarrow b = r^{\frac{1}{d}}$.

126. Given $\log_3 2, \log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in A.P.

$\Rightarrow 2\log_3(2^x - 5) = \log_3\left(2^x - \frac{7}{2}\right) + \log_3 2$

$\Rightarrow (2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$

Let $z = 2^x$, then

$z^2 - 10x + 25 = 2z - 7 \Rightarrow z^2 - 12z + 32 \Longrightarrow z = 4, 8$

$\Rightarrow x = 2, 3$, however, if $x = 2$ then $2^x - 5 < 0$ so only acceptable value of x is 3.

127. Let $\log_2 7$ is a rational number i.e. $\log_2 7 = \frac{p}{q}$, where $p, q \in \mathbb{I}$

$\Rightarrow 7 = 2^{\frac{p}{q}} \Rightarrow 7^q = 2^q$

However, integral power of 7 is an odd number while that of 2 is an even number. Thus, by contradiction $\log_2 7$ is irrational number.

128. Given, $\log_{0.5}(x - 2) < \log_{0.25}(x - 2)$

$\Rightarrow (x - 2)^2 > x - 2 \Rightarrow (x - 2)(x - 3) > 0$

Thus, $x > 3$ for logarithm function to be defined.

Answers of Chapter 2
Progressions

1. Given $t_n = 2n^2 + 1 \Rightarrow t_{n-1} = 2(n-1)^2 + 1$

   $\therefore d = t_n - t_{n-1} = 4n - 2$, which is not constant. Hence, the sequence is not in A.P.

2. Given, $t_1 = 1, t_2 = 2$ and $t_{n+2} = t_n + t_{n+1}$

   $\therefore t_3 = t_1 + t_2 = 3, t_4 = t_2 + t_3 = 5, t_5 = t_3 + t_4 = 8$.

3. Given $t_n = 3n + 5 \Rightarrow t_1 = 3 \times 1 + 5 = 8, t_2 = 3 \times 2 + 5 = 11, t_3 = 3 \times 3 + 5 = 14$. So the seuquence is $8, 11, 14, \ldots, 3n + 5$.

4. Given $t_n = 2n^2 + 3 \Rightarrow t_1 = 2 \times 1^2 + 3 = 5, t_2 = 2 \times 2^2 + 3 = 11, t_3 = 2 \times 3^2 + 5 = 23$. So the sequence is $5, 11, 23, \ldots, 2n^2 + 3$.

5. Given, $t_n = \frac{3n}{2n+4} \Rightarrow t_1 = \frac{3\times1}{2\times1+4} = \frac{3}{6} = \frac{1}{2}, t_2 = \frac{3\times2}{2\times2+4} = \frac{6}{8} = \frac{3}{4}, t_3 = \frac{3\times3}{2\times3+4} = \frac{9}{10}$. So the sequence is $\frac{1}{2}, \frac{3}{4}, \frac{9}{10}, \cdots, \frac{3n}{2n+4}$.

6. Given, $t_1 = 2, t_{n+1} = \frac{2t_n+1}{t_n+3} \Rightarrow t_2 = \frac{2t_1+1}{t_1+3} = \frac{2\times1+1}{1+3} = \frac{3}{4}, t_3 = \frac{2t_2+1}{t_2+3} = \frac{2\times\frac{3}{4}+1}{\frac{3}{4}+3} = \frac{10}{15} = \frac{2}{3}$.

   So the sequence is $2, \frac{3}{4}, \frac{2}{3}, \cdots$.

7. Given, $t_n = 4n^2 + 1 \Rightarrow t_{n-1}4(n-1)^2 + 1$

   $\therefore d = t_n - t_{n-1} = 8n - 4$, which is not constant. Hence the sequence is not in A.P.

8. Given $t_n = 2an + b \Rightarrow t_{n-1} = 2a(n-1) + b$

   $\therefore d = t_n - t_{n-1} = 2a$. which is a constant. Hence the sequence will be an A.P.

9. Given, $t_1 = 3, t_2 = 3, t_3 = 6$ and $t_{n+2} = t_n + t_{n+1}$

   $\therefore t_4 = t_2 + t_3 = 3 + 6 = 9$ and $t_5 = t_3 + t_4 = 6 + 9 = 15$.

10. $t_1 = 1 = a + b + c, t_2 = 5 = 4a + 2b + c$ and $t_3 = 11 = 9a + 3b + c$

    $\therefore t_2 - t_1 = 4 = 3a + b$ and $t_3 - t_2 = 6 = 5a + b$

    $\Rightarrow 2a = 2 \Rightarrow a = 1 \Rightarrow b = 1 \Rightarrow c = -1$

    $\Rightarrow t_{10} = 1 \times 10^2 + 1 \times 10 - 1 = 109$.

11. Difference between successive terms i.e. commond difference, $d = 12 - 9 = 15 - 12 = 18 - 15 = 3$ which is a constant, hence, the given sequence is an A.P.

    Here first term $t_1 = 9$ and $d = 3 \therefore t_{16} = 9 + (16-1)3 = 54$ and $t_n = 9 + (n-1)3 = 3(n+2)$.

12. $t_1 = \log a, t_2 = \log(ab) = \log a + \log b, t_3 = \log(ab^2) = \log a + 2\log b$

$t_2 - t_1 = t_3 - t_2 = \log b$. Clearly, $t_1 = \log a$, $d = \log b$ which is constant so the sequence is an A.P.

$\therefore t_n = \log a + (n-1)\log b = \log(ab^{n-1})$.

13. Given, $t_n = 5 - 6n \Rightarrow t_1 = 5 - 6 = -1$

$S_n = \frac{n}{2}[t_1 + t_n] = n(2 - 3n)$.

14. $d = 7 - 3 = 11 - 7 = 4, t_n = 407 = 3 + (n-1)d \Rightarrow n = \frac{404}{4} + 1 = 102$.

15. Since a, b, c, d, e are in A.P. $\therefore a + e = b + d = 2c = k$(say)

$\therefore a - 4b + 6c - 4d + e = (a + e) - 4(b + d) + 3.2c = k - 4k + 3k = 0$.

16. Let a be the first term and d be the common difference of the given A.P.

Given, $5t_5 = 8t_8 \Rightarrow 5a + 20d = 8a + 56d \Rightarrow 3a = -36d \Rightarrow a = -12d$

$\Rightarrow t_{13} = a + 12d = 0$.

17. Let nth term be the smallest positive number. From the sequence we obtain that $t_1 = 25$ and $d = -2\frac{1}{4} = -\frac{9}{4}$.

Then $t_n > 0 \Rightarrow 25 - (n-1)\frac{9}{4} > 0 \Rightarrow n < \frac{25\times4}{9} + 1 \Rightarrow n = 12$.

18. The given pay scale represents an A.P. with $t_1 = 700$, $d = 40$ and $t_n = 1500$.

$\therefore t_n = t_1 + (n-1)d \Rightarrow n = \frac{t_n - t_1}{d} + 1 = \frac{1500-700}{40} + 1 = 21$.

Thus, the person will reach maximum payment in 21 years.

19. Let a be the first term and d be the common difference of the A.P. According to the question,

$t_7 = a + 6d = 34$ and $t_{13} = a + 12d = 64$

Subtracting $6d = 30 \Rightarrow d = 5 \Rightarrow a = 4$. So the A.P. is $4, 9, 14, \ldots$.

20. If 55 is the nth term then n will have to be an integer. From the given sequence $a = 1, d = 3 - 1 = 5 - 3 = 2$.

$55 = 1 + (n-1)2 \Rightarrow n = 28$, which is an integer and hence, 55 will be 28th term of the A.P.

21. From the given sequence $a = 2000, d = 1995 - 2000 = 1990 - 1995 = -5$.

Let nth term be first negative term, then, $a + (n-1)d < 0 \Rightarrow 2000 - (n-1)5 < 0$

$\Rightarrow n > 401 \Rightarrow n = 402 \Rightarrow t_{402} = 2000 - (402 - 1)5 = -5$.

22. Common different of the sequence 2, 4, 6, 8, ... is 2 and common difference of the seqquence 3, 6, 9, ... is 3.

    Thus, common terms will have a common different which is L.C.M. of these two commond differences i.e. 6.

    Last term of first sequence is 200 and last term of second sequence is 240. Clearly, last identical(common) number will be less than 200. We also observe that 6 is the first identical term. Let there be n such terms. Then

    $6+(n-1)6 \leq 200 \Rightarrow n \leq \frac{194}{6}+1 \Rightarrow n=33$. Thus there will be 33 identical terms in the two given A.P.

23. Clearly the first number of three digits divisible by 5 is 100; while the last such number is 995. Since these numbers are all divisible by 5 they will form an A.P. with common difference 5.

    Clearly, $t_1=100, t_n=995, d=5$ and we have to find n.

    $t_n=995=100+(n-1)5 \Rightarrow n=180$.

24. Given sequence is 4, 9, 14, ... So $a=4, d=9-4=14-9=5$. Let 105 be nth term of this A.P. then n has to be an integer for this assumption to be true.

    $105=4+(n-1)5 \Rightarrow n=\frac{106}{5}$ which is not an integer and therefore 105 is not a term in the given A.P.

25. This problem is same as problem 21 and has been left as an exercise.

26. This problem is same as problem 22 and has been left as an exercise.

27. Let a be the first term and d be the common difference of the A.P. Given,

    $mt_m=nt_n \Rightarrow ma+(m-1)md=na+(n-1)nd \Rightarrow (m-n)a=(n^2-n-m^2+m)d$

    $\Rightarrow a=-(m+n-1)d \therefore t_{m+n}=a+(m+n-1)d=0.$

28. Let x be the first term and y be the common difference of the A.P. Then,

    $a=x+(p-1)y, b=x+(q-1)y, c=x+(r-1)y$

    We have to prove that $a(q-r)+b(r-p)+c(p-q)=0$.

    Substituting the values of a, b and c in the above equation

    L.H.S. $=[x+(p-1)y](q-r)+[x+(q-1)y](r-p)+[x+(r-1)y](p-q)$

    $=x(q-r+r-p+p-q)+y[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$

    $=0=$ R.H.S.

29. First number after 100 which is divisible by 7 is 105. The last number divisible by 7 before 1000 is 994.

Let n be the numbers divisible by 7 between 100 and 1000. Then $994 = 105 + (n-1)7$

$\Rightarrow n = 128$. Then no. of numbers not divisible by 7 is $1000 - 100 - 128 = 772$.

30. Let x be the first term and y be the common difference of the A.P. Then,

$a = x + (p-1)y, b = x + (q-1)y, c = x + (r-1)y$

We have to prove that $(a-b)r + (b-c)p + (c-a)q = 0$

Substituting the values of a, b and c in the above equation

L.H.S. $= (p-q)yr + (q-r)yp + (r-p)yq = 0 =$ R.H.S.

31. Let the numbers in A.P. be $a-d, a$ and $a+d$. Given their sum is 27 and sum of squares is 293.

$\therefore a - d + a + a + d = 27 \Rightarrow a = 9$

$\therefore (a-d)^2 + a^2 + (a+d)^2 = 293 \Rightarrow 3a^2 + 2d^2 = 293 \Rightarrow 3 \times 81 + 2d^2 = 293$

$\Rightarrow 2d^2 = 50 \Rightarrow d = \pm 5$

So the numbers are 4, 9, 14 or 14, 9, 4.

32. Let the numbers in A.P. be $a-3d, a-d, a+d, a+3d$. Given their sum is 24 and product is 945.

$\therefore a - 3d + a - d + a + d + a + 3d = 24 \Rightarrow 4a = 24 \Rightarrow a = 6$

Also, $(a-3d)(a-d)(a+d)(a+3d) = 945 \Rightarrow (a^2 - 9d^2)(a^2 - d^2) = 945$

$\Rightarrow a^4 - 10a^2d^2 + 9d^4 = 945 \Rightarrow 9d^4 - 360d^2 + 1296 - 945 = 0$

$\Rightarrow 9d^4 - 360d^2 + 351 = 0 \Rightarrow d^4 - 40d^2 + 39 = 0$

$\Rightarrow (d^2 - 1)(d^2 - 39) = 0$. Since the numbers are integers $\Rightarrow d^2 \neq 39$.

$\Rightarrow d = \pm 1$. So the numbers are 3, 5, 7, 9 or 9, 7, 5, 3.

33. Let a be the first term and d be the common ratio of the A.P. Given,

$t_p = a + (p-1)d = q$ and $t_q = a + (q-1)d = p$

$\Rightarrow (p-q)d = q - p \Rightarrow d = -1 \Rightarrow a = p + q - 1$

$\Rightarrow t_{p+q} = a + (p+q-1)d = p + q - 1 - (p+q-1) = 0$.

34. Let a be the first term and d be the common ratio of the A.P.

$\Rightarrow t_m = a + (m-1)d, t_{2n+m} = a + (2n+m-1)d$

$\Rightarrow t_m + t_{2n+m} = 2a + (2m+2n-2)d = 2[a + (m+n-1)d] = 2t_{m+n}$

35. Let the three numbers be $a-d, a, a+d$. Given that their sum is 15 and sum of their square is 83.

$\Rightarrow a-d+a+a+d=15 \Rightarrow 3a=15 \Rightarrow a=5$

$\Rightarrow (a-d)^2+a^2+(a+d)^2=83 \Rightarrow 3a^2+2d^2=83 \Rightarrow 3\times 5^2+2d^2=83^2$

$\Rightarrow d=\pm 2$. So the numbers are $3, 5, 7$ or $7, 5, 3$.

36. This problem is similar to previous problem and has been left as an exercise.

37. Let the three numbers be $a-d, a, a+d$. Given their sum as 12 and sum of cubes as 408.

$\therefore a-d+a+a+d=12 \Rightarrow 3a=12 \Rightarrow a=4$

$\therefore (a-d)^3+a^3+(a+d)^3=3a^3+6ad^2=408 \Rightarrow 24d^2=216 \Rightarrow d=\pm 3$

Hence, the numbers are $1, 4, 7$ or $7, 4, 1$.

38. Let the numbers in A.P. be $a-3d, a-d, a+d, a+3d$. Given their sum is 24 and product of first and fourth to product of second and third is $2:3$.

$\therefore a-3d+a-d+a+d+a+3d=20 \Rightarrow 4a=20 \Rightarrow a=5$

$\therefore \frac{(a-3d)(a+3d)}{(a-d)(a+d)}=\frac{2}{3}$

$\Rightarrow 3a^2-27d^2=2a^2-2d^2 \Rightarrow a^2=25d^2 \Rightarrow d=\pm 1.$

Therefore numbers are $2, 4, 6, 8$ or $8, 6, 4, 2$.

39. Let the three numbers be $a-d, a, a+d$. Given their sum is -3 and product is 8.

$\therefore a-d+a+a+d=-3 \Rightarrow 3a=-3 \Rightarrow a=-1$

$\therefore (a-d).a.(a+d)=8 \Rightarrow a^2-d^2=-8 \Rightarrow d=\pm 3$

Hence the numbers are $-4, -1, 2$ or $2, -1, -4$.

40. This problem is similar to problem 38 and has been left as an exercise.

41. Given $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

Adding 2 to each term will give us another A.P. [refer properties of A.P.]

$\therefore \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ will be in A.P.

Dividing each term with $a+b+c$ will yield another A.P.

$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be in A.P.

42. Given a, b, c are in A.P.

Dividing each term by abc will yield another A.P.

$\therefore \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ will be in A.P.

Multiplying each term with $abc + 1$ will yield another A.P.

$\therefore a + \frac{1}{bc}, b + \frac{1}{ca}, c + \frac{1}{ab}$ will be in A.P.

43. Given a, b, c are in A.P. $\therefore b - a = c - b$

$\Rightarrow \frac{1}{b-a} = \frac{1}{c-b} \Rightarrow \frac{ab+bc+ca}{b-a} = \frac{ab+bc+ca}{c-b}$

$\Rightarrow ab(b-a) + c(b^2 - a^2) = bc(c-a) + a(c^2 - b^2)$

$\Rightarrow b^2a + b^2c - a^2b - a^2c = c^2a + c^2b - b^2c - b^2a \Rightarrow b^2(a+c) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

$\therefore a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

44. We will prove this in reverse. We assume that $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

$\Rightarrow \frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} + \frac{1}{\sqrt{c}+\sqrt{a}}$

$\Rightarrow \frac{2}{\sqrt{c}+\sqrt{a}} = \frac{1}{\sqrt{b}+\sqrt{c}} + \frac{1}{\sqrt{a}+\sqrt{b}}$

$\Rightarrow \frac{2}{\sqrt{c}+\sqrt{a}} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{b}+\sqrt{c}}{(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b})}$

$\Rightarrow 2(\sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b}) = (\sqrt{c} + \sqrt{a})(\sqrt{a} + 2\sqrt{b} + \sqrt{c})$

$\Rightarrow 2(\sqrt{ab} + b + \sqrt{ac} + \sqrt{bc}) = \sqrt{ac} + 2\sqrt{bc} + c + a + 2\sqrt{ab} + \sqrt{ac}$

$\Rightarrow 2b = a + c$, which implies that a, b, c are in A.P. So the reverse is also true.

45. Given a, b, c are in A.P.

Dividing each term by abc will yield another A.P.

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ will be in A.P.

Multiplying each term with $ab + bc + ca$ will yield another A.P.

$\Rightarrow \frac{ab+ca}{bc} + 1, \frac{ab+bc}{ca} + 1, \frac{bc+ca}{ab} + 1$ will be in A.P.

Subtracting 1 from each term yields desired terms in A.P.

46. We have to prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

i.e. $\frac{1}{c-a} - \frac{1}{b-c} = \frac{1}{a-b} - \frac{1}{c-a}$

$\Rightarrow \frac{b-2c+a}{(c-a)(b-c)} = \frac{c-2a+b}{(a-b)(c-a)}$

$\Rightarrow (a+b-2c)(a-b) = (b+c-2a)(b-c)$

Now, given that $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

$\Rightarrow (c-a)^2 - (b-c)^2 = (a-b)^2 - (c-a)^2$

$\Rightarrow (b-a)(2c-a-b) = (c-b)(2a-b-c)$

Thus, we have proven the desierd result.

47. Given a, b, c are in A.P.

Subtracting a, b, c from each term will yield another A.P.

$\Rightarrow -(b+c), -(c+a), -(a+b)$ will be in A.P.

Multiplying each term with -1 will yield the desired A.P.

48. We have to prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

i.e. $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$\Rightarrow \frac{b-a}{(b+c)} = \frac{c-b}{(a+b)}$

$\Rightarrow b^2 - a^2 = c^2 - b^2 \Rightarrow a^2, b^2, c^2$ are in A.P.

Thus, we have proven the desired result in reverse.

49. Given that a, b, c are in A.P. $\Rightarrow b-a = c-b = k$ (say)

$\Rightarrow c-a = 2k \Rightarrow 2(a-b) = a-c = 2(b-c) = -2k.$

50. Given that a, b, c are in A.P. Let $b = a+d \Rightarrow c = a+2d$

Now, $(a-c)^2 = 4d^2, 4(b^2-ac) = 4[(a+d)^2 - a(a+2d)] = 4d^2$

$\Rightarrow (a-c)^2 = 4(b^2-ac)$

51. Let $n = 2m+1$ where $m \in N. \Rightarrow S_1 = \frac{n}{2}[t_1 + t_n]$ where d is the commond difference.

For S_2 the no. of terms will be $m. \Rightarrow S_2 = \frac{m}{2}[t_2 + t_{n-1}]$

We know that $t_1 + t_n = t_2 + t_{n-1}$

$\therefore \frac{S_1}{S_2} = \frac{n}{m} = \frac{n}{\frac{n-1}{2}} = \frac{2n}{n-1}.$

52. The degree is the highest power of x which will be $1 + 6 + 11 + \cdots + 101$.

Clearly, the above sequence is an A.P. having first term 1, common difference 5 and last term as 101.

$n = \frac{t_n - t_1}{d} + 1 = \frac{101-1}{5} + 1 = 21.$

$\Rightarrow S = \frac{21}{2}[t_1 + t_n] = \frac{21}{2}[1 + 101] = 21 \times 51 = 1071$

Therefore, the degree of the polynomial will be 1071.

53. Consider an A.P. with first term as a, commond difference as d and no. of terms as n. Then sum is given by

$S = \frac{n}{2}[2a + (n-1)]d = \frac{n^2d^2}{2} + \frac{(2a-d)n}{2}$

which is of the form $An^2 + Bn$ where $A = \frac{d^2}{2}$ and $B = \frac{2a-d}{2}$.

54. Let the common difference of the A.P. be d.

L.H.S. $= a_1^2 - a_2^2 + a_3^2 - a)4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2$

$= (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \cdots + (a_{2n-1} - a_{2n})(a_{2n-1} + a_{2n})$

$= -d(a_1 + a_2 + a_3 + a_4 + \cdots + a_{2n-1} + a_{2n})$

$= -\frac{2nd}{2}[a_1 + a_{2n}]$

$= \frac{n}{2n-1}(a_1^2 - a_{2n}^2)\left[\because d = \frac{a_{2n} - a_1}{2n-1}\right]$

55. We know that sum of equidistant terms from start and end of an A.P. is equal.

$\therefore a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15} = k$ (say)

$\therefore a_1 + a_5 + a_{10} + a_{15} + a_{24} = 3k = 225 \Rightarrow k = 75$

Sum of first 24 terms $S = a_1 + a_2 + \cdots + a_{24} = \frac{24}{2}[a_1 + a_{24}] = 12 \times 75 = 600.$

56. Let a be the first term and d be the common difference. Also let S_1 denote the sum of first $3n$ terms and S_2 denote the sum of next n terms.

$S_1 = \frac{3n}{2}[2a + (3n-1)d], S_2 = \frac{n}{2}[2a + 6nd + (n-1)d][\because t_{3n+1} = a + 3nd]$

Given, $S_1 = S_2 \Rightarrow \frac{3n}{2}[2a + (3n-1)d] = \frac{n}{2}[2a + 6nd + (n-1)d]$

$\Rightarrow 6a + (9n-3)d = 2a + (7n-1)d \Rightarrow 2a + (n-2)d = 0$

Let S_3 be sum of first $2n$ terms and S_4 be sum of next $2n$ terms, then

$\frac{S_3}{S_4} = \frac{\frac{2n}{2}[2a+(2n-1)d]}{\frac{2n}{2}[2a+4nd+(2n-1)]d}$

$\Rightarrow = \frac{nd}{5nd} = \frac{1}{5}[\because 2a + (n-1)d = 0xs]$

57. Given $S_n = 5n^2 + 3n \Rightarrow t_n = S_n - S_{n-1} = 5n^2 + 3n - 5(n-1)^2 - 3(n-1)$

$= 10n - 5 + 3 = 10n - 2 \Rightarrow d = t_n - t_{n-1} = 10n - 2 - 10(n-1) + 2 = 10,$

Since common difference is a constant the series is in A.P.

58. Common difference of the series $d = (a^2+b^2) - (a+b)^2 = (a-b)^2 - (a^2+b^2) = -2ab$

$S = \frac{n}{2}[2(a+b)^2 - (n-1)2ab] = \frac{n}{2}[2a^2 + 2b^2 - 2(n+1)ab]$

$= n[a^2 + b62 - (n+1)ab].$

59. There will be two cases. First n being odd and second n being even.

Case I: When n is odd i.e. $n = 2m + 1$, where $m = 0, 1, 2, ...$

$S = 1 + 5 + 9 + \cdots$ up to $m + 1$ terms $-3 - 7 - 11$ up to m terms

$= \frac{m+1}{2}[2 + 4m] - \frac{m}{2}[6 + 4m - 4] = (m+1)(1+2m) - m(2m+1)$

$= 2m^2 + 3m + 1 - 2m^2 - m = 2m + 1 = n.$

Case II: When n is even i.e. $n = 2m$, where $m = 1, 2, 3, ...$

$S = 1 + 5 + 9 + \cdots$ up to m terms $-3 - 7 - 11$ up to m terms

$= \frac{m}{2}[2 + 4m - 4] - \frac{m}{2}[6 + 4m - 4] = -2m = -n.$

60. Let there be n sides of the polygon. From geometry, we know that sum of angles of the polygon $= (n-2)180^\circ$

From the formula for sum of an A.P. $S = \frac{n}{2}[2 \times 120^\circ + (n-1)5^\circ] = (n-2)180^\circ$

$\frac{n}{[}240^\circ + (n-1)5^\circ] = (n-2)360^\circ \Rightarrow n[48^\circ + (n-1)] = (n-2)72^circ$

$\Rightarrow n^2 - 25n + 144 = 0 \Rightarrow n = 9, 16$

61. To water first tree the gardener will have to travel 10 m. To water second tree he will have tp travel back 10 m to well and then 15 m to the tree i.e. 25 m. Similarly, for third tree he will have to travel 15 m to well and 20 m i.e a total of 35 m.

Thus, total distance travelled will be $10 + 25 + 35 + \cdots$

Clearly, 25 will be the first term of the A.P. and there will be 24 such terms because distance travelled for first tree is noty part of the A.P. Note that common difference would be 10.

Total distance travelled $= 10 + \frac{24}{2}[2 \times 25 + (24-1)10] = 10 + 3360 = 3370$ m.

62. Let d be the common difference. Given $S_p = 0 \Rightarrow \frac{p}{2}[2a + (p-1)d] = 0$

$\Rightarrow 2a + (p-1)d = 0 \Rightarrow d = \frac{2a}{1-p}$

$p+1$th term $t_{p+1} = a + pd$, so the sum of next q terms $S = \frac{q}{2}[2a + 2pd + (q-1)d]$

$= \frac{q}{2}[2a + (2p+q-1)d] = \frac{q}{2}\left[2a + (2p+q-1).\frac{2a}{1-p}\right]$

$= \frac{q}{2}\left[\frac{2a.(p+q)}{1-p}\right] = -\frac{a(p+q)}{p-1}q.$

63. Sum of first p terms, $S_p = \frac{p}{2}[2a+(p-1)d]$; sum of first q terms $S_q = \frac{q}{2}[2a+(q-1)d]$

$2ap + (p^2 - p)d = 2aq + (q^2 - q)d \Rightarrow 2a(p-q) = (q^2 - p^2 + p - q)d$

$2a = (1-p-q)d$

Sum of $(p+q)$ terms, $S_{p+q} = \frac{p+q}{2}[2a + (p+q-1)d] = \frac{p+q}{2}[(1-p-q)d + (p+q-1)d] = 0.$

64. Sum of latter half of $2n$ terms means $n+1$th term to $2n$th term. $t_{n+1} = a + nd$ and $t_{2n} = a + (2n-1)d$ where a and d are the first term and common difference respectively.

Sum of latter half of terms, $S = \frac{n}{2}[t_{n+1} + t_{2n}] = \frac{n}{2}[2a + (3n-1)d]$

Sum of first $3n$ terms, $S_{3n} = \frac{3n}{2}[2a + (3n-1)d]$

Clearly, $S/S_{3n} = 1:3.$

65. Let S_r be the rth A.P. whose first term is r and common difference is also r.

$S_r = \frac{n}{2}[2r + (n-1)r] = \frac{n}{2}[(n+1)r] = \frac{n(n+1)r}{2}$

$$S_1 + S_2 + S_3 + \cdots + S_p = \sum_{r=1}^{p} S_r$$

$$= \frac{n(n+1)}{2}\sum_{r=1}^{p} r = \frac{np}{4}(n+1)(p+1)\left[\because \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right].$$

66. Let x be the first term and y be the common difference of the A.P.

Then, according to the question $a = \frac{p}{2}[2x + (p-1)y]$, $b = \frac{q}{2}[2x + (q-1)y]$, $c = \frac{r}{2}[2x + (r-1)y]$

We have to prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$

L.H.S. $=x(q-r+r-p+p-q)+\frac{y}{2}[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$

$=0.$

67. Let a be the first term and d be the common difference of the A.P.

Given, $S_m=\frac{1}{2}S_{m+n}\Rightarrow\frac{m}{2}[2a+(m-1)d]=\frac{1}{2}.\frac{m+n}{2}[2a+(m+n-1)d]$

Let $2a+(m-1)d=x$, then the above equation can be written as

$mx=\frac{m+n}{2}[x+nd]\Rightarrow 2mx=(m+n)[x+nd]\Rightarrow mx=n(x+nd)+mnd$

$\Rightarrow(m-n)x=(m+n)nd$

Similarly, $(m-p)x=(m+p)pd$

Dividing, we get

$(m-n)(m+p)p=(m+n)(m-p)n$

Dividing both sides with mnp we arrive at the desired result.

68. Let a be the first term and d be the common difference of the A.P. For odd terms, the no. of terms will be $n+1$, first term will be a and common difference will be $2d$.

$\therefore S_{odd}=\frac{n+1}{2}[2a+2nd]$

For even terms, the no. of terms will be n, first term will be $a+d$ and common difference will be $2d$.

$\therefore S_{even}=\frac{n}{2}[2a+2d+2(n-1)d]=\frac{n}{2}[2a+2nd]$

$\therefore\frac{S_{odd}}{S_{even}}=\frac{n+1}{n}.$

69. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of the two series in A.P.

Given, $\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]}=\frac{3n-12}{5n+21}$

$\Rightarrow\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{3n-13}{5n+21}$

We need to find ratio of the 24th terms i.e. $\frac{a_1+23d_1}{a_2+23d_2}=\frac{2a_1+46d_1}{2a_2+46d_2}$

Putting $n=47$ in the ratio of sums, we have

$\frac{2a_1+46d_1}{2a_2+46d_2}=\frac{3\times 47-13}{5\times 47+21}=\frac{1}{2}$

70. Let a be the first term and d be the common difference of the A.P.

Given, $t_m = a + (m-1)d = \frac{1}{n}, t_n = a + (n-1)d = \frac{1}{m}$

Subtracting, we get $(m-n)d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn} \Rightarrow a = \frac{1}{mn}$

$\therefore S_{mn} = \frac{mn}{2}\left[\frac{2}{mn} + \frac{mn-1}{mn}\right] = \frac{mn+1}{2}.$

71. Let a be the first term and d be the common difference of the A.P.

Given, $S_m = n = \frac{m}{2}[2a + (m-1)d] \Rightarrow 2a + (m-1)d = \frac{2n}{m}$

and $S_n = m = \frac{n}{2}[2a + (n-1)d] \Rightarrow 2a + (n-1)d = \frac{2m}{n}$

$\Rightarrow d = -\frac{2(m+n)}{mn} \Rightarrow a = \frac{m^2+n^2+mn-m-n}{mn}$

$\Rightarrow S_{m+n} = \frac{m+n}{2}[2a + (m+n-1)d] = -(m+n).$

72. Let a be the first term and d be the common difference of the A.P.

$\therefore S = \frac{2n+1}{2}[2a + 2nd]$

For S_1 first term would be a, common difference would be $2d$ and no. of terms would be $n+1$.

$\therefore S_1 = \frac{n+1}{2}[2a + 2nd]$

$\therefore \frac{S}{S_1} = \frac{2n+1}{n+1}.$

73. Let d be the common difference, then $b = a + 2d \Rightarrow d = \frac{b-a}{2}$

$c = a + (n-1)d \Rightarrow n - 1 = \frac{c-a}{d} = \frac{2(c-a)}{b-a}$

$\Rightarrow n = \frac{2(c-a)}{b-a} + 1$

$\therefore S = \frac{n}{2}[2a + (n-1)d] = \frac{1}{2}\left[\frac{2(c-a)}{b-a} + 1\right]\left[2a + \frac{2(c-a)}{b-a} . \frac{b-a}{2}\right]$

$= \frac{c+a}{2} + \frac{c^2-a^2}{b-a}.$

74. Let a_1, a_2 be the first terms and d_1, d_2 be the common differences of the two series in A.P.

According to the question $\frac{2a_1+(n-1)a_1}{2a_2+(n-1)d_2} = \frac{3n+8}{7n+15}.$

We have to find ratio of 12th terms i.e. $\frac{a_1+11d_1}{a_2+11d_2} = \frac{2a_1+22d_1}{2a_2+22d_2}$

Putting $n = 23$ in previous equation, we get

$\frac{2a_1+22d_1}{2a_2+22d_2} = \frac{77}{176} = \frac{7}{16}$.

75. Let a be the first term and d be the common difference of the A.P.

Given, $\frac{S_m}{S_n} = \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2}$

$\Rightarrow \frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n}$

$\Rightarrow 2a(n-m) + [(m-1)n - (n-1)m]d = 0 \Rightarrow a = \frac{d}{2}$

We have to find $\frac{t_m}{t_n} = \frac{a+(m-1)d}{a+(n-1)d} = \frac{2m-1}{2n-1}$

76. Let n be the no. of terms. Clearly, common ratio $r = \frac{20}{5} = \frac{80}{20} = 4$

Then $t_n = 5120 = 5.r^{n-1} \Rightarrow 4^{n-1} = 1024 = 4^5 \Rightarrow n = 6$.

77. Let n be the no. of terms. Clearly, common ratio $r = \frac{0.06}{0.03} = \frac{0.12}{0.06} = 2$

Then $t_n = 3.84 = 0.03r^{n-1} \Rightarrow 2^{n-1} = 128 \Rightarrow n = 8$.

78. From the question we deduce that it is a G.P. with $a = 1, r = 2, n = 20$. We have to find t_{20}.

$t_{20} = 1.2^{20-1} = 524288$.

79. This is a G.P. with $a = 20000, r = 1.02, n = 11$. We have to find t_{11}.

$t_{11} = 20000 \times (1.02)^{11-1} = 24380$.

80. Given, $S_n = 2^n - 1 \Rightarrow t_n = S_n - S_{n-1} = 2^n - 1 - (2^{n-1} - 1) = 2^{n-1}$

$r = \frac{t_n}{t_{n-1}} = \frac{2^{n-1}}{2^{n-2}} = 2$, which is a constant and hence the sequence is in G.P.

81. Let the first term of the G.P. be a and common ratio is r.

Then $t_2 = ar = 24$ and $t_5 = ar^4 = 81$, Dividing, we have $r^3 = \frac{81}{24} = \frac{27}{8}$

$\Rightarrow r = \frac{3}{2} \Rightarrow a = 16$.

Hence the G.P. is $16, 24, 36, 54, 81, ...$

82. Let the first term of the G.P. be a and common ratio is r.

Given $t_7 = 8t_4 \Rightarrow ar^6 = 8ar^3 \Rightarrow r = 2$. Also given, $t_5 = 48 \Rightarrow ar^4 = 48$

$\Rightarrow a = 3$. Hence, the G.P. is $3, 6, 12, 24, ...$

83. Let the first term of the G.P. be a and common ratio is r.

Given, $t_5 = ar^4 = 48$ and $t_8 = ar^7 = 384 \Rightarrow r^3 = 8 \Rightarrow r = 2$

$\Rightarrow a = 3$. Hence, the G.P. is 3, 6, 12, 24, ...

84. Let the first term of the G.P. be a and common ratio is r.

Given $t_6 = ar^5 = \frac{1}{16}$ and $t_{10} = ar^9 = \frac{1}{256} \Rightarrow r = \pm\frac{1}{2}$

$\Rightarrow a = \pm 2$. Hence the G.P. is $2, 1, \frac{1}{2}, \ldots$ or $-2, 1, -\frac{1}{2}, \ldots.$

85. Let the first term of the G.P. be x and common ratio is y. Then

$a = xy^{p-1}, b = xy^{q-1}, c = xy^{r-1}$

Taking log of both sides for these three terms

$\log a = \log x + (p-1)\log y, \log b = \log x + (q-1)\log y, \log c = \log x + (r-1)\log y$

Clearly, $(q-r)\log a + (r-p)\log b + (p-q)\log r = 0.$

86. Let the first term of the G.P. be x and common ratio is r.

Given, $t_{p+q} = a = xr^{p+q-1}$ and $t_{p-q} = b = xr^{p-q-1}$

Multiplying the two terms, we have

$x^2r^{2p-2} = (xr^{p-1})^2 = t_p^2 = ab \Rightarrow t_p = \sqrt{ab}.$

87. Let a be the first term and b be the common ratio. Then,

$x = ab^{p-1}, y = ab^{q-1}, z = ab^{r-1}$

We have to prove that $x^{q-r}.y^{r-p}.z^{p-q} = 1$

L.H.S. $= (ab^{p-1})^{q-r}.(ab^{q-1})^{r-p}.(ab^{r-1})^{p-q}$

$= a^{(q-r+r-p+p-q)}b^{[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]}$

$= a^0b^0 = 1 =$ R.H.S.

88. Let r be the common ratio and first term is given as 1.

$t_3 + t_5 = 90 \Rightarrow r^4 + r^2 = 90 \Rightarrow r^2 = 9 \Rightarrow r = pm3.$

r^2 cannot be -10 as that would mean that it is an imaginary number.

89. Let a be the first term and r be the common ratio of the G.P.

Gibem $t_5 = ar^4 = 2$ and we have to find the product of the first nine terms. Let the required product be S.

$S = a.ar.ar^2.\ldots.ar^8 = a^9r^{1+2+\cdots+8} = a^9r^{\frac{8.9}{2}} = a^9r^{36} = (ar^4)^9 = 2^9 = 512.$

90. Let a be the first term, r be the common ratio and n be the number of terms.

Given, $t_4 = ar^3 = 10, t_7 = ar^6 = 80, t_n = ar^{n-1} = 2560$

$\therefore \frac{t_7}{t_4} = r^3 = 8 \Rightarrow r = 2 \Rightarrow a = \frac{10}{8}$

$\Rightarrow \frac{10}{8}2^{n-1} = 2560 \Rightarrow 2^{n-1} = 2048 \Rightarrow n = 12.$

91. Let the three numbers in G.P. be a, ar, ar^2. According to question, on doubling ar the numbers form an A.P.

$\Rightarrow 2ar - a = ar^2 - 2ar \Rightarrow r^2 - 4r + 1 = 0 \Rightarrow r = \frac{4\pm\sqrt{12}}{2} = 2 \pm \sqrt{3}.$

92. Given, p, q, r are in A.P. i.e. $q - p = r - q$.

Let x be the first term and y be the common ratio of the G.P. We have to prove that t_p, t_q, t_r are in G.P.

$\Rightarrow \frac{t_q}{t_p} = \frac{t_r}{t_q} \Rightarrow \frac{xy^{q-1}}{xy^{p-1}} = \frac{xy^{r-1}}{xy^{q-1}}$

$\Rightarrow y^{q-p} = y^{r-q}$ which is true from the condition for A.P.

93. Let r be the common ratio of the G.P. Then, $b = ar, c = ar^2, d = ar^3$

L.H.S. $= (a.ar + ar.ar^2 + ar^2.ar^3)^2 = a^4r^2(1 + r^2 + r^4)^2$

R.H.S. $= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6) = a^2(1 + r^2 + r^4).a^2r^2(1 + r^2 + r^4)$

$= a^2r^4(1 + r^2 + r^4)^2 =$ L.H.S.

94. Given a, b, c are in A.P. $\Rightarrow 2b = a + c$

If we increase a by 1 then they are in G.P. $\Rightarrow b^2 = (a + 1)c \Rightarrow b^2 = (a + 1)(2b - a)$

$\Rightarrow b^2 = 2ab - a^2 + 2b - a \Rightarrow (a - b)^2 = 2b - a$

If we increase c by 2 then again they are in G. P $\Rightarrow b^2 = a(c + 2) = a(2b - a + 2)$

$\Rightarrow b^2 = 2ab - a^2 + 2a \Rightarrow (a - b)^2 = 2a \Rightarrow 2b - a = 2a \Rightarrow 2b = 3a$

$\Rightarrow \left(a - \frac{3a}{2}\right)^2 = 2a \Rightarrow a = 8 \Rightarrow b = 12 \Rightarrow c = 16.$

95. Let the three numbers in G.P. be $\frac{a}{r}, a, ar$. Then,

$\frac{a}{r} + a + ar = 70$ and $10a = \frac{4a}{r} + 4ar \Rightarrow \frac{10a}{4} = \frac{a}{r} + ar$

$\Rightarrow \frac{10a}{4} + a = 70 \Rightarrow a = 20$

$\Rightarrow \frac{20}{r} + 20r = 50 \Rightarrow r = 2, \frac{1}{2}$

So the numbers are $10, 20, 40$ or $40, 20, 10$.

96. Let the three numbers in G.P. be $\frac{a}{r}, a, ar$. Given that product of these numbers is 216.

$\Rightarrow \frac{a}{r}.a.ar = 216 \Rightarrow a^3 = 216 \Rightarrow a = 6$

Also, given that their sum is 19 $\Rightarrow \frac{6}{r} + 6 + 6r = 19$

$\Rightarrow 6r^2 - 13r + 6 = 0 \Rightarrow r = \frac{2}{3}, \frac{3}{2}.$

So the numbers are $9, 6, 4$ or $4, 6, 9$.

97. Let the number be $100a + 10ar + ar^2$.

According to question $a + ar^2 = 2ar + 1$ and $a + ar = \frac{2}{3}(ar + ar^2)$

$\Rightarrow a(r-1)^2 = 1$ and $3 + 3r = 2r + 2r^2 \Rightarrow r = -1, \frac{3}{2}$

If $r = -1, a = \frac{1}{4}$, but a cannot be a fraction.

If $r = \frac{3}{2} \Rightarrow a = 4$ and the number is 469.

98. Given that three of four numbers are in A.P. and so we choose them as $a - d, a, a + d$. Also, since first number is same as first so the numbers are $a + d, a - d, a, a + d$. The first three are in G.P. Given $d = 6$

$\therefore (a - d)^2 = a(a + d) \Rightarrow (a - 6)^2 = a(a + 6) \Rightarrow 18a = 36 \Rightarrow a = 2.$

So the numbers are $8, -4, 2, 8$.

99. Let the three numbers are a, ar, ar^2. The sum is given as 21 $\Rightarrow a + ar + ar^2 = 21$.

Also, sum of squares is given as 189 $\Rightarrow a^2 + a^2r^2 + a^2r^4 = 189$

$\Rightarrow \frac{441(1+r^2+r^4)}{(1+r+r^2)^2} = 189$

$\Rightarrow 7(1 + 2r^2 + r^4 - r^2) = 3(r + r + r^2)^2 \Rightarrow 7(1 - r + r^2) - 3(1 + r + r^2)$

$\Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow r = 2, \frac{1}{2}$

When $r = 2, a = 3$ and so the numbers are $3, 6, 12$.

When $r = \frac{1}{2}, a = 12$ and so the numbers are $12, 6, 3$.

100. Let the terms in G.P. be $\frac{a}{r}, a, ar$. Given that the product of these is -64.

$\therefore \frac{a}{r}.a.ar = -64 \Rightarrow a^3 = -64 \Rightarrow a = -4.$

Also given that the first term is four times the third. $\Rightarrow \frac{a}{r} = 4.ar \Rightarrow r^2 = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2}.$

If $r = \frac{1}{2}$, the terms will be $-8, -4, -2$. If $r = -\frac{1}{2}$, the terms will be $8, -4, 2$.

101. Let the numbers be $a-d, a, a+d$. Given that sum is 15. $\Rightarrow a-d+a+a+d = 15 \Rightarrow a = 5$.

Also given that if $1, 4, 19$ are added to them then they are in G.P.

$\Rightarrow (5+4)^2 = (5-d+1)(5+d+19) \Rightarrow 81 = (6-d)(24+d)$

$\Rightarrow d^2 + 18d - 63 = 0 \Rightarrow d = -21, 3.$

If $d = -15$, the numbers will be $26, 5, -16$ and if $d = 3$ the numbers will be $2, 5, 8$.

102. Let the two sets of three numbers in G.P. are $a_1, a_1r_1, a_1r_1^2$ and $a_2, a_2r_2, a_2r_2^2$.

Given that the difference is also in G.P.

$\Rightarrow (a_1r_1 - a_2r_2)^2 = (a_1r_1^2 - a_2r_2^2)(a_1 - a_2)$

$\Rightarrow a_1^2r_1^2 + a_2^2r_2^2 - 2a_1a_2r_1r_2 = a_1^2r_1^2 - a_1a_2r_2^2 - a_1a_2r_1^2 + a_2^2r_2^2$

$\Rightarrow 2a_1a_2r_1r_2 = a_1a_2r_2^2 + a_1a_2r_1^2 \Rightarrow 2r_1r_2 = r_1^2 + r_2^2 \Rightarrow (r_1 - r_2)^2 = 0$

$\Rightarrow r_1 = r_2$ which implies that they have same common ratio.

103. Let r be the common ratio. Then $b = ar, c = ar^2, d = ar^3$

L.H.S. $= (b-c)^2 + (c-a)^2 + (d-b)^2 = (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2(r - r^2)^2 + a^2(r^2-1)^2 + a^2(r^3 - r)^2 = a^2(r^2 + r^4 - 2r^3 + r^4 + 1 - 2r^2 + r^6 + r^2 - 2r^4)$

$= a^2(r^6 - 2r^3 + 1) = (ar^3 - a)^2 = (d-a)^2 =$ R.H.S.

104. This problem can be solved like previous problem.

105. Given that x, y, z are in G.P. Let p be the first term and r be the common ratio of this G.P.

Also given, $a^x = b^y = c^z \Rightarrow x \log a = y \log b = z \log c$

$\Rightarrow \frac{\log a}{\log b} = \frac{y}{x}$ and $\frac{\log b}{\log c} = \frac{z}{y}$. Clearly, $\frac{y}{x} = \frac{z}{y} = r \Rightarrow \log_b a = \log_c b$.

106. Let $\frac{a}{r}, a, ar$ be the terms in G.P. Given that continued product is 216 i.e.

$\frac{a}{r}.a.ar = 216 \Rightarrow a^3 = 216 \Rightarrow a = 6$

Sum of products when taken in pair is given as 156.

$\Rightarrow \frac{a}{r}.a + a.ar + \frac{a}{r}.ar = 156 \Rightarrow \frac{1}{r} + r + 1 = \frac{26}{6}$

$\Rightarrow 6r^2 - 20r + 6 = 0 \Rightarrow r = \frac{1}{3}, 3$

So the numbers are $18, 6, 2$ or $2, 6, 18$.

107. Let r be the common ratio. Then, $\frac{(b+c)^2}{(a+b)^2} = \frac{(ar+ar^2)^2}{(a+ar)^2} = r^2$.

Similarly, $\frac{(c+d)^2}{(b+c)^2} = r^2 = \frac{(b+c)^2}{(a+b)^2}$.

Thus, $(a+b)^2, (b+c)^2, (c+d)^2$ are also in G.P.

108. This problem can be solved like previous problem.

109. This problem can be solved like previous problem.

110. This problem can be solved like previous problem.

111. Let r be the common ratio. Then, $a(b-c)^3 = a(ar-ar^2)^3 = a^4r^3(1-r)^3$ and $d(a-b)^3 = ar^3(a-ar)^3 = a^4r^3(1-r)^3$.

Thus, $a(b-c)^3 = d(a-b)^3$.

112. We have to prove that $(a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(b+c)^2$ where a, b, c, d are in G.P.

Now, $(a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(a+b)(c+d)$ so it is enough to prove that $(a+b)(c+d) = (b+c)^2$.

$(a+b)(c+d) = (a+ar)(ar^2+ar^3) = a^2r^2(1+r)^2$ and $(b+c)^2 = (ar+ar^2)^2 = a^2r^2(1+r)^2$ which proves the required equality.

113. Let r be the common ratio. L.H.S. $= a^2b^2c^2\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = \frac{b^2c^2}{a} + \frac{a^2c^2}{b} + \frac{a^2b^2}{c}$

$= a^3r^6 + a^3r^3 + a^3 = a^3 + b^3 + c^3 =$ R.H.S.

114. Let r be the common ratio. L.H.S. $= (a^2-b^2)(b^2+c^2) = (a^2-ar^2)(a^2r^2+a^2r^4) = r^2(a^2-a^2r^2)(a^2+a^2r^2) = (a^2r^2-a^2r^4)(a^2+a^2r^2) = (b^2-c^2)(a^2+b^2) =$ R.H.S.

115. Let r be the common ratio. Given a, b, c are in G.P. i.e. a, ar, ar^2 are in G.P.

Taking log of a, b, c, we have

$\log a, \log a + \log r, \log a + 2\log r$ are in A.P. with $\log a$ being the first term and $\log r$ be the common difference.

116. Given series is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ to n terms. Let S be the sum, $a = 1, r = \frac{1}{2}$, then

$S = \frac{a(1-r^n)}{1-r} = 2\left(\frac{2^n-1}{2^n}\right)$

117. Given series is $1 + 2 + 4 + 8 + \cdots$ to 12 terms. First term $a = 1$, common ratio $r = 2$ and no. of terms $n = 12$. Let S be the sum of the series. Then,

$S = \frac{a(r^n-1)}{r-1} = \frac{1(2^{12}-1)}{2-1} = 4095.$

118. Given series is $1-3+9-27+\cdots$ to 9 terms. First terms $a=1$, common ratio $r=-3$ and no. of terms $n=9$. Let S be the sum of the series. Then,

$S=\frac{a(1-r^n)}{1-r}=\frac{1-(-3)^9}{1-(-3)}=4921$

119. This problem is similar to 115, and has been left as an exercise.

120. Given series is $(a+b)+(a^2+2b)+(a^3+3b)+\cdots$ to n terms. We can rewrite the series as $a+a^2+a^3+\cdots$ to n terms $+\ b+2b+3b+\cdots$ to n terms.

We know that $a+a^2+a^3+\cdots$ to n terms $=\frac{a(a^n-1)}{a-1}$ and for the second series applying the A.P. formula, $b+2b+3b+\cdots$ to n terms $=\frac{n}{2}[2b+(n-1)b]=\frac{n}{2}[(n+1)b]=\frac{n(n+1)b}{2}$.

121. Clearly the given situation forms a G.P. with $a=1$, common ratio $r=2$ and $n=120$. Let S be the sum which he gets at the end of 120 days. Then,

$S=\frac{a(r^n-1)}{r-1}=2^{120}-1=1329227995784915872903807060280344575.$

122. Given series is $S=8+88+888+\cdots=\frac{8}{9}[9+99+999+\cdots]$

$=\frac{8}{9}[(10-1)+(100-1)+(1000-1)+\cdots]$

$=\frac{8}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]=\frac{8}{81}[10^{n+1}-10-9n].$

123. This problem can be solved like previous problem.

124. This problem can be solved like previous problem.

125. This problem can be solved like previous problem.

126. Let $S=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$ to n terms. Clearly, $a=1$ and $r=-\frac{1}{2}$.

$\Rightarrow S=\frac{a(1-r^n)}{1-r}=\frac{1-(-1)^n\frac{1}{2^n}}{1-(-)\frac{1}{2}}=\frac{2}{3}.\frac{2^n-(-1)^n}{2^n}.$

127. When we make 1000 per day for 31 days total amount received will be 31,000.

When we receive 1 for the first day and doubling every day then that would be a G.P. with $a=1, r=2, n=31 \Rightarrow S=\frac{a(r^n-1)}{r-1}=2^{31}-1=2{,}147{,}483{,}647$ which is clearly way more than we make in the first case so we will happily take the second option.

128. We assume that n terms of the series $1+3+3^2+\cdots$ make for 3280. Then

$S=\frac{1(3^n-1)}{3-1}\Rightarrow 3^n=6561\Rightarrow n=8.$

129. Let $S=1+3+3^2+\cdots+3^{n-1}\Rightarrow S=\frac{3^n-1}{3-1}>1000\Rightarrow 3^n>2001\Rightarrow n=7.$

130. Let the sum be S. Clearly it is a G.P. with $a = 1, r = \frac{1}{2}$. We know that when $|r| < 1$ the sum of an infinite G.P. is given by $S = \frac{a}{1-r}$. Thus, $S = \frac{1}{1-\frac{1}{2}} = 2$.

131. Clearly, it is a G.P. with $a = 1, r = 3$ and $n = 20$. Thus sum is given by $S = \frac{3^{20}-1}{3-1} =$ 1,743,392,200.

132. We can represent the given series as three series like $(x^2 + x^4 + x^6 + \cdots)$ to n terms $+\left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \cdots\right)$ to n terms $+2 + 5 + 8 + \cdots$ to n terms. Let the sum be S.

$S = x^2 \frac{(x^2)^n - 1}{x^2 - 1} + \frac{1}{x^2} . \frac{\frac{1}{(x^2)^n - 1}}{\frac{1}{x^2} - 1} + \frac{n}{2}[3n + 1]$.

133. Let n be the no. of terms required to make the sum of given G.P. with $a = 1, r = 2$ equal to 511.

$511 = \frac{2^n - 1}{2-1} \Rightarrow 2^n = 512 \Rightarrow n = 9$.

134. Let the sum be $S.S = 1 + 2 + 2^2 + \cdots + 2^{n-1} = \frac{2^n - 1}{2-1} \geq 300 \Rightarrow 2^n \geq 301 \Rightarrow n = 9$.

135. Let r be the common ratio. $a_n = ar^{n-1} = 96.S = \frac{a_1(r^n - 1)}{r-1} = \frac{a_n r - a_1}{r-1} = \frac{96r-3}{r-1} = 189 \Rightarrow$ $32r - 1 = 63r - 63 \Rightarrow r = 2 \Rightarrow n = 6$.

136. $0.4\dot{2}\dot{3} = 0.4232323 \ldots$ to $\infty = \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \cdots$ to ∞

$= \frac{4}{10} + \frac{23}{100}\left[1 + \frac{1}{100} + \frac{1}{10000} + \cdots \text{ to } \infty\right] = \frac{4}{10} + \frac{23}{100}\frac{1}{1-\frac{1}{100}} = \frac{419}{990}$.

137. Given series can be written as $S = \frac{1}{5} + \frac{1}{5^2} + \cdots$ to $\infty + \frac{1}{7} + \frac{1}{7^2} + \cdots$ to ∞

$= \frac{1}{5} . \frac{1}{1-\frac{1}{5}} + \frac{1}{7} . \frac{1}{1-\frac{1}{7}} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

138. Let the sum be S, then $S = (10 + 1) + (100 + 3) + (1000 + 5) + \cdots$ to n terms

$= \frac{10(10^n - 1)}{10-1} + \frac{n}{2}[2 + (n-1)2] = \frac{10}{9}(10^n - 1) + n^2$.

139. The general term of the series is $t_n = \left(x^n + \frac{1}{x^n}\right)^2 = x^{2n} + \frac{1}{x^{2n}} + 2$ so we can write it as three series and solve like problem 132.

140. Let a be the first term and r be the common ratio of the G.P. Then,

$S = \frac{a(r^n - 1)}{r-1}, P = a.ar.ar^2 \ldots ar^{n-1} = a^n r^{\frac{n(n-1)}{2}}, R = \frac{1}{a}\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}} = \frac{1}{a}\frac{r^n - 1}{r-1} . \frac{1}{r^{n-1}}$

$P^2 = a^{2n} r^{n(n-1)}, \frac{S}{R} = a^2 . r^{n-1} \therefore \left(\frac{S}{R}\right)^n = P^2$.

141. Clearly, the given series is a G.P. with $a = 1, r = \frac{x}{1+x} \Rightarrow S = \frac{1}{1-\frac{x}{1+x}} = 1 + x$.

142. We consider the n-th term. $t_n = ar^{n-1}$, where a is the first term. Sum of all succeeding terms $S = \frac{ar^n}{1-r}$ $\therefore \frac{t_n}{S} = \frac{1-r}{r}$. Hence proven.

143. $S_1 = \frac{1}{1-\frac{1}{2}} = 2, S_2 = \frac{2}{1-\frac{1}{3}} = 3, S_3 = \frac{3}{1-\frac{1}{4}}, \ldots, S_p = \frac{p}{1-\frac{1}{p+1}} = p + 1$.

Clearly, $S_1, S_2, \ldots, S_p$ forms an A.P. with 2 as first term and 1 as c.d.

$S_1 + S_2 + \cdots + S_p = \frac{p}{2}[2.2 + (p-1)] = \frac{p(p+3)}{2}$.

144. $x = \frac{1}{1-a} \Rightarrow a = 1 - \frac{1}{x} = \frac{x-1}{x}$ and similarly $b = \frac{y-1}{y}$.

$1 + ab + a^2b^2 + \cdots$ to $\infty = \frac{1}{1-ab} = \frac{1}{1-\frac{x-1}{x}.\frac{y-1}{y}} = \frac{xy}{x+y-1}$.

145. Let S be the sum, then $S = \frac{1}{1-r} + \frac{a}{1-r} + \frac{a^2}{1-r} + \cdots$ to ∞

$\Rightarrow S = \frac{a}{1-r} \cdot \frac{1}{1-a} = \frac{a}{(1-r)(1-a)}$.

146. When the ball is dropped it will first travel 120 mts. Then it will bounce back $120.\frac{4}{5} = 96$ m and fall 96 m as well. It will then bounce back $96.\frac{4}{5}$ m and fall the same distance as well.

Thus, total distance travelled $120 + 120 \times 2 \times \frac{4}{5} + 120 \times 2 \times \frac{4^2}{5^2} + \cdots$ to ∞

$= 120 + 192\left[1 + \frac{4}{5} + \frac{4^2}{5^2} + \cdots\right]$ to $\infty = 120 + 192.\frac{1}{1-\frac{4}{5}} = 120 + 960 = 1080$ meters.

147. Let r be the common ratio. Then $b = ar^{n-1} \Rightarrow (ab)^n = a^{2n}r^{n(n-1)}$

$p = a.ar.ar^2.ar^3.\ldots ar^{n-1} = a^n r^{1+2+3+\cdots+(n-1)} = a^n r^{\frac{n(n-1)}{2}}$

$\Rightarrow p^2 = (ab)^n$.

148. Let the first terms are a and b; and the common ratio is r. Ratio of sums would be $a : b$ which is equal to $ar^{n-1} : br^{n-1}$ i.e. ratio of nth terms.

149. Let a be the first term. Then, $S_1 = \frac{a(r^n-1)}{r-1}, S_2 = \frac{a(r^{2n}-1)}{r-1}$ and $S_3 = \frac{a(r^{3n}-1)}{r-1}$.

$S_2 - S_1 = \frac{a(r^{2n}-r^n)}{r-1} = \frac{ar^n(r^n-1)}{r-1}$

$S_1(S_3 - S_2) = \frac{a(r^n-1)}{r-1}\left(\frac{ar^{2n}(r^n-1)}{r-1}\right) = \frac{a^2r^{2n}(r^n-1)^2}{(r-1)^2}$

$\Rightarrow S_1(S_3 - S_2) = (S_2 - S_1)^2$.

150. $S_1 = a, S_2 = \frac{a(r^2-1)}{r-1}, S_2 = \frac{a(r^3-1)}{r-1}, \cdots, S_{2n-1} = \frac{a(r^{2n-1}-1)}{r-1}$

$S_1+S_2+S_3+\cdots+S_{2n-1} = \frac{a}{r-1}\left[r+r^2+r^3+\cdots+r^{2n-1}-(1+1+\cdots+\text{to } 2n-1\text{terms})\right]$

$= \frac{a}{r-1}\left[\frac{r(r^{2n-1}-1)}{r-1} - (2n-1)\right]$.

151. Given, $S_n = a.2^n - b; t_n = S_n - S_{n-1} = a.2^n - b - a.2^{n-1} + b = a.2^{n-1}; r = \frac{t_n}{t_{n-1}} = 2$ which is a constant independent of n hence the given series is in G.P.

152. Given $x \geq 0 \therefore \frac{2x}{1+x^2} < 1$ therefore we can apply the sum formula of a G.P. for infinite terms.

Let S be the required sum, then $S = \frac{1}{1+x^2} \cdot \frac{1}{1-\frac{2x}{1+x^2}} = \frac{1}{(1-x)^2}$.

153. Let a be the first term and r be the common ratio. Then given, $a + ar = 24$ and $S_\infty = \frac{a}{1-r} = 32$

$a = \frac{24}{1+r}$ and $a = 32(1-r) \Rightarrow 1 - r^2 = \frac{24}{32} = \frac{3}{4} \Rightarrow r = \pm\frac{1}{2}$

If $r = \frac{1}{2}$ then series is 16, 8, 4, ... If $r = -\frac{1}{2}$ then series is 48, -24, 12, -6, ...

154. Let a be the first term and r be the common ratio. Sum of this G.P. $\frac{a}{1-r} = 4$ and sum of squares of terms $\frac{a^2}{1-r^2} = \frac{16}{3}$.

$\Rightarrow \frac{16(1-r)^2}{1-r^2} = \frac{16}{3} \Rightarrow \frac{1-r}{1+r} = \frac{1}{3} \Rightarrow r = \frac{1}{2} \Rightarrow a = 2$. So the G.P. is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

155. $p(x) = \frac{\frac{x^{2n}-1}{x^2-1}}{\frac{x^n-1}{x-1}} = \frac{x^n+1}{x+1}$ so clearly n is an odd number for $p(x)$ to be a polynomial in x.

156. $x = a + \frac{a}{r} + \frac{a}{r^2} + \cdots$ to $\infty = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$. Similarly $y = \frac{br}{r+1}$ and $z = \frac{cr^2}{r^2-1}$, $\therefore \frac{xy}{z} = \frac{ab}{c}$.

157. Let a be the first term, r be the common ratio and $2n$ be the no. of terms. Then sum of all terms $S = \frac{a(r^{2n}-1)}{r-1}$ and sum of odd terms $S_{\text{odd}} = \frac{a(r^{2n}-1)}{r^2-1}$.

Given, $S = 5S_{\text{odd}} \Rightarrow r = 4$.

158. $S_n = 3 - \frac{3^{n+1}}{4^{2n}} \Rightarrow t_n = S_n - S_{n-1} = \frac{3^n}{4^{2(n-1)}} - \frac{3^{n+1}}{4^{2n}} = \frac{16.3^n - 3^{n+1}}{4^{2n}} = \frac{3^n.13}{4^{2n}}$.

$\Rightarrow r = \frac{t_n}{t_{n-1}} = \frac{3}{16}$.

159. Let a be the first term and r be the common ratio; then $t_n = ar^{n-1}$. Let the sum of all terms succeeding t_n be S. Then $S = \frac{ar^n}{1-r}$.

$\frac{t_r}{S} = \frac{1-r}{r}$. If $\frac{1-r}{r} > 1$ then $r < \frac{1}{2}$, if $\frac{1-r}{r} = 1$ then $r = \frac{1}{2}$ and $\frac{1-r}{r} < 1$ then $r > \frac{1}{2}$.

160. $666 \ldots n$ digits $= \frac{6}{9}(10^n - 1) = \frac{2}{3}(10^n - 1)$.

$888 \ldots n$ digits $= \frac{8}{9}(10^n - 1)$. $\Rightarrow$ L.H.S. $= \frac{4}{9}(10^{2n} - 2.10^n + 1 - 2.10^n - 2) = \frac{4}{9}(10^{2n} - 1)$

R.H.S. $= 444 \ldots 2n$ digits $= \frac{4}{9}(10^{2n} - 1) =$ L.H.S.

161. Let $S = (x+y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \cdots$ to n terms

$S = \frac{1}{x-y}[(x^2 - y^2) + (x^3 - y^3) + (x^4 + y^4) + \cdots]$ to n terms

$= \frac{1}{x-y}\left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}\right]$.

162. $S = \frac{1}{1-r} \Rightarrow r = \frac{S-1}{S}$. Let $S' = \sum_{n=0}^{\infty} r^{2n}$ then $S' = \frac{1}{1-r^2} = \frac{S^2}{2S-1}$.

163. Let a be the first term and r be the common ratio. Then $t_m = ar^{m-1} = \frac{1}{n^2}$ and $t_n = ar^{n-1} = \frac{1}{m^2} \Rightarrow \frac{t_m}{t_n} = r^{m-n} = \frac{m^2}{n^2} \Rightarrow r = \sqrt[m-n]{\frac{m^2}{n^2}}$.

$ar^{m-1} = \frac{1}{n^2} \Rightarrow a = \frac{1}{n^2}\left(\frac{n^2}{m^2}\right)^{\frac{m-1}{m-n}}$

$\Rightarrow t_{\frac{m+n}{2}} = ar^{\frac{m+n-2}{2}} = \frac{1}{n^2}\left(\frac{n^2}{m^2}\right)^{\frac{m-1}{m-n}} . \left(\frac{m^2}{n^2}\right)^{\frac{m+n-2}{2(m-n)}} = \frac{1}{mn}$.

This can be alternatively computed with G.M. formula i.e. $t_{\frac{m+n}{2}} = \sqrt{t_m t_n} = \frac{1}{mn}$.

164. Given condition is $c > 4b - 3a \Rightarrow c - 4b + 3a > 0 \Rightarrow r^2 - 4r + 3 < 0$ $[\because a < 0] \Rightarrow r > 3$ or $r < 1$.

165. Given, $(1-k)(1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5) = 1 - k^6 \Rightarrow (1-k)\frac{64x^6-1}{x-1} = 1 - k^6 \Rightarrow k = 2x \Rightarrow \frac{k}{x} = 2$.

166. Given, $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) \leq (ab + bc + cd)^2 \Rightarrow (b^2 - ac)^2 + (c^2 - ad)^2 + (ad - bc)^2 \leq 0$

Since a, b, c, d are non-zero real numbers therefore the above conditiion leads to equality if and only if $b^2 = ac, c^2 = ad, ad = bc$ i.e. a, b, c, d are in G.P.

167. This problem is generalization of previous problem and can be solved similarly.

168. Let r be the common ratio, then $\beta = \alpha r, \gamma = \alpha r^2, \delta = \alpha r^3$.

From roots of quadratic equaiton $\alpha + \beta = 3, \alpha\beta = a, \gamma + \delta = 12, \gamma\delta = b$

$\frac{\gamma+\delta}{\alpha+\beta} = r^2 = 4 \Rightarrow r = 2$ because G.P. is increasing so we discard the negative root.

$\Rightarrow \alpha = 1 \Rightarrow a = 2, \Rightarrow b = 32.$

169. Let a be the first term of the A.P. Then $t_{2n+1} = a + 4n$. So the first term of the G.P. is $a + 4n$.

Middle term of A.P. $t_{n+1} = a + 2n$ and middle term of G.P. $= \frac{a+4n}{2^n}$

Given, $a + 2n = \frac{a+4n}{2^n}$ thus, a can found and hence $a + 4n$ which is the mid term can be deduced.

170. $f(x) = 2x + 1, f(2x) = 4x + 1, f(4x) = 8x + 1$. Given that $f(x), f(2x), f(4x)$ are in G.P.

$\Rightarrow \frac{f(2x)}{f(x)} = \frac{f(4x)}{f(2x)} \Rightarrow (4x + 1)^2 = (2x + 1)(8x + 1) \Rightarrow 8x + 1 = 10x + 1 \Rightarrow x = 0.$

171. Let r be the common ratio then $a + b + c = xb \Rightarrow 1 + r + r^2 = xr \Rightarrow x = \frac{1+r+r^2}{r} = \frac{1}{r} + 1 + r$. We know that if $r > 0, r + \frac{1}{r} > 2 \Rightarrow x > 3$ and if $r < 0, r + \frac{1}{r} < -2 \Rightarrow x < -1$.

172. $x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c} \Rightarrow \frac{1}{x} = 1 - a, \frac{1}{y} = 1 - b, \frac{1}{z} = 1 - c$

Thus, $\because a, b, c$ are in A.P. where $|a|, |b|, |c| < 1 \therefore x, y, z$ are also in A.P.

173. $p = \frac{1}{1+\tan^2 x} = \cos^2 x; q = \frac{1}{1+\cot^2 y} = \sin^2 y$

$$\sum_{k=0}^{\infty} \tan^{2k} x \cot^{2k} y = \frac{1}{1-\tan^2 x \cot^2 y}$$

$$\frac{1}{\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}} = \frac{\cos^2 x \sin^2 y}{\cos^2 x+\sin^2 y-1}$$

Dividing numerator and denominator with $\cos^2 x \sin^2 y$, we get

$$= \frac{1}{\csc^2 y+\sec^2 x-\csc^2 y \sec^2 x} = \frac{1}{\tan^2 x+\cot^2 y+2-1-\tan^2 x-cot^2 y-\tan^2 x cot^2 y} = \sum_{k=0}^{\infty} \tan^{2k} x \cot^{2k} y.$$

174. We know that area of an equilateral triangle is $\frac{\sqrt{3}}{4}a^2$, where a is one of the sides. In this case $\Delta = \frac{3}{4}$.

Now the area of sides joining mid-point will have side $\frac{a}{2}$ and terefore area will be $\frac{1}{4}$th of the original triangle. This ratio of $\frac{1}{4}$ will continue and areas of all triangles will form a G.P. with common ratio of $\frac{1}{4}$. Thus sum of areas of all these triangles $= \frac{\frac{3}{4}}{1-\frac{1}{4}} = 1$.

175. $1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \cdots$ to $\infty = \frac{1}{1-|\cos x|} = p(\text{let})$.

$\Rightarrow e^{p.\log_e 4} = 4^p$. Now given equation is $t^2 - 20t + 64 = 0 \Rightarrow t = 4, 16 \Rightarrow p = 1, 2 \Rightarrow |\cos x| = 0, 1/2 \Rightarrow x = \pi/2, \pi/3, 2\pi/3$.

176. $1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \cdots$ to $\infty = \frac{1}{1-|\cos x|} \Rightarrow \frac{1}{1-|\cos x|} = 2 \Rightarrow |\cos x| = \frac{1}{2} \Rightarrow \cos x = \pm\frac{1}{2} \Rightarrow S = \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$.

177. $\sin^2 x + \sin^4 x + \cdots$ to $\infty = \frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x$

Roots of $x^2 - 9x + 8 = 0$ are $1, 8$ i.e. $2^0, 2^3 \Rightarrow \tan x = 0, \sqrt{3}$ (rejecting $-\sqrt{3}$ as for $0 < x < \frac{\pi}{2}$, $\tan x$ cannot be negative.)

$\frac{\cos x}{\cos x+\sin x} = \frac{1}{1+\tan x} = 1, \frac{1}{1+\sqrt{3}}$.

178. $S_\lambda = \frac{\lambda}{\lambda-1}$ [Hint: It is a G.P.] $\sum_{\lambda=1}^{n}(\lambda - 1)S_\lambda = \sum_{\lambda=1}^{n}\lambda = \frac{n(n+1)}{2}$.

179. Let $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ are in G.P. $\Rightarrow \frac{2^{bx+1}}{2^{ax+1}} = \frac{2^{cx+1}}{2^{bx+1}} \Rightarrow (b-a)x = (c-b)x \Rightarrow b - a = c - b$

which implies that a, b, c are in A.P. which is a given and hence we have proven required condition in reverse.

180. Given $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} \Rightarrow ab - ace^x + b^2e^x - bce^{2x} = ab + ace^x - b^2e^x - bce^{2x} \Rightarrow 2ace^x = b^2e^x \Rightarrow 2ac = b^2$, which implies a, b, c are in G.P. Similarly it can be proven that b, c, d are in G.P. making a, b, c, d are in G.P.

181. Given, $2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z \Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-zx}$

But we are also given that $y^2 = zx \Rightarrow 2y = x + z \Rightarrow x, y, z$ are in A.P. Now $4y^2 = (x+z)^2 = 2(x+z) \Rightarrow x = z = y$ but the common values are not necessarily 0.

182. Given, $b - c = a - b$ [$\because a, b, c$ are in A.P.]. From second condition $(c-b)^2 = (b-a)a \Rightarrow (a-b)^2 = (a-b)a \Rightarrow 2a = b \Rightarrow 3a = c \Rightarrow a : b : c = 1 : 2 : 3$.

183. Since a, b, c are in G.P. $\Rightarrow b^2 = ac$. From second condition, $2(\log 2b - \log 3c) = \log 3c - \log 2b \Rightarrow 3\log 2b = 3\log 3c \Rightarrow 2b = 3c \Rightarrow b = \frac{2a}{3}, c = \frac{4a}{9}$. Clearly, a is the greatest side. Using cos rule,

$\cos A = \frac{b^2+c^2-a^2}{2bc} = -\frac{1}{2}$ and thus $A > 90^\circ$ making the triangle obtuse-angled triangle.

184. Let α, β, γ are the roots. Then $\alpha + \beta + \gamma = -\frac{b}{c}, \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}$. Let r be the common ratio of the G.P. then $\beta = \alpha r, \gamma = \alpha r^2$. Also let $\alpha = x$.

$\frac{c^3}{b^3} = \frac{c^3}{a^3} \cdot \frac{a^3}{b^3} = -\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^3}{(\alpha+\beta+\gamma)^3} = -\left(\frac{x^2r+x^2r^3+x^2r^2}{x+xr+xr^2}\right)^3 = -x^3r^3 = -\alpha\beta\gamma = \frac{d}{a} \Rightarrow c^3a = b^3d.$

185. Clearly $t_n = \frac{1}{2n-1} \Rightarrow t_{100} = \frac{1}{199}$.

186. The corresponding pth and qth term in the A.P.would be $\frac{1}{qr}$ and $\frac{1}{rp}$. Let a be the first term and d be the commond difference of this A.P. Then, $a+(p-1)d = \frac{1}{qr}$ and $a+(q-1)d = \frac{1}{rp}$. Subtracting $(p-q)d = \frac{p-q}{pqr} \Rightarrow d = \frac{1}{pqr}$.

$\Rightarrow a = \frac{1}{qr} - \frac{p-1}{pqr} = \frac{1}{pqr}. \Rightarrow t_r = \frac{1}{pqr} + \frac{r-1}{pqr} = \frac{1}{pq}$. Therefore rth term in H.P. would be pq.

187. Corrsponding pth, qth and rth term of the A.P. would be $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$. Let x be the first term and y be the c.d. of this A.P. Then,

$x+(p-1)y = \frac{1}{a}, x+(q-1)y = \frac{1}{b}, x+(r-1)y = \frac{1}{c}$

$(p-q)y = \frac{b-a}{ab} \Rightarrow (p-q)ab = \frac{b-a}{y}$. Similarly, $(q-r)bc = \frac{c-b}{y}$ and $(r-p)ca = \frac{c-a}{y}$. Clearly, $(q-r)bc+(r-p)ca+(p-q)ab = 0$.

188. We have to prove that $\frac{a-b}{b-c} = \frac{a}{c} \Rightarrow ac-bc = ab-ac \Rightarrow 2ac = ab+bc$ which prove that a, b, c are in H.P. Thus required equality is proven in reverse.

189. Given $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P. Let p be the c.d. of this A.P. $\Rightarrow \frac{1}{b} - \frac{1}{a} = p \Rightarrow ab = \frac{a-b}{p}$. Similarly, $bc = \frac{b-c}{p}$, $cd = \frac{c-d}{p}$. Adding these we have $ab+bc+cd = \frac{a-d}{p}$. Now $\frac{1}{d} - \frac{1}{a} = 3p \Rightarrow 3ad = \frac{a-d}{p}$. Thus, $ab+bc+cd = 3ad$.

190. Let d be the common difference of the corresponding $A.P.$ Then, $\frac{1}{x_n} - \frac{1}{x_1} = (n-1)d \Rightarrow \frac{x_1-x_n}{d} = (n-1)x_1x_n =$ R.H.S.

Now, $\frac{1}{x_1} - \frac{1}{x_2} = d \Rightarrow \frac{x_1-x_2}{d} = x_1x_2$. Similarly, $\frac{x_2-x_3}{d} = x_2x_3$ and so on till $\frac{x_{n-1}-x_n}{d} = x_{n-1}x_n$. Adding these and comparing with R.H.S. we get the required equality.

191. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{a+b+c}{a} - 1, \frac{a+b+c}{b} - 1, \frac{a+b+c}{c} - 1$ are in A.P.

$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

192. a^2, b^2, c^2 are in A.P. $\Rightarrow a^2+ab+bc+ca, b^2+ab+bc+ca, c^2+ab+bc+ca$ are in A.P.

$\Rightarrow (a+b)(c+a), (b+c)(a+b), (c+a)(b+c)$ are in A.P.

Dividing each term by $(a+b)(b+c)(c+a)$, we have

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow b+c, c+a, a+b$ are in H.P.

193. If $t_n = \frac{1}{3n-2}$ then the sequence is $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \cdots$

Let us assume that it is in H.P. then corresponding nth term in A.P. is $3n-2$. Thus, c.d. $= 3n-2-(3n-1)-2 = 3$ which is a constant so the sequence is in A.P. Thus our assumption is correct and given sequence is in H.P.

194. Let a be the first term and d be the c.d. of the corresponding A.P. Then,

$a+(m-1)d = \frac{1}{n}$ and $a+(n-1)d = \frac{1}{m}$. Subtracting, $(m-n)d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn} \Rightarrow a = \frac{1}{n} - \frac{m-1}{mn} = \frac{1}{mn}$.

Then $t_{m+n} = \frac{1}{mn} + (m+n-1)\frac{1}{mn} = \frac{m+n}{mn}$ thus corrsponding term in H.P. would be $\frac{mn}{m+n}$. Also, $t_{mn} = \frac{1}{mn} + \frac{mn-1}{mn} = 1$ and hence corresponding term in H.P. is 1.

195. Let the three numbers in H.P. are a, b, c then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be in A.P. Given, $a+b+c = 37, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{4}$. Let d be the c.d. of the A.P. then $\frac{3}{b} = \frac{1}{4} \Rightarrow b = 12$

$\Rightarrow \frac{12}{1-12d} + 12 + \frac{12}{1+12d} = 37 \Rightarrow d = \frac{1}{60}$. So the numbers are 15, 12, 10.

196. $\because a, b, c$ are in H.P. $\therefore b = \frac{2ac}{a+c}$.

L.H.S. $= \frac{1}{b-a} + \frac{1}{b-c} = \frac{a+c}{ac-a^2} + \frac{a+c}{ac-c^2} = \frac{a+c}{ac} = \frac{1}{a} + \frac{1}{c} =$ R.H.S.

197. $\because a, b, c$ are in H.P. $\therefore b = \frac{2ac}{a+c}$.

L.H.S. $= \frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{a^2+3ac}{ac-a^2} + \frac{c^2+3ac}{ac-c^2} = \frac{3ac^2+a^2c-3a^2c-ac^2}{ac(c-a)} = \frac{2ac^2-2a^2c}{ac(c-a)} = 2 =$ R.H.S.

198. Let d be the c.d. of corresponding A.P., then $\frac{1}{x_2} - \frac{1}{x_1} = d \Rightarrow x_1x_2 = \frac{x_1-x_2}{d}$ and similarly, $x_2x_3 = \frac{x_2-x_3}{d}, x_3x_4 = \frac{x_3-x_4}{d}, x_4x_5 = \frac{x_4-x_5}{d}$.

Adding toegther, $\frac{x_1-x_5}{d} = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = \frac{x_1x_5}{d}\left[\frac{1}{x_1} - \frac{1}{x_5}\right] = 4x_1x_5$. Hence proved.

199. Like previous problem $x_1 - x_3 = 2x_1x_3d$ and $x_2 - x_4 = 2x_2x_4d$ so L.H.S. $= 4x_1x_2x_3x_4d^2$

And $x_1 - x_2 = x_1x_2d$ and $x_3 - x_4 = x_3x_4d$ so R.H.S. $= 4x_1x_2x_3x_4d^2$ and thus L.H.S. $=$ R.H.S.

200. Given $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Multiplying with $a+b+c$ and then subtracting 1 from each term we get required condition.

201. Given $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Multiplying each term with $a+b+c$ and then subtracting $ab+bc+ca$ from each term we get the required condition.

202. Given that a, b, c are in A.P. Dividing each term by abc, we get that $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P. Multiplying each term with $ab+bc+ca$ and then subtracting 1 from each term we get the desired condition.

203. Given that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. Multiplying each term with $a+b+c$ and then subtracting 2 from each term we get the desired condition.

204. Given that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. Multiplying each term with $a+b+c$ and then subtracting 1 from each term we get the desired condition.

205. Let d be the c.d. of the A.P. and r be the common ratio of the G.P.

$\Rightarrow b-c=-d, c-a=2d, a-b=-d$ and $y=xr, z=xr^2$.

L.H.S. $= x^{b-c}y^{c-a}z^{a-b} = x^{-d}(xr)^{2d}(xr^2)^{-d} = x^0y^0 = 1$.

206. Let a be the first term and d be the c.d. of the A.P. Then,

$\frac{a+(q-1)d}{a+(p-1)d} = \frac{a+(r-1)d}{a+(q-1)d} = \frac{a+(s-1)d}{a+(r-1)d}$

$\Rightarrow \frac{[a+(q-1)d]-[a+(r-1)d]}{[a+(p-1)d]-[a+(q-1)d]} = \frac{[a+(r-1)d]-[a+(s-1)d]}{[a+(q-1)d]-[a+(r-1)d]}$

$\Rightarrow \frac{q-r}{p-q} = \frac{r-s}{q-r}$ which proves the required condition.

207. Let x be the first term and d be the c.d. of the A.P. Then $a = x+(p-1)d, b = x+(q-1)d, c = x+(r-1)d$

$\Rightarrow b-c = (q-r)d, c-a = (r-p)d$ and $a-b = (p-q)d$

Also let m be the first term and n be the common ratio of the G.P. Then $a = mn^{p-1}, b = mn^{q-1}, c = mn^{r-1}$

L.H.S. $= a^{b-c}b^{c-a}c^{a-b} = (mn^{p-1})^{(q-r)d}(mn^{q-1})^{(r-p)d}(mn^{r-1})^{(p-q)d} = m^0n^0 = 1 =$ R.H.S.

208. Given, a, b, c are in A.P. $\Rightarrow 2b = a+c$ and b, c, d are in H.P. $\Rightarrow c = \frac{2bd}{b+d}$

$\Rightarrow bc = \frac{a+c}{2}\frac{2bd}{b+d} = \frac{(a+c)bd}{b+d} \Rightarrow b^2c + bcd = abbd + bcd \Rightarrow bc = ad.$

209. Given $a^x = b^y = c^z = p(\text{let}) \Rightarrow a = p^{\frac{1}{x}}, b = p^{\frac{1}{y}}, c = p^{\frac{1}{z}}$.

Also given, a, b, c are in G.P. $\Rightarrow \frac{b}{a} = \frac{c}{b} \Rightarrow p^{\frac{1}{y}-\frac{1}{x}} = p^{\frac{1}{z}-\frac{1}{y}} \Rightarrow \frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y}$

$\therefore \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in H.P.

210. $\because \frac{x+y}{2}, y, \frac{y+z}{2}$ are in H.P. $\therefore y = \frac{2\left(\frac{x+y}{2}\right)\left(\frac{y+z}{2}\right)}{\frac{x+y}{2}+\frac{y+z}{2}}$

$\Rightarrow xy + 2y^2 + yz = xy + y^2 + zx + yz \Rightarrow y^2 = zx \Rightarrow a, b, c$ are in G.P.

211. $\because x, y, z$ are in G.P. $\therefore y^2 = zx$. Also, $x + a, y + a, z + a$ are in H.P. $\Rightarrow y + a = \frac{2(x+a)(z+a)}{x+a+z+a} \Rightarrow xy + yz + 2ay + ax + az + 2a^2 = 2(zx + az + ax + a^2) \Rightarrow (y - a)(x + z - 2y)$

But $x + z - 2y \neq 0$ else $x + z = 2y$ i.e. x, y, z are in A.P. $\Rightarrow x = y = z \therefore y = a$.

212. $\because a, b, c$ are in A.P., G.P. and H.P. $\therefore 2b = a + c, b^2 = ac, b = \frac{2ac}{a+c} \Rightarrow \left(\frac{a+c}{2}\right)^2 = ac \Rightarrow (a + c)^2 = 4ac \Rightarrow a = c = b$.

213. $\because a, b, c$ are in A.P. $\Rightarrow 2b = a + c$. $\because b, c, d$ are in G.P. $\therefore c^2 = bd$. $\because c, d, e$ are in H.P. $\therefore d = \frac{2ce}{c+e}$.

$c^2 = bd = \frac{a+c}{2} . \frac{2ce}{c+e} \Rightarrow c(c + e) = (a + c)e \Rightarrow c^2 = ae \Rightarrow a, c, e$ are in G.P.

214. $\because a, b, c$ are in A.P. $\therefore 2b = a + c$. $\because a^2, b^2, c^2$ are in H.P. $\therefore b^2 = \frac{2a^2c^2}{a^2+c^2}$

$\Rightarrow \left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2+c^2} \Rightarrow (a^2 + c^2)(a + c)^2 = 8a^2c^2 \Rightarrow (a - c)^2[(a + c)^2 + 2ac] = 0$

If $(a - c)^2 = 0 \Rightarrow a = c \Rightarrow a = b = c$ else $(a + c)^2 + 2ac = 0 \Rightarrow ac = -2b^2 \Rightarrow b^2 = -\frac{a}{2}.c \Rightarrow -\frac{a}{2}, b, c$ are in G.P.

215. $a^b b^c c^a = a^c b^a c^b \Rightarrow a^{b-c} b^{c-a} c^{a-b} = 1$ which has been proved previously.

216. Let a be the first terms of both the A.P. and G.P. d be c.d. of the A.P. and r be the common ratio of the G.P. Given,

$a + a = 1 \Rightarrow a = \frac{1}{2}, a + d + ar = \frac{1}{2} \Rightarrow d = -ar \Rightarrow 2d = -r$ and $a + 2d + ar^2 = 2 \Rightarrow -r + \frac{r^2}{2} = \frac{3}{2} \Rightarrow r^2 - 2r + 3 = 0$. Now r and sum of fourth term can be easily found.

217. $\because p, q, r$ are in A.P. $\therefore 2q = p + r$. Also, let $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} = f$

$\therefore p = \frac{a}{fx} - \frac{1}{f}, q = \frac{a}{fy} - \frac{1}{f}, r = \frac{a}{fz} - \frac{1}{f}$. Substituting these in $2q = p + r$

$\frac{2a}{fy} - \frac{2}{f} = \frac{a}{fx} - \frac{1}{f} + \frac{a}{fy} - \frac{1}{f} \Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z} \Rightarrow x, y, z$ are in H.P.

218. Let d be c.d. of the A.P. and d' be the c.d. of the A.P. corrsponding to H.P. then,

$b = a + (n-1)d$ and $\frac{1}{b} = \frac{1}{a} + (n-1)d' \Rightarrow d = \frac{b-a}{n-1}, d' = \frac{a-b}{ab(n-1)}$

Product of the rth term of the A.P. and $(n-r+1)$ th term of the H.P. $= \left[a + (r-1)\frac{b-a}{n-1}\right] \cdot \frac{1}{\frac{1}{a}+(n-r)\cdot\frac{a-b}{ab(n-1)}} = ab.$

219. Let a, b, c be three consecutive terms of an H.P. then $b = \frac{2ac}{a+c}$.

Terms after subtraction will be $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$. The condition for these to be in G.P. is $b^2 = (2a-b)(2c-b) = 4ac - 2b(a+c) + b^2 \Rightarrow b = \frac{2ac}{a+c}$ which is a given.

220. $\because y-x, 2(y-a), y-z$ are in H.P. $\therefore \frac{1}{2(y-z)} - \frac{1}{y-x} = \frac{1}{y-z} - \frac{1}{2(y-a)} = \frac{2a-y-z}{(y-x)} = \frac{y+z-2a}{y-z}$

$= \frac{(x-a)+(y-a)}{(x-a)-(y-a)} = \frac{(y-a)+(z-a)}{(y-a)-(z-a)} = \frac{x-a}{y-a} = \frac{y-a}{z-a}$ Hence, $x-a, y-a, z-a$ are in G.P.

221. From given conditions we have $2b = a+c, q = \frac{2pr}{p+r}$ and $b^2q^2 = acpr$. Substituting the values of b and q in third equations, we arrive at

$\left[\left(\frac{a+c}{2}\right)^2\left(\frac{2pr}{p+r}\right)^2\right] = acpr = \frac{(a+c)^2}{(r+p)^2}.p^2r^2 \Rightarrow \frac{pr}{(r+p)^2} = \frac{ac}{(a+c)^2}$

$\Rightarrow \frac{(r+p)^2}{pr} = \frac{(a+c)^2}{ac} \Rightarrow \frac{p}{r} + \frac{r}{p} = \frac{a}{c} + \frac{c}{a}$.

222. From given conditions we have, $2b = a+x, b^2 = ay$ and $\frac{2}{b} = \frac{1}{a} + \frac{1}{x} \Rightarrow x = 2b-a, y = \frac{b^2}{a}$ and $z = \frac{ab}{2a-b}$

Now we can substitute in the required result and prove the equality.

223. From given equations $2 = x+z$ and $4 = zx$, we have to prove that $4 = \frac{2zx}{x+z}$. Substituting the values from given conditions to required equality we find that equality holds.

224. Given that $t_n = 12n^2 - 6n + 5$ then $S_n = 12\sum_{i=1}^{n} i^2 - 6\sum_{i=1}^{n} i + 5\sum_{i=1}^{n} 1$

$= 12.\frac{n(n+1)(2n+1)}{6} - 6\frac{n(n+1)}{2} + 5n = n\left[4n^2 + 6n + 2 - 3n - 3 + 5\right] = n(4n^2 + 3n + 4).$

225. Clearly $t_n = (2n-1)^2 = 4n^2 - 4n + 1 \Rightarrow S_n = 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$

$= 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n = n\left[\frac{4n^2+6n+2-6n-6+3}{3}\right] = \frac{n(4n^2-1)}{3}.$

226. Clearly, $t_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n \Rightarrow S_n = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$ $= \left[\frac{n(n+1)}{2}\right]^2 + 3.\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n + 1 + 1\right] = \frac{n(n+1)}{2}.\frac{n^2+5n+4}{2} = \frac{n(n+1)^2(n+4)}{4}$.

227. rth term of the series, $t_r = r(n-r+1) \Rightarrow S_n = n\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$ $= \frac{n.n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}\left[n - \frac{2n+1}{3} + 1\right] = \frac{n(n+1)}{2}\left[\frac{3n-2n-1+3}{3}\right] = \frac{n(n+1)(n+2)}{6}$.

228. If you see carefully this series is same as previous problem hence sum will be same. $t_n = 1 + 2 + 3 + \cdots + n = \frac{n^2+n)}{2} \Rightarrow t_n = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$ $= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{n(n+1)}{4}\left[\frac{2n+1}{3} + 1\right] = \frac{n(n+1)(n+2)}{6}$.

229. First term contains 1 integer, second term contains 2 and so on. So before t_n we will have $1 + 2 + \cdots + (n-1)$ integers i.e. $\frac{n(n-1)}{2}$ integers. So t_n will start with $\frac{n(n-1)+2}{2}$ and will have n integers. So $t_n = \frac{n^2-n+2}{2}$ and now it is trivial to find the sum, which will be $S_n = \frac{1}{2}\sum_{i=1}^{n} i^2 - \frac{1}{2}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = \frac{n(n+1)(2n+1)}{12} - \frac{n(n+1)}{2} + n$ simplification is left to you.

230. Let nt_n represent numerator and dt_n be the denominator of the nth term t_n. Then $nt_n = \left[\frac{n(n+1)}{2}\right]^3$ and $dt_n = \frac{n}{2}[2 + (n-1)2] = n^2$ $\Rightarrow t_n = \left(\frac{n+1}{2}\right)^2 = \frac{n^2+2n+1}{2} \Rightarrow S_n = \frac{1}{2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i + \frac{1}{2}\sum_{i=1}^{n} 1 = \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{2} + \frac{n}{2}$. Simplify and put $n = 16$ to arrive at the answer.

231. $t_n = [(2n+1)^3 - (2n)^3] = 12n^2 + 6n + 1 \Rightarrow S_n = 12\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = 12\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} + n = 2n(n+1)(2n+1) + 3n(n+1) + n$. Simplify and put $n = 10$ to get the answer.

232. $t_1 = \frac{1}{1} - \frac{1}{2}, t_2 = \frac{1}{2} - \frac{1}{3} \cdots t_n = \frac{1}{n} - \frac{1}{n+1}$. Adding $S_n = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$.

233. $t_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{2}\left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right]$

Then, $t_1 = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.3}, t_2 = \frac{1}{2.2} - \frac{1}{3} + \frac{1}{2.4}, t_3 = \frac{1}{2.3} - \frac{1}{4} + \frac{1}{2.5}, \ldots, t_{n-2} = \frac{1}{2(n-1)} - \frac{1}{n-1} + \frac{1}{2n}, t_{n-1} = \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}, t_n = \frac{1}{2.n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

$\Rightarrow S_n = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.2} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \Rightarrow S_\infty = \frac{1}{4}$

234.
$$\begin{aligned} S_n &= 1 + 5 + 11 + 19 + \cdots + t_{n-1} + t_n \\ S_n &= \quad 1 + 5 + 11 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, we get $t_n = 1 = [4 + 6 + 8 + \cdots$ to $(n-1)$ terms $] = 1 + \frac{n-1}{2}[2.4 + (n-2)2] = n^2 + n - 1 \Rightarrow S_n = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n = \frac{n(n^2+3n-1)}{3}$.

235. First person gets 1 repee, second person gets $1 + 1 = 2$ rupee, third person gets $2 + 2 = 4$ rupee, fourth person gets $4 + 3 = 7$ rupee and so on.

$$\begin{aligned} S_n &= 1 + 2 + 4 + 7 + \cdots + t_n \\ S_n &= \quad 1 + 2 + 4 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, we get $t_n = 1 + [1 + 2 + 3 + \cdots$ to $(n-1)$ terms$] = 1 + \frac{n-1}{2}[2.1 + (n-2)] = \frac{n^2-n+2}{2} = 67 \Rightarrow n^2 - n - 132 = 0 \Rightarrow n = 12$.

236. First term contains 1 integer, second term contains 2 and so on. So before t_n we will have $1 + 2 + \cdots + (n-1)$ integers i.e. $\frac{n(n-1)}{2}$ integers. So t_n will start with $\frac{n(n-1)+2}{2}$ and will have n integers. So $t_n = \frac{n^2-n+2}{2}$. This will be the first number in nth group. So sum of nth group $= \frac{n}{2}[n^2 - n + 2 + n - 1] = \frac{n(n^2+1)}{2}$.

237.
$$\begin{aligned} S_n &= 1 + 3 + 7 + 15 + \cdots + t_n \\ S_n &= \quad 1 + 3 + 7 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, we have $t_n = 1 + [2 + 4 + 8 + \cdots$ to $(n-1)$ terms$] = 1 + \frac{2(2^{n-1}-1)}{2-1} = 2^n - 1 \Rightarrow S_n = (2-1) + (2^2-1) + (2^3-1) + \cdots + (2^n-1) = \frac{2(2^n-1)}{2-1} - n = 2^{n+1} - 2 - n$.

238.
$$\begin{aligned} S_n &= 1 + 2x + 3x^2 + \cdots + t_n \\ xS_n &= \quad 1.x + 2x^2 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting we get $(1-x)S_n = 1 + x + x^2 + \cdots$ to n terms $- xt_n = \frac{1-x^n}{1-x} - x.nx^{n-1} \Rightarrow S_n = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}$.

239. Given
$$\begin{aligned} S_{100} &= 1 + 2.2 + 3.2^2 + 4.3^3 + \cdots + 100.2^{99} \\ 2.S_{100} &= \quad 1.2 + 2.2^2 + 3.2^3 + \cdots + 99.2^{99} + 100.2^{100} \end{aligned}$$

Subtracting, we get $-S_n = 1 + [2 + 2^2 + 2^3 + \cdots$ to 99 terms$] - 100.2^{100}$

$S_n = 100.2^{100} - \frac{2^{100}-1}{2-1} = 99.2^{100} + 1$.

240. Clearly
$$S = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots \text{ to } \infty$$
$$xS = \quad x + 2^2x^2 + 3^2x^3 + \cdots \text{ to } \infty$$

Subtracting, we get
$$(1-x)S = 1 + 3x + 5x^2 + 7x^3 + \cdots \text{ to } \infty$$
$$x(1-x)S = \quad x + 3x^2 + 5x^3 + \cdots \text{ to } \infty$$

Again subtracting, $(1-x)^2S = 1 + 2x + 2x^2 + 2x^3 + \cdots \text{ to } \infty = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x} \Rightarrow S = \frac{1+x}{(1-x)^2}$.

241. $S_n = 2n^2 + 4, t_n = S_n - S_{n-1} = 2n^2 + 4 - 2(n-1)^2 - 4 = 4n - 2 \Rightarrow d = t_n - t_{n-1} = 4n - 2 - 4(n-1) + 2 = 4$ which is constant therefore the given sequence is in A.P.

Hint: Any sequence which is of the for which sum is of the form $an^2 + bn + c$ will lead to an A.P.

242. Given $t_n = n(n-1)(n+1) = n^3 - n \Rightarrow S_n = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} i = \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)}{2} = \frac{n(n+1)(n^2+n-2)}{4}$.

243. Clearly, $t_n = (2n-1)^3 = 8n^3 - 12n^2 + 6n - 1 \Rightarrow S_n = 8\sum_{i=1}^{n} i^3 - 12\sum_{i=1}^{n} i^2 + 6\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$; simplification is left to you.

244. Clearly, $t_n = (3n-2)^2 = 9n^2 - 12n + 4 \Rightarrow S_n = 9\sum_{i=1}^{n} i^2 - 12\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} 1 = \frac{3n(n+1)(2n+1)}{2} - 6n(n+1) + 4n$; simplification is left to you.

245. Given series is $1^2 + 3^2 + 5^2 + \cdots$ to n terms $+ 2 + 4 + 6 + \cdots$ to n terms.

$\Rightarrow t_n = (2n-1)^2 + \frac{n}{2}[2.2 + (n-1)2] = 4n^2 - 4n + 1 + n^2 + n = 5n^2 - 3n + 1$

$\Rightarrow S_n = 5\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = \frac{5n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n$; simplification is left to you.

246. **Case I:** When n is even. Let $n = 2m$ then $S = 1^2 + 3^2 + 5^2 + \cdots$ to m terms $-$ $[2^2 + 4^2 + 6^2 + \cdots$ to m terms$]$

$= \sum_{i=1}^{m}(2i-1)^2 - \sum_{i=1}^{m}(2i)^2 = -4\sum_{i=1}^{m} i + \sum_{i=1}^{m} 1 = -2m(m+1) + 4m = -2m^2 + 2m$ and then we substitute $m = \frac{n}{2}$.

Case II: When n is odd. Let $n = 2m+1$, then $S = 1^2 + 3^2 + 5^2 + \cdots$ to $(m+1)$ terms $-$ $[2^2 + 4^2 + 6^2 + \cdots$ to m terms$]$

$= \sum_{i=1}^{m+1}(2i-1)^2 - \sum_{i=1}^{m}(2i)^2 = \frac{4(m+1)(m+2)(2m+3)}{6} - 2(m+1)(m+2) + (m+1) - \frac{2m(m+1)(2m+1)}{3}$; put $m = \frac{n-1}{2}$ and simplify.

247. Clearly, $t_n = (2n-1)(2n+1) = 4n^2 - 1 \Rightarrow S_n = 4\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} 1 = \frac{2n(n+1)(2n+1)}{3} - n$; simplification is left to you.

248. Clearly, $t_n = n(n+1) \Rightarrow S_n = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$; simplification is left to you.

249. Clearly, $t_n = n(n+1)^2 = n^3 + 2n^2 + n \Rightarrow S_n = \sum_{i=1}^{n} i^3 + 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}$; simplification is left to you.

250. Clearly, $t_n = (n+1)n^2 = n^3 + n^2 \Rightarrow S_n = \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$; simplification is left to you.

251. $t_n = 1 + 3 + 5 + \cdots$ upto n terms $= \frac{n}{2}[2.1 + (n-1)2] = n^2 \Rightarrow S_n = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

252. $t_n = 1^2 + 2^2 + 3^2 + \cdots$ upto n terms $= \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3+3n^2+n}{6}$.

$S_n = \frac{1}{6}\left[\sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{6}\left[\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$; simplification is left to you.

253. $t_n = n(n+1)(2n+1) = 2n^3 + 3n^2 + n \Rightarrow S_n = 2\sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}$; simplification is left to you.

254. $t_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n \Rightarrow S_n = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + n(n+1)$; simplification is left to you.

255. $t_n = n(2n+1)^2 = 4n^3 + 4n^2 + n \Rightarrow S_n = 4\sum_{i=1}^{n} i^3 + 4\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = n^2(n+1)^2 + \frac{2n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}$; put $n = 20$ and simplify.

256. $t_r = r(n^2 - r^2) = n^2 r - r^3 \Rightarrow S = n^2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^3 = \frac{n^3(n+1)}{2} - \frac{n^2(n+1)^2}{4}$; simplification is left to you.

257. $t_n = (2n+1)^3 - (2n)^3 = 12n^2 + 6n + 1 \Rightarrow S_n = 12\sum_{i=1}^{n} i^2 + 6\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = 2n(n+1)(2n+1) + 3n(n+1) + n$; put $n = 10$ to get the answer.

258. $t_n = \frac{1}{1+2+3+\cdots \text{ to } n \text{ terms}} = \frac{2}{n(n+1)} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$

$t_1 = 2\left[1 - \frac{1}{2}\right], t_2 = 2\left[\frac{1}{2} - \frac{1}{3}\right], t_3 = 2\left[\frac{1}{3} - \frac{1}{4}\right], \ldots, t_n = 2\left[\frac{1}{n} - \frac{1}{n+1}\right].$

Adding, $S = 2\left[1 - \frac{1}{n+1}\right] = \frac{2n}{n+1}$.

259. $S = \frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} + \ldots = 2\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + \cdots \text{ to } \infty\right] = 1.$

260.
$$\begin{aligned} S &= 2 + 6 + 12 + 20 + \cdots + t_n \\ S &= \quad\; 2 + 6 + 12 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, $t_n = 2 + 4 + 6 + 8 + \cdots$ to n terms $= \frac{n}{2}[2.2 + (n-1)2] = n(n+1) = n^2 + n \Rightarrow S = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$; simplification is left to you.

261.
$$\begin{aligned} S &= 3 + 6 + 11 + 18 + \cdots + t_n \\ S &= \quad\; 3 + 6 + 11 + 18 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, $t_n = 3 + [3 + 5 + 7 + \cdots \text{ to } (n-1) \text{ terms}] = 3 + \frac{n-1}{2}[2.3 + (n-2)2] = 3 + n^2 - 1 = n^2 + 2$

$S = \frac{n(n+1)(2n+1)}{6} + 2n$; simplification is left to you.

262.
$$\begin{aligned} S &= 1 + 9 + 24 + 46 + 75 + \cdots + t_n \\ S &= \quad\; 1 + 9 + 24 + 46 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting $t_n = 1 + 8 + 15 + 22 + 29 + \cdots$ to $n$terms $= \frac{n}{2}[2 + (n-1)7] = \frac{7n^2 - 5n}{2}$.

$\Rightarrow S = \frac{7n(n+1)(2n1)}{12} - \frac{5n(n+1)}{4}$.

263.
$$\begin{aligned} S &= 2 + 4 + 7 + 11 + 16 + \cdots + t_n \\ S &= \quad\; 2 + 4 + 7 + 11 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, $t_n = 2 + [2 + 3 + 4 + 5 + \cdots \text{ to } (n-1) \text{ terms}] = 2 + \frac{n-1}{2}[2.2 + n - 1] = 2 + \frac{n^2+2n-3}{2} = \frac{n^2-2n+1}{2}$.

264.
$$\begin{aligned} S &= 1 + 3 + 6 + 10 + \cdots + t_n \\ S &= \quad\; 1 + 3 + 6 + \cdots + t_{n-1} + t_n \end{aligned}$$

Subtracting, $t_n = 1 + 2 + 3 + 4 + \cdots$ to $n$terms $= \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

$\Rightarrow S = \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$. Put $n = 10$ to get the answer.

265. First group contains 2 odd numbers, second group contains 4 odd numbers, third group contains 6 odd numbers so $(n-1)$th group will contain $2n-2$ odd numbers.

Total no. of odd numbers till $(n-1)$th group will be $n(n-1)$. So last no. in $(n-1)$th group will be $1+(n^2-n-1)2 = 2n^2-2n-1$ and hence first number in nth group will be $2n^2-2n+1$ and there will be $2n$ odd numbers. So sum of $2n$ odd numbers starting from $2n^2-2n+1$ is given by $\frac{2n}{2}[4n^2-4n+2+(2n-1)2] = 4n^3$.

266. Groups contain 1, 3, 5, ... number of terms so nth group will contain $2n-1$ numbers starting from n. So sum will be $\frac{2n-1}{2}[2n+2n-2] = (2n-1)^2$ which is square of odd positive integer.

267.
$$\begin{aligned} S &= 2+5+14+41+\cdots+t_n \\ S &= \quad\; 2+5+14+\cdots+t_{n-1}+t_n \end{aligned}$$

Subtracting $t_n = 2+[3+3^2+\cdots \text{ to } (n-1)\text{ terms}] = 2+\frac{3(3^{n-1}-1)}{3-1} = \frac{3^n+1}{2}$.

$\Rightarrow S = \frac{1}{2}\left[\frac{3(3^n-1)}{2}+n\right]$.

268.
$$\begin{aligned} S &= 1.1+2.3+4.5+8.7+\cdots+t_n \\ 2S &= \qquad 2.1+4.3+8.5+\cdots+t_{n-1}+2^n(2n-1) \end{aligned}$$

Subtracting, $-S = 1.1+[2.2+4.2+8.2+\cdots \text{ to } (n-1) \text{ terms}] - 2^n(2n-1)$

$S = 2^n(2n-1)-1-4(2^{n-1}-1)$.

269. Clearly, $a_{2n}-a_1 = (2n-1)d \Rightarrow d = \frac{a_{2n}-a_1}{2n-1}$

Now, $a_1^2-a_2^2+a_3^2-a_4^2+\cdots+a_{2n-1}^2-a_{2n}^2 = (a_1-a_2)(a_1+a_2)+(a_3-a_4)(a_3+a_4)+\cdots+(a_{2n-1}-a_{2n})(a_{2n-1}+a_{2n})$

$= -d(a_1+a_2+a_3+a_4+\cdots+a_{2n-1}+a_{2n}) = -\frac{a_{2n}-a_1}{2n-1}.\frac{2n}{2}[a_1+a_{2n}] = \frac{n}{2n-1}(a_1^2-a_{2n}^2)$.

270. $d = \alpha_2-\alpha_1 = \alpha_3-\alpha_2 = \cdots = \alpha_n-\alpha_{n-1}$

$\sin d \sec\alpha_1 \sec\alpha_2 = \frac{\sin(\alpha_2-\alpha_1)}{\cos\alpha_1\cos\alpha_2} = \tan\alpha_2-\tan\alpha_1$. Similarly, $\sin d\sec\alpha_2\sec\alpha_3 = \tan\alpha_3-\tan\alpha_2$ and so on. $\sin d\sec\alpha_{n-1}\sec\alpha_n = \tan\alpha_n-\tan\alpha_{n-1}$

Adding we get L.H.S. = R.H.S.

271. L.H.S. $= \frac{1}{a_1+a_n}\left[\frac{a_1+a_n}{a_1a_n}+\frac{a_1+a_n}{a_2a_{n-1}}+\cdots+\frac{a_1+a_n}{a_na_1}\right] = \frac{1}{a_1+a_n}\left[\frac{a_1+a_n}{a_1a_n}+\frac{a_2+a_{n-1}}{a_2a_{n-1}}+\cdots+\frac{a_1+a_n}{a_na_1}\right]$

$= \frac{1}{a_1+a_n}\left[\frac{1}{a_1}+\frac{1}{a_n}+\frac{1}{a_2}+\frac{1}{a_{n-1}}+\cdots+\frac{1}{a_n}+\frac{1}{a_1}\right] = \frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$.

272. $\frac{1}{a_1} - \frac{1}{a_2} = \frac{a_2 - a_1}{a_1 a_2} = \frac{d}{a_1 a_2} \Rightarrow \frac{1}{a_1 a_2} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)$. Similarly $\frac{1}{a_2 a_3} = \frac{1}{d}\left(\frac{1}{a_2} - \frac{1}{a_3}\right)$ and so on.

$\therefore S = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{n+1}}\right) = \frac{n}{a_1 a_{n1}}$

273. $\because a_1 = 0$ then $a_2 = d, a_3 = 2d, \ldots, a_n = (n-1)d$ where d is the c.d. of the A.P.

L.H.S. $= \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n-1}{n-2} - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-3}\right)$

$= (1+1) + \left(1 + \frac{1}{2}\right) + \cdots + \left(1 + \frac{1}{n-2}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-3}\right)$

$= n - 2 + \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-2}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-3}\right)\right]$

$= n - 2 + \frac{1}{n-2} = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}} =$ R.H.S.

274. L.H.S. $= \sum_{k=1}^{n} \frac{a_k a_{k+1} a_{k+2}}{(a_{k+1} - d) + (a_{k+1} + d)} = \frac{1}{2}\sum_{k=1}^{n} a_k a_{k+2} = \frac{1}{2}\sum_{i=1}^{k}(a_{k+1}^2 - d^2) = \frac{1}{2}\sum_{k=1}^{n}[(a_1 + kd)^2 - d^2] = \frac{1}{2}\sum_{k=1}^{n}[a_1^2 + 2a_1 dk + (k^2 - 1)d^2]$

$= \frac{1}{2}\left[\sum_{k=1}^{n} a_1^2 + 2a_1 d \sum_{k=1}^{n} k + d^2 \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} d^2\right] = \frac{1}{2}\left[na_1^2 + 2a_1 d \frac{n(n+1)}{2} + d^2 \frac{n(n+1)(2n+1)}{6} - nd^2\right]$

$= \frac{n}{2}\left[a_1^2 + (n+1)a_1 d + \frac{(n-1)(2n+5)}{6} d^2\right] =$ R.H.S.

275. Given, $x^{18} = y^{21} \Rightarrow 18 \log x = 21 \log y \Rightarrow \log_y x = \frac{7}{6}$

Similarly $y^{12} = z^{28} \Rightarrow \log_z y = \frac{4}{3}$ and $x^{18} = y^{28} \Rightarrow \log_x z = \frac{9}{14}$

Now it is trivial to prove that $3, 3\log_y x, \log_z x, 7\log_x z$ are in A.P.

276. Given, $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x} dx$. Since we have to prove that $I_1, I_2, I_3, \ldots$ are in A.P. we can simply prove that I_n, I_{n+1}, I_{n+2} are in A.P. which will be enough to prove the entire sequence. So it is enough to prove that $I_n + I_{n+2} - 2I_{n+1} = 0$

L.H.S. $= \sin_0^{\frac{\pi}{2}} \frac{\sin^2(n+2)x + \sin^2 nx - \sin^2(n+1)x}{\sin^2 x} dx$

$= \int_{i=0}^{\frac{\pi}{2}} \frac{1 - \cos(2n+4)x + 1 - \cos 2nx - 2 + 2\cos(2n+2)x}{2\sin^2 x} dx$

$= \int_{i=0}^{\frac{\pi}{2}} \frac{2\cos(2n+2)x - 2\cos(2n+2)x \cos 2x}{2\sin^2 x} dx$

$$= \int_{i=0}^{\frac{\pi}{2}} \frac{2\cos(2n+2)x.2\sin^2 x}{2\sin^2 x}dx = \int_{i=0}^{\frac{\pi}{2}} 2\cos(2n+2)dx = \left[\frac{\sin(2n+2)x}{n+1}\right] = 0.$$

277. Let $a_1, a_2, a_3, ...$ be an A.P. which are distinct primes. Clearly $a_1 \geq 1$. $d = a_2 - a_1 \geq 1$. Now $(a_1+1)th$ term $= a_1 + a_1 d = a_1(1+d)$ which is a composite number. Thus, there cannot be such an A.P.

278. Let the four distinct integers in A.P. be $a, a+d, a+2d, a+3d$ where $d > 0$. Obviously, the term which is sum of squares of remaining terms will be $a+3d$.

Let $a+3d = a^2+(a+d)^2+(a+2d)^2 = 3a^2+6ad+5d^2 \Rightarrow 5d^2+a(6d-1)+5d^2-3d = 0$

$\Rightarrow 9(2a-1)^2 - 20(3a^2-a) \geq 0$ [$\because d$ is real] $\Rightarrow -24a^2 - 16a + 9 \geq 0$

Corresponding roots are $-\frac{4\pm\sqrt{70}}{12} \Rightarrow -\frac{4-\sqrt{70}}{12} \leq a \leq -\frac{4+\sqrt{70}}{12} \therefore a = -1, 0$[$\because a$ is an integer].

$\Rightarrow a = 1$ other roots are not acceptable. Numbers are $-1, 0, 1, 2$.

279. Given, $t_n = p+q$ and $t_{n+1} = p-q \Rightarrow d = -2q$. We also know that

$t_1 + t_{2n} = t_2 + t_{2n-1} = \cdots = t_n + t_{n+1} = 2p$

$t_1^3 + t_{2n}^3 = (t_1+t_{2n})^3 - 3t_1t_{2n}(t_1+t_{2n}) = 8p^3 - 6pt_1t_{2n} = 8p^3 + \frac{6p}{4}[(t_1+t_{2n})^2 - (t_1-t_{2n})] = 8p^3 - \frac{3p}{2}[4p^2 - (2n-1)^2d^2] = 2p^3 + 6pq^2(2n-1)^2$

$S = 2np^3 + 6pq^2[1^2+3^2+\cdots+(2n-1)^2]$ (we have found $\sum_{i=1}^{n}(2i-1)^2$ so we will use that result)

$= 2np^3 + 2pq^2.n(2n+1)(2n-1) = 2np[p^2 + (4n^2-1)q^2]$.

280. Let a be the first term and d be the c.d. of the A.P. Then,

$S = \frac{n}{2}[2a+(n-1)d]$

$S_n = a^3 + (a+d)^3 + (a+2d)^3 + \cdots + [a+(n-1)d]^3$

$= na^3 + 3a^2d[1+2+3+\cdots+(n-1)] + 3ad^2[1^2+2^2+3^2+\cdots+(n-1)^2] + d^3[1^3+2^3+3^3+\cdots+(n-1)^3]$

$= na^3 + 3a^2d\frac{n(n-1)}{2} + 3ad^2\frac{n(n-1)(2n-1)}{6} + d^3\frac{n^2(n-1)^2}{4}$

$= \frac{n}{2}[2a+(n-1)d][a^2+(n-1)ad+\frac{n(n-1)}{2}d^2] = S[a^2+(n-1)ad+\frac{n(n-1)}{2}d^2]$.

Hence, S is a factor of S_n.

281. Let r be a positive integer greater than 1. If possible, let $m^r = (2k+1)+(2k+3)+\cdots+(2k+2m-1) = \frac{m}{2}[2k+1+2k+2m-1] = 2k+m \Rightarrow k = \frac{m^{r-1}-m}{2}$

Clealry for $r > 1, m^{r-1}$ and m are both odd or both even. $\therefore m^{r-1}-m$ is an even number. Thus such an integer k exists.

Also, the first odd inetger $= 2k+1 = m^{r-1}-m+1$.

282. Let x be the first term and d be the c.d. of the A.P. Then,

$x+(x+d)+(x+2d)+\cdots+[x+(n-1)d] = a$

$$a = nx + \frac{dn(n-1)}{2}] \tag{2.1}$$

Also, $x^2+(x+d)^2+(x+2d)^2+\cdots+[x+(n-1)d]^2 = b^2$

$= nx^2+2xd[1+2+3+\cdots+(n-1)]+d^2[1^2+2^2+\cdots+(n-1)^2]$

$$b^2 = nx^2 + xdn(n-1) + d^2\frac{(n-1)n(2n-1)}{6} \tag{2.2}$$

Sqauring Eq. 2.1, we have

$a^2 = n^2x^2+n^2xd(n-1)+\frac{n^2d^2(n-1)^2}{4} = a^2$

$$nx^2 + nxd(n-1) + \frac{nd^2(n-1)^2}{4} = a^2 \tag{2.3}$$

Eq. 2.2 - Eq. 2.3

$\Rightarrow d^2\frac{n(n-1)(n+1)}{12} = \frac{nb^2-a^2}{n} \Rightarrow d = \pm\frac{2\sqrt{3(nb^2-a^2)}}{n\sqrt{n^2-1}}$

Now you can find x trivially.

283. $d = a_2-a_1 = a_3-a_2 = \cdots = a_n-a_{n-1}$. We have to find

$\sin d[\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \cdots + \csc a_{n-1}\csc a_n]$

$= \sin d[\frac{1}{\sin a_1 \sin a_2} + \frac{1}{\sin} a_2 \sin a_3 + \cdots + \frac{1}{\sin a_{n-1}\sin a_n}]$

$= \frac{\sin(a_2-a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3-a_2)}{\sin a_2 \sin a_3} + \cdots + \frac{\sin(a_n-a_{n-1})}{\sin a_{n-1}\sin a_n}$

$= \frac{\sin a_2 \cos a_1 - \sin a_1 \cos a_2}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \sin a_2 \cos a_3}{\sin a_2 \sin a_3} + \frac{\sin a_n \cos a_{n-1} - \sin a_{n-1}\cos a_n}{\sin a_{n-1}\sin a_n}$

$= \cot a_1 - \cot a_2 + \cot a_2 - \cot a_3 + \cdots + \cot a_{n-1} - \cot a_n = \cot a_1 - \cot a_n$.

284. Let d be common difference of the A.P.

L.H.S. $= \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$

$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + \cdots + \frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$

$= -\frac{1}{d}[\sqrt{a_1} - \sqrt{a_n}]$ $[\because d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}]$

$= -\frac{n-1}{(n-1)d}\frac{a_1-a_n}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$ $[\because a_n = a_1 + (n-1)d]$.

285. Let d be the common difference of the A.P., then

L.H.S. $= \sum_2^n \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \sum_2^n \tan^{-1} \frac{a_n-a_{n-1}}{1+a_{n-1}a_n} = \sum_2^n \tan^{-1} a_n - \tan^{-1} a_{n-1} \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}\right]$

$= \tan^{-1} a_2 - \tan^{-1} a_1 + \tan^{-1} a_3 - \tan^{-1} a_2 + \cdots + \tan^{-1} a_n - \tan^{-1} a_{n-1} = \tan^{-1} a_n - \tan^{-1} a_1 = \tan^{-1} \frac{a_n-a_1}{1+a_1a_n} =$ R.H.S.

286. Given, $S_n = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} - \cdots + \frac{1}{a_{n-1}a_n}$

$= \frac{1}{d}\left[\frac{a_2-a_1}{a_1a_2} + \frac{a_3-a_2}{a_2a_3} + \cdots + \frac{a_n-a_{n-1}}{a_{n-1}a_n}\right]$ $[\because d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}]$

$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \cdots + \frac{1}{a_{n-1} - \frac{1}{a_n}}\right]$

$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_n}\right] = \frac{a_n-a_1}{da_1a_n} = \frac{(n-1)d}{da_1a_n}$ $[\because a_n = a_1 + (n-1)d]$

$\Rightarrow a_{a_n}S_n = n - 1$, which does not depend on a or d.

287. We know that $S = \frac{n}{2}[t_1 + t_n]$ so

$S_1 = \frac{n}{2}[a_1 + a_n] = \frac{n}{2}[2a + (n-1)d]$

$S_2 = \frac{n}{2}[a_{n+1} + a_{2n}] = \frac{n}{2}[2a + (3n-1)d]$

$S_3 = \frac{n}{2}[a_{2n+1} + a_{3n}] = \frac{n}{2}[2a + (5n-1)d]$

… …

$S_r = \frac{n}{2}[2a + \{(2r-1)n - 1\}d]$

Clearly, $S_2 - S_1 = S_3 - S_2 = \cdots = S_{r+1} - S_r = n^2d$ which is an A.P.

288. Let d be the c.d. of the A.P. then $\frac{b-c}{a-b} = \frac{-d}{-d} = 1$ which is a rational number.

289. $\tan 70^\circ = \tan(50^\circ + 20^\circ) = \frac{\tan 70^\circ + \tan 20^\circ}{1 - \tan 50^\circ \tan 20^\circ}$

$\Rightarrow \tan 70^\circ - \tan 70^\circ \tan 50^\circ \tan 20^\circ = \tan 50^\circ + \tan 20^\circ$

$\Rightarrow \tan 70^\circ - \cot(90^\circ - 70^\circ)\tan 50^\circ \tan 20^\circ = \tan 50^\circ + \tan 20^\circ$

$\Rightarrow \tan 70^\circ - \tan 50^\circ = \tan 50^\circ + \tan 20^\circ \Rightarrow \tan 80^\circ = 2\tan 50^\circ + \tan 20^\circ$

Adding $\tan 20^\circ$ to both sides, we have

$\tan 70^\circ + \tan 20^\circ = 2(\tan 50^\circ + \tan 20^\circ)$ and thus required condition is proved.

290. Given $\log_l x, \log_m x, \log_n x$ are in A.P. Therefore $2\log_m x = \log_l x + \log_n x$

$\Rightarrow \frac{2\log x}{\log m} = \frac{\log x}{\log l} + \frac{\log x}{\log n} \Rightarrow \frac{2}{\log m} = \frac{\log ln}{\log l \log n}$

$\Rightarrow 2\log n = \frac{\log ln \log m}{\log l}$ (multiplying with $\log m \log n$ on both sides)

$\Rightarrow \log n^2 = \log_l m \log ln = \log ln^{\log_l m} \Rightarrow n^2 = (ln)^{\log_l m}$; hence proved.

291. Let b, p, h be base, perpendicular, hypotenuse of the triangle. Let b be smallest then $2p = h + b \Rightarrow h = 2p - b$

We know that for a right angle triangle $h^2 = b^2 + p^2$. Substituting for h,

$4p^2 - 4bp + b^2 = b^2 + p^2 \Rightarrow 3p^2 = 4bp \Rightarrow 3p = 4b \Rightarrow h^2 = \frac{16b^2}{9} + b^2 \Rightarrow h = \frac{5b}{3}$

$\Rightarrow b : p : h = 3 : 4 : 5.$

292. Let $5^x = t$ then for condition for A.P. gives us $a = 5t + \frac{5}{t} + t^2 + \frac{1}{t^2}$

We know that $x + \frac{1}{x} \geq 2 \therefore a \geq 12.$

293. Given $\log 2, \log(2^x - 1), \log(2^x + 3)$ are in G.P. Therefore, $2\log(2^x - 1) = \log 2 + \log(2^x + 3)$

$\Rightarrow (2^x - 1)^2 = 2.2^2 + 6 \Rightarrow 2^{2x} - 4.2^x - 5 = 0 \Rightarrow 2^x = 5, -1$ however, $2^x \neq -1$ so $2^x = 5 \Rightarrow x = \log_2 5.$

294. Let d be the c.d. of the A.P. $\therefore \log_y x = 1 + d \Rightarrow x = y^{1+d}, \log_z y = 1 + 2d \Rightarrow y = z^{1+2d}, -15\log_x z = 1 + 3d \Rightarrow z = x^{-\frac{1+3d}{15}}$

$\because x = y^{1+d} = z^{(1+2d)(1+d)} = x^{-\frac{(1+d)(1+2d)(1+3d)}{15}} \Rightarrow (1+d)(1+2d)(1+3d) = -15 \Rightarrow (d+2)(6d^2 - d + 8) = 0$

Discriminant of $6d^2 - d + 8$ is less than 0 and thus $d = -2$.

$\Rightarrow x = z^3, y = z^{-3}.$

295. Let $\sqrt{2}, \sqrt{3}, \sqrt{5}$ be pth, qth and rth term of an A.P. whose c.d. is d.

$\sqrt{3} - \sqrt{2} = (q - p)d$ and $\sqrt{5} - \sqrt{3} = (r - q)d$. Dividing, we get

$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}-\sqrt{3}} = \frac{q-p}{r-q} = x$, which will be a rational number as p, q, r are integers.

Sqauring $5-2\sqrt{6}=x^2(8-2\sqrt{15}) \Rightarrow \sqrt{15}x^2-\sqrt{6}=(8x^2-5)/2=y$ (which will again be a rational number)

Squaring again $15k^4+6-2\sqrt{90}k^2=y^2 \Rightarrow 15k^4+6-y^2=2\sqrt{90}k^2$

L.H.S. is a rational number while R.H.S. is irrational thus our assumption is wrong.

296. Area of rth circle $A_r=\pi r^2$ and area of $(r+1)$th circle is $A_{r+1}=\pi(r+1)^2$ so the difference is $D_r=\pi(2r+1)$ therefore c.d. $=D_{r+1}-D_r=2\pi$ which is a constant and hence the successive areas of each color is in A.P.

297. $\because x, y, z$ are in A.P. $\therefore 2y=x+z$. Similarly, $2\tan^{-1}y=\tan^{-1}x+\tan^{-1}z$

$\Rightarrow \frac{2y}{1-y^2}=\frac{x+z}{1-xz} \Rightarrow \frac{x+z}{1-\frac{(x+z)^2}{4}} \Longrightarrow 1-zx=1-\frac{(x+z)^2}{4} \Rightarrow (z-x)^2=0 \Rightarrow x=z=y.$

298. From given conditiion $\frac{\cos^4\theta}{\cos^2\alpha}+\frac{\sin^4\theta}{\sin^2\alpha}=1=\cos^2\theta+\sin^2\theta$

$\Rightarrow \frac{\cos^4\theta}{\cos^2\alpha}(\cos^2\theta-\cos^2\alpha)=\frac{\sin^2\theta}{\sin^2\alpha}(sin^2\alpha-\sin^2\theta)=\frac{\sin^2\theta}{\sin^2\alpha}(sin^2\theta-\sin^2\alpha)$

$\Rightarrow \frac{\cos^2\theta}{\cos^2\alpha}=\frac{\sin^2\theta}{\sin^2\alpha}$ and thus we prove the required condition because $\frac{\cos^{2n+2}\theta}{\cos^{2n\alpha}}=\cos^2\theta.$

299. $a_{n+1}-a_n=\int_0^\pi \frac{\sin(2n+2)x-\sin 2nx}{\sin x}dx=\int_0^\pi \frac{2\cos(2n+1)x\sin x}{\sin x}dx$

$=\left[\frac{2\sin(2n+1)x}{2n+1}\right]_0^\pi=0$

Hence, c.d. is 0 making all terms equal and in A.P.

300. $l_n+l_{n+2}=\int_0^{\frac{\pi}{4}}(\tan^n x+\tan^{n+2}xdx)=\left[\frac{\tan^{n+1}x}{n+1}\right]_0^{\frac{\pi}{4}}=\frac{1}{n+1}.$

Thus, $\frac{1}{l_2+l_4}=3, \frac{1}{l_3+l_5}=4, \frac{1}{l_4+l_5}=5, \cdots$, which is an A.P. with a c.d. of 1.

301. $I_{n+1}-I_n=\int_0^\pi \frac{\cos 2nx-\cos(2n+2)x}{\sin^2 x}dx=2\int_0^\pi \frac{\sin x\sin(2n+1)x}{\sin^2 x}dx$

$D_n=2\int_0^\pi \frac{\sin(2n+1)x}{\sin x}dx$

$D_{n+1}-D_n=2\int_0^\pi \frac{\sin(2n+3)x-sin(2n+1)x}{\sin x}dx=4\int_0^\pi \frac{\sin x\cos(2n+2)x}{\sin x}dx=2\left[\frac{\sin 2(n+1)x}{n+1}\right]_0^\pi=0$

$\Rightarrow D_1=\pi \Rightarrow I_{n+1}-I_n=\pi$ which is a constant and hence $I_1, I_2, I_3, \ldots$ are in A.P.

302. $\because \alpha, \beta, \gamma$ are in A.P. $\therefore 2\beta=\gamma+\alpha$

$2\sin(\alpha+\gamma)=\sin(\beta+\gamma)+\sin(\alpha+\beta) \Rightarrow 2\sin 2\beta=2\sin\left(\frac{\alpha+\beta+2\beta}{2}\right).\cos\frac{\gamma-\alpha}{2}$

$\Rightarrow \cos\frac{\gamma-\alpha}{2} = 1 = \cos 0 \Rightarrow \gamma = \alpha = \beta$ and hence $\tan\alpha = \tan\beta = \tan\gamma$.

303. Let d be the c.d then we have $2b = a + c$ and $abc = 4 \Rightarrow ac(a + c) = 4$. We know that A.M. $\geq$ G.M $\Rightarrow \frac{a+c}{2} \geq \sqrt{ac} \Rightarrow \frac{(a+c)^2}{4}(a + c) \geq 4 \Rightarrow b^3 \geq 4$ and hence proved.

304. Let $S = \log a + \log\frac{a^3}{b} + \log\frac{a^5}{b^2} + \log\frac{a^7}{b^3} + \cdots$

$= (\log a + 3\log a + 5\log a + \cdots) - (\log b + 2\log b + \cdots) = \frac{n}{2}[2\log a + (n-1)2\log a] - \frac{n-1}{2}[2\log b + (n-2)\log b] = \frac{n}{2}[2n\log a] - \frac{n-1}{2}[2n\log b]$

$= \log a^{n^2} - \log b^{n(n-1)} = \log\frac{a^{n^2}}{b^{(n(n-1))}}$.

305. $b = a + d \Rightarrow d = b - a$ and $n = \frac{c-a}{b-a} + 1 = \frac{b+c-2a}{b-a}$

$S_n = \frac{n}{2}[a + c] = \frac{(b+c-2a)(a+c)}{2(b-a)}$.

306. Let a be the first term and d be the c.d. of the A.P.

$S_{n+3} = \frac{n+3}{2}[2a + (n+2)d]$ and $3(S_{n+2} - S_{n+1}) + S_n = 3t_{n+2} + \frac{n}{2}[2a + (n-1)d] = 3[a + (n+1)d] + \frac{n}{2}[2a + (n-1)d]$

$= \frac{1}{2}[2an + n(n-1)d + 6a + 6(n+1)d] = \frac{1}{2}[2a(n+3) + (n^2 + 5n + 6)d] = S_{n+3}$.

307. Observe that $2ab = (a+b)^2 - (a^2+b^2), 2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2)$. Similarly it can be observed that $2\sum_{r<s} a_r a_s = \left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2$

Now, $\left(\sum_{i=1}^{n} a_i\right)^2 = \left[\frac{n}{2}(2a_1 + (n-1)d)\right]^2$

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \frac{n}{2}[4a_1^2 + 4a_1(n-1)d + (n-1)^2 d^2] \tag{2.4}$$

and $\sum_{i=1}^{n} a_i^2 = a_1^2 + (a_1+d)^2 + (a_1+2d)^2 + \cdots + [a_1 + (n-1)d]^2$

$$\sum_{i=1}^{n} a_i^2 = na_1^2 + a_1 dn(n-1) + \frac{d^2(n-1)n(2n-1)}{6} \tag{2.5}$$

Adding Eq. 2.4 and Eq. 2.5, we get the desired answer.

308. Let there be n rows in the equilateral triangle. Then $S = \frac{n(n+1)}{2}$. Now according to given facts, $\frac{n(n+1)}{2} + 669 = (n-8)^2 \Rightarrow n = 55 \Rightarrow S = 1540$.

309. Required sum $= \frac{(1+2+3+\cdots+n)^2 - (1^2+2^2+3^2+\cdots+n^2)}{2}$

$$= \frac{\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}}{2} = \frac{\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - \frac{2n+1}{3}\right)}{2} = \frac{1}{24}n(n^2-1)(3n+2).$$

310. Let a be the first term and d be the c.d. for the given A.P. Let S, S' represent the sum for first 24 days and last 18 days. Then,

$S = \frac{24}{2}[2a + 23d]$, $S' = \frac{18}{2}[2(a + 24d) + 17d]$ and

$\frac{24}{2}[2a+23d] + \frac{18}{2}[2a+65d] = \frac{42}{2}[2a+41d]$ and $S = S' \Rightarrow \frac{24}{2}[2a+23d] = \frac{18}{2}[2a+65d]$

Solving these two equations yield the answer as 12096.

311. Let a be the first term and d be the c.d. for the given A.P. Then,

$S_n = \frac{n}{2}[2a + (n-1)d] = n^2p$ and $S_m = \frac{m}{2}[2a + (m-1)d] = m^2p$

$\Rightarrow 2a + (n-1)d = 2np$ and $2a + (m-1)d = 2mp \Rightarrow (n-m)d = 2p(n-m) \Rightarrow d = 2p$

Substituting this in equation for S_n, $2a + 2(n-1)p = 2np \Rightarrow a = p$

$\Rightarrow S_p = \frac{p}{2}[2p + 2(p-1)p] = p^3$.

312. Let $S_1, S_2, \dots, S_n$ denote the sum of A.P. with c.d. $1, 2, \dots, n$. Then,

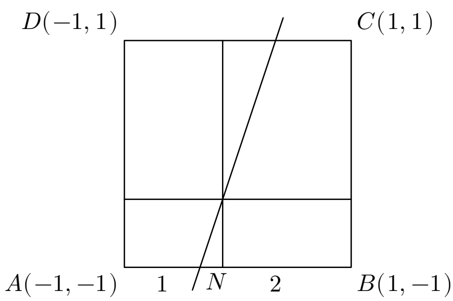
$t_r = 1 + (n-1)r$

$$S_1 + S_2 + \dots + S_n = \sum_{r=1}^{n} t_r = n + (n-1)\frac{n(n+1)}{2} = \frac{n}{2}(n^2+1).$$

313. $S_r = \frac{n}{2}[2r + (n-1)(2r-1)] = \frac{n}{2}[2r + 2rn - 2r - n + 1] = \frac{n}{2}[2rn - n + 1]$

$$S_1 + S_2 + \dots + S_m = \sum_{r=1}^{m} S_r = \frac{n^2m(m+1)}{2} - \frac{n(n-1)m}{2} = \frac{1}{2}[m^2n^2 + mn^2 - mn^2 + mn] =$$
$\frac{mn}{2}(mn+1)$.

314. Given below is the diagram for the problem:



Let the inclines straight line passing through origin cuts AB at N such that $AN : NB = 1 : 2$. Let the coordinates of A, B, C, D are $(-1,-1), (1,-1), (1,1), (-1,1)$. Then $N = (-1/3, -1)$. Thus equation of line would be $y = 3x$. Let (x_1, y_1) be the point from where we have drawn perpendiculars to the sides. Then length of $\perp$ to $AB = \frac{3x_1+1}{2}$, length of $\perp$ to $AD = x_1 + 1$, length of $\perp$ to $BC = \frac{3x_1-1}{2}$ and length of $\perp$ to $CD = x_1 - 1$. It is now trivial to observe that these lengths are in A.P.

315. Let p, b, h be the perpedicular, base, hypotenuse of the right angle triangle such that $b < p < h$ and r be the common ratio of the G.P. such that $r > 1$. Clearly $h^2 = p^2 + b^2 \Rightarrow b^2r^4 = b^2r^2 + b^2 \Rightarrow r^2 = \frac{1+\sqrt{5}}{2}$.

Clearly, the greater acute angle will be opposite to p which we let as θ, then

$\cos\theta = \frac{b}{h} = \frac{1}{r^2} = \frac{1}{1+\sqrt{5}}$.

316. Let $27, 8, 12$ be the pth, qth, kth terms respectively of a G.P. whose first term is a and common ratio is r then $27 = ar^{p-1}, 8 = ar^{q-1}, 12 = ar^{k-1}$.

$\Rightarrow \frac{27}{8} = r^{p-q} = \left(\frac{3}{2}\right)^3, \frac{12}{8} = r$ & $k - q = \frac{3}{2} \Rightarrow r^{p-q} = r^{3(k-q)} \Rightarrow p + 2q - 3k = 0.$

The system of solutions of this equation is $p = 4t, q = t, k = 2t$ where $t \in \mathbb{P}$.

317. Let $10, 11, 12$ be the pth, qth, kth terms respectively of a G.P. whose first term is a and common ratio is r then $10 = ar^{p-1}, 11 = ar^{q-1}, 12 = ar^{k-1}$.

$\Rightarrow \frac{11}{10} = r^{q-p}$ and $\frac{12}{11} = r^{k-q} \Rightarrow \left(\frac{11}{10}\right)^{k-q} = r^{(q-p)(k-q)}$ and $\left(\frac{12}{11}\right)^{q-p} == r^{(k-q)(q-p)}$

$\Rightarrow \left(\frac{11}{10}\right)^{k-q} = \left(\frac{12}{11}\right)^{q-p} \Rightarrow (11)^{k-q+q-p} = 10^{k-q}12^{q-p} = 5^{k-q}w^{k+q-2p}3^{q-p}$

This is possible only if $k - p = 0, k - q = 0, k + q - 2p = 0$ and $q - p = 0$ i.e. $p = q = k = 0$ which is not possible as they are distinct.

318. We have $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos(nx)dx, I_{n+1} = \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos[(n+1)x]dx$

$$I_{n+1} = \int_0^{\frac{\pi}{2}} \cos^n x[\cos x \cos[(n+1)x]dx$$

$\cos nx = \cos[(n+1)x - x] = \cos(n+1)x\cos x + \sin(n+1)x\sin x \Rightarrow \cos(n+1)x\cos x = \cos nx - \sin(n+1)x\sin x$

$$I_{n+1} = \int_0^{\frac{\pi}{2}} \cos^n x[\cos nx - \sin(n+1)x\sin x]dx = I_n - \int_0^{\frac{\pi}{2}} \cos^n x \sin x \sin(n+1)xdx$$

$= I_n + \left[\frac{\cos^{n+1} x\sin(n+1)x}{n+1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos^{n+1} x\cos(n+1)xdx$ [we take $u = \sin(n+1)x$ and $v = \cos^n x\sin x$]

$= I_n + 0 - 0 - I_{n+1} \Rightarrow \frac{I_{n+1}}{I_n} = 2$ and thus, $I_1, I_2, I_3, \ldots$ are in G.P.

319. $I_1, I_2, I_3, \ldots$ will be both in A.P. and G.P.if and only if $I_1 = I_2 = I_3 = \cdots = I_n$

$$I_{n+1} - I_n = \int_0^{\pi} \frac{\sin(2n+1)x}{\sin x}dx - \int_0^{\pi} \frac{\sin(2n-1)x}{\sin x}dx = \int_0^{\pi} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}dx$$

$$= \int_0^{\pi} \frac{2\cos 2nx\sin x}{\sin x}dx = 2\int_0^{\pi} \cos 2nxdx = \frac{2}{2n}[\sin 2nx]_0^{\pi} = 0$$

So $I_{n+1} = I_n$ also, $I_1 = \int_0^{\pi} \frac{\sin x}{\sin x}dx = \pi$. Hence, $I_1 = I_2 = I_3 = \cdots = I_n = \pi$ which proves that the terms are both in A.P. and G.P.

320. Let a, ar, ar^2 be the sides of the triangle. If $r > 1$ then from the properrties of the triangle we have $ar^2 < a + ar \Rightarrow r^2 - r - 1 < 0 \Rightarrow r < \frac{1+\sqrt{5}}{2}$. If $r < 1$ the the triangle will be formed if $ar + ar^2 < a \Rightarrow r^2 + r - 1 > 0 \Rightarrow r > \frac{-1+xs5}{2}$. Hence we have required inequality.

321. $111\ldots1(91 \text{ digits}) = 10^{90} + 10^{89} + \cdots + 10 + 1 = \frac{10^{91}-1}{10-1}$.

Since $91 = 13 \times 7$ we use 7 to multiply and divide with $10^7 - 1$ which gives us

$\frac{10^{91}-1}{10^7-1} \cdot \frac{10^7-1}{10-1} = (10^{84} + 10^{83} + \cdots + 10 + 1)(10^6 + 10^5 + \cdots + 10 + 1)$, which is a composite number.

322. $f(a+k) = f(a) + f(k) \because f(x+y) = f(x)f(y)\ \forall\ x, y \in \mathbb{N}$

$$\Rightarrow \sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)f(k) = f(a)[f(1) + f(2) + \cdots + f(n)]$$

Given, $f(1) = 2$, $f(2) = f(1) + f(1) = f(1)f(1) = 2^2$, $f(3) = f(1) + f(2) = f(1)f(2) = 2^3, \cdots, f(n) = 2^n$ and $f(a) = 2^a$

$\Rightarrow \sum_{k=1}^{n} f(a+k) = 16[2^n - 1] \Rightarrow 2^a[2 + 2^2 + \cdots + 2^n] = 2^a 2(2^n - 1) = 16(2^n - 1) \Rightarrow a = 3.$

323. Number of students giving wrong answers to at least i questions $= 2^{n-i}$.

Number of students giving wrong answers to at least $i+1$ questions $= 2^{n-i-1}$.

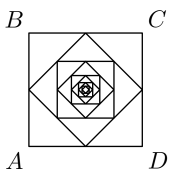
$\therefore$ Number of students giving wrong answers to exactly i questions $= 2^{n-i} - 2^{n-i-1}$. Also, total no. of students giving wrng answers to exactly n questions $= 2^{n-n} = 1$

$\therefore$ Total no. of wrong answers $= 1(2^{n-1} - 2^{n-2}) + 2.(2^{n-2} - 2^{n-3}) + \cdots + (n-1)(2^1 - 2^0) + n(2^0) = 2^{n-1} + 2^{n-2} + \cdots + 2^0 = 2^n - 1 = 2047 \Rightarrow n = 11.$

324. $S_1 = \frac{1}{1-\frac{1}{2}} = 2$, $S_2 = \frac{2}{1-\frac{1}{3}} = 3$, $S_3 = \frac{3}{1-\frac{1}{4}} = 4, \cdots$ and so on.

We have $S_1^2 + S_2^2 + \cdots + S_{2n-1}^2 = 2^2 + 3^2 + \cdots + (2n-1)^2 = 1^2 + 2^2 + 3^2 + \cdots + (2n)^2 - 1 = \frac{2n(2n+1)(4n+1)}{6} - 1 = \frac{n(n+1)(6n+1)}{3} - 1.$

325.



Let $ABCD$ be the first square and length of sides are a. Clearly, sides of second square $= \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \frac{a}{\sqrt{2}}$ $\therefore$. Area of second square $= \frac{a^2}{2}$. Area of third square $= \frac{a^2}{4}$ and so on.

Total area of innser squares $= \frac{\frac{a^2}{2}}{1-\frac{1}{2}} = a^2 =$ Sum of first square.

326. Let $y = 7 + 2x\log 25 - 5^{x-1} - 5^{2-x} \Rightarrow \frac{dy}{dx} = 4\log 5 - 5^{x-1}\log 5 + 5^{2-x}\log 5 = \frac{\log 5}{5^{x+1}}(5^x - 25)(5^x + 5)$

Now $y' > 0$ if $x > 2$ and $y' < 0$ if $x < 2$. Since y has only one local maxima at $x = 2$ and has no local minima, therefore y has greatest value at $x = 2 \Rightarrow a = 2$ which is first term of G.P.

$r = \lim_{x\to 0} \int_0^x \frac{t^2}{x^2\tan(\pi + x)}dt = \lim_{x\to 0} \frac{\int_0^x t^2 dt}{x^2 \tan x}$

$= \lim_{x\to 0} \frac{x^3}{3x^2\tan x} = \frac{1}{3}$ $\therefore \lim_{n\to\infty} \sum_{n=1}^{n} ar^{n-1} = \frac{2}{1-\frac{1}{3}} = 3.$

327. Let x be the first term and y be the common ratio of the G.P. Then $a = xy^{p-1}$, $b = xy^{q-1}$, $c = xy^{r-1}$

$(\log a).\vec{\imath} + (\log b)\vec{\jmath} + (\log c)\vec{k} = (\log x - 1).(\vec{\imath} + \vec{\jmath} + \vec{k}) + p\log y.\vec{\imath} + q\log y.\vec{\jmath} + r\log y.\vec{k}$

Dot products of given vectors $= (\log x - 1)(q - r + r - p + p - q) + \log y[p(q-r) + q(r-p) + r(p-q)] = 0$

And therefore the vectors are perpendicular to each other.

328. Pollution after first day $= 20(1 - .8) = 4\%$ and after second day $= 4(1 - .8) = .8$. Let us say that it takes n days then $20(1 - .8)^n < .01 \Rightarrow \frac{1}{5^n} < \frac{1}{2000} \Rightarrow 5^n > 2000 \Rightarrow n = 5$

329. Let the sides of the triangle are a, ar, ar^2 where $a > 0, r > 1$ then from properties of the triangle

$ar^2 < ar + a \Rightarrow r^2 - r - 1 < 0 \Rightarrow r = \frac{1 \pm \sqrt{5}}{2} \Rightarrow r > \frac{-1+\sqrt{5}}{2}$

Given that largest angle is twice the smallest one. $\Rightarrow \frac{a}{\sin\theta} = \frac{ar^2}{\sin 2\theta}$

$\Rightarrow 2\cos\theta = r^2 \Rightarrow r < \sqrt{2}$ so the range is $(1, \sqrt{2})$.

330. Let r be the common ratio then $b = ar, c = ar^2, d = ar^3$ then $\frac{ax^3 + arx^2 + ar^2x + ar^3}{ax^2 + ar^2} = x + r$ leaving no remainder thus given condition is satisfied.

331. Given, $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0 \Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \leq 0$

However, sum of squares cannot be less than zero. $\Rightarrow p = \frac{b}{a} = \frac{c}{b} == \frac{d}{c}$ thus a, b, c, d are in G.P. with common ratio p.

332. $\because \log_y x, \log_z y, \log_x z$ are in G.P. $\therefore \left(\frac{\log y}{\log z}\right)^2 = \frac{\log x}{\log y} \cdot \frac{\log z}{\log x} = \frac{\log z}{\log y} \Rightarrow \log y = \log z \Rightarrow y = z$

$2x^4 = 2y^4 \Rightarrow x = y$ and $xyz = 8 \Rightarrow x^3 = 8 \Rightarrow x = 2 \Rightarrow x = y = z = 2$.

333. If a, b, c, d are both in A.P. and G.P. then $a = b = c = d$ $\because b = 2$ $\therefore$ number of such sequences is 1.

334. We have $\log_x a, a^{x/2}, \log_b x$ are in G.P. $\therefore a^x = \log_x a \log_b x = \frac{\log a \log x}{\log x \log b} = \log_b a$

Taking log of both sides with base a, we get $x = \log_a(\log_b a)$.

335. Let a be the first term and r be the common ratio of the G.P. then

$t_{m+n} = ar^{m+n-1} = p$ and $t_{m-n} = ar^{m-n-1} = q$

Dividing $r^{2n} = \frac{p}{q} \Rightarrow r = \left(\frac{p}{q}\right)^{\frac{1}{2n}}$

$\Rightarrow a = p.r^{1-m-n} = p.\left(\frac{p}{q}\right)^{\frac{1-m-n}{2n}}$

$t_m = ar^{m-1} = p.\left(\frac{p}{q}\right)^{\frac{1-m-n}{2n}}.\left(\frac{p}{q}\right)^{\frac{m-1}{2n}} = p.\left(\frac{p}{q}\right)^{\frac{-n}{2n}} = \sqrt{pq}.$

$t_n = ar^{n-1} = p.\left(\frac{p}{q}\right)^{\frac{1-m-n}{2n}}.\left(\frac{p}{q}\right)^{\frac{n-1}{2n}} = p.\left(\frac{q}{p}\right)^{\frac{m}{2n}}.$

336. Let a be the first term and d be the c.d. of the A.P. then terms are $a+(p-1)d, a+(q-1)d, a+(r-1)d$, which are in G.P. Let $a+(p-1)d = x, a+(q-1)d = xy, a+(r-1)d = xy^2$ where x is the first term and y is the c.r. of the G.P.

$(p-q)d = x(1-r)$ and $(q-r) = xr(1-r)$. Dividing $r = \frac{q-r}{p-q}$.

337. Let a be the first term and r be the c.r. of the G.P. Then,

$S_1 = a + ar^2 + ar^4 + \cdots + ar^{2n-2} = \frac{a(r^{2n}-1)}{r^2-1}, S_2 = ar + ar^3 + \cdots + ar^{2n-1} = \frac{ar(ar^{2n}-1)}{r^2-1}$

Dividing $S_2/S_2 = r$, which is c.r. of the G.P.

338. $S_n = \frac{a(r^n-1)}{r-1} \Rightarrow rS_n = \frac{ar(r^n-1)}{r-1}$

$$\sum_{n=1}^{n} S_n = S_1 + S_2 + \cdots + S_n = \frac{a(r-1)}{r-1} + \frac{a(r^2-1)}{r-1} + \cdots + \frac{a(r^{n-1}-1)}{r-1}$$

$$(1-r)\sum_{n=1}^{n} S_n = a(1-r) + a(1-r^2) + \cdots + a(1-r^{n-1}) = na + \frac{ar(1-r^n)}{1-r}$$

$\Rightarrow rS_n + (1-r)\sum_{n=1}^{n} S_n = na.$

339. The series is $1 + x + xy + x^2y + x^2y^2 + \cdots = [1 + xy + x^2y^2 + \cdots] + x[1 + xy + x^2y^2 + \cdots]$

$= \frac{(x^ny^n-1)}{xy-1} + \frac{x(x^ny^n-1)}{xy-1} = \frac{(x^ny^n-1)(1+x)}{xy-1}.$

340. $49 = (4\times 10) + 9, 4489 = (4\times 10^3 + 4\times 10^2) + (8\times 10) + 9$ and so on.

$t_k = 4\frac{10^k-1}{9}.10^k + 8.\frac{10^k-1}{9} + 1 = 4\frac{10^k-1}{9}10^k - 4\frac{10^k-1}{9} + 12\frac{10^k-1}{9} + 1$

$= 36\frac{10^{2k}-2.10^k+1}{81} + 12\frac{10^k-1}{9} + 1 = \left(6\frac{10^k-1}{9}+1\right)^2.$

341. $S_m = a + ar + ar^2 + \cdots + ar^{m-1} = \frac{a(r^m-1)}{r-1}$. Let S be required sum then

$S = \frac{(\sum a_i)^2 - \sum a_i^2}{2} = \frac{\left(\frac{a(r^m-1)}{r-1}\right)^2 - [a^2 + a^2r^2 + \cdots + a^2r^{2(m-1)}]}{2}$

$2S = \frac{a^2(r^m-1)}{r-1}\left[\frac{r^m-1}{r-1} - \frac{r^m+1}{r+1}\right] = \frac{r}{r+1}.\frac{a(r^m-1)}{r-1}.\frac{a(r^{m-1}-1)}{r-1} = \frac{r}{r+1}S_mS_{m-1}.$

342. $y = \log_{10} x + \log_{10}(x)^{\frac{1}{2}} + \log_{10}(x)^{\frac{1}{4}} + \cdots = \log_{10} x + \frac{1}{2}\log_{10} x + \frac{1}{4}\log_{10} x + \cdots$

$y = \frac{\log_{10} x}{1-\frac{1}{2}} = 2\log_{10} x$

$\frac{1+3+5+(2y-1)}{4+7+10+\cdots+3y+1} = \frac{20}{7\log_{10} x} \Rightarrow \frac{y^2}{\frac{y}{2}[8+(y-1).3]} = \frac{40}{7y}$

$\Rightarrow y = 10, x = 10^5.$

343. Let $a = a_1$ be the first term and r to be the common ratio of the G.P., then

$$S = \frac{a(r^n-1)}{r-1}, P = a^n r^{1+2+\cdots+(n-1)} = a^n r^{\frac{n(n-1)}{2}}, T = \frac{1}{a}\cdot\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}} = \frac{1}{a}\cdot\frac{r^n-1}{r-1}\cdot\frac{1}{r^{n-1}}$$

Clearly, $P^2 = \left(\frac{S}{T}\right)^n$.

344. Let x be the first term and y be the c.r. of the G.P. Then $a = xy^{n-1}$. The next n terms will start from $xy^n \Rightarrow b = xy^n.y^{n-1}$ and similarlry $c = xy^{2n}y^{n-1}$

It is clear that $b^2 = ac$ i.e. a, b, c are in G.P.

345. $S_1 = a = \frac{a(1-r)}{1-r}, S_2 = \frac{a(1-r^2)}{1-r}, \cdots, S_n = \frac{a(1-r^n)}{1-r}$

$S_1 + S_2 + \cdots + S_n = \frac{a}{1-r}[1 + 1 + \cdots + \text{ to } n \text{ terms}] - \frac{ar}{1-r}[1 + r + r^2 + \cdots + r^{n-1}]$

$= \frac{na}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}.$

346. $S_1 = a = \frac{a(1-r)}{1-r}, S_3 = \frac{a(1-r^3)}{1-r}, \cdots, S_{2n-1} = \frac{a(1-r^{2n-1}}{1-r}$

$S_1 + S_3 + \cdots + S_{2n-1} = \frac{a}{1-r}[1 + 1 + \cdots + \text{ to } n \text{ terms}] - \frac{ar}{1-r^2}[1 + r^2 + r^4 + \cdots + r^{2(n-1)}]$

$= \frac{na}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}.$

347. Let a be the first term and r be the common ratio. Then,

$s = \frac{a}{1-r}, \sigma = \frac{a^2}{1-r^2}, S_n = \frac{a(1-r^n)}{1-r}$

$$s\left[1 - \left(\frac{s^2-\sigma^2}{s^2+\sigma^2}\right)^n\right] = \frac{a}{1-r}\left[1 - \left(\frac{\frac{a^2}{(1-r)^2} - \frac{a^2}{1-r^2}}{\frac{a^2}{(1-r)^2} + \frac{a^2}{1-r^2}}\right)^n\right] = \frac{a}{1-r}\left[1 - \left(\frac{\frac{1}{1-r} - \frac{1}{1+r}}{\frac{1}{1-r} + \frac{1}{1+r}}\right)^n\right].$$

$= \frac{a(1-r^n)}{1-r} = S_n.$

348. $\sum_{i<j} a_i a_j = \frac{1}{2}[(a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)]$

$= \frac{1}{2}\left[(a + ar + \ldots + ar^{n-1})^2 - (a^2 + a^2r^2 + \ldots + a^2r^{2(n-1)})\right]$

$$= \frac{1}{2}\left[\frac{a^2(1-r^n)^2}{(1-r)^2 - \frac{a^2(1-r^{2n})}{1-r^2}}\right] = \frac{1}{2}\left[\frac{a^2(1-2r^n+r^{2n})}{(1-r)^2} - \frac{a^2(1-r^{2n})}{1-r^2}\right] = \frac{a^2r(1-r^{n-1})(1-r^n)}{(1-r)^2(1+r)}$$

349. Let a be the first term and r be the common ratio. Then,

$$\text{L.H.S.} = \frac{1}{a^2-a^2r^2} + \frac{1}{a^2r^2-a^2r^4} + \frac{1}{a^2r^4-a^2r^6} + \ldots + \frac{1}{a^2r^{2(n-2)}-a^2r^{2(n-1)}}$$

$$= \frac{1}{a^2(1-r^2)}\left[1 + \frac{1}{r^2} + \frac{1}{r^4} + \ldots + \frac{1}{r^{2(n-2)}}\right] = \frac{1}{a^2(1-r^2)}.\frac{1-\frac{1}{r^{2(n-1)}}}{1-\frac{1}{r^2}} = \frac{1}{a^2(1-r^2)}.\frac{1-r^{2n-2}}{1-r^2}.\frac{r^2}{r^{2n-2}}.$$

350. Let a be the first term and r be the common ratio. Then,

$$\text{L.H.S.} = \frac{1}{a^m+a^mr^m} + \frac{1}{a^mr^m+a^mr^{2m}} + \ldots + \frac{1}{a^mr^{m(n-2)}+a^mr^{m(n-1)}}$$

$$= \frac{1}{a^m(1+r^m)}\left[1 + \frac{1}{r^m} + \frac{1}{r^{2m}} + \ldots + \frac{1}{r^{m(n-2)}}\right] = \frac{1}{a^m(1+r^m)}.\frac{1-\frac{1}{r^{m(n-1)}}}{1-\frac{1}{r^m}} = \frac{r^{mn-m}-1}{a^m(1+r^m)(r^{mn-m}-r^{mn-2m})}.$$

351. Let a be the first term and r be the common ratio. Then,

$$\text{L.H.S.} = \sqrt{a^2r} + \sqrt{a^2r^5} + \sqrt{a^2r^9} + \ldots + \sqrt{a^2r^{4n-3}} = a\sqrt{r}(1+r^2+r^4+\ldots+r^{2(n-1)}) = a\sqrt{r}.\frac{(r^{2n-1})}{r^2-1}$$

$$\sqrt{a_1 + a_3 + \ldots + a_{2n-1}} = \sqrt{a(1+r^2+\ldots+r^{2n-2})} = \sqrt{a.\frac{r^{2n-1}}{r^2-1}}$$

$$\sqrt{a_2 + a_4 + \ldots + a_{2n}} = \sqrt{ar(1+r^2+\ldots+r^{2n-2})} = \sqrt{a\sqrt{r}.\frac{r^{2n-1}}{r^2-1}}$$

$$\therefore \sqrt{a_1a_2} + \sqrt{a_3a_4} + \sqrt{a_5a_6} + \ldots + \sqrt{a_{2n-1}a_{2n}} = \sqrt{a_1 + a_3 + \ldots + a_{2n-1}}\sqrt{a_2 + a_4 + \ldots + a_{2n}}.$$

352. Given $1 + x + x^2 + \ldots + x^{23} = 0, 1 + x + x^2 + \ldots + x^{19} = 0$

$\frac{x^{24}-1}{x-1} = 0, \frac{x^{20}-1}{x-1} = 0 \Rightarrow x^{24} - 1 = 0, x^{20} - 1 = 0 \therefore x^{20}.x^4 - 1 = 0 \Rightarrow x^4 - 1 = 0$

Thus, roots are $-1, \pm i$.

353. \$$a$ will become $a + r.(a) = a(1+r)$ at the end of second year, $a + ar + r(a+ar) = a + 2ar + ar^2 = a(1+r)^2$ at the end of third year, $a + 2ar + ar^2 + r(a+2ar+ar^2) = a + 3ar + 3ar^2 + ar^3 = a(1+r)^3$ and so on. So amount received for \$$a$ will be $a(1+r)^{n+1}$

Similarly, amount receoved for \$$2a$ will be $2a(1+r)^n$ and so on.

Thus, total amount received will be $S = a(1+r)^{n+1} + 2a(1+r)^n + 3a(1+r)^{n-1} + \ldots + na(1+r)$

$\frac{S}{1+r} = a(1+r)^n + 2a(1+r)^{n-1} + \ldots + (n-1)(1+r) + na$

Writing first term of second sum against second term of first sum, second term of second sum against third term of first sum and so on and subtracting, we get $\frac{rS}{1+r} = a(1+r)^{n+1} + a(1+r)^n + a(1+r)^{n-1} + \ldots + a(1+r) - na$

$\frac{rS}{1+r} = a(1+r)[(1+r)^n + (1+r)^{n-1} + \ldots + 1]) - na$

$S = \frac{a(1+r)^2[(1+r)^n-1]}{r^2} - \frac{na(1+r)}{r}$.

354. $\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right) = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2} \Rightarrow (0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right)} = \left(\frac{4}{25}\right)^{\frac{\log_5 \frac{1}{2}}{2}} = \left(\frac{1}{2}\right)^{\frac{\log_5 \frac{4}{25}}{2}} = \left(\frac{1}{2}\right)^{-2} = 4.$

355. $A = 1 + r^a + r^{2a} + \ldots$ to $\infty = \frac{1}{1-r^a} \Rightarrow r = \left(\frac{A-1}{A}\right)^{\frac{1}{a}}$

$B = 1 + r^b + r^{2b} + \ldots$ to $\infty = \frac{1}{1-r^b} \Rightarrow r = \left(\frac{B-1}{B}\right)^{\frac{1}{b}}$.

356. $s_1 = \frac{1}{1-\frac{1}{2}} = 2, s_2 = \frac{2}{1-\frac{1}{3}} = 3, \ldots, s_n = \frac{n}{1-\frac{1}{n+1}} = n+1$

$s_1 + s_2 + \ldots + s_n = 2 + 3 + \ldots + (n+1) = \frac{1}{2}n(n+3)$.

357. $S_1 = \frac{1}{1-\frac{1}{2}} = 2, S_2 = \frac{2}{1-\frac{1}{3}} = 3, \ldots S_n = \frac{n}{1-\frac{1}{n+1}} = n+1$

General term of numerator $t_i = S_iS_{n-i+1} = (i+1)(n-i+2) = (n+1)i - i^2 + (n+1)$

$\therefore$ Sum for numerator $= \sum_{i=1}^{n} t_i = \sum_{i=1}^{n}[(n+1)i - i^2 + (n+1)] = \frac{n(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{6} + n(n+1)$

Sum for denominator $= 1^2 + 2^2 + \ldots + (n+1)^2 - 1 = \frac{(n+1)(n+2)(2n+3)}{6} - 1$

Upon simplification $\lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + \ldots + S_nS_1}{S_1^2 + S_2^2 + \ldots + S_n^2} = \frac{1}{2}$.

358. $f'(x) = 3x^2 + 3$ which yields imaginary roots implying that there is no local maxima. However, $3x^2 + 3$ is positive for all values of x which means that $f(x)$ is monotonically increasing in $[-5, 3]$ implying that maximum value will be at $x = 3$

$f(3) = 27$, also let a to be the first term and r to be the common ratio then given, $a - ar = f'(0) = 3$. The sum is given as $\frac{a}{1-r} = 27$ solving these yields $r = \frac{2}{3}, -\frac{4}{3}$ but the series is decreasing so $r = \frac{2}{3}$.

359. Let $S = \frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \ldots\infty$

$$= \frac{5}{9}\left[\frac{10-1}{13}+\frac{100-1}{13^2}+\frac{1000-1}{13^3}+\ldots\infty\right] = \frac{5}{9}\left[\frac{10}{13}+\frac{10^2}{13^2}+\frac{10^3}{13^3}+\ldots\infty-\frac{1}{13}-\frac{1}{13^2}-\frac{1}{13^3}-\ldots\infty\right]$$

$$= \frac{5}{9}\left[\frac{\frac{10}{13}}{1-\frac{10}{13}}-\frac{\frac{1}{13}}{1-\frac{1}{13}}\right] = \frac{5}{9}\left[\frac{10}{13}.\frac{13}{3}-\frac{1}{13}.\frac{13}{12}\right] = \frac{65}{36}$$

360. $S = \cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + \ldots$

$$= \frac{\cos x}{1-\frac{2}{3}\sin^2 x} = \frac{3\cos x}{3-2\sin^2 x} = \frac{3\cos x}{2+\cos 2x}$$

The term $\frac{3\cos x}{2+\cos 2x}$ is finite for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

361. Let a be the first term, b be the last term and n be the number of terms of A.P. and G.P.

Then c.d. of A.P. $= \frac{b-a}{n-1}$ and c.r. of the G.P. $= \left(\frac{b}{a}\right)^{n-1}$. Let S be the sum of n terms of A.P. and S' the sum of n terms of G.P. then $S = \frac{n}{2}(a+b)$

$S' = a(1+r+r^2+\ldots+r^{n-1}),\ S' = a(r^{n-1}+r^{n-2}+\ldots+1)$

$\therefore S' = \frac{a}{2}[(1+r^{n-1})+(r+r^{n-2})+(r^k+r^{n-k-1})+\ldots+(r^{n-1}+1)]$

Now, $(r^k+r^{n-k-1})-(r^{n-1}+1) = (r^k-1)+r^{n-1}(r^{-k}-1)$

$= (r^k-1)\left(1-\frac{r^{n-1}}{r^k}\right) = (r^k-1)(1-r^{n-k-1}) \le 0$

$\therefore S' \le \frac{an}{2}(1+r^{n-1}) = \frac{an}{2}\left(1+\frac{b}{a}\right) = \left(\frac{a+b}{2}\right)n = S$

$\therefore S \ge S'$.

362. Given $a, a_1, a_2, a_3, \ldots$ are in G.P. so $\log a, \log a_1, \log a_2, \ldots$ are in A.P. Let the common difference of this A.P. be d_1. Now $\log a_n = \log a + nd_1$. Further if d be the common difference of the A.P. $b, b_1, b_2, \ldots$ then $b_n = b + nd$

$\therefore \frac{\log a_n - \log a}{b_n - b} = \frac{nd_1}{nd} = \frac{d_1}{d}$

Let $\log x = \frac{d_1}{d}$ for a fixed positive real number x.

$\Rightarrow \frac{\log a_n - \log a}{b_n - b} = \log x \Rightarrow b_n - b = \log_x\left(\frac{a_n}{a}\right) \Rightarrow \log_x a_n - \log_x a = b_n - b \Rightarrow \log_x a_n - b_n = \log_x a - b$

363. Given $a+md, a+nd, a+rd$ are in G.P., where a is the first term and d is the c.d. of A.P.

$\Rightarrow (a+nd)^2 = (a+md)(a+rd) \Rightarrow d(n^2d+2an) = d(am+ar+mrd) \Rightarrow (n^2-mr)d = a(m+r-rn)$

$\frac{d}{a} = \frac{m+r-2n}{n^2-mr}$

Given, m, n, r are in H.P. $\therefore n = \frac{2mr}{m+r} \Rightarrow m + r = \frac{2mr}{n}$

$\therefore \frac{d}{a} = \frac{\frac{2mr}{n} - 2n}{n^2 - mr} = -\frac{2}{n} \therefore \frac{a}{d} = -\frac{n}{2}$

364. Let r be the common ratio of the G.P., then $b = ar, c = ar^2$. Given, $a - b, c - a, b - c$ are in H.P.

$\therefore c - a = \frac{2(a-b)(b-c)}{a-b+b-c}$

$(c-a)^2 = 2(a-b)(b-c) \Rightarrow (ar^2 - a)^2 = 2(a - ar)(ar - ar^2)$

$a^2(r^2-1)^2 = -2a^2(1-r)r(1-r) \Rightarrow (r+1)^2 = -2r \Rightarrow 1 + 4r + r^2 = 0$

$\Rightarrow a + 4ar + ar^2 = 0 \Rightarrow a + 4b + c = 0.$

365. Let d_1, d_2, d_3 be the common differences of the A.P.'s.

$\Rightarrow S_1 = \frac{n}{2}[2 + (n-1)d_1] \Rightarrow d = \frac{2(S_1-n)}{n(n-1)}$

Similalrly $d_2 = \frac{2(S_2-n)}{n(n-1)}, d_3 = \frac{2(S_3-n)}{n(n-1)}$

$\because d_1, d_2, d_3$ are in H.P. $\therefore \frac{1}{d_2} - \frac{1}{d_1} = \frac{1}{d_3} - \frac{1}{d_2}$

$\Rightarrow \frac{n(n-1)}{2(S_2-n)} - \frac{n(n-1)}{2(S_1-n)} = \frac{n(n-1)}{2(S_3-n)} - \frac{n(n-1)}{2(S_2-n)}$

$\Rightarrow \frac{1}{S_2-n} - \frac{1}{S_1-n} = \frac{1}{S_3-n} - \frac{1}{S_2-n} \Rightarrow \frac{S_1-S_2}{(S_1-n)(S_2-n)} = \frac{S_2-S_3}{(S_3-n)(S_2-n)}$

$\Rightarrow n = \frac{2S_3S_1 - S_1S_2 - S_2S_3}{S_1 - 2S_2 + S_3}.$

366. Let the digits at hundreds, tens and units places be a, ar and ar^2 and the required number be x, then $x = 100a + 10a + ar^2$

Let $y = x - 400 \Rightarrow y = 100(a-4) + 1 - ar + ar^2$ In the number y, the digit at hundreds place is $a - 4$. Clearly

$1 \leq a - 4 \leq 5$ [$\because 1 \leq a \leq 9$ and $a - 4 \geq 1$] $\Rightarrow 5 \leq a \leq 9$

According to question $a - 4, ar, ar^2$ are in A.P. $\therefore 2ar = a - 4 + ar^2 \Rightarrow a(r-1)^2 = 4 \Rightarrow r - 1 = \pm\frac{2}{\sqrt{a}}$

$\because a$ and ar are integers. $\therefore r$ is a rational number. Thus, a must be a perfect square. $\therefore a = 9$

Thus, $r = \frac{5}{3}, \frac{1}{3}$ but $r \neq \frac{5}{3}$ othereise $ar = 15 \therefore r = \frac{1}{3} \therefore ar = 3, ar^2 = 1$

Hence required number is 931.

367. Given a, b, c are in G.P. Let r be the common ratio of this G.P. then $b = ar$ and $c = ar^2$.

Given, $\log_c a, \log_b c, \log_a b$ are in A.P.

$\Rightarrow \frac{\log a}{\log c}, \frac{\log c}{\log b}, \frac{\log b}{\log a}$ are in A.P.

$\Rightarrow \frac{\log a}{\log a+2\log r}, \frac{\log a+2\log r}{\log a+\log r}, \frac{\log a+\log r}{\log a}$ are in A.P.

$\frac{1}{1+2x}, \frac{1+2x}{1+x}, 1+x$ are in A.P. where $\frac{\log r}{\log a} = x$

$2\left(\frac{1+2x}{1+x} = \frac{1}{1+2x} + 1 + x\right) \Rightarrow x(2x^2 - 3x - 3) = 0$

$2x^2 - 3x - 3 = 0 [\because x \neq 0, \text{else } \log r = 0 \Rightarrow r = 1 \text{ which is not possible as } a, b, c \text{ are distinct}]$

$2d = 1 + x - \frac{1}{1+2x} = \frac{2x^2+3x}{1+2x} = \frac{3x+3+3x}{1+2x} = 3 \Rightarrow d = \frac{3}{2}.$

368. Let the two numbers be a and b. Since n A.M.'s have been inserted between a and b $\therefore$ common difference of A.P., $d = \frac{b-a}{n+1}$

Now $p =$ first A.M. $=$ 2nd term of A.P. $= a + d = \frac{an+b}{n+1}$

Similarly for harmonic series $q = \frac{ab(n+1)}{bn+a}$

We know that x will not lie between α and β if $(x-\alpha)(x-\beta) > 0$

$q - p = -\frac{n(a-b)^2}{(bn+a)(n+1)}$

$q - \left(\frac{n+1}{n-1}\right)^2 p = -\frac{(n+1)(a+b)^2 n}{(n-1)^2(bn+a)}$

$\Rightarrow (q-p)\left[q - \left(\frac{n+1}{n-1}\right)^2 p\right] = \frac{n^2(a-b)^2(a+b)^2}{(n-1)^2(bn+a)^2} > 0.$

369. Common difference of A.P. $= q - p$ and common ratio of G.P. $= \frac{q}{p} < 1$

$s = \frac{p}{1-\frac{q}{p}} = \frac{p^2}{p-q}$. Let S_n be the sum of n terms of A.P., then

$S_n = \frac{n}{2}[2p + (n-1)d] = np + \frac{n(n-1)d}{2} = np + \frac{n(n-1)(q-p)p^2}{2p^2} = np - \frac{n(n-1)}{2} \cdot \frac{p^2}{s}.$

370. $\because \log_x y, \log_z x, \log_y z$ are in G.P.

$\Rightarrow (\log_z x)^2 = \log_x y . \log_y z \Rightarrow \left(\frac{\log x}{\log z}\right)^2 = \frac{\log y}{\log x} \cdot \frac{logz}{\log y}$

$\Rightarrow (\log x)^3 = (\log z)^3 \Rightarrow x = z \Rightarrow x = y = z = 4 \because xyz = 64$ and $2y^3 = x^3 + z^3$.

371. $2(x+2y) = x+2x+y \Rightarrow 3y = x, (xy+5)^2 = (y+1)^2(x+1)^2 \Rightarrow (3y^2+5) = \pm(y+1)(3y+1)$

$\Rightarrow y = 1, \frac{-1\pm 2\sqrt{2}i}{3}, x = 3, -1\pm 2\sqrt{2}i.$

372. Let $a = 3$ be the first term and d be the common difference of the G.P. then, given

$(a+9d)^2 = (a+d)(a+33d) \Rightarrow a^2+18ad+81d^2 = a^2+34ad++33d^2 \Rightarrow d = \frac{a}{3} = 1$

So the A.P. is 3, 4, 5, ...

373. Given, $\sqrt{ab} = \sqrt{cd}, \frac{a^2+b^2}{2} = \frac{c^2+d^2}{2} \Rightarrow ab = cd, a^2+b^2 = c^2+d^2$

$\Rightarrow (a-b)^2 = (c-d)^2, (a+b)^2 = (c+d)^2 \Rightarrow a = c, b = d$

Thus, arithmetic mean of a^n and b^n is equal to the arithmetic mean of c^n and d^n for every integral value of n.

374. Let a be the first term and d be the common difference of A.P. and thus d will be the first term and a be the common ratio of the G.P. Given,

$155 = \frac{10}{2}[2a+(10-1)d] \Rightarrow 2a+9d = 31$

$d+ad = 9 \Rightarrow a = \frac{25}{2}, 2 \Rightarrow d = \frac{2}{3}, 3$

Thus, A.P. is 2, 5, 8, ... or $\frac{25}{2}, \frac{79}{6}, \frac{83}{6}$, ... and the G.P. is 3, 6, 12, ... or $\frac{2}{3}, \frac{25}{3}, \frac{625}{6}$, ...

375. Since a, b, c are in H.P. therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow \frac{2}{b} = \frac{1}{a}+\frac{1}{c} \Rightarrow b = \frac{2ac}{a+c} \Rightarrow \frac{3}{b}-\frac{2}{c} = \frac{1}{a}+\frac{1}{b}-\frac{1}{c}$ and $\frac{3}{b}-\frac{2}{a} = \frac{1}{b}+\frac{1}{c}-\frac{1}{a}$

$\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right) = \left(\frac{3}{b}-\frac{2}{c}\right)\left(\frac{3}{b}-\frac{2}{a}\right)$

$= \frac{9ac-6ab-6bc+4b^2}{acb^2} = \frac{4}{ac}+\frac{9}{b^2}-\frac{6b(a+c)}{acb^2}$

$= \frac{4}{ac}+\frac{9}{b^2}-\frac{6b}{acb^2}.\frac{2}{b} = \frac{4}{ac}-\frac{3}{b^2}.$

376. Because a, b, c are in H.P. therefore $\frac{2}{b} = \frac{1}{a}+\frac{1}{c}$

$\frac{a+b}{2a-b}+\frac{b+c}{2c-b} = \frac{\frac{1}{b}+\frac{1}{a}}{\frac{2}{b}-\frac{1}{a}}+\frac{\frac{1}{b}+\frac{1}{c}}{\frac{2}{b}-\frac{1}{c}} = \frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c} = \frac{c^2+a^2}{ac}+\frac{a+c}{b}$

$= \frac{c^2+a^2}{ac}+\frac{(a+c)^2}{2ac} = \frac{c^2+a^2}{ac}-2+\frac{(a+c)^2}{2ac}-2+4 = \frac{(c-a)^2}{ac}+\frac{(a-c)^2}{2ac}+4 \geq 4.$

377. $b-\frac{a+b}{1-ab} = \frac{b+c}{1-bc}-b \Rightarrow \frac{b-ab^2-a-b}{1-ab} = \frac{b+c-b+b^2c}{1-bc}$

$\Rightarrow \frac{-a(1+b^2)}{1-ab} = \frac{c(1+b^2)}{1-bc} \Rightarrow -a(1-bc) = c(1-ab) \Rightarrow a+c = 2abc \Rightarrow 2b = \frac{a+c}{ac}$

$\therefore a, b^{-1}, c$ are in H.P.

378. $x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$

a, b, c are in A.P. $\Rightarrow 1-a, 1-b, 1-c$ are in A.P.

$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in H.P. $\Rightarrow x, y, z$ are in H.P.

379. Let $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k \Rightarrow a = k^x, b = k^y, c = k^z$

$\because a, b, c$ are in G.P. $\Rightarrow b^2 = ac \Rightarrow k^{2y} = k^{x+z} \Rightarrow 2y = x+z$

$\therefore x, y, z$ are in A.P.

380. $2b = a+c, m = \frac{2ln}{l+n}, b^2m^2 = acln \Rightarrow \left(\frac{a+c}{2}.\frac{2ln}{l+n}\right)^2 = acln$

$\Rightarrow \frac{ln}{(l+n)^2} = \frac{ac}{(a+c)^2} \Rightarrow \frac{(a+c)^2}{ac} = \frac{(l+n)^2}{ln}$

$\Rightarrow \frac{a}{c} + \frac{c}{a} = \frac{l}{n} + \frac{n}{l} \Rightarrow a : c = \frac{1}{n} : \frac{1}{l}$

Now it can be proven that $a : b : c = \frac{1}{n} : \frac{1}{m} : \frac{1}{l}$.

381. The common difference of A.P. $= b-a$, common ratio of G.P. is b/a and common difference for corresponding A.P. of H.P. is $(a-b)/ab$

$n+2$th term of A.P. $= a + (n+1)(b-a) = (n+1)b - na$

$n+2$th term of G.P. $= ar^{n+1} = \frac{b^{n+1}}{a^n}$

$n+2$th term of H.P. $= \frac{1}{\frac{1}{a} + \frac{(n+1)(a-b)}{ab}} = \frac{ab}{(n+1)a-nb}$

These will be in G.P. if

$\frac{[(n+1)b-na]ab}{(n+1)a-nb} = \frac{b^{2n+2}}{a^{2n}} \Rightarrow (n+1)a^{2n+1}b^2 - na^{2n+2}b = (n+1)ab^{2n+2} - nb.b^{2n+2}$

$\Rightarrow (n+1)ab^2[a^{2n} - b^{2n}] = nb[a^{2n+2} - b^{2n+2}] \Rightarrow \frac{b^{2n+2}-a^{2n+2}}{ab(b^{2n}-a^{2n})} = \frac{n+1}{n}$.

382. $ar^n - a - nd = a\left(1+\frac{d}{a}\right)^n - a - nd\left[\because r = \frac{a+d}{a}\right]$

$= a\left[1 +^n C_1\left(\frac{d}{a}\right) +^n C_2\left(\frac{d}{a}\right)^2 + ... +^n C_n\left(\frac{d}{a}\right)^n\right] - a - nd$

$= a\left[^nC_2\frac{d^2}{a^2} +^n C_3\frac{d^3}{a^3} + ... +^n C_n\frac{d^n}{a^n}\right] > 0\left(\because \frac{d}{a} > 0\right)$.

383. $A = \frac{a+b}{2}, H = \frac{2ab}{a+b}, G = \sqrt{ab} \Rightarrow A = kH \Rightarrow (a+b)^2 = 4kab \Rightarrow A = kG^2$

Let $b = ma \Rightarrow a^2(1+m^2) = 4kma^2 \Rightarrow 1 + m^2 = 4km \Rightarrow m = \frac{4k \pm \sqrt{16k^2-4}}{2} = 2k \pm \sqrt{4k^2-1}$

Also, $(a+b)^2 = 4kab \Rightarrow (a-b)^2 = 4kab - 4ab \because (a-b)^2 \geq 0 \therefore k \geq 1.$

384. Since n means are inserted therefore total no. of terms will be $n+2$. Let d be the c.d. of A.P. and d' be the c.d of H.P.

$\Rightarrow d = \frac{b-a}{n+1}, d' = \frac{a-b}{(n+1)ab} \Rightarrow p = a + rd = \frac{(n+1)a+r(b-a)}{n+1}, \frac{1}{q} = \frac{1}{a} + r\frac{a-b}{(n+1)ab} \Rightarrow q = \frac{(n+1)ab}{r(a-b)+(n+1)b}$

$\frac{p}{a} + \frac{b}{q} = \frac{(n+1)a+r(b-a)}{a(n+1)} + \frac{r(a-b)+(n+1)b}{(n+1)a} = \frac{a+b}{a}$ which is independent of n and r.

385. Let s be the distance between P and Q.

Time taken by train $A = \frac{s}{2x} + \frac{s}{2y} = \frac{s(x+y)}{2xy} = \frac{s}{\text{H.M of } x \text{ and } y}$

Time taken by train $B = \frac{2s}{x+y} = \frac{s}{\text{A.M of } x \text{ and } y}$

So, second train wil reach earlier as A.M. $\geq$ H.M.

386. Let d be the common difference of corresponding A.P. Also, let H_1 and H_n be first and last H.M.

$\Rightarrow d = \frac{\frac{1}{c}-\frac{1}{a}}{n+1} = \frac{ac}{ac(n+1)}$

$\frac{1}{H_1} = \frac{1}{a} + \frac{a-c}{ac(n+1)} \Rightarrow H_1 = \frac{ac(n+1)}{nc+a}$

$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-n)}{ac(n+1)} \Rightarrow H_n = \frac{ac(n+1)}{na+c}$

$H_1 - H_n = \frac{ac(n+1)}{nc+a} - \frac{ac(n+1)}{na+c} = \frac{ac(n^2-1)(a-c)}{(n^1+1)ac+n(a^2+c^2)}$

Also, given that n is a root of equation $x^2(1-ac) - x(a^2+c^2) - (1+ac) = 0$

$\therefore n^2(1-ac) - n(a^2+c^2) - 1 - ac = 0 \Rightarrow n^2 - 1 = (n^2+1)ac + n(a^2+c^2) \therefore H_1 - H_n = ac(a-c).$

387. Let d be the common difference for A.P. and d' be the common difference for H.P., then

$d = \frac{b-a}{n+1}, d' = \frac{\frac{1}{b}-\frac{1}{a}}{n+1} = \frac{a-b}{(n+1)ab}$

$A_r = a + rd = a + \frac{r(b-a)}{n+1} = \frac{(n-r+1)a+rb}{n+1}$

$\frac{1}{H_{n-r+1}} = \frac{1}{a} + \frac{(n-r+1)(a-b)}{(n+1)ab} = \frac{(n-r+1)a+rb}{(n+1)ab}$

$\Rightarrow H_{n-r+1} = \frac{(n+1)ab}{(n-r+1)a+rb} \Rightarrow A_r H_{n-r+1} = ab.$

388. Consider the equation $(x-1)(x-2)(x-3)\ldots(x-100) = 0$. Its roots are $1, 2, 3, \ldots, 100$

So the equation is a polynomial of x of degree 100. Coefficient of $x^{100} = 1$

Now sum of roots of equation taken one at a time

$1+2+3+\ldots+100 = (-1)^1 \frac{\text{coeff. of } x^{99}}{\text{coeff. of } x^{100}} = -\text{coeff. of } x^{99}$

$\therefore \text{coeff. of } x^{99} = -(1+2+3+\ldots+100) = -5050$

Sum of products of roots taken two at a time $=$ coeff. of $x^{98} = \frac{1}{2}[(1+2+3+\ldots+100)^2 - (1^2+2^2+\ldots+100^2)]$

$= \frac{1}{2}\left[5050^2 - \frac{100\times101\times102}{6}\right] = 12582075.$

389. $t_1 = 12, 40, 90, 168, 280, 432, \ldots$ $\Delta t_1 = 28, 50, 78, 112, 152, \ldots,$ $\Delta^2 t_1 = 22, 28, 34, 40, \ldots, \Delta^3 t_1 = 6, 6, 6, \ldots$

$t_n = 12 + 28^{n-1}C_1 + 22.^{n-1}C_2 + 6.^{n-1}C_3$

$S_n = \sum_{n=1}^{n}(12 + 28^{n-1}C_1 + 22.^{n-1}C_2 + 6.^{n-1}C_3)$

$S_n = 12n + 28.^nC_2 + 22.^nC_3 + 6.^nC_4$

$= 12n + 28.\frac{n(n-1)}{2!} + 22.\frac{n(n-1)(n-2)}{3!} + 6.\frac{n(n-1)(n-2)(n-3)}{4!}$

$= \frac{n}{12}(n+1)(3n^2+23n+46).$

390. The series and the successive order differences are:

10, 23, 60, 169, 494, ...

13, 37, 109, 325, ...

24, 72, 216, ...

Here second order differences are in G.P. whose common ratio is 3. Let $t_n = a + bn + c.3^{n-1}$

$\therefore a+b+c = t_1 = 10, a+2b+3c = t_2 = 23, a+3b+9c = t_3 = 60$

$\Rightarrow a = 3, b = 1, c = 6 \Rightarrow t_n = 3 + n + 6.3^{n-1}$

$S_n = \sum_{n=1}^{n} t_n = \frac{1}{2}(n^2+7n-6) + 3^{n+1}.$

391. Here one factor of the terms is in G.P. i.e. x.

Now the series of the coeff. of terms together with successive order differences are 3, 5, 9, 15, 23, 33, ...

2, 4, 6, 8, 10, ...

2, 2, 2, ,2, ...

0, 0, 0, ...

Hence third order differences are constant. Now,

$S = 3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + \ldots \infty$

$-3xS = -9x - 15x^2 - 27x^3 - 45x^4 - 69x^5 - \ldots$

$3x^2S = 9x^2 + 15x^3 + 27x^4 + 45x^5 + \ldots$

$-x^3S = -3x^3 - 5x^4 - 9x^5 - \ldots$

Adding, we get $(1-x)^3S = 3 - 4x + 3x^2$

$\therefore S = \frac{3-4x+3x^2}{(1-x)^3}$.

392. Let t_r denote the rth term of the series $\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + \ldots + \frac{n-2}{2.3}$, then

$t_1 = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$, $t_2 = \frac{2}{n-2} - \frac{2}{n-1} = \frac{2}{n-2} - \frac{1}{n-1} - \frac{1}{n-1}$, $t_3 = \frac{3}{n-3} - \frac{3}{n-2} = \frac{3}{n-3} - \frac{2}{n-2} - \frac{1}{n-2}, \ldots, t_{n-2} = \frac{n-2}{2} - \frac{n-2}{3} = \frac{n-2}{2} - \frac{n-3}{3} - \frac{1}{3}$

$t_1 + t_2 + \ldots t_n = \frac{n-2}{2}\left(-\frac{1}{n} - \frac{1}{n-1} - \frac{1}{n-2} - \ldots - \frac{1}{3}\right)$

$= \frac{n+1}{2} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$

$\therefore H_n' = \frac{n+1}{2} - (t_1 + t_2 + \ldots + t_n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} = H_n$.

393. $\tan^{-1}\left(\frac{x}{1+1.2x^2}\right) = \tan^{-1}\left(\frac{2x-x}{1+x.2x}\right) = \tan^{-1}2x - \tan^{-1}x$

$\tan^{-1}\left(\frac{x}{1+2.3x^2}\right) = \tan^{-1}\left(\frac{3x-2x}{1+2x.3x}\right) = \tan^{-1}3x - \tan^{-1}2x$

...

$\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right) = \tan^{-1}\left(\frac{(n+1)x-nx}{1+nx.(n+1)x}\right) = \tan^{-1}(n+1)x - \tan^{-1}nx$

Adding, we get

$L.H.S. = \tan^{-1}(n+1)x - \tan^{-1}x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right) = R.H.S.$

394. The nth term of the given series is $t_n = \frac{n}{1+n^2+n^4} = \frac{n}{(1+n^2)^2-n^2} = \frac{1}{2}\left(\frac{1}{1+n^2-n} - \frac{1}{1+n^2+n}\right)$

$\therefore t_1 = \frac{1}{2}\left(1-\frac{1}{3}\right), t_2 = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right), t_3 = \frac{1}{2}\left(\frac{1}{7}-\frac{1}{13}\right), \dots, t_n = \frac{1}{2}\left(\frac{1}{1+n^2-n} - \frac{1}{1+n^2+n}\right)$

Adding, we get

$S = \frac{1}{2}\left(1 - \frac{1}{1+n^2+n}\right) = \frac{n(n+1)}{2(1+n+n^2)}.$

395. $t_n = \tan^{-1}\frac{2n}{2+n^2+n^4} = \tan^{-1}\frac{2n}{1+1+n^2+n^4} = \tan^{-1}\frac{2n}{1+1+(n^2+1)^2-n^2} = \tan^{-1}\frac{2n}{1+(n^2+n+1)(n^2-n+1)} = \tan^{-1}\frac{(n^2+n+1)-(n^2-n+1)}{1+(n^2+n+1)(n^2-n+1)} = \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$

$\therefore t_1 = \tan^{-1}3 - \tan^{-1}1, t_2 = \tan^{-1}7 - \tan^{-1}3, \dots, t_{n-1} = \tan^{-1}(n^2-n+1) - \tan^{-1}[(n-1)^2-(n-1)+1]$

$t_n = \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$

Adding, we get $S_n = \tan^{-1}(n^2+n+1) - \tan^{-1}1 = \tan^{-1}\frac{n^2+n}{n^2+n+2}.$

396. $t_n = \frac{n^4}{4n^2-1} = \frac{1}{16}\left[\frac{16n^4}{4n^2-1}\right] = \frac{1}{16}\left[\frac{16n^4-1+1}{4n^2-1}\right] = \frac{1}{16}\left[4n^2+1+\frac{1}{(2n-1)(2n+1)}\right]$

$= \frac{1}{16}\left[4n^2+1+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$

$S_n = \sum t_n = \frac{1}{4}\sum n^2 + \frac{1}{16}\sum 1 + \frac{1}{32}\sum\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right) = \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{n}{16} + \frac{1}{32}\left(1-\frac{1}{2n+1}\right)$

$= \frac{n}{48}(4n^2+6n+5) + \frac{1}{16}\frac{n}{2n+1} = \frac{n(4n^2+6n+5)}{48} + \frac{n}{16(2n+1)}.$

397. $t_k = a_k a_{k+1} \dots a_{k+r-1}, t_{k+1} = a_{k+1}a_{k+2}\dots a_{k+r} \therefore a_{k+r}t_k = a_k t_{k+1}$

$[a_1+(k+r-1)d]t_k = [a_1+(k-1)d]t_{k+1} \Rightarrow [a_1+(k-2)d]t_k - [a_1+(k-1)d]t_{k+1} = -(1+r)dt_k$

Thus,

$(a-d)t_1 - (a_1+0d)t_2 = -(1+r)dt_1$

$(a+0d)t_2 - (a_1+d)t_3 = -(1+r)dt_2$

...

$[a_1+(n-2)d]t_n - [a_1+(n-1)d]t_{n+1} = -(1+r)dt_n$

$(a-d)t_1 - [a_1+(n-1)d]t_{n+1} = -(1+r)d[t_1+t_2+\dots+t_n]$

$\therefore t_1+t_2+\dots+t_n = \frac{a_n a_{n+1}\dots a_{n+r} - a_0 a_1 \dots a_r}{(r+1)d}.$

398. Let a be the first term and d be the common difference of A.P. Let t_k be the kth term of the given sequence. Then,

$t_k = \frac{1}{a_k a_{k+1} \ldots a_{k+r-1}}, t_{k+1} = \frac{1}{a_{k+1} a_{k+2} \ldots a_{k+r}} \Rightarrow a_k t_k = a_{k+r} t_{k+1}$

$[a+(k-1)d]t_k - (a+kd)t_{k+1} = d(r-1)t_{k+1} \therefore (a+0d)t_1 - (a+d)t_2 = d(r-1)t_2$
$(a+d)t_2 - (a+2d)t_3 = d(r-1)t_3$

...

$[a+(n-2)d]t_{n-1} - [a+(n-1)d]t_n = d(r-1)t_n$

Adding, we get

$at_1 - [a+(n-1)d]t_n = d(r-1)[t_2 + t_3 + \ldots + t_n]$

$[a+(r-d)d]t_1 - [a+(n-1)d]t_n = d(r-1)S[t_1 + t_2 + \ldots + t_n]$

$t_1 + t_2 + \ldots + t_n = \frac{1}{(r-1)d}\left(\frac{a_r}{a_1 a_2 \ldots a_r} - \frac{a_n}{a_n a_{n+1} \ldots a_{n+r-1}}\right)$

$S_n = \frac{1}{(r-1)(a_2-a_1)}\left(\frac{1}{a_1 a_2 \ldots a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \ldots a_{n+r-1}}\right).$

399. Let t_i be the ith term of the series, then

$t_i = \frac{1}{i(i+1)(i+2)(i+3)}, t_{i+1} = \frac{1}{(i+1)(i+2)(i+3)(i+4)}$

$\Rightarrow it_i = (i+4)t_{i+1} \Rightarrow it_i - (i+1)t_{i+1} = 3t_{i+1}$

$\therefore 1.t_1 - 2t_2 = 3t_2, 2.t_2 - 3.t_3 = 3t_3, \ldots, (n-1).t_i - nt_n = 3t_n$

Adding, we get

$t_1 - nt_n = 3(t_1 + t_2 + \ldots + t_n) \Rightarrow 4t_1 - nt_n = 3[t_1 + t_2 + \ldots + t_n]$

$t_1 + t_2 + \ldots + t_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}.$

400. $t_n = \frac{n+2}{n(n+1)(n+3)} = \frac{(n+2)^2}{n(n+1)(n+2)(n+3)}$

$= \frac{n^2+4n+4}{n(n+1)(n+2)(n+3)} = \frac{n(n+4)}{n(n+1)(n+2)(n+3)} + \frac{4}{n(n+1)(n+2)(n+3)}$

$= \frac{n(n+1)+3n}{n(n+1)(n+2)(n+3)} + \frac{4}{n(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{3}{(n+1)(n+2)(n+3)} + \frac{4}{n(n+1)(n+2)(n+3)}$

Now that we have found t_n we can find S_n like previous problem.

$S_n = \frac{29}{36} - \frac{1}{n+3} - \frac{3}{2(n+2)(n+3)} - \frac{4}{3(n+1)(n+2)(n+3)}.$

401. $t_n = \frac{n}{1.3.5.7\ldots(2n-1)(2n+1)} = \frac{1}{2}\left[\frac{1}{1.3.5.7\ldots(2n-1)} - \frac{1}{1.3.5.7\ldots(2n+1)}\right]$

$\therefore t_1=\frac{1}{2}\left(1-\frac{1}{1.3}\right), t_2=\frac{1}{2}\left(\frac{1}{1.3}-\frac{1}{1.3.5}\right), \dots, t_n=\frac{1}{2}\left(\frac{1}{1.3.5.7...(2n-1)}-\frac{1}{1.3.5.7...(2n+1)}\right)$

$S_n=\frac{1}{2}\left[1-\frac{1}{1.3.5.7...(2n+1)}\right].$

402. $t_n=\frac{n+1}{(2n-1)(2n+1)}\cdot\frac{1}{3^n}=\frac{1}{4}\left[\frac{3}{2n-1}-\frac{1}{2n+1}\right]\cdot\frac{1}{3^n}=\frac{1}{4}\left[\frac{1}{2n-1}\cdot\frac{1}{3^{n-1}}-\frac{1}{2n+1}\cdot\frac{1}{3^n}\right]$

$\therefore t_1=\frac{1}{4}\left(\frac{1}{1.1}-\frac{1}{3}\cdot\frac{1}{3}\right), t_2=\frac{1}{4}\left(\frac{1}{3.3}-\frac{1}{5}\cdot\frac{1}{3^2}\right), t_3=\frac{1}{4}\left(\frac{1}{5}\cdot\frac{1}{3^2}-\frac{1}{7}\cdot\frac{1}{3^3}\right), \dots, t_n=\frac{1}{4}\left(\frac{1}{2n-1}\cdot\frac{1}{3^{n-1}}-\frac{1}{2n+1}\cdot\frac{1}{3^n}\right)$

$S_n=\frac{1}{4}\left[1-\frac{1}{2n+1}\cdot\frac{1}{3^n}\right].$

403. $t_n=\frac{2n-1}{3.7.11...(4n-1)}=\frac{1}{2}\left[\frac{1}{3.7.11...(4n-5)}-\frac{1}{3.7.11...(4n+1)}\right]$

$t_2=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{3.7}\right), t_3=\frac{1}{2}\left(\frac{1}{3.7}-\frac{1}{3.7.11}\right), \dots, t_n=\frac{1}{2}\left(\frac{1}{3.7.11...(4n-5)-\frac{1}{3.7.11...(4n-1)}}\right)$

$t_1+t_2+...t_n=\frac{1}{3}+\frac{1}{2}\left[\frac{1}{3}-\frac{1}{3.7.11...(4n-1)}\right]$

$S_n=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{3.7.11...(4n-1)}.$

404. $t_n=n(1-a)(1-2a)...[a-(n-1)a], t_n=-\frac{1}{a}(1-na-1)(1-a)(1-2a)...[a-(n-1)a]=-\frac{1}{a}[(1-a)(1-2a)...(1-na)-(1-a)(1-2a)...\{a+(n-1)a\}]$

$\therefore t_1=-\frac{1}{a}[(1-a)-1], t_2=-\frac{1}{qa}[(1-a)(1-2a)-(1-a)], ...$

Adding, we get

$S_n=\frac{1}{a}[1-(1-a)(1-2a)...(1-na)].$

405. $t_1=1,\ t_2=\frac{x}{b_1}=\frac{(x+b_1)-b_1}{b_1}=\frac{x+b_1}{b_1}-1,\ t_3=\frac{x(x+b_1)}{b_1b_2}=\frac{[(x+b_2)-b_2](x+b_1)}{b_1b_2}=\frac{(x+b_1)(x+b_2)}{b_1b_2}-\frac{x+b_1}{b_1}$

...

$t_{n+1}=\frac{(x+b_1)...(x+b_n)}{b_1b_2...b_n}-\frac{(x+b_1)...(x+b_{n-1})}{b_1b_2...b_{n-1}}$

$\therefore S_n=\frac{(x+b_1)...(x+b_n)}{b_1b_2...b_n}.$

406. $nS_k(n)=n[1^k+2^k+...+n^k]=1^k+(1^k+2.2^k)+(1^k+2^k+3.3^k)+...+(1^k+2^k+...+n.n^k)$

$=1^{k+1}+[S_k(1)+2^{k+1}]+[S_k(2)+3^{k+1}]+...+[S_k(n-1)+n^{k+1}]=S_k(1)+S_k(2)+...+S_k(n-1)+S_{k+1}(n).$

407. $n^3 > 100 \Rightarrow n > 4, n^3 < 100000 \Rightarrow n < 22$

So $S = 5^3 + 6^3 + ... + 21^3, S' = 1^3 + 2^3 + 3^3 + 4^3$

$S' + S - S' = 1^3 + 2^2 + ... + 21^3 - (1^3 + 2^3 + ... + 4^3) = 53261.$

408. $S = a + (a+1) + ... + (a+n-1), = na + \frac{n(n-1)}{2}$

$S^2 = n^2a^2 + n^2(n-1)a + \frac{n^2(n-1)^2}{4}$

$t = a^2 + (a+1)^2 + ... + (a+n-1)^2 \Rightarrow nt = n^2a^2 + n^2(n-1)a + n\sum_{i=1}^{n-1} i^2$

Clearly, $nt - S^2$ is independent of a.

409. $\sum_{x=5}^{n+5} 4(x-3) = \sum_{x=1}^{n+5} 4(x-3) - \sum_{x=1}^{4} 4(x-3) = \frac{4(n+5)(n+6)}{2} - 12(n+5) - \frac{4.4.5}{2} + 12.4 = 2n^2 + 10n + 8$

$\therefore P + Q = 12.$

410. Let S be the sum of series, then

$S = 5^3 + 7^3 + 9^3 + ...$ to n terms $+ 2^5(3^3 + 4^3 + 5^3 + ...$ to n terms$)$

$= 1^3 + 3^3 + 5^3 + ...$ to $(n+2)$ terms $-1^3 - 3^3 + 2^5(1^3 + 3^3 + 5^3 + ...$ to $n+1$ terms$) - 2^5$

$= \sum_{i=1}^{n+2}(2i-1)^3 - 28 + 2^5\sum_{i=1}^{n+1}(2i-1)^3 - 32 = n(10n^3 + 96n^2 + 243n + 540).$

411. Let S be the sum of the series and $x = \frac{2n+1}{2n-1}$, then

$S = x + 3x^2 + 5x^3 + ...$

$xS = x^2 + 3x^3 + ... + (2n-1)x^{n+1}$

$(1-x)S = x + 2x^2 + 2x^3 + ... = x + 2x^2(1 + x + x^2 + ...$ to $n-1$ terms$) - (2n-1)x^{n+1} = x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$ $S = \frac{x}{1-x} + \frac{2x^2(1-x^{n-1})}{(1-x)^2} - \frac{(2n-1)x^{n+1}}{1-x} = \frac{x^2 - x + 2x^{n+1} - 2x^2 + (x-1).(2n-1)x^{n+1}}{(x-1)^2} = n(2n+1).$

412. Let S be the sum to n terms and $x = \frac{4n+1}{4n-3}$, then

$S = 1 + 5x + 9x^2 + 13x^3 + ...$

$xS = x + 5x^2 + 9x^3 + ... + (4n+1)x^n$

$(1-x)S = 1 + 4x + 4x^2 + 4x^3 + ... + 4x^{n-1} - (4n+1)x^n$

$S = \frac{1}{x-1} + \frac{4x(x^{n-1}-1)}{(x-1)^2} - \frac{(4n+1)x^n}{(x-1)} = 4n^2 - 3n.$

413. $t_n = 1.10^{2n} + 2.10^{2n-1} + 3.10^{n-2} + ... + n.10^{n+1} + (n+1)10^n + n.10^n + (n-1)10^{n-2} + ... + 3.10^2 + 2.10 + 1$

$= 10^{2n}\Big[1 + 2.\frac{1}{10} + 3.\frac{1}{10^2} + ... + n.\frac{1}{10^{n-1}}\Big] + (1 + 2.10 + 3.10^2 + ... + n.10^{n-1} + (n+1)10^n) = 10^{2n}S_1 + S_2$ $S_1 = 1 + 2.\frac{1}{10} + 3.\frac{1}{10^2} + ... + n.\frac{1}{10^{n-1}}$

$\frac{S_1}{10} = \frac{1}{10} + 2\frac{1}{10^2} + ... + (n-1)\frac{1}{10^{n-1}} + n.\frac{1}{10^n}$

$S_1 = \frac{100}{81}\Big(1 - \frac{1}{10^n}\Big) - \frac{90n}{81.10^n}$

$S_2 = 1 + 2.10 + 3.10^2 + ... + (n+1)10^n$

$10S_2 = 10 + 2.10^2 + ... + n.10^n + (n+1)10^{n+1}$

$S_2 = \frac{1-10^{n+1}}{81} + \frac{(n+1)10^{n+1}}{9}$

Substituting S_1 and S_2 we obtain t_n as

$t_n = \Big(\frac{10^{n+1}-1}{9}\Big)^2$. Thus, the numbers in the sequence will be square of odd positive integer.

414. $t_n = \frac{2n+1}{1^2+2^2+...+n^2} = \frac{2n+1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$

$\therefore t_1 = \frac{6}{1.2} = 6\Big(1 - \frac{1}{2}\Big), t_2 = \frac{6}{2.3} = 6\Big(\frac{1}{2} - \frac{1}{3}\Big), ..., t_n = \frac{6}{n(n+1)} = 6.\Big(\frac{1}{n} - \frac{1}{n+1}\Big)$

Adding, we get

$S = \frac{6n}{n+1}$.

415. $t_n = \frac{1}{(1+nx)[1+(n+1)x]} = \frac{1}{x}\Big(\frac{1}{1+nx} - \frac{1}{1+(n+1)x}\Big)$

$t_1 = \frac{1}{x}\Big(\frac{1}{1+x} - \frac{1}{1+2x}\Big), t_2 = \frac{1}{x}\Big(\frac{1}{1+2x} - \frac{1}{1+3x}\Big), ...$

Adding, we get

$S_n = \frac{1}{x}\Big(\frac{1}{1+x} - \frac{1}{1+(n+1)x}\Big) = \frac{n}{(1+x)[1+(n+1)x]}$.

416. $t_n = \frac{a^{n-1}}{(1+a^{n-1}x)(1+a^n x)} = \frac{1}{(a-1)x}\Big(\frac{1}{1+a^{n-1}x} - \frac{1}{1+a^n x}\Big)$

$t_1 = \frac{1}{(a-1)x}\Big(\frac{1}{1+x} - \frac{1}{1+ax}\Big), t_2 = \frac{1}{(a-1)x}\Big(\frac{1}{1+ax} - \frac{1}{1+a^2x}\Big), ...$

Adding, we get

$S = \frac{1}{(a-1)x}\Big(\frac{1}{1+x} - \frac{1}{1+a^n x}\Big)$.

417. $t_n = \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}} = \frac{\sqrt{2n+1}-\sqrt{2n-1}}{2}$

$\therefore t_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}, t_2 = \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2}, \ldots$

Adding, we get

$S = \frac{\sqrt{2n+1}-1}{2}.$

418. $t_k = a_k a_{k+1}, t_{k+1} = a_{k+1}a_{k+2}$

$a_{k+2}t_k = a_k t_{k+1}$

$[a_1 + (k+1)d]t_k - [a_1 + (k-1)d]t_{k+1} = 0$

$[a_1 + (k-2)d]t_k - [a_1 + (k-1)d]t_{k+1} = -3dt_k$

$\therefore (a_1 - d)t_1 - (a_1 + 0d)t_2 = -3dt_1$

$(a_1 + 0d)t_2 - (a_1 + d)t_3 = -3dt_2$

...

$[a_1 + (n-2)d]t_n - [a_1 + (n-1)]t_{n+1} = -3dt_n$

Adding, we get

$-3d(t_1 + t_2 + \ldots + t_n) = (a_1 - d)t_1 - [a_1 + (n-1)]t_{n+1}$

$S = \frac{[a+(n-1)d](a+nd)[a+(n+1)d]-(a-d)a(a+d)}{3d} = \frac{n}{3}[3a^2 + 3nad + (n^2-1)d^2].$

419. $t_k = a_k a_{k+1} a_{k+2}, t_{k+1} = a_{k+1}a_{k+2}a_{k+3}$

$a_{k+3}t_k = a_k t_{k+1}$

$[a_1 + (k+2)d]t_k = [a_1 + (k-1)d]t_{k+1}$

$[a_1 + (k-2)d]t_k - [a_1 + (k-1)d]t_{k+1} = -4dt_k$

$(a_1 - d)t_1 - (a_1 + 0d)t_2 = -4dt_1$

$(a_1 + 0d)t_2 - (a_1 + d)t_3 = -4dt_2$

...

$[a_1 + (n-2)d]t_n - [a_1 + (n-1)]t_{n+1} = -4dt_n$

Adding, we get

$-4d(t_1 + t_2 + \ldots + t_n) = (a_1 - d)t_1 - [a_1 + (n-1)]t_{n+1}$

$S = \frac{[a+(n-1)d](a+nd)[a+(n+1)d][a+(n+2)d]-(a-d)a(a+d)(a+2d)}{4d}$

$= \frac{n}{4}[4a^3 + 6(n+1)a^2d + 2(2n^2+3n-1)ad^2 + (n^3-2n^2-n-2)d^3].$

420. $t_n = \frac{2n+1}{n^2.(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$

$t_1 = \frac{1}{1} - \frac{1}{2^2}, t_2 = \frac{1}{2^2} - \frac{1}{3^2}, ...$

Adding, we get

$S = 1 - \frac{1}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}.$

421. $t_n = n(n+1), S_n = \sum(n^2+n) = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$\Rightarrow S_n = \frac{n(n+1)(n+2)}{3}$

We have proved in earlier that $\sigma_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$

$\therefore \sigma_{n-1} = \frac{1}{18} - \frac{1}{3n(n+1)(n+2)}$

Now it is trivial to prove that $18S_n\sigma_{n-1} - S_n = -2$.

422. $t_n = \frac{2n+3}{n(n+1)} \cdot \frac{1}{3^n} = \left(\frac{3}{n} - \frac{1}{n+1}\right) \cdot \frac{1}{3^n}$

$\therefore t_1 = \left(3 - \frac{1}{2}\right) \cdot \frac{1}{3}, t_2 = \left(\frac{3}{2} - \frac{1}{3}\right) \cdot \frac{1}{3^2}, t_3 = \left(\frac{3}{3} - \frac{1}{4}\right) \cdot \frac{1}{3^3}, ...$

Adding, we get

$S_n = 1 - \frac{1}{n+1} \cdot \frac{1}{3^n}.$

423. $S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\infty, S' = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + ...\infty \Rightarrow 4S' = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\infty$

$4S' = S \Rightarrow S' = \frac{S}{4} \therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\infty \Rightarrow S - S' = \frac{3}{4}S = \frac{\pi^2}{8}.$

424. In previous problem we have proved that $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + ...\infty = \frac{\pi^2}{24}$ and $\frac{1}{1^2} + \frac{1}{3^3} + \frac{1}{5^2} + ...\infty = \frac{\pi^2}{8}$

$\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\infty = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$

425. $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}, = n - n + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

$= n - (1-1) - \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{3}\right) + ... + \left(1 - \frac{1}{n}\right)$

$= n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + ... + \frac{n-1}{n}\right).$

426. We can rewrite the question like $\frac{1}{x+1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} - ... - \frac{2^n}{x^{2^n}+1} = \frac{2^{n+1}}{x^{2^{n+1}}-1}$

L.H.S. $= \left(\frac{1}{x-1} - \frac{1}{x+1}\right) - \frac{2}{x^2+1} - \frac{4}{x^4+1} - ... - \frac{2^n}{x^{2^n}+1}$

$= \left(\frac{2}{x^2-1} - \frac{2}{x^2+1}\right) - \frac{4}{x^4+1} - \dots - \frac{2^n}{x^{2^n}+1}$

$= \left(\frac{4}{x^4-1} - \frac{4}{x^4+1}\right) - \dots - \frac{2^n}{x^{2^n}+1}$. Progreessing similarly we obtain R.H.S.

427. Multiplying and dividing by $1 - \frac{1}{3}$, we get L.H.S. $= \frac{\left(1-\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)}\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\dots\left(1 + \frac{1}{3^{2^n}}\right)$

$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1 - \frac{1}{3^2}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\dots\left(1 + \frac{1}{3^{2^n}}\right)$

$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1 - \frac{1}{3^4}\right)\left(1 + \frac{1}{3^4}\right)\dots\left(1 + \frac{1}{3^{2^n}}\right)$

Proceeding similarly we obtain the R.H.S.

428. Since A.M $\geq$ G.M.

$\therefore \frac{x+y}{2} \geq \sqrt{xy}, \frac{y+z}{2} \geq \sqrt{yz}, \frac{x+z}{2} \geq \sqrt{zx}$

$\frac{(x+y)(y+z)(z+x)}{8} \geq xyz \Rightarrow (1-x)(1-y)(1-z) \geq 8xyz.$

429. Since A.M $\geq$ H.M.

$\therefore \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$

430. Taking A.M. and G.M of 7 numbers $\frac{a}{2}, \frac{a}{2}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2}$, we get

$\frac{2.\frac{a}{2}+3.\frac{b}{3}+2.\frac{c}{2}}{7} \geq \left[\left(\frac{a}{2}\right)^2\left(\frac{b}{3}\right)^3\left(\frac{c}{2}\right)^2\right]^{\frac{1}{7}} \Rightarrow \frac{3}{7} \geq \left(\frac{a^2b^3c^2}{2^2 3^3 2^2}\right)^{\frac{1}{7}} \Rightarrow \frac{3^7}{7^7} \geq \frac{a^2b^3c^2}{2^2 3^3 2^2} \Rightarrow a^2b^3c^2 \leq \frac{3^{10}2^4}{7^7}.$

431. $\sum_{i=1}^{n} a_ib_i = \sum_{i=1}^{n} a_i(1-a_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i^2 = na - \sum_{i=1}^{n}(a_i - a + a)^2$

$= na - \sum_{i=1}^{n}[(a_i-a)^2 + a^2 + 2a(ai-a)] = na - \sum_{i=1}^{n}(ai-a)^2 - na^2 + 2a\sum_{i=1}^{n}(a_i - na)$

$= na(1-a) - \sum_{i=1}^{n}(a_i-a)^2 = nab - \sum_{i=1}^{n}(a_i-a)^2, \because na+nb = \sum_{i=1}^{n}(a_i+b_i) = n \therefore a+b = 1.$

432. Let a_{n+1} be a number such that $|a_{n+1}| = |a_n + 1|$

Squaring all the numbers, we get

$a_1^2 = 0$, $a_2^2 = a_1^2 + 2a_1 + 1$, $a_3^2 = a_2^2 + 2a_2 + 1, \dots, a_n^2 = a_{n-1}^2 + 2a_{n-1} + 1$, $a_{n+1}^2 = a_n^2 + 2a_n + 1$

Adding, we get

$a_1^2 + a_2^2 + ... + a_n^2 + a_{n+1}^2 = a_1^2 + a_2^2 + ... + a_n^2 + 2(a_1 + a_2 + ... + a_n) + n$

$\Rightarrow 2(a_1 + a_2 + ... + a_n) = -n + a_{n+1}^2 \geq -n \Rightarrow (a_1 + a_2 + ... + a_n)/n \geq -1/2.$

433. We know that A.M. $\geq$ G.M.

$\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}, \frac{b+c}{2} \geq \sqrt{bc}, \frac{a+c}{2} \geq \sqrt{ac}$

Multiplying, we get $(a+b)(b+c)(c+a) \geq 8abc$.

434. We know that A.M $\geq$ H.M.

$\Rightarrow \frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{a}.$

435. We know that A.M $\geq$ G.M. $\Rightarrow \frac{1+3+5+...+(2n-1)}{n} \geq (1.3.5. ... (2n-1))^{\frac{1}{n}}$

$\Rightarrow \frac{n^2}{n} \geq (1.3.5. ... (2n-1))^{\frac{1}{n}} \Rightarrow n^n \geq 1.3.5 ... (2n-1).$

436. We consider seven numbers five of which are $2+x$ and remaining four are $7-x$. Now, we know that A.M $\geq$ G.M.

$\Rightarrow \frac{4.\frac{7-x}{4}+5.\frac{2+x}{5}}{9} \geq \left[\left(\frac{7-x}{4}\right)^4 \left(\frac{2+x}{5}\right)^5\right]^{\frac{1}{9}} \Rightarrow \frac{9}{9} \geq \left[\left(\frac{7-x}{4}\right)^4 \left(\frac{2+x}{5}\right)^5\right]^{\frac{1}{9}}$

$\Rightarrow (7-x)^4(2+x)^5 \leq 4^4.5^5$. So the greatest value would be $4^4.5^5$.

437. We know that A.M $\geq$ H.M.

$\Rightarrow \frac{a+b}{2} \geq \frac{2ab}{a+b}, \frac{b+c}{2} \geq \frac{2bc}{b+c}, \frac{c+a}{2} \geq \frac{2ca}{c+a}$

$\Rightarrow \frac{a+b+c}{2} \geq \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}.$

438. $(a-b)^2 \geq 0, (b-c)^2 \geq 0, (c-a)^2 \geq 0$

$\Rightarrow \frac{(a-b)^2}{ab} \geq 0, \frac{(b-c)^2}{bc} \geq 0, \frac{(c-a)^2}{ac} \geq 0 \Rightarrow \frac{a^2+b^2}{ab} \geq 2, \frac{b^2+c^2}{bc} \geq 2, \frac{c^2+a^2}{ca} \geq 2$

$\Rightarrow \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} \geq 6 \Rightarrow \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6.$

439. We know that A.M. $\geq$ H.M. $\frac{x_1+x_2+...+x_n}{n} \geq \frac{n}{\left(\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}\right)}$

$\Rightarrow (x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \geq n^2$

440. We know that A.M $\geq$ G.M.Considering 1 and $x^{2n} \Rightarrow \frac{1+x^{2n}}{2} \geq \sqrt{1.x^{2n}} = x^n$ Considering 1 and $y^{2m} \Rightarrow \frac{1+y^{2m}}{2} \geq \sqrt{1.y^{2m}} = y^m$

Myltiplying. we get

$(1+x^{2n})(1+y^{2m}) \geq 4x^ny^m \Rightarrow \frac{x^ny^m}{(1+x^{2n})(1+y^{2m})} \leq \frac{1}{4}$.

441. Let $b-c=x, c-a=y$ and $a-b=z, \Rightarrow x+y+z=0$. This also implies that $a+b-2c=x-y, b+c-2a=y-z, c+a-2b=z-x$

Clearly, $x+y+z=0$

Given, $\frac{(x-y)^2+(y-z)^2+(z-x)^2}{3}=\frac{x^2+y^2+z^2}{3} \Rightarrow x^2+y^2+z^2-2xy-2yz-2zx=0$

$\Rightarrow (x+y+z)^2=4(xy+yz+zx) \Rightarrow xy+yz+zx=0 \Rightarrow (c-a)(a-b)+(a-b)(b-c)+(c-a)(b-c)=0$

$\Rightarrow ca-bc-a^2+ab+ab-ca-b^2+bc+bc-c^2-ab+ca=0 \Rightarrow ab+bc+ca-a^2-b^2-c^2=0 \Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0 \Rightarrow a=b=c$

Answers of Chapter 3
Complex Numbers

1. Let $z = 7 + 8i$, and $\sqrt{z} = \sqrt{7 + 8i} = x + iy$. Squaring $7 + 8i = (x^2 - y^2) + 2ixy$ Comparing real and imaginary parts $x^2 - y^2 = 7$, $xy = 4 \Rightarrow x^2 + y^2 = \sqrt{113}$. We discaard $-\sqrt{113}$ as that will make x, y complex.

   $\Rightarrow x = \frac{\sqrt{7+\sqrt{113}}}{2}, y = \frac{\sqrt{\sqrt{113}-7}}{2}$.

2. Let $\sqrt{a^2 - b^2 + 2abi} = x + iy$, then on squaring and comparison of real and imaginary parts, we have $x^2 - y^2 = a^2 - b^2$, $xy = ab \Rightarrow x^2 + y^2 = a^2 + b^2 \Rightarrow x = a, y = b$.

3. $\sqrt[4]{81i^2} = \sqrt{\pm 9i}$ and now we can solve it like previous problems.

4. Let $z = \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{31}{16} = \left(\frac{x}{y} + \frac{y}{x}\right)^2 - 2\frac{i}{4}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{i^2}{4} = \left(\frac{x}{y} + \frac{y}{x} - \frac{i}{4}\right)^2$

   $\therefore$ square root $= \pm\left(\frac{x}{y} + \frac{y}{x} - \frac{i}{4}\right)$.

5. We know that $i^4 = 1$. Let $z = i^{n+80} + i^{n+50} = i^{n+4.20} + i^{n+12.4+2} = i^n + i^{n+2} = i^n - i^n = 0$.

6. Let $z = \left(i^{17} + \frac{1}{i^{15}}\right)^3 = \left(i^{4.4+1} + \frac{1}{i^{4.4-1}}\right)^3 = (i + i)^3 = 8i^3 = -8i$.

7. Let $z = \frac{(1+i)^2}{2+3i} = \frac{2i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{-6+4i}{13}$.

8. Let $z = \left(\frac{1}{1+i} + \frac{1}{1-i}\right)\frac{7+8i}{7-8i} = \frac{2}{1-i^2}\frac{(7+8i)(7+8i)}{(7-8i)(7+8i)} = \frac{2}{2}\frac{-15+112i}{49+64} = \frac{-15+112i}{113}$.

9. Let $z = \frac{(1+i)^{4n+7}}{(1-i)^{4n-1}} = \frac{(1+i)^{4(n+2)-1}}{(1-i)^{4n-1}} = \frac{1-i}{1+i} = \frac{(1-i)^2}{1-i^2} = \frac{1-2i+i^2}{2} = -i$.

10. Let $z = \frac{1}{1-\cos\theta+2i\sin\theta} = \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+1+3\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$.

11. Let $z = \frac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cot u+i)(i+\tan v)}$. Using Euler's formula, we have $z = \frac{e^{ix}.e^{iy}}{\frac{e^{iu}}{\sin u}.\frac{e^{iv}}{\cos v}} = \sin u\cos v.e^{i(x+y-u-v)} = \sin u\cos v\cos(x+y-u-v) + i\sin u\cos v\sin(x+y-u-v)$.

12. $i^5 = i^{4+1} = i$.

13. $i^{67} = i^{64+3} = i^3 = -i[\because i^2 = -1]$.

14. $i^{-59} = \frac{1}{i^{15.4-1}} = i$.

15. $i^{2014} = i^{4.503+2} = i^2 = -1$.

16. $|a| = -a \Rightarrow \sqrt{ab} = \sqrt{|a|b}\,i$.

17. Let $z = i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n + i.i^n - i^n - i.i^n = 0$.

18. $\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{i=1}^{13} i^n + \sum_{i=1}^{13} i^{n+1} = (i + i^2 + i^3 + \cdots + i^{13}) + (i^2 + i^3 + i^4 + \cdots + i^{14}) = i - 1$.

19. $\frac{2^n}{(1+i)^{2n}} + \frac{(1+i)^{2n}}{2^n} = \frac{2^n}{(1+i^2+2i)^n} + \frac{(1+i^2+2i)^n}{2^n} = \frac{1}{i^n} + i^n = \frac{i^n}{i^{2n}} + i^n = i^n\left(\frac{1}{(-1)^n} + 1\right) = i^n[(-1)^n + 1]$.

20. Let $z = i^n + \frac{1}{i^n} = \frac{i^{2n}+1}{i^n}$. Substituting $n = 1, 2, 3, 4,\ z = 0, \pm 2$ i.e. there exists three different solutions.

21. $4x + (3x - y)i = 3 - 6i$. Comparing real and imaginary parts, $4x = 3, 3x - y = -6 \Rightarrow x = \frac{3}{4} \Rightarrow \frac{9}{4} - y = -6 \Rightarrow y = \frac{33}{4}$.

22. $\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) - \left(-\frac{4}{3} + i\right) = \left(\frac{1}{3} + 4 + \frac{4}{3}\right) + i\left(\frac{7}{3} + \frac{1}{3} - 1\right) = \frac{17}{3} + i\frac{5}{3}$.

23. $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i \Rightarrow [(1+i)x - 2i](3-i) + [(2-3i)y + i](3+i) = i(3+i)(3-i) \Rightarrow (4x + 9y - 3) + i(2x - 7y - 3) = 10i$. Equating real and imaginary parts, $4x + 9y = 3, 2x - 7y = 13 \Rightarrow x = 3, y = -1$.

24. The multiplicative inverse is $\frac{1}{z} = \frac{1}{4-3i} = \frac{1}{4-3i}\cdot\frac{4+3i}{4+3i} = \frac{4+3i}{25}$.

25. Let $x_1 = 2, y_1 = 3, x_2 = 1$ and $y_2 = 12$. $\therefore \frac{z_!}{z_2} = \frac{[(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)]}{x_2^2+y_2^2} = \frac{8-i}{5}$.

26. $z_1 = z_2 \Rightarrow 9y^2 - 4 - 10xi = 8y^2 - 20i$. Equating real and imaginary parts, $9y^2 - 4 = 8y^2 \Rightarrow y = \pm 2$ and $-10x = -20 \Rightarrow x = -2 \Rightarrow z = x + iy = -2 \pm 2i$.

27. Let $z = x + iy$ then $|x + iy + 1| = x + iy + 2(1 + i) \Rightarrow \sqrt{(x+1)^2 + y^2} = (x + 2) + i(y + 2)$. Equating real and imaginary parts, $y + 2 = 0 \Rightarrow y = -2$ and $(x + 1)^2 + y^2 = (x + 2)^2 \Rightarrow x^2 + 2x + 5 = 4 = x^2 + 4x + 4 \Rightarrow x = \frac{1}{2} \Rightarrow z = \frac{1-4i}{2}$.

28. Let $z = \frac{1+2i}{1-3i} = \frac{(1+2i)(1+3i)}{1-(3i)^2} = \frac{1+3i+2i+6i^2}{1+9} = \frac{-5+5i}{10} = -\frac{1}{2} + \frac{1}{2}i$

$\Rightarrow |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \frac{1}{2^2}} = \frac{1}{\sqrt{2}}$

$\tan\theta = \frac{\frac{1}{2}}{-\frac{1}{2}} \Rightarrow \theta = \tan^{-1} - 1 = \frac{3\pi}{4}$.

29. Given, $\frac{x-3}{3+i} + \frac{y-3}{3-i} = i(3-i)(3+i) \Rightarrow (x-3)(3-i) + (y-3)(3+i) = 10i \Rightarrow 3x - 9 + i(3 - x) + (3y - 9) + i(y - 3) = 10i$

Comparing real and imaginary parts, we get $3x + 3y - 18 = 0$ and $y - x = 10 \Rightarrow x = -2, y = 8$.

30. $(1+i)^2 = 1+2i-i = 2i \Rightarrow (1+i)^{50} = (2i)^{25} = 2^{25}i^{4.6+1} = 2^{25}i$ Thus, real part will be 0.

31. Let $z = x+iy$ then $x+iy+\sqrt{x^2+y^2} = 2+8i$, Comparing real and imaginary parts, we get $y = 8$ and $x+\sqrt{x^2+y^2} = 2 \Rightarrow \sqrt{x^2+y^2} = 2-x$

    $\Rightarrow x^2+64 = 4-4x+x^2 \Rightarrow x = -15 \Rightarrow z = -15+8i$.

32. $S = i+2i^2+3i^3+\ldots+100i^{100} \Rightarrow iS = i^2+2i^3+\ldots+99i^{100}+100i^{101}$

    $\Rightarrow S(1-i) = i+i^2+\ldots+i^{100}-100i^{101} = \frac{i(1-i^{101})}{1-i} - 100i^{101}$

    $S = \frac{i(1-i^{101})}{(1-i)^2} - \frac{100i^{101}}{1-i}$.

33. Consider $t_1 = \frac{1}{1+i}+\frac{1}{1-i}+\frac{1}{-1+i}+\frac{1}{-1-i} = \frac{1+i+1-i}{1^2-i^2}+\frac{-1+i-1-i}{(-1)^2-i^2} = \frac{2}{2}+\frac{-2}{2} = 0$

    $t_2 = 2\left(\frac{1}{1+i}+\frac{1}{1-i}+\frac{1}{-1+i}+\frac{1}{-1-i}\right) = 0$

    Similarly all other terms and sum will be zero.

34. Given, $z^2-z-5+5i = 0 \Rightarrow D = (-1)^2-4.1.(-5+5i) = 21-20i$ and we will need $\sqrt{D}$

    $\sqrt{D} = \sqrt{b^2-4ac} = \sqrt{21-20i} = \pm\left[\sqrt{\frac{x^2+y^2+x}{2}} - i\sqrt{\frac{x^2+y^2-x}{2}}\right] = \pm(5-2i)$

    $z = \frac{1+5-2i}{2}$ or $z = \frac{1-5+2i}{2} \Rightarrow z = 3-i, i-2$

    Thus, product of real parts $= -2\times 3 = -6$

35. Given, $z^3 = -\bar{z} \Rightarrow |z|^3 = |z| \Rightarrow |z|(|z|-1)(|z|+1) = 0 \Rightarrow |z| = 0, |z| = 1[\because |z|+1>0]$

    If $|z| = 0$, then $z = 0$. If $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1 \Rightarrow z^3+\frac{1}{z} = 0 \Rightarrow z^4+1 = 0$, which has four distinct roots. Thus, given equation has five roots.

36. Since we have to find real roots, let $z = x$, a real value. The given equation becomes $x^3+ix-1 = 0 \Rightarrow x^3 = 1, x = 0$ which is not possible. So there are no real solutions.

37. Let $z = x+iy$, then $\sqrt{x^2+y^2} > 1$, because point A is outside circle.

    $\frac{1}{z} = \frac{x-iy}{\sqrt{x^2+y^2}}$ so $\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{x^2+y^2} < 1$

    This leads to the fact that point E is reciprocal of point A.

38. $z = (3p-7q)+i(3q+7p)$, which is purely imaginary, $\Rightarrow 3p-7q = 0$

    $\Rightarrow \frac{p}{q} = \frac{7}{3} \Rightarrow \frac{p}{q}+i = \frac{7}{3}+i \Rightarrow \frac{p+iq}{q} = \frac{7+3i}{3}$

    $\Rightarrow p+iq = 7+3i \Rightarrow z = 21+9i+49i-21 = 58i \Rightarrow |z|^2 = 3364$.

39. Given, $\alpha = \left(\frac{a-ib}{a+ib}\right)^2 + \left(\frac{a+ib}{a-ib}\right)^2 = \frac{(a-ib)^4+(a+ib)^4}{(a-ib)^2(a+(ib)^2)}$

$= \frac{a^4-4a^3.ib+6a^2i^2b^2-4ai^3b^3+b^4+a^4+4a^3ib+6a^2i^2b^2+4ai^3b^3+b^4}{(a^2+b^2)^2} = \frac{2a^4-12a^2b^2+2b^4}{(a^2+b^2)^2}$, which is purely real.

40. Let $z = x + iy$ then given $|z| = 1 \Rightarrow x^2 + y^2 = 1$

Let $\beta = \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} . \frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{x^2-1+y^2+iy(x+1-x+1)}{(x+1)^2+y^2} = \frac{2iy}{(x+1)^2+y^2}$ which is purely imaginary.

41. Let $z = x + iy \Rightarrow x^2 + (y-3)^2 = 9 \Rightarrow x = 3\cos\theta, y = 3\sin\theta + 3$

$z = 3[\cos\theta + i(\sin\theta + 1)] = 3\left[\sin\left(\frac{\pi}{2} - \theta\right) + i\left(1 + \cos\left(\frac{\pi}{2} - \theta\right)\right)\right]$

$= 3\left[2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$

$= 6\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right] = 6\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)e^{i\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}$

$\cot(\arg(z)) = \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

$\frac{6}{z} = \sec\left(\frac{\pi}{4} - \frac{\theta}{2}\right)e^{-i\left(\frac{\pi}{4} + \frac{\theta}{2}\right)} = \sec\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2} - i\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)\right]$

$= \tan\left(\frac{\pi}{4} - \frac{A}{2}\right) - i \Rightarrow \cot(\arg(z)) - \frac{6}{z} = i.$

42. Let $z = r(\cos\theta + i\sin\theta) = \frac{-16}{1+\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} . \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{-16(1-i\sqrt{3})}{1+3}$

$= -4 + i4\sqrt{3}$ then $r\cos\theta = 4, r\sin\theta = 4\sqrt{3} \Rightarrow r^2 = 64 \Rightarrow r = 4, \cos\theta = \frac{-1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$

$\Rightarrow z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$

43. Let $z = r(\cos\theta + i\sin\theta)$ then because $\arg(z) + \arg(w) = \pi \Rightarrow \arg(w) = \pi - \theta$

$\Rightarrow w = r(-\cos\theta + i\sin\theta) = -r(\cos\theta - i\sin\theta) \therefore r = -\overline{w}.$

44. $x - iy = \sqrt{\frac{a-ib}{c-id}} \Rightarrow x^2 - y^2 - 2ixy = \frac{a-ib}{c-id} = \frac{(a-ib)(c+id)}{c^2+d^2} \Rightarrow x^2 - y^2 - 2ixy = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$

Comparing real and imaginary parts, we get $x^2 - y^2 = \frac{ac+bd}{c^2+d^2}, 2xy = \frac{bc-ad}{c^2+d^2}$

$\Rightarrow (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)} = \frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2} = \frac{a^2+b^2}{c^2+d^2}.$

45. We know that for two complex numbers z_1 and z_2, $|z_1| + |z_2| \geq |z_1 - z_2|$

$|z| + |z-2| \geq |z - (z-2)| = |2| = 2$. Therefore, minimum value is 2.

46. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6 \leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$

$< 1 + 2 + 3 + 6 = 12$. Thus, maximum value of $|z_1 + z_2 + z_3|$ is 12.

47. $|\alpha + \beta|^2 = (\alpha + \beta)(\overline{\alpha + \beta}) = (\alpha + \beta)(\overline{\alpha} + \overline{\beta}) = \alpha\overline{\alpha} + \alpha\overline{\beta} + \overline{\alpha}\beta + \beta\overline{\beta} = |\alpha|^2 + |\beta|^2 + \alpha\overline{\beta} + \overline{\alpha}\beta$

Similarly, $|\alpha - \beta|^2 = |\alpha|^2 + |\beta|^2 - \alpha\overline{\beta} - \overline{\alpha}\beta$

Thus, $|\alpha|^2 + |\beta|^2 = \frac{1}{2}(|\alpha + \beta|^2 + |\alpha - \beta|^2)$

48. If $|z| = 0$ then $\sqrt{x^2 + y^2} = 0 \Rightarrow x^2 + y^2 = 0$

Above is possible if and only if $x = 0$ and $y = 0 \Rightarrow z = 0$.

49. $\frac{z_1 z_2}{\overline{z_1}} = \frac{(1-i)(2+7i)}{1+i} = \frac{2+7-2i+7i}{1+i} = \frac{9+5i}{1+i} = \frac{9+5i}{1+i} \cdot \frac{1-i}{1-i} = \frac{9+5+5i-9i}{2} = 7 - 2i \therefore Im\left(\frac{z_1 z_2}{\overline{z_1}}\right) = -2$.

50. $|z + 12 - 6i| \leq |z - i| + |12 - 5i| < 1 + 13 = 14$.

51. Given, $|z + 6| = |2z + 3|$, let $z = x + iy \Rightarrow (x+6)^2 + y^2 = (2x+3)^2 + 4y^2 \Rightarrow x^2 + 12x + 36 + y^2 = 4x^2 + 12x + 9 + 4y^2$

$\Rightarrow 3x^2 + 2y^2 = 27 \Rightarrow x^2 + y^2 = 9 \Rightarrow |z| = 3$.

52. Given $\sqrt{a - ib} = x - iy$, squaring we get $a - ib = x^2 - y^2 - 2ixy$. Comparing real and imaginary parts, we get $a = x^2 - y^2, b = 2xy \Rightarrow a + ib = x^2 - y^2 + 2ixy = x^2 + i^2y^2 + 2ixy \Rightarrow \sqrt{a + ib} = x + iy$.

53. $x_1 x_2 x_3 \ldots \infty = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\ldots\infty = \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\infty\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \ldots\infty\right)$

$= \cos\frac{\pi}{2} \cdot \frac{1}{1-\frac{1}{2}} + i\sin\frac{\pi}{2} \cdot \frac{1}{1-\frac{1}{2}} = \cos\pi + i\sin\pi = -1$.

54. Given, $\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{i^5\left(\frac{1}{i}\sin\theta + \cos\theta\right)^5}$

$= \frac{(\cos\theta + i\sin\theta)^4}{i(\cos\theta - i\sin\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{i(\cos\theta + i\sin\theta)^{-5}} = \frac{1}{i}(\cos\theta + i\sin\theta)^9 = \sin 9\theta - i\cos 9\theta$.

55. $z = \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^5 + \left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]^5$

$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} = 2\cos\frac{5\pi}{6} \;\therefore Im(z) = 0$.

56. $z = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}}$, thus general root is $\cos\frac{2n\pi+\pi}{4} + i\sin\frac{2n\pi+\pi}{4}$

Thus, substituting $n = 0, 1, 2, 3$ we find four roots and the product is

$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$= \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= \left(-\frac{1}{2} - \frac{1}{2}\right)\left(\frac{-1}{2} - \frac{1}{2}\right) = -1. - 1 = 1.$

57. Let $z_1 = r_1(\cos x + i\sin x)$ and $z_2 = r_2(\cos y + i\sin y)$ Then $(r_1\cos x + r_2\cos y)^2 + (r_1\sin x + r_2\sin y)^2 = r_1^2 + r_2^2 + 2r_2r_2$

$\Rightarrow 2r_1r_2(\cos x\cos y + \sin x\sin y) = 2r_2r_2 \Rightarrow \cos(x - y) = 1 \Rightarrow x - y = 0 \Rightarrow \arg(z_1) - \arg(z_2) = 0.$

58. Let $z = 1 - \sin\alpha + i\cos\alpha = r(\cos\theta + i\sin\theta)$, then $r = \sqrt{(1 - \sin\alpha)^2 + \cos^2\alpha} = \sqrt{2 - 2\sin\alpha}$

$\tan\theta = \frac{\cos\alpha}{1-\sin\alpha} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}-2\tan\frac{\alpha}{2}} = \frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}} = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \Rightarrow \theta = \frac{\pi}{4} - \frac{\alpha}{2}.$

59. Let $z = \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right] = \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right] . \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}\right]$

$= \frac{\left(1+\sin\frac{\pi}{8}\right)^2 - \cos^2\frac{\pi}{8} + 2i\left(1+\sin\frac{\pi}{8}\right)\cos\frac{\pi}{8}}{\left(1+\sin\frac{\pi}{8}\right)^2 + \cos^2\frac{\pi}{8}} = \frac{2\sin\frac{\pi}{8} + 2\sin^2\frac{\pi}{8} + 2i\left(1+\sin\frac{\pi}{8}\right)\cos\frac{\pi}{8}}{2+2\sin\frac{\pi}{8}}$

$= \sin\frac{\pi}{8} + i\cos\frac{\pi}{8} = i\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right) \Rightarrow z^8 = i^8(\cos\pi - i\sin\pi) = -1.$

60. $z_1z_2z_3z_4z_5 = \cos\left(\frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5} + \frac{10\pi}{5}\right) + i\sin\left(\frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5} + \frac{10\pi}{5}\right)$

$= \cos\frac{30\pi}{5} + i\sin\frac{30\pi}{5} = \cos 6\pi + i\sin 6\pi = 1.$

61. $z_n = \cos\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right).\frac{\pi}{2} + i\sin\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right).\frac{\pi}{2}$

$\therefore z_1z_2z_3 \ldots \infty = \cos\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} \ldots \infty\right).\frac{\pi}{2} + i\sin\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} \ldots \infty\right).\frac{\pi}{2}$

$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}.$

62. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \Rightarrow |az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (ax_1 - bx_2)^2 + (ay_1 - by_2)^2 + (bx_1 + ax_2)^2 + (by_1 + ay_2)^2$

$= a^2x_1^2 + b^2x_2^2 - 2abx_1x_2 + a^2y_1^2 + b^2y_2^2 - 2aby_1y_2 + b^2x_1^2 + a^2x_2^2 + 2abx_1x_2 + b^2y_1^2 + a^2y_2^2 + 2aby_1y_2 = (a^2 + b^2)(x_1^2 + y_1^2 + x_2^2 + y_2^2) = (a^2 + b^2)(|z_1|^2 + |z_2|^2).$

63. Let $x = y + iz$, then given expression becomes $\frac{A^2}{y+iz-a} + \frac{B^2}{y+iz-b} + \ldots + \frac{H^2}{y+iz-h} = y + iz + l$

$\frac{A^2(y-a-iz)}{(y-a)^2+z^2} + \frac{B(y-b-iz)}{(y-b)^2+z^2} + \ldots + \frac{H^2(y-iz-h)}{(y-h)^2+z^2} = y + iz + l$. Comparing imaginary parts, we have $-iz\left[\frac{A^2}{(y-a)^2+z^2} + \frac{B^2}{(y-a)^2+z^2} + \ldots + \frac{H^2}{(y-a)^2+z^2}\right] = iz \Rightarrow iz\left[1 + \frac{A^2}{(y-a)^2+z^2} + \frac{B^2}{(y-a)^2+z^2} + \ldots + \frac{H^2}{(y-a)^2+z^2}\right] = 0$

Clearly the term inside brackets is non-zero. So $z = 0$.

64. Let $2^{-x} = p$, then $|1 + 4i - p| \leq 5 \Rightarrow (1-p)^2 + 16 \leq 25$

$1 - p \leq \pm 3 \Rightarrow p \geq 4, -2 \Rightarrow x \geq -2 \because p \nless 0 \Rightarrow p \in [-2, \infty]$.

65. A unimodular number has a modulus of 1. $\cos\theta + i\sin\theta = \frac{c+i}{c-i} = \frac{c+i}{c-i} . \frac{c+i}{c-i} = \frac{c^2-1+2ic}{c^2+1}$

Comparing real and imaginary parts, $\cos\theta = \frac{c^2-1}{c^2+1} \Rightarrow c = \pm\cot\frac{\theta}{2}$

and $\sin\theta = \frac{2c}{c^2+1} \Rightarrow c = \cot\frac{\theta}{2}, \tan\frac{\theta}{2}$. So the common value is $c = \cot\frac{\theta}{2}$.

66. $(z^3+3)^2 = -16 = 16i^2 \Rightarrow z^3 = -3 \pm 4i \Rightarrow |z^3| = 5 \Rightarrow |z| = 5^{1/3}$.

67. $z = \frac{\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x}{1+2i\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x}{1+2i\sin\frac{x}{2}} . \frac{1-2i\sin\frac{x}{2}}{1-2i\sin\frac{x}{2}}$

Since it is real so imaginary part of this will be 0. $\Rightarrow -\tan x - 2\sin\frac{x}{2}\cos\frac{c}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} = 0$

$2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos x} = 0 \Rightarrow \sin\frac{x}{2} = 0 \Rightarrow x = 2n\pi$ where $n = 0,1,2,3\ldots$

or $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\cos x + \cos\frac{x}{2} = 0 \Rightarrow \tan^3\frac{x}{2} - \tan\frac{x}{2} - 2 = 0$

If α is a solution of above then the set of possible values are $x = 2n\pi + 2\alpha$. Solving the cubic equation is left to you.

68. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $|z_1 + z_2|^2 + |z_1 - z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= 2(x_1^2 + y_1^2 + x_2^2 + y_2^2) = 2(|z_1|^2 + |z_2|^2)$.

69. Given, $x^2 - x + 1 = 0 \Rightarrow x = -\omega, -\omega^2$

$$\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2 = \sum_{n=1}^{5}\left(x^{2n} + \frac{1}{x^{2n}} + 2\right)$$

$$= \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right) + \left(x^6 + \frac{1}{x^6} + 2\right) + \left(x^8 + \frac{1}{x^8} + 2\right) + \left(x^{10} + \frac{1}{x^{10}} + 2\right)$$

$$= (x^2 + x^4 + x^6 + x^8 + x^{10}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \frac{1}{x^8} + \frac{1}{x^{10}}\right) + 10$$

$= (\omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^{10}) + \left(\frac{1}{\omega^2} + \frac{1}{\omega^4} + \frac{1}{\omega^6} + \frac{1}{\omega^8} + \frac{1}{\omega^{10}}\right) + 10$

$= -1 - 1 + 10 = 8.$

70. $3^{49}(x + iy) = \left[i\sqrt{3}\left(\frac{1-i\sqrt{3}}{2}\right)\right]^{100} = i^{100}3^{50}(-\omega)^{100} \Rightarrow 3^{49}(x + iy) = 3^{50}.\omega$

$x + iy = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \Rightarrow x = -\frac{3}{2}, y = \frac{3\sqrt{3}}{2}.$

71. $|z_1 + z_2|^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2x_1x_2 + 2y_1y_2 = |z_1|^2 + |z_2|^2 + 2(x_1x_2 + y_1y_2)$

Now, $2Re(z_1\overline{z_2}) = 2Re[(x_1 + iy_1)(x_2 - iy_2)] = 2\Re[x_1x_2 + y_1y_2 - i(x_1y_2 + x_2y_1)] = 2(x_1x_2 + y_1y_2)$

Similalry, $2\Re(\overline{z_1}z_2) = 2(x_1x_2 + y_1y_2)$.

72. R.H.S. $= \left|\frac{1}{z_1} + \frac{1}{z_2}\right| = \left|\frac{z_2+z_1}{z_1z_2}\right|$

Since $|z_1| = |z_2| = 1 \therefore |z_1z_2| = 1$ and thus $|z_1 + z_2| = \left|\frac{1}{z_1} + \frac{1}{z_2}\right|$.

73. Let $z = x + iy$, then $x^2 - 4x + 4 + y^2 = 4x^2 - 8x + 4 + 4y^2 \Rightarrow 3x^2 + 3y^2 = 4x$

$\Rightarrow 3|z|^2 = 4Re(z) \Rightarrow |z|^2 = \frac{4}{3}Re(z).$

74. Given $\sqrt[3]{a + ib} = x + iy \Rightarrow a + ib = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$

Comparing real and imaginary parts, we have $a = x^3 - 3xy^2, b = 3x^2y - y^3 \Rightarrow \frac{a}{x} = x^2 - 3y^2, \frac{b}{y} = 3x^2 - y^2$

$\therefore \frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2).$

75. $x + iy = \sqrt{\frac{a+ib}{c+id}} \Rightarrow (x + iy)^2 = \frac{a+ib}{c+id} \Rightarrow |(x + iy)^2| = \left|\frac{a+ib}{c+id}\right| = \frac{|a+ib|}{|c+id|} \Rightarrow (x^2 + y^2)^2 = \frac{a^2+b^2}{c+d^2}.$

76. Let $z = 1 = \cos 0^\circ + i\sin 0^\circ = e^{i2r\pi}\ \forall i \in N \Rightarrow \sqrt[n]{z} = e^{\frac{i.2r\pi}{n}}$. Clearly, $|z_k| = |z_{k+1}| = 1$.

77. $z^n = (z + 1)^n \Rightarrow \frac{z}{z+1} = 1^{1/n}$

This means $\frac{z}{z+1}$ is nth root of unity. $\Rightarrow \left|\frac{z}{z+1}\right| = 1$

$\Rightarrow |z| = |z + 1| \Rightarrow x^2 + y^2 = x^2 + 2x + 1 + y^2 \Rightarrow x = -\frac{1}{2} \Rightarrow Re(z) < 0.$

78. Roots of $1 + x + x^2 = 0$ are ω and ω^2. Let $f(x) = x^{3m} + x^{3n-1} + x^{3r-2}$

$f(x) = x^{3m} + \frac{x^{3n}}{x} + \frac{x^{3r}}{x^2} \Rightarrow f(\omega) = 1 + \frac{1}{\omega} + \frac{1}{\omega^2} = \frac{1+\omega+\omega^2}{\omega^2} = 0$

Similarly $f(\omega^2) = 0$. Thus, we see that $f(x)$ has same roots as $1 + x + x^2 = 0$. Hence, $f(x)$ will be divisible by $1 + x + x^2$.

79. $\sqrt{3} + i = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2e^{i\frac{\pi}{6}}$

Similarly, $\sqrt{3} - i = 2e^{-i\frac{\pi}{6}}$

Since imaginary part is what prevents equality we need to get rid of it and the least value for which it will happen is when argument is π. Thus, we need to raise to the power by 6 making $n = 6$.

80. $\sqrt{3} - i = 2.\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

Thus, $(\sqrt{3} - i)^n = 2^n \Rightarrow 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{\pi}{6}\right) = 2^n$

$\Rightarrow \cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6} = 1 \Rightarrow \frac{n\pi}{6} = 2k\pi\ \forall k \in I \Rightarrow n = 12k$

Thus, n is a multiple of 12.

81. Given, $z^4 + z^3 + 2z^2 + z + 1 = 0 \Rightarrow z^2(z^2 + z + 1) + z^2 + z + 1 = 0$

$\Rightarrow (z^2 + 1)(z^2 + z + 1) = 0$. If $z^2 + 1 = 0 \Rightarrow z = i \Rightarrow |z| = 1$

If $z^2 + z + 1 = 0 \Rightarrow z = \omega, \omega^2 \Rightarrow |z| = 1$.

82. $\because z = \sqrt[7]{-1} \Rightarrow z^7 = -1 \Rightarrow z^{86} + z^{175} + z^{289} = (z^7)^{14}.z^2 + (z^7)^{25} + (z^7)^{41}z^2 = z^2 - 1 - z^2 = -1$

83. Given, $z^3 + 2z^2 + 3z + 2 = 0 \Rightarrow z^3 + z^2 + 2z + z^2 + z + 2 = 0 \Rightarrow (z+1)(z^2 + z + 2) = 0$

If $z + 1 = 0 \Rightarrow z = -1$, which is real and is of no interest for us.

If $z^2 + z + 2 = 0 \Rightarrow z = \frac{-1+i\sqrt{7}}{2}$ which are complex roots of the given equation.

84. $z = \sqrt[5]{1} \Rightarrow z^5 = 1$

$2^{|1+z+z^2+z^{-2}-z^{-1}|} = 2^{|1+z+z^2+z^3-z^4|}[\because z^4 = 1 \Rightarrow z^{-1} = \frac{z^5}{z} = z^4]$

$= 2^{|1+z+z^2+z^3+z^4-2z^4|} = 2^{\left|\frac{1-z^5}{1-z}-2z^4\right|} = 2^{|2z^4|} = 2^2 = 4[\because |z| = 1]$.

85. Let $S = 1 + 3z + 5z^2 + ... + (2n-1)z^{n-1}$

$\Rightarrow zS = z + 3z^2 + 5z^3 + ... + (2n-3)z^{n-1} + (2n-1)z^n$

$\Rightarrow (1-z)S = 1 + 2z + 2z^2 + 2z^3 + ... + 2z^{n-1} + (2n-1)z^n$

$\Rightarrow (1-z)S = 1 + 2n - 1 + 2[z + z^2 + ... z^{n-1}][\because z^n = 1]$

$= 2n + 2. - 1[\because 1 + z + z^2 + ... + z^{n-1} = 0] \Rightarrow S = \frac{2(n-1)}{1-z}$.

86. Let $z = \sqrt{-1 - \sqrt{-1 - \sqrt{-1 - \infty}}} \Rightarrow z = \sqrt{-1 - z}$

$\Rightarrow z^2 = -1 - z \Rightarrow z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm i\sqrt{3}}{2} \Rightarrow z = \omega, \omega^2.$

87. Given, $z = e^{\frac{i2\pi}{n}}$, which is nth root of unity.

$\therefore x^n - 1 = (x-1)(x-z)(x-z^2(x-z^3)\ldots(x-z^{n-1})$

Putting $x = 11$, $(11-z)(11-z^2)\ldots(11-z^{n-1}) = \frac{11^n - 1}{10}$.

88. Given, $\frac{3}{2+\cos\theta+i\sin\theta} = a + ib \Rightarrow a + ib\frac{3(2+\cos\theta-i\sin\theta)}{5+4\cos\theta}$

Comparing real and imaginary parts, we get $a = \frac{6+3\cos\theta}{5+4\cos\theta}, b = \frac{-3\sin\theta}{5+4\cos\theta} \Rightarrow a^2 + b^2 = \frac{36+36\cos\theta+9\cos^2\theta+9\sin^2\theta}{(5+4\cos\theta)^2}$

$= \frac{45+36\cos\theta}{(5+\cos\theta)^2} = \frac{9(5+4\cos\theta)}{(5+4\cos\theta)^2} = \frac{9}{5+4\cos\theta}, 4a - 3 = \frac{24+12\cos\theta-15-12\cos\theta}{5+4\cos\theta} = \frac{9}{5+4\cos\theta} \Rightarrow a^2 + b^2 = 4a - 3.$

89. Let $z = x + iy, \Rightarrow |(2x-1) + 2iy| = |(x-2) + iy| \Rightarrow 4x^2 - 4x + 1 + 4y^2 = x^2 - 4x + 4 + y^2 \Rightarrow 3x^2 + 3y^2 = 3 \Rightarrow x^2 + y^2 = 1 \Rightarrow |z| = 1.$

90. Given, $\frac{1-ix}{1+ix} = m + in \Rightarrow m + in = \frac{1-ix}{1+ix} \cdot \frac{1-ix}{1-ix}$

$m + in = \frac{1-x^2-2ix}{1+x^2}$, Comparing real and imaginary parts, $m = \frac{1-x^2}{1+x^2}, n = \frac{-2x}{1+x^2}$

$\Rightarrow m^2 + n^2 = \frac{(1-x^2)^2+4x^2}{(1+x^2)^2} = 1.$

91. We know that the equation of a straight line is given by $\begin{bmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{bmatrix} = 0$

$\Rightarrow z(\overline{z_1} - \overline{z_2}) - \overline{z}(z_1 - z_2) + z_1\overline{z_2} - \overline{z_1}z_2 = 0$

$\Rightarrow z(1 + i - 1 - i) - \overline{z}(1 + i - 1 + i) + (1+i)^2 - (1-i)^2 = 0 \Rightarrow z + \overline{z} - 2 = 0.$

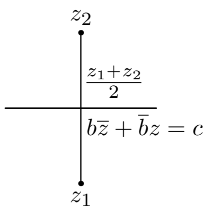
92. Given, $5z_1 - 13z_2 + 8z_3 = 0 \Rightarrow z_2 = \frac{5z_1+8z_3}{5+8}$

This means z_1 divides the line segment joining z_1 and z_2 in the ratio of $5:8$ which also implies that these three points are collinear. Thus, $\begin{bmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{bmatrix} = 0$

93. We know that length of perpendicular from z_1 to $\overline{a}z + a\overline{z} + b = 0$ is given by $\frac{|\overline{a}z_1+a\overline{z_1}+b|}{2|a|}$.

Thus desired length $= \frac{|(2-3i)(3+4i)+(2+3i)(3-4i)+9|}{2|3-4i|} = \frac{45}{10} = \frac{9}{2}$.

94.



Since mid-point lies on the given line, therefore $b\left(\frac{\overline{z_1}+\overline{z_2}}{2}\right)+\overline{b}\left(\frac{z_1+z_2}{2}\right)=c$

Since line segment joining z_1 and z_2 is perpedicular to the given line therefore, Slope of z_1z_2 + Slope of line = 0

$\Rightarrow \frac{z_2-z_1}{\overline{z_2}-\overline{z_1}}-\frac{b}{\overline{b}}=0$

Solving these two equations, we get $\overline{b}z_2+b\overline{z_1}=c$.

95. Let $z=2-i$ then after rotation new point would be $z.e^{i\pi/2}=(2-i)\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=(2-i)i=1+2i$.

96. Coordinate of z_0 after moving 5 points horizontally and 3 points vertically away from starting pont would be $6+5i$.

It then moves in the direction of vecor $\hat{\imath}+\hat{\jmath}$ for $\sqrt{2}$ units. This vector makes angle $\pi/4$ with x-axis. So new coordinate would be $6+\sqrt{2}\cos\pi/4+5+\sqrt{2}\sin\pi/4=7+6i$.

It then rotates by angle $\pi/2$ so new coordinate would be $(7+6i)e^{i\pi/2}=(7+6i)i=-6+7i$.

97. North-East direction makes angle of $\pi/4$ with x-axis. So coordinates of point 3 units from origin in North-East direction $=3.e^{i\pi/4}=3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{3}{\sqrt{2}}+i\frac{3}{\sqrt{2}}$.

North-West direction makes angle of $3\pi/4$ with x-axis. A disaplacement of 4 units in this direction will mean a shift in coordinates by $4.e^{i3\pi/4}=4\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-\frac{4}{\sqrt{2}}+i\sin\frac{4}{\sqrt{2}}$.

Thus, final coordiate would be sum of the above two i.e. $-\frac{1}{\sqrt{2}}+i\frac{7}{\sqrt{2}}$.

98. Given, $\frac{z_1-z_3}{z_2-z_3}=\frac{1-i\sqrt{3}}{2}=\frac{1-i\sqrt{3}}{2}.\frac{1+i\sqrt{3}}{2}$

$=\frac{1+3}{2(1+i\sqrt{3})}=\frac{2}{1+i\sqrt{3}}$

$\Rightarrow \frac{z_2-z_3}{z_1-z_3}=\frac{1+i\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$\Rightarrow \left|\frac{z_2-z_3}{z_1-z_3}\right|=1$ and $\arg\left(\frac{z_2-z_3}{z_1-z_3}\right)=\frac{\pi}{3}$

Hence, the triangle is equilateral.

99. Since sides of an equilateral triangle make an angle of 60° with each other, therefore $\frac{z_3-z_1}{z_2-z_1}=\cos 60^\circ\pm\sin 60^\circ=\frac{1\pm i\sqrt{3}}{2}$

$\Rightarrow 2z_3-2z_1+z1-z_2=\pm i(z_2-z_1)\sqrt{3}\Rightarrow(2z_3-z_1-z_2)^2=3(z_2-z_1)^2\Rightarrow z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

$\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 - z_z^2 - z_2^2 - z_3^2 + z_1z_2 - z_1z_2 + z_2z_3 - z_2z_3 + z_1z_3 - z_1z_3 = 0$

$\Rightarrow (z_1 - z_2)(z_2 - z_3) + (z_2 - z_3)(z_3 - z_1) + (z_3 - z_1)(z_1 - z_2) = 0 \Rightarrow \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$

100. Since it is an equilateral triangle, therefore centroid and circumcenters would be identical. $\therefore z_0 = \frac{z_1+z_2+z_3}{3}$

Since it is an equilateral triangle, we have just proven that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

From first equation, we have $\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$

$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2) \Rightarrow 3z_0^2 = z_1^2 + z_2^2 + z_3^2.$

101. Since right angle is at z_3, therefore $\frac{z_2 - z_3}{z_1 - z_3} = e^{i\pi/2} = i \Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2 \Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2 + 2z_1z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = -2z_3^2 + 2z_2z_3 + 2z_1z_3 - 2z_1z_2 \Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$

102. Clearly, $|z - z_0|^2 = r^2 \Rightarrow (z - z_0)(\overline{z - z_0}) = r^2 \Rightarrow (z - z_0)(\overline{z} - \overline{z_0}) = r^2$

$\Rightarrow z\overline{z} - \overline{z}z_0 - z\overline{z_0} + z_0\overline{z_0} = r^2.$

103. Given, $z = 1 - t + i\sqrt{t^2 + t + 2}$; comparing real and imaginary parts, we get $x = 1 - t, y = \sqrt{t^2 + t + 1} \Rightarrow y^2 = t^2 + t + 2$

$\Rightarrow y^2 = (1 - x)^2 + (1 - x) + 2 = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$, which is equation of a hyperparabola.

104. Given, $\overline{z} = \overline{a} + \frac{r^2}{z - a} \Rightarrow (\overline{z} - \overline{a})(z - a) = r^2$, which is equation of a circle with center at a and radius r.

105. Since z_1 and z_2 are ends of diameter $\Rightarrow |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2 \Rightarrow k = |z_1 - z_2|^2 = |2 + 3i - 4 - 3i|^2 = 4.$

106. $z = x + iy$, then $|(x + 1) + iy| = \sqrt{2}|(x - 1) + iy|$

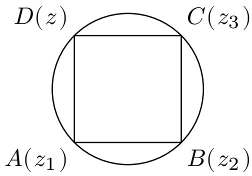
Squaring both sides, we get $(x + 1)^2 + y^2 = 2[(x - 1)^2 + y^2] \Rightarrow x^2 + y^2 - 6x + 1 = 0$, which is equation of a circle.

107. Given, $\left|\frac{z-1}{z-i}\right| = 1 \Rightarrow |z - 1| = |z - i|$

Let $z = x + iy$, then we have $|(x - 1) + iy| = |x + i(y - 1)|$

Squaring both sides, we get $\Rightarrow (x - 1)^2 + y^2 = x^2 + (y - 1)^2 \Rightarrow 2x = 2y \Rightarrow x = y$, which is equation of a straight line.

108.



$\angle z_1 = \arg\left(\frac{z_1-z_2}{z_1-z_4}\right)$, $\angle z_2 = \arg\left(\frac{z_3-z_2}{z_1-z_2}\right)$, $\angle z_3 = \arg\left(\frac{z_3-z_4}{z_3-z_2}\right)$, and $\angle z_4 = \arg\left(\frac{z_1-z_4}{z_3-z_4}\right)$

$\angle z_1 + \angle z_3 = \pi \Rightarrow \arg\frac{z_1-z_2}{z_1-z_4} + \arg\left(\frac{z_3-z_4}{z_3-z_2}\right) = \pi$

$\Rightarrow \arg\left(\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}\right) = \pi \Rightarrow \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is real number.

109. Given, $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{z_2-z_1}{z_3-z_1} = -\frac{z_2}{z_3} \Rightarrow \arg\left(\frac{z_2-z_1}{z_3-z_1}\right) = \pi - \arg\frac{z_3}{z_2}$

$\Rightarrow \arg\left(\frac{z_2-z_1}{z_3-z_1}\right) + arg\left(\frac{z_3-0}{z_2-0}\right) = \pi$ Thus, the given points and the origin are concyclic.

110. From the equation of circle, $r^2 = |\omega - \omega^2|^2 \Rightarrow r^2 = |i\sqrt{3}|^2 = 3 \Rightarrow r = \sqrt{3}$.

111. Let $z = x + iy \Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2 \Rightarrow x^2 - 8x + 16 < x^2 - 4x + 4 \Rightarrow 4x > 12 \Rightarrow x > 3$.

112. Given, $2z_1 - 3z_2 + z_3 = 0 \Rightarrow z_2 = \frac{2z_1+z_3}{3} = \frac{2z_1+z_3}{2+1}$

Thus, z_1 divides the line segement z_1z_3 in the ratio of $2:1$ i.e. all three points are collinear.

113. Given, $|z+1| = |z-1| \Rightarrow (x+1)^2 + y^2 = (x-1)^2 + y^2 \Rightarrow x = 0$

Also, given that $\arg\frac{z-1}{z+1} = \frac{\pi}{4} \Rightarrow z - 1 = (z+1)e^{i\pi/4} \Rightarrow -1 + iy = (1+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\Rightarrow -1 + iy = (1+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \Rightarrow y = \sqrt{2} + 1.$

114. Given, $|z|^8 = |z-1|^8 \Rightarrow |z| = |z-1|, \Rightarrow x^2 + y^2 = (x-1)^2 + y^2 \Rightarrow x = \frac{1}{2}, y \in (\infty, \infty)$, which is equation of straight line parallel to y-axis at $x = 1/2$.

115. Given, $z\overline{z} + a\overline{z} + \overline{a}z + b = 0 \Rightarrow z\overline{z} + a\overline{z} + \overline{a}z + a\overline{a} = a\overline{a} - b$

$(z+a)(\overline{z}+\overline{a}) = |a|^2 - b$, which is equation of a circle if $|a|^2 - b > 0 \Rightarrow |a|^2 > b$.

116. Let $z = x + iy$, comparing real and imaginary part gives us $x = \lambda + 3, y = \sqrt{3 - \lambda^2} \Rightarrow y^2 = 3 - \lambda^2$

$\Rightarrow (x-3)^2 + y^2 = 3$, which is equation of a circle with center $(3,0)$ and radius $\sqrt{3}$.

117. Let $z = x + iy$, then $|Re(z)| + |Im(z)| = k$ will give us four equations. $x + y = k, x - y = k, -x + y = k$ and $-x - y = k$

These lines will intersect at $(k,0), (0,k), (-k,0), (0-k)$ giving us a square as locus of z.

118. $z_2 = z_1^2 + i = i$, $z_3 = z_2^2 + i = i - 1$, $z_4 = z_3^2 + i = (i-1)^2 + i = -i$, $z_5 = z_4^2 + i = i - 1$, $z_6 = z_5^2 + i = -i$

Thus, we see that it is a cycle between $-i$ and $i-1$ starting at z_3. $\Rightarrow z_{111} = z_3 = i - 1 \Rightarrow |z_{111}| = \sqrt{2}$

119. Given, $z\overline{z}^3 + z^3\overline{z} = 350 \Rightarrow z\overline{z}(\overline{z}^2 + z^2) = 350$

Let $z = x + iy$, then given equation becomes $2(x^2+y^2)(x^2-y^2) = 350 \Rightarrow (x^2 + y^2)(x^2-y^2) = 175$

Prime factors of 175 are $5, 5, 7$ so the only solution which yields integers for x and y are $x^2+y^2 = 25, x^2-y^2 = 7$

$\Rightarrow x = \pm 4, y = \pm 3$ which gives a rectangle with four points and digonal with a length of 10 units.

120. We know that $z_1 + z_2$ and $z_1 - z_2$ are the diagonals of a quadrilateral. Now diagonals of a parallelogram does not intersect at angle $\pi/2$ and diagonals of a square and rectangle are equal. Only rhombus satisfies the given criteria of diagonals meeting at right angle and having different lengths. Thus, the given conditions represent a rhombus but not a square.

121. Let $\arg(z_1) = \theta, \arg(z_2) = \theta + \alpha \Rightarrow \frac{az_1}{bz_2} = \frac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} = e^{-i\alpha}$

$\Rightarrow \frac{bz_2}{az_1} = e^{i\alpha} \Rightarrow \frac{az_1}{bz_2} + \frac{bz_2}{az_1} = e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$

Thus, it will lie on the line segment $[-2, 2]$ of the real axis.

122. Since z_1, z_2, z_3 are roots of the equation $z^3 + 3\alpha z^2 + 3\beta z + \gamma = 0 \Rightarrow z_1 + z_2 + z_3 = -3\alpha, z_1z_2 + z_2z_3 + z_3z_1 = 3\beta, z_1z_2z_3 = \gamma$

We know that for a triangle to be equilateral $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$\Rightarrow (z_1+z_2+z_3)^2 = 3(z_1z_2 + z_2z_3 + z_3z_1) \Rightarrow 9\alpha^2 = 3.3\beta \Rightarrow \alpha^2 = \beta.$

123. Given, $z_1^2 + z_2^2 + 2z_1z_2\cos\theta = 0$ Dividing both sides with z_2^2, we get $\left(\frac{z_1}{z_2}\right)^2 + 1 + 2\frac{z_1}{z_2}\cos\theta = 0$

The above equation is a quadratic equation in $\frac{z_1}{z_2}, \therefore \frac{z_1}{z_2} = \frac{-2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2}$

$\Rightarrow \frac{z_1}{z_2} = -\cos\theta \pm i\sin\theta \Rightarrow \left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2| \Rightarrow |z_1 - 0| = |z_2 - 0|$

Thus, z_1, z_1 and the origin form an isosceles triangle.

124. Since origin is circumcenter $\Rightarrow |z_1| = |z_2| = |z_3| = |z| \Rightarrow z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} = z\overline{z}$

$\because AP \perp BC \therefore \frac{z-z_1}{\overline{z}-\overline{z_1}} + \frac{z_2-z_3}{\overline{z_2}-\overline{z_3}} = 0 \Rightarrow \frac{z-z_1}{\frac{z\overline{z_1}}{z}-\overline{z_1}} + \frac{z_2-z_3}{\frac{z_3\overline{z_3}}{z}-\overline{z_3}} = 0$

$\Rightarrow \frac{z(z-z_1)}{z_1\overline{z_1}-z\overline{z_1}} + \frac{z_2(z_2-z_3)}{z_3\overline{z_3}-z_2\overline{z_3}} = 0 \Rightarrow \frac{-z(z_1-z)}{\overline{z_1}(z_1-z)} - \frac{z_2(z_3-z_2)}{\overline{z_3}(z_3-z_2)} = 0 \Rightarrow \frac{-z}{z_1} - \frac{z_2}{z_3} = 0 \Rightarrow z = -\frac{z_1z_2}{z_3}.$

125. Given $OA = OB$, $\Rightarrow |z_1| = |z_2| = l$ (let). Also given, $\arg(z_1) = \alpha + \arg(z_2) \Rightarrow z_1 = le^{i(\alpha+\arg(z_2))} = le^{i\arg(z_2)}.e^{i\alpha} = z_2e^{i\alpha}$

Now, $z_1z_2 = q \Rightarrow z_2^2e^{i\alpha} = q$ and $z_1 + z_2 = -p \Rightarrow z_2(1+e^{i\alpha}) = -p \Rightarrow 2z_2\cos\frac{\alpha}{2}.e^{i\alpha/2} = -p \Rightarrow p^2 = 4z_2^2\cos^2\frac{\alpha}{2}.e^{i\alpha} \Rightarrow p^2 = 4q\cos^2\frac{\alpha}{2}$.

126. Let $z + iy$, then $\Re\left(\frac{z+4}{2x-i}\right) = \Re\left(\frac{x+4+iy}{2x+i(2y-1)}\right) \Rightarrow \Re\left(\frac{[(x+4)+iy][(2x-i(2y-1))]}{4x^2+(2y-1)^2}\right) = \frac{1}{2}$

$\Rightarrow \frac{2x(x+4)+y(2y-1)}{4x^2+(2y-1)^2} = \frac{1}{2} \Rightarrow 16x + 2y - 1 = 0$, which is equation of a straight line.

127. Since the circle is inscribed in $|z| = 2$ so center is origin. Also, since z_1, z_2 and z_3 are in clockwise direction $z_2 = z_1e^{-i120^\circ}$, $z_3 = z_2e^{-i120^\circ}$

$\Rightarrow z_2 = (1+\sqrt{3}i)[(\cos.-120^\circ + i.\sin-120^\circ)] = 1-\sqrt{3}i \Rightarrow z_3 = -2$.

128. Given $z_1 = \frac{a}{1-i} \Rightarrow z_1 = \frac{a+ia}{2}$, $z_2 = \frac{b}{2+i} = \frac{2b-ib}{5}$ Also given, $z_1 - z_2 = 1 \Rightarrow 5a + i5a - 4b + i2b = 10$

Comparing real and imaginary parts, we get $5a - 4b = 10, 5a + 2b = 0 \Rightarrow a = \frac{2}{3}, b = -\frac{5}{3}$
Cnetroid is $\frac{z_1+z_2+z_3}{3} = \frac{1}{3}(1+7i)$.

129. From the quadratic equation we have $z_1 + z_2 = -1$ and $z_1z_2 = \frac{\lambda}{2}$. Since $0, z_1, z_2$ form an equilateral triangle, $\Rightarrow z_1z_2 + z_2.0 + z_1.0 = z_1^2 + z_2^2 + 0^2$

$\Rightarrow (z_1+z_2)^2 = 3z_1z_2 \Rightarrow (-1)^2 = 3.\frac{\lambda}{2} \Rightarrow \lambda = \frac{2}{3}$.

130. Let A, B, C represent a,b,c and U, V, W represent u, v, w. $\Rightarrow AB = b - c, BC = c - b = (a-b)(1-r), CA = a - c = r(a-b)$

$\Rightarrow UV = v - u, VW = w - v = (u-v)(1-r), WU = u - w = r(u-v) \Rightarrow \frac{AB}{UV} = \frac{BC}{VW} = \frac{CA}{WU}$ Thus, the triangles are similar.

131. Let z_1 and z_2 be points on real axis which circle cuts with. Since these are on real axis and if z represents this points then $z = \overline{z}[\because z = x + i.0]$

Substituting $z = \overline{z}$ in the equation of the circle, we get $z^2 + (\overline{\alpha}+\alpha)z + r = 0$ Since z_1, z_2 are the roots $\therefore z_1 + z_2 = -\alpha, z_1z_2 = r$

Length of intercept $= |z_1 - z_2| = \sqrt{(z_1-z_2)^2} = \sqrt{(z_1+z_2)^2 - 4z_1z_2} = \sqrt{(\overline{\alpha}+\alpha)^2 - 4r}$.

132. Clearly, $a = e^{i\alpha}, b = e^{i\beta}, c = e^{i\gamma}$. Also given, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1 \Rightarrow e^{i(\alpha-\beta)} + e^{i(\beta-\gamma)} + e^{i(\gamma-\alpha)} = 1$.

Comparing real parts, we get $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = 1$.

133. Let $A(z_1), B(z_2)$ be the centers of given circles and P be the center of the variable circle which touches given circles externally, then

$|AP| = a + r$ and $|BP| = b + r$ where r is the radius of the variable circle. Clearly, $|AP| - |BP| = a - b \Rightarrow ||AP| - |BP|| = |a - b|$ =a constant.

Hence, locus of P is a right bisector if $a = b$, a hyperbola if $|a - b| < |AB|$ an empty set of $|a - b| > |AB|$, set of all points on line AB except those which lie between A and B if $|a - b| = |AB| \neq 0$.

134. Let $a + ib = re^{i\theta}, r^2 = a^2 + b^2 \Rightarrow a - ib = e^{-i\theta}, \tan\theta = \frac{b}{a}\,\frac{a-ib}{a+ib} = e^{-2i\theta} \Rightarrow i\log\left(\frac{a-ib}{a+ib}\right) = i\log e^{-2i\theta} = 2\theta$

$\Rightarrow \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2b/a}{1-b^2/a^2} = \frac{2ab}{a^2-b^2}$.

135. Given, $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1\ \Re(z_1\overline{z_2}) = 0 \Rightarrow \Re[(a + ib)(c - id)] = 0 \Rightarrow ac + bd = 0$

$a^2 + b^2 = c^2 + d^2 \Rightarrow (a + ic)^2 = (d - ib)^2 [\because ac == bd] \Rightarrow a + ic = d - ibor - d + ib \Rightarrow a = d$ and $c = -b$ or $a = -d, c = b$

$\Rightarrow a^2 + c^2 = b^2 + d^2 = 1 \Rightarrow |w_1| = |w_2| = 1 \Rightarrow \Re(w_1\overline{w2}) = \Re[(a + ic)(b - id)] = ab + cd = 0.$

136. Let $z_1 = r(\cos\theta + i\sin\theta)$. Given, $\left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2| = r$. Also given, $\arg(z_1z_2) = 0 \Rightarrow \arg(z_1) + \arg(z_2) = 0$

$\Rightarrow \arg(z_2) = -\theta \Rightarrow z_2 = r[\cos(-\theta) + i\sin(-\theta)] = r[\cos\theta - i\sin\theta] = \overline{z_1} \Rightarrow \overline{z_2} = z_1 \Rightarrow |z_2|^2 = z_1z_2$.

137. $t_n = (n + 1)\left(n + \frac{1}{\omega}\right)\left(n + \frac{1}{\omega^2}\right) = n^3 + n^2\left(1 + \frac{1}{\omega} + \frac{1}{\omega^2}\right) + n\left(1 + \frac{1}{\omega} + \frac{1}{\omega^2}\right) + 1$

$= n^3 + n^2(1 + \omega + \omega^2) + n(1 + \omega + \omega^2) + 1 = n^3 + 1 \therefore S_n = \sum_{i=1}^{n} t_i = \sum_{i=1} (i^3 + 1) = \frac{n^2(n+1)^2}{4} + 1.$

138. Given $|z_1 + iz_2| = |z_1 - iz_2| \Rightarrow (z_1 + iz_2)(\overline{z_1} - i\overline{z_2}) = (z_1 - iz_2)(\overline{z_1 + i\overline{z_2}})$

$\Rightarrow \overline{z_1}z_2 = z_1\overline{z_2} \Rightarrow \frac{z_1}{z_2} = \frac{\overline{z_1}}{\overline{z_2}}$. Thus, $\frac{z_1}{z_2}$ is purely real.

139. $z = -2 + 2\sqrt{3}i = 4\omega \Rightarrow z^{2n} + 2^{2n}z^n + 2^{4n} = 4^{2n}[\omega^{2n} + \omega^n + 1]$

The above expression has value of 0 if n is not a multiple of 3 and 3.4^{2n} if n is multiple of 3.

140. $x + \frac{1}{x} = 2\cos\theta, \Rightarrow x^2 - 2\cos\theta x + 1 = 0 \Rightarrow x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

Similarly, $y = e^{\pm i\phi} \Rightarrow \frac{x}{y} + \frac{y}{z} = 2\cos(\theta - \phi)$ and $xy + \frac{1}{xy} = 2\cos(\theta + \phi)$.

141. Given, $|z_1| = |z_2|, \mathfrak{R}(z_1) > 0$ and $\mathfrak{I}(z_1) < 0$ $\mathfrak{R}\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{1}{2}\left(\frac{z_1+z_2}{z_1-z_2} + \frac{\overline{z_1}+\overline{z_2}}{\overline{z_1}-\overline{z_2}}\right)$

$= \frac{1}{2}\left(\frac{2(|z_1|^2-|z_2|^2)}{|z_1-z_2|^2}\right) = 0$ Thus, $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

142. Given, $\frac{AB}{BC} = \sqrt{2} \Rightarrow \frac{z_1-z_2}{z_3-z_2} = \frac{|z_1-z_2|}{|z_3-z_2|}.e^{i\pi/4}$

$= \frac{AB}{BC}.e^{i\pi/4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 1+i \Rightarrow z_1-z_2 = (1+i)(z_3-z_2) \Rightarrow z_2 = z_3+i(z_1-z_3).$

143. Given, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11 \Rightarrow z_1^3 - 3z_1z_2^2 + iz_2(3z_1^2 - z_2^2) = 2 + 11i \Rightarrow (z_1 + iz_2)^3 = 2 + 11i$, and

$\Rightarrow z_1^3 - 3z_1z_2^2 - iz_2(3z_1^2 - z_2^2) = 2 - 11i \Rightarrow (z_1 - iz_2)^3 = 2 - 11i$

Multiplying above equations, we get $(z_1^2 + z_2^2)^3 = 4 + 121 = 125 \Rightarrow z_1^2 + z_2^2 = 5.$

144. Given $\sqrt{1 - c^2} = nc - 1 \Rightarrow 1 - c^2 = n^2c^2 - 2nc + 1 \Rightarrow \frac{c}{2n} = \frac{1}{1+n^2}$

$\frac{c}{2n}(1 + nz)\left(1 + \frac{n}{z}\right) = \frac{1}{1+n^2}\left[1 + n^2 + n\left(z + \frac{1}{z}\right)\right]$

$= \frac{1}{1+n^2}\left[1 + n^2 + 2\cos\theta + n\right] = 1 + \frac{2n}{1+n^2}\cos\theta = 1 + c\cos\theta.$

145. If $P(z)$ is any point of the ellipse, then equation of ellipse is given by $|z-z_1|+|z-z_2| = \frac{|z_1-z_2|}{e}$

If we put z_1 or z_2 in the above equation then L.H.S. becomes $|z_1 - z_2|$. Thus, for any interior point of the ellipse, we have $|z - z_1| + |z - z_2| < \frac{|z_1-z_2|}{e}$

If $P(z)$ lies on the ellipse, we have $|z - z_1| + |z - z_2| = \frac{|z_1-z_2|}{e}$. It is given that origin is an internal point, so $|0 - z_1| + |0 - z_2| < \frac{|z_1-z_2|}{e} \Rightarrow e \in \left(0, \frac{|z_1-z_2|}{|z_1|+|z_2|}\right)$.

146. Let $z = x + iy$, then we have $|(x - 2) + i(y - 1)| = |z|\left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right|$ where, $\theta = \arg(z)$

$\Rightarrow \sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}|x - y|$, which is equation of a parabola.

147. Since $|z - z_1| = |z - z_2|$, therefore z will be one of the vertices of the isosceles triangle where base will be formed by z_1 and z_2.

Also, since $\left|z - \frac{z_1+z_2}{2}\right| \leq r$ so z will lie on the circle whose center is $\frac{z_1+z_2}{2}$ and radius is r. Thus, the distance between segment z_1z_2 will be r. Thus, the maximum area of the triangle will be $\frac{1}{2}|z_1 - z2|.r$.

148. Given $|z_1| = 1 \Rightarrow a_1^2 + b_1^2 = 1, |z_2| = 2 \Rightarrow a_2^2 + b_2^2 = 4$. Also given $\mathfrak{R}(z_1z_2) = 0 \Rightarrow a_1a_2 - b_1b_2 = 0 \Rightarrow a_1a_2 = b_1b_2$

$\Rightarrow a_2^2 + b_2^2 = 4a_1^2 + 4b_1^2 \Rightarrow a_2^2 - 4a_1^2 = 4b_1^2 - b_2^2 \Rightarrow a_2^2 - 4a_1^2 + 4ia_1a_2 = 4b_1^2 - b_2^2 + 4ib_1b_2$
$\Rightarrow (a_2 + 2ia_1)^2 = (2b_1 + ib_2)^2 \Rightarrow a_2 = \pm 2b_1$

$\omega_1 = a_1 + \frac{ia_2}{2} = a_1 \pm b_1 \Rightarrow |\omega_1| = \sqrt{a_1^2 + b_1^2} = 1$ $\omega_2 = 2b_1 + ib_2 = \pm a_2 + ib_2 \Rightarrow |\omega_2| = \sqrt{a_2^2 + b_2^2} = 2$ $\Re(\omega_1\omega_2) = 2a_1b_1 - 2a_2b_2 = 0$.

149. Given $z^2 + az + a^2 = 0 \Rightarrow z = a\omega, a\omega^2$ where ω is cube-root of unity.

Thus, it represents a pair of straight lines and $|z| = |a|$. $\arg(z) = \arg(a) + \arg(\omega)$ or $\arg(a) + \arg(\omega^2) = \pm\frac{2\pi}{3}$.

150. Given $x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0 \therefore x = -\omega, -\omega^2$. Now, for $x = -\omega, p = \omega^{4000} + \frac{1}{\omega^{4000}} = \omega + \frac{1}{\omega} = -1$

Similarly, for $x = -\omega^2$, $p = -1 \Rightarrow 2^{2^n} = 2^{4k} = 16^k =$ a number with last digit as $6 \Rightarrow q = 6 + 1 = 7 \Rightarrow p + q = -1 + 7 = 6$.

151. $A(z_1) = \frac{2i}{\sqrt{3}}, B(z_2) = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 1 - \frac{i}{\sqrt{3}}, C(z_3) = \frac{2}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -1 - \frac{i}{\sqrt{3}}$

Clearly, the points lie on the circle $z = 2/\sqrt{3}$ and $\triangle ABC$ is equilateral and its centroid coincides with circumcentre. Hence,

$z_1 + z_2 + z_3 = 0$ and $\overline{z_1} + \overline{z_2} + \overline{z_3} = 0$. Clearly, radius of incircle $= \frac{1}{\sqrt{3}}$ hence any point on circle is $\frac{1}{\sqrt{3}}(\cos\alpha + i\sin\alpha)$. $AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\overline{z_1} + \overline{z}z_1)$

$\Rightarrow AP^2 + BP^2 + CP^2 = 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\overline{z_1} + \overline{z_2} + \overline{z_3}) - \overline{z}(z_1 + z_2 + z_3)$
$= 3 \times \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{4}{3} - 0 - 0 = 5$.

152. Let O be the center of the polygon and $z_0, z_1, \ldots, z_{n-1}$ represent the vertices $A_1, A_2, \ldots, A_n$. $\therefore z_0 = 1, z_1 = \alpha, z_2 = \alpha^2, \ldots, z_{n-1} = \alpha^{n-1}$ where $\alpha = e^{i2\pi/n}$

$|A_1A_2|^2 = |\alpha^r - 1|^2 = |1 - \alpha^r|^2 = \left|1 - \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right|^2 = \left(1 - \cos\frac{2r\pi}{n}\right)^2 + \sin^2\frac{2r\pi}{n} = 2 - 2\cos\frac{2r\pi}{n}$

$\sum_{r=1}^{n} |A_1A_2|^2 = 2(n-1) - 2\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{3} + \ldots + \cos\frac{2(n-1)\pi}{n}\right] = 2(n-1) - 2$. real part of $(\alpha + \alpha^2 + \ldots + \alpha^{n-1}) = 2n[\because 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0]$

$|A_1A_2||A_1A_3|\ldots|A_1A_n| = |1 - \alpha||1 - \alpha^2|\ldots|1 - \alpha^{n-1}| = |(1 - \alpha)(1 - \alpha^2)\ldots(1 - \alpha^{n-1})|$

Since $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are roots of $z^n - 1 = 0$. $(z-1)(z-\alpha)(z-\alpha^2)\ldots(z-\alpha^{n-1}) = z^n - 1 \Rightarrow (z - \alpha)(z - \alpha^2)\ldots(z - \alpha^{n-1}) = \frac{z^n - 1}{z - 1} = 1 + z + z^2 + \ldots + z^{n-1}$

Putting $z = 1$, we get $|(1 - \alpha)(1 - \alpha^2)\ldots(1 - \alpha^{n-1})| = n \Rightarrow \frac{a}{b} = 2$.

153. Let L.H.S. $= z_1$ and R.H.S. $= z_2$ then $\overline{z_1} = \overline{z_2} \Rightarrow z_1\overline{z_1} = z_2\overline{z_2} \Rightarrow z_1^2 = z_2^2$

$\Rightarrow \left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)... = A^2+B^2.$

154. Given, $x+iy+\alpha\sqrt{(x-1)^2+y^2}+2i=0$. Equating real and imaginary parts, we get

$y+2=0 \Rightarrow y=-2$ and $x+\alpha\sqrt{(x-1)^2+y^2}=0$. Substituting the value of y, we get $\alpha\sqrt{x^2-2x+5}=-x \Rightarrow (\alpha^2-1)x^2-2\alpha^2x+5\alpha^2=0$

Because x is real, the discriminant has to be greater than zero. $\Rightarrow 4\alpha^4-20\alpha^2(\alpha^2-1)\geq 0$ $\Rightarrow \alpha^2-5\alpha^2+5\geq 0 \Rightarrow -\frac{\sqrt{5}}{2}\leq \alpha \leq \frac{\sqrt{5}}{2}$.

155. Let $z=x+iy \Rightarrow 2\sqrt{x^2+y^2}-4a(x+iy)+1+ia=0$. Equating real and imaginary parts, we get

$2\sqrt{x^2+y^2}-4ax+1=0$ and $-4ay+a=0 \Rightarrow y=\frac{1}{4} \Rightarrow 2\sqrt{x^2+\frac{1}{16}}-4ax+1=0 \Rightarrow$ $4\left(x^2+\frac{1}{16}\right)=16a^2x^2-8ax+1$

$x^2(4-16a^2)+8ax-\frac{3}{4}=0 \Rightarrow x=\frac{-a}{1-4a^2}\pm\frac{1}{4}\frac{\sqrt{4a^2+3}}{1-4a^2}$.

156. $(x+iy)^5=(x^5-10x^3y^2+5xy^4)+i(5x^4y-10x^2y^3+y^5)$. Taking modulus and squaring, we get $(x^2+y^2)^5=(x^5-10x^3y^2+5xy^4)+(5x^4y-10x^2y^3+y^5)^2$.

157. $(x+ia)(x+ib)(x+ic)=[(x^2-ab)+i(a+b)x](x+ic)=(x^3-abx-acx-bcx)+$ $i(cx^2-abc+ax^2+bx^2)$

Taking modulus and squaring, we get $(x^2+a^2)(x^2+b^2)(x^2+c^2)=[x^3-(ab+bc+ca)x]+[(a+b+c)x^2-abc]^2$.

158. Given, $(1+x)^n=a_0+a_1x+a_2x^2+...+a_nx^n$. Substituting $x=i$,we get

$(1+i)^n=a_o+ia_1-a_2-ia_3+a_4+...=(a_0-a_2+a_4-...)+i(a_1-a_3+a_5-...)$

Taking modulus and squaring, we get $2^n=(a_0-a_2+a_4-...)^2+(a_1-a_3+a_5-...)^2$.

159. Let $f(z)=m(z-i)+i$ and $f(z)=n(z+i)+1+i$ where m and n are quotients upon division. Substituting $z=i$ in the first equation and $z=-i$ in the second we obtain $f(i)=i$ and $f(-i)=1+i$.

Let $g(z)$ be the quotient and $az+b$ be the remainder upong division of $f(z)$ by z^2+1. Hence we have $f(z)=g(z)(z^2+1)+az+b$. Substituting $z=i$ and $z=-i$, we get

$f(i)=i=ai+b$ and $f(-i)=1+i=-ai+b$. Adding, we get $2b=1+2i \Rightarrow b=$ $\frac{1+2i}{2} \Rightarrow ai=i-\frac{1+2i}{2}$.

160. Let $z=r_1e^{i\theta_1}, w=r_2e^{i\theta_2}$. $\because |z|\leq 1$ and $|w|\leq 1 \Rightarrow r_1\leq 1$ and $r_2\leq 1$

$|z-w|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2 = r_1^2 + r_2^2 - 2r_2r_2\cos(\theta_1 - \theta_2) = (r_1 - r_2)^2 + 2r_2r_2 - 2r_2r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 4r_1r_2\sin\left(\frac{\theta_1-\theta_2}{2}\right)^2 \leq (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2 [\because r_1, r_2 \leq 1 \text{ and } \sin\theta \leq \theta]$
$= (|z| - |w|)^2 + [\arg(z) - \arg(w)]^2$.

161. Let $z = re^{i\theta}$, then $\frac{z}{|z|} = e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \left|\frac{z}{|z|} - 1\right| = |(\cos\theta - 1) + i\sin\theta| = \sqrt{\cos\theta^2 - 2\cos\theta + 1 + \sin^2\theta}$

$= \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}} = 2\sin\frac{\theta}{2} \leq \theta \Rightarrow \left|\frac{z}{|z|} - 1\right| \leq |arg(z)|$.

162. Clearly, $|z-1| = |z - |z| + |z| - 1| \leq |z - |z|| + ||z| - 1| = |z|\left|\frac{z}{|z|} - 1\right| + ||z| - 1|$

Using the result of previous problem, we get $|z-1| \leq ||z| - 1| + |z||argz|$.

163. Let $z = r(\cos\theta + i\sin\theta)$, then $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$, $\left|z + \frac{1}{z}\right| = \left|\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right|$

$\Rightarrow \left(r + \frac{1}{r}\right)^2\cos^2\theta + i\left(r - \frac{1}{r}\right)^2\sin^2\theta = a^2 \Rightarrow \left(r - \frac{1}{r}\right)^2 = a^2 - 4\cos^2\theta$

r will be greatest when $r - \frac{1}{r}$ will be greatets i.e. $\cos\theta = 0 \Rightarrow r - \frac{1}{r} = a \Rightarrow r_{max} = \frac{a+\sqrt{a^2+4}}{2}$

Similarly, for lowest value of r, $\cos\theta = 1 \Rightarrow r - \frac{1}{r} = a^2 - 4 \Rightarrow r^2 - (a^2-4)r - 1 = 0$
$r_{min} = \frac{a^2-4-\sqrt{a^4-8a^2+20}}{2}$.

164. We have to prove that $|z_1 + z_2|^2 < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2 \Rightarrow (z_1+z_2)(\overline{z_1} + \overline{z_2}) < (1+c)|z_1|^2 + \left(1+\frac{1}{c}\right)|z_2|^2$

$\Rightarrow |z_1|^2 + z_1\overline{z_2} + z_2\overline{z_1} + |z_1|^2 < (1+c)|z_1|^2 + \left(1+\frac{1}{c}\right)|z_2|^2 \Rightarrow z_1\overline{z_2} + z_2\overline{z_1} < (1+c)|z_1|^2 + \left(1+\frac{1}{c}\right)|z_2|^2$

$\Rightarrow (x_1 + iy_1)(x_2 - iy_2) + (x_2 + iy_2)(x_1 - iy_1) < \frac{1}{c}[c^2(x_1^2 + y_1^2) + (x_2^2 + y_2^2)] \Rightarrow 2cx_1x_2 + 2cy_1y_2 < c^2x_1^2 + c^2y_1^2 + x_2^2 + y_2^2$

$\Rightarrow (cx_1 - x_2)^2 + (cy_1 - y_2)^2 > 0$ which is true.

165. Given $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1 \Rightarrow |z_1 - z_2|^2 = |z_1 + z_2|^2 \Rightarrow (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = (z_1+z_2)(\overline{z_1} + \overline{z_2})$

$\Rightarrow 2z_1\overline{z_2} = -2z_2\overline{z_1} \Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = -\frac{z_1}{z_2} \Rightarrow \frac{z_1}{z_2} =$ purely imaginary $\Rightarrow i\frac{z_1}{z_2} =$ real $= x$

Now $\frac{z_1+z_2}{z_1-z_2} = \frac{z_1/z_2+1}{z_1/z_2-1} = \frac{-ix+1}{-ix-1} = \frac{-1+x^2+2ix}{1+x^2}$. If θ is the angle between given lines then $\tan\theta = \arg\frac{z_1+z_2}{z_1-z_2} = \frac{2x}{x^2-1}$.

166. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Also let $a = r\cos\alpha, b = r\sin\alpha$. $|az_1 + bz_2|^2 = |rr_1(\cos\theta_1 + i\sin\theta_1)\cos\alpha + rr_2(\cos\theta_2 + i\sin\theta_2)\sin\alpha|^2$

$= r^2(r1\cos\theta_1\cos\alpha + r_2\cos\theta_2\sin\alpha)^2 + r^2(r_1\sin\theta_1\cos\alpha + r_2\sin\theta_2\sin\alpha)^2 = r^2[r_1^2\cos^2\alpha + r_2^2\sin^2\alpha + 2r_1r_2\cos\alpha\sin\alpha\cos(\theta_1 - \theta_2)]$

$= \frac{r^2}{2}[r_1^2(1 + \cos 2\alpha) + r_2^2(1 - \cos 2\alpha) + 2r_1r_2\sin 2\alpha\cos(\theta_1 - \theta_2)]$ $\frac{2|az_1+bz_2|^2}{a^2-b^2} = r_1^2 + r_2^2 + (r_1^2 - r_2^2)\cos 2\alpha + 2r_2r_2\cos(\theta_1 - \theta_2)\sin 2\alpha$

$= A + B\cos 2\alpha + C\sin 2\alpha$ where $A = r_1^2 + r_2^2$, $B = r_1^2 - r_2^2$, $C = 2r_1r_2\cos(\theta_1 - \theta_2)$ Clearly, $-\sqrt{B^2 + C^2} \le B\cos 2\alpha + C\sin 2\alpha \le \sqrt{B^2 + C^2}$

$\therefore A - \sqrt{B^2 + C^2} \le A + B\cos 2\alpha + C\sin 2\alpha \le A + \sqrt{B^2 + C^2}$ $\therefore A - \sqrt{B^2 + C^2} \le \frac{2|az_1+bz_2|^2}{a^2+b^2} \le A + \sqrt{B^2 + C^2}$

Now $B^2 + C^2 = r_1^4 + r_2^4 - 2r_1^2r_2^2 + 4r_1^2r_2^2\cos^2(\theta_1 - \theta_2)$. Again $|z_1^2 + z_2^2| = |r_1^2(\cos 2\theta_1 + i\sin 2\theta_1) + r_2^2(\cos 2\theta_2 + i\sin 2\theta_2)| = \sqrt{(r_1^2\cos 2\theta_1 + r_2^2\cos 2\theta_2)^2 + (r_1^2\sin 2\theta_1 + r_2^2\sin 2\theta_2)^2}$

$= \sqrt{r_1^4 + r_2^4 + 2r_1^2r_2^2\cos 2(\theta_1 - \theta_2)} = \sqrt{r_1^4 + r_2^4 + 2r_1^2r_2^2[2\cos^2(\theta_1 - \theta_2) - 1]} = \sqrt{B^2 + C^2}$

$A = r_1^2 + r_2^2 = |z_1|^2 + |z_2|^2$ Hence, $|z_1|^2 + |z_2|^2 - |z_1^2 + z_2^2| \le 2\frac{|az_1+bz_2|^2}{a^2+b^2} \le |z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2|$.

167. Given $z = \frac{b+ic}{1+a} \therefore iz = \frac{-c+ib}{1+a} \Rightarrow \frac{1}{iz} = \frac{1+a}{-c+ib}$. Using componendo and dividendo, we get $\Rightarrow \frac{1+iz}{1-iz} = \frac{1+a-c+ib}{1+a+c-ib}$. Also, given $a^2 + b^2 + c^2 = 1 \Rightarrow a^2 + b^2 = 1 - c^2$

$\Rightarrow (a + ib)(a - ib) = (1 + c)(1 - c) \Rightarrow \frac{a+ib}{1-c} = \frac{1+c}{a-ib} = \frac{1}{u}(\text{say}) \therefore \frac{1+iz}{1-iz} = \frac{a+ib+1-c}{1+c+a-ib} = \frac{a+ib+u(a+ib)}{1+c+u(1+c)} = \frac{a+ib}{1+c}$.

168. We can write that $(x - a)(x - b)\ldots(x - k) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

Substituting $x = i$, we get $(i - a)(i - b)\ldots(i - k) = i^n + p_1i^{n-1} + p_2i^{n-2} + \ldots + p_{n-1}i + p_n$. Dividing both sides by i^n, we get $(1 + ia)(1 + ib)\ldots(1 + ik) = 1 + \frac{p_1}{i} + \frac{p_2}{i^2} + \ldots$

Taking modulus and squaring, we get $(1 + a^2)(1 + b^2)\ldots(1 + k^2) = (1 - p_2 + p_4 + \ldots)^2 + (p_1 - p_3 + \ldots)^2$.

169. $3 + 2i$ is one value of x for which $f(3 + 2i) = a + ib \Rightarrow x = 3 + 2i \Rightarrow x^2 - 6x + 13 = 0$

$f(x) = x^4 - 8x^3 + 4x^2 + 4x + 39 = (x^2 - 6x + 13)(x^2 - 2x - 21) - 96x + 312$ $\Rightarrow f(3 + 2i) = -96(3 + 2i) + 312 = 24 - 192i = a + ib \Rightarrow a : b = 1 : -8$.

170. Given $\frac{A}{B} + \frac{B}{A} = 1 \Rightarrow A^2 - AB + B^2 = 0$. $A = \frac{B\pm\sqrt{3}iB}{2} = -\omega B, -\omega^2 B \Rightarrow |A| = |B|$

$|A - B| = |-\omega B - B|$ or $|-\omega^2 B - B| = |\omega^2 B|$ or $|\omega B| \Rightarrow |A - B| = |B|$. Thus, $|A| = |B| = |A - B|$ making the triangle equilateral.

171. Given $z^n = (z+1)^n \Rightarrow |z|^n = |z+1|^n \Rightarrow |z| = |z+1| \Rightarrow x^2 = (x^2+2x+1) \Rightarrow 2x+1 = 0$, which is the equation of a straight line on which roots of the given equation will lie.

172. Let z_1, z_2, z_3, z_4 be represented by the points A, B, C, D respectively. $\therefore AD = |z_1 - z_4|$ and $BC = |z_2 - z_3|$

Let $a = (z_1 - z_4)(z_2 - z_3)$, $b = (z_2 - z_4)(z_3 - z_1)$ and $c = (z_3 - z_4)(z_1 - z_2)$ $b + c = (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = -(z_1 - z_4)(z_2 - z_3) = -a$

$|a| = |b + c| \leq |b| + |c| \Rightarrow |-(z_1 - z_4)(z_2 - z_3)| = |(z_2 - z_4)(z_3 - z_1)| + |(z_3 - z_4)(z_1 - z_2)| \Rightarrow AD.BC \leq BD.CA + CD.AB.$

173. Euqation of a line joining points a and ib is $\begin{bmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ ib & i\overline{b} & 1 \end{bmatrix} = 0$ or $(\overline{a} + i\overline{b})z - (a - ib)\overline{z} - i(a\overline{b} + \overline{a}b) = 0$

$\Rightarrow (a + ib)z - (a - ib)\overline{z} - 2abi = 0[\because a, b \in R \therefore a = \overline{a}, b = \overline{b}] \Rightarrow (a + ib)z - (a - ib)\overline{z} = 2abi \Rightarrow \left(\frac{1}{2a} - \frac{i}{2b}\right)z + \left(\frac{1}{2a} + \frac{i}{2b}\right)\overline{z} = 1.$

174. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

Then $r_1 - r_2 = \sqrt{(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2}$

$\Rightarrow 2r_1 r_2 = 2r_1 r_2 \cos(\theta_1 - \theta_2) \Rightarrow \cos(\theta_1 - \theta_2) = \cos 2n\pi \Rightarrow \arg(z_1) - \arg(z_2) = 2n\pi.$

175. $\triangle ABC$ and $\triangle DOE$ will be similar if $\frac{AC}{AB} = \frac{DE}{DO}$ and $\angle BAC = \angle ODE$

$\Rightarrow \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{z_5 - z_4}{0 - z_4}\right|$ and $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{z_5 - z_4}{0 - z_4}\right)$

$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \frac{z_5 - z_4}{0 - z_4}$. Solving this yields $(z_3 - z_2)z_4 = (z_1 - z_2)z_5$ and hence triangles are similar.

176. Given $OA = 1$ and $|z| = 1 = OP \Rightarrow OA = OP$. $OP_0 = |z_0|$ and $OQ = |z\overline{z_0}| = |z||\overline{z_0}| = |z_0|$

$\Rightarrow OP_0 = OQ$. Also given that $\angle P_0OP = \arg\frac{z_0}{z}$. $\angle AOQ = \arg\left(\frac{1}{z\overline{z_0}}\right) = \arg\left(\frac{\overline{z}}{\overline{z_0}}\right) [\because z\overline{z} = 1]$

$= -\arg\left(\frac{\overline{z_0}}{\overline{z}}\right) = -\arg\overline{\left(\frac{z_0}{z}\right)} = \arg\left(\frac{z_0}{z}\right) = \angle P_0OP$ and thus the triangles are congruent.

177. $P = \frac{az_2 + bz_1}{a + b}, Q = \frac{az_2 - bz_1}{a - b}$ $OP^2 = \left|\frac{az_2 + bz_1}{a + b}\right|^2 = \left(\frac{az_2 + bz_1}{a + b}\right)\left(\frac{a\overline{z_2} + b\overline{z_1}}{a + b}\right)$

$= \frac{1}{a^2 + b^2}[a^2|z_2|^2 + b^2|z_1|^2 + ab(z_1\overline{z_2} + \overline{z_1}z_2)]$. Similarly OQ^2 can be computed and the sum be found.

178. Let $c \neq 0$, then $c = -(a + b)$ so we can write $az_1 + bz_2 - (a + b)z_3 = 0 \Rightarrow z_3 = \frac{az_1 + bz_2}{a + b}$.

Thus, we see that z_3 divides line segment z_1z_2 in the ratio of $a : b$ making all three of them collinear.

179. Equation of a line passing through origin is $a\overline{z} + \overline{a}z = 0$. Let us assume that all the points lie on the same side of the above line, so we have

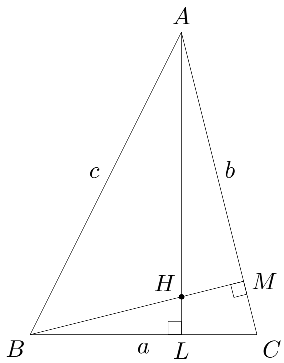
$a\overline{z_i} + \overline{a}z_i > 0$ or < 0 for $i = 1, 2, 3, \ldots, n$. Thus, $a\sum_{i=1}^{n}\overline{z_i} + \overline{a}\sum_{i=1}^{n} z_i > 0$ or < 0

But it is given that $\sum_{i=1}^{n} z_i = 0 \Rightarrow \sum_{i=1}^{n}\overline{z_i} = 0 \therefore a\sum_{i=1}^{n}\overline{z_i} + \overline{a}\sum_{i=1}^{n} z_i = 0$, which is in contradiction with equation above. So all points cannot lie on the same side of line.

180. Let OA and OB be the unit vectors representing z_1 and z_2, then we have $\overrightarrow{OA} = \frac{z_1}{|z_1|}, \overrightarrow{OB} = \frac{z_2}{|z_2|}$

Therefore equation of bisector will be $z = t\left(\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right) = \frac{6}{5}t$, where is an arbitrary positive integer.

181. The diagram is given below:



Let AL be perpendicular on BC and H be orthocenter of the $\triangle ABC$.

$\frac{BL}{LC} = \frac{c\cos B}{b\cos C} = \frac{c\sec C}{b\sec B}$, thus L divides BC internally in the ratio of $c\sec C : b\sec B$, $L = \frac{z_3 c\sec C + z_2 b\sec B}{c\sec C + b\sec B}$

$\frac{AH}{HL} = \frac{\Delta ABH}{\Delta HBL} = \frac{\frac{1}{2}AB.BH\sin\angle ABM}{\frac{1}{2}BL.BH.\sin\angle MBC} = \frac{c\cos A}{c\cos B\cos C}[\because \angle ABM = 90^\circ - A, \angle MBC = 90^\circ - C]$

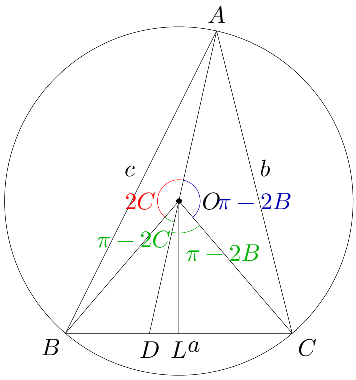
$= \frac{a\cos A}{a\cos B\cos C} = \frac{(b\cos C + c\cos B)\cos A}{a\cos B\cos C} = \frac{b\sec B + c\sec C}{a\sec A}$

$H = \frac{z_1 a\sec A + z_2 b\sec B + z_3 c\sec C}{a\sec A + b\sec B + c\sec C}$

Since the above expression is similar w.r.t. A, B and C, therefore it will also lie on the perpendiculars from B and C to opposing sides as well. Thus, orthocenter $H = \frac{z_1 a\sec A + z_2 b\sec B + z_3 c\sec C}{a\sec A + b\sec B + c\sec C}$

$H = \frac{z_1 k\sin A\sec A + z_2 k\sin B\sec B + z_3 k\sin C\sec C}{k\sin A\sec A + k\sin B\sec B + k\sin C\sec C}$, $H = \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$.

182. The diagram is given below:



Let O be the circumcenter of $\triangle ABC$ where $A = z_1$, $B = z_2$ and $C = z_3$. $\frac{BD}{DC} = \frac{\frac{1}{2}BD.OL}{\frac{1}{2}DC.OL} = \frac{\Delta BOD}{\Delta COD}$

$= \frac{\frac{1}{2}OB.OD.\sin(\pi-2C)}{\frac{1}{2}OC.OD\sin(\pi-2C)} = \frac{\sin 2C}{\sin 2B}$. Thus, D divides BC internally in the ratio $\sin 2C : \sin 2B \Rightarrow D = \frac{z_3\sin 2C + z_2\sin 2B}{\sin 2C+\sin 2B}$

The complex number dividing AD internally in the ratio $\sin 2B + \sin 2C : \sin 2A$ is $\frac{z_1\sin 2A+z_2\sin 2B+z_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}$

Since the above expression is similar w.r.t. A, B and C, therefore it will also lie on the perpendicular bisectors on AC and AB as well.

Let BO produced meet AC at E and CO produced meet AB at F. We can show that, the complex numner representing the point dividing the line segment BE internally in the ratio $(\sin 2C + \sin 2A) : \sin 2B$ and the complex number representing the point dividing the line segment CF internally in the ratio $(\sin 2A + \sin 2B) : \sin 2C$ will be each $= \frac{z_1\sin 2A+z_2\sin 2B+z_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}$

Thus, circumcenter is $\frac{z_1\sin 2A+z_2\sin 2B+z_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}$

183. Let z be the circumcenter of the triangle represented by $A(z_1), B(z_2)$ and $C(z_3)$ respectively, then $|z-z_1| = |z-z_2| = |z-z_3|$ so we have $|z-z_1| = |z-z_2| \Rightarrow |z-z_1|^2 = |z-z_2|^2 \Rightarrow (z-z_1)(\overline{z}-\overline{z_1}) = (z-z_2)(\overline{z}-\overline{z_2})$

$\Rightarrow z\overline{z}+z_1\overline{z_1}-\overline{z}z_1-z\overline{z_1} = z\overline{z}+z_2\overline{z_1}-\overline{z}z_2-z\overline{z_2} \Rightarrow z(\overline{z_1}-\overline{z_2})+\overline{z}(z_1-z_2) = z_1\overline{z_1}-z_2\overline{z_2}$

Similarly considering $|z-z_1| = |z-z_3|$, we will have $\Rightarrow z(\overline{z_1}-\overline{z_3})+\overline{z}(z_1-z_3) = z_1\overline{z_1}-z_3\overline{z_3}$

We have to eliminate $\overline{z}$ from equation (1) and (2) i.e. multiplying equation (1) with (z_1-z_3) and (2) with (z_1-z_2), we get following

$z[\overline{z_1}(z_2-z_3)+\overline{z_2}(z_3-z_1)+\overline{z_3}(z_1-z_2)] = z_1\overline{z_1}(z_2-z_3)+z_2\overline{z_2}(z_3-z_1)+z_3\overline{z_3}(z_1-z_2) \Rightarrow z = \frac{\sum z_1\overline{z_1}(z_2-z_3)}{\sum\overline{z_1}(z_2-z_3)}$.

184. Let z be the orthocenter of $\triangle A(z_1)B(z_2)C(z_3)$ i.e. the intersection point of perpendiculars on sides from opposite vertices.

Since $AH \perp BC \therefore \arg\left(\frac{z_1-z}{z_3-z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \frac{z_1-z}{z_3-z_2}$ is purely imaginary.

$\Rightarrow \overline{\left(\frac{z_1-z}{z_3-z_2}\right)} = -\left(\frac{z_1-z}{z_3-z_2}\right) \Rightarrow \frac{\overline{z_1}-\overline{z}}{\overline{z_3}-\overline{z_2}} = \frac{z-z_1}{z_3-z_2} \Rightarrow \overline{z_1}-\overline{z} = \frac{(z-z_1)(\overline{z_3}-\overline{z_2})}{z_3-z_2}$

Similarly for $BH \perp AC$, $\overline{z_2}-\overline{z} = \frac{(z-z_2)(\overline{z_1}-\overline{z_2})}{z_1-z_3}$

Eliminating $\overline{z}$ like last problem we arrive at the desired result.

185. We have $\angle CBA = \frac{2\pi}{3}$, therefore $\frac{z_3-z_2}{z_1-z_2} = \frac{|z_3-z_2|}{|z_1-z_2|}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\,[\because BC = AB]$

$z_3 + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)z_1 = \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right)z_2$

Solving this yields $2\sqrt{3}\,z_2 = (\sqrt{3} - i)\,z_1 + (\sqrt{3} + i)\,z_3$. Also, since diagonals bisect each other $\Rightarrow \frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$, $z_4 = z_1 + z_3 - z_2$ Substituting the value of z_2, we get $2\sqrt{3}\,z_4 = (\sqrt{3} + i)\,z_1 + (\sqrt{3} - i)\,z_3$.

186. Since $\angle PQR = \angle PRQ = \frac{1}{2}(\pi - \alpha) \therefore PQ = PR$ Also, $\angle QPR = \pi - 2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \alpha$
$\therefore \arg\frac{z_3-z_1}{z_2-z_1} = \alpha \Rightarrow \frac{z_3-z_1}{z_2-z_1} = \frac{PR}{RQ}(\cos\alpha + i\sin\alpha)$

$\Rightarrow \frac{z_3-z_1}{z_2-z_1} - 1 = (\cos\alpha - 1) + i\sin\alpha \Rightarrow \frac{z_3-z_2}{z_2-z_1} = -2\sin^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$\Rightarrow \left(\frac{z_3-z_2}{z_2-z_1}\right)^2 = -4\sin^2\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]^2 = -4\sin^2\frac{\alpha}{2}\left[\cos\alpha + i\sin\alpha\right] = -4\sin^2\frac{\alpha}{2}\cdot\frac{z_3-z_1}{z_2-z_1}$

$\Rightarrow (z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$.

187. Let C be the center of a regular polygon of n sides. Let $A_1(z_1)$, $A_2(z_2)$ and $A_3(z_3)$ be its three consecutive vertices.

$\angle CA_2A_1 = \frac{1}{2}\left(\pi - \frac{2\pi}{n}\right) \therefore A_1A_2A_3 = \pi - \frac{2\pi}{n}$

Case I: When z_1, z_2, z_3 are in anticlockwise order. $\Rightarrow z_1 - z_2 = (z_3 - z_2)\,e^{i(\pi-2\pi/n)}[\because A_1A_2 = A_3A_2]$

$z_1 - z_2 = (z_2 - z_3)\,e^{-i2\pi/n}[\because e^{i\pi} = -1] \Rightarrow z_3 = z_2 - (z_1 - z_2)\,e^{i2\pi/n}$

Case II: When z_1, z_2, z_3 are in clockwise order. $\Rightarrow z_3 - z_2 = (z_1 - z_2)\,e^{i(\pi-i2\pi/n)}$

$z_3 = z_2 + (z_2 - z_1)\,e^{-i2\pi/n}$.

188. Let O be the origin and the complex number representing A_1 be z, then A_2, A_3, A_4 will be represented by $ze^{i2\pi/n}$, $ze^{i4\pi/n}$, $ze^{i6\pi/n}$. Let $|z| = a$

$A_1A_2 = \left|z - ze^{i2\pi/n}\right| = |z|\left|1 - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right| = a\sqrt{\left(1 - \cos\frac{2\pi}{n}\right)^2 + \sin^2\frac{2\pi}{n}} = a\sqrt{2\left(1 - \cos\frac{2\pi}{n}\right)} = 2a\sin\frac{\pi}{n}$

Similarly, $A_1A_3 = 2a\sin\frac{2\pi}{n}$ and $A_1A_4 = 2a\sin\frac{3\pi}{n}$

Given $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4} \therefore \frac{1}{2a\sin\frac{\pi}{n}} = \frac{1}{2a\sin\frac{2\pi}{n}} + \frac{1}{2a\sin\frac{3\pi}{n}} \Rightarrow \sin\frac{\pi}{n}\left(\sin\frac{3\pi}{n} + \sin\frac{2\pi}{n}\right) = \sin\frac{2\pi}{n}\sin\frac{3\pi}{n}$

$\Rightarrow \sin\frac{3\pi}{n} + \sin\frac{2\pi}{n} = 2\cos\frac{2\pi}{n}\sin\frac{3\pi}{n} = \sin\frac{4\pi}{n} + \sin\frac{2\pi}{n} \Rightarrow \sin\frac{3\pi}{n} = \sin\frac{4\pi}{n} \Rightarrow \frac{3\pi}{n} = m\pi + (-1)^n\frac{4\pi}{n}, m = 0, \pm 1, \pm 2, \ldots$

If $m = 0 \Rightarrow \frac{3\pi}{n} = \frac{4\pi}{n} \Rightarrow 3 = 4$ (not possible). If $m = 1 \Rightarrow \frac{3\pi}{n} = \pi - \frac{4\pi}{n} \Rightarrow n = 7$. If $m = 2, 3 \ldots, -1, -2, \ldots$ gives values of n which are not possible. Thus $n = 7$.

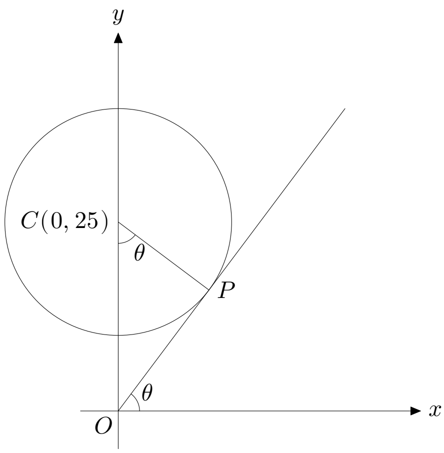
189. Given, $|z| = 2$. Let $z_1 = -1 + 5z \Rightarrow z_1 + 1 = 5z$.

$|z_1 + 1| = |5z| = 5|z| = 10 \Rightarrow z_1$ lies on a circle with center $(-1, 0)$ having radius 10.

190. Given, $|z - 4 + 3i| \leq 2 \Rightarrow ||z| - |4 - 3i|| \leq 2 \Rightarrow ||z| - 5| \leq 2 \Rightarrow -2 \leq |z| - 5 \leq 2 \Rightarrow 3 \leq |z| \leq 7$.

191. $|z - 6 - 8i| \leq |4| \Rightarrow -4 \leq ||z| - |6 + 8i|| \leq 4 \Rightarrow -4 \leq |z| - 10 \leq 10 \Rightarrow 6 \leq |z| \leq 14$.

192. The diagram is given below:



Given $z - 25i \leq 15$, which represents a circle having center $(0, 25)$ and a radius 15. Let OP be tangent to the circle at point P, then $\angle XOP$ will represent least value of $\arg(z)$.

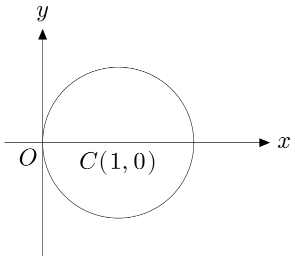
Let $\angle XOP = \theta$ then $\angle OCP = \theta$. Now $OC = 25$, $CP = 15 \therefore OP = 20 \therefore \tan\theta = \frac{OP}{CP} = \frac{4}{3}$. $\therefore$ Least value of $\arg(z) = \theta = \tan^{-1}\frac{4}{3}$

193. Given, $|z - z_1|^2 + |z - z_2|^2 = k \Rightarrow |z|^2 + |z_1|^2 - 2z\overline{z_1} + |z|^2 + |z_2|^2 - 2z\overline{z_2} = k$

$\Rightarrow 2|z|^2 - 2z(\overline{z_1} + \overline{z_2}) = k - (|z_1|^2 + |z_2|^2) \Rightarrow |z|^2 - 2z\left(\frac{\overline{z_1 + z_2}}{2}\right) + \frac{1}{4}|z_1 + z_2|^2 = \frac{k}{2} + \frac{1}{4}[|z_1 + z_2|^2 - 2|z_1|^2 - 2|z_2|^2]$

$\Rightarrow \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}\left[k - \frac{1}{2}|z_1 - z_2|^2\right]$. The above equation represents a circle with center at $\frac{z_1 + z_2}{2}$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{|z_1 - z_2|^2}{2}$.

194. Since $|z - 1| = 1$, z represents a circle with center $(1, 0)$ and a radius of of 1. It is shown below:



Now $|z-1| = 1$. Let $z = x + iy$ then $x^2 + y^2 = 2x$. Also, $\frac{z-2}{z} = \frac{x-2+iy}{x+iy} = \frac{x^2-2x+y^2+2iy}{x^2+y^2} = i\frac{y}{x}$

Case I. When z lies in the first quadrant. This implies $\arg(z) = \theta$, where $\tan\theta = \frac{y}{x} \therefore i\tan[\arg(z)] = i\tan\theta = i\frac{y}{x}$.

Case II. When z lies in the fourth quadrant. Thus, $\arg(z) = 2\pi - \theta$, where $\tan\theta = \frac{-y}{x} \therefore i\tan[\arg(z)] = i\tan(2\pi-\theta) = i\frac{y}{x}$.

195. Let $z = x + iy$. Now we have $\frac{z-1}{z+1} = \frac{(x^2-1)+y^2}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2}$

$\therefore \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \tan\left(\arg\left(\frac{z-1}{z+1}\right)\right) = \frac{2y}{x^2-1+y^2}$

$\Rightarrow x^2 + y^2 - 1 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 2$, which is equation of a circle having center at $(0,1)$ and radius $\sqrt{2}$.

196. Let $z = x + iy$. Now, $u + iv = (z-1)(\cos\alpha - i\sin\alpha) + \frac{1}{z-1}(\cos\alpha + i\sin\alpha) = (x-1)\cos\alpha + y\sin\alpha + i[y\cos\alpha - (x-1)\sin\alpha] + \frac{x-1-iy}{(x-1)^2+y^2}(\cos\alpha + i\sin\alpha) = 0$

Equating imaginary parts, we get $v = y\cos\alpha - (x-1)\sin\alpha + \frac{(x-1)\sin\alpha - y\cos\alpha}{(x-1)^2+y^2} = 0 \Rightarrow [y\cos\alpha - (x-1)\sin\alpha][(x-1)^2 + y^2] = 0$

$\therefore$ Either $y\cos\alpha - (x-1)\sin\alpha = 0 \Rightarrow y = \tan\alpha(x-1)$, which is a straight line passing through $(1,0)$ or $(x-1)^2 + y^2 - 1 = 0$ which is a circle with center $(1,0)$ and unit radius.

197. Given, $1 + a_1z + a_2z^2 + \cdots + a_nz^n = 0 \Rightarrow |a_1z| + |a_2z^2| + \cdots + |a_nz^b| \geq 1$ and

L.H.S. $< 2|z| + 2|z|^2 + \cdots$ to $\infty[\because |a_n| < 2]$.

Let $|z| < 1$ then $\frac{2|z|}{1-|z|} < 1 \Rightarrow |z| > \frac{1}{3}$

When $|z| > 1$, clearly $|z| > \frac{1}{3}$; hence, z does not lie in the interior of the circle with radius $\frac{1}{3}$.

198. Given, $z^n\cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + \cos\theta_n = 2 \Rightarrow 2 = |z^n\cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + \cos\theta_n|$

$< |z^n\cos\theta_0| + |z^{n-1}\cos\theta_1| + \cdots + |\cos\theta_n| = |z^n||\cos\theta_0| + |z^{n-1}||\cos\theta_1| + \cdots + |\cos\theta_n|$

$\leq |z|^n + |z|^{n-1} + \cdots + 1 < 1 + |z| + |z|^2 + \cdots$ to $\infty \Rightarrow 2 = \frac{1}{1-|z|} \Rightarrow |z| > \frac{1}{2}$[when $|z| < 1$]

Hence z lies outside the circle $|z| = \frac{1}{2}$. Thus all roots of the given equation lie outside the circle $|z| = \frac{1}{2}$.

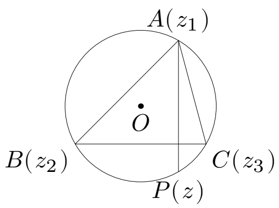
199. Recall that points z_1, z_2, z_3 are concyclic if $\left(\frac{z_2-z_4}{z_1-z_4}\right)\left(\frac{z_1-z_3}{z_2-z_3}\right)$ is real. We assume that z_4 is origin.

Given, $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} = \frac{z_2+z_3}{z_2z_3} \therefore z_1 = \frac{2z_2z_3}{z_1+z_3}$.

Putting the value of z_1 and z_4 in the concyclic condition expression we obtain

$\left(\frac{z_2-z_4}{z_1-z_4}\right)\left(\frac{z_1-z_3}{z_2-z_3}\right) = \frac{1}{2}$. Thus, z_1, z_2, z_3 lie on a circle passing through origin.

200. The diagram given below:

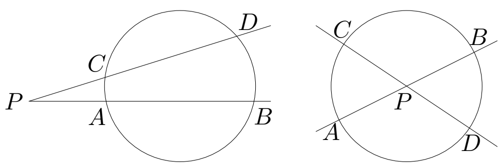


We have $OP = OA = OB = OC \therefore |z| = |z_1| = |z_2| = |z_3| \Rightarrow |z|^2 = |z_1|^2 = |z_2|^2 = |z_3|^2 \Rightarrow z\overline{z} = z_1\overline{z_1} = z\overline{z_2} = z\overline{z_3}$.

Since AP is perpendicular to BC, $\therefore \arg\left(\frac{z_1-z}{z_2-z_3}\right) = \frac{\pi}{2}$ or $\frac{-\pi}{2} \Rightarrow \frac{z_1-z}{z_2-z_3}$ is purely imaginary.

$\Rightarrow \overline{\left(\frac{z_1-z}{z_2-z_3}\right)} = -\frac{z_1-z}{z_2-z_3}$. Solving the above equation gives $z = \frac{z_2z_3}{z_1}$.

201. The diagram is given below:



Let $P(z)$ be the point of intersection and A, B, C, D represent points a, b, c, d respectively. Clearly, P, A, B are collinear. Thus,

$\begin{bmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{bmatrix} = 0 \Rightarrow z(\overline{a} - \overline{b}) - \overline{z}(a - b) + (a\overline{b} - \overline{a}b) = 0$

Similarly, P, C, D are collinear and thus $\Rightarrow z(\overline{c} - \overline{d}) - \overline{z}(c - d) + (c\overline{d} - \overline{c}d) = 0$

Eliminating $\overline{z}$ because we have to find z, we have $z(\overline{a} - \overline{b})(c - d) - z(\overline{c} - \overline{d})(a - b) = (c\overline{d} - \overline{c}d)(a - b) - (a\overline{b} - \overline{a}b)(c - d)$.

$\because a$, b, c, d lie on the circle. $|a| = |b| = |c| = |d| = r \Rightarrow a^2 = b^2 = c^2 = d^2 = r^2$ $\Rightarrow a\overline{a} = b\overline{b} = c\overline{c} = d\overline{d} = r^2$

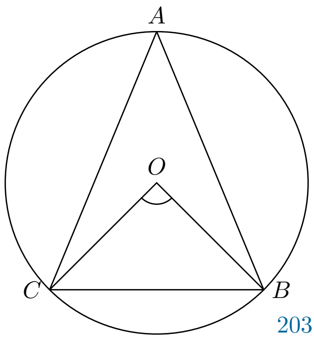
$\Rightarrow \overline{a} = \frac{r^2}{a}, \overline{b} = \frac{r^2}{b}, \overline{c} = \frac{r^2}{c}, \overline{d} = \frac{r^2}{d}$

Putting these values in the equation we had obtained, $z\left(\frac{r^2}{a} - \frac{r^2}{b}\right)(c - d) - z\left(\frac{r^2}{c} - \frac{r^2}{d}\right)(a - b) = \left(\frac{cr^2}{d} - \frac{dr^2}{c}\right)(a - b) - \left(\frac{ar^2}{b} - \frac{br^2}{a}\right)(c - d)$

Solving this for z, we arrive at desired answer.

202. Given $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = 0 \Rightarrow a^3+b^3+c^3-3abc = 0 \Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$

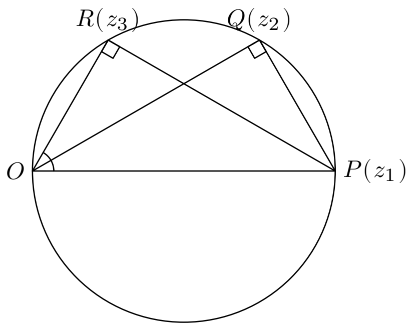
$\because z_1, z_2, z_3$ are three non-zero complex numbers, hence $a^2+b^2+c^2-ab-bc-ca=0$ $\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2 = 0 \Rightarrow a=b=c$. This can be represented by following diagram:



Now $OA = OB = OC$, where O is the origin and A, B and C are the points representing z_1, z_2 and z_3 respectively. $\therefore O$ is the circumcenter of $\triangle ABC$.

Now $\arg\left(\frac{z_3}{z_2}\right) = \angle BOC = 2\angle BAC = \arg\left(\frac{z_3-z_1}{z_2-z_1}\right)^2$.

203. The diagram is given below:



$z_2 = \frac{OQ}{OP} z_1 e^{i\theta} = \cos\theta z_1 e^{i\theta}$ and $z_3 = \frac{OR}{OP} z_1 e^{i2\theta} = \cos 2\theta z_1 e^{i2\theta}$

$\Rightarrow z_2^2 = \cos^2\theta z_1^2 e^{i2\theta} \Rightarrow z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$.

204. Given circles are $|z| = 1 \Rightarrow x^2+y^2-1=0$ and $|z-1| = 4 \Rightarrow x^2-2x+y^2-15=0$.

Let the circles cut by these two orthogonally is $x^2+y^2+2gx+2fy+c=0$. Since first circle cuts this family of circles orthoginally, therefore

$2g.0 + 2f.0 = c-1 \Rightarrow c = 1$ and $2g(-1)+2f.0 = c-15 \Rightarrow g = 7$. Thus, required circles are $x^2+y^2+14x+2fy+1=0 \Rightarrow |z+7+if| = \sqrt{48+f^2}$.

205. Given, $|z+3| = t^2-2t+6$ which is equation of a circle having center $(-3,0)$ and radius t^2-2t+6. Let $A = (-3,0)$ and $r_1 = t^2-2t+6$. In this case z lies on the circle.

Also, $|z - 3\sqrt{3}i| < t^2$ implies z lies on the interior of the circle having center $(0, 3\sqrt{3})$ and radius t^2. Let $B = (0, 3\sqrt{3})$ and $r_2 = t^2$. $AB = \sqrt{3^2 + 27} = 6$. $r_2 - r_1 = 2(t - 3)$

Clearly, when the two circles are disjoint or touching each other no solution is possible. This leads to following cases:

Case I: When $t > 3$ i.e. $r2 > r_1$. In this case at least one z is possible if $AB < r_1 + r_2 \Rightarrow 6 < 2(t^2 - t + 3) \Rightarrow t < 0$ or $t > 1 \Rightarrow 3 < t < \infty$

Case II: When $t \leq 3$ i.e. $r_1 > r_2$. In this case at least one z will be possible if $|r_1 - r_2| \leq AB < r_1 + r_2$

$2(3 - t) \leq 6 < 2(t^2 - t + 3)$ i.e. $t \leq 0$ and $t < 0$ or $t > 1$ Combining all solutions we gace $1 < t < \infty$.

206. Let $z = x + iy$. $\frac{az+b}{cz+d} = \frac{ax+b+iay}{cx+d+icy} = \frac{(ax+b+iay)(cx+d-icy)}{(cx+d)^2+c^2y^2}$

$\Im\left(\frac{az+b}{cz+d}\right) = \frac{ay(cx+d)-cy(ax+b)}{(cx+d)^2+c^2y^2} = \frac{ady-bcy}{(cx+d)^2+c^2y^2}$

$\because ad > bc$, therefore the signs of imaginary parts of z and $\frac{az+b}{cz+d}$ are the same.

207. Given, $z_1 = \frac{i(z_2+1)}{z_2-1} \Rightarrow x_1 + iy_1 = \frac{-y_2+i(x_2+1)}{(x_2-1)+iy_2} = \frac{[-y_2+i(x_2+1)][(x_2-1)+iy_2]}{(x_2-1)^2+y_2^2}$

Comparing real and imaginary parts, we have

$x_1 = \frac{-y_2(x_2-1)-(x_2+1)y_2}{(x_2-1)^2+y_2^2} = \frac{-2x_2y_2}{(x_2-1)^2+y_2^2}$ and $y_1 = \frac{x_2^2-1-y_2^2}{(x_2-1)^2+y_2^2}$

Substituting for x_1 and y_1 in $x_1^2 + y_1^2 - x_1$ we will arrive at the desired result.

208. $(\cos 3\theta - i\sin 3\theta)^6 = (e^{-i3\theta})^6 = e^{-i18\theta}$ and $(\cos 2\theta + i\sin 2\theta)^5 = (e^i 2\theta)^5 = e^{i10\theta}$

$(\sin\theta - i\cos\theta)^3 = [(-i)^3(\cos\theta + i\sin\theta)^3] = i.e^{i3\theta}$ and $\frac{(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3}{(\cos 2\theta + i\sin 2\theta)^5} = i.e^{-i25\theta} = \sin 25\theta + i\cos 25\theta$.

209. Let $z = x + iy$, then we have $x^2 - y^2 + 2ixy + \sqrt{x^2 + y^2} = 0$

Equating imaginary parts, we have $2xy = 0$ i.e. either $x = 0$ or $y = 0$.

If $x = 0$, then $-y^2 + \sqrt{y^2} = 0 \Rightarrow y^4 - y^2 = 0 \Rightarrow y = 0, y = \pm 1$.

If $y = 0$, then $x^2 + \sqrt{x^2} = 0$ Since x is real only one solution is possible i.e. $x = 0$. Hence, $z = 0, \pm i$.

210. Clearly $z = 0$ is one of the solutions. For other solutions divide both sides by $|z|^2$ which gives us $t^2 + t + 1 = 0$ where $t = \frac{z}{|z|}$.

The equation $t^2 + t + 1 = 0$ has two roots i.e. $t = \omega, \omega^2 \Rightarrow \frac{z}{|z|} = \omega, \omega^2 \Rightarrow z = k\omega, k\omega^2$ where $k = |z|$ is a non-negative real number.

211. Let $z = x + iy$, then $(x + iy)\sqrt{x^2 + y^2} + a(x + iy) + 1 = 0$. Comparing real and imaginary parts, we get

$y\sqrt{x^2 + y^2} + ay = 0 \Rightarrow y = 0 \because \sqrt{x^2 + y^2} + a \neq 0\ [\because a > 0]$ and $\therefore x\sqrt{x^2 + 0} + ax + 1 = 0 \Rightarrow x^2 + ax + 1 = 0 \Rightarrow x = \frac{-a \pm \sqrt{a^2 - 4}}{2}$

Clearly, both the values of x are negative, so z is a negative real number.

212. Let $z = x + iy$, then $x^2 + y^2 - 2i(x + iy) + 2a(1 + i) = 0$. Comparing real and imaginary parts, we get

$x^2 + y^2 + 2y + 2a = 0 \Rightarrow x^2 + (y - 1)^2 = 1 - 2a$ and $-2x + 2a = 0 \Rightarrow x = a$

$\Rightarrow (y - 1)^2 = 1 - 2a - a^2 \Rightarrow y = 1 \pm \sqrt{1 - 2a - a^2}$. However $1 - 2a - a^2 > 0$. Roots of equivalent quadratic equation is $a = \frac{2 \pm \sqrt{8}}{-2} \Rightarrow -1 \pm \sqrt{2}$ but $a > 0$ so the range for a is $0 < a < \sqrt{2} - 1$.

213. i. We have $(3 + 4i)^x = 5^{\frac{x}{2}}$. Squaring both sides $(-7 + 24i)^x = 5^x \Rightarrow \left(\frac{-7+24i}{5}\right)^x = 1$ which is possible only if $x = 0$.

ii. Given $(1 - i)^x = 2^x \Rightarrow \left(\frac{1-i}{2}\right)^x = 1$ which is possible only if $x = 0$.

iii. Given $(1 - i)^x = (1 + i)^x \Rightarrow \left(\frac{1-i}{1+i}\right)^x = 1 \Rightarrow (-i)^x = 1 \Rightarrow x = 0, 4, 8, \ldots, 4n\ \forall\ 4n \in I$.

214. $z^3 + 2z^2 + 2z + 1 = 0 \Rightarrow (z + 1)(z^2 + z + 1) = 0 \Rightarrow z = -1, \omega, \omega^2$.

When $z = -1$, $z^{1985} + z^{100} + 1 = -1 + 1 + 1 = 1 \neq 0$, when $z = \omega$, $\omega^{1985} + \omega^{100} + 1 = \omega^2 + \omega + 1 = 0$ and when $z = \omega^2$, $\omega^{1985*2} + \omega^{200} + 1 = \omega + \omega^2 + 1 = 0$. Thus common roots are ω, ω^2.

215. Adding all equations $\alpha + \beta + \gamma = 3z_1 \Rightarrow z_1 = \frac{\alpha+\beta+\gamma}{3}$. Similarly, multiplying second equatin with ω and third equation with ω^2, and then adding we have $z_3 = \frac{\alpha+\beta\omega+\gamma\omega^2}{3}$. Similarly, $z_2 = \frac{\alpha+\beta\omega^2+\gamma\omega}{3}$.

$|\alpha|^2 = \alpha\overline{\alpha} = (z_1 + z_2 + z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3})$, $|\beta|^2 = \beta\overline{\beta} = (z_1 + z_2\omega + z_3\omega^2)(\overline{z_1} + \overline{z_2}\omega^2 + \overline{z_3}\omega)$ and $|\gamma|^2 = \gamma\overline{\gamma} = (z_1 + z_2\omega^2 + z_3\omega)(\overline{z_1} + \overline{z_2}\omega + \overline{z_3}\omega^2)$ $[\because \overline{\omega} = \omega^2\ \&\ \overline{\omega^2} = \omega]$

$\Rightarrow |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2) + z_1[\overline{z_2}(1 + \omega + \omega^2) + \overline{z_3}(1 + \omega + \omega^2)] + z_2[\overline{z_1}(1 + \omega + \omega^2) + \overline{z_2}(1 + \omega + \omega^2)] + z_3[\overline{z_1}(1 + \omega + \omega^2) + \overline{z_2}(1 + \omega + \omega^2)] = 3(|z_1|^2 + |z_2|^2 + |z_3|^2) =$ R.H.S.

216. Let $f(x) = (x + 1)^n - x^n - 1.x^3 + x^2 + x = 0 \Rightarrow x(x^2 + x + 1) = 0 \Rightarrow x = 0, \omega, \omega^2$. So for $x^3 + x^2 + x$ to be a factor of $f(x)$, $f(0) = 0$, $f(\omega) = 0$, $f(\omega^2) = 0$.

$f(0) = 1^n - 1 = 0$, $f(\omega) = (\omega + 1)^n - \omega^n - 1 = -\omega^{2n} - \omega^n - 1$ $[\because n$ is odd. $] = -(1 + \omega^n + \omega^{2n}) = 0$. Similarly, $f(\omega^2) = 0$. Hence proved.

217. Let $f(x,y) = (x+y)^n - x^n - y^n.xy(x+y)(x^2+xy+y^2) = 0 \Rightarrow x = 0, y = 0, x = -y, y = x\omega, y = x\omega^2$. When $x = 0, f(x,y) = 0; y = 0, f(x,y) = 0; y = -x \Rightarrow f(x,y) = -x^n - (-x)^n = 0[\because n = 2m+1\ \forall\ m \in \mathbb{I}], y = xw \Rightarrow f(x,y) = [x^n(1+\omega)^n - x^n - x^n\omega^n] = -x^n\omega^{2n} - x^n - x^n\omega^n = 0$, and similarly when $y = x\omega^2, f(x,y) = 0$. Hence proved.

218. R.H.S. $= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right| = \left|\frac{\overline{z_1}}{|z_1|^2} + \frac{\overline{z_2}}{|z_2|^2} + \cdots + \frac{\overline{z_n}}{|z_n|^2}\right|$

$= |\overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}| = |\overline{z_1 + z_2 + \cdots + z_n}| = |z_1 + z_2 + \cdots + z_n| =$ L.H.S.

219. For any two complex numbers z_1 and z_2, we know that $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. Let $z_1 = \alpha + \sqrt{\alpha^2-\beta^2}$ and $z_2 = \alpha - \sqrt{\alpha^2-\beta^2}$.

Now $(|z_1|+|z_2|)^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = 2|\alpha|^2 + 2|\alpha^2-\beta^2| + 2|\beta|^2 = |\alpha+\beta|^2 + |\alpha-\beta|^2 + 2|\alpha+\beta||\alpha-\beta|$

$= (|\alpha+\beta| + |\alpha-\beta|)^2 \Rightarrow |z_1| + |z_2| = |\alpha+\beta| + |\alpha-\beta| =$ R.H.S.

220. $|z_1| = |z_1| = 1 \Rightarrow a^2+b^2 = c^2+d^2 = 1, z_1\overline{z_2} = ac+bd+i(bc-ad) \because \Re(z_1\overline{z_2}) = 0 \Rightarrow ac+bd = 0 \Rightarrow \frac{a}{d} = -\frac{b}{c} = k$ (say). $\therefore a = kd, b = -kc$.

$\therefore k^2d^2 + k^2c^2 = 1 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$. Now $|\omega_1| = \sqrt{a^2+c^2} = \sqrt{a^2+b^2} = 1, |\omega_2| = \sqrt{b^2+d^2} = \sqrt{a^2+b^2} = 1, \omega_1\overline{\omega_2} = (a+ic)(b-id) \therefore \Re(\omega\overline{\omega_2}) = ab+cd = 0$.

221. Given, $\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right| < 1 \Leftrightarrow \left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|^2 < 1 \Leftrightarrow |z_1-z_2|^2 < |1-\overline{z_1}z_2|^2$

$\Leftrightarrow (z_1-z_2)\overline{(z_1-z_2)} < (1-\overline{z_1}z_2)\overline{(1-\overline{z_1}z_2)} \Leftrightarrow (z_1-z_2)(\overline{z_1}-\overline{z_2}) < (1-\overline{z_1}z_2)((1-z_1\overline{z_2}))$

$\Leftrightarrow |z_1|^2 + |z_2|^2 > 1 + |z_1|^2|z_2|^2 \Leftrightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 > 0 \Leftrightarrow (1-|z_1|^2)(1-|z_2|^2) > 0 \Rightarrow (1+|z_1|)(1-|z_1|)(1+|z_2|)(1-|z_2|) > 0$

$\Leftrightarrow (1-|z_1|)(1-|z_2|) > 0$ which is true as $|z_1| < 1$ and $|z_2| < 1$.

222. Let $z = x+iy$ then $\frac{z-z_1}{z-z_2} = \frac{(x-10)+i(y-6)}{(x-4)+i(y-6)}$. Rationalizing $\frac{x^2-14x+40+(y-6)^2}{(x-4)^2+(y-6)^2} + \frac{i6(y-6)}{(x-4)^2+(y-6)^2} = a+ib$ (say)

$\because \arg(a+ib) = \frac{\pi}{4} \Rightarrow x^2 - 14x + 40 + (y-6)^2 = 6(y-6) \Rightarrow x^2+y^2-14x-18y+112 = 0 \Rightarrow |z-7-9i|^2 = 18$. Hence proved.

223. Let $z = x+iy$ then $\frac{3z-6-3i}{2z-8-6i} = \frac{x-6+i(3y-3)}{2x-8+i(2y-6)}$. Rationalizing $\frac{6x^2+6y^2-36x-24y+66+i(12x-12y-12)}{(2x-8)^2+(2y-6)^2} = a+ib$ (let)

$\because \arg(a+ib) = \frac{\pi}{4} \Rightarrow 6x^2+6y^2-36x-24y+66 = 12x-12y-12 \Rightarrow x^2+y^2-8x-2y+13 = 0$. Also given, $|z-3+i| = 3 \Rightarrow x = -2y+6$. Substituting this in previously obtained equation, we have

$5y^2 - 10y + 1 = 0 \Rightarrow y = 1 \pm \frac{2}{\sqrt{5}} \Rightarrow x = 4 \mp \frac{4}{\sqrt{5}}$. Hence we have our z.

224. Let $|z| = r_1$, $|w| = r_2$, $\arg(z) = \theta_1$ and $\arg(w) = \theta_2$. Then, $|z-w|^2 = (r_1\cos\theta_1 - r_1\sin\theta_1)^2 + (r_2\cos\theta_2 - r_2\sin\theta_2)^2 = (r_1-r_2)^2 + 2r_1r_2 - 2r_1r_2\cos(\theta_1-\theta_2)$

$= (r_1-r_2)^2 + 4r_1r_2\sin^2\frac{\theta_1-\theta_2}{2} \le (r_1-r_2)^2 + 2.1.1.2\left(\frac{\theta_1-\theta_2}{2}\right)^2 = (|z|-|w|)^2 + (\theta_1-\theta_2)^2$. Hence proved.

225. Let $z = r(\cos\theta + i\sin\theta) \Rightarrow \frac{z}{|z|} = \cos\theta + i\sin\theta \therefore \left|\frac{z}{|z|} - 1\right| = |(\cos\theta - 1) + i\sin\theta| = \sqrt{(\cos\theta-1)^2 + \sin^2\theta} = \sqrt{4\sin^2\frac{\theta}{2}} = 2|\sin\frac{\theta}{2}| \le |\theta|$.

Now, $|z - |z|| = |z - 1 - (|z|-1)| \ge |z-1| - ||z|-1| \therefore |z-1| - ||z|-1| \le |z-|z||$

$\Rightarrow |z-|z|| = |r(\cos\theta + i\sin\theta) - r| = \sqrt{4r^2\sin^2\frac{\theta}{2}} \le 2r\left|\frac{\theta}{2}\right| = r|\theta| = |z||\arg(z)|$

$\Rightarrow |z-1| - ||z|-1| \le |z||\arg(z)| \Rightarrow |z-1| \le ||z|-1| + |z||\arg(z)|$.

226. Let $z = r(\cos\theta + i\sin\theta)$ then $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$. Given $\left|z + \frac{1}{z}\right| = a \Rightarrow \left|\left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta\right| = a$

$\Rightarrow \left(r+\frac{1}{r}\right)^2\cos^2\theta + \left(r-\frac{1}{r}\right)^2\sin^2\theta = a^2 \Rightarrow \left(r-\frac{1}{r}\right) = a^2 - 4\cos^2\theta$. Clearly, r will be greatest if $\cos\theta = 0 \Rightarrow r^2 - ar - 1 = 0 \Rightarrow r = \frac{a\pm\sqrt{a^2+4}}{2}$. This also implies that z is a purely imaginary number.

227. $|z_1+z_2|^2 < |z_1|^2 + c|z_1|^2 + |z_2|^2 + \frac{1}{c}|z_2|^2 \Rightarrow (z_1+z_2)(\overline{z_1}+\overline{z_2}) < |z_1|^2 + |z_2|^2 + \frac{c^2|z_1|^2+|z_2|^2}{c} \Rightarrow z_2\overline{z_1} + z_1\overline{z_2} < \frac{1}{c}(c^2|z_1|^2 + |z_2|^2)$

$\Rightarrow (x_2+iy_2)(x_1-iy_1) + (x_1+iy_1)(x_2-iy_2) < \frac{1}{c}[c^2(x_1^2+y_1^2) + x_2^2 + y_2^2] \Rightarrow (cx_1 - x_2)^2 + (cy_1-y_2)^2 > 0$ which is true.

228. Given, $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1 \Rightarrow |z_1-z_2|^2 = |z_1+z_2|^2 \Rightarrow (z_1-z_2)(\overline{z_1}-\overline{z_2}) = (z_1+z_2)(\overline{z_1}+\overline{z_2})$

$\Rightarrow -z_2\overline{z_1} - z_1\overline{z_2} = z_2\overline{z_1} + z_1\overline{z_2} \Rightarrow z_1\overline{z_2} = -2z_2\overline{z_1} \Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = -\frac{z_1}{z_2}$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary $\Rightarrow \frac{iz_1}{z_2}$ is real, which we take as x.

$\frac{z_1+z_2}{z_1-z_2} = \frac{z_1/z_2+1}{z_1/z_2-1} = \frac{-ix+1}{-ix-1} = \frac{-1+x^2+2ix}{1+x^2}$

If θ is the angle between the lines joining the origin to the points z_1+z_2 and z_1-z_2, then $\tan\theta = \left|\arg\left(\frac{z_1+z_2}{z_1-z_2}\right)\right| = \left|\frac{2x}{x^2-1}\right|$.

229. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Let $\sqrt{a^2+b^2} = r$. Let $a = r\cos\alpha, b = r\cos\alpha$. Now $|az_1 + bz_2|^2 = |rr_1(\cos\theta_1 + i\sin\theta_1)\cos\alpha + rr_2(\cos\theta_2 + i\sin\theta_2)\sin\alpha|^2$

$= r^2[r_1^2\cos^2\alpha + r_2^2\sin^2\alpha + 2r_1r_2\cos\alpha\sin\alpha\cos(\theta_1-\theta_2)] = \frac{r^2}{2}[r_1^2 + r_2^2 + (r_1^2 - r_2^2)\cos 2\alpha + 2r_2r_2\cos(\theta_1-\theta_2)\sin 2\alpha]$

Thus, $|az_1+bz_2|^2 = \frac{r^2}{2}[A+B\cos 2\alpha + C\sin 2\alpha] \Rightarrow \frac{2|az_1+bz_2|^2}{r^2}[A+B\cos 2\alpha+C\sin 2\alpha]$, where $A = r_1^2+r_2^2$, $B = r_1^2 - r_2^2$ and $C = 2r_1r_2\cos(\theta_1-\theta_2)$.

Since $A - \sqrt{B^2+C^2} \leq A + B\cos 2\alpha + C\sin 2\alpha \leq A + \sqrt{B^2+C^2}$

$B^2+C^2 = r_1^4 + r_2^4 - 2r_1^2r_2^2 + 4r_1^2r_2^2\cos^2(\theta_1-\theta_2)$.

$|z_1^2+z_2^2| = |r_1^2(\cos 2\theta_1 + i\sin 2\theta_1) + r_2^2(\cos 2\theta_2 + i\sin 2\theta_2)| = \sqrt{B^2+C^2}$. Hence proved.

230. Given $z = \frac{b+ic}{1+a} \Rightarrow iz = \frac{-c+ib}{1+a} \Rightarrow \frac{1+iz}{1-iz} = \frac{1+a-c+ib}{1+a+c-ib}$

Given, $a^2+b^2+c^2 = 1 \Rightarrow (a+ib)(a-ib) = (1+c)(1-c) \Rightarrow \frac{1+iz}{1-iz} = \frac{a+ib}{1+c}$.

231. Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$. L.H.S. $= |az_1-bz_2|^2 + |bz_1-az_2|^2 = (ax_1-bx_2)^2 + (ay_1-by_2)^2 + (bx_1-ax_2)^2 + (by_1-by_2)^2$

$= (a^2+b^2)(x_1^2+y_1^2) + (a^2+b^2)(x_2^2+y_2^2) = (a^2+b^2)(|z_1|^2+|z_2|^2) =$ R.H.S.

232. Let $\alpha = x_1+iy_1$ and $\beta = x_2+iy_2$. Then $|\alpha+\beta|^2 = (x_1+x_2)^2 + (y_1+y_2)^2 = x_1^2+x_2^2+y_1^2+y_2^2+2x_1x_2+2y_1y_2$.

$|\alpha|^2 = x_1^2+y_1^2$, $|\beta|^2 = x_2^2+y_2^2$, $\Re(\alpha\overline{\beta}) = x_1x_2+y_1y_2$ and $\Re(\overline{\alpha}\beta) = x_1x_2+y_1y_2$. Now it is trivial to prove the equality.

233. $|1-\overline{z_1}z_2|^2 - |z_1-z_2|^2 = (1-\overline{z_1}z_2)(1-z_1\overline{z_2}) - (z_1-z_2)(\overline{z_1}-\overline{z_2}) = (1-\overline{z_1}z_2 - z_1\overline{z_2} + |z_1|^2|z_2|^2) - (|z_1|^2 - \overline{z_1}z_2 - z_1\overline{z_2} + |z_2|^2) = 1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 = (1-|z_1|^2)(1-|z_2|^2) =$ R.H.S.

234. Consider two complex numbers $z_1 = a_1+ib_1$ and $z_2 = a_2+ib_2$. Now we have to prove $|z_1+z_2| \leq |z_1|+|z_2|$ which can be further extended to prove the result.

$\Rightarrow \sqrt{(a_1+a_2)^2+(b_2+b_2)^2} \leq \sqrt{a_1^2+b_1^2} + \sqrt{a_2^2+b_2^2}$.

Squaring both sides and simplifying

$\Rightarrow a_1a_2+b_1b_2 \leq \sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)} \Rightarrow (a_1a_2+b_1b_2)^2 - (a_1^2+b_1^2)(a_2^2+b_2^2) \leq 0 \Rightarrow -(a_1b_2-a_2b_1)^2 \leq 0$.

235. Given, $\left|\frac{\overline{z_1}-2\overline{z_2}}{2-z_1\overline{z_2}}\right| = 1 \Rightarrow |\overline{z_1}-2\overline{z_2}|^2 = |2-z_1\overline{z_2}|^2$

$\Rightarrow (\overline{z_1}-2\overline{z_2})(z_1-2z_2) = (2-z_1\overline{z_2})(2-\overline{z_1}z_2) \Rightarrow |z_1|^2 - 2z_1\overline{z_2} - 2\overline{z_1}z_2 + 4|z_2|^2 = 4 - 2z_1\overline{z_2} - 2\overline{z_1}z_2 + |z_1|^2|z_2|^2$

$\Rightarrow |z_1|^2|z_2|^2 - 4|z_2|^2 - |z_1|^2 - 4 = 0 \Rightarrow |z_2| = 2 \because |z_1| \neq 1$.

236. $\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|$

$=\frac{1}{2}\left|(\sqrt{z_1}+\sqrt{z_2})^2\right|+\frac{1}{2}\left|(\sqrt{z_1}-\sqrt{z_2})^2\right|=|z_1|+|z_2|$

237. We have proven that $|a+\sqrt{a^2-b^2}|+|a-\sqrt{a^2-b^2}|=|a+b|+|a-b|$. Substituting $a=\beta$ and $b=\sqrt{\alpha\gamma}$ we have

$|\beta+\sqrt{\alpha\gamma}|+|\beta-\sqrt{\alpha\gamma}|=|\alpha|\Big(|\frac{\beta}{\alpha}+\sqrt{\frac{\gamma}{\alpha}}|+|\frac{\beta}{\alpha}-\sqrt{\frac{\gamma}{\alpha}}|\Big)$

$=|\alpha|\big(|-z_1-z_2+\sqrt{z_1z_2}|+|-z_1-z_2-\sqrt{z_1z_2}|\big)=|\alpha|(|z_1|+|z_2|)$.

238. We have $|a|=1\Rightarrow|a|^2=1\Rightarrow a\bar{a}=1\Rightarrow\bar{a}=\frac{1}{a}$. Thus, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\bar{a}+\bar{b}+\bar{c}=0[\because a+b+c=0]$

239. $|z+4|\leq 3\Rightarrow -3\leq z+4\leq 4\Rightarrow 0\leq z+1\leq 6$.

240. We have to prove that $(|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|\leq 2|z_1+z_2|$. Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$. Then

$(|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|=(r_1+r_2)|(\cos\theta_1+\cos\theta_2)+i(\sin\theta_1+\sin\theta_2)|=(r_1+r_2)\sqrt{2+2\cos(\theta_1-\theta_2)}$

Also, $4|z_1+z_2|^2=4[(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2]=4[r_1^2+r_2^2+r_1r_2\cos(\theta_1-\theta_2)]$ and squaring L.H.S. we have $2(r_1+r_2)^2[1+\cos(\theta_1-\theta_2)]^2$. Clearly, L.H.S. $\leq$ R.H.S.

241. Given equation is $z^2+az+b=0$. Let p,q are two of its roots. Then we have $p+q=-a$ and $pq=b$. Taking modulus of both we have $|p+q|=|a|$ and $|pq|=b$. Now it is required that $|p|=|q|=1$. Therefore we have $|p+q|\leq|p|+|q|=2\therefore|a|\leq 2$. Similarly, $|b|=|pq|=|p||q|=1$. Since p,q have unit modulii, we can have them as $p=cos\theta_1+isin\theta_1$ and $q=cos\theta_2+isin\theta_2$.

$\arg(b)=\arg(pq)=\arg(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))=\theta_1+\theta_2$

$\arg(a)=\arg(p+q)=\arg[(\cos\theta_1+\cos\theta_2)+i(\sin\theta_1+\sin\theta_2)]=\arg\Big[\Big(\cos^2\frac{\theta_1}{2}+i^2\sin\frac{\theta_1}{2}+2i\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\Big)+\Big(\cos^2\frac{\theta_2}{2}+i^2\sin\frac{\theta_2}{2}+2i\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2}\Big)\Big]$

$=\arg\Big[\cos\frac{\theta_1+\theta_2}{2}+i\sin\frac{\theta_1+\theta_2}{2}\Big]=\frac{\theta_1+\theta_2}{2}$ and hence $\arg(b)=2\arg(a)$.

242. Let $z=x+iy$. First we consider first two inequalities $|z|\leq|\Re(z)|+|\Im(z)|\Rightarrow\sqrt{x^2+y^2}\leq x+y$. Squaring, we have $x^2+y^2\leq x^2+y^2+2xy\Rightarrow 2xy\geq 0$,

which is true. Now we consider last two inequalities, $|\Re(z)|+|\Im(z)|\leq\sqrt{2}|z|\Rightarrow x+y\leq\sqrt{2(x^2+y^2)}$. Squaring, we have $x^2+y^2+2xy\leq 2(x^2+y^2)\Rightarrow(x-y)^2\geq 0$, which is also true.

243. $\left|z-\frac{4}{z}\right|=2\Rightarrow|z|-\frac{4}{|z|}\geq 2\Rightarrow|z|^2-2|z|-4\geq 0$. The greatest root of this equation is $\sqrt{5}+1$. Hence proven.

244. Since $\alpha,\beta,\gamma,\delta$ are roots of the equation. $\therefore a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=ax^4+bx^3+cx^2+dx+e$. Substituting $x=i$, we get following

$a(i-\alpha)(i-\beta)(i-\gamma)(i-\delta)=ai^4+bi^3+ci^2+di+e\Rightarrow a(1+i\alpha)(1+i\beta)(1+i\gamma)(1+i\delta)=a-ib-c+id+e$.

Taking modulus and squaring we get our desired result.

245. $\because \alpha_1,\alpha_2,\ldots,\alpha_n$ are the roots of the given equation. $\therefore (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)=x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0$.

Substituting $x=i$, we get following $(i-\alpha_1)(i-\alpha_2)\cdots(i-\alpha_n)=i^n+a_1i^{n-1}+a_2i^{n-2}+\ldots+a_{n-1}i+a_n$.

Taking modulus and squaring we get our desired result.

246. Let $|z_1|=|z_2|=|z_3|=R$. $\therefore$ Origin is the circumcenter of triangle. Since triangle is also equilateral circumcenter and centroid coincide. Therefore, origin is also centroid. Thus,

$\frac{z_1+z_2+z_3}{3}=0\Rightarrow z_1+z_2+z_3=0$.

247. $z_1+z_2+z_3=0$ implies centroid of the triangle is the origin. Circumcenter is also origin as Z_i lies on the circle $|z|=1$. Hence, circumcenter is same as centroid making the triangle an equilateral triangle having circumcircle with unit radius.

248. Since the triangle is equilateral therefore the circumcenter and centroid will be same i.e. $z_0=\frac{z_1+z_2+z_3}{3}$. Also for equilateral triangle, $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$.

Squaring the first equation $9z_0^2=z_1^2+z_2^2+z_3^2+2(z_1z_2+z_2z_3+z_3z_2)=z_1^2+z_2^2+z_3^2+2(z_1^2+z_2^2+z_3^2)\Rightarrow z_1^2+z_2^2+z_3^2=3z_0^2$.

249. Since z_1,z_2 and origin form an equilateral triangle we have $z_1^2+z_2^2+0^2-z_1z_2-z_2.0-z_1.0=0$. Hence, proven.

250. From previous probelm z_1,z_2 and origin will form a triangle if $z_1^2+z_2^2-z_1z_2=0$. Therefore, $(z_1+z_2)^2=3z_1z_2\Rightarrow a^2=3b$.

251. Since z_1,z_2,z_3 are roots of the equation $z^3+3\alpha z^2+3\beta z+\gamma=0\Rightarrow z_1+z_2+z_3=-3\alpha$, $z_1z_2+z_2z_3+z_3z_1=3\beta$ and $z_1z_2z_3=-\gamma$.

Centroid is given by $\frac{z_1+z_2+z_3}{3}=-\alpha$. Triangle will be equilateral if $z_1^2+z_2^2+z_3^3=z_1z_2+z_2z_3+z_3z_1\Rightarrow(z_1+z_2+z_3)^2=3(z_1z_2+z_2z_3+z_3z_1)\Rightarrow\alpha^2=\beta$.

252. Given $2z_2=z_1+z_3$. Clearly, from section formula we can deduce that z_2 divides line segment joining z_1 and z_3 in two equal segments hence the complex numbers are collinear.

253. If z_1, z_2, z_3 are collinear then either z_2 divides z_1z_3 internally/externally or z_3 divides z_1z_2 internally/externally. Now we can apply the condition for collinearity i.e. $\begin{vmatrix} z_1 & z_2 & z_3 \\ \overline{z_1} & \overline{z_2} & \overline{z_3} \\ 1 & 1 & 1 \end{vmatrix} = 0$ and hence we can show desired conditions.

254. z represents the ring between the concentric circles whose center is at $(3, 4i)$ having radii 1 and 2.

255. Let $z = x + iy \Rightarrow |z|^2 = x^2 + y^2, |z-1|^2 = (x-1)^2 + y^2, |z+1|^2 = (x+1)^2 + y^2$. From given inequailty $|z+1|^2 = 16 + |z-1|^2 - 8|z-1| \Rightarrow 4x = 16 - 8|z-1| \Rightarrow 4|z-1|^2 = (4-x)^2 \Rightarrow 3x^2 + 4y^2 = 12$, which is an equation of an ellipse.

256. Let $z = x + iy$, then $x = t + 5 \Rightarrow x - 5 = t$ and $y = \sqrt{4 - t^2} \Rightarrow y^2 = 4 - t^2 \Rightarrow (x-5)^2 + y^2 = 4$, which is a circle with center $(5, 0)$ and radius 2.

257. Let $z = x + iy$, then $\frac{z^2}{z-1} = \frac{(x^2-y^2+2ixy)[(x-1)-iy]}{(x-1)^2+y^2}$. Since it is real, we can equate the imaginary part to zero.

$\Rightarrow y(y^2 - x^2) + 2x^2y - 2xy = 0 \Rightarrow y = 0$ or $x^2 + y^2 - 2x = 0 \Rightarrow (x-1)^2 + y^2 = 1$. However, $y \neq 0$ else z won't remain a complex number. $\Rightarrow x^+y^2 - 2x = (x-1)^2 + y^2 = 1$, which represents a circle with center at $(1, 0)$ and unit radius.

258. Let $z = x + iy$, then $|z^2 - 1| = |z|^2 + 1 \Rightarrow (x^2 - y^2 - 1)^+ 4x^2y^2 = (x^2 + y^2 + 1)^2 \Rightarrow x = 0$. Hence, locus of z is a straight line specifically imginary axis.

259. Let $z = x + iy$ then $\frac{y}{x} \geq \tan\frac{\pi}{3} \Rightarrow y \geq \sqrt{3}x$. Similarly, $\frac{y}{x} \leq \tan\frac{3\pi}{2} = -\infty$.

This represents the set of straight lines whose slope is greater than $\sqrt{3}$ and less than or equal to $-\infty$.

260. Let $z = x + iy$, then $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3} \Rightarrow \arg\left(\frac{x-2+iy}{x+2+iy}\right) = \frac{\pi}{3}$

$\Rightarrow \arg\left(\frac{x^2+y^2-4+4iy}{(x+2)^2+y^2}\right) = \frac{\pi}{3} \Rightarrow \frac{4y}{x^2+y^2-4} = \sqrt{3}$, which is equation of a circle.

261. Let $z = x + iy$. Given, $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2} \Rightarrow \arg\left(\frac{(x-1)+iy}{(x+1)+iy}\right) = \frac{\pi}{2} \Rightarrow \frac{2y}{x^2+y^2-1} = \infty$.

The above equation implies $x^2 + y^2 - 1 = 0$ and $y > 0$ which is circle at $(0, 0)$ with unit circle above x-axis. The points $(-1, 0)$ and $(1, 0)$ are excluded because that will make the above equation indeterminate.

262. $\log_{\sqrt{3}} \frac{|z|^2-|z|+1}{2+|z|} < 2 \Rightarrow \frac{|z|^2-|z|+1}{2+|z|} < (\sqrt{3})^2 \Rightarrow |z|^2 - 4|z| - 5 < 0 \Rightarrow |z| < 5$.

263. Clearly A is $(1, 0)$ or $(-1, 0)$. Let A is (1, 0). Then $z = \cos 0^\circ + i\sin 0^\circ$. Clearly, B and C would be $\cos 120^\circ + i\sin 120^\circ$ and $\cos 240^\circ + i\sin 240^\circ$. Similarly, B and C can be found if A is $(-1, 0)$.

264. Let z represent A, then $\frac{z-(2-i)}{1+i-(2-i)} = \frac{AM}{MD}e^{\frac{2\pi i}{2}} \Rightarrow z = (2-i) + \frac{i}{2}(-1+2i) \Rightarrow z = 1 - \frac{3}{2}i$ or $3 - \frac{i}{2}$.

265. $\frac{z_1 - z_2}{z_3 - z_2} = r^{\frac{i\pi}{2}} = i \Rightarrow z_3 = -iz_1 + z_2(1+i)$. Similarly, z_4 can be found.

266. $z_1 = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2(\cos 60^\circ + i \sin 60^\circ)$. Therefore, $z_2 = 2(\cos 180^\circ + i \sin 180^\circ) = -2$ and $z_3 = 2(\cos 300^\circ + i \sin 300^\circ)$.

267. We know that three vertices represent an equilateral triangle if $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_1 z_3 = 0$. Substituting the respective values, we get

$a^2 - 1 + 2ai + 1 - b^2 + 2bi - a + b - abi - i = 0 \Rightarrow a^2 - b^2 - a + b = 0 \Rightarrow (a-b)(a+b+1) = 0$. So either $a = b$ or $a + b = -1$ but if we choose $a + b = -1$ then the other part leads us to $ab = 3$ which is not possible.

Choosing $a = b$, the imaginary part becomes $2a + 2b - ab - 1 = 0 \Rightarrow a = 2 \pm \sqrt{3}$. But $a = 2 + \sqrt{3}$ does not make triangle equilateral. So $a = b = 2 - \sqrt{3}$.

268. Let $O = z$ represent center of the sqsuare then $z = \frac{A+C}{2} \Rightarrow C = 4 + 0i = 4$. $AC = AB.\sqrt{2}.e^{\pi/4} \Rightarrow B = 1 + 2i$ and $AD = AB.e^{\pi/2} = 6 + 3i$.

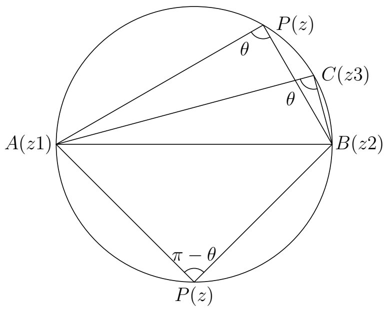
269. Let O be the origin and A_1 the vertex z_1. Let the vertex adjacent to A_1 be A_2. Then $z_2 = z_1 e^{2\pi i/n} \because \angle A_1 O A_2 = \frac{2\pi}{n}$. Similarly, $z_3, z_4, \ldots, z_n$ are other vertices in order, then $z_3 = e^{4\pi i/n}$, $z_4 = e^{6\pi i/n}$, ... Thus, all vertices are given by $z_{r+1} = z_1 e^{2\pi r i/n} = z_1(\cos 2r\pi/n + i \sin 2r\pi/n)$, ..., where $r = 1, 2, \ldots, n-1$.

270. z_1, z_2, z_3 are collinear if $\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$. Substituting a, b, c in this and expnading the determinant it is trivial to obtain the given condition.

271. $PA^2 = 4PB^2 \Rightarrow |z - 6i|^2 = 4|z-3|^2 \Rightarrow x^2 + (y-6)^2 = 4[(x-3)^2 + y^2] \Rightarrow x^2 + y^2 - 8x + 4y = 0$, which represents a circle with center at $(4-2)$ and radius $\sqrt{20}$.

$x^2 + y^2 - 8x + 4y = 0 \Rightarrow x^2 + y^2 = 4(2x) + 2i(2iy) \Rightarrow |z|^2 = 4(z + \overline{z}) + 2i(z - \overline{z}) = (4+2i)z + (4-2i)\overline{z}$.

272. The diagram is given below:



Let three non-collinear points be $A(z_1), B(z_2)$ and $C(z_3)$. Let $P(x)$ be any point on the circle.

Then either $\angle ACB = \angle APB$ (when they are in the same segment) or $\angle ACB + \angle APB = \pi$ (when they are in the opposite segment).

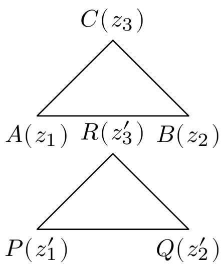
$\arg\left(\frac{z_3-z_2}{z_3-z_1}\right) - \arg\left(\frac{z-z_2}{z-z_1}\right) = 0$ or $\arg\left(\frac{z_3-z_2}{z_3-z_1}\right) + \arg\left(\frac{z-z_1}{z-z_2}\right) = \pi$

$\arg\left[\left(\frac{z_3-z_2}{z_3-z_1}\right)\left(\frac{z-z_1}{z-z_2}\right)\right] = 0$ or $\arg\left[\left(\frac{z_3-z_2}{z_3-z_1}\right)\left(\frac{z-z_1}{z-z_2}\right)\right] = \pi$

In any case, we get $\frac{(z_3-z_2)}{(z_3-z_1)}\frac{(z-z_1)}{(z-z_2)}$ is purely real. Hence, proved.

273. Following from previous problem we have one equation for the condition for the four vertices to be cyclic. Also, sum of all four angles of the quadrilateral is equal to be 2π. From these two equations, the results can be deduced.

274. Consider the following diagram:



$\triangle ABC$ and $\triangle PQR$ will be similar if all their angles are equal and ratios of sides as well.

$arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = arg\left(\frac{z_3'-z_1'}{z_2'-z_1'}\right)$

$\frac{AB}{PQ} = \frac{AC}{PR}$ or $\frac{AC}{AB} = \frac{PR}{PQ}$ or $\frac{z_3-z_1}{z_2-z_1} = \frac{z_3'-z_1'}{z_2'-z_1'}$

Simplifying these two equations gives us our determinant.

275. From these two equations we have $r = \frac{c-a}{b-a}$ and $r = \frac{\omega-u}{v-u}$. Equating these two equations and taking modulus and argument, it follows from the previous problem that the two triangles are similar.

276. We know that points on a perpendicular bisector is equidistant from the two points of the line to which it is perperndicular bisector.

$\Rightarrow |z-z_1| = |z-z_2| \Rightarrow |z-z_1|^2 = |z-z_2|^2 \Rightarrow (z-z_1)(\overline{z}-\overline{z_1}) = (z-z_2)(\overline{z}-\overline{z_2})$, which can be written in the form of $\overline{a}z + a\overline{z} + b = 0$, which is equation of a straight line.

277. Mid-point of such a diameter is $\frac{z_1+z_2}{2}$. Let P be a point lying on this circle, which, is represented by complex number z. Thus, the equation of circle is $\left|z - \frac{z_1+z_2}{2}\right| = \left|z_1 - \frac{z_1+z_2}{2}\right|$ or $\left|z - \frac{z_1+z_2}{2}\right| = \left|z_2 - \frac{z_1+z_2}{2}\right|$. Square and simplify to arrive at the equation.

278. The equation can be written as $|z-z_1| = c|z-z_2|$, which, when substituted with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ gives following

$|(x-x_1)+i(y-y_1)| = c|(x-x_2)+i(y-y_2)| \Rightarrow (x-x_1)^2+(y-y_1)^2 = c^2\{(x-x_2)^2+(y-y_2)^2\}$, which is equation of a circle.

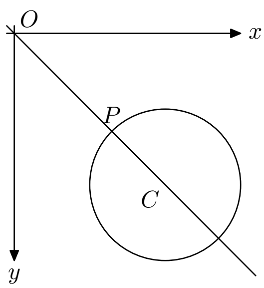
279. Given, $|z| = 1 \Rightarrow 2z\overline{z} = 2 \Rightarrow \frac{2}{z} = 2\overline{z}$ which gives us a circle.

280. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then L.H.S. $= |z_1+z_2|$ $\Rightarrow |z_1+z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)$.

Similarly, $(|z_1|+|z_1|)^2 = \left(r_1^2+r_2^2+2r_1r_2\right)$.

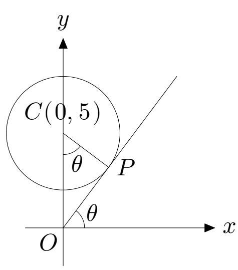
Thus, $\cos(\theta_1-\theta_2) = 0 \Rightarrow \arg(z_1) - \arg(z_2) = 2n\pi$.

281. The diagram is given below:



The equation $|z-2+2i| = 1$ represents a circle with center at $(2, -2i)$ with unity radius. Since, the line between $(2, -2i)$ and origin will make an angle of 45°. Therefore, P is $2 - \frac{1}{\sqrt{2}} + i(\frac{1}{\sqrt{2}} - 2)$.

282. The diagram is given below:



Given equation is a circle with center $(0, 5)$ and radius $3 \therefore OC = 5, CP = 3$.

The point having least argument will have a tangent from origin which makes $\triangle OCP$ right angle triangle.

$\Rightarrow CP = 4 \Rightarrow \tan\theta = \frac{4}{3}$. Therefore, the point would be $4(\cos\theta + i\sin\theta) = \frac{12}{5} + \frac{16i}{5}$.

283. From given equation, $\left(\frac{|z-1|+4}{3|z-1|-2}\right) < \frac{1}{2}$

$\Rightarrow |z-1| > 10$. This represents area which lies outside a circle with center at $(1, 0)$ and radius 10.

284. Let $z = x+iy$ then the equation becomes $x^2 - y^2 + x + 1 + iy(1+2x) = 0$. Clearly, imaginary part has to be zero i.e. either $y = 0$ or $x = -\frac{1}{2}$. So, it is real and positive for all points on the x-axis. When, $x = -\frac{1}{2}$ the real part becomes $y^2 = \frac{3}{4}$. Thus, for points $x = -\frac{1}{2}$ and $-\frac{\sqrt{3}}{2} < y < \frac{\sqrt{3}}{2}$ the required condition is satisfied.

285. First equation represents a circle whose center is at $(0, ia)$ and radius equal to $\sqrt{a+4}$. The second equation represents interior of a circle with center at $(2,0)$ and radius unity. Now, for the possibility of existence of z the two circles must intersect each other.

$\Rightarrow \sqrt{a^2+4} \leq a+4+1 \Rightarrow a \geq -\frac{21}{10}$ and $a+4-1 \leq \sqrt{a^2+4} \Rightarrow a \leq -\frac{5}{6}$. Combining these two gives us the range for values of a.

286. Let $z = x+iy$ then $|z+\sqrt{2}| = \sqrt{x^2+2\sqrt{2}x+2+y^2} = t^2-3t+2$ and $|z+i\sqrt{2}| = \sqrt{x^2+y^2+2\sqrt{2}y+2} < t^2$.

Because $|z+\sqrt{2}| > 0 \Rightarrow t^2-3t+2 > 0 \Rightarrow t < 1, t > 2$ and $t > 0$. Both the equations are circles so they must intersect for t to exist. The distance between centers i.e. $(-\sqrt{2}, 0)$ and $(0, -i\sqrt{2})$ is 2.

$\Rightarrow r_1 + r_2 > 2 \Rightarrow 2t^2-3t+2 > 2 \Rightarrow t(2t-3) > 0 \Rightarrow t < 0, t > \frac{3}{2}$ and $r_1 < r_2 + 2 \Rightarrow t^2-2t+2 < t^2+2 \Rightarrow t > 0$. Combining all the inequalities, $t > 2$.

287. Let $z = x+iy$ then $\sqrt{x^2+8x+16+y^2} = \sqrt{a^2-12a+28}$ and $\sqrt{x^2-8\sqrt{3}x+48+y^2} < 1$.

Becaise $|z+4| > 0 \Rightarrow a^2-12a+28 > 0 \Rightarrow a > 6+2\sqrt{2}, a < 6-2\sqrt{2}$ and $a > 0$. Both the equations are circles so they must intersect for a to exist. The distance between centers i.e. $(0, -4i)$ and $(4\sqrt{3}, 0)$ is 8.

$\Rightarrow r_1 + r_2 > 8 \Rightarrow \sqrt{a^2-12a+28} + a > 8 \Rightarrow a > 9$ and $r_1 < r_2 + 8 \Rightarrow a < -\frac{9}{7}$. Combining all these inequalities we have $a > 9$.

288. Let $z = x+iy \Rightarrow (1+i)z^2 = (1+i)(x^2-y^2+2ixy) \Rightarrow \Re[(1+i)z^2] = x^2-y^2-2xy > 0 \Rightarrow x$ has two limits $y(1 \pm \sqrt{2})$.

289. Let $z = x+iy$ then $2z = |z| + 2i \Rightarrow 2(x+iy) = \sqrt{x^2+y^2} + 2iy$. Equating real and imaginary parts, $y = 1, 2x = \sqrt{x^2+1}$. Squaring $4x^2 = x^2+1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$.

290. We have earlier proven that if there are two non-parallel lines cutting a circle at a, b and c, d then their point of intersection is given by $\frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$. Now if c and d coincide then that line will become a tangent. So putting $d = c$ we have

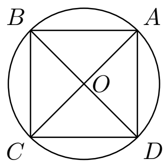
$z = \frac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}$.

291. Given $a_1z^3 + a_2z^2 + a_3z + a_4 = 3 \Rightarrow |a_1z^3 + a_2z^2 + a_3z + a_4| = 3 \Rightarrow |a_1z^3| + |a_2z^2| + |a_3z| + |a_4| \geq 3$

$\Rightarrow |a_1||z^3| + |a_2||z^2| + |a_3||z| + |a_4| \geq 3 \Rightarrow |z|^3 + |z|^2 + |z| + 1 \geq 3\ [\because |a_i| \leq 1]$

$\Rightarrow 1 + |z| + |z|^2 + |z|^3 + \cdots$ to $\infty > 3 \Rightarrow \frac{1}{1-|z|} > 3 \Rightarrow |z| > \frac{2}{3}$, which shows that roots lie outside the circle with center origin and radius $\frac{2}{3}$.

292. The diagram is given below:



Given, $b_1z_1 + b_3z_3 = -(b_2z_2 + b_4z_4)$ and $b_1 + b_3 = -(b_2 + b_4)$ $\therefore \frac{b_1z_1+b_3z_3}{b_1+b_3} = \frac{b_2z_2+b_4z_4}{b_2+b_4}$.

This means that the point dividing AC in the ratio $b_3 : b_1$ also divides BC in the ratio $b_4 : b_2$. Let this point be O. Let $b_1b_2|z_1 - z_2|^2 = b_3b_4|z_3 - z_4|^2$

$\Rightarrow b_1b_2(b_3^2 + b_4^2 - 2b_3b_4\cos\alpha) = b_3b_4(b_2^2 + b_1^2 - 2b_1b_2\cos\alpha)$

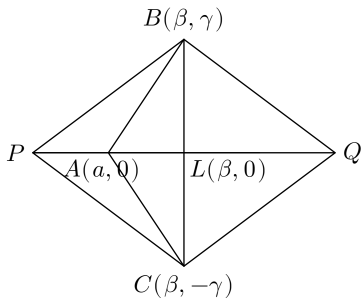
$\Rightarrow \frac{b_3}{b_4} + \frac{b_4}{b_3} = \frac{b_1}{b_2} + \frac{b_2}{b_1} \Rightarrow \frac{b_3}{b_4} = \frac{b_1}{b_2}$ or $\frac{b_2}{b_1}$

If $\frac{b_3}{b_4} = \frac{b_1}{b_2}$, then $\frac{b_3}{b_1} = \frac{b_4}{b_1} \Rightarrow \frac{AO}{CO} = \frac{BO}{DO}$

$\Rightarrow \triangle AOB \sim \triangle BCO \Rightarrow \angle BAO = \angle CDO \Rightarrow AB \parallel CD$ which is not possible.

If $\frac{b_3}{b_4} = \frac{b_2}{b_1}$ then $\frac{AO}{BO} = \frac{DO}{CO} \Rightarrow \triangle ADO \sim \triangle BCO \Rightarrow \angle DAO = \angle OBC \Rightarrow A, B, C, D$ are concyclic.

293. The diagram is given below:



Let $f(x) = k(x-a)(x-\beta-i\gamma)(x-\beta+i\gamma) = k(x-a)[(x-\beta)^2+\gamma^2]$

$\Rightarrow f'(x) = k[3x^2 - 2(a+2\beta)x + \beta^2 + \gamma^2 + 2a\beta]$. Discriminant of $f'(x)$ is given by $D = 4[(a+2\beta)^2 - 3(\beta^2+\gamma^2+2a\beta)] = 4(a^2+\beta^2-3\gamma^2-2a\beta)$

$BC = 2|\gamma| \Rightarrow PL = \sqrt{3}|\gamma|$. If A lies inside the equilateral triangle having BC as base, then $|\beta - a| < \sqrt{3}\gamma \Rightarrow (\beta-a)^2 < 3\gamma^2 \Rightarrow a^2 + \beta^2 - 3\gamma^2 - 2a\beta < 0 \Rightarrow D < 0$. Thus roots will be complex numbers.

294. Let $a = \alpha + i\beta$ and $z = x + iy$, then $\overline{a}z + a\overline{z} = 0$ becomes as $\alpha x + \beta y = 0$ or $y = \left(\frac{-\alpha}{\beta}\right)x$.

Its reflection in the x-axis is $y = \frac{\alpha}{\beta}x$ or $\alpha x - \beta y = 0 \Rightarrow \left(\frac{a+\overline{a}}{2}\right)\left(\frac{z+\overline{z}}{2}\right) - \left(\frac{a-\overline{a}}{2}\right)\left(\frac{z-\overline{z}}{2}\right) = 0$

$\Rightarrow az + \overline{az} = 0$

295. We have $z = \frac{\alpha+\beta t}{\gamma+\delta t} \Rightarrow t = \frac{\alpha-\gamma z}{\delta z-\beta}$. As t is real, $\frac{\alpha-\gamma z}{\delta z-\beta} = \frac{\overline{\alpha}-\overline{\gamma z}}{\overline{\delta z}-\overline{\beta}}$

$\Rightarrow\Rightarrow (\alpha - \gamma z)(\overline{\delta z} - \overline{\beta}) = (\overline{\alpha} - \overline{\gamma z})(\delta z - \beta)$

$\Rightarrow (\overline{\gamma}\delta - \gamma\overline{\delta})z\overline{z} + (\gamma\overline{\beta} - \overline{\alpha}\delta)z + (\alpha\overline{\delta} - \beta\overline{\gamma})\overline{z} = (\alpha\overline{\beta} - \overline{\alpha}\beta)$

Since $\frac{\gamma}{\delta}$ is real, $\frac{\gamma}{\delta} = \frac{\overline{\gamma}}{\overline{\delta}}$ or $\gamma\overline{\delta} - \delta\overline{\gamma} = 0$.

Thus, $\overline{a}z + a\overline{z} = c$, where $a = i(\alpha\overline{\delta}) - \beta\overline{\gamma}$ and $c = i(\overline{\alpha}\beta - \alpha\overline{\beta})$.

Note that $a \neq 0$ for if $a = 0$ then $\alpha\overline{\delta} - \beta\overline{\gamma} = 0 \Rightarrow \frac{\alpha}{\beta} = \frac{\overline{\gamma}}{\overline{\delta}} = \frac{\gamma}{\delta} \Rightarrow \alpha\delta - \beta\gamma = 0$, which is against the hypothesis.

Also, note that $c = i(\overline{\alpha}\beta - \alpha\overline{\beta})$ is a purely real number. Thus, $z = \frac{\alpha+\beta t}{\gamma+\delta t}$ represents a straight line.

296. The solutions are given below:

   i. L.H.S. $= (3+3\omega+5\omega^2)^6 - (2+6\omega+2\omega^2)^3 = [(3+3\omega+3\omega^2+2\omega^2)^6 - (2+2\omega+2\omega^2+4\omega)^3] = [\{3(1+\omega+\omega^2)+2\omega^2\}^6] - [\{2(1+\omega+\omega^2)+4\omega\}^3]$

   $= 64\omega^{12} - 64\omega^3 = 0 =$ R.H.S. $[\because 1+\omega+\omega^2 = 0]$.

   ii. L.H.S. $= (2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2) = [(2-\omega)(2-\omega^2)]^2$

   $= (4-2\omega-2\omega^2+\omega^3)^2 = [5-2(\omega+\omega^2)]^2 = (5+2)^2 = 49 =$ R.H.S.

   iii. L.H.S. $= (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = (1-\omega)^2(1-\omega^2)^2 = (1-\omega-\omega^2+\omega^3)^2$

   $= [2-(-1)]^2 = 9 =$ R.H.S.

   iv. L.H.S. $= (1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5 = -32(\omega+\omega^2) = 32 =$ R.H.S.

   v. L.H.S. $= 1+\omega^n+\omega^{2n}$, where $n = 3m\ \forall\ m \in \mathbb{I}$ L.H.S. $= 1+\omega^{3m}+\omega^{6m} = 1+(\omega^3)^m+(\omega^3)^{2m} = 1+1+1 = 3 =$ R.H.S.

   vi. We have to prove that $1+\omega^n+\omega^{2n} = 0$. If $n = 3m+1\ \forall\ m \in \mathbb{I}$ then L.H.S. $= 1+\omega^{3m+1}+\omega^{6m+2} = 1+\omega+\omega^2 = 0 =$ R.H.S.

   If $n = 3m+2$, $\forall\ m \in \mathbb{I}$ then L.H.S. $= 1+\omega^{3m+2}+\omega^{6m+4} = 1+\omega^2+\omega = 0 =$ R.H.S.

297. We have $a^2+b^2+c^2-ab-bc-ca = a^2+\omega^3b^2+\omega^3c^2+(\omega+\omega^2)ab+(\omega+\omega^2)bc+(\omega+\omega^2)ca$

   $= (a^2+ab\omega+ca\omega^2)+(ab\omega^2+b^2\omega^3+bc\omega)+(ca\omega+bc\omega^2+c^2\omega^3)$

   $= a(a+b\omega+c\omega^2)+b\omega^2(a+b\omega+c\omega^2)+c\omega(a+b\omega+c\omega^2)$

   $= (a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$.

298. $x^3+y^3+z^3 = (a+b)^3+(a\omega+b\omega^2)^3+(a\omega^2+b\omega)^3 = a^3+b^3+3a^2b+3ab^2+a^3\omega^3+b^3\omega^6+3a^2b\omega^4+3ab^2\omega^5+a^3\omega^6+b^3\omega^3+3a^2b\omega^5+3ab^2\omega^4 = 3[a^3+b^3+3a^2b(1+\omega+\omega^2)+3ab^2(1+\omega+\omega^2)] = 3(a^3+b^3)$ =R.H.S.

   $xyz = (a+b)(a\omega+b\omega^2)(a\omega^2+b\omega) = (a+b)(a^2+ab\omega+ab\omega^2+b^2) = (a+b)(a^2+b^2-ab) = a^3+b^3 =$ R.H.S.

299. Given below are the factorization of the expressions:

i. $a^2 - ab + b^2 = a^2 + (\omega + \omega^2)ab + b^2\omega^3 = (a + b\omega)(a + b\omega^2)$.

ii. $a^2 + ab + b^2 = a^2 - (\omega + \omega^2)ab + b^2\omega^3 = (a - b\omega)(a - b\omega^2)$.

iii. $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)(a + b\omega)(a + b\omega^2)$.

iv. $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = (a + b)(a - b\omega)(a - b\omega^2)$.

v. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

300. $x^{3p} + x^{3q+1} + x^{3r+2}$ will be divisible by $x^2 + x + 1$ only if all the factors of $x^2 + x + 1$ satisfy $x^{3p} + x^{3q+1} + x^{3r+2}$.

$x^2 + x + 1 = 0 \Rightarrow x = \omega, \omega^2$. If $x = \omega$ then $x^{3p} + x^{3q+1} + x^{3r+2} = (\omega^3)^p + (\omega^3)^q.\omega + (\omega^3)^r.\omega^2 = 1 + \omega + \omega^2 = 0$.

If $x = \omega^2$ then $x^{3p} + x^{3q+1} + x^{3r+2} = (\omega^6)p + (\omega^6)^q.\omega^2 + (\omega^6)^r.\omega^4 = 1 + \omega^2 + \omega = 0$. Hence proved.

301. Following like previous problem $x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1) = 0 \Rightarrow x = -1, \pm i$.

If $x = -1$ then $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3} = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3} = 1 - 1 + 1 - 1 = 0$.

If $x = i$, then $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3} = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i - 1 - i = 0$.

If $x = -i$, then $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3} = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3} = 1 - i - 1 + i = 0$. Hence proved.

302. $p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) = (p + q + r)(p + q\omega + r\omega^2)(p + q\omega^2 + r\omega)$

$p + q + r = 3a + b(1 + \omega + \omega^2) + c(1 + \omega^2 + \omega) = 3a$. Similarly, $p + q\omega + r\omega^2 = 3c$ and $p + q\omega^2 + r\omega = 3b$. Hence, $p^3 + q^3 + r^3 - 3pqr = 27abc$, proved.

303. Let $p = (a + b\omega + c\omega^2)$ and $q = (a + b\omega^2 + c\omega)$ then we know that $p^3 + q^3 = (p + q)(p + q\omega)(p + \omega^2)$.

$p + q = 2a - b - c$, $p + q\omega = 2b - c - a$, $p + q\omega^2 = 2c - a - b$, and hence

$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a - b - c)(2b - a - c)(2c - a - b)$.

304. The solutions are given below:

i. $(a^2 + b^2 + c^2 - ab - bc - ca)(x^2 + y^2 + z^2 - xy - yz - zx) = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

$= (a + b\omega + c\omega^2)(x + y\omega + z\omega^2)[(a + b\omega^2 + c\omega)(x + y\omega^2 + z\omega)]$

$= (ax + cy\omega^3 + bz\omega^3 + cx\omega^2 + by\omega^2 + za\omega^2 + bx\omega + ay\omega + cz\omega^4)(ax + cy\omega^3 + bz\omega^3 + cx\omega + by\omega^4 + az\omega + bz\omega^2 + ay\omega^2 + cz\omega^2)$

$= [(ax + cy + bz)(cx + by + az)\omega^2 + (bx + ay + cz)\omega][(ax + cy + bz)(cx + by + az)\omega + (bx + ay + cz)\omega^2]$

$= (X + Y\omega^2 + Z\omega)(X + Y\omega + Z\omega^2) = (X^2 + Y^2 + Z^2 - YZ - ZX - XY).$

ii. We just introduce two new factors to previous problem $a + b + c$ and $x + y + z$ and then it is only a matter of simplification to obtain the result.

305. L.H.S. $= \left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \left(\frac{\cos\theta + i\sin\theta}{i(\cos\theta - i\sin\theta)}\right)^4 = \frac{e^{i4\theta}}{e^{-i4\theta}} = e^{i8\theta} = \cos 8\theta + i\sin 8\theta =$ R.H.S.

306. Roots of the quadratic equation $z^2 - 2z\cos\theta + 1 = 0$ are given by $z = \cos\theta \pm i\sin\theta$.

$\Rightarrow z^2 + z^{-2} = \cos 2\theta \pm i\sin 2\theta + \cos 2\theta \mp i\sin 2\theta = 2\cos 2\theta =$ R.H.S.

307. $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $(1 - i) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

L.H.S. $= (1 + i)^n + (1 - i)^n = (\sqrt{2})^n .2\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}.cos\frac{n\pi}{4} =$ R.H.S.

308. $\displaystyle\sum_{k=1}^{6}\left(sin\frac{2\pi k}{7} - icos\frac{2\pi k}{7}\right) = -i\sum_{k=1}^{6}\left(cos\frac{2\pi k}{7} + isin\frac{2\pi k}{7}\right)$

$= -i\sum_{k=1}^{6} e^{\frac{i2\pi k}{7}} = -i\left[e^{\frac{i2\pi}{7}} + e^{\frac{i4\pi}{7}} + .. + e^{\frac{i12\pi}{7}}\right] = -i\left[\left(\frac{1-e^{2\pi}}{1-e^{\frac{i2\pi}{7}}}\right) - 1\right] = -i[0 - 1] = i.$

309. Let $cot^{-1}p = \theta$, then $cot\theta = p$. Now, L. H. S. is

$e^{2mi\theta}\left(\frac{icot\theta+1}{icot\theta-1}\right)^m = e^{2mi\theta}\left[\frac{i(cot\theta-i)}{i(cot\theta+i)}\right]^m$

$= e^{2mi\theta}\left(\frac{cos\theta - isin\theta}{cos\theta + isin\theta}\right)^m$

$= e^{2mi\theta}\left(\frac{e^{-i\theta}}{e^{i}\theta}\right)^m = e^{2mi\theta}.e^{-2mi\theta} = e^0 = 1 =$ R.H.S.

310. Let $1 + \sin\phi + i\cos\phi = r(\cos\theta + i\sin\theta) \therefore 1 + \sin\phi = r\cos\theta$ and $\cos\phi = r\sin\theta$

Now $(1 + \sin\phi + i\cos\phi)^n = r^n(\cos n\theta + i\sin n\theta)$. Taking conjugates, we get $(1 + \sin\phi - i\cos\phi)^n = r^n(\cos n\theta - i\sin n\theta)$

From these two, we get $\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n = \frac{\cos n\theta + i\sin n\theta}{\cos n\theta - i\sin n\theta} = \frac{e^{in\theta}}{e^{-in\theta}}$

$= e^{2in\theta} = \cos 2n\theta + \sin 2n\theta$

$\tan\theta = \frac{\cos\phi}{1+\sin\phi} = \frac{\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}}{\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\right)^2} = \frac{\cos\frac{\phi}{2} - \sin\frac{\phi}{2}}{\cos\frac{\phi}{2} + \sin\frac{\phi}{2}} = \frac{1-\tan\frac{\phi}{2}}{1+\tan\frac{\phi}{2}} = \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$

$\therefore \theta = \frac{\pi}{4} - \frac{\phi}{2} \therefore 2n\theta = \left(\frac{n\pi}{2} - n\phi\right)$. Hence, proved.

311. Let $a = \cos\alpha + i\sin\alpha, b = \cos\beta + i\sin\beta, c = \cos\gamma + i\sin\gamma$

Now, $a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i.0 = 0$

Now, $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$ $[\because a+b+c=0]$

$\therefore a^3+b^3+c^3 = 3abc \therefore \cos 3\alpha+\cos 3\beta+\cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and $\sin 3\alpha+\sin 3\beta+\sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$.

312. Proceeding similarly as last problem and with an extra calculation we have

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = (\cos\alpha+\cos\beta+\cos\gamma) - i(\sin\alpha+\sin\beta+\sin\gamma) = 0$

$\therefore a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = (a+b+c)^2 - 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$\Rightarrow 0^2 - 2abc.0 = 0 \therefore L.H.S. = (\cos 2\alpha+\cos 2\beta+\cos 2\gamma) + i(\sin 2\alpha+\sin 2\beta+\sin 2\gamma) = 0$

Equating real and imaginary parts we have our desired result.

313. $t^2 - 2t + 2 = 0 \Leftrightarrow t = \frac{2\pm\sqrt{4-8}}{2} = 1 \pm i$

Let $\alpha = 1+i$ and $\beta = 1-i$ $\therefore x+\alpha = (x+1)+i, x+\beta = (x+1)-i$ and $\alpha-\beta = 2i$

Let $x+1 = r\cos\phi$ and $1 = r\sin\phi$. We have, $\frac{(x+\alpha)^n-(x+\beta)^n}{(\alpha-\beta)} = \frac{\sin\theta}{\sin^n\theta}$

$\Leftrightarrow \frac{r^n(\cos n\phi+i\sin n\phi)-r^n(\cos n\phi-i\sin n\phi)}{2i} = \frac{\sin\theta}{\sin^n\theta} \Leftrightarrow r^n\sin n\phi = \frac{\sin\theta}{\sin^n\theta}$

$\Leftrightarrow \frac{\sin n\phi}{\sin^n\phi} = \frac{\sin\theta}{\sin^n\theta} \Leftrightarrow$ one of the values of ϕ is θ. $\left[\because r\sin\phi = 1 \Rightarrow r^n = \frac{1}{\sin^n\phi}\right]$

$\therefore x+1 = r\cos\theta$ and $1 = r\sin\theta$. Dividing and evaluating we get $x = \cot\theta - 1$.

314. Given, $(1+x)^n = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$. Putting $x = i$, we get $(1+i)^n = p_0 + p_1i + p_2i^2 + \cdots + p_ni^n$

$= (p_0 - p_2 + p_4 - \cdots) + i(p_1 - p_3 + p_5 - \cdots) \Rightarrow \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = (p_0 - p_2 + p_4 - \cdots) + i(p_1 - p_3 + p_5 - \cdots)$

Equating real and imaginary parts, we have $p_0 - p_2 + p_4 \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$ and $p_1 - p_3 + p_5 - \cdots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$.

315. Given, $(1-x+x^2)^n = a_0 + a_1 + a_2x^2 + \cdots a_{2n}x^{2n}$. Putting $x = 1, \omega$ and ω^2, we get

$1 = a_0 + a_1 + a_2 + \cdots + a_{2n}$, $(-2\omega)^n = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{2n}\omega^{2n}$, $(-2\omega^2)^n = a_0 + a_1\omega^2 + a_2\omega^4 + \cdots + a_{2n}\omega^{4n}$

Adding these we get, $3(a_0 + a_3 + a_6 + \cdots) = 1 + (-2)^n(\omega^n + \omega^{2n})$. Now $\omega = \frac{-1+\sqrt{3}i}{2} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\omega^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$. Now $\omega^2 = \frac{-1-\sqrt{3}i}{2} = \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right) \therefore \omega^n + \omega^{2n} = 2\cos\frac{2n\pi}{3} = 2\cos\left(n\pi - \frac{n\pi}{3}\right)$

$= 2(-1)^n\cos\frac{n\pi}{3}$. Thus, $3(a_0 + a_3 + a_6 + \cdots) = 1 + (-2)^n 2(-1)^n \cos\frac{n\pi}{3} = 1 + 2^{n+1}\cos\frac{n\pi}{3}$.

$a_0 + a_3 + a_6 + \cdots = \frac{1}{3}\left(1 + 2^{n+1}\cos\frac{n\pi}{3}\right)$.

316. Given, $(1+x)^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$. Putting $x = 1$ and $x = -1$, we get $2^n = c_0 + c_1 + c_2 + \cdots + c_n$

and $0 = c_0 - c_1 + c_2 - \cdots + (-1)^n c_n$. Adding these two, we get $2^n = 2(c_0 + c_2 + c_4 + \cdots)$ or $c_0 + c_2 + c_4 + \cdots = 2^{n-1}$

Putting $x = i$, we get $(1+i)^n = c_0 + c_1i + c_2i^2 + c_3i^3 + \cdots + c_ni^n \Rightarrow \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = (c_0 - c_2 + c_4 - \cdots) + i(c_1 - c_3 + \cdots)$

$\Rightarrow 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{i\pi}{4}\right) = (c_0 - c_2 + c_4 - \cdots) + i(c_1 - c_3 + \cdots)$

Equating real parts, we get $c_0 - c_2 + c_4 - \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$. Adding this result with the one obtained previously, we have $2[c_0 + c_4 + c_8 + \cdots] = 2^{n-1} + 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$.

317. $z^8 + 1 = 0 \Rightarrow z^8 = -1 = \cos\pi + i\sin\pi \therefore z = (\cos\pi + i\sin\pi)^{\frac{1}{8}} = \cos\frac{2r\pi+\pi}{8} + i\sin\frac{2r\pi+\pi}{8}$, $r = 0, 1, 2, \ldots, 7$

$\therefore z = \cos\frac{\pi}{8} \pm \sin\frac{\pi}{8}, \cos\frac{3\pi}{8} \pm \sin\frac{3\pi}{8}, \cos\frac{5\pi}{8} \pm \sin\frac{5\pi}{8}, \cos\frac{7\pi}{8} \pm \sin\frac{7\pi}{8}$

Now, quadratic equation whose roots are $\cos\frac{\pi}{8} \pm \sin\frac{\pi}{8}$, is $z^2 - 2\cos\frac{\pi}{8}z + 1 = 0$

Similarly, we can find the quadratic equations for remaining three pairs of roots. Thus,

$z^8 + 1 = \left(z^2 - 2\cos\frac{\pi}{8}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{8}z + 1\right)\left(z^2 - 2\cos\frac{5\pi}{8}z + 1\right)\left(z^2 - 2\cos\frac{7\pi}{8}z + 1\right)$

Dividing both sides by z^4, we get

$z^4 + \frac{1}{z^4} = \left(z + \frac{1}{z} - 2\cos\frac{\pi}{8}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{8}\right)\left(z + \frac{1}{z} - 2\cos\frac{5\pi}{8}\right)\left(z + \frac{1}{z} - 2\cos\frac{7\pi}{8}\right)$

Putting $z = \cos\theta + i\sin\theta$, so that $z^n + \frac{1}{z^n} = 2n\cos n\theta$, we get

$2\cos 4\theta = 2\left(\cos\theta - \cos\frac{\pi}{8}\right)2\left(\cos\theta - \cos\frac{3\pi}{8}\right)2\left(\cos\theta - \cos\frac{5\pi}{8}\right)2\left(\cos\theta - \cos\frac{5\pi}{8}\right)$

$\therefore \cos 4\theta = 8\left(\cos\theta - \cos\frac{\pi}{8}\right)\left(\cos\theta - \cos\frac{3\pi}{8}\right)\left(\cos\theta - \cos\frac{5\pi}{8}\right)\left(\cos\theta - \cos\frac{7\pi}{8}\right)$

318. Let $z = \cos\theta + i\sin\theta$, then $z^7 = \cos 7\theta + i\sin 7\theta$. If

$\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \frac{7\pi}{7}, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$ then $z^7 = \cos 7\theta + i\sin 7\theta = 1$ or $z^7 + 1 = 0$

Thus, $z = \cos\theta + i\sin\theta$, where $\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \frac{7\pi}{7}, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$ are the roots of the equation.

Also, when $\theta = \pi, z = -1$. Now, $z^7 + 1 = 0 \Rightarrow (z+1)(z^6 - z^5 + z^4 - z^3 + z^2 - z + 1) = 0$

Root of equation $z + 1 = 0$ is $\cos\theta + i\sin\theta$, where $\theta = \pi$

Roots of equation $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0 \quad (1)$

are $\cos\theta + i\sin\theta$, where $\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \frac{7\pi}{7}, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$

Let $x = \cos\theta$, then $z + \frac{1}{z} = \cos\theta + i\sin\theta + \frac{1}{\cos\theta + i\sin\theta} = 2\cos\theta = 2x$

But $\cos\left(\frac{13\pi}{7}\right) = cos\left(2\pi - \frac{\pi}{7}\right) = \cos\frac{\pi}{7}, \cos\frac{11\pi}{7} = \cos\frac{3\pi}{7}, \cos\frac{9\pi}{7} = \cos\frac{5\pi}{7}$

Dividing (1) by z^3, we get $z^3 - z^2 + z - 1 + \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} = 0$

$\left(z^3 + \frac{1}{z^3}\right) - \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 1 = 0$

$\left(z + \frac{1}{z}\right)^3 - 3z.\frac{1}{z}\left(z + \frac{1}{z}\right) - \left[\left(z + \frac{1}{z}\right)^2 - 2z.\frac{1}{z}\right] + z + \frac{1}{z} - 1 = 0$

$\Rightarrow 8x^3 - 4x^2 - 4x + 1 = 0$. Roots of this equation are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$.

319. Given, $z^{10} - 1 = 0 \Rightarrow z^{10} = 1 = \cos 0 + i\sin 0 \therefore z = (\cos 0 + i\sin 0)^{\frac{1}{10}} = \cos\frac{2r\pi}{10} + i\sin\frac{2r\pi}{10}$

$= \pm 1, \cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}, \cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5}, \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}, \cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}$

Quadratic equation whose roots are ± 1 is $z^2 - 1 = 0$. And quadratic equation whose roots are $\cos\frac{\pi}{5} \pm \sin\frac{\pi}{5}$ is $z^2 - 2\cos\frac{\pi}{5}z + 1 = 0$. Thus,

$z^{10} - 1 = (z^2 - 1)\left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$

Dividing both sides by z^5, we get

$z^5 - \frac{1}{z^5} = \left(z - \frac{1}{z}\right)\left(z + \frac{1}{z} - 2\cos\frac{\pi}{5}\right)\left(z + \frac{1}{z} - 2\cos\frac{2\pi}{5}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{5}\right)\left(z + \frac{1}{z} - 2\cos\frac{4\pi}{5}\right)$

Putting $z = \cos\theta + i\sin\theta$ in the above equation, so that $z^5 - \frac{1}{z^5} = 2i\sin 5\theta$, we get

$2i\sin 5\theta = 2i\sin\theta.2\left(\cos\theta - \cos\frac{\pi}{5}\right)2\left(\cos\theta - \cos\frac{2\pi}{5}\right)2\left(\cos\theta - \cos\frac{3\pi}{5}\right)2\left(\cos\theta - \cos\frac{4\pi}{5}\right)$

$\therefore \sin 5\theta = 16\sin\theta\left(\cos\theta - \cos\frac{\pi}{5}\right)\left(\cos\theta - \cos\frac{2\pi}{5}\right)\left(\cos\theta - \cos\frac{3\pi}{5}\right)\left(\cos\theta - \cos\frac{4\pi}{5}\right)$

$= 16\sin\theta\left(\cos\theta - \cos\frac{\pi}{5}\right)\left(\cos\theta + \cos\frac{\pi}{5}\right)\left(\cos\theta - \cos\frac{2\pi}{5}\right)\left(\cos\theta + \cos\frac{2\pi}{5}\right)$

$= 16\sin\theta\left(\cos^2\theta - \cos^2\frac{\pi}{5}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{5}\right)$

$= 16\sin\theta\left(\sin^2\frac{\pi}{5} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{5} - \sin^2\theta\right)$

$= 16\sin\theta\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5}\left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{5}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{2\pi}{5}}\right)$

$= 16\sin\theta\sin^2 36^\circ\sin^2 72^\circ\left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{5}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{2\pi}{5}}\right)$

$= 16\sin\theta\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2\left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{5}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{2\pi}{5}}\right)$

Thus, $\sin 5\theta = 5\sin\theta\left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{5}}\right)\left(1 - \frac{\sin^\theta}{\sin^2\frac{2\pi}{5}}\right)$.

320. Given, $x^7 + 1 = 0$ or $x^7 = -1 = \cos\pi + i\sin\pi$

$\therefore x = (\cos\pi + i\sin\pi)^{\frac{1}{7}} = \cos\frac{2r\pi+\pi}{7} + i\sin\frac{2r\pi+\pi}{7}, r = 0, 1, 2, \dots, 6$

$= \cos\frac{\pi}{7} \pm i\sin\frac{\pi}{7}, \cos\frac{2\pi}{7} \pm i\sin\frac{2\pi}{7}, \cos\frac{3\pi}{7} \pm i\sin\frac{3\pi}{7}, \cos\pi + i\sin\pi(= -1)$

$x^7 + 1 = (x+1)\left(x^2 - 2\cos\frac{\pi}{7}x + 1\right)\left(x^2 - 2\cos\frac{2\pi}{7}x + 1\right)\left(x^2 - 2\cos\frac{3\pi}{7}x + 1\right)$. Putting $x = i$, we get

$i^7 + 1 = (1+i)\left(-2i\cos\frac{\pi}{7}\right)\left(-2i\cos\frac{2\pi}{7}\right)\left(-2i\cos\frac{3\pi}{7}\right)$

$1 - i = 8(1+i)\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = -8(1-i)\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$

$\therefore \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = -\frac{1}{8}$.

321. $(\cos\alpha + i\sin\alpha)^n = \cos^n\alpha + i.{}^nC_1\cos^{n-1}\alpha\sin\alpha + i^2.{}^nC_2\cos^{n-2}\alpha\sin^2\alpha + \cdots + i^n.{}^nCn\sin^n\alpha$

$\Rightarrow \cos n\alpha + i\sin n\alpha = (\cos^n\alpha -{}^n C_2\cos^{n-2}\alpha\sin^2\alpha) + i({}^nC_1\cos^{n-1}\alpha\sin\alpha)$. Equating imaginary parts, we get

$\therefore \sin n\alpha ={}^n C_1\cos^{n-1}\alpha\sin\alpha -{}^n C_3\cos^{n-3}\alpha\sin^3\alpha + \cdots$

$\therefore \sin(2n+1)\alpha ={}^{2n+1} C_1\cos^{2n}\alpha\sin\alpha -{}^{2n+1} C_3\cos^{2n-2}\alpha\sin^3\alpha + \cdots$

$\Rightarrow \sin(2n+1)\alpha = \sin^{2n+1}\alpha[{}^{2n+1}C_1\cot^{2n}\alpha -{}^{2n+1} C_3\cot^{2n-2}\alpha + \cdots]$

when$\alpha = \frac{\pi}{2n+1}, \frac{2\pi}{2n+1}, \cdots, \frac{n\pi}{2n+1}, \sin(2n+1)\alpha = 0$

$\therefore \cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \cdots, \cot^2\frac{n\pi}{2n+1}$ are the roots of the equation. From the second term coefficient we get sum of roots in a polynomial.

$\therefore \cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \cdots + \cot^2\frac{n\pi}{2n+1} = \frac{{}^{2n+1}C_3}{{}^{2n+1}C_1}$.

322. Let $C = \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cdots + cos^n\theta\cos n\theta$ and

$S = \cos\theta\sin\theta + \cos^2\theta\sin 2\theta + \cdots + cos^n\theta\sin n\theta$

Now, $C + iS = \cos\theta(cos\theta + i\sin\theta) + \cos^2\theta(\cos 2\theta + i\sin 2\theta) + \cdots + \cos^n\theta(\cos n\theta + i\sin n\theta)$

$= \cos\theta.e^{i\theta} + \cos^2\theta.e^{2i\theta} + \cdots + \cos^n\theta.e^{ni\theta} = x + x^2 + \cdots + x^n$, where $x = \cos\theta e^{i\theta} = \frac{x(x^n-1)}{x-1} = \frac{\cos\theta e^{i\theta}(\cos^n\theta e^{in\theta}-1)}{\cos\theta e^{i\theta}-1}$

$= \frac{\cos\theta[\cos^n\theta(\cos n\theta+i\sin n\theta)-1]}{\cos\theta-e^{-i\theta}} = \frac{\cos\theta[(\cos^n\theta\cos n\theta-1)+icos^n\theta\sin n\theta]}{\cos\theta-(\cos\theta-i\sin\theta)}$

$= -i\cot\theta(\cos^n\theta\cos n\theta - 1) + i\cos^n\theta\sin n\theta$

Equating imaginary parts, we get

$S = -\cot\theta(\cos^n\theta\cos n\theta - 1) = \cot\theta(1 - \cos^n\theta\cos n\theta)$.

323. L.H.S. $= -3 - 4i = 5\left(-\frac{3}{5} - i\frac{4}{5}\right) = 5\left(\cos\left(\pi + \tan^{-1}\frac{4}{5}\right) + i\sin\left(\pi + \tan^{-1}\frac{4}{5}\right)\right)$

$= 5e^{i\left(\pi+\tan^{-1}\frac{4}{5}\right)} =$ R.H.S.

324. Putting $x^4 = \frac{\sqrt{3}-1}{2\sqrt{2}} + i\frac{\sqrt{3}+1}{2\sqrt{2}}$ in polar form we get

$x^4 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12} \therefore x = \cos\frac{(24r+5)\pi}{48} + i\sin\frac{(24r+5)\pi}{48}, r = 0, 1, 2, 4.$

325. L.H.S. $= z_1z_2z_3 \ldots = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\cdots$

$= \cos\left(\frac{\frac{\pi}{3}}{1-\frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1-\frac{1}{3}}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i =$ R.H.S.

326. Given $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$, prove that $p_1\sin\theta + p_2\sin 2\theta + \cdots + p_n = 0 \Rightarrow p_0(\cos n\theta + i\sin n\theta) + p_1[\cos(n-1)\theta + \sin(n-1)\theta] + p_2[\cos(n-2)\theta + i\sin(n-2)\theta] + \cdots + p_n = 0$ [$\because \cos\theta + i\sin\theta$] is a solution.

Dividing both sides by $\cos n\theta + i\sin n\theta$, we have

$p_0 + p_1(\cos\theta - i\sin\theta) + p_2(\cos 2\theta - i\sin 2\theta) + \cdots + p_n(\cos n\theta - i\sin n\theta) = 0$. Equating real and imaghinary parts we have required equations.

327. L.H.S. $= \left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n = \left(\frac{(1+\cos\phi+i\sin\phi)(1+\cos\phi+i\sin\phi)}{(1+\cos\phi)^2+\sin^2\phi}\right)^n$

$= \left(\frac{1+2\cos\phi+\cos^2\phi-\sin^2\phi+2i\sin\phi(1+\cos\phi)}{1+2\cos\phi+\cos^2\phi+\sin^2\phi}\right)^n = \left(\frac{2(1+\cos\phi)+2i\sin\phi(1+\cos\phi)}{2\cos\phi(1+\cos\phi)}\right)^n$

$= (\cos\phi + i\sin\phi^n) = \cos n\phi + i\sin n\phi =$ R.H.S.

328. Given $2\cos\theta = x + \frac{1}{x} \Rightarrow x^2 - 2\cos\theta x + 1 = 0 \Rightarrow x = \cos\theta \pm i\sin\theta$. Similarly, $y = \cos\phi \pm i\sin\phi$.

   i. $\frac{x}{y} = \cos(\theta - \phi) \pm i\sin(\theta - \phi)$ and $\frac{y}{x} = \cos(\phi - \theta) \pm i\sin(\phi - \theta)$

   $\therefore$ L.H.S. $= 2\cos(\theta - \phi) =$ R.H.S. $[\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$

   ii. $xy = \cos(\theta + \phi) \pm i\sin(\theta + \phi), \frac{1}{xy} = \cos(\theta + \phi) \mp i\sin(\theta + \phi)$

   $\therefore$ L.H.S. $= 2\cos(\theta + \phi) =$ R.H.S.

   iii. $x^m y^n = (\cos m\theta \pm i\sin m\theta)(\cos n\phi \pm i\sin n\phi) = \cos(m\theta + n\phi) \pm i\sin(m\theta + n\phi)$ and $\frac{1}{x^m y^n} = \cos(m\theta + n\phi) \mp i\sin(m\theta + n\phi)$ $\therefore$ L.H.S. $= 2\cos(m\theta + n\phi) =$ R.H.S.

   iv. $\frac{x^m}{y^n} = \cos(m\theta - n\phi) \pm i\sin(m\theta - n\phi)$ and $\frac{y^n}{x^m} = \cos(n\phi - m\theta) \pm i\sin(n\phi - m\theta)$ $\therefore$ L.H.S. $= 2\cos(m\theta - n\phi) =$ R.H.S.

329. Given equation is $x^2 - 2x + 4 = 0$ whose roots are $\alpha, \beta = 1 \pm i\sqrt{3} = 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right) \Rightarrow \alpha^n, \beta^n = 2\left(\cos\frac{n\pi}{3} \pm i\sin\frac{n\pi}{3}\right)$

   $\therefore \alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3} =$ R.H.S.

330. Given equation is $x^2 - 2x\cos\theta + 1 = 0$, whose roots are $\cos\theta \pm i\sin\theta$, nth power of which are $\cos n\theta \pm i\sin n\theta$. Therefore, the equation having these roots are $x^2 - 2\cos n\theta + 1 = 0$.

331. L.H.S. $= A(\cos 2\theta + i\sin 2\theta) + B(\cos 2\theta - i\sin 2\theta) = 5\cos 2\theta + 7i^2\sin 2\theta$.

   $\Rightarrow A + B = 5, A - B = 7i \Rightarrow A = \frac{5+7i}{2}, B = \frac{5-7i}{2}$.

332. Given $x = \cos\theta + i\sin\theta$ and $\sqrt{1 - c^2} = nc - 1$. Squaring the second equaiton $n^2c^2 + c^2 - 2nc = 0 \Rightarrow c = \frac{2n}{n^2+1}$. We have to prove that $1 + \cos\theta = \frac{c}{2n}(1 + nx)\left(1 + \frac{n}{x}\right)$.

   R.H.S. $= \frac{1}{n^2+1}\left(1 + n^2 + 2n\cos\theta\right) = 1 + \frac{2n}{n^2+1}\cos\theta = 1 + c\cos\theta =$ L.H.S.

333. From the given equality, we have $\left(\frac{1+z}{1-z}\right)^n = 1 \Rightarrow 1 + z = (1 - z)(\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n})$

   Let $\frac{2r\pi}{n} = \theta$ then $1 + z = (1 - z)(\cos\theta + i\sin\theta) \Rightarrow z((1 + \cos\theta) + i\sin\theta) = (\cos\theta - 1) + i\sin\theta \Rightarrow z = \frac{(\cos\theta - 1) + i\sin\theta}{(1 + \cos\theta) + i\sin\theta}$

   $z = i\tan\frac{\theta}{2} = i\tan\frac{2\pi}{n},\ r = 0, 1, 2, ..., (n - 1)$

   Clearly, the above equation is invalid if n is even and $r = \frac{n}{2}$ as it will cause the value of tan function to reach infinity.

334. L.H.S. $= \frac{xy(x+y)-(x+y)}{xy(x-y)+(x-y)}$. Dividing numerator and denominator by xy

$= \frac{x+y-\frac{1}{x}-\frac{1}{y}}{x-y+\frac{1}{y}-\frac{1}{x}} = \frac{\cos\alpha+i\sin\alpha+\cos\beta+i\sin\beta-\cos\alpha+i\sin\alpha-\cos\beta+i\sin\beta}{\cos\alpha+i\sin\alpha-\cos\beta-i\sin\beta-\cos\alpha+i\sin\alpha+\cos\beta-i\sin\beta} = \frac{\sin\alpha+\sin\beta}{sin\alpha-\sin\beta} =$ R.H.S.

335. $(1+x)^n = ^n C_0 + ^n C_1 x + ^n C_3 x^2 + ^n C_3 x^3 + \cdots$

We know that $\omega, \omega^2 = \frac{-1\pm\sqrt{3}i}{2} = -cos\frac{\pi}{3} \pm \sin\frac{\pi}{3}$.

Putting $x = 1, \omega, \omega^2$ and adding we get

$2^n + 2\cos\frac{n\pi}{3} = 3[^n C_0 + ^n C_3 + ^n C_6 + \cdots] \Rightarrow ^n C_0 + ^n C_3 + ^n C_6 + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$.

336. Proceeding like previous question,

$2^n = ^n C_0 + ^n C_1 + ^n C_2 + ^n C_3 + ^n C_4 + ^n C_5 + \cdots$

$(-\omega^2)^n = ^n C_0 + ^n C_1\omega + ^n C_2\omega^2 + ^n C_3\omega^3 + ^n C_4\omega^4 + ^n C_5\omega^5 + \cdots$

$\Rightarrow (-\omega^2)^n\omega^2 = ^n C_0\omega^2 + ^n C_1\omega^3 + ^n C_2\omega^4 + ^n C_3\omega^5 + ^n C_4\omega^6 + ^n C_5\omega^7 + \cdots$

and $(-\omega)^n = ^n C_0 + ^n C_1\omega^2 + ^n C_2\omega^4 + ^n C_3\omega^6 + ^n C_4\omega^8 + ^n C_5\omega^{10} + \cdots$

$\Rightarrow (-\omega)^n\omega = ^n C_0\omega + ^n C_1\omega^3 + ^n C_2\omega^5 + ^n C_3\omega^7 + ^n C_4\omega^9 + ^n C_5\omega^{11} + \cdots$

Adding $2^{n-2} + 2\cos\frac{(n-2)\pi}{3} = 3[^n C_1 + ^n C_4 + ^n C_7 + \cdots] \Rightarrow ^n C_1 + ^n C_4 + ^n C_7 + \cdots = \frac{1}{3}\left[2^{n-2} + 2\cos\frac{(n-2)\pi}{3}\right]$

337. This problem can be solved like previous problem. Put $x = 1, \omega, \omega^2$ and multiply with $1, \omega, \omega^2$ and then add to obtain the result.

338. $C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \cdots = (1+x)^{4n}$. Putting $x = 1, -1, i, -i$ and adding

$4[C_0 + C_4 + C_8 + \cdots] = 2^{4n} + (1+i)^{4n} + (1-i)^{4n}$

$\Rightarrow C_0 + C_4 + C_8 + \cdots = 2^{4n-2} + (-1)^n 2^{2n-1}$.

339. Given $(1-x+x^2)^{6n} = a_0 + a_1x + a_2x^2 + \cdots$. Putting $x = 1, \omega, \omega^2$

$1^{6n} = a_0 + a_1 + a_2 + a_3 + \cdots$

$(-2\omega)^{6n} = 2^{6n} = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \cdots$

$(-2\omega^2)^{6n} = 2^{6n} = a_0 + a_1\omega^2 +_2 \omega^4 + a_3\omega^6 + \cdots$

Adding $2^{6n+1} + 1 = 3[a_0 + a_3 + a_6 + \cdots] \Rightarrow a_0 + a_3 + a_6 + \cdots = \frac{1}{3}[2^{6n+1} + 1]$.

340. Proceeding like previous problem we obtain $3[a_0 + a_3 + a_6 + \cdots]$.

R.H.S. becomes $1^n + (-2\omega)^n + (-2\omega^2)^n$ but $-\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and $-\omega^2 = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$ and hence we have R.H.S.

341. Clearly, $x'' = \frac{AA'+BB'+CC'}{3}$, $y'' = \frac{AA'+BB'\omega^2+CC'\omega}{3}$ and $z'' = \frac{AA'+BB'\omega+CC'\omega^2}{3}$,

and $AA' + BB' + CC' = (x + y + z)(x' + y' + z') + (x + y\omega + z\omega^2)((x' + y'\omega + z'\omega^2) + (x + y\omega^2 + z\omega)((x' + y'\omega^2 + z'\omega) = 3(xx' + zy' + yz')$. Analogously $y'' = yy' + zx' + xz'$, $z'' = zz' + xy' + yz'$.

342. We have the identity $(\alpha\delta - \beta\gamma)(\alpha'\delta' - \beta'\gamma') = (\alpha\alpha' + \beta\gamma')(\gamma\beta' + \delta\delta') - (\alpha\beta' + \beta\delta')(\gamma\alpha' + \gamma\alpha' + \delta\gamma')$

Putting $\alpha = x + yi, \beta = z + ti, \gamma = -(z - ti), \delta = x - yi, \alpha' = a + bi, \beta' = c + di, \gamma' = -(c - di)$ and $\delta' = a - bi$ then

$\alpha\delta - \beta\gamma = x^2 + y^2 + z^2 + t^2$ and $\alpha'\delta' - \beta'\gamma' = a^2 + b^2 + c^2 + d^2$

$\Rightarrow \alpha\alpha' + \beta\gamma' = (ax - by - ca - dt) + i(bx + ay + dz - ct), \gamma\beta' + \delta\delta' = \overline{\beta\gamma'} + \overline{\alpha\alpha'} = \overline{\beta\gamma' + \alpha\alpha'}$

$\therefore \alpha\beta' + \beta\delta' = (cx - dy + az + bt) + i(dx + cy - bz + at), \gamma\alpha' + \delta\gamma' = -(cx - dy + az + bt) + i(dx + cy - bz + at)$

Thus, $-(\alpha\beta' + \beta\delta')(\gamma\alpha' + \delta\gamma') = (cx - dy + az + bt)^2 + (dx + cy - bz + at)^2$

Substituting obtained expression in the original idendity we have the required result.

343. $(\cos\theta + i\sin\theta)^n = \cos^n\theta + iC_1^n\cos^{n-1}\theta\sin\theta + i^2C_2^n\cos^{(n-2)}\theta\sin^2\theta + \cdots + i^rC_r^n\cos(n - r + 1)\theta\sin^{r-1}\theta + \cdots$

Separating real part, $\cos n\theta = \cos^n\theta - C_2^n\cos^{n-2}\theta\sin^2\theta + \cdots$

Taking into account the parity of n and dividing both members of these equalities by $\cos^n\theta$, we get the required formulas.

344. Replacing real part with imaginary part in previous problem we arrive at required formula.

345. $\cos\theta = \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{2}$. Let $\cos\theta + i\sin\theta = z$ then $\cos\theta - i\sin\theta = z^{-1}$.

$\therefore \cos^{2m}\theta = \left(\frac{z+z^{-1}}{2}\right)^{2m} = \frac{1}{2^{2m}}\sum_{k=0}^{2m} C_k^{2m} z^{2m-k}.z^{-k}$

Moreover $2^{2m}\cos^{2m}\theta = \sum_{k=0}^{m-1} C_k^{2m} z^{2(m-k)} + C_m^{2m} + \sum_{k=m+1}^{2m} C_k^{2m} z^{2(m-k)}$

Putting $m - k = -(m - k')$, we rewrite the sum $\sum_{k'=m-1}^{0} C_{2m-k'}^{2m} z^{-2(m-k')} = \sum_{k=0}^{m-1} C_k^{2m} z^{-2(m-k)}$

And so $2^{2m}\cos^{2m}\theta = \sum_{k=0}^{m-1} C_k^{2m}(z^{2(m-k)} + z^{-2(m-k)}) + C_m^{2m}$.

However, $z^{2(m-k)} + z^{-2(m-k)} = 2\cos 2(m-k)$.

$$\therefore 2^{2m}\cos^{2m}\theta = \sum_{k=0}^{m-1} 2\binom{2m}{k}\cos 2(m-k)x + \binom{2m}{m}.$$

346. Putting $\theta = \frac{\pi}{2} - \theta$ in the previous problem, we get the required formula.

347. This is deduced like previous problem.

348. This is deduced like previous problem.

349. We have the expression $u_n + iv_n = (\cos\alpha + i\sin\alpha) + r[\cos(\alpha+\theta) + i\sin(\alpha+\theta)] + \cdots + r^n[\cos(\alpha+n\theta) + i\sin(\alpha+n\theta)]$

$= (\cos\alpha + i\sin\alpha)[1 + (\cos\theta + i\sin\theta) + \cdots + r^n(\cos n\theta + i\sin n\theta)]$. Putting $z = \cos\theta + i\sin\theta$, then

$u_n + iv_n = (\cos\alpha + i\sin\alpha)[1 + rz + \cdots + r^n z^n] = e^{i\alpha}\frac{(rz)^{n+1}-1}{rz-1}$

Transforming $\frac{(rz)^{n+1}-1}{rz-1}$, separating real part from the imaginary one.

$\frac{(rz)^{n+1}-1}{rz-1} = \frac{[(rz)^{n+1}-1][\overline{rz}-1]}{(rz-1)(\overline{rz}-1)}$

$= \frac{r^{n+2}[\cos n\theta + i\sin n\theta] - r[\cos\theta - i\sin\theta]}{1-2r\cos\theta+r^2} + \frac{-r^{n+1}[\cos(n+1)\theta + i\sin(n+1)\theta]+1}{1-2r\cos\theta+r^2}$

Multiplying above with $(\cos\alpha + i\sin\alpha)$ and separating real and imaginary parts we have

$u_n + iv_n = \frac{\cos\alpha - r\cos(\alpha-\theta) - r^{n+1}\cos[\alpha+(n+1)\theta] + r^{n+2}\cos(\alpha+n\theta)}{1-2r\cos\theta+r^2} +$

$i\frac{\sin\alpha - r\sin(\alpha-\theta) - r^{n+1}\sin[\alpha+(n+1)\theta] + r^{n+2}\sin(\alpha+n\theta)}{1-2r\cos\theta+r^2}$.

Note: Putting $\alpha = 0, r = 1$, we obtain $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin\frac{n+1}{2}\theta\cos\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$

and $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\sin\frac{n+1}{2}\theta\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$.

350. $S + iS' = \sum_{k=0}^{n} C_k^n(\cos k\theta + i\sin k\theta) = \sum_{k=0}^{n}(\cos\theta + i\sin\theta)^k$

$= (1 + \cos\theta + i\sin\theta)^n = \left[2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n = 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n$

$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right)$.

Equating real and imaginary parts we have S and S'.

351. Put $S = \sin^{2p}\alpha + \sin^{2p}2\alpha + \cdots + \sin^{2p}2\alpha = \sum_{l=1}^{n}\sin^{2p}l\alpha$

But we have proved earlier $\sin^{2p}l\alpha = \frac{1}{2^{2p-1}})(-1)^p\sum_{k=0}^{p-1}C_k^{2p}\cos 2(p-k)l\alpha + \frac{1}{2^{2p}}C_p^{2p}$, therefore

$$S = \frac{(-1)^p}{2^{2p-1}}\sum_{k=0}^{p-1}(-1)^k C_k^{2p}\sum_{l=1}^{n}\cos 2(p-k)l\alpha + \frac{1}{2^{2p}}C_p^{2p}$$

Put $2(p-k)\alpha = \theta$, $\sum_{l=1}^{n}\cos 2(p-k)\alpha = \cos\theta + \cdots + \cos n\theta = \frac{\sin\frac{n\theta}{2}\cos\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}$

Denoting $\frac{\sin\frac{n\theta}{2}\cos\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}} = \sigma_k$, we can prove that $\sigma_k = 0$ if k is of the same parity as $p\{k \equiv p(\text{mod } 2)\}$ and $\sigma_k = -1$ if k and p are of different parity $\{k \equiv p+1(\text{mod})\ 2\}$, and we get

$$S = \frac{(-1)^{p+1}}{2^{2p-1}}\sum_{\substack{k=0\\k\equiv p+1(\text{mod }2)}}^{p-1}(-1)^k C_k^{2p} + \frac{n}{2^{2p}}C_p^{2p}.$$

Hence, $S = \frac{1}{2^{2p-1}}\sum_{\substack{k=0\\k\equiv p+1(\text{mod }2)}}^{p-1}C_k^{2p} + \frac{n}{2^{2p}}C_p^{2p}.$

But we can prove that $\sum_{\substack{k=0\\k\equiv p+1(\text{mod }2)}}^{p-1}C_k^{2p} = 2^{2p-2}$ (check binomial theorem chapter) and hence our formula is deduced.

352. Considering the given expression as a polynomial in y we see that at $y = 0$ the polynomial vanishes. Therefore, our polynomial is divisible by y. Since it is symmetrical both w.r.t. to x and y this must also be true for x i.e. the polynomial being divisible by x. Hence, the polynomial is divisible by xy. Putting $y = -x$(we do this for checking divisibility by $x+y$), we have $(x-x)^n - x^n - (-x)^n = 0$. Consequently, the polynomial is divisible by $x+y$.

Now it remains to prove that the polynomial is divisible by x^2+xy+y^2. Expansind this into linear factors we have $x^2+xy+y^2 = (y-x\omega)(y-x\omega^2)$ where ω is cube root of unity, which leads to $1+\omega+\omega^2 = 0$.

Since $n = 3m+1, 3m+2\ \forall\ m \in \mathbb{I}$, we substitute $y = x\omega$ and $y = x\omega^2$ and find that it vanishes for both. Consequently, we have proven the divisibility condition.

353. Let the quantities $-x, -y$ and $x+y$ be the roots of the cubic equation $x^3 - rx^2 - px - q = 0$. Then $r = -x-y+x+y = 0, -p = xy - x(x+y) - y(x+y), q = xy(x+y)$ reducing our equation to $x^3 - px + q = 0$.

Putting $(x+y)^n - x^n - y^n = S_n$ we find that between successive values of S_n their exists relationship $S_{n+3} = pS_{n+1} + qS_n$. We will use mathematical induction to prove that S_n is divisible by p^2 with the knowledge that $S_1 = 0$.

Let S_n be divisble by p^2 then let S_{n+6} be also divisible by p^2. We have, $S_{n+6} = pS_{n+4} + qS_{n+3}, S_{n+4} = pS_{n+2} + qS_{n+1}$. Therefore,

$S_{n+6} = p(pS_{n+2} + qS_{n+1}) + q(pS_{n+1} + qS_n) = p^2 S_{n+2} + 2pqS_{n+1} + q^2 S_n.$

Since by supposition, S_n is divisible by p^2, it suffices to prove that S_{n+1} is divisible by p. Thus, we only have to prove that given expression is divisible by $x^2 + xy + y^2$ if $n \equiv 2 (\bmod 6)$, which can be proved by proceeding like previous problem.

354. Let $f(x) = (\cos\theta + x\sin\theta)^n - \cos n\theta - x\sin n\theta$. But $x^2 + 1 = (x+i)(x-i)$ and $f(i) = \cos n\theta + i\sin n\theta - \cos n\theta - i\sin n\theta = 0$. Similarly, $f(-i) = 0$. And hence, required condition is proved.

355. Roots of the equation $x^2 - 2px\cos\theta + p^2 = 0$ are $p(\cos\theta \pm \sin\theta)$. Let $f(x) = x^n \sin\theta - p^{n-1}x\sin n\theta + p^n \sin(n-1)\theta$, then

$f[p(\cos\theta + i\sin\theta)] = p^n(\cos n\theta + i\sin n\theta)\sin\theta - p^n(\cos\theta + i\sin\theta)\sin n\theta + p^n\sin(n-1)\theta$. Separating real and imaginary parts

$\cos n\theta \sin\theta - \cos\theta \sin n\theta + \sin(n-1)\theta = -\sin(n-1)\theta + \sin(n-1)\theta = 0$

and $\sin\theta\sin n\theta - \sin\theta\sin n\theta = 0$. Hence, $f(x)$ is divisible by $p(\cos\theta + i\sin\theta)$ and similarly we can prove it for the other root.

356. Let $x^4 + 1 = (x^2 + px + q)(x^2 + p'x + q') = x^4 + (p+p')x^3 + (pp' + q + q')x^2 + (pq' + p'q)x + qq'$ which gives us four equations $p + p' = 0, pp' + q + q' = 0, pq' + p'q = 0$ and $qq' = 1$.

Assuming $p = 0, p' = 0, q + q' = 0, qq' = 1, q^2 = -1, q = \pm i, q' = \mp i$.

Consequently, corresponding factorization has form $x^4 + 1 = (x^2 + i)(x^2 - i)$.

Let $q = q', q^2 = 1, q = \pm 1$. First let $q = q' = 1$. Then $pp' = -2, p + p' = 0, p^2 = 2, p = \pm\sqrt{2}, p' = \mp\sqrt{2}$. The corresponding factorization is $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$.

Then we assume $q = q' = -1, p + p' = 0, pp' = 2, p = \pm\sqrt{2}i, p' = \mp\sqrt{2}i$.

The factorization will be $(x^2 + \sqrt{2}ix - 1)(x^2 - \sqrt{2}ix - 1)$.

357. Let $S = \sum_{k=1}^{n-1} x_k^p = \sum_{k=1}^{n-1} z^{kp}$ where $z = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Thus, $\sum_{k=1}^{n-1} x_k^p = 1 + z^p + z^{2p} + \dots + z^{(n-1)p}$ but $z^p = \cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}$. Obviously $z^p = 1$ if and only if p is divisible by n, in which case $S = n$. If $z^p \neq 1$, then $S = \frac{z^{np-1}}{z^p - 1} = 0 \because z^{np} = 1$.

358. We have $\sum_{k=0}^{n-1} |A_k|^2 = \sum_{k=0}^{n-1} A_k \overline{A_k}$.

But $A_k\overline{A_k} = (x + y\epsilon^k + z\epsilon^{2k} + \ldots + w\epsilon^{(n-1)k})(\overline{x} + \overline{y}\epsilon^{-k} + \overline{z}\epsilon^{-2k} + \cdots + \overline{w}\epsilon^{-(n-1)k})$

$= (x\overline{x} + y\overline{y} + \cdots + w\overline{w}) + x(\overline{y}\epsilon^{-k} + \overline{z}\epsilon^{-2k} + \cdots + \overline{w}\epsilon^{-(n-1)k}) + y\epsilon^k(\overline{x} + \overline{x}\epsilon^{-2k} + \cdots + \overline{w}\epsilon^{-(n-1)k}) + \cdots + w\epsilon^{(n-1)k}(\overline{x} + \overline{y}\epsilon^{-k} + \cdots + \overline{u}\epsilon^{-(n-2)k})$

Therefore, $\sum_{k=0}^{n-1} |A_k|^2 = n(|x|^2 + |y|^2 + \cdots + |w|^2) + x\sum_{k=1}^{n-1}(\overline{y}\epsilon^{-k} + \overline{z}\epsilon^{-2k} + \cdots + \overline{w}\epsilon^{-(n-1)k}) + y\sum_{k=0}^{n-1}(\overline{x}\epsilon^k + \overline{z}\epsilon^{-k} + \cdots + \overline{w}\epsilon^{-(n-2)k}) + \cdots + w\sum_{k=0}^{n-1}(\overline{x}\epsilon^{(n-1)k} + \overline{y}\epsilon^{(n-2)k} + \cdots + \overline{u}\epsilon^k)$

But $\sum_{k=0}^{n-1}\epsilon^{lk} = 0$ if l is not divisible by n from previous problem. Therefore all the sums in the right vanish and we get

$$\sum_{k=0}^{n-1} |A_k|^2 = n(|x|^2 + |y|^2 + \ldots + |w|^2).$$

359. Considering $2n$th root of unity $x_s = \cos\frac{2s\pi}{n} + i\sin\frac{2s\pi}{n}$ $(s = 1, 2, 3, \ldots, n)$.

Therefore, $x^{2n} - 1 = \prod_{s=1}^{2n}(x - x_s) = \prod_{s=1}^{n-1}(x - x_s)\prod_{s=n+1}^{2n-1}(x - x_s)(x^2 - 1)$ $\because x_n = -1, x_{2n} = 1$. But $x_{2n-s} = \overline{x_s}$, consequently,

$$x^{2n} - 1 = (x^2 - 1)\prod_{s=1}^{n-1}(x - x_s)(x - \overline{x_s}) = (x^2 - 1)\prod_{s=1}^{n-1}(x^2 - 2x\cos\tfrac{s\pi}{n} + 1).$$

360. Considering $2n + 1$th root of unity $x_s = \cos\frac{2(2s+1)\pi}{2n+1} + i\sin\frac{(2s+1)\pi}{2n+1}$ $(s = 1, 2, 3, \ldots, n)$.

Therefore $x^{2n+1} - 1 = \prod_{s=1}^{2n+1}(x - x_s)$. However, $x_{2n+1} = 1$, therefore

$x^{2n+1} - 1 = (x - 1)\prod_{s=1}^{2n}(x - x_s)$, but $x_{2n-s} = \overline{x_s} \Rightarrow x^{2n+1} - 1 = (x - 1)\prod_{x=1}^{n}(x - x_s)(x - \overline{x_s}) = (x + 1)\prod_{k=1}^{n}\Big(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1\Big)$.

361. Considering $2n + 1$th root of -1, $x_s = -\cos\frac{2(2s+1)\pi}{2n+1} + i\sin\frac{(2s+1)\pi}{2n+1}$ $(s = 1, 2, 3, \ldots, n)$.

Therefore $x^{2n+1} + 1 = \prod_{s=1}^{2n+1}(x - x_s)$. However, $x_{2n+1} = -1$, therefore

$x^{2n+1}+1=(x+1)\prod_{s=1}^{2n}(x-x_s)$, but $x_{2n-s}=\overline{x_s} \Rightarrow x^{2n+1}+1=(x+1)\prod_{x=1}^{n}(x-x_s)(x-\overline{x_s})=(x+1)\prod_{k=1}^{n}\left(x^2+2x\cos\frac{2k\pi}{2n+1}+1\right)$.

362. This problem can be solved like previous problem.

363. We have proven that $x^{2n}-1=(x^2-1)\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)$

$\Rightarrow x^{2n-2}+x^{2n-4}+\cdots+x^2+1=\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)$

Putting $x=1$, we have $n=\prod_{k=1}^{n-1}\left(2-2\cos\frac{k\pi}{n}\right)=\prod_{k=1}^{n-1}4\sin^2\frac{k\pi}{2n}=2^{2(n-1)}\sin^2\frac{\pi}{2n}\sin^2\frac{2\pi}{2n}\cdots\sin^2\frac{(n-1)\pi}{2n}$

$\Rightarrow \sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}}$.

364. This problem can be solved like previous problem.

365. Since $\cos\alpha+i\sin\alpha$ is the root of the given equation, we have $\sum_{k=0}^{n}p_k(\cos\alpha+i\sin\alpha)^{n-k}=0\ \ (p_0=1)$

$\Rightarrow (\cos\alpha+i\sin\alpha)^n\sum_{k=0}^{n}p_k(\cos\alpha+i\sin\alpha)^{-k}=0 \Rightarrow \sum_{k=0}^{n}p_k(\cos\alpha k-i\sin\alpha k)=0$.

Hence, $\sum_{k=0}^{n}p_k\sin\alpha k=p_1\sin\alpha+p_2\sin 2\alpha+\cdots+p_n\sin n\alpha=0$.

366. The roots of the equation $x^7=1$ are $\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}\ \ (k=0,1,2,\ldots,6)$.

Therefore, the roots of the equation $x^6+x^5+x^4+x^3+x^2+x+1=0$ will be $x_k=\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}\ \ (k=1,2,3,\ldots,6)$.

Putting $x+\frac{1}{x}=y$, then $x^2+\frac{1}{x^2}=y^2-2$ and $x^3+\frac{1}{x^3}=y^3-3y$. Rewriting the above equation $\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0$.

Clearly, $x_1=\overline{x_6}, x_2=\overline{x_5}, x_3=\overline{x_4}, x_k+\frac{1}{x_k}=x_k+\overline{x_k}=2\cos\frac{2k\pi}{7}$.

Hence we can say that quantities $2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7}$ are the rootss of the equation $y^3+y^2-2y-1=0$.

Let the roots of the cubic equation $x^3 - ax^2 + bx - c = 0$ be α, β, γ. Then $\alpha + \beta + \gamma = a$, $\alpha\beta + \beta\gamma + \gamma\alpha = b$, $\alpha\beta\gamma = c$.

Let the equation, whose roots are $\sqrt[3]{\alpha}$, $\sqrt[3]{\beta}$, $\sqrt[3]{\gamma}$, be $x^3 - Ax^2 + Bx - C = 0$. Then,

$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = A$, $\sqrt[3]{\alpha\beta} + \sqrt[3]{\beta\gamma} + \sqrt[3]{\gamma\alpha} = B$, $\sqrt[3]{\alpha\beta\gamma} = C$.

We know that $(m + p + q)^3 = m^3 + p^3 + q^3 + 3(m + p + q)(mp + mq + pq) - 3mpq$. Substituting $\sqrt[3]{\alpha}$, $\sqrt[3]{\beta}$, $\sqrt[3]{\gamma}$ and $\sqrt[3]{\alpha\beta}$, $\sqrt[3]{\beta\gamma}$, $\sqrt[3]{\gamma\alpha}$ for m, p, q we obtain

$A^3 = a + 3AB - 3C, B^3 = b + 3BCA - 3C^2$. In our case, $a = -1, b = -2, c = 1, C = 1$. Hence, $A^3 = 3AB - 4$, $B^3 = 3AB - 5$.

Multiplying these equations and putting $AB = z$, we find

$z^3 - 9z^2 + 27z - 20 = 0 \Rightarrow (z - 3)^3 + 7 = 0 \Rightarrow z = 3 - \sqrt[3]{7}$

But $A^3 = 3z - 4 \Rightarrow A = \sqrt[3]{5 - 3\sqrt[3]{7}}$ and hence

$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{1}{2}(5 - 3\sqrt[3]{7})}$.

367. This problem can be solved like previous problem.

368. Squaring the first trimonial, $A^2 = (x_1^2 + 2x_2x_3) + (x_3^2 + 2x_1x_2)\omega + (x_2^2 + 2x_1x_3)\omega^2$.

Then $A^3 = (x_1^2 + x_2^2 + x_3^2 + 6x_1x_2x_3) + (3x_1^2x_2 + 3x_2^2x_1 + 3x_2^2x_3)\omega + (3x_1^2x_3 + 3x_2^2x_1 + 3x_3^2x_2)\omega^2$

Putting $3\alpha = 3x_1^2x_2 + 3x_2^2x_1 + 3x_2^2x_3$ and $3\beta = 3x_1^2x_3 + 3x_2^2x_1 + 3x_3^2x_2$.

Now $x_1^3 + x_2^3 + x_3^3 = -(px_1 + q) - (px_2 + q) - (px_3 + q) = -3q$ since $x_1 + x_2 + x_3 = 0$. Moreover, $x_1x_2x_3 = -q$, therefore

$A^3 = -9q + 3\alpha\omega + 3\beta\omega^2$, we also find $B^3 = -9q + 3\alpha\omega^2 + 3\beta\omega$.

Hence, $A^3 + B^3 = -18q - 3\alpha - 3\beta = -27q$, and similarly, $A^3B^3 = -27p^3$.

369. Let $f(x) = \frac{5x^4+10x^2+1}{x^4+10x^2+5}$ then the equation takes the form $f(x).f(a) = ax$.

$x - f(x) = \frac{(x-1)^5}{x^4+10x^2+5}$ and $x + f(x) = \frac{(x+1)^5}{x^4+10x^2+5}$. Dividing,

$\frac{x-f(x)}{x+f(x)} = \left(\frac{x-1}{x+1}\right)^5$. Let $\frac{x-1}{x+1} = y$ and $\frac{a-1}{a+1} = b$.

$\Rightarrow x - f(x) = y^5x + y^5f(x), x(1 - y^5) = f(x)(1 + y^5) \Rightarrow \frac{f(x)}{x} = \frac{1-y^5}{1+y^5}$.

Similarly, $\frac{f(a)}{a} = \frac{1-b^5}{1+b^5}$. So we can write the equation as $\frac{1-y^5}{1+y^5} = \frac{1+b^5}{1-b^5} \Rightarrow y^5 = -b^5$.

The last equation has five roots. $y_k = -b\epsilon^k$, where $\epsilon = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

But $x = \frac{1+y}{1-y} \Rightarrow x_k = \frac{(a+1)-(a-1)\epsilon^k}{(a+1)+(a-1)\epsilon^k} = \frac{\cos\frac{k\pi}{5} - ia\sin\frac{k\pi}{5}}{a\cos\frac{k\pi}{5} - i\sin\frac{k\pi}{5}}$.

370. $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$

Further $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n = \frac{(-1)^n}{2^n}(1-i\sqrt{3})^n = \frac{(-1)^n}{2^n}[1 + C_1^n(-i\sqrt{3}) + C_2^n(-i\sqrt{3})^2 + C_3^n(-i\sqrt{3})^3 + \cdots]$

$= \frac{(-1)^n}{2^n}[1 - 3C_2^n + \cdots] - i\sqrt{3}[C_1^n - 3C_3^n + 3^2C_5^n - 3^3C_7^n + \cdots]$

Equating coefficient of i in both the equations, $S = (-1)^{n+1}\frac{2^n}{\sqrt{3}}\sin\frac{2n\pi}{3}$.

371. We have $(1+i)^n = 1 + C_1^n i + C_2^n i^2 + C_3^n i^3 + \cdots = 1 + C_1^n i - C_2^n - C_3^n i + \cdots$

But $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Therefore, $\sigma = 1 - C_2^n + C_4^n - C_6^n + \cdots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$,

$\sigma' = C_1^n - C_3^n + C_5^n - C_7^n + \cdots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$.

Hence, if $n = 0(\operatorname{mod} 4)$ i.e. $n = 4m\ \forall\ m \in \mathbb{I}$, then $\sigma = (-1)^m\, 2^{2m}$, $\sigma' = 0$. If $n = 4m+1$, then $\sigma = \sigma' = (-1)^m\, 2^{2m}$, for $n = 4m+2$, $\sigma = 0$, $\sigma' = (-1)\, 2^{2m+1}$ and for $n = 4m+3$, $\sigma = (-1)^{m+1}\, 2^{2m+1}$, $\sigma' = (-1)^m\, 2^{2m+1}$.

Answers of Chapter 4

Polynomials and Theory of Equations

1. $x + x^9 + x^{25} + x^{49} + x^{81} = x(1 + x^8 + x^{24} + x^{48} + x^{80}) = x[(x^{80} - 1) + (x^{48} - 1) + (x^{24} - 1) + (x^8 - 1) + 5]$.

   All terms are divisible by $x(x^2 - 1)$ except last term $5x$, and hence, $5x$ is the remainder.

2. Let $P = x^{9999} + x^{8888} + x^{7777} + \cdots + x^{1111} + 1$ and $Q = x^9 + x^8 + x^7 + \cdots + x + 1$, then $P - Q = x^9[(x^{10})^{999} - 1] + x^8[(x^{10})^{888} - 1] + \cdots + x[(x^{10})^{100} - 1]$

   But $(x^{10})^n - 1$ is divisible by $x^{10} - 1\ \forall\ n \geq 1. \therefore P - Q$ is divisible by $x^{10} - 1$.

   Because $x^9 + x^8 + x^7 + \cdots + x + 1 | x^{10} - 1 \Rightarrow x^9 + x^8 + x^7 + \cdots + x + 1 | P - Q \Rightarrow x^9 + x^8 + x^7 + \cdots + x + 1 | P$.

3. We will prove this by contradiction. Suppose that $f(n) = 0$, then $f(x - n)$ divides $f(x)$ i.e. $f(x) = (x - n)g(x)$, where $g(x)$ is another polynomial with integral coefficients. Now $f(1) = (1 - n)g(1)$ and $f(2) = (2 - n)g(2)$. Both of these should be odd numbers but that is not possible as $1 - n$ and $2 - n$ are consecutive integers. Thus, either $f(1)$ or $f(2)$ should be even, which is a contradiction, and hence, the result.

4. Suppose that there exists such an integer b, such that $f(b) = 1993$. Let $g(x) = f(x) - 1991$. Now, g is a polynomial with integer coefficients and $g(a_i) = 0$ for $i = 1, 2, 3, 4$.

   Thus, $(x - a_1), (x - a_2), (x - a_3)$ and $(x - a_4)$ are all factors of $g(x)$. So $g(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)h(x)$, where $h(x)$ is a polynomial with integer coefficients. $g(b) = f(b) - 1991 = 2$ so $g(b) = (b - a_1)(b - a_2)(b - a_3)(b - a_4)h(b) = 2$.

   Thus, $(b - a_1)(b - a_2)(b - a_3)(b - a_4)$ are all divisors of 2 and distinct. Such values are $1, -1, -2, 2$ and $h(b)$ is an integer.

   $\therefore g(b) = 4.h(b) = 2$, which is not possible. Hence, such an integer does not exist.

5. We know that when coefficients of a polynomial are integers then quadratic surds as roots appear in pairs. Therefore, the other root would be $-\sqrt{5}$ giving us a second degree polynomial $x^2 - 5$. Therefore, we can write the polynomial is of the form $ax^2 - 5a$.

   Second method: Since the order of the surd $\sqrt{5}$ is 2, we can expect a polynomial of the lowest degree to be a polynomial of degree 2. Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{Q}$. $f(\sqrt{5}) = 5a + \sqrt{5}a + c = 0$ But $\sqrt{5}$ is irrational so $5a + c = 0$ and $b = 0 \Rightarrow c = -5a$ so the polynomial is of the form $ax^2 - 5a$ giving us second root at $-\sqrt{5}$.

6. Let $f(x) = x - (\sqrt{5} + \sqrt{2}) = [(x - \sqrt{5}) - \sqrt{2}]$. Using conjugate as the other zero, we have $f_1(x) = [(x - \sqrt{5}) - \sqrt{2}][(x - \sqrt{5}) + \sqrt{2}] = (x^2 + 3 - 2\sqrt{5}x) \Rightarrow f_2(x) = [(x^2 + 3) - 2\sqrt{5}x][(x^2 + 3) + 2\sqrt{5}x] = x^4 - 14x^2 + 9 \Rightarrow f(x) = ax^4 - 14x^2 + 9a$, where $a \in \mathbb{Z}, a \neq 0$.

7. Putting $x = 0, 0 = -f(0) \Rightarrow f(0) = 0$. Putting $x = 1, f(0) = -3f(1) \Rightarrow f(1) = 0$. Similalrly, $f(2) = f(3) = 0$. Let is assume $f(x) = x(x-1)(x-2)(x-3)g(x)$, where $g(x)$ is some polynomial. Now using the given relation we have $x(x-1)(x-2)(x-3)(x-4)g(x-1) = x(x-1)(x-2)(x-3)(x-4)g(x)$

   $\Rightarrow g(x-1) = g(x)\ \forall\ x \in \mathbb{R} - \{0, 1, 2, 3, 4\} \Rightarrow g(x-1) = g(x)\ \forall\ x \in \mathbb{R}$ from identity theorem.

   $\Rightarrow g(x)$ is periodic. $\Rightarrow g(x) = c \Rightarrow f(x) = cx(x-1)(x-2)(x-3)$

8. Because $f(x)$ is a monic coefficient of highest degree will be 1. Let $g(x) = f(x) - x$, where $g(x)$ is also a cubic polynomial.

   $g(1) = 0, g(2) = 0, g(3) = 0 \Rightarrow g(x) = (x-1)(x-2)(x-3) \Rightarrow f(x) = (x-1)(x-2)(x-3) + x \Rightarrow f(4) = 10$.

9. Let $f(x) = x - (\sqrt{3} + \sqrt{7}) = [(x - \sqrt{3}) - \sqrt{7}]$. Using conjugate as the other zero, we have $f_1(x) = [(x - \sqrt{3}) - \sqrt{7}][(x - \sqrt{3}) + \sqrt{7}] = (x^2 - 4 - 2\sqrt{3}x) \Rightarrow f_2(x) = [(x^2 - 4) - 2\sqrt{3}x][(x^2 - 4) + 2\sqrt{3}x] = x^4 - 8x^2 + 16 - 12x^2 = x^4 - 20x^2 + 16 = 0$.

10. Clearly, we will have conjugate roots for the given surds as roots, which would be $2 - \sqrt{3}$ and $3 - \sqrt{2}$. Therefore, the polynomial would be

    $f(x) = [(x-2) - \sqrt{3}][(x-2) + \sqrt{3}][(x-3) - \sqrt{2}][(x-3) + \sqrt{2}] = (x^2 - 4x + 4 - 3)(x^2 - 6x + 9 - 2) = (x^2 - 4x + 1)(x^2 - 6x + 7) = x^4 - 10x^3 + 32x^2 - 34x + 7 = 0$.

11. Let $y = \sqrt[3]{2}$, then $x = y + 3y^2 = y(3y+1)$. Cubing both sides $x^3 = y^3(27y^3 + 27y^2 + 9y + 1) = 2(9x + 55) \Rightarrow x^3 - 18x - 110 = 0$. This is the minimal polynomial as $[\mathbb{Q}(\sqrt[3]{2}) : Q] = 3$.

12. $x^n - nx + n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) - n(x-1) = (x-1)[(x^{n-1} - 1) + (x^{n-2} - 1) + \cdots + (x-1)]$, which clearly has a factor $(x-1)^2$.

13. Because a, b, c, d, e are all zeroes of the polynomial $6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$, therefore, $6(x-a)(x-b)(x-c)(x-d)(x-e) = 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$.

    Putting $x = 1, -6(1+a)(1+b)(1+c)(1+d)(1+e) = -6 + 5 - 4 + 3 - 2 + 1 = -3 \Rightarrow (1+a)(1+b)(1+c)(1+d)(1+e) = \frac{1}{2}$.

14. Because $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the roots of the equation $x^n - 1 = 0$, therefore, $(x-1)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}) = x^n - 1 \Rightarrow (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}) = x^{n-1} + x^{n-2} + \cdots + x + 1$.

    Putting $x = 1$, in the above equation, we deduce the desired result.

15. Consider a function $g(x) = f(x) - 10x$, then $g(1) = g(2) = g(3) = 0$ i.e. $(x-1)(x-2)(x-3)$ would divide $g(x)$. Since $f(x)$ has a degree of 4 so $g(x)$ will also have a degree of 4. Let $g(x) = (x-t)(x-1)(x-2)(x-3)$ so $f(x) = 10x + (x-t)(x-1)(x-2)(x-3)$.

Now for $x = 12$, $(x-1)(x-2)(x-3) = 990$ and for $x = -8$, $(x-1)(x-2)(x-3) = -990$.

$\therefore \frac{f(12)+f(-8)}{10} = \frac{10(12-8)+(12-t)990+(-8-t).-990}{10} = 1984$.

16. Roots of x^2+x+1 are ω, ω^2. Since given polynomial is not divisible by x^2+x+1, so these roots won't satisfy the given polynomial. Thus,

    $\omega^{2k}+1+(1+\omega)^{2k} = \omega^{2k}+1+(\omega^2)^{2k} = 1+\omega^k+\omega^{2k} \neq 0$. We know that $1+\omega^k+\omega^{2k} = 3$ when $k = 3n, n \in \mathbb{N}$. Hence, $k = 3, 6, 9, \ldots$.

17. Putting $x = 1, -7P(2) = 0 \Rightarrow P(2) = 0$. Putting $x = 8, 0 = 56P(8) \Rightarrow P(8) = 0$.

    $\Rightarrow P(x) = (x-2)(x-8)Q(x) \Rightarrow P(2x) = (2x-2)(2x-8)Q(2x)$

    $\Rightarrow (x-8)(2x-2)(2x-8)Q(2x) = 8(x-1)(x-2)(x-8)Q(x) \Rightarrow \frac{Q(2x)}{Q(x)} = \frac{2x-4}{x-4} \Rightarrow Q(x) = x-4 \Rightarrow P(x) = (x-2)(x-4)(x-8)$.

18. If $(x-1)^3$ divides $f(x)+1$, then $(x-1)^2$ divides $f'(x)$ and if $(x+1)^3$ divides $f(x)-1$ then $(x+1)^3$ divides $f'(x)$. Since we have to find $f(x)$ of degree 5, $f'(x)$ will be of degree 4. So $f'(x) = k(x-1)^2(x+1)^2 = k(x^4-2x^2+1)$.

    Integrating both sides, $f(x) = K\left(\frac{x^5}{5} - \frac{2x^3}{3} + x\right) + c$, where $c \in \mathbb{R}$. Also, $(x-1)^3$ divides $f(x)+1 \Rightarrow f(1)+1 = 0 \Rightarrow f(1) = -1$ and $(x+1)^3$ divides $f(x)-1 \Rightarrow f(-1)-1 = 0 \Rightarrow f(-1) = 1$.

    Putting $x = 1$ in the equation for $f(x), \Rightarrow f(1) = K\left(\frac{1}{5} - \frac{2}{3} + 1\right) + c = -1$, and putting $x = -1 \Rightarrow f(-1) = K\left(\frac{-1}{5} + \frac{2}{3} - 1\right) + c = 1$.

    From these two equations we deduce $K = -\frac{15}{8}, c = 0$. Thus, our required polynomial is $f(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x$.

19. Since the polynomial equation has rational coefficients the complex roots must appear in conjugate pairs. So we have at least two more roots i.e. $3-2i$ and $2-3i$ making out polynomial equation of at least having a degree of 4. Let us find out the polynomial equation to test if the coefficients with these roots are rational.

    $f(x) = a[(x-3-2i)(x-3+2i)][x-2-3i][x-2+3i] = a(x^4-10x^3+50x^2-130x+169), a \in \mathbb{Q} \setminus \{0\}$.

20. Since all the roots are rational, so they are divisors of -30. The divisors or -30 are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15$, and ±30. By applying remainder theorm, we find the roots as $-1, -2, -3$ and 5.

21. Let the roots be of the form $\frac{p}{q}$, where $(p, q) = 1$ and $q > 0$. Since $q \mid 2$, q must be 1 or 2 and $p \mid 6 \Rightarrow p = \pm1, \pm2, \pm3, \pm6$.

Applying remainder theorem, $f\left(\frac{1}{2}\right) = f\left(\frac{-2}{1}\right) = f\left(\frac{3}{1}\right) = 0$. So the three roots of the equation are $\frac{1}{2}, -2$, and 3.

22. $x^3 - 3x^3 + 5x - 15 = (x^2+5)(x-3) = 0 \Rightarrow x = 3, \sqrt{5}i, -\sqrt{5}i$.

23. Let the roots be of the form $\frac{p}{q}$, where $(p,q) = 1$ and $q > 0$. Since $q \mid 1 \Rightarrow q = \pm 1$, also $p \mid 1 \Rightarrow p = \pm 1 \Rightarrow \frac{p}{q} = \pm 1$. But $f(\pm 1) \neq 0$.

    Hence, the given equation has no real roots.

24. Let α and β be the two roots of the given equation, where $\alpha \in \mathbb{Z}$. Then,

    $\alpha + \beta = -a$ and $\alpha\beta = b + 1 \Rightarrow \beta = -a - \alpha$ is an integer. Also, since $b + 1 \neq 0, \beta \neq 0$. From these equations $a^2 + b^2 = (\alpha+\beta)^2 + (\alpha\beta - 1)^2 = (1+\alpha^2)(1+\beta)^2$. Hence, $a^2 + b^2$ is a composite number.

25. Let α and β bet the roots of the given equation, then $\alpha + \beta = p, \alpha\beta = p - 1$.

    $(\alpha^2+\beta^2) = (\alpha+\beta)^2 - 2\alpha\beta = p^2 - 2p + 2 = (p-1)^2 + 1$. For the sum to be minimum $(p-1)^2$ has to be minimum, which is minimum at $p = 1$.

26. Let $x^3 + ax^2 + bx + c = 0$ be the polynomial, of which α,β and $\alpha\beta$ are the roots and a, b and c are all rationals.

    From Vieta's relations $\alpha + \beta + \alpha\beta = -a, \alpha\beta + \alpha^2\beta + \alpha\beta^2 = b, \alpha^2\beta^2 = -c$. $b = \alpha\beta(1+\alpha+\beta) = \alpha\beta(1-a-\alpha\beta) = (1-a)\alpha\beta - \alpha^2\beta^2 = (1-a)\alpha\beta + c$. As $a \neq -1, \alpha\beta = \frac{b-c}{1-a}$ and since a, b, c are rational $\alpha\beta$ is rational.

    Note that $a = 1 \Rightarrow 1 + \alpha + \beta + \alpha\beta = 0 \Rightarrow (1+\alpha)(1+\beta) = 0 \Rightarrow \alpha = -1$ or $\beta = -1$, which is not the case.

27. Let the roots be $\alpha, 2\alpha$ and β, then from Vieta's relations we have $3\alpha + \beta = \frac{27}{9} = 3 \Rightarrow \beta = 3(1-\alpha), 2\alpha^2 + 3\alpha\beta = \frac{26}{9}$ and $2\alpha^2\beta = \frac{8}{9}$.

    From first two equations, we get $2\alpha^2 + 3\alpha.3(1-\alpha) = \frac{26}{9} \Rightarrow \alpha = \frac{13}{21}$ or $\frac{2}{3}$. If $\alpha = \frac{13}{21}$ then $beta = \frac{8}{7}$ but then $2\alpha^2\beta = 2 \times \frac{169}{144} \times \frac{8}{7} \neq \frac{8}{9}$, which is a contradiction.

    So taking $\alpha = \frac{2}{3} \Rightarrow \beta = 1$. Hence, $\alpha + 2\alpha + \beta = 3, 2\alpha^2 + 3\alpha\beta = \frac{26}{9}$ and $2\alpha^2\beta = \frac{8}{9}$. Hence, the roots are $\frac{2}{3}, \frac{4}{3}$ and 1.

28. Suppose the roots are $\alpha, \beta, \gamma, \delta$ and $\alpha\beta = 1$. Now $\alpha + \beta + \gamma + \delta = \frac{-24}{6} = -4, (\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = \frac{31}{4} \Rightarrow (\alpha+\beta)(\gamma+\delta) + \gamma\delta = \frac{27}{4}, \gamma\delta(\alpha+\beta) + \alpha\beta(\gamma+\delta) = \frac{-3}{2} \Rightarrow \gamma\delta(\alpha+\beta) + \gamma + \delta = \frac{-3}{2}, \alpha\beta\gamma\delta = -2 \Rightarrow \gamma\delta = -2$.

From second and fourth equation, we have $(\alpha+\beta)(\gamma+\delta)=\frac{35}{6}$ from third and fourth equation, we have $-2(\alpha+\beta)+\gamma+\delta=\frac{-3}{2}\Rightarrow 3(\alpha+\beta)=\frac{15}{2}\Rightarrow \alpha+\frac{1}{\alpha}=\frac{5}{2}\Rightarrow \alpha=2,\frac{1}{2}$. Hence, $\beta=\frac{1}{2},2$. Now it is trivial to find γ and δ, which can be found to be $\frac{-1}{2}$ and 4.

29. Since the coefficients are rational, where $3+\sqrt{2}$ is a root, so $3-\sqrt{2}$ is also a root. Thus, if two other roots are α and β, we have

$\sigma_1=\alpha+\beta+3+\sqrt{2}+3-\sqrt{2}=-(-5)=5\Rightarrow \alpha+\beta=-1.$

$\sigma_2=(\alpha+\beta)(3+\sqrt{2}+3-\sqrt{2})+\alpha\beta+(3+\sqrt{2})(3-\sqrt{2})=a\Rightarrow 6(\alpha+\beta)+\alpha+\beta+7=a\Rightarrow \alpha\beta=a-1.$

$\sigma_3=\alpha\beta(3+\sqrt{2}+3-\sqrt{2})+(3+\sqrt{2})(3-\sqrt{2})(\alpha+\beta)=-b\Rightarrow 6\alpha\beta-7=b\Rightarrow \alpha\beta=\frac{7-b}{6}$

$\sigma_4=7\alpha\beta=c\Rightarrow \alpha\beta=\frac{c}{7}.$

We take $\alpha+\beta=-1,\alpha\beta=k$. α and β are roots of the equation $x^2+x+k=0$. Since the roots of the given equation are real $\Rightarrow 1-4k\geq 0\Rightarrow k\leq\frac{1}{4}$. Now for $a, k=a-1\Rightarrow a\leq\frac{5}{4}$. So the greatest value of a is $\frac{5}{4}$. For $b, k=\frac{7-b}{6}\Rightarrow b\geq\frac{11}{2}$ so least value of b will be $\frac{11}{2}$. For $c, k=\frac{c}{7}\Rightarrow c\leq\frac{7}{4}$ So the maximum value of c will be $\frac{7}{4}$.

The two other roots can be found as $-\frac{1}{2}$, which is a repeated root.

30. Let the rational roots be of the form $\frac{p}{q}$, then $q\mid 1\Rightarrow q=\pm1$ and $p\mid 1\Rightarrow p=\pm1\Rightarrow\frac{p}{q}=\pm1$. But we see that $x=-1$ does not satisfy the equation so $x=1$ is the only root.

Second method: You can observe by looking at the coefficients that it is expansion of $(x-1)^4$ as the coefficients are from binomoal theorem. Hence, the root is 1.

31. Let $\alpha,\beta,\gamma,\delta$ are the roots of the equation, then from Vieta's relations $\alpha+\beta+\gamma+\delta=-10$. From question $\alpha+\beta=\gamma+\delta\Rightarrow\alpha+\beta=\gamma+\delta=-5$.

Let the roots be of the form $\frac{p}{q}$ then $q\mid 1\Rightarrow q=\pm1$ and $p\mid 24\Rightarrow p=\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24$. Clearly, ±12 and ±24 are not possible values. Testing with other values we find roots as $-1,-2,-3,-4$.

32. Let the rational roots be of the form $\frac{p}{q}$, then $q\mid 6\Rightarrow q=\pm1,\pm2,\pm3,\pm6$ and $p\mid -4\Rightarrow p=\pm1,\pm2,\pm4$.

We find that $-\frac{1}{2}$ and $\frac{4}{3}$ satisfy the given equation and the given equation becomes $(2x+1)(3x-4)(x^2+x+1)=0$, which has two more roots ω,ω^2, which are cube roots of unity, and are not rational roots.

33. Let the rational roots be of the form $\frac{p}{q}$, then $q\mid 6\Rightarrow q=\pm1,\pm2,\pm3,\pm6$ and $p\mid 2\Rightarrow p=\pm1,\pm2$. We see that all coefficients are positive so positive values of $\frac{p}{q}$ will not satisfy the given equation.

From negative values we see that only $x=-1$ satisfies the given equation.

34. Let α, β, γ are the roots of the given equation, then according to the questions $\alpha+\beta=0 \Rightarrow \alpha=-\beta$.

    From Vieta's relations $\alpha+\beta+\gamma=-\frac{a}{4} \Rightarrow \gamma=-\frac{a}{4}, \alpha\beta+\beta\gamma+\alpha\gamma=\alpha\beta=-\beta^2=-\frac{1}{4} \Rightarrow \beta=\pm\frac{1}{2} \Rightarrow \alpha=\mp\frac{1}{2}$ and $\alpha\beta\gamma=-\frac{b}{4} \Rightarrow a+4b=0$, where $b\in\mathbb{Q}$.

35. Let the roots be α, $\alpha.r$, $\alpha.r^2$ be the roots of the given equation, then from Vieta's relations, we have

    $\frac{\alpha}{r}+\alpha+\alpha.r=-a, \frac{\alpha^2}{r}+\alpha^2+\alpha^2.r=b$ and $\alpha^3=8 \Rightarrow \alpha=2$.

    From first two equations, $\alpha=-\frac{b}{a}=2 \Rightarrow b=-2a$. Substituting the value of α in the first equation, we have

    $2r^2+(a+2)r+2=0$, but r is real so $D\geq 0 \Rightarrow a^2+4a-12=0 \Rightarrow a\in(-\infty,6)\cup(2,\infty)$.

36. $2x^6+12x^5+30x^4+60x^3+80x^2+30x+45=2(x^3+3x^2)^2+12\left(x^2+\frac{5}{2}x\right)^2+5(x+3)^2=0$, but it could be zero only if

    $(x^3+3x^2)=\left(x^2+\frac{5}{2}x\right)=x+3=0.$

    The last and first condition simplifies to $x=-3$, but it contradicts the seccond. Thus, given polynomial has no real roots.

    Second method: Let the roots be of the form $\frac{p}{q}$ then $q\div 2 \Rightarrow q=\pm1,\pm2$ and $p\div 45 \Rightarrow p=\pm1,\pm3,\pm5,\pm9,\pm15,\pm45$. Clearly, the roots have to be negative as all coefficients are positive. But none of the combinations of $\frac{p}{q}$ satisfy the given equation, hence, it has no real roots.

37. $\sin 30^\circ=3\sin 10^\circ-4\sin^3 10^\circ \Rightarrow \sin 10^\circ$ is a root of $6x-8x^3=1$. By the rational root theorem, this equation has no rational roots. Therefore, $\sin 10^\circ$ is not rational. Since 3 is prime, this equation is the one with least degree having $\sin 10^\circ$ as a root.

    Second Method: $\sin 10^\circ=\cos 80^\circ=\cos\frac{4\pi}{9}$. Let $\omega=e^{2i\pi/9}$, then $\omega^6+\omega^3+1=0$, from which we can calculate that $\omega+\frac{1}{\omega}, \omega^2+\frac{1}{\omega^2}$ and $\omega^4+\frac{1}{\omega^4}$ are the roots of $x^3-3x+1=0$. Since $2\cos 80^\circ$ is such a root so $8x^3-6x+1=0$ is the equation.

38. Following like previous problem $\sin 60^\circ=3\sin 20^\circ-4\sin^3 20^\circ$. Putting $x=\sin 20^\circ$ and squaring, $64x^6-96x^4+36x^2-3=0$ is the required equation.

39. Following like previous problem $\cos 30^\circ=4\cos^3 10^\circ-3\cos 10^\circ \Rightarrow \frac{\sqrt{3}}{2}=4\cos^3 10^\circ-3\cos 10^\circ \Rightarrow 64x^6-96x^4+36x^2-3=0$ is the required equation.

40. Following like previous problems $\cos 60^\circ=4\cos 20^\circ-3\cos 20^\circ \Rightarrow 8x^3-6x-1=0$.

41. Following like previous problems $\tan 30^\circ = \frac{3\tan 10^\circ - \tan^3 10^\circ}{1-3\tan^2 10^\circ} \Rightarrow \frac{1}{\sqrt{3}} = \frac{3x-x^3}{1-3x^2}$. Squaring, we get $3x^6 - 27x^4 + 33x^2 - 1 = 0$.

42. Following like previous problems $\tan 60^\circ = \frac{3\tan 20^\circ - \tan^3 20^\circ}{1-3\tan^2 20^\circ} \Rightarrow \sqrt{3} = \frac{3x-x^3}{1-3x^2}$. Squaring, we get $x^6 - 33x^4 + 27x^2 - 3 = 0$.

43. We have found the equations for $\sin 10^\circ$ and $\cos 20^\circ$ are $8x^3 - 6x + 1 = 0$ and $8x^3 - 6x - 1 = 0$. Therefore, the equation having these two as roots must be $(8x^3 - 6x + 1)(8x^3 - 6x - 1) = 0 \Rightarrow 64x^6 - 96x^4 - 36x^2 - 1 = 0$.

44. From Vieta's relations $p + q + r = 6$, $pq + qr + rp = 3$, $pqr = -1 \Rightarrow p^2 + q^2 + r^2 = 30$, $p^3 + q^3 + r^3 = 159$, $p^3q^3 + q^3r^3 + r^3p^3 = 84$.

    Let $A = p^2q + q^2r + r^2p$ and $B = p^2r + q^2p + r^2q$, then $A + B = 6(p^2 + q^2 + r^2) - (p^3 + q^3 + r^3) = 21$ and $AB = -(p^3 + q^3 + r^3)(p^3q^3 + q^3r^3 + r^3p^3) + 3 = 72$.

    Thus, possible value of A are $24, -3$.

45. Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation such that $\alpha\beta = -32$, then from Vietas relations $\alpha + \beta + \gamma + \delta = 18, \alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \alpha\delta + \beta\delta = k, \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -200$ and $\alpha\beta\gamma\delta = -1984$.

    $\therefore \gamma\delta = \frac{\alpha\beta\gamma\delta}{\alpha\beta} = \frac{-1984}{-32} = 62$.

    $\therefore -32 + \beta\gamma + 62 + \alpha\gamma + \alpha\delta + \beta\delta = k \Rightarrow \beta\gamma + \alpha\gamma + \alpha\delta + \beta\delta = k - 30$. Let $p = \alpha + \beta$ and $q = \gamma + \delta$.

    $\therefore -200 = -32q + 62p$ and $p + q = 18 \Rightarrow p = 4, q = 14 \Rightarrow \frac{\alpha+\beta}{2}\frac{\gamma+\delta}{2} = k - 30 \Rightarrow k = 86$.

46. $x^2 + y^2 = 1 - 2xy \Rightarrow (x^2 + y^2)^2 = (1 - 2xy)^2 \Rightarrow x^4 + y^4 = 2x^2y^2 - 4xy + 1 \Rightarrow 2x^2y^2 - 4xy + 1 - c = 0 \Rightarrow xy = \frac{4\pm\sqrt{16+8c-8}}{4} = 1 \pm \sqrt{\frac{1+c}{2}}$

    Now, $x^2 + y^2 = 1 - 2\left(1 \pm \sqrt{\frac{1+c}{2}}\right) = -1 \pm \sqrt{2(1+c)}$,

    and $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 2 \pm \frac{3}{2}\sqrt{2+2c}$.

47. Let $x + y = \alpha$ and $xy = \beta$, then $x^2 + y^2 = \alpha^2 - 2\beta$.

    Now, $x^3 + y^3 = (x+y)(x^2 + y^2 - xy) = \alpha(\alpha^2 - 3\beta) = 7 \Rightarrow \alpha^3 - 3\alpha\beta = 7$,

    and $x^2 + y^2 + x + y + xy = 4 \Rightarrow \alpha^2 - 2\beta + \alpha + \beta = 4 \Rightarrow \beta = \alpha^2 + \alpha - 4$.

    From these two equations $\alpha^3 - 3\alpha(\alpha^2 + \alpha - 4) = 7 \Rightarrow f(\alpha) = 2\alpha^3 + 3\alpha^2 - 12\alpha + 7 = 0$.

    Since sum of coefficients is zero, therefore, $\alpha = 1$ must be a solution. $\Rightarrow f(1) = 0 \Rightarrow f(\alpha) = (\alpha - 1)^2(2\alpha + 7) = 0 \Rightarrow \alpha = 1, -\frac{7}{2}$.

When $\alpha = 1, \beta = -2$ and when $\alpha = -\frac{7}{2}, \beta = \frac{19}{4}$. Thus, when $\alpha = 1, \beta = -2$ we find that (x, y) is $(-2, 1)$ or $(1, -2)$. But when $\alpha = -\frac{7}{2}$ and $\beta = \frac{19}{4}$, then x, y are roots of $4t^2 + 14t + 19 = 0$, whose discriminant is less than 0 and hence no real roots are possible. Thus, value of x, y is $-2, 1$ or $1, -2$.

48. From Vieta's relations $\alpha+\beta+\gamma = \sum\alpha = 0, \alpha\beta+\beta\gamma+\gamma\alpha = \sum\alpha\beta = p, \alpha\beta\gamma = \prod\alpha = q$.

    Since α, β, γ are roots of $x^3+px+q = 0 \Rightarrow \alpha^3+p\alpha+q = 0, \beta^3+p\beta+q = 0, \gamma^3+p\gamma+q = 0$

    Adding these equations, we have $\sum\alpha^3 + p\sum\alpha + 3q = 0 \Rightarrow \sum\alpha^3 = -3q[\because \sum\alpha = 0]$

    $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta = 0^2 - 2p = -2p.$

    Multiplying the given equation by x^2, we get $x^5 + px^3 + qx^2 = 0$. Putting $x = \alpha, \beta, \gamma$ and adding, we have

    $\sum\alpha^5 + p\sum\alpha^3 + q\sum\alpha^2 = 0 \Rightarrow \sum\alpha^5 = 5pq \Rightarrow \frac{1}{5}\sum\alpha^5 = pq = \frac{1}{3}\sum\alpha^3.\frac{1}{2}\sum\alpha^2.$

    Hence, proved.

49. Following like previous problem and using results from previous problem, multiplying the given equation by x, we have $x^4 + px^2 + qx = 0 \Rightarrow \sum\alpha^4 + p\sum\alpha^2 + q\sum\alpha = 0 \Rightarrow \sum\alpha^4 = -p\sum\alpha^2.$

    Multiplying the given equation by x^4, we get $x^7 + px^5 + qx^4 = 0 \Rightarrow \sum\alpha^7 + p\sum\alpha^5 + q\sum\alpha^4 = 0 \Rightarrow \sum\alpha^7 = -p\sum\alpha^5 - q\sum\alpha^4 = -5p^2q + pq\sum\alpha^2 = -7p^2q \Rightarrow \frac{\sum\alpha^7}{7} = pq.(-p) = \frac{\sum\alpha^5}{5}.\frac{\sum\alpha^2}{2}$

    $\Rightarrow \frac{\alpha^7+\beta^7+\gamma^7}{7} = \frac{\alpha^5+\beta^5+\gamma^5}{5} \times \frac{\alpha^2+\beta^2+\gamma^2}{2}.$

50. Since $\alpha+\beta+\gamma = 0$, therefore, α, β, γ are the roots of the equation $x^3 + px + q = 0. \Rightarrow \sum\alpha\beta = p$ and $\prod\alpha = -q$ as shown in previous problems.

    $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = 0^2 - 2p = -2p$ and $\sum\alpha^3 = 3\alpha\beta\gamma = -3q$.

    Multiplying $x^3 + px + q = 0$ with x, we have $x^4 + px^2 + qx = 0$. Putting $x = \alpha, \beta, \gamma$ and adding, we have

    $\sum\alpha^4 + p\sum\alpha^2 + q\sum\alpha = 0 \Rightarrow \sum\alpha^4 = -p\sum\alpha^2 = 2p^2.$

    Similarly, $x^5 + px^3 + qx^2 = 0 \Rightarrow \sum\alpha^5 = -p\sum\alpha^3 - q\sum\alpha^2 = -5pq.$

    $\therefore 3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = 3\times -2p \times -5pq = 5\times(-3q)\times -2p^2 = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4).$

    Hence, proved.

51. Suppose that $a^3+b^3=c^3+d^3$ and $a+b=c+d=m$(say), then $(a+b)^3=(c+d)^3 \Rightarrow 3ab(a+b)=3cd(c+d) \Rightarrow ab=cd=n$ (say).

    If a, b are the roots of a quadratic equation, then the equation is $x^2-mx+n=0$. But $a+b=m$ and $ab=n$. So a and b are roots of this equation, and thus, c and d are also the roots of the equation. But a quadratic equation can have at most two distinct roots.

    Hence, our supposition is incorrect. Hence, proved.

52. Let x, y, z be the roots of the cubic equation $t^3-at^2+bt-c=0$, then $x+y+z=a, xy+yz+zx=b \Rightarrow 2xy+2yz+2zx=2b=(x+y+z)^2-(x^2+y^2+z^2)=9-3 \Rightarrow b=3$.

    Substituting x, y, z in our equation and adding, we get $(x^3+y^3+z^3)-a(x^2+y^2+z^2)+b(x+y+z)-3c=0 \Rightarrow c=1$.

    Thus, our equation becomes $t^3-3t^2+3t-1=0 \Rightarrow (t-1)^3=0$, thus roots are $1, 1, 1$. And hence, $x=y=z=1$.

53. $xy+yz+zx=\frac{1}{2}[(x+y+z)^2-(x^2+y^2+z^2)]=2$.

    We know that $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx) \Rightarrow xyz=-\frac{2}{3}$.

    $x^4+y^4+z^4=(x^2+y^2+z^2)^2-2[(xy)^2+(yz)^2+(zx)^2]=25-2[(xy+yz+zx)^2-2(xy^2z+zxy^2+xyz^2)]=25-2[4-2xyz(x+y+z)]=9$.

54. From question $\alpha+\beta=a+d$ and $\alpha\beta=ad-bc$.

    $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(a+d)^3-3(ad-bc)(a+d)=a^3+d^3+3a^2d+3ad^2-3a^2d-3ad^2+3abc+3bcd=a^3+d^3+3abc+3bcd$ and $\alpha^3\beta^3=(ad-bc)^3$.

    Thus, equation whose roots are α^3 and β^3 is $x^2-(a^3+d^3+3abc+3bcd)x+(ad-bc)^3=0$.

55. $a^3+b^3+c^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a) \Rightarrow (a+b)(b+c)(c+a)=0$, which implies that one of $a+b, b+c, c+a=0$.

    In any case $a^{2n+1}+b^{2n+1}+c^{2n+1}=(a+b+c)^{2n+1}\ \forall\ n \in \mathbb{N}$ and for $n=2$, $a^5+b^5+c^5=(a+b+c)^5$.

56. We know that $p^3+q^3+r^3=(p+q+r)[(p+q+r)^2-3(pq+qr+rp)]+3pqr$.

    From Vieta's relations, we have $p+q+r=1, pq+qr+rp=1$ and $pqr=2$, therefore,

    $p^3+q^3+r^3=1[1-3]+6=4$.

57. Let a, b, c be the roots of the equation, then from Vieta's relations $a+b+c=0, ab+bc+ca=3, abc=-9$.

    Given equation is $x^3+3x+9=0$, putting $x=a, b, c$, and adding $a^3+b^3+c^3+3(a+b+c)+27=0 \Rightarrow a^3+b^3+c^3=-27$.

Multiplying given equation with x^2, putting $x = a, b, c$, and adding $a^5 + b^5 + c^3 + 3(a^3 + b^3 + c^3) + 9(a^2 + b^2 + c^2) = 0$

$\Rightarrow a^5 + b^5 + c^5 = 81 - 9[(a + b + c)^2 - 2(ab + bc + ca)] = 81 + 18 \times 3 = 135.$

58. Let a, b, c are the roots of the equation $x^3 - 7x^2 + 4x - 3 = 0$, then from Vieta's relations $a + b + c = 7, ab + bc + ca = 4$, and $abc = 3$.

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 7^2 - 2.4 = 41.$

Putting $x = a, b, c$ and adding, we get $a^3 + b^3 + c^3 = 7(a^2 + b^2 + c^2) - 4(a + b + c) + 9 = 7 \times 41 - 4 \times 7 + 9 = 287 - 28 + 9 = 268.$

Multiplying given equation with x, putting $x = a, b, c$, and adding $a^4 + b^4 + c^4 = 7(a^3 + b^3 + c^3) - 4(a^2 + b^2 + c^2) + 3(a + b + c) = 7 \times 268 - 4 \times 41 + 3 \times 7 = 1733.$

Multiplying given equation with x^2, putting $x = a, b, c$, and adding $a^5 + b^5 + c^5 = 7(a^4 + b^4 + c^4) - 4(a^3 + b^3 + c^3) + 3(a^2 + b^2 + c^2) = 7 \times 1733 - 4 \times 268 + 3 \times 41 = 11182.$

59. From Vieta's relations we have $\alpha + \beta + \gamma = 0, \alpha\beta + \beta\gamma + \gamma\alpha = -9, \alpha\beta\gamma = -9.$

Putting $x = \frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ in the given equation and adding, we have $\alpha^{-3} + \beta^{-3} + \gamma^{-3} = 9\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 27 = 9\frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} - 27 = -18.$

Multiplying the given equation by x^2 and putting $x = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, and adding

$\alpha^{-5} + \beta^{-5} + \gamma^{-5} = 9(\alpha^{-3} + \beta^{-3} + \gamma^{-3}) - 9(\alpha^{-2} + \beta^{-2} + \gamma^{-2}) = \frac{4}{9}.$

60. Let the cubic equation be $x^3 + ax^2 + bx + c = 0$, then from Vieta's relation $a = -(\alpha + \beta + \gamma) = -9$. We also have $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{(\alpha+\beta+\gamma)^2 - (\alpha^2+\beta^2+\gamma^2)}{2} = \frac{81-29}{2} = 26$ and hence $b = \alpha\beta + \beta\gamma + \gamma\alpha = 26$.

Putting $x = \alpha, \beta, \gamma$ in the given equation, and adding

$(\alpha^3 + \beta^3 + \gamma^3) - 9(\alpha^2 + \beta^2 + \gamma^2) + 26(\alpha + \beta + \gamma) + 3c = 0 \Rightarrow c = -24$, and hence, our equation is $x^3 - 9x^2 + 26x - 24 = 0$

Multiplying the given equation with x, putting $x = \alpha, \beta, \gamma$ and adding, we have

$\alpha^4 + \beta^4 + \gamma^4 = 9(\alpha^3 + \beta^3 + \gamma^3) - 26(\alpha^2 + \beta^2 + \gamma^2) + 9(\alpha + \beta + \gamma) = 353.$

61. Let the cubic equation be $x^3 + ax^2 + bx + c = 0$, then from Vieta's relation $a = -(\alpha + \beta + \gamma) = -4$. We also have $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{(\alpha+\beta+\gamma)^2 - (\alpha^2+\beta^2+\gamma^2)}{2} = \frac{16-7}{2} = \frac{9}{2}$ and hence $b = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{9}{2}$.

Putting $x = \alpha, \beta, \gamma$ in the given equation, and adding

$(\alpha^3 + \beta^3 + \gamma^3) - 4(\alpha^2 + \beta^2 + \gamma^2) + \frac{9}{2}(\alpha + \beta + \gamma) + 3c = 0 \Rightarrow 3c = -28 + 4 \times 7 - \frac{9}{2}4 = -18 \Rightarrow c = -6.$

Now we can multiply the given equation with x and x^2 and put $x = \alpha, \beta, \gamma$ and add to find $\alpha^4 + \beta^4 + \gamma^4$ and $\alpha^5 + \beta^5 + \gamma^5$ as $\frac{209}{2}$ and 334.

62. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) \Rightarrow a^2 = a^2 + +2(xy + yz + zx) \Rightarrow xy + yz + zx = 0$

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \Rightarrow a^3 - 3xyz = a(a^2 - 0) \Rightarrow xyz = 0 \Rightarrow x = 0$ or $y = 0$ or $z = 0$.

If $x = 0 \Rightarrow y + z = a$ and $y^2 + z^2 = a^2 \Rightarrow (a - z)^2 + z^2 = a^2 \Rightarrow 2z^2 - 2za = 0 \Rightarrow z = 0, a \Rightarrow y = a, 0$.

When $y = 0, z = 0$, then $x = a$. Thus the solution is $(x, y, z) = (a, 0, 0), (0, a, 0), (0, 0, a)$.

63. On multiplying given equation with $x - 1$ we have $x^3(x-1) + (x-1)(x^3 + x^2 + x + 1) = 0 \Rightarrow 2x^4 - x^3 - 1 = 0 \Rightarrow \frac{1}{x^3} = 2x - 1$. Thus, required equation becomes

$E = (2\beta + 2\gamma - 2\alpha - 1)(2\beta + 2\alpha - 2\gamma - 1)(2\alpha + 2\gamma - 2\beta - 1)$.

From Vieta's relations $\alpha + \beta + \gamma = -\frac{1}{2}$. So the expression becomes

$E = -8(2\alpha + 1)(2\beta + 1)(2\gamma + 1) = -16$.

64. Observe that $x = 2, y = 3$ or $x = 3, y = 2$ are two possible roots. Then,

$(x^2 - 5x + 19)(x - 2)(x - 3) = 0$ so the roots of $x^2 - 5x + 19 = 0$ are $\frac{5 \pm \sqrt{51}i}{2}$ complex conjugates.

65. Let $\sqrt[4]{97 - x} = a$ and $\sqrt[4]{x} = b$, then $a + b = 5$ and $a^4 + b^4 = 97$.

Now, $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = [(a + b)^2 - 2ab]^2 - 2a^2b^2 = 625 - 100ab + 2a^2b^2 = 97$

$\Rightarrow (ab - 25)^2 = 361 \Rightarrow ab = 44$ which is impossible and $ab = 6$ which gives x as $16, 81$.

66. The HCF of given polynomials is $x^2 - 2$, and hence, the common roots of the given polynomials are the roots of $x^2 - 2 = 0$ i.e. $\pm\sqrt{2}$.

67. The HCF is $4(x^2 - 5x + 6)$ and hence the common roots are $x = 2, 3$. If two other roots of first equation are α and β, then $\alpha + \beta + 5 = -5$ and $6\alpha\beta = 132 \Rightarrow \alpha\beta = 22$.

Therefore, the having α, β as roots is $x^2 + 10x + 22 = 0$, whose roots are $-5 \pm \sqrt{3}$.

Similarly, let α_1 and β_1 be two other roots of the second equation then $\alpha_1 + \beta_1 + 5 = -1$ and $6\alpha_1\beta_1 = 24 \Rightarrow \alpha_1\beta_1 = 4$. Thus, α_1 and β_1 are roots of $x^2 + 6x + 4 = 0$, whose roots are $-3 \pm \sqrt{5}$.

68. If $k = 1, p_1(x) = x^9 + x^3 + x^2 + x + 1 = x^9 - x^4 + x^4 + x^3 + x^2 + x + 1 = x^4(x^5 - 1) + (x^4 + x^3 + x^2 + x + 1)$

$= (x^4 + x^3 + x^2 + x + 1)[x^4(x-1)+1]$. Thus, $x^4 + x^3 + x^2 + x + 1$ is a non-tirvial polynomial divisor of $p_1(x)$.

$p_k(x) = x^4(x^{5k}-1) + x^4 + x^3 + x^2 + x + 1$. $x^5 - 1$ divides $x^{5k} - 1, x^4 + x^3 + x^2 + x + 1$ divides $x^5 - 1$, and hence $x^{5k} - 1$. Therefore, $x^4 + x^3 + x^2 + x + 1$ divides $p_k(x)$ for all k.

69. The HCF of given equations is $x + 2$, and hence, common root is -2.

70. The HCF of given equations is $x^2 - 4$, and hence, common roots are $2, -2$.

71. $\because d, e, f$ are in G.P. $\therefore df = e^2$. Discriminant of second equation is $D = 4e^2 - 4fd = 4e^2 - 4e^2 = 0$.

Thus, second equation will have one, repeated root $x = -\frac{2e}{2d} = -\frac{e}{d}$. This is the common root with first equation. Thus,

$\frac{ae^2}{d^2} - \frac{2be}{d} + c = 0 \Rightarrow \frac{adf}{d^2} - \frac{2be}{d} + c = 0 \Rightarrow \frac{af}{d} - \frac{2be}{d} + c = 0 \Rightarrow \frac{af}{d} + c = \frac{2be}{d} \Rightarrow \frac{a}{d} + \frac{c}{f} = \frac{2be}{df} = \frac{2b}{e}$.

Thus, $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in H.P.

72. $x^2 + px + q = (x-\alpha)(x-\beta) \Rightarrow \alpha + \beta = -p$ and $\alpha\beta = q$.

$x^{2n} + p^n x^n + q^n = (x - \alpha^n)(x - \beta^n) \Rightarrow \alpha^n + \beta^n = -p^n$ and $\alpha^n\beta^n = q^n$

$f\left(\frac{\alpha}{\beta}\right) = \frac{\left(1+\frac{\alpha^n}{\beta^n}\right)}{1+\frac{\alpha^n}{\beta^n}} = \frac{(\alpha+\beta)^n}{\alpha^n+\beta^n} = \frac{(-p)^n}{-p^n} = -1.$

73. Over $\mathbb{Q}: x^4 + 4 = x^4 + 4x^2 - 4x^2 + 4 = (x^2+2)^2 - (2x)^2 = (x^2+2x+2)(x^2-2x+2)$. Over $\mathbb{R}$ it is same.

Over $\mathbb{C}$. We need further factorization of $x^2 + 2x + 2$ and $x^2 - 2x + 2.x^2 + 2x + 2 = 0 \Rightarrow x = -1 \pm i$ and $x^2 - 2x + 2 = 0 \Rightarrow x = 1 \pm i$.

Thus, $x^4 + 4 = (x+1-i)(x+1+i)(x-1-i)(x-1+i)$.

74. $x^4 + x^3 - x - 1 = x^3(x+1) - (x+1) = (x^3-1)(x+1) = (x-1)(x+1)(x^2+x+1)$. Hence, it is reducible over $\mathbb{Z}$.

75. As it is a cubic polynomial, if this is reducible then it would have to have a linear factor $x - \alpha$, hence a root $(\alpha \in \mathbb{Z})$. But by integer root theorem α would have been an integer divisor of constant 3, hence it would have to be $1, -1, 3$ or -3, however, none of these is a root, and hence the polynomial is irreducible.

76. Following like previous problem using integer root theorem we have no integral roots, and hence, no linear factors. However, it might be a product of two quadratics. Consider:

$x^4 + x^3 - x + 1 = (x^2 + ax + b)(x^2 + cx + d)$. Now equating coefficients, $a + c = 1, b + ac + d = 0, ad + bc = -1, bd = 1$. Since a, b, c, d are all integers, we have either $b = d = 1$ or $b = d = -1$.

In the first case the other equations become $a + c = 0, ac = -2, a + c = -1$, which is impossible. And in the second case we obtain $a + c = 1, ac = 2$ which has no integer solution. Thus, there is no factorization, and the poynomial is irreducible.

77. Suppose f can be faactored then $f(x) = (x - n)g(x)$ or $f(x) = (x^2 - bx + c)g(x)$.

In the first case, $f(n) = n^5 - n + a$. Now $n^5 \equiv n (\bmod 5)$ by Fermat's little theorem $5 \mid (b - b^5) = a$, contradiction.

In the second case, $f(x) = x^5 - x + a$ by $x^2 - bx + c$, we get the remainder $(b^4 + 3b^2c + c^2 - 1)(b^3c + 2bc^2 + a)$. Since $x^2 - bx + c$ is a factor of $f(x)$, both coefficients of remainder equal to 0. That is $b^4 + 3b^2x + c^2 - 1 = 0$ and $b^3c + 2bc^2 + a = 0 \Rightarrow b(b^4 + 3b^2x + c^2 - 1) - 3(b^3c + 2bc^2 + a) = b^5 - b - 5bc^2 - 3a = 0 \Rightarrow 3a = b^5 - b - 5bc^2$ is divisible by $5 \Rightarrow 5 \mid a$, which is a contradiction.

78. Let α be any complex zero of f.

Case I: Consider $|\alpha| \leq 1$, then $|a_0| = |a_1\alpha + \cdots + a_n\alpha^n| \leq |a_1| + \cdots + |a_n|$, which is a contradiction.

Case II: Therefore, all the zeros of f satisfy the condition $|\alpha| > 1$. Let us assume that $f(x) = g(x)h(x)$, where g and h are non-constant integer polynomials. Then $a_0 = f(0) = g(0)h(0)$. Since a_0 is a prime, one of $|g(0), h(0)|$ equals 1.. Say $|g(0)| = 1$, and let b be the leading coefficient of g.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of g, then $|\alpha_1\alpha_2 \ldots \alpha_k| = \frac{1}{|b|} \leq 1 (\because b \in \mathbb{Z} - \{0\} \Rightarrow |b| \geq 1)$.

But α_1, α_2, ... ,α_k are also zeroes of f, and from case 1 have magnitude of each $\alpha_i \geq 1 \Rightarrow |\alpha_1\alpha_2 \ldots \alpha_k| \geq 1$, which is a contradiction.

Hence, f is irreducible.

79. The given polynomial is irreducible by Eisenstein's criterion with 7 being the prime p. 7 does not divide the leading coefficient but it divides all others, and its square 49, does not divide 175. Note that using prime 5 is not valid because 25 divides the constant coefficinet 175.

80. Let $f(x) = x^3 - 3x^2 + 3x + 22$. Eisenstein's criteria does not apply since there is no suitable prime. Substituting $x - 1$ for x gives the polynomial $x^3 - 6x^2 + 6x + 21$ to which we can apply Eisenstein's criteria with $p = 3$. Writing $f(x)$ for the original polynomial, we deduce that $f(x - 1)$ is irreducible. But a factorization of $f(x)$ would give a factorization of $f(x - 1)$, hence $f(x)$ is irreducible over $\mathbb{Z}$.

81. $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = \frac{x^p - 1}{x - 1}$.

Consider $\Phi_p(x + 1) = \frac{(x+1)^p - 1}{x+1-1} = \frac{x^p + C_1^p x^{p-1} + C_2^p x^{p-1} + C_P^{p-1} x^2 + C_P^{p-1} x}{x} = x^{p-1} + C_1^p x^{p-1} + \cdots + C_p^{p-2} x + C_p^{p-1}$.

As $p \mid C_i^p\ \forall\ i = 1, 2, 3, \ldots, p-1$, so all the lower coefficients are divisible by p and the constant coefficient is exactly p, so it is not divisible by p. Thus, Eisenstein's criteria apply, and $\Phi_p(x+1)$ is irreducible. Certainly if $\Phi_p(x) = g(x)h(x)$ then $\Phi_p(x+1) = g(x+1)h(x+1)$ gives a factorization of $\Phi_p(x+1)$. Thus, Φ_p is irreducible.

82. Taking prime $p = 3$, clearly $3 \mid a_i\ \forall\ i = 0, 1, 2, \ldots, n-2$; $3^2 \nmid a_0 = 3$, $3 \nmid a_{n-1} = 5$. Hence, by extended Eisenstein's criterion f has an irreducible factor of degree at least $n-1$. If possible, let us take one factor of degree $n-1$ then other must be linear and monic as f is monic. This implies that f has integral roots. By integer root theorem this root must be an integer divisor of 3, hence would have to be $1, -1, 3$ or -3. However, none of these are roots of the given equation, and hence, f is irreducible.

83. We treat the polynomial as $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n = 0, a_i = 0\ \forall\ i \in \{1, 2, \ldots, n-1\}$ and $a_0 = -p$.

    Clearly if we consider the prime p given in the polynomial then $p \mid a_i$ for $0 \le i \le n-1$, $p \nmid 1$ and $p^2 \nmid -p$. Thus, $f(x)$ is irreducible over $\mathbb{Z}$.

84. Considering prime $p = 3$, we have $p \mid a_i$ for $0 \le n-1$, $p \nmid 1$ and $p^2 \nmid 24$. Thus, the polynomial is irreducible over $\mathbb{Z}$.

85. Considering prime $p = 2$, we have $p \mid a_i$ for $0 \le n-1$, $p \nmid 1$ and $p^2 \nmid 2$. Thus, the polynomial is irreducible over $\mathbb{Z}$, which implies that it cannot be represented as product of two given polynomials.

86. Considerung prime $p = 3$, we have $p \mid a_i$ for $0 \le n-1$, $p \nmid 1$ and $p^2 \nmid 2$. Thus, the polynomial is irreducible over $\mathbb{Z}$.

87. There is no suitable prime for $x^3 + 3x^2 + 3x + 5$. Substituting $x-3$ for x gives the polynomial $x^3 - 6x^2 + 14x - 10$ to which Eisenstein does apply, with $p = 2$. Writing $f(x)$ for the original polynomial, we deduce that $f(x-3)$ is irreducible. But a factorization of $f(x)$ would give a factorization of $f(x-3)$, hence $f(x)$ is irreducible over $\mathbb{Z}$.

88. We see that our prime p will divide the coefficient of x but it won't divide $p-1$ for $p > 2$ making the prime p unsuitable for Eisenstein criteria. However, if $p = 2$ the the polynomial is $x^2 + 2x + 1 = (x+1)^2$. So if the polynomial has to be reducible for some prime then it must be 2.

89. Given polynomial is irreducible over $\mathbb{Z}$. Substituting $x = \frac{1}{x}$ we obtain the desired polynomial and find that it is irreducible over $\mathbb{Z}$.

    If we substitute $x = \frac{1}{x}$ in $21x^5 - 49x^3 + 14x^2 - 4$ then it becomes $4x^5 - 14x^3 + 49x^2 - 21$ for which Eisenstein's ceriteria is satisfied for $p = 7$.

90. If the polynomial were reducible over $\mathbb{Z}$, then there would exist two monic polynomials $P(x)$ and $Q(x)$ such that $P(x)Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - 1$. Consequently, $P(a_i)Q(a_i) = -1\ \forall i \in \{1, 2, \ldots, n\}$ but $P(a_i)$ and $Q(a_i)$ are integers, so there are only two possibilities:

$P(a_i) = 1, Q(a_i) = -1$ and $P(a_i) = -1, Q(a_i) = 1$. In any case, $P(a_i) + Q(a_i) = 0$, but it is impossible because $P(x) + Q(x) \neq 0$ (sum of monic polynomials) has degree less than n, so according to fundamental theorem of algebra, it cannot have n different roots. Hence, there do not exist such polynomials making the given polynomial irreducible.

91. This problem is similar to 81 and can be solved similarly.

92. $x = \frac{p-2}{2p(p-2)}$. If $p = 0$ or $p = 2$ then the equation is undefined. However, if $p = 0$, then the equation becomes $0 = -2$, which is inconsistent. Hence, no value of x will satisfy it and there is no solution for $p = 0$.

If $p = 2$ then the equation becomes $0 = 0$. Thus, every value from the domain of x will satisfy the equationm, and hence, there exists infinite number of solutions.

If $p \neq 0, p \neq 2$, then the equation is well defined and $x = \frac{1}{2p}$.

93. Substituting the roots we have $ax_1^2 + bx_1 + c = 0, -ax_2^2 + bx_2 + c = 0$ and $f(x_1) = \frac{a}{2}x_1^2 + bx_1 + c, f(x_2) = \frac{a}{2}x_2^2 + bx_2 + c$.

$\therefore f(x_1) + \frac{a}{2}x_1^2 = ax_1^2 + bx_1 + c = 0$ and $f(x_2) - \frac{3}{2}ax_2^2 = -ax_2^2 + bx_2 + c = 0$

$\because f(x_1)$ and $f(x_2)$ have opposite signs, and hence, $f(x)$ must have a root between x_1 and x_2.

94. Let $P(x) = x^2 + ax + b = (x - \alpha)(x - \beta)$, where $\alpha + \beta = -a$ and $\alpha\beta = b$.

Now $P(n)P(n+1) = (n-\alpha)(n-\beta)(n+1-\alpha)(n+1-\beta) = (n-\alpha)(n+1-\beta)(n-\beta)(n+1-\alpha) = [n^2 - (\alpha+\beta-1)n + \alpha\beta - \alpha][n^2 + (\alpha+\beta-1)n + \alpha\beta - b] = [n^2 + (a+1)n + b - \alpha][n^2 + (a+1)n + b - \beta] = (M-\alpha)(M-\beta)$, where $M = n^2 + (a+1)n + b$.

95. Let there be a rational root $\frac{p}{q}$, where $(p, q) = 1$. Then, $a\frac{p^2}{q^2} + b\frac{p}{q} + c = 0 \Rightarrow ap^2 + bpq + cq^2 = 0$

Now p, q both may be odd or one of the p, q be even. In both the cases $ap^2 + bpq + cq^2$ cannot be equal to zero. Thus, the equation cannot have rational roots.

96. Given, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{a+b+c} - \frac{1}{c} \Rightarrow (a+b)(b+c)(c+a) = 0 \Rightarrow a = -b$ or $b = -c$ or $c = -a$.

If $a = -b$, then $a^n = -b^n$ for odd $n \Rightarrow \frac{1}{a^n} = -\frac{1}{b^n} \Rightarrow \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n+b^n+c^n}$. Similarly, it can be proved for other two cases as well.

97. We have $\frac{a^3}{(a-b)(a-c)} = -\frac{a^3}{(a-b)(c-a)}, \frac{b^3}{(b-a)(b-c)} = -\frac{b^3}{(a-b)(b-c)}$ and $\frac{c^3}{(c-a)(c-b)} = -\frac{c^3}{(c-a)(b-c)}$.

$\frac{a^3}{(a-b)(a-c)} = \left[\frac{(b-c)a^3 + (c-a)b^3 + (a-b)c^3}{(a-b)(b-c)(c-a)}\right]$

Numerator of RHS is a cyclic symmetric expression in a, b, c in 4th degree and writting $b = c$, we get $(c-a)b^3 + (a-b)c^3 = 0$. So $b-c$ and hence $c-a$ and $a-b$ are factors. Since it is a 4th degree symmetric expression $a+b+c$ is also a factor. Thus, we have

$$k(a+b+c)(a-b)(b-c)(c-a) = (b-c)a^3 + (c-a)b^3 + (a-b)c^3.$$

If $a = 1, b = -1$ and $c = 2$, we get $k = -1$.

Thus, the expression $\frac{(a+b+c)(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} = a+b+c$.

98. $x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n = 0 \Rightarrow -x^n\left[-1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\right] = 0.$

Let $f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}$.$f(x)$ is a decreasing function as x increases in $(0, \infty)$, $f(x)$ decreases in $(\infty, 0)$. Hence there exists a unique positive real number R such that $f(R) = \frac{a_1}{R} + \frac{a_2}{R^2} + \cdots + \frac{a_n}{R^n} = 1$. Thus, for $x = R$, we get

$-R^n\left[-1 + \frac{a_1}{R} + \frac{a_2}{R^2} + \cdots + \frac{a_n}{R^n}\right] = 0$. Therefore, R is a root of the given equation.

99. Considering the polynomial $\pm P(\pm x)$ we may assume without loss of generality that $a, b \geq 0$.

Case I: If $c, d \geq 0$, then $|a| + |b| + |c| + |d| = P(1) \leq 1 < 7$

Case II: If $c \geq 0$ and $d \leq 0$, then $|a| + |b| + |c| + |d| = a + b + c - d = (a+b+c+d) - 2d = P(1) - 2P(0) \leq 1 + 2 = 3 < 7$

Case III: If $c < 0$ and $d \geq 0$, then $|a| + |b| + |c| + |d| = a + b - c + d = \frac{4}{3}P(1) - \frac{1}{3}P(-1) - \frac{8}{3}P\left(\frac{1}{2}\right) + \frac{8}{3}P\left(-\frac{1}{2}\right) \leq \frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{8}{3} = 7$

Case IV: If $c, d < 0$, then $|a| + |b| + |c| + |d| = a + b - c - d = \frac{5}{3}P(1) - 4P\left(\frac{1}{2}\right) + \frac{4}{3}P\left(-\frac{1}{2}\right) \leq \frac{5}{3} + 4 + \frac{4}{3} = 7$.

100. In one hour, the minute hand makes one complete revolution, i.e., it moved through 60 divisions and the hour hand moves through 5 divisions. Suppose, when the man went out, the hour hand was x divisions ahead of the point labeled 12 of the dial, where $20 < x < 25$ as he went out between 4 p.m. and 5 p.m. Also, suppose, when the man the hour hand was y divisions ahead of 0 mark and $25 < y < 30$.

Given that minute hand and hour hand exchanged their places, the minute hand was at y when he went out and at x when he returned. Because minute hand moves 12 times faster than hour hand,

$y = 12(x - 20)$ and $x = 12(y - 25) \Rightarrow y = 12[12(y-25) - 20] \Rightarrow y = \frac{3840}{143}$. Since the man went out when the hand was at y, the man went at $\frac{3840}{143}$ minutes past 4 p.m.

101. Sum of roots is $\alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1} + \alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2} = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9 + \alpha^{10} + \alpha^{11} + \alpha^{12}$ $(\because \alpha^{13} = 1) = (1 + \alpha + \alpha^2 + \cdots + \alpha^{12}) - 1 = \frac{1-\alpha^{13}}{1-\alpha} - 1 = -1.$

Product of roots is $(\alpha+\alpha^3+\alpha^4+\alpha^{-4}+\alpha^{-3}+\alpha^{-1})(\alpha^2+\alpha^5+\alpha^6+\alpha^{-6}+\alpha^{-5}+\alpha^{-2}) = 3(\alpha + \alpha^2 + \cdots + \alpha^{12}) = -3.$

Thus, the quadratic equation having these roots is $x^2 + x - 3 = 0$.

102. $1 + x^n + x^{2n} + \cdots + x^{mn} = \frac{x^{(m+1)n}-1}{x^n-1}$ and

$1 + x + x^2 + \cdots + x^m = \frac{x^{m+1}-1}{x-1}$.

We have to find (m, n) such that $\frac{x^{(m+1)n}-1}{x^n-1} \div \frac{x^{m+1}-1}{x-1} = \frac{(x^{(m+1)n}-1)(x-1)}{(x^n-1)(x^{m+1}-1)}$ is a polynomial.

Now if k and l are relatively prime, then $(x^k - 1)$ and $(x^l - 1)$ have just one common factor $x - 1$. By De'moivre's theorem, the roots of $x^k - 1 = 0$ are $\cos\frac{2n\pi}{k} + i\sin\frac{2n\pi}{k}$ for $n = 0, 1, 2, \ldots, k-1$ and those of $x^l - 1 = 0$ are $\cos\frac{2n\pi}{l} + i\sin\frac{2n\pi}{l}$ for $n = 0, 1, 2, \ldots, n-1$. If l and k are co-prime integer other than zero, these roots will be different.

Since all the factors of $x^{(m+1)n} - 1$ are distinct, $x^m - 1, x^n - 1$ cannot have any common factors other than $x - 1$. Thus, $m + 1$ and n must be relatively prime.

Again, $x^{(m+1)n} - 1 = (x^n)^{m+1} - 1 = (x^{m+1})^n - 1$. So $x^{(m+1)n} - 1$ is divisible by both the factors in denominator. Thus, it is sufficient for our result for $m + 1$ and n to be relatively prime.

103. Given, $(a-b)^2 + (a-c)^2 = (b-c)^2 \Rightarrow 2a^2 - 2ab - 2ac + 2bc = 0 \Rightarrow a^2 - a(b+c) + bc = 0 \Rightarrow (a-b)(a-c) = 0 \Rightarrow a = b$ or $a = c$.

However, it contradicts with the given fact that a, b, c are all distinct, and hence, has no solution.

104. $2^m = (1+1)^m = C_0^m + C_1^m + C_2^m + \cdots + C_m^m$ for $m = 1, 2, \ldots, n+1$.

Now consider the polynomial $f(x) = 2[C_0^{x-1} + C_1^{x-1} + C_2^{x-1} + \cdots + C_n^{x-1}]$, clearly, $f(x)$ is of degree n.

So $f(x) = 2.2^{x-1}$ for all $x = 1, 2, \ldots, n+1$. Thus, $f(x+2) = 2[C_0^{x+1} + C_1^{x+1} + C_2^{x+1} + \cdots + C_n^{x+1}] = 2[2^{x+1} - 1] = 2^{x+2} - 2$.

105. Given, $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da \Rightarrow (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 = 0 \Rightarrow a = b, b = c, c = d, d = a \Rightarrow a = b = c = d$.

106. Given, $2x^2 + y^2 + 2x^2 - 8x + 2y - 2xy + 2xz - 16z = 35 = 0 \Rightarrow (x-y)^2 + (x+z)^2 + z^2 - 16x - 8x + 2y + 35 = 0 \Rightarrow (x-y-1)^2 + (x+z-3)^2 + z^2 - 10z + 25 = 0 \Rightarrow (x-y-1)^2 + (x+z-3)^2 + (z-5)^2 = 0 \Rightarrow x - y = 1, x + z = 3, z = 5 \Rightarrow x = -2, y = -3.$

107. We know that $x^8 + y^8 + 6 = 8xy \Rightarrow x$ and y must be of the same sign otherwise LHS > 0 and RHS < 0. Moreover (x, y) is a solution then $(-x, -y)$ is also a solution, also WLOG $x, y > 0$.

Now, $x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1 = 8xy$. By AM-GM inequality $x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1 \geq 8\sqrt[8]{x^8 + y^8} = 8|xy|$. But, by this hypothesis, equality holds. Hence, all eight terms are equal. Therefore, $x^8 = y^8 = 1$. Hence, $(x, y) = (1, 1) = (-1, -1)$ is the solution set.

By observation you can see that the genral equation is $x^{2n} + y^{2n} = 2nxy + 2n - 2$, where $n = 1, 2, ...$ which will have the same solution set.

108. Given that, $5x\left(1 + \frac{1}{x^2+y^2}\right) = 12 \Rightarrow 25x^2 = \frac{144}{\left(1+\frac{1}{x^2+y^2}\right)^2}$. Similarly, we can find from the second equation $25y^2 = \frac{16}{\left(1-\frac{1}{x^2+y^2}\right)^2}$.

Adding, $25(x^2 + y^2) = \frac{144}{\left(1+\frac{1}{x^2+y^2}\right)^2} + \frac{16}{\left(1-\frac{1}{x^2+y^2}\right)^2}$. Let $\frac{1}{x^2+y^2} = t \Rightarrow x^2 + y^2 = \frac{1}{t}$

$\therefore \frac{25}{t} = \frac{144}{(1+t)^2} + \frac{16}{(1-t)^2} \Rightarrow 32t(5t^2 - 8t + 5) = 25(t^4 - 2t^2 + 1)$. Dividing both sides by t^2, we arrive at $32\left[5\left(t + \frac{1}{t}\right) - 8\right] = 25\left[\left(t + \frac{1}{t}\right)^2 - 4\right]$

Putting $t + \frac{1}{t} = \alpha$, we have $25\alpha^2 - 160\alpha + 156 = 0 \Rightarrow \alpha = \frac{6}{5}, \frac{26}{5} \Rightarrow t + \frac{1}{t} = \frac{6}{5}, \frac{26}{5} \Rightarrow 5t^2 - 6t + 5 = 0$ or $5t^2 - 26t + 5 = 0$. Since the discriminant of $5t^2 - 6t + 5 = 0$ is $36 - 100 < 0$, there is no real root. $5t^2 - 26t + 5 = 0$, the roots are $5, \frac{1}{5}$.

If $x^2 + y^2 = 5$ then $5x\left(1 + \frac{1}{5}\right) = 12$ and $5y\left(1 - \frac{1}{5}\right) = 4 \Rightarrow x = 2, y = 1$. If $x^2 + y^2 = \frac{1}{5}$ then $5x(1 + 5) = 12$ and $5y(1 - 5) = 4$. Thus, $x = \frac{2}{5}$ and $y = -\frac{1}{5}$.

109. Adding all $2(x + y + z)^2 = 48 + 2L \Rightarrow x + y + z = \sqrt{24 + L}$. Dividing all the equations with $x + y + z = \sqrt{24 + L}$, we get $x + y = \frac{18}{\sqrt{24+L}}, y + z = \frac{30}{\sqrt{24+L}}, z + x = \frac{2L}{\sqrt{24+L}}$.

Solving these, we get $x = \frac{L-6}{\sqrt{24+L}}, y = \frac{24-L}{\sqrt{24+L}}, z = \frac{L+6}{\sqrt{24+L}}$, where $6 < L < 24$.

110. From eq. (1), $(x - z) = (4 - y) \Rightarrow x^2 - 2zx + z^2 = 16 - 8y + y^2 \Rightarrow (x^2 + z^2 - y^2) - 2zx + 8y - 16 = 0 \Rightarrow zx = 2(2y - 5)\,[\because x^2 + y^2 - z^2 = -4] \cdots (4)$

From eq. (3) and (4), we get $y \times 2(2y - 5) = 6 \Rightarrow y = -\frac{1}{2}, 3$. Putting $y = -\frac{1}{2}$ in eq. (1) and (3), we get $x - z = \frac{9}{2}$ and $zx = -12$

$(x + z)^2 = (x - z)^2 + 4zx = \frac{81}{4} - 48 < 0$. So $y = 3$ is the only valid solution for y

$(x - z) = 1, zx = 2 \Rightarrow x + z = \pm 3 \Rightarrow x = 2, -1$ and $z = 1, -2$.

111. Multiplying given equations, we get $3xy(x+y-2)(x+y-1)=18xy \Rightarrow 3xy[(x+y-1)(x+y-2)-6]=0 \Rightarrow 3xy(x+y-4)(x+y+1)=0$.

So $x=0$ or $y=0$ or $x+y=4$ or $x+y=-1$. Putting $x+y=4$ in eq. (1), we get $6x=2y \Rightarrow y=3x \Rightarrow x=1, y=3$.

Putting $x+y=-1$ in eq. (1), we get $y=-\frac{9x}{2} \Rightarrow -\frac{7}{2}x=-1 \Rightarrow x=\frac{2}{7}, y=-\frac{9}{7}$.

112. Adding 1 to both sides of (1), $xy+x+y+1=24 \Rightarrow (x+1)(y+1)=24$. Similarly, $(y+1)(z+1)=32$ and $(z+1)(x+1)=48$.

$\Rightarrow (x+1)^2(y+1)^2(z+1)^2=24\times 32\times 48 \Rightarrow (x+1)(y+1)(z+1)=\pm(24\times 8) \Rightarrow z+1=\pm 8, x+1=\pm 6, y+1=\pm 4$.

Thus, $x=5, y=3, z=7$ and $x=-7, y=-5, z=-9$.

113. If $x>1$, then $y=x^3+3x(x^2-1)>x^3>x>1, z=y^3+3y(y^2-1)>y^3>y>1$ and $x=z^3+3z(z^2-1)>z^3>z>1$.

Thus, $z>y>x>z$, which is impossible. Thus, $x\le 1$ and again, $x<-1$ leads to $x>y>z>x$ so $xgeq-1$. So $|x|\le 1, |y|\le 1, |z|\le 1$.

And hence we can write $x=\cos\theta$, where $0\le\theta\le\pi$.

Now, $y=4\cos^3\theta-3\cos\theta=\cos 3\theta, z=4\cos^3 3\theta-3\cos 3\theta=\cos 9\theta$ and $x=4z^3-3z=4\cos^3 9\theta-3\cos 9\theta=\cos 27\theta$.

Since trigonometric functions are periodic, it is possible. Thus, $\cos\theta=\cos 27\theta \Rightarrow \cos\theta-\cos 27\theta=0 \Rightarrow 2\sin 14\theta\cos 13\theta=0 \Rightarrow \theta=\frac{k\pi}{13}$, where $k=0,1,2,\dots,13$ or $\theta=\frac{k\pi}{14}$, where $k=1,2,\dots,13$. The solution is $(x,y,z)=(\cos\theta,\cos 3\theta,\cos 9\theta)$ where θ takes all the above values.

114. Consider the equation $p+qt+rt^2+st^3=t^4 \Rightarrow t^4-st^3-rt^2-qt-p=0$. Given that a_1, a_2, a_3, a_4 are the solutions of this equation, and hence,

$\sigma_1=a_1+a_2+a_3+a_4=s, \sigma_2=(a_1+a_2)(a_3+a_4)+a_1a_2+a_3a_4=-r, \sigma_3=a_1a_2(a_3+a_4)+a_3a_4(a_1+a_2)=q, \sigma_4=a_1a_2a_3a_4=-p$.

The second system of equation is $(t^2)^4-w(t^2)^3-z(t^2)^2-y(t^2)-x=0$. Putting $t^2=u$, we have $u^4-wu^3-zu^2-yu-x=0$ and the roots would be $a_1^2, a_2^2, a_3^2, a_4^2$.

$$\sigma_1=a_1^2+a_2^2+a_3^2+a_4^2=w \Rightarrow w=\left(\sum a_i\right)^2-2\sum_{i<j}a_ia_j=s^2+2r.$$

$$\sigma_2=_{i<j}a_i^2a_j^2=-z \Rightarrow z=-\left(\sum_{i,j}a_ia_j\right)^2-2\sum a_i\sum_{i<j<k}a_ia_ja_k-2a_1a_2a_3a_4=-r^2+2qs+2p.$$

$$\sigma_3=a_1^2a_2^2a_3^2+a_1^2a_2^2a_4^2+a_1^2a_3^2a_4^2+a_2^2a_3^2a_4^2=y \Rightarrow y=q^2-2pr$$

$$\sigma_4=-x=-p^2.$$

115. We observe that both (x, y, z) and $(-x, -y, z)$ satisfy the given system of equations. Since there has to be only one solution, we deduce $x = y = 0$ and so $z^2 = 4 \Rightarrow z = \pm 2$.

From equations (1) and (2), $z = a, z = b$. So either $a = b = 2$ or $a = b = -2$.

If $a = b = 2$, we have $xyz + z = 2, xyz^2 + z = 2, x^2 + y^2 + z^2 = 4 \Rightarrow xyz(z-1) = 0$ so either $x = 0$ or $y = 0$ or $z = 0$ or $z = 1$. If $z = 0$ the from last equation $0 = 4$, which is not possible. If $z = 1$, then x, y are not zero, which gives more than one solution of the equation. Hence, $a = b = 2$ does not satisfy the condition.

If $a = b = -2$, we have $xyz + z = -2, xyz^2 + z = -2, x^2 + y^2 + z^2 = 4$. Following like above when $z = 0$, we have $0 = -2$, which is not possible. If $z = 1$, then $xy + 1 = -2 \Rightarrow xy = -3$ and $x^2 + y^2 = 3 \Rightarrow (x + y)^2 = -3$, which is not posssible for real x, y, and hence, $z \neq 1$.

Thus, we have a unique solution $(0, 0, -2)$.

116. Clearly, $a^2 + ab + \frac{b^2}{3} = \frac{b^2}{3} + c^2 + c^2 + ca + a^2 \Rightarrow 2c^2 + ac - ab = 0 \Rightarrow a + 2c = \frac{ab}{c}$.

Also, $25 - 9 + 16 = 32 \Rightarrow 2a + b + c = \frac{32}{a}$.

$ab + 2bc + 3ca = b(a + 2c) + 3ca = \frac{b \times ab}{c} + 3ca = \frac{3a}{c}\left(\frac{b^2}{3} + c\right) = \frac{27a}{c}$.

Again, $ab + 2bc + 3ca = 2c^2 + ac + 2bc + 3ca = 2c(c + b + 2a)$.

$\Rightarrow \frac{32}{a} = \frac{27a}{c} \cdot \frac{1}{2c} = \frac{27a}{2c^2} \Rightarrow \frac{a^2}{c^2} = \frac{64}{27} \Rightarrow ab + 2bc + 3ca = 27 \times \frac{a}{c} = 24\sqrt{3}$.

117. We have $\log_3(\log_2 x) + \log_{1/3}(\log_{1/2} y) = 1 \Rightarrow \log_3(\log_2 x) - \log_3(\log_{1/2} y) = 1 \Rightarrow \frac{\log_2 x}{\log_{1/2} y} = 3 \Rightarrow \log_2 x = -3\log_2 y \Rightarrow \log_2 xy^3 = 0 \Rightarrow xy^3 = 1 \Rightarrow y = \frac{1}{4}, x = 64$.

118. We know that $\log_a x = \log_{(a^n)}(x^n)$. So $\log_2 x = \log_4 x^2, \log_3 y = \log_9 y^2, \log_4 z = \log_{16} z^2$.

Now, $\log_2 x + \log_4 y + \log_4 z = 2 \Rightarrow x^2yz = 16$. Similarly, $y^2zx = 81$ and $z^2xy = 256$.

$\Rightarrow x^2yz \times y^2zx \times z^2xy = (xyz)^4 = 16 \times 81 \times 256 \Rightarrow xyz = 24$.

$\Rightarrow x = \frac{2}{3}, y = \frac{27}{8}, z = \frac{32}{3}$.

119. By observation one solution is $x = 3, y = 2$. As $\log_3 3 + \log_2 2 = 2$ and $3^3 - 2^2 = 23$.

If $x < 3$, then $\log_3 x < 1$. Since, $\log_3 x + \log_2 y = 2 \Rightarrow \log_2 y > 2 \Rightarrow y > 2$. Hence, $3^x < 3^3 = 27$ and $2^y > 2^2 = 4 \Rightarrow 3^x - 2^y < 27 - 4 = 23$. Hence, x cannot be less than 3.

Similarly, x cannot be greater than 3. Thus, $x = 3, y = 2$.

120. Given that α, β, γ are roots of the equation $x^3 - x^2 - 1 = 0$, so from Vieta's relations, we have $\alpha + \beta + \gamma = 1, \alpha\beta + \beta\gamma + \gamma\alpha = 0$ and $\alpha\beta\gamma = 1$. Now,

$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1+\gamma} = \frac{3-(\alpha+\beta+\gamma)-(\alpha\beta+\beta\gamma+\gamma\alpha)+3\alpha\beta\gamma}{1-(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma} = -5$.

121. $x^{m+1} - x^m - x + 1 = (x^m - 1)(x - 1) = (x^{m-1} + x^{m-2} + \cdots + x^2 + x + 1)(x - 1)^2$, and hence, $(x-1)^2$ is a factor of $x^{m+1} - x^m - x + 1$.

122. $x^{10} - x^8 + 8x^6 - 24x^4 + 32x^2 - 48 = (x^2 - 2)(x^8 + x^6 + 10x^4 - 4x^2 + 24) = 0$.

If $x^2 - 2 = 0$ then $x = \pm\sqrt{2}$. Now $x^8 + x^6 + 10x^4 - 4x^2 + 24 = x^8 + x^6 + 9x^4 + x^4 - 4x^2 + 4 + 20 = x^8 + x^6 + 9x^4 + (x^2 - 2)^2 + 20 > 0$.

Thus, the given equation has two solutions $\pm\sqrt{2}$.

123. Given, $3 + 2x + \cdots + 3x^{96} + 2x^{97} + 3x^{98} + 2x^{99} = 0 \Rightarrow 3\frac{x^{100}-1}{x^2-1} + 2x\frac{x^{100}-1}{x^2-1} = 0 \Rightarrow 3 + 2x = 0$ as $x \neq 1 \Rightarrow x = -\frac{3}{2}$.

124. $1 + x^{111} + x^{222} + x^{333} + x^{444} = \frac{x^{555}-1}{x^{111}-1}$ and $1 + x^{111} + x^{222} + x^{333} + \cdots + x^{999} = \frac{x^{1110}-1}{x^{111}-1}$.

Now $x^{1110} = x^{2*555} \Rightarrow x^{1110} - 1 = (x^{555} - 1)(x^{555} + 1)$, and thus, we see that required condition is satisfied.

125. Given, $\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \Rightarrow zx + yz - xy = 0$

$(x + y - z)^2 = x^2 + y^2 + z^2 - zx - yz + xy = x^2 + y^2 + z^2$, and hence, $\sqrt{x^2 + y^2 + z^2} = \pm(x + y - z)$, which is rational.

126. Given, $ax^2 = by^2 = cz^2 = k(\text{say})$, then also given, $a^2x^3 + b^2y^3 + c^2z^3 = p^5 \Rightarrow \frac{k^2}{x} + \frac{k^2}{y} + \frac{k^2}{z} = p^5 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{p^5}{k^2} = \frac{1}{p} \Rightarrow k = p^3$.

Now, $\sqrt{a} + \sqrt{b} + \sqrt{c} = \frac{\sqrt{p^3}}{x} + \frac{\sqrt{p^3}}{y} + \frac{\sqrt{p^3}}{z} = \sqrt{p^3}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{\sqrt{p^3}}{p} = \sqrt{p}$.

127. Given, $ax^3 = by^3 = cz^3 = k(\text{say})$. Then, $\sqrt[3]{ax^2 + by^2 + cz^2} = \sqrt[3]{\frac{k}{x} + \frac{k}{y} + \frac{k}{z}} = \sqrt[3]{k} = \sqrt[3]{k}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$.

128. Clearly, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z} \Rightarrow (x + y)(y + z)(z + x) = 0$. Clearly, at least one of the terms on L.H.S. has to be zero automatically making third as 'a'.

129. Since $a, k \in \mathbb{R}$, the root will have a complex conjugate pair. Thus, second root will be $\frac{1}{2}(a - 5i)$. Sum of the roots would be $a = 3$. Product of the roots is $\frac{k}{2} = \frac{1}{4}(9 + 25) \Rightarrow k = 17$.

130. Let the given equation have roots a, a, b, where a has the multiplicity of 2. From Vieta's relations $2a + b = -p$, $a^2 + 2ab = 0 \Rightarrow a = -2b$ and $a^2b = -q$.

We have to prove that $4p^3 + 27q = 0 \Rightarrow -4(2a + b)^3 - 27a^2b = 0 \Rightarrow -4(-3b)^3 - 27(-2b)^2b = 0$. Hence proved.

131. Let the quadratic polynomial be $ax^2 + bx + c = 0$. Given, $f(0) = 6 \Rightarrow c = 6$; $f(1) = 1 \Rightarrow a + b + c = 1$, and $f(2) = 0 \Rightarrow 4a + 2b + c = 0$.

$\Rightarrow 2b + 3c = 4 \Rightarrow b = -7, \Rightarrow a = 2.$

$\Rightarrow f(3) = 18 - 21 + 6 = 3.$

132. Given, $ac = 2(b + d)$. Discriminant of given equations are $a^2 - 4b$ and $c^2 - 4d$. For roots of the equation to be real the discriminant have to be greater or equal to zero. We assume that it is the case. Thus,

$a^2 + c^2 \geq 4(b + d) \Rightarrow a^2 + c^2 \geq 2ac \Rightarrow (a - c)^2 \geq 0$. Thus, the assumption is correct, and hence proven.

133. Suppose we have four real numbers $0 < A < B < C < D$. If $1 < 4BC$, then $1 < 4BC < 4BD < 4CD$; so we can choose B, C, D. If $1 \geq 4BC$, then $1 \geq 4BC \geq 4AC \geq 4AB$; so we can choose A, B,C. In first case all roots are imaginary, and in second case all roots are real.

134. We can rewrite the given equations as $(x^2 - 3)^2 - x^3 - 2x > 0\ \forall\ x < 0$. And thus, the given equation has no negative roots.

135. Given equation is $3x^2 - (a + c + 2b + 2d) + (ac + 2bd) = 0$, whose discriminant is $(a + c + 2b + 2d)^2 - 12(ac + 2db) = [(a + 2d) - (c + 2b)]^2 + 8(c - b)(d - a) > 0$, and hence, the roots will be real and distinct.

136. Let $f(x) = x^4 + x^3 + x^2 - x - 1$. We notice that there is one sign change so it can have at most one positive root. Now, $f(-x) = x^4 - x^3 + x^2 + x - 1$, and there are three sign changes so it can have at most treee negative roots.

137. Dicriminanats are $b^2 - 4ac$ and $b^2 + 4ac$, and thus, one of these have to be positive, making either $P(x)$ or $Q(x)$ to have two real roots. And hence, $P(x)Q(x)$ will have two real roots.

138. Let α, β, γ are the roots of the equation $f(x) = 0$, then roots of $g(x) = 0$ will be $\alpha^2, \beta^2, \gamma^2$. It is given that $f(x) = x^3 + x + 1$, and hence, $\alpha + \beta + \gamma = 0, \alpha\beta + \beta\gamma + \gamma\alpha = 1, \alpha\beta\gamma = -1$. Also, since $g(0) = -1 \Rightarrow \alpha^2\beta^2\gamma^2 = -1$.

Thus, $g(x) = x^3 - (\alpha^2 + \beta^2 + \gamma^2)x^2 + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)x - \alpha^2\beta^2\gamma^2 \Rightarrow g(9) = 729 - 81[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] + 9[(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - \alpha\beta\gamma(\alpha + \beta + \gamma)] - \alpha^2\beta^2\gamma^2 = 729 + 162 + 9 - 1 = 899.$

139. The dicriminants of the terms are $p^2 - 12q$, $r^2 - 4q$ and $s^2 + 8q$. Clearly, p^2, r^2, s^2 are positive. Now if q is positive then $s^2 + 8q > 0$ giving us two real roots. However, if $q < 0$, then both $p^2 - 12q$ and $r^2 - 4q$ are positive giving us four real roots.

140. Let a be the first term and d be the common difference. Then, $\frac{1}{q} = a + (p - 1)d$ and $\frac{1}{p} = a + (q - 1)d \Rightarrow d = \frac{1}{pq} \Rightarrow a = \frac{1}{pq}$.

Now, $t_{pq} = \frac{1}{pq} + \frac{pq-1}{pq} = 1$. Clearly, 1 is a root of the given equation.

141. $D = \sqrt{4p^2 - 8q} = 2\sqrt{p^2 - 2q}$. The term under square root is odd minus even. Let this be a perfect square so that we have rational roots. Now since the term under question

is odd the square root will be odd. Let it be r. Now, $p^2 - r^2 = 2q$. Let $p = 2m+1$ and $r = 2n+1$, then $(2m+1)^2 - (2n+1)^2 = 2q \Rightarrow (2m+2n+2)(m-n) = q$. L.H.S. is even while R.H.S. is odd and thus our supposition is wrong giving us no rational roots.

142. If for some n division is possible then the g.c.d. of $n^3 - n + 3$ and $n^3 + n^2 + n + 2$ must be $n^3 - n + 3$. Using Euclidean algorithm, $(n^3 + n^2 + n + 2, n^3 - n + 2) = (n^3 - n + 2, n^2 + 2n - 1) = (n^2 + 2n - 1, -2n^2 + 3) = (n^2 + 2n - 1, 4n + 5) = (4n^2 + 8n - 4, 4n + 5) = (4n+5, 3n-4) = (3n-4, n+9) = (-31, n+9) \in \{\pm 1, \pm 31\}$. So we have four possible values for $n^3 - n + 2$ i.e. $\pm 1, \pm 31$. But for none of these values division is possible.

143. $s_n = \frac{q^{n+1}-1}{q-1}$, $S_n = \frac{\left(\frac{1+q}{2}\right)^{n+1}-1}{\frac{1+q}{2}-1} = \frac{(1+q)^{n+1}-2^{n+1}}{2^n(q-1)} \Rightarrow 2^n S_n = \frac{(1+q)^{n+1}-2^{n+1}}{(q-1)}$.

We have to find $\binom{n+1}{1} + \binom{n+1}{2} s + \binom{n+1}{3} s^2 + \cdots + \binom{n+1}{n+1} s^n$. The rth term is given by $t_r = C_r^{n+1} \frac{q^{r+1}}{q-1} - C_r^{n+1} \frac{1}{q-1}$.

Therefore, sum is given by $\sum t_r = \frac{1}{q-1}\left[\sum C_r^{n+1} q^{r+1} - \sum C_r^{n+1}\right] = \frac{1}{q-1}\left[(1+q)^{n+1} - 1 - 2^{n+1} + 1\right]$. Hence proved.

144. $\frac{1}{a} = \frac{1}{x} + \frac{1}{y}, \frac{1}{b} = \frac{1}{y} + \frac{1}{z}, \frac{1}{c} = \frac{1}{x} + \frac{1}{z} \Rightarrow \frac{a-b}{ab} = \frac{z-x}{xz}, \frac{1}{c} = \frac{z+x}{zx} \Rightarrow z = \frac{2abc}{ab-bc+ca}, x = \frac{2abc}{ab+bc-ca}, y = \frac{2abc}{bc+ca-ab}$.

145. Adding all equations $(x+y+z)^2 = 0 \Rightarrow x = \frac{0}{0}$. Thus, x, y, z can assume any value as long as $x + y + z = 0$ so if $x = a, y = b$ then $z = -a - b$.

146. $z^2 - x^2 + yz - xy = 12 \Rightarrow (z-x)(x+y+z) = 12, y^2 - z^2 + xy - zx = 4 \Rightarrow (y-z)(x+y+z) = 4, y^2 - x^2 + yz - zx = 16 \Rightarrow (y-x)(x+y+z) = 16$.

$\Rightarrow z - x = 3(y-z), y - x = 4(y-z), 3(y-x) = 4(z-x) \Rightarrow (x, y, z) = (-1, 3, 2), (1, -3, -2), \left(-\frac{5}{\sqrt{13}}, \frac{11}{\sqrt{13}}, \frac{7}{\sqrt{13}}\right), \left(\frac{5}{\sqrt{13}}, -\frac{11}{\sqrt{13}}, -\frac{7}{\sqrt{13}}\right)$.

147. Let $a = x^2 + 3x - 4, b = 2x^2 - 5x + 3$, the the given equation is of the form $a^3 + b^3 = (a+b)^3 \Rightarrow 3ab(a+b) = 0$.

This implies that $x^2 + 3x - 4 = 0$ or $2x^2 - 5x + 3 = 0$ or $3x^2 - 2x - 1 = 0$. Thus solutions are $x = -4, 1, \frac{3}{2}, -\frac{1}{3}$.

148. $n^4 + 2n^3 + 2n^2 + 2n + 1 = (n^2+1)(n+1)^2$ so for the given number to be a perfect square $n^2 + 1$ will have to be perfect square. Now we see that perfect squares are 1, 4, 9, 16, ... i.e. difference between perfect squares is more than 1 so $n^2 + 1$ cannot a perfect square.

Another way to solve this is $n^2 + 1$ can only be $(n+1)^2 \Rightarrow n = 0$. Hence, there is no such positive number.

149. $a^{x/y} = a^{y/z} = a^{z/x} \Rightarrow x = y = z \Rightarrow a = 3x \Rightarrow (x, y, z) = \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$.

150. Suppose the given polynomial can be factored into polynomials with integer coefficients. One such factor must be of the form $px + q$ then from rational root theorem $q = \pm 1, \pm 2$ and $p = \pm 1, \pm 5$. But we see that no such value satisfies the given polynomial and thus $px + q$ cannot be a factor.

151. Let $\alpha, \beta, \gamma, \delta$ be four roots such that $\alpha\beta = -200$. From Vieta's relations $\alpha + \beta + \gamma + \delta = -7, \alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \alpha\delta + \beta\delta = -240, \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -k, \alpha\beta\gamma\delta = 2000$.

 $\Rightarrow \gamma\delta = -10 \Rightarrow 200(\gamma + \delta) + 10(\alpha + \beta) = k \Rightarrow (\alpha + \beta)(\gamma + \delta) = -30 \Rightarrow (\gamma + \delta)^2 + 7(\gamma + \delta) - 30 = 0 \Rightarrow \gamma + \delta = -10, 3 \Rightarrow \alpha + \beta = 3, -10 \Rightarrow k = -1970, 500$.

152. $x^4 - 20x^3 + kx^2 + 590x - 1992 = (x^2 + ax + 24)(x^2 + bx - 83)$. Comparing coefficients $a + b = -20, ab - 59 = k, 24b - 83a = 590 \Rightarrow a = -10, b = -10 \Rightarrow k = 41$.

153. Let the roots be x_1, x_2, x_3, x_4, x_5, x_6, then from Vieta's relations $\sum x_i = 0$ and $\sum x_i x_j = 0 \Rightarrow \sum x_i^2 = 0$. However, if all the roots are real then $\sum x_i^2 \neq 0$. Hence, all roots cannot be real.

154. $ax^2 + 2bx + c \geq 0 \Rightarrow a > 0$ and $D \leq 0 \Rightarrow b^2 \leq ac$. Similarly, $q^2 \leq pq$. Multiplying $b^2q^2 \leq acpr \Rightarrow b^2q^2 \leq 4acpr$ which makes $apx^2 + bqx + cr \geq 0$.

155. Let $P(x) = x^n + (2 + x)^n + (2 - x)^n$. First we assume n to be even, then $P(x) = 3x^n + \cdots + 2^{n+1}$. From rational root theorem if there is a root $\frac{p}{q}$, then $p = \pm 1, \pm 2$ and $q = \pm 1, \pm 3$. Since we need integral roots, therefore $\frac{p}{q} = \pm 1$. Now, $P(\pm 1) = 2 + 3^n \neq 0$. Thus, our assumption is wrong and n is an odd number.

 Also, if x is odd then $P(x)$ is odd, hence, x must be even. Let $x = 2y, Q(y) = \frac{P(2x)}{2^n} = y^n + (1 + y)^n + (1 - y)^n = 2 + \cdots + y^n$. Again, from rational root theorem, $y = \pm 1, \pm 2$. The values of $Q(y)$ for these values are $3^n - 2^n - 1, 2^n - 1, 2^n + 1, 3^n + 2^n - 1$. None of these vanish unless $n = 1$, which is our first case. When $n = 1, P(x) = x + 4$, and hence, $x = -4$ is the only solution.

156. Since the equation is symmetrical so we can assume $\gamma = 90^\circ, \alpha = \theta$, and $\beta = 90^\circ - \theta$ because they are angles of a right-angle triangle. Substituting these in the given equation

 $\sin\theta\cos\theta\sin(2\theta - 90^\circ) + \cos\theta\sin(-\theta) + \sin\theta\cos\theta + \sin(2\theta - 90^\circ)\sin(-\theta)\sin(90^\circ - \theta) = -\sin\theta\cos\theta\cos 2\theta - \sin\theta\cos\theta + \sin\theta\cos\theta + \cos 2\theta\sin\theta\cos\theta = 0$.

157. Since the roots are complex $D < 0 \Rightarrow (a + b + c)^2 - 4(ab + bc + ca) < 0 \Rightarrow a^2 + b^2 + c^2 < 2(ab + bc + ca)$. Since a, b, c are real $a^2 + b^2 + c^2 > 0$. $2\alpha = a + b + c \Rightarrow a + b + c > 0$. Thus, $a, b, c > 0$.

 We assume that $a \leq b \leq c$, so it is enough to prove that $\sqrt{a} + \sqrt{b} \geq \sqrt{c} \Rightarrow a + b + 2\sqrt{ab} \geq c \Rightarrow 4ab \geq a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$, which is true.

158. Since $x + 1$ divides $ax^2 + bx + c$, therefore, $x = -1$ must be its root. Thus, $a - b + c = 0 \Rightarrow b = a + c$.

Clearly, $b > 2$. So we observe that for $b = 3, 4$ we have 2 pairs of solutions for a and c, for $b = 5, 6$, we have 4 and so on. This becomes an A.P. summing which, we get 498002.

159. For $x^2 + 2ax + b = 0$, $D = a^2 - b = (a - k)^2 = a^2 - 2ak + k^2$. Setting $b = k^2 - 2ak$ we have discriminant of first equation as $a^2 - 4b = (a - 4k)^2 - 3(2k)^2$. Thus, we can say that if there are infinite no. of rational points on $x^2 - 3y^2 = 1$ such that (a, b) is relatively prime then it is proved. Now we choose $(2, 1)$ as (a, b), which are relatively prime. The infinte no. of points is given by intersection of the curve with $y = m(x - 2) + 1$, where $m \in \mathbb{Q}$.

160. Putting $x = 0, 1, \frac{1}{2}$, we get $|c| \leq 1, |a + b + c| \leq 1, |a + 2b + 4c| \leq 4$.

Thus, $|b + 3c| \leq 5 \Rightarrow |b| \leq 8 \Rightarrow |a + b + 3c| \leq 3 \Rightarrow |a| \leq 8 \Rightarrow |a| + |b| + |c| \leq 17$.

161. When $x = 0$, $|c| \leq 1$. Setting $x = \pm 1$, $|a + b + c| \leq 1$ and $|a - b + c| \leq 1 \Rightarrow |a + c| \leq 1 \Rightarrow |a| \leq 2$.

Considering $-1 \leq x \leq 1$, if $a > 0$ then, $2ax + b \leq 2a + b = a + b + c + a - c \leq 1 + a - c \leq 1 + 1 + 2 = 4$ and $-4 = -2 - 1 - 1 \leq -(a + c) - 1 \leq (-a + c) - a + b - c \leq -2a + b \leq 2ax + b$.

Similarly, it can be proven for $a < 0$.

162. $|a| = max\{-a, a\}$ implies $\frac{|p(1)+p(-1)|}{2} = \frac{|1+p+q|+|1-pq|}{2} = \frac{1}{2}\max\{(1 + p + q + 1 - p - q), (1 + p + q - (1 - p + q))\} = \max\{|1 + q|, |p|\}$.

Clearly, minimum value of $\max\{|1 + q|, |p|\}$ will be when $p = 0$ and $q = -\frac{1}{2}$. Thus, our polynomial is $x^2 - \frac{1}{2}$.

163. Clearly, $|a+b+c| \leq 1, |a-b+c| \leq 1, |c| \leq 1 \Rightarrow |a| \leq 2$. Now, $\frac{8}{3}a^2 + 2b^2 = \frac{2}{3}[4a^2 + 3b^2] = \frac{2}{3}[2(a + b)^2 + 2(a - b)^2 - b^2]$, which will be maximum if $b = 0$. Thus, $a = \pm 2, c = \pm 1$.

164. Let the roots are $-p, -q, -r$ so that $p, q, r > 0$. From Vieta's relations, we have $p+q+r = a < 3, b+c = pq+qr+rp+pqr$. By AM-GM inequality, $pqr \leq \left(\frac{p+q+r}{3}\right)^3 < 1$. Also, $p^2 + q^2 + r^2 \geq pq + qr + rp \Rightarrow (p + q + r)^2 \geq 3(pq + qr + rp) \Rightarrow pq + qr = rp \leq 3 \Rightarrow pq + qr + rp + pqr < 3 + 1 = 4$.

165. Set $x = 0, p(0) = 1$. Setting $x = 1, p(1) = 0$. We continue till $x = 29$ to find $p(29) = 0$. Let $p(x) = x(x - 1)(x - 2) \cdots (x - 29)Q(x)$, where $Q(x)$ is some polynomial. Then,

$x(x-1)(x-2) \cdots (x-30)Q(x-1) = x(x-1)(x-2) \cdots (x-30)Q(x) \Rightarrow Q(x-1) = Q(x) \Rightarrow Q(x)$ is periodic. $\Rightarrow Q(x) = c \Rightarrow p(x) = cx(x - 1)(x - 2) \cdots (x - 29)$.

166. We observe that $p(0) = 0, p(1) = 0, p(2) = 0$ and so on. Hence, $p(x)$ has infinite roots. Thus, $p(x) = 0$.

167. We will prove this by contradiction. Suppose $f(f(x)) = x$ has a real root a i.e. $f(f(a)) = a$, then $f(x) = b \neq a$ since $f(x) = x$ has no real root. WLOG, we may

assume that $b > a$, then $f(a) = b, f(b) = a$. Let $g(x) = f(x) - x$ for all x. It follows that $g(a) = b - a > 0$ and $g(b) = a - b < 0$. Since f is a quadratic poplynomial, f is continuous. This means that between a and b there must exist one x_0 such that $f(x_0) = 0$ making it a root, which is a contradiction.

168. We set $x = 0, \pm1, \pm2$, we have $7|e, 7|a+b+c+d+e, 7|a-b+c-d+e, 7|16a+8b+4c+2d+e, 7|16a-8b+4c-2d+e$, which simplifies to $7|a+c, 7|32a+8c$, which implies the given result.

169. $(2a+b-3)^2 + 3(b-1)^2 \geq 0$.

170. $p(x^5) = x^{20}+x^{15}+x^{10}+x^5+1 = x^{20}-x^{15}+2x^{15}-2x^{10}+3x^{10}-3x^5+4x^5-4+5 = (x^5-1)(x^{15}+2x^{10}+3x^5+4)+5 = f(x)(x-1)(x^{15}+2x^{10}+3x^5+4)+5$. Thus, remainder would be 5.

171. Roots of x^2+1 are $\pm i$ and roots of x^2+x+1 are ω, ω^2. Let $x^{2025} = f(x)g(x) + ax^3+bx^2+cx+d$, where $g(x)$ is divisible by $(x^2+1)(x^2+x+1)$.

Setting $x = i, i = -ai - b + ci + d$. Comparing real and imaginary parts, $b = d$ and $c - a = 1$. Setting $x = \omega, \omega = a + b\omega^2 + c\omega + d = a + b\left(\frac{-1+i\sqrt{3}}{2}\right) + c\left(\frac{-1-i\sqrt{3}}{2}\right) + d$. Comparing real and imaginary parts, $2a+2d-b-c = 2$ and $b = c$. Thus, $a = 1, b = 2, c = 2, d = 2$. So the remainder is x^3+2x^2+2x+2.

172. We may assume that the leading coefficient in $p(x)$ is positive, so that $p(x) \to \infty$ when $n \to \infty$, and $p(n) > 1$ for $n > N$. If $x > N$ and $p(x) = a_nx^n + \cdots = y > 1$ then $p(ry+x) = a_n(ry+x)^n + \cdots$ is divisible by y for every integral r; and $p(ry+x)$ tends to infinity with r. Hence there are infinitely many composite values off(n).

173. Given, $x + \sqrt{a+\sqrt{x}} = a \Rightarrow (a-x)^2 = a + \sqrt{x} \Rightarrow a^2 - (2x+1)a + x^2 - \sqrt{x} = 0$, which is a quadratic equation in a. Thus, $a = \frac{2x+1\pm(2\sqrt{x})+1}{2}$. Let $z = \sqrt{x}$, then $a = z^2+z+1$ and $a = z^2 - z$. First we consider second root. From the given equation $\sqrt{a+\sqrt{x}} = a - x \geq 0 \Rightarrow a \geq x \Rightarrow a \geq z^2$. Substituting from first equation $z^2 - z = a \geq z^2 \Rightarrow -z \geq 0 \Rightarrow -\sqrt{x} \geq 0$. This is true only if $x = 0 \Rightarrow a = 0$.

Now we consider the first root. $a = z^2+z+1 \Rightarrow z = \frac{-1\pm\sqrt{4a-3}}{2}$. Since $z > 0$, we discard negative root. $\Rightarrow z = \frac{-1\pm\sqrt{4a-3}}{2} \Rightarrow 4a-3 \geq 0 \cap z \geq 0 \Rightarrow a \geq 1$. Thus, $x = z^2 = \frac{2a-1-\sqrt{4a-3}}{2}$.

174. Given, $x^2 - \sqrt{a-x} = a \Rightarrow a^2 - (2x^2+1)a + x^4 + x = 0 \Rightarrow a = x^2+x, x^2-x+1 \Rightarrow x = \frac{-1\pm\sqrt{1+4a}}{2}, \frac{1\pm\sqrt{4a-3}}{2}$. Clearly, $a \geq \frac{3}{4}$ and we also need to discard negative root in the first one.

175. Given, $\sqrt{a-\sqrt{a+x}} = x \Rightarrow a^2 - (2x^2+1)a + x^4 - x = 0 \Rightarrow a = x^2+x+1, x^2-x \Rightarrow x = \frac{-1+\sqrt{4a}}{2}, \frac{1\pm\sqrt{1+4a}}{2}$. Clearly, if $a \geq 0$, then x will be real.

176. Let the three rational roots of the equation are $\frac{\alpha_2}{\alpha_1}, \frac{\beta_2}{\beta_1}, \frac{\gamma_2}{\gamma_1}$, then $ax^3 + bx^2 + cx + d = k(\alpha_1 x + \alpha_2) + (\beta_1 x + \beta_2) + (\gamma_1 x + \gamma_2) \Rightarrow a = \alpha_1\beta_1\gamma_1, b = \alpha_1\beta_1\gamma_2 + \alpha_1\beta_2\gamma_1 + \alpha_2\beta_1\gamma_1, c = \alpha_1\beta_2\gamma_2 + \alpha_2\beta_1\gamma_2 + \alpha_2\beta_2\gamma_1, d = \alpha_2\beta_2\gamma_2$. As ad is odd, all of $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ must be odd. This implies that b and c must be odd as well, which is a contradiction.

177. We observe that $f(x) = f\left(\frac{1}{x}\right)$. Dividing the given equation by x^2, we have $x^2 + ax + b + \frac{a}{x} + \frac{1}{x^2} = 0 \Rightarrow \left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) + b - 2 = 0$. Substituting $x + \frac{1}{x} = z \geq 2$, we have $z^2 + az + b - 2 = 0 \Rightarrow D = \frac{-a \pm \sqrt{a^2 - 4b + 8}}{2} \geq 2 \Rightarrow a^2 - 4b + 8 \geq a^2 + 8a + 16 \Rightarrow -b \geq 2a + 2 \Rightarrow a^2 + b^2 \geq 5a^2 + 8a + 4$.

We know that for a quadratic equation minimum value is $-\frac{D^2}{4a}$, so minimum value of $a^2 + b^2$ is $\frac{4}{5}$.

178. $(x-2)^2 = x^2 - 4x + 4 \Rightarrow (x-2)^2 - 2 = x^2 - 4x + 2$ thus we see that the term involving power of x^0 will always become 4 as after the expansion and before squaring the constant term will always be 2, and hence, $a_0 = 4$.

a_1 is coefficient of x. Again $(x-2)^2 - 2 = x^2 - 4x + 2$. Now if we perform a square then we see that coefficient of x will be four times because it will get multiplied with 2 twice and get added. Thus, $a_1 = -4^k$.

Similarly, we find that on first expansion $a_2 = 1 = \frac{4-1}{3}$, on second expansion it is $a_2 = 4 + 4^2 = \frac{4^3 - 4^1}{3}$, and on third expansion $a_2 = 4^2 + 4^3 + 4^4 = \frac{4^{2k-1} - 4^{k-1}}{3}$, proceeding we find that $a_k = 4^{k-1} + 4^k + \cdots + 4^{2k-2} = \frac{4^{2k-1} - 4^{k-1}}{3}$.

The coefficient of $a_{2^k} = 1$ becauase it is the coefficient of highest power of x, which will remain constant on any number of squaring.

179. We have $x^2 - 3xy + 2y^2 + x - y = 0 \Rightarrow (x-y)(x-2y+1) = 0 \Rightarrow x = y$ or $x = 2y - 1$. Substituting $x = y$ in the second equation, we have $y = 0$, which satisfy the third equation. Setting $x = 2y - 1$ in the second equation, we have $y^2 - 5y + 6 = 0 \Rightarrow y = 2{,}3 \Rightarrow x = 3, 5$. Both these pairs of value satisfy the third equation.

180. From first equation $y = 2x + a$. Setting this in the second equation we have $3x^2 + 3ax + (a^2 - b) = 0 \Rightarrow x = \frac{-3a \pm \sqrt{12b - 3a^2}}{3}$.

Given that roots are rational, $\therefore 12b - 3a^2 = c^2\ c \in \mathbb{I}$. c will be a multiple of 3 becaause L.H.S. is a multiple of 3. If a is odd, then c is odd and if a is even then c is even. Thus, a and c has same parity, and $-3a \pm c$ is always even, and hence is a multiple of 6. This implies that x and y are integers.

181. We can assume that $a \geq b \geq c \geq d \geq e$, then $3a \leq 3e \Rightarrow a = e \Rightarrow 3a = (3a)^3 \Rightarrow a = 0, \frac{1}{3}, -\frac{1}{3}$. Similarly, it can be proven if $a \leq b \leq c \leq d \leq e$.

182. Setting $y = xt$, we have $x^3t^2 = 15x^2 + 17x^2t + 15x^2t^2$ and $x^3t^2 = 20x^2 + 3x^2t^2$. Multiplying second equation by t, and subtracting $3x^2(t-5)(t^2+1) = 0 \Rightarrow x = 0, t = 5 \Rightarrow (x, y) = (19, 95)$.

183. Subtracting we have $y^2 - x^2 = 7 \Rightarrow (y-x)(y+x) = 7$. Now there are four possibilities $y - x = 1, y + x = 7 \Rightarrow y = 4, x = 3$, $y - x = 7, y + x = 1 \Rightarrow y = 3, -4$ which does not satisfy the given equations, $y - x = -1, y + x = -7 \Rightarrow y = -4, x = -3$, and $y - x = -7, y + x = -1 \Rightarrow y = -4, x = 3$, which does not satisfy the given equations.

184. $a_1 + a_2 + a_3 = 40, a_2 + a_3 + a_4 = 40 \Rightarrow a_1 = a_4$ or $a_{i+1} = a_{i+4}$. Similarly, $a_{3i} = a_{3i+3} \Rightarrow a_{2013} = a_3 = 10$.

185. $t_1 = 2007, t_2 = 53, t_3 = 34, t_4 = 25, t_5 = 29, t_6 = 85, t_7 = 89, t_8 = 145, t_9 = 42, t_{10} = 20, t_{11} = 4, t_{12} = 16, t_{13} = 37, t_{14} = 58, t_{15} = 89$, so we see that 89 has recurred, and hence, the sequence will repeat. The sum of first 7 terms is 2322. Now we will have this repetition of next 8 terms. No. of such repetitions is $\frac{2013-7}{8} = 250 + \frac{6}{8}$. Sum of these 8 terms is 411 and sum of first 6 terms is 264. Hence, required sum is $2322 + 411 \times 250 + 264 = 105336$.

186. Let the sequence be $a_1, a_2, \ldots, a_{16}$. Then, $a_1 + a_2 + \cdots + a_7 = -1, a_2 + a_3 + \cdots + a_8 = -1, \cdots, a_{10} + a_{11} + \cdots + a_{16} = -1$, and $a_1 + a_2 + \cdots + a_{11} = 1, a_2 + a_3 + \cdots + a_{12} = 1, \cdots, a_6 + a_7 + \cdots + a_{16} = 1$. Also, $a_1 = a_{16}, a_2 = a_{15}, \cdots, a_8 = a_9$. Solving these we get the numbers as 5, 5, −13, 5, 5, 5, −13, 5, 5, −13, 5, 5, 5, −13, 5, 5.

187. Because of the inaccuracies of the balance if x is the weight in left pan and y is the weight in right pan then there will be two constants m and n such that $y = mx + n$. Thus, $A_2 = mA_1 + n, B_2 = mB_1 + n \Rightarrow m = \frac{B_2 - A_2}{B_1 - A_1}, n = \frac{A_2B_1 - A_1B_2}{B_1 - A_1}$.

Thus, $C_2 = mC_1 + n = \frac{C_1(B_2 - A_2) + A_2B_1 - A_1B_2}{B_1 - A_1}$.

188. Let a, b, c, d be the roots of $x^4 + x^3 - 1 = 0$. From Vieta's relations, we have $a + b + c + d = -1$, $ab + bc + cd + ac + ad + bd = 0$, $abc + abd + acd + bcd = 0$ and $abcd = -1$. Thus, $ab = -\frac{1}{cd}, c + d = -1 - a - b$. Also, $ab + (a+b)(c+d) + cd = 0 \Rightarrow ab + (a+b)(-1-a-b) - \frac{1}{ab} = 0$. Let $a + b = m$ and $ab = n$, then $n + m(-1-m) - \frac{1}{n} = 0$. Also, $abc + abd + acd + bcd = 0 \Rightarrow n(-1-m) - \frac{n}{m} = 0 \Rightarrow n = -\frac{m^2}{m^2+1}$. Thus, $\frac{n^6+n^4+n^3-n^2-1}{n(n^2+1)^2} = 0$. Thus, ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

189. Let l, m, n be three distinct fifth roots of unity. Then, $P(1) + lQ(1) + l^2R(1) = \frac{l^5-1}{l-1}S(l) = 0$, $P(1) + mQ(1) + m^2R(1) = \frac{m^5-1}{m-1}S(m) = 0 \Rightarrow mQ(1) + m^2R(1) = nQ(1) + n^2R(1) \Rightarrow Q(1) = -(l+m)R(1)$. By symmetry, $Q(1) = -(l+n)R(1)$. Since $l \neq n$, it follows that $Q(1) = R(1) = 0$. Then as above $P(1) + lQ(1) + l^2R(1) = 0$, so $(x-1)$ is a factor of $P(x)$ as asked for.

190. We have to prove that $x^6 \geq 2a - 1 \Rightarrow x^6 - 2x^5 + 2x^3 - 2x + 1 \geq 0 \Rightarrow x^6 - 2x^5 + x^4 - x^4 + 2x^3 - x^2 + x^2 - 2x + 1 \geq 0 \Rightarrow x^4(x^2 - 2x + 1) - x^2(x^2 - 2x + 1) + x^2 - 2x + 1 \geq 0 \Rightarrow (x^2 - 2x + 1)(x^4 - x^2 + 1) \geq 0$, which is true.

191. From Vieta's relations, $x_1+x_2+x_3=0, x_1x_2+x_2x_3+x_3x_1=a, x_1x_2x_3=-a$. Also, because x_1, x_2, x_3 are roots of $x^3+ax+a=0$, we will have $x_1^3=-(ax_1+a), x_2^3=-(ax_2+a), x_3^3=-(ax_3+a)$.

We have to prove that $\frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+\frac{x_3^2}{x_1}=-8 \Rightarrow \frac{x_1^3x_3+x_2^3x_1+x_3^3x_2}{x_1x_2x_3}=-8 \Rightarrow \frac{-a(x_1x_2+x_2x_3+x_3x_1+x_1+x_2+x_3)}{-a}=-8 \Rightarrow a=-8 \Rightarrow x^3-8x-8=0$. Solving Vieta's relations we have roots as $-2, 1\pm\sqrt{5}$.

192. Let $f(x)$ be a function such that $f(x)=2007-x$, then $g(x)=p(x)-f(x)=x(x-1)(x-2)\cdots(x-2007) \Rightarrow p(x)=x(x-1)(x-2)\cdots(x-2007)+(2007-x)$.

193. $xP(x)-1=k(x-1)(x-2)\cdots(x-n-1)$. Setting $x=0, k=\frac{(-1)^n}{(n+1)!} \Rightarrow (n+2)P(n+2)-1=(-1)^n \Rightarrow P(n+2)=\frac{1+(-1)^n}{n+2}$.

194. $Q(x)=(x+1)P(x)-x=kx(x-1)(x-2)\cdots(x-n)$. Setting $x=-1, k=\frac{(-1)^{n+1}}{(n+1)!} \Rightarrow Q(n+1)=(-1)^{n+1} \Rightarrow P(n+1)=\frac{n+1+(-1)^{n+1}}{n+2}$.

195. If P is a polynomial with integral coefficients then $a-b \mid P(a)-P(b)$. Clearly, $a-b \mid P(a)-P(b)=b-c \mid P(b)-P(c)=c-a \mid P(c)-P(a)$ so $a-b, b-c, c-a$ must be equal in magnitude. Let us say that two of them, $a-b$ and $b-c$, are equal. Then $0=|a-b+b-c+c-a|=|2(a-b)+(c-a)|\geq 2|a-b|-|c-a|=|a-b|$ so $a=b=P(a)$, and $c=P(b)=P(a)=b$, so a, b, c are equal.

196. The proof will follow from following Lemma:

Lemma: If $0\leq m\leq 1\leq n$ then $(2+m)(2+n)\geq 3(2+mn)$

Proof: $(2+m)(2+n)\geq 3(2+mn) \Rightarrow 4+2m+2n+mn\geq 6+3mn \Rightarrow 0\geq 2-2m-2n+2mn \Rightarrow 0\geq (m-1)(n-1)$.

Now we solve the problem. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n real roots then $P(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

Let $\beta_i=-\alpha_i$, then without loss of generality we can assume that $\beta_1\leq\beta_2\leq\ldots\leq\beta_n$. As $\beta_1\beta_2\ldots\beta_n=1$ if $\beta_n\geq 1$, then $\beta_1\beta_2\ldots\beta_{n-1}\leq 1$.

$P(2)=(2+\beta_1)(2+\beta_2)\cdots(2+\beta_n)$. Now we repeat the lemma

$(2+\beta_1)(2+\beta_2)\cdots(2+\beta_n)\geq(3+\beta_1\beta_2)(2+\beta_3)\cdots(2+\beta_n)\geq 3^2(2+\beta_1\beta_2\beta_3)\ldots(2+\beta_n)\geq 3^{n-1}(2+\beta_1\beta_2\ldots\beta_b)\geq 3^n$.

Aliter: If $\beta_1, \beta_2, \ldots, \beta_n$ are non-negative numbers, then by the AM-GM inequality: $2+\beta_i=1+1+\beta_i\geq 3\sqrt[3]{\beta_i}$. And thus, $(2+\beta_1)(2+\beta_2)\cdots(2+\beta_n)\geq 3^n\sqrt[3]{\beta_1\beta_2\ldots\beta_n}=3^n$.

197. Let the polynomial be $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0=0$ such that $a_i\in\{1,-1\}$. Let the roots be α_i, where $i=1,2,3,\ldots,n$.

WLOG, let $a_n = 1$, then from Vieta's relations $\sum_{i=1}^{n} \alpha_i = -a_{n-1}$, $\sum_{1\leq ij\leq n} \alpha_i\alpha_j = a_{n-2}$, and $\prod_{i=1}^{n} \alpha_i = (-1)^n a_0$.

$\sum_{i=1}^{n} \alpha_i^2 = \left(\sum_{i=1}^{n} \alpha_i\right) - 2\left(\sum_{1\leq ij\leq n} \alpha_i\alpha_j\right) = a_{n-1}^2 - 2a_{n-2} = 1 - 2a_{n-2}$. However, all the roots are real, therefore $\sum_{i=1}^{n} \alpha_i^2 \geq 0$, and hence, $a_{n-1} = -1 \Rightarrow \sum_{i=1}^{n} \alpha_i^2 = 3$.

We also have that $\sum_{i=1}^{n} |\alpha_i^2| = \sum_{i=1}^{n} \alpha_i^2 = 3$, and $\prod_{i=1}^{n} |\alpha_i| = \left|\prod_{i=1}^{n} \alpha_i\right| = |(-1)^n a_0| = 1$.

Using RMS-GM inequality, $1 = \left(\prod_{i=1}^{n} |\alpha_i|\right)^{\frac{1}{n}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} |\alpha_i|^2\right)^{\frac{1}{2}} = \sqrt{\frac{3}{n}}$.

Thus, maximum value of n is 3. Now we find out the equations as $x-1, x+1, x^2-x-1, x^2+x-1, x^3-x^2-x+1, x^3+x^2-x-1$, and their negatives.

198. WLOG we assume that at the three different integer a, b, c the polynomial $p(x)$ assumes the value 1. Let d be one of its roots. Then $p(d) = (d-a)(d-b)(d-c)q(d)+1$, where $q(x)$ has integer coefficients. We know for suree that one of $|d-a|, |d-b|, |d-c|$ is greater than 1. Thus, $p(d) = kq(d)+1 = 0 \Rightarrow q(d) = -\frac{1}{k}$, but d is an integer and q has integer coefficients so we have a contradiction, and hence, $p(x)$ cannot have integral roots.

199. We have $\alpha+\beta = 6, \alpha\beta = 1$, and $\alpha^n+\beta^n = (\alpha+\beta)(\alpha^{n-1}+\beta^{n-1}) - \alpha\beta(\alpha^{n-2}+\beta^{n-2})$, which is a recurrence relation which will eventually come down to $\alpha+\beta$ and $\alpha\beta$ both of which are in $\mathbb{Z}$, and hence, $\alpha^n+\beta^n \in \mathbb{Z}$. Similarly, second part can be proven.

200. It is given that $P(x) \geq 0$. If it has a root of odd multiplicity, then it changes sign at that root. So any real roots must have even multiplicity. If $x = \alpha+i\beta$ is a complex root so is $x = \alpha - i\beta$, because the coefficients are real. Let x_i be the even no. of rela roots.

Therefore, $P(x) = \prod_i (x-x_i)^2 \prod_j ((x-\alpha_j)^2+\beta_j^2)$. Now $\prod_j ((x-\alpha_j)^2+\beta_j^2) = (x-\alpha_j+i\beta_j)(x-\alpha_j-i\beta_j) = R(x)^2+I(x)^2$.

201. Let $F(x) = f(g(h(x))) = \prod_{i=1}^{8}(x-i) \Rightarrow F'(x) = 4(2x-9)(x^6-27x^5+288x^4-1539x^3+4299x^2-5886x+3044)$. We see that 6 degree equation is irreducible. However, $F'(x) = f'(g(h(x))).g'(h(x)).h'(x)$, which is not possible. Hence, the required condition is not possible.

202. For $|z| = 1$, we have $\bar{z} = \frac{1}{z} \Rightarrow |P(z)|^2 = P(z)\overline{P(z)}$. Expanding, $|P(z)|^2 = 4 + 2\Re(a\bar{b}z + a\bar{z}z^2 + a\bar{c}z^3 + b\bar{c}z + b\bar{d}z^2 + c\bar{d}z)$.

Let $\omega = e^{2\pi i/3}$ as third root of unity, we have $|P(z)|^2 + |P(z\omega)|^2 + |P(z\omega^2)|^2 = 12 + 6(a\bar{d}z^3)$. We can choose a z_0 such that $a\bar{d}z_0^3 = 1$, then one of the terms on left must be greater than or equal to 6.

203. From the question it is clear that both the functions cross x-axis at a and b respectively. This means, a and b are roots of $f(x)$ and $g(x)$. Let a and b repeat evenly for $f(x)$, then we have to prove that

$(\alpha - a)^2(\alpha - b)^2 < (\alpha - a)(\alpha - b) \Rightarrow (\alpha - a)(\alpha - b)[(\alpha - a)(\alpha - b) - 1] < 0$. Substituting $\alpha = b + c$, $(2 + c)c[(2 + c)c - 1] < 0 \Rightarrow c(c^3 + 4c^2 + 3c - 2) < 0$ which is true for very small c.

Let a has a multiplicity of 3 and b has of 1, then we have to prove that

$(x - a)^3(x - b) < (x - a)(x - b) \Rightarrow (x - a)(x - b)[(x - a)^2 - 1] < 0$, which is true between a and b because $b - a > 2$.

204. Putting $x = 2\cos t$ we find $P_n(x) = 2\cos 2^n t$. Then $P_n(x) = x \Rightarrow \cos 2^n t = \cos t$, which has 2^n solutions giving 2^n distinct solutions in x.

205. If α is a root then α^2 is also a root. Similarly, if $\alpha - 1$ is a root then $(\alpha - 1)^2$ is also a root. If $|\alpha| > 1$, then it implies that there are infinitely many roots. If $\alpha \in (-1, 0)$ or $\alpha \in (0, 1)$, then it implies that there are infinitely many roots. Thus, $f(x)$ can have only 0 and 1 as roots. We also see that if we have have root of one kind then it must also have root of the other kind.

Setting $f(x) = kx^a(x - 1)^b$, where $k \neq 0$. Using the identity we get that $-k = x^{a-b}(x + 1)^{b-a}$, but k is a constant. Hence, $a = b$ and $k = -1$. Thus, $-x^n(x - 1)^n$ is one form of function.

We see that complex numbers, which are root of unity, and satisfy the property $z_1 = z_2^2$ will also be roots of this identity. For example, cube roots of unity. Deducing similarly we find that $-(x^2 + x + 1)^n$ is another form of funciton.

If $f(x) = c$ then $c = -1$ or $c = 0$.

206. $P(2x^2) = \frac{P(2x^3+x)}{P(x)}$ is an even function so either both $P(2x^3 + x)$ and $P(x)$ are even functions or both are odd functions. We also see that $P(0)P(0) = P(0) \Rightarrow P(0) = 0$ or 1.

Consider $P(0) = 0$ and $P(x)$ is even, let $a_{2n}x^{2n}$ be the minimum power of x. Then, $(P'(x) + a_{2n}x^{2n})(P'(2x^2) + a_{2n}(2x^2)^{2n}) = P'(2x^3 + x) + a_{2n}(2x^3 + x)^{2n}$. Considering only lower powers of x, we find that $a_{2n} = 0$. Similarly, we find that of $P(0) = 0$ and $P(x)$ odd, we can prove that $P(x)$ cannot also be odd.

So $P(0) = 1$ and $P(x)$ is even then $P(x) = 1 + \sum_{k=1}^{n} a_{2n}x^{2n}$, putting $n = 1, 2, ...$ and so on we find that $P(x) = (1 + x^2)^n$.

207. If $f(x)$ is constant k, then $k^2 = k \Rightarrow f(x) = 0$ and $f(x) = 1$. Given, $f(x)f(x + 1) = f(x^2 + x + 1)$. Setting $x = x - 1$, we have $f(x)f(x - 1) = f(x^2 - x + 1)$. Suppose $f(x)$ is not a constant. Assume that it has at least one complex root. Let z be at

maximum distance from O. From our equations $f(z^2+z+1) = f(z^2-z+1) = 0$. Thus, $z \neq 0$. If also $z^2+1 \neq 0$, then $z, z^2+z+1, z^2-z+1, -z$ are vertices of a parallelogram. Thus, either of z^2+z+1 or z^2-z+1 is greater than $|z|$, which is a contradiction with the choice of z. Thus, $z^2+1=0$ is a factor of f. Hence, $f(x) = (x^2+1)^m g(x), m \in \mathbb{N}, x^2+1 \nmid g(x)$.

Putting this in our equation we see that it is satisfied. We also see that $g(x)$ satisfies our equation. Since it is not divisible by x^2+1, we must have $g(x)=1$. Thus the solution is $f(x) = (x^2+1)^m \ \forall m \in \mathbb{N}$.

208. We see that if α is a root then α^2 is also a root. Thus if $|\alpha| > 1$ or $0 < |\alpha| < 1$ then there will be infinitely many roots. Thus, all roots must lie on unit circle.

If $f(x)$ is constant then $c^2 - c^2 = 0 \Rightarrow c = 0$ or $c = 1$. Let us assume that $f(x) = ax+b$. then $(ax+b)(b-ax) = ax^2+b \Rightarrow a = -1, b = 0$ or $b = 1$. Thus, $f(x) = -x$ and $f(x) = 1-x$.

If $f(x) = ax^2+bx+c$, then $(ax^2+bx+c)(ax^2-bx+c) = (ax^4+bx^2+c) \Rightarrow a = 1, c^2 = c \Rightarrow c = 0, 1$, and $2ac - b^2 = b$. If $c = 0 \Rightarrow b = 0, -1$. If $c = 1 \Rightarrow b = 1, -2$. So we have $f(x) = x^2, x^2-x, x^2-2x+1 = (x-1)^2, x^2+x+1$. We can rewrite the second and third in the form $f(x) = -x(1-x)$ and $f(x) = (1-x)^2$. Now we can write a general solution $f(x) = (-x)^p(1-x)^q(x^2+x+1)^r, p, q, r \in \mathbb{Z}$.

209. Suppose $p(x), q(x), r(x) \in \mathbb{Z}[x]$ with $p(x) = q(x)r(x)$ where $0 \leq \deg(q) \leq 3 < \deg(r) \leq \deg(p) = 7$. Given that for $n_1, n_2, \ldots, n_7$ distinct integers $|p(n_i)| = 1$ for $1 \leq i \leq 7$. Observe that $q(x)$ and $r(x)$ are both polynomials with integer coefficients.

Then $q(n_i), r(n_i) \in \mathbb{Z}$ for each i and $|q(n_i)r(n_i)| = 1 \Rightarrow q(n_i) = \pm 1$. We see that the values of 1 or -1 is taken at least 4 times but the degree of $q(x)$ is at most 3 thus, $q(x) = \pm 1$. Hence, $p(x)$ is irreducible.

210. Let $f(x) = (x-a_1)^2(x-a_2)^2 \cdots (x-a_n)^2 + 1$, and suppose that it factors non-trivially as $f(x) = g(x)h(x)$ over $\mathbb{Z}$.

By Gauss's lemma we may assume that g, h are monic polynomials with integeral coefficients. Let $g(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0, h(x) = x^l + c_{l-1}x^{l-1} + \cdots + c_0 \in \mathbb{Z}[x]$

Observe that the polynomial functions g, h satisfy $g(r), h(r) > 0$ for $r \in \mathbb{r}$ and that $g(a_i) = h(a_i) = 1$. Suppose $k < l$ then the polynomial g has the value 1 then for n distinct values $a_1, \ldots, a_n$ and, because it has degree $k < n$, that the polynomial is the constant 1, and the factorization is trivial which is a contradiction. When $k = l = n$, then $f = g^2 \Rightarrow 1 = [g(x)] + (x-a_1)\cdots(x-a_n)[g(x)] - (x-a_1)\cdots(x-a_n)$, which is again trivial factorization. In both the cases the assumed non-triviality of factorization leads to contradiction, and thus f is actullay irreducible.

211. For roots to be equal the discriminant has to be zero.

$D = 4(1+3m)^2 - 4(1+m)(1+8m) = 0 \Rightarrow 4(1+9m^2+6m-1-9m-8m^2) = 0 \Rightarrow m^2 - 3m = 0 \therefore m = 0, 3$

212. Discriminant of the equation is: $D = (c+a-b)^2 - 4(b+c-a)(a+b-c) = 4(b^2-4ac)$

Given $a+b+c=0 \Rightarrow b=-(a+c)$. Substituting in above equation, $D=4\{(a+c)^2-4ac\}=4(a-c)^2=$ a perfect square and thus roots are rational.

213. Discriminant of the equation is: $D=4(ac+bd)^2-4(a^2+b^2)(c^2+d^2)=-4(ad-bc)^2$. Roots are real if $D\geq 0$ i.e. $-4(ad-bc)^2\geq 0 \Rightarrow (ad-bc)^2\leq 0$

But since $(ac-bd)^2 \nless 0 \therefore (ad-bc)^2=0$ i.e. $D=0$ (because roots are real). Thus, if roots are real they are equal.

214. Let $A=a(b-c), B=b(c-a)$ and $c=c(a-b)$ Clearly, $A+B+C=0$. Since roots are equal i.e. $D=0 \therefore B^2-4AC=0$

Substituting for B, $[-(A+C)^2-4AC]=(A-C)^2=0 \Rightarrow A=C \Rightarrow 2ac=ab+cb \Rightarrow b=\frac{2ac}{a+c}$.

Thus, a, b, c are in H. P.

215. Given equation is $(b-x)^2-4(a-x)(c-x)=0 \Rightarrow -3x^2+2(2a+2c-b)x+b^2-4ac=0$

Discriminant of the above equation is: $D=4(2a+2c-b)^2+12(b^2-4ac)=8[(a-b)^2+(b-c)^2+(c-a)^2] \because a, b, c$ are real $\therefore D>0$ unless $a=b=c$.

Hence, roots are real unless $a=b=c$.

216. Discriminant of the equations are p^2-4q and r^2-4s.

Adding them we have $p^2+r^2-4(q+s)=p^2+r^2-2pr=(p-r)^2\geq 0$.

Thus, at least one of the discriminant is greater than zero and that equation has real roots.

217. Since $x^2-2px+q=0$ has equal roots $D=0 \Rightarrow 4p^2-4q=0 \Rightarrow p^2=q$.

Discriminant of the second equation: $D=4(p+y)^2-4(1+y)(q+y)=4[p^2+2y+y^2-q-qy-y-y^2]$

Substituting for q, $D=-4y(p-1)^2$. Roots of the equation will be real and distinct only if $D\geq 0$ but $(p-1)\geq 0$ if $p\neq 1$. Thus, y has to be negative as well.

218. Since roots of equation $ax^2+2bx+c=0$ are equal $\therefore 4b^2-4ac\geq 0$. Discriminant of the equation $ax^2+2mbx+nc=0$ is $4m^2b^2-4anc$.

Since $m^2>n>0$ and $b^2\geq ac$ $4m^2b^2-4anc>0$. Thus, roots of the second equation are real.

219. Given $ax+by=1 \Rightarrow y=\frac{1-ax}{b}$, substituting this in second equation, $cx^2+d\left(\frac{1-ax}{b}\right)^2=\frac{b^2cx^2+d(1-ax)^2}{b^2}=1$

$\Rightarrow (b^2c+da^2)x^2-2adx+d-b^2=0$. Since first two equations have one solution this equation will also have only one solution which means roots will be equal i.e. $D=0$

$\Rightarrow 4a^2d^2 - 4(b^2c + a^d)(d - b^2) = 0 \Rightarrow b^2(b^2c + a^2d - cd) = 0 \because b^2 \neq 0 \therefore b^2c + a^2d - cd = 0 \Rightarrow b^2c + a^d = cd$

Dividing both sides by cd we have

$\frac{b^2}{d} + \frac{a^2}{c} = 1 \Rightarrow x = \frac{2ad}{2(b^2c + a^2d)} = \frac{a}{c}$. Substituting for y, we get $y = \frac{b}{d}$.

220. Let the roots of the equation be α and $r\alpha$.

Sum of roots $= \alpha + r\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{a(r+1)}$.

Product of roots $= r\alpha^2 = \frac{rb^2}{a^2(1+r)^2} = \frac{c}{a} \Rightarrow \frac{b^2}{ac} = \frac{(r+1)^2}{r}$.

221. Let the roots of the equation be α and 2α.. Sum of roots $= 3\alpha = -\frac{l}{l-m} \Rightarrow \alpha = -\frac{l}{l-m}$.

Product of roots $= 2\alpha^2 = \frac{1}{l-m}$. Substituting for α, $\frac{2l^2}{9(l-m)^2} = \frac{1}{l-m} \Rightarrow 2l^2 - 9l + 9m = 0[\because l \neq m$ else it would not be a quadratic equation$]$.

Since l is real, therefore discriminant of this equation would be $\geq 0, \Rightarrow 81 - 72m \geq 0 \therefore m \leq \frac{9}{8}$.

222. Let the roots be α and α^n, then sum of roots $= \alpha + \alpha^n = -\frac{b}{a}$ and product of roots $= \alpha^{n+1} = \frac{c}{a}$.

From products, we have $\alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$. From sum we have $a\alpha^n + a\alpha + b = 0$.

Substituting value of α from above $\Rightarrow a\left(\frac{c}{a}\right)^{\frac{n}{n+1}} + a\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + b = 0$. From this we arrive at our desired equation.

223. Let the roots be $p\alpha$ and $q\alpha$.

Sum of roots $= (p+q)\alpha = -\frac{b}{a}$ and product of roots $= pq\alpha^2 = \frac{c}{a}$.

From equation for product of roots, we have $\alpha^2 = \frac{c}{apq} \therefore \alpha = \sqrt{\frac{c}{apq}}$.

Substituting this in sum of roots and solving we arrive at desired equation.

224. The questions are solved below:

i. $\alpha + \beta = -p$ and $\alpha\beta = q$. Now, $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$

$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{p(3q - p^2)}{q}$.

ii. $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta) = \omega^3\alpha^2 + \omega^4\alpha\beta + \omega^2\alpha\beta + \omega^3\beta^2$

$= \alpha^2 + \omega\alpha\beta + \omega^2\alpha\beta + \beta^2 = \alpha^2 - \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 3\alpha\beta = p^2 - 3q$.

225. Rewriting the equation we have $(A+cm^2)x^2+Amx+Am^2=0$.

Sum of roots $=\alpha+\beta=-\frac{Am}{A+cm^2}$ and product of roots $=\alpha\beta=\frac{Am^2}{A+cm^2}$

The expression to be evaluated is $A(\alpha^2+\beta^2)+A\alpha\beta+c\alpha^2\beta^2$.

$=A[(\alpha+\beta)^2-2\alpha\beta]+A\alpha\beta+c(\alpha\beta)^2$.

$=A\left[\frac{A^2m^2}{(A+cm^2)^2}-\frac{2Am^2}{A+cm^2}\right]+\frac{A^2m^2}{A+cm^2}+\frac{cA^2m^4}{(A+cm^2)^2}=0.$

226. Sum of roots $=\alpha+\beta=-\frac{b}{a}$ and product of roots $=\alpha\beta=\frac{c}{a}$.

Now, $a\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)+b\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)=\frac{a(\alpha^3+\beta^3)}{\alpha\beta}+\frac{b(\alpha^2+\beta^2)}{\alpha\beta}$

$=a\frac{[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)]}{\alpha\beta}+\frac{b[(\alpha+\beta)^2-2\alpha\beta]}{\alpha\beta}$. Substituting for sum and product of the roots $=\frac{a\left[\left(-\frac{b}{a}\right)^3-3.\frac{c}{a}\left(-\frac{b}{a}\right)\right]}{\frac{c}{a}}+\frac{b\left[\left(-\frac{b}{a}\right)^2-2\frac{c}{a}\right]}{\frac{c}{a}}$

Solving this we get the desired result.

227. Since a and b are the roots of the equation $x^2+px+1=0$ we have $a+b=-p$ and $ab=1$.

Similarly, since c and d are the roots of the equation $x^2+qx+1=0$ we have $c+d=-p$ and $cd=1$.

Now $(a-c)(b-c)(a+d)(b+d)=(ab-bc-ac+c^2)(ab+bd+ad+d^2)=[ab-c(a+b)+c^2].[ab+d(a+b)+d^2]$

$=[1+pc+c^2].[1-pd+d^2]$(putting the values of $a+b$ and ab) $=1+cp+c^2-pd-cdp^2-c^2pd+d^2+cpd^2+c^2d^2$

$=1+(c^2+d^2)+c^2d^2-cdp^2+p(c-d)+cpd(d-c)=1+[(c+d)^2-2cd]+c^2d^2-cdp^2+p(c-d)+cpd(d-c)$.

Substituting for $c+d$ and cd, $1+q^2-2+1-p^2+p(c-d)+p(d-c)=q^2-p^2$.

228. Let α and β be the roots of the equation $x^2+px+q=0$ then $\alpha+\beta=-p$ and $\alpha\beta=q$.

Also, let γ and δ be the roots of the equation $x^2+qx+p=0$ then $\gamma+\delta=-q$ and $\gamma\delta=p$.

Now, given is that roots differ by the same quantity so we can say that, $\alpha-\beta=\gamma-\delta\Rightarrow(\alpha-\beta)^2=(\gamma-\delta)^2$

$(\alpha+\beta)^2-4\alpha\beta=(\gamma+\delta)^2-4\gamma\delta\Rightarrow p^2-4q=q^2-4p\Rightarrow p^2-q^2+4(p-q)=0\Rightarrow(p-q)(p+q+4)=0$

Clearly, $p\neq q$ else equations would be same $\therefore p+q+4=0$.

229. Since α, β are the roots of the equation $ax^2+bx+c=0 \therefore a\alpha^2+b\alpha+c=0$ and $a\beta^2+b\beta+c=0$.

and $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. Also, given $S_n=\alpha^n+\beta^n$. Now, $aS_{n+1}+bS_n+cS_{n-1}$

$=a(\alpha^{n+1}+\beta^{n+1})+b(\alpha^n+\beta^n)+c(\alpha^{n-1}+\beta^{n-1})=\alpha^{n-1}(a\alpha^2+b\alpha+c)+\beta^{n-1}(a\beta^2+b\beta+c)=\alpha^{n-1}.0+\beta^{n-1}.0$

$\therefore S_{n+1}=-\frac{b}{a}S_n-\frac{c}{a}S_{n-1}$

Substituting $n=4$ we have

$S_5=-\frac{b}{a}S_4-\frac{c}{a}S_3=-\frac{b}{a}(-\frac{b}{a}S_3-\frac{c}{a}S-2)-\frac{c}{a}S_3=\left(\frac{b^2}{a^2}-\frac{c}{a}\right)S_3+\frac{bc}{a^2}S_2$

Proceeding similarly we have the solution as

$=-\frac{b}{a^5}(b^2-2ac)^2+\frac{(b^2-ac)bc}{a^4}$.

230. Let α and β be the roots of the equation $ax^2+bx+c=0$. Given, $\alpha+\beta=\frac{1}{\alpha^2}+\frac{1}{\beta^2}\Rightarrow$ $\alpha+\beta=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$

$-\frac{b}{a}=\frac{\frac{b^2}{a^2}-2\frac{c}{a}}{\frac{c^2}{a^2}}=\frac{b^2-2ac}{c^2}\Rightarrow -bc^2=ab^2-2a^2c\Rightarrow ca^2=\frac{ab^2+bc^2}{2}$

Thus, bc^2, ca^2, ab^2 are in A. P.

231. Rewriting the equation $m^2x^2+(2m-m^2)x+3=0$.

Since α and β are the roots of the equation $\alpha+\beta=-\frac{2m-m^2}{m^2}=\frac{m-2}{m}$ and $\alpha\beta=\frac{3}{m^2}$

Given, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{3}\Rightarrow\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{4}{3}$

$3(\alpha^2+\beta^2)=4\alpha\beta\Rightarrow 3[(\alpha+\beta)^2-2\alpha\beta]=4\alpha\beta\Rightarrow 3(\alpha+\beta)^2-10\alpha\beta=0\Rightarrow 3\left[\left(\frac{m-2}{m}\right)^2-\frac{10}{m^2}\right]=0$

$\Rightarrow m^2-4m-6=0$

Since m_1, m_2 are two values of m we have $m_1+m_2=4$ and $m_1m_2=-6$. Now, $\frac{m_1^2}{m_2}+\frac{m_2^2}{m_1}=\frac{m_1^3+m_2^3}{m_1m_2}=\frac{(m_1+m_2)^3-3m_1m_2(m_1+m_2)}{3m_1m_2}=-\frac{68}{3}$.

232. Let α and β be the roots of the equation $ax^2+bx+c=0$; γ and δ are the roots of the equation $a_1x^2+b_1x+c_1=0$, then

$\alpha+\beta=-\frac{b}{a}, \alpha\beta=\frac{c}{a}$ and $\gamma+\delta=-\frac{b_1}{a_1}, \gamma\delta=\frac{c_1}{a_1}$

According to question, $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$. By componendo and dividendo,

$\frac{\alpha-\beta}{\alpha+\beta}=\frac{\gamma-\delta}{\gamma+\delta}$. Squaring both sides

$\Rightarrow\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2=\left(\frac{\gamma-\delta}{\gamma+\delta}\right)^2\Rightarrow\frac{(\alpha+\beta)^2-4\alpha\beta^2}{(\alpha+\beta)}=\frac{(\gamma+\delta)^2-4\gamma\delta}{(\gamma+\delta)^2}$

$\Rightarrow\frac{b^2-4ac}{b^2}=\frac{b_1^2-4a_1c_1}{b_1^2}\Rightarrow -4acb_1^2=-4a_1c_1b^2\Rightarrow\left(\frac{b}{b_1}\right)^2=\frac{ac}{a_1c_1}$.

233. Since irrational roots appear in pairs and are conjugate. Thus, if first root is $\alpha=\frac{1}{2+\sqrt{5}}$

$\alpha=\frac{1}{2+\sqrt{5}}\frac{2-\sqrt{5}}{2-\sqrt{5}}=\frac{2-\sqrt{5}}{4-5}=-2+\sqrt{5}$

Then second root would be $\beta=-2+\sqrt{5}\Rightarrow\alpha+\beta=-4$ and $\alpha\beta=-1$

Therefore, the equation is $x^2-(\alpha+\beta)x+\alpha\beta=0\Rightarrow x^2+4x-1=0$.

234. Since α and β are the roots of the equation $\therefore\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. Sum of the roots for which quadratic equation is to be found $=\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}$

$=\frac{a(\alpha+\beta)+2b}{a^2\alpha\beta+ab(\alpha+\beta)+b^2}=\frac{a\left(-\frac{b}{a}\right)+2b}{a^2.\frac{c}{a}+av\left(-\frac{b}{a}\right)}+b^2=\frac{b}{ac}$

Product of the roots $=\left(\frac{1}{a\alpha+b}\right)\left(\frac{1}{a\beta+b}\right)=\frac{1}{a^2\alpha\beta+ab(\alpha+\beta)+b^2}=\frac{1}{a^2.\frac{c}{a}+ab\left(-\frac{c}{a}\right)+b^2}=\frac{1}{ac}$.

Therefore, the equation is $x^2-\frac{b}{ac}x+\frac{1}{ac}=0\Rightarrow acx^2-bx+1=0$.

235. Given equation is $(x-a)(x-b)-k=0\Rightarrow x^2-(a+b)x+ab-k=0$.

Since c,d are roots of this equation $\Rightarrow c+d=a+b$ and $cd=ab-k$.

The equation where roots are a,b is $x^2-(a+b)x+ab=0\Rightarrow x^2-(c+d)x+cd+k=0$.

236. Correct equation is $x^2+13x+q=0$ and incorrect equation is $x^2+17x+q=0$.

Roots of correct incorrect equation are -2 and -15. Thus $q=30$.

Therefore, correct equation is $x^2+13x+30=0$ and thus roots are $-3,-10$.

237. Clearly, $\alpha+\beta=-p$ and $\alpha\beta=q$. Substituting $x=\frac{\alpha}{\beta}$ in the given equation we have

$q\frac{\alpha^2}{\beta^2}-(p^2-2q)\frac{\alpha}{\beta}+q=0\Rightarrow q\alpha^2-(p^2-2q)\alpha\beta+q\beta^2=0$

$q(\alpha^2+\beta^2)-(p^2-2q)q=0\Rightarrow q[(\alpha+\beta)^2-2\alpha\beta]-(p^2-2q)q=0$

$q(p^2-2q)-(p^2-2q)q=0\Rightarrow 0=0$. Thus, $\frac{\alpha}{\beta}$ is a root of the given equation.

238. Let α and β be the roots of $x^2-ax+b=0$ and α be the common and equal root from the second equation $x^2-px+q=0$.

Thus, $\alpha+\beta=a, \alpha\beta=b$ and $2\alpha=p, \alpha^2=q \Rightarrow b+q=\alpha\beta+\alpha^2=\alpha(\beta+\alpha)=\frac{p}{2}a=\frac{ap}{2}$.

239. Let α be the common root. Then, we have $a\alpha^2+2b\alpha+c=0$ and $a_1\alpha^2+2b_1\alpha+c_1=0$.

Solving equations by cross-multiplication we have $\frac{\alpha^2}{2(bc_1-b_1c)}=\frac{\alpha}{(ca_1-a_1c)}=\frac{1}{2(ab_1-a_1b)}$.

From first two we have α as $\alpha=\frac{2(bc_1-b_1c)}{ca_1-a_1c}$ and from last two we have α as $\alpha=\frac{ca_1-ac_1}{2(ab_1-a_1b)}$

Equating we get, $\frac{2(bc_1-b_1c)}{ca_1-a_1c}=\frac{ca_1-ac_1}{2(ab_1-a_1b)} \Rightarrow (ca_1-ac_1)^2=4(ab_1-a_1b)(bc_1-b_1c)$

Given, $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in A. P., let d be the common difference.

$\left(\frac{c}{c_1}-\frac{a}{a_1}\right)^2 c_1^2a_1^2=4\left(\frac{a}{a_1}-\frac{b}{b_1}\right)a_1b_2\left(\frac{b}{b_1}-\frac{c}{c_1}\right)b_1c_1$

$(2d)^2c_1^2a_2^2=4(-d)a_1b_1(-d)b_1c_1 \Rightarrow 4d^2c_1^2a_1^2=4d^2a_1c_1b_1^2 \Rightarrow c_1a_1=b_1^2$.

Thus, a_1, b_1, c_1 are in G. P.

240. Let α be the common root between first two, β be the common root between last two and γ be the common root between first and last equations.

Thus, α and β are the roots of the first equation. $\Rightarrow \alpha+\gamma=-p_1, \alpha\gamma=q_1$

Similarly, $\alpha+\beta=-p_2, \alpha\beta=q_2 \Rightarrow \beta+\gamma=-p_3, \beta\gamma=q_3$

L.H.S. $=(p_1+p_2+p_3)^2=4(\alpha+\beta+\gamma)^2$ and R.H.S. $=4(p_1p_2+p_2p_3+p_1p_3-q_1-q_2-q_3)$

$=4[(\alpha+\gamma)(\alpha+\beta)+(\alpha+\beta)(\beta+\gamma)+(\alpha+\gamma)(\beta+\gamma)-\alpha\gamma-\alpha\beta-\beta\gamma]$

$=4(\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\alpha\gamma+2\beta\gamma)=4(\alpha+\beta+\gamma)^2$.

Hence, proven that L.H.S. = R.H.S.

241. Let α be the common root then we have, $\alpha^2+c\alpha+ab=0$ and $\alpha^2+b\alpha+ca=0$.

By cross-multiplication, we get the solution as $\frac{\alpha^2}{ac^2-ab^2}=\frac{\alpha}{ab-ac}=\frac{1}{b-c}$.

From first two we have $\alpha=\frac{ac^2-ab^2}{ab-ac}=-(b+c)$. From last two we have $\alpha=a$.

Equating these two we get $a=-(b+c) \Rightarrow a+b+c=0$. Let the other root of the equations be β and β_1 then we have

$\alpha\beta=ab$ and $\alpha\beta_1=ca \therefore \beta=b$ and $\beta_1=c$. Equation whose roots are β and β_1 is

$x^2-(\beta+\beta_1)x+\beta\beta_1=0 \Rightarrow x^2-(b+c)+bc=0 \Rightarrow x^2+ax+bc=0$.

242. Clearly, root of the equation $x^2+2x+9=0$ are imaginary and since they appear in pairs both the roots will be common and thus the ratio of the coefficients of the terms will be equal. $\Rightarrow a:b:c=1:2:9$.

243. Let α be a common root. Then, we have $3\alpha^2 - 2\alpha + p = 0$ and $6\alpha^2 - 17\alpha + 12 = 0$.

Solving by cross-multiplication $\frac{\alpha^2}{-24+17p} = \frac{\alpha}{6p-36} = \frac{1}{-39}$.

From first two we have $\alpha = \frac{17p-24}{6p-36}$ and from last two we have $\alpha = \frac{6p-36}{-39} = -\frac{2p-12}{13}$.

Equating these two and solving for p we get $p = -\frac{15}{4}, -\frac{8}{3}$.

244. When $x = 0$, $|x|^2 - |x| - 2 = |0|^2 - |0| - 2 = -2 \neq 0$. Since it is not satisfied by $x = 0$ it is an equation.

245. When $x = -a$ the equation is satisfied. Similarly, it is satisfied by values of x being $-b$ and $-c$. The highest power of x occurring is 2 and is true for three distinct values of x therefore it cannot be equation but an identity.

246. Since both the equations have only one common root so the roots must be rational as irrational and complex roots appear in pairs. Thus, the roots of these two equations must be rational and therefore the discriminants must be perfect squares. Therefore, $b^2 - ac$ and $b_1^2 - a_1c_2$ must be perfect squares.

247. Equating the coefficients for similar powers of x, we get, coefficient of $x^2 : a^2 - 1 = 0 \Rightarrow a = \pm 1$.

Coefficient of $x : a - 1 = 0 \Rightarrow a = 1$. Constant term: $a^2 - 4a + 3 = 0 \Rightarrow a = 1, 3$.

The common value of a is 1 which will make this an identity.

248. Given, $\left(x + \frac{1}{x}\right)^2 = 4 + \frac{3}{2}\left(x - \frac{1}{x}\right) \Rightarrow \left(x + \frac{1}{x}\right)^2 - 4 - \frac{3}{2}\left(x - \frac{1}{x}\right) = 0 \Rightarrow \left\{\left(x - \frac{1}{x}\right)^2 + 4x\frac{1}{x}\right\} - \frac{3}{2}\left(x - \frac{1}{x}\right) - 4 = 0$

Substituting $a = x - \frac{1}{x} \Rightarrow a^2 - \frac{3}{2}a = 0 \Rightarrow 2a^2 - 3a = 0 \therefore a = 0, \frac{3}{2}$

$x - \frac{1}{x} = 0 \Rightarrow x = \pm 1 \Rightarrow x - \frac{1}{x} - \frac{3}{2} \Rightarrow x = 2, -\frac{1}{2}$.

249. Given equation is $(x+4)(x+7)(x+8)(x+11) + 20 = 0$.

Rewriting the equation, $[(x+4)(x+11)][(x+7)(x+8)] + 20 = 0$

$\Rightarrow (x^2 + 15x + 44)(x^2 + 15x + 56) + 20 = 0$. Substituting $a = x^2 + 15x$, we get $(a+44)(a+56) + 20 = 0 \Rightarrow a = -46, -54$

If $a = -46 \Rightarrow x^2 + 15x + 46 = 0 \Rightarrow x = \frac{-15 \pm \sqrt{41}}{2}$. If $a = -54 \Rightarrow x^2 + 15x + 54 = 0 \Rightarrow x = -6, -9$.

250. Given equation is $3^{2x+1} + 3^2 = 3^{x+3} + 3^x$. Let $3^x = a$, then we have $3a^2 + 9 = 28a \Rightarrow 3a^2 - 28a + 9 = 0$.

$\Rightarrow a = \frac{1}{3}, 9$. If $a = \frac{1}{3} \Rightarrow x = -1$. If $a = 9 \Rightarrow x = 2$.

251. Clearly, $(5+2\sqrt{6})^{x^2-3}(5-2\sqrt{6})^{x^2-3}=1$. Let $(5+2\sqrt{6})^{x^2-3}=1$ then $(5-2\sqrt{6})^{x^2-3}=\frac{1}{y}$.

The given equation becomes $y+\frac{1}{y}=10$ where $y=(5+2\sqrt{6})^{x^2-3}\Rightarrow y^2-10y+1=0$.

Solving the equation we have roots as $y=5\pm 2\sqrt{6}\therefore x^2-3=\pm 1\Rightarrow x=\pm 2,\pm\sqrt{2}$.

252. Let the speed of the bus $=x$ km/hour $\therefore$ the speed of car $=x+25$ km/hour.

Time taken by bus $=\frac{500}{x}$ hours and by car $=\frac{500}{x+25}$ hours. Given, $\frac{500}{x}=\frac{500}{x+25}+10\Rightarrow$ $x^2-25x+1250=0$.

$x=-50,25$ but x cannot be negative as it is a scalar quantity. Thus, speed of car $=$ 50 km/hour.

253. Given equation is $(a+b)^2x^2-2(a^2-b^2)x+(a-b)^2=0$. Discriminant $=4(a^2-b^2)^2-4(a+b)^2(a-b)^2=0$. Since discriminant is zero, roots are equal.

254. Given equation is $3x^2+7x+8=0$. Discriminant $D=49-96<0$.

Since it is negative roots will be complex and conjugate pair.

255. Given equation is $3x^2+(7+a)+8-a=0$. Discriminant $D=(7+a)^2+12a$

For roots to be equal it has to be zero. $\Rightarrow a^2+26a+49=0\Rightarrow a=13\pm 6\sqrt{6}$.

256. It is given that roots are equal i.e. discriminant is zero. $\Rightarrow 4(ac+bd)^2-4(a^2+b^2)(c^2+d^2)=0\Rightarrow a^2c^2+b^2d^2-2abcd-a^2c^2-a^2d^2-b^2c^2-b^2d^2=0$

$\Rightarrow (ad-bc)^2=0\Rightarrow ad=bc\Rightarrow \frac{a}{b}=\frac{c}{d}$.

257. Discriminant is $4(c-a)^2-4(b-c)(a-b)$

$=c^2+a^2-2ac-ab+b^2+ac-bc=a^2+b^2+c^2-ab-bc-ac=\frac{1}{2}[(a-b)^2(b-c)^2(c-a)^2]$.

Clearly the above expression is either greater than zero or equal to zero. Hence, roots are real.

258. Given equation is $x^2-x+x^2-(a+1)x+a+x^2-ax=0\Rightarrow 3x^2-2(a+1)+a=0$.

Discriminant $D=4(a+1)^2-12a=a^2+2a+1-3a=a^2-a+1=(a-1)^2+a$

which is greater than zero for all a and hence roots are real.

259. Discriminant of the equation $D=b^2-4ac$. Given, $a+b+c=0\Rightarrow b=-(a+c)$.

Substituting value of b, $D=(a+c)^2-4ac=(a-c)^2$, which is either zero or positive. Hence, roots are rational.

260. $D = (c+a-2b)^2 - 4(b+c-2a)(a+b-2c) = c^2 + a^2 + 4b^2 + 2ac - 4bc - 4ab - 4ba - 4b^2 + 8bc - 4ca - 4bc + 8c^2 + 8a^2 + 8ab - 8ca$

$\Rightarrow 9a^2 + 9c^2 - 18ca = 9(a-c)^2 \geq 0$ which is a perfect square. Hence, roots are rational.

261. Given $r = k + \frac{s}{k} \Rightarrow r^2 = k^2 + \frac{s^2}{k^2} + 2s$

$\Rightarrow r^2 - 4s = k^2 + \frac{s^2}{k^2} + 2s - 4s \Rightarrow r^- 4s = k^2 + \frac{s^2}{k^2} - 2s = \left(k - \frac{s}{k}\right)^2$

Clearly, $r^2 - 4s \geq 0$ if r, s, k are rationals which is discriminant of the given equation. Thus, roots will be rational provided given condition is met.

262. The given equation is $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

$\Rightarrow 3x^2 - (a+b+b+c+c+a)x + ab + bc + ca = 0 \Rightarrow D = 4(a+b+c)^2 - 12(ab+bc+ca)$

$= 4a^2 + 4b^2 + 4c^2 - 4ab - 4bc - 4ac = 2[(a-b)^2 + (b-c)^2 + (c-a)^2].$

This cannot be zero unless $a = b = c$, which is the required condition for the roots to be equal.

263. Given equation is $a^2(b^2-c^2)x^2 + b^2(c^2-a^2)x + c^2(a^2-b^2) = 0$

$D = b^4(c^2-a^2)^2 - 4a^2c^2(b^2-c^2)(a^2-b^2) = b^4c^4 + b^4a^4 - 2b^4a^2c^2 - 4a^4b^2c^2 + 4a^2b^4c^2 - 4a^4c^4 + 4a^2b^2c^4$

$= b^4c^4 + b^4a^4 + 2b^4a^2c^2 - 4a^4b^2c^2 - 4a^4c^4 + 4a^2b^2c^4 = (b^2c^2 + b^2a^2 - 2a^2c^2)^2 \geq 0,$ which is a perfect square, and thus, roots will be rational.

264. $D = 16a^2b^2c^2d^2 - 4(a^4+b^4)(c^4+d^4) = 4[4a^2b^2c^22d^2 - a^4c^4 - a^4d^4 - b^4c^4 - b^4d^4]$

$= -4[(a^2c^2 + b^2d^2)^2(a^2c^2 + b^2d^2)^2]$. Thus, if the roots are real then discriminant has to be zero because else it can be only negative and then roots wont remain real.

265. $D = 4q^2 - 4pr = 4(q^2 - pr)$. Since p, q, r are in H. P. $\Rightarrow q = \frac{2pr}{p+r}$

Substituting for q, we get $D = 4\left[\frac{4p^2r^2}{(p+r)^2} - pr\right] = 4\left[\frac{4p^2r^2 - p^3r - pr^3 - 2p^2r^2}{(p+r)^2}\right]$

$= 4\left[\frac{2p^2r^2 - p^3 - r^3}{(p+r)^2}\right] = 4\left[\frac{pr(2pr - p^2 - r^2)}{(p+r)^2}\right]$

$= 4\left[\frac{-pr(p-r)^2}{(p+r)^2}\right]$. Since p and r have the same sign discriminant is bound to be negative and roots will be complex numbers.

266. Discriminant of $bx^2 + (b-c)x + (b-c-a) = 0$, $D_1 = (b-c)^2 - 4b(b-c-a) = b^2 + c^2 - 2bc - 4b^2 + 4bc + 4ab$

Discriminant of $ax^2 + 2bc + b = 0$, $D_2 = 4b^2 - 4ab$. Now, if $D_2 < 0$

$D_1 = (b+c)^2 - (4b^2 - 4ab) > 0$ and thus roots will be real. However, if $D_1 < 0$ i.e. roots are imaginary then we have

$D_1 = (b+c)^2 - (4b^2 - 4ab) < 0 \Rightarrow 4b^2 - 4ab > 0 \because [(b+c)^2 > 0]$.

Then roots of equation $ax^2 + 2bx + b = 0$ will be real.

267. From first equation $x = \sqrt{\frac{1-by^2}{a}}$ and from second equation $x = \frac{1-by}{a}$.

Equating the values obtained $\left(\frac{1-by}{a}\right)^2 = \frac{1-by^2}{a}$

$1 + b^2y^2 - 2by = a - aby^2 \Rightarrow (b^2 + ab)y^2 - 2by + 1 - a = 0$

Values of x will be equal if values of y are equal i.e. discriminant of above equation is zero.

$\Rightarrow 42b^2 - 4(b^2 + ab)(1-a) = 0 \Rightarrow 4b^2 - 4b^2 + 4b^2a - 4ab + 4a^2b = 0$

$(a^2b + ab^2 - ab) = 0 \Rightarrow ab(a+b) = ab \Rightarrow a + b = 1$.

268. Substituting $y = mx + c$ in $x^2 + y^2 = a^2$, we get $x^2 + m^2x^2 + 2cmx + c^2 - a^2 = 0$

For roots to be equal, discriminant must be zero. $D = 4c^2m^2 - 4(1+m^2)(c^2 - a^2) = 0$

$\Rightarrow c^2m^2 - c^2 + a^2 - c^2m^2 + a^2m^2 = 0 \Rightarrow c^2 = a^2(1+m^2)$.

269. Clearly, roots are $\alpha, \alpha+1$. Sum of roots $= \alpha + \alpha + 1 = \frac{5a+1}{4} \Rightarrow \alpha = \frac{5a-3}{8}$.

Product of roots $= \alpha(\alpha+1) = \frac{5a}{4}$. Substituting value of α from above

$\left(\frac{5a-3}{8}\right)^2 + \frac{5a-3}{8} = \frac{5a}{4} \Rightarrow \frac{25a^2-30a+9+40a-24-80a}{64} = 0$

$\Rightarrow 25a^2 - 70a - 15 = 0 \Rightarrow 5a^2 - 14a - 3 = 0 \Rightarrow a = 3, -\frac{1}{5}$.

If $a = 3 \Rightarrow \alpha = \frac{3}{2}$ else if $a = -\frac{1}{5} \Rightarrow \alpha = -\frac{1}{2}$.

Now it is trivial to calculate the value of β.

270. Let one of the roots is α then second root is $\frac{1}{\alpha}$.

Product of roots $= \alpha * \frac{1}{\alpha} = \frac{k}{5} \Rightarrow k = 5$.

271. (a) The equation is :math:$(5+4m)x^2 - (4+2m)x + 2 - m = 0$

For roots to be equal discriminant has to be zero.

$4(2+m)^2 - 4(5+4m)(2-m) = 0 \Rightarrow 4 + 4m + m^2 - 10 - 3m + 4m^2 = 0$

$5m^2 - m - 6 = 0 \Rightarrow m = 1, -\frac{6}{5}$

(b) Product of roots $= \frac{2-m}{5+4m} = 2 \Rightarrow 2 - m = 10 + 8m \Rightarrow -\frac{8}{9}$

(c) Sum of roots $= \frac{4+2m}{5+4m} = 6 \Rightarrow m = -\frac{13}{11}$

272. Let one root be α then the second root is $n\alpha$.

Sum of roots $(n+1)\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{(n+1)a}$

Product of roots $n\alpha^2 = \frac{c}{a}$

Substituting value of α from the earlier equation

$\frac{nb^2}{(n+1)^2a^2} = \frac{c}{a} \Rightarrow (n+1)^2ca = nb^2.$

273. Following from previous problem $n = \frac{3}{4}$ and substituting in final solution

$\left(\frac{3}{4}+1\right)^2 ca = \frac{3}{4}b^2 \Rightarrow 12b^2 = 49ac.$

274. From earlier problem, we have $a = 4, b = a, c = 3$ and $n = \frac{1}{2}$

Substituting in the final relation we have, $\frac{9}{4}.3.4 = \frac{1}{2}a^2 \Rightarrow a^2 = 54.$

Discriminant of the second equation, $D = 9 - 4(a^2 - 2a) < 0$, and thus roots are imaginary.

275. Let α, β be the roots of the given equation.

Sum of roots, $\alpha + \beta = p$ and product of the roots $\alpha\beta = q$

Given, $\alpha + \beta = m(\alpha - \beta)$. Squaring, $(\alpha + \beta)^2 = m^2(\alpha - \beta)^2$

$p^2 = m^2(\alpha + \beta)^2 - 4m^2\alpha\beta = m^2p^2 - 4m^2q \Rightarrow p^2(m^2 - 1) = 4m^2q.$

276. Let α, β be the roots of the given equation. Sum of roots, $\alpha + \beta = p$ and product of the roots $\alpha\beta = q$

Given, $\alpha - \beta = 1$. Squaring we have,

$\Rightarrow (\alpha - \beta)^2 = 1 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1 \Rightarrow p^2 - 4q = 1$. Also, $[(\alpha - \beta)^2 + 2\alpha\beta]^2 = (1 + 2q)^2$

$\Rightarrow (\alpha^2 + \beta^2)^2 = \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = \alpha^4 + \beta^4 - 2\alpha^2\beta^2 + 4\alpha^2\beta^2 = (\alpha^2 - \beta^2)^2 + 4q^2$

$\Rightarrow [(\alpha + \beta)^2(\alpha - \beta)^2] + 4q^2 = p^2 + 4q^2.$

277. The given equation is $a(x - b) + b(x - a) = m(x - a)(x - b) \Rightarrow mx^2 - xm(a + b) - mab - ax + ab - bx + ab = 0$

$\Rightarrow mx^2 - x(m + 1)(a + b) - ab(m - 2) = 0$. If roots are equal in magnitude but opposite in sign then sum would be zero.

$\Rightarrow (m + 1)(a + b) = 0 \Rightarrow m = -1$ or $a + b = 0$.

278. Let α, β be the roots of the equation.

Sum of roots, $\alpha + \beta = -\frac{b}{a}$ and product of roots, $\alpha\beta = \frac{c}{a}$.

Difference of roots, $\alpha - \beta = k$ as given.

Squaring we get, $(\alpha - \beta)^2 = k^2 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = k^2$

$\frac{b^2}{a^2} - 4\frac{c}{a} = k^2 \Rightarrow b^2 - 4ac = k^2a^2$.

279. Let α be one of the roots of the equation $ax^2 + bx + c = 0$. Clearly, α^2 will be the other root.

Sum of roots, $\alpha + \alpha^2 = -\frac{b}{a}$ and product of the roots $\alpha^3 = \frac{c}{a}$. Cubing sum of roots,

$\frac{b^3}{a^3} = -\alpha^3(\alpha + 1)^3 = -\frac{c}{a}(\alpha^3 + 3\alpha(\alpha + 1) + 1)$

$\frac{b^3}{a^3} = -\frac{c}{a}\left(\frac{c}{a} - \frac{3b}{a} + 1\right)$

Simplifying we get the desired relationship.

280. Let α be one of the roots of the equation $ax^2 + bx + c = 0$. Clearly, α^2 will be the other root.

Sum of roots, $\alpha + \alpha^2 = -p$ and product of roots $\alpha^3 = 1$.

Thus, α is cube root of unity. If $\alpha = -1$ then $p = -2$

else if it is one of the complex numbers then we know that $1 + \omega + \omega^2 = 0$ which makes $p = 1$.

281. Let α be one of the roots of the equation $ax^2 + bx + c = 0$. Clearly, α^2 will be the other root.

Sum of roots, $\alpha + \alpha^2 = -p$ and product of roots $\alpha^3 = q$

$p^3 = -\alpha^3(\alpha + 1)^3 = -q(\alpha^3 + 3\alpha(\alpha + 1) + 1) = -q(q - 3p + 1)$

$\Rightarrow p^3 - q(3p - 1) + q^2 = 0$.

282. The solution is given below:

i. $\alpha + \beta = -\frac{3}{2}$ and $\alpha\beta = \frac{4}{2} = 2$.

$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} - 4 = -\frac{7}{4}$.

ii. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

Substituting for numerator from previous part,

$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = -\frac{7}{8}$.

283. Sum of roots, $\alpha+\beta=-\frac{b}{a}$ and product of roots, $\alpha\beta=\frac{c}{a}$

$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{-\frac{b^3}{c^3}+\frac{3c}{a}\frac{b}{a}}{\frac{c}{a}}=\frac{3abc-b^3}{a^2c}.$

284. Sum of roots, $\alpha+\beta=-\frac{b}{a}$ and product of roots, $\alpha\beta=\frac{b}{a}$

Given expression is, $\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{b}{a}}=\frac{\alpha+\beta}{\sqrt{\alpha\beta}}+\sqrt{\frac{b}{a}}=\frac{-\frac{b}{a}}{\sqrt{\frac{b}{a}}}+\sqrt{\frac{b}{a}}=0.$

285. Product of the roots of the first equation is b^2 and sum of roots of the second equation is $2b$.

Geometric mean of the roots of the first equation = square root of product of roots = $\sqrt{b^2}=b$.

Arithmetic mean of the roots of the second equation = half of sum of roots = $\frac{2b}{2}=b$ and thus both are equal.

286. Let α,β be the roots of the equation.

Sum of roots, $\alpha+\beta=-\frac{q}{p}$ and product of roots, $\alpha\beta=\frac{r}{p}$.

Given, sum of roots is equal to sum of square of roots. $\therefore\alpha+\beta=\alpha^2+\beta^2$

$-\frac{q}{p}=(\alpha+\beta)^2-2\alpha\beta=\frac{q^2}{p^2}-\frac{2r}{p}\Rightarrow 2pr=pq+q^2.$

287. Let α,β be the roots of the equation. Sum of roots, $\alpha+\beta=p$ and product of roots, $\alpha\beta=q$.

$\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{\alpha^4+\beta^4}{(\alpha\beta)^2}=\frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}=\frac{[(\alpha+\beta)^2-2\alpha\beta]^2}{\alpha^2\beta^2}-2$

$=\frac{(p^2-2q)^2}{q^2}-2=\frac{p^4}{q^2}-\frac{4p^2}{q}+2.$

288. Let α,β be the roots of the equation. Sum of roots, $\alpha+\beta=-\frac{b}{a}$ and product of roots, $\alpha\beta=\frac{c}{a}$

$\Rightarrow\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}=\frac{(a\alpha+b)^2+(a\beta+b)^2}{[(a\alpha+b)(a\beta+b)]^2}$

$\Rightarrow\frac{a(\alpha^2+\beta^2)+2ab(\alpha+\beta)+2b^2}{(a^2\alpha\beta+2ab(\alpha+\beta)+b^2)^2}$

Substituting for sum of roots, product of roots and $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ and simplifying

$=\frac{b^2-2ac}{c^2a^2}.$

289. Rewriting the equation we have $\lambda x^2 + x(1-\lambda) + 5 = 0$.

Since α and β are the roots therefore, we have $\alpha + \beta = \frac{\lambda - 1}{\lambda}$ and $\alpha\beta = \frac{5}{\lambda}$.

Given, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$

$\frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} \Rightarrow \frac{(\lambda-1)^2-10\lambda}{5\lambda} = \frac{4}{5}$

$\Rightarrow (\lambda-1)^2 - 10\lambda = 4\lambda \Rightarrow \lambda^2 - 16\lambda + 1 = 0 \therefore \lambda_1 + \lambda_2 = 16$ and $\lambda_1\lambda_2 = 1$.

i. $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = \frac{(\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2}$

   Substituting the values for sum and product we have, result as 254.

ii. $\frac{\lambda_1^2}{\lambda_2} + \frac{\lambda_2^2}{\lambda_1} = \frac{\lambda_1^3+\lambda_2^3}{\lambda_1\lambda_2} = \frac{(\lambda_1+\lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}$

   $= 4048$.

290. For the first equation $\alpha + \beta = -p$ and $\alpha\beta = q$ and similarly for the second $\gamma + \delta = -r$ and $\gamma\delta = s$.

i. $(\alpha+\gamma)(\alpha+\delta)(\beta+\gamma)(\beta+\delta) = [\alpha^2 + \alpha(\gamma+\delta) + \gamma\delta][\beta^2 + \beta(\gamma+\delta) + \gamma\delta]$

   $= (\alpha^2 - r\alpha + s)(\beta^2 - r\beta + s) = (\alpha^2\beta^2 - r\alpha\beta^2 + s\beta^2 - r\alpha^2\beta - r^2\alpha\beta - rs\beta + s\alpha^2 - rs\alpha + s^2)$

   $= q^2 - r\alpha\beta(\alpha+\beta) + s(\alpha^2+\beta^2) + r^2p - rs(\alpha+\beta) + s^2 = q^2 + prs + s(p^2-2q) + r^2p - rsq + s^2$

ii. $(\alpha-\gamma)(\beta-\delta) + (\beta-\gamma)(\alpha-\delta) = \alpha\beta - \alpha\delta - \beta\gamma + \gamma\delta + \alpha\beta - \beta\delta - \alpha\gamma + \gamma\delta$

   $= 2\alpha\beta + 2\gamma\delta - (\alpha+\beta)(\gamma+\delta) = 2q + 2s - pr$.

iii. $(\alpha-\gamma)^2 + (\beta-\delta)^2 + (\beta-\gamma)^2 + (\alpha-\delta)^2$

   $= 2(\alpha^2+\beta^2+\delta^2+\gamma^2) - 2(\alpha+\beta)(\gamma+\delta) = 2[(\alpha+\beta)^2 - 2\alpha\beta + (\gamma+\delta)^2 - 2\gamma\delta] - 2(\alpha+\beta)(\gamma+\delta)$

   $= 2[p^2 + r^2 - 2q - 2s] - 2pr$.

291. $\alpha + \beta = p$ and $\alpha\beta = q$

Now, R.H.S. $= (\alpha+\beta)(\alpha^n+\beta^n) - \alpha\beta(\alpha^{n-1}+\beta^{n-1}) = \alpha^{n+1} + \beta^{n+1} =$ L.H.S.

292. $\alpha + \beta = \gamma + \delta = -p, \alpha\beta = -q$ and $\gamma\delta = r$ Also, since α, β are roots of $x^2 + px + q = 0, \therefore \alpha^2 + p\alpha + q = 0$ and $\beta^2 + p\beta + q = 0$.

Now, $(\alpha-\gamma)(\alpha-\delta) = \alpha^2 - \alpha(\gamma+\delta) + \gamma\delta = \alpha^2 + p\alpha - r = -q - r = -(q+r)$, and similarly, $(\beta-\gamma)(\beta-\delta) = -(q+r)$.

293. Clearly, $\alpha+\beta=2p, \alpha\beta=q$ and $\gamma+\delta=2r, \gamma\delta=s$

i. $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$. By componendo and dividendo

$\Rightarrow \frac{\alpha+\beta}{\alpha-\beta}=\frac{\gamma+\delta}{\gamma-\delta}$

Squaring, $\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2=\left(\frac{\gamma+\delta}{\gamma-\delta}\right)^2$

$1-\frac{4\alpha\beta}{(\alpha+\beta)^2}=1-\frac{4\gamma\delta}{(\gamma+\delta)^2} \Rightarrow \frac{q}{p^2}=\frac{s}{r^2}$.

ii. Since $\alpha, \beta, \gamma, \delta$ are in G. P. Hence, $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$ and then we can proceed like previous part.

iii. Since $\alpha, \beta, \gamma, \delta$ are in A. P. Hence, $\alpha-\beta=\gamma-\delta$

$\Rightarrow (\alpha+\beta)^2-4\alpha\beta=(\gamma+\delta)^2-4\gamma\delta \Rightarrow 4p^2-4q=4r^2-4s \Rightarrow s-q=r^2-p^2$.

294. Clearly, $\alpha+\beta=-\frac{2b}{a}$ and $\alpha\beta=\frac{c}{a}$ for $ax^2+2bx+c=0$ and $\alpha+\beta+2k=-\frac{2B}{A}$ and $(\alpha+k)(\beta+k)=\frac{C}{A}$ for $AX^2+2Bx+C=0$.

Given expression can be rewritten as $\frac{b^2}{a^2}-\frac{c}{a}=\frac{B^2}{A^2}-\frac{C}{A}$

$\frac{(\alpha+\beta)^2}{4}-\alpha\beta=\frac{(\alpha+\beta+2k)^2}{4}-(\alpha+k)(\beta+k) \Rightarrow (\alpha-\beta)^2=(\alpha+k-\beta-k)^2$, which is true.

295. Proceeding like previous problem, we have to prove that $\frac{b^2-4ac}{B^2-4AC}=\frac{a^2}{A^2} \Rightarrow \frac{b^2}{a^2}-\frac{4c}{a}=\frac{B^2}{A^2}-\frac{4C}{A} \Rightarrow (\alpha+\beta)^2-4\alpha\beta=(\alpha+\beta+2k)^2-4(\alpha+k)(\beta+k)$

$\Rightarrow (\alpha-\beta)^2=(\alpha+k-\beta-k)^2$, which is true.

296. Let α, β be the roots of $x^2+2px+q=0$ and γ, δ be the roots of $x^2+2qx+p=0$

$\alpha+\beta=-2p$ and $\gamma+\delta=-2q$. Also, $\alpha\beta=q$ and $\gamma\delta=p$

Given that roots differ by a constant term say k. $\therefore \alpha+k=\gamma$ and $\beta+k=\delta$

Thus, $\alpha+\beta+2k=-2q \Rightarrow -2p+2k=-2q \Rightarrow k=p-q \Rightarrow \gamma\delta=\alpha\beta+(\alpha+\beta)k+k^2=p$

Also, $q-2pk+k^2=p \Rightarrow -2p+k=1 \Rightarrow p+q+1=0$.

297. Clearly, $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

i. Sum of these roots is $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{b^2-2ac}{ac}$

Product of these roots is 1. Therefore, such an equation is $x^2-\frac{b^2-2ac}{ac}x+1=0$.

ii. Sum of these roots is $\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{3abc-b^3}{a^2c}$.

Product of these roots is $\alpha\beta = \frac{c}{a}$. Therefore, an equation whose roots were these is $x^2 - \frac{3abc-b^3}{a^2c}x + \frac{c}{a} = 0$.

iii. Sum of these roots is $(\alpha+\beta)^2 + (\alpha-\beta)^2 = 2(\alpha+\beta)^2 - 4\alpha\beta = \frac{2b^2}{a^2} - \frac{4c}{a}$.

Product of these roots is $(\alpha+\beta)^2(\alpha-\beta)^2 = (\alpha+\beta)^2[(\alpha+\beta)^2 - 4\alpha\beta] = \frac{b^2}{a^2}\left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$.

So the equation is $x^2 - \left(\frac{2b^2}{a^2} - \frac{4c}{a}\right)x + \frac{b^2}{a^2}\left(\frac{b^2}{a^2} - \frac{4c}{a}\right) = 0$.

iv. Sum of these roots is $\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} = \frac{1+\beta-\alpha-\alpha\beta+1+\alpha-\beta-\alpha\beta}{1+(\alpha+\beta)+\alpha\beta}$

$= \frac{2-2\alpha\beta}{1+(\alpha+\beta)+\alpha\beta} = \frac{2\left(1+\frac{b}{a}\right)}{1-\frac{b}{a}+\frac{c}{a}} = \frac{2(a+b)}{a-b+c}$.

Product of these roots is $\frac{1-\alpha}{1+\alpha}.\frac{1-\beta}{1+\beta} = \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta} = \frac{1+\frac{b}{a}+\frac{c}{a}}{1-\frac{b}{a}+\frac{c}{a}} = \frac{a+b+c}{a-b+c}$.

Therefore, the equation is $(a-b+x)x^2 - 2(a+b)x + (a+b+c) = 0$.

v. Sum of these roots is $\frac{1}{(\alpha+\beta)^2} + (\alpha-\beta)^2 = \frac{a^2}{b^2} + [(\alpha+\beta)^2 - 4\alpha\beta] = \frac{a^2}{b^2} + \left[\frac{b^2}{a^2} - \frac{4c}{a}\right]$.

Product of these roots is $\frac{1}{(\alpha+\beta)^2}.(\alpha-\beta)^2 = \frac{1}{(\alpha+\beta)^2}.[(\alpha+\beta)^2 - 4\alpha\beta] = \frac{a^2}{b^2}\left[\frac{b^2}{a^2} - \frac{4c}{a}\right] = \frac{b^2-4ac}{b^2}$.

Now it is trivial to deduce the equation.

298. Let the roots of the equation $ax^2 + bx + c = 0$ are p and q, then $p + q = -\frac{b}{a}$ and $pq = \frac{c}{a}$.

(a) The reciprocal of roots are $\frac{1}{p}$ and $\frac{1}{q}$. Sum of these is $\frac{p+q}{pq} = -\frac{b}{c}$ and product is $\frac{1}{pq} = \frac{a}{c}$. Therefore, the equation is $cx^2 + bx + a = 0$.

(b) Let one of the roots is p then the other will be $-p$. Sum will be 0 and product will be $-\frac{c}{a}$. Therefore, the equation is $ax^2 - c = 0$.

299. Clearly, $\alpha + \beta = -p$ and $\alpha\beta = q$.

(a) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2) - 2\alpha^2\beta^2 = [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2 = [p^2 - 2q]^2 - 2q^2 = p^4 - 4p^2q + 2q^2$.

(b) $\alpha^{-4} + \beta^{-4} = \frac{\alpha^4+\beta^4}{\alpha^4\beta^4} = \frac{p^4-4p^2q+2q^2}{q^4}$.

300. Clearly, $\alpha + \beta = p$ and $\alpha\beta = q$.

i. Sum of these roots is $\frac{q}{p-\alpha}+\frac{q}{p-\beta}=\frac{2pq-q(\alpha+\beta)}{p^2-p(\alpha+\beta)+\alpha\beta}=\frac{pq}{q}=p.$

Product of these roots is $\frac{q}{p-\alpha}.\frac{q}{p-\beta}=\frac{q^2}{q}=q.$

Thus the eqation of these new roots remain same i.e. $x^2-px+q=0.$

ii. Sum of these roots is $\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}=\alpha+\beta+\frac{\alpha+\beta}{\alpha\beta}=p+\frac{p}{q}=\frac{p(1+q)}{q}.$

Product of these roots is $\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)=\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}=q+\frac{1}{q}+\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{q^2+1}{q}+\frac{p^2-2q}{q}.$

Now deducing the equation is trivial.

301. Because $5+3i$ is a complex root the other root will be complex conjugate i.e. $5-3i$. Thus, equation having these complex roots will be $x^2-10x+34=0.$

302. Because $3+4i$ is a complex root the other root will be complex conjugate i.e. $3-4i$. Thus, equation having these complex roots will be $x^2-6x+25=0.$

303. Roots are given by $\frac{-2\pm\sqrt{4+16}}{6}=\frac{-1\pm\sqrt{5}}{4}$. Now $\frac{\sqrt{5}-1}{4}=\cos 72^\circ$ and $-\frac{\sqrt{5}+1}{4}=-\cos 36^\circ=\cos 216^\circ=\cos(3.72^\circ)$

Now, $\cos 3x=4\cos^3 x-3\cos x$, therefore if one root is α then the other would be $4\alpha^3-3\alpha$.

304. Clearly, by observation α,β are roots of the eqation $x^2-5x+3=0.\Rightarrow\alpha+\beta=5$ and $\alpha\beta=3$.

Now, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{5(\alpha+\beta)-6}{3}=\frac{19}{3}.$

305. Correct value of $p=-11$. q is $4\times 6=24$. Hence, the correct equation is $x^2-11x+24=0$. Hence roots are $8,3$.

306. Correct value of q is 2. p is $-(6-1)=5$. Hence, the correct equation is $x^2-5x+2=0.$

307. From first student the correct value of $q=6\times 2=12$. From second student the correct value of $p=-(2+-9)=7$. Hence the correct equation is $x^2+7x+12=0$ giving us $3,4$ as correct roots.

308. We have $\alpha+\beta=-p, \alpha\beta=q, \alpha_1+\beta_1=p, \alpha_1\beta_1=q.$

Now, $\frac{1}{\alpha_1\beta}+\frac{1}{\alpha\beta_1}+\frac{\alpha\alpha_1}{+}\beta\beta_1=\frac{(\alpha+\beta)(\alpha_1+\beta_1)}{\alpha\beta\alpha_1\beta_1}=\frac{pq}{qp}=1$

and $\left(\frac{1}{\alpha_1\beta}+\frac{1}{\alpha\beta_1}\right)\left(\frac{1}{\alpha\alpha_1}+\frac{1}{\beta\beta_1}\right)=\frac{1}{\alpha_1^2\alpha\beta}+\frac{1}{\alpha_1\beta_1\beta^2}+\frac{1}{\alpha_1\beta_1\alpha^2}+\frac{1}{\alpha\beta\beta_1^2}$

$=\frac{1}{\alpha\beta}\left[\frac{1}{\alpha_1^2}+\frac{1}{\beta_1^2}\right]+\frac{1}{\alpha_1\beta_1}\left[\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right]=\frac{1}{q}\left[\frac{\alpha_1^1+\beta_1^2}{\alpha_1^2\beta_1^2}\right]+\frac{1}{p}\left[\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right]$

$=\frac{p^3+q^3-pq}{p^2q^2}$. Therefore, the equation with these as roots is

$x^2 - x + \frac{p^3+q^3-pq}{p^2q^2} = 0.$

309. We know that complex roots always appear in pair and as $2+\sqrt{3}i$ is a complex root the other root will be its complex conjugate i.e. $2-\sqrt{3}i$. Hence, $p=-4$ and $q=13$ makring the equation $x^2-4x+13=0$.

310. $\frac{1}{2+\sqrt{3}} = 2-\sqrt{3}$ which is an irrational root and the other root will be its conjugate i.e. $2+\sqrt{3}$ hence the equation will be $x^2-4x+1=0$

311. Since α, β are roots of the equation $x^2-px+q=0$, $\alpha+\beta=p$ and $\alpha\beta=q$.

Let us assume that $\alpha+\frac{1}{\beta}$ is a root of $qx^2-p(1+q)x+(1+q)^2=0$ then it must satisfy the equation. Substituting the values we have

$\alpha\beta\frac{(\alpha\beta+1)^2}{\beta^2} - \frac{(\alpha+\beta)(1+\alpha\beta)(\alpha\beta+1)}{\beta} + (1+\alpha\beta)^2 = 0$

$(\alpha\beta+1)^2[\alpha\beta-(\alpha+\beta)\beta-\beta^2]=0$

$\because$ L.H.S. = R.H.S. it is proven that $\alpha+\frac{1}{\beta}$ is a root of the given equation.

312. One of the given equations is $2x^2+3x-2=0 \Rightarrow (2x-1)(x+2)=0$ so the roots are $x=\frac{1}{2}, -2$. Putting these two in the equation $3x^2+4mx+2=0$ we obtain two values $-\frac{7}{4}, -\frac{11}{8}$ for m.

313. Let p be the common root then it must satisfy both the equations i.e. $p^2-11p+a=0$ and $p^2-14p+2a=0$. Equating a from both equations $11p-p^2=\frac{14p-p^2}{2} \Rightarrow p^2-8p=0 \Rightarrow p=0,8 \Rightarrow a=0,24$.

314. The condition for having common roots is obtained by cross-multiplication:

$(ba-c^2)(ca-b^2)=(a^2-bc)^2 \Rightarrow a^2bc-ab^3-ac^3+b^2c^2=a^4-2a^2bc+b^2c^2 \Rightarrow 3a^2bc-ab^3-ac^3-a^4=0$

$a(3abc-b^3-c^3-a^3)=0 \because a \neq= 0 \Rightarrow a^3+b^3+c^3-3abc=0 \Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$

$\Rightarrow a+b+c=0$ or $a=b=c$.

315. Proceeding as in last example, condition for common root is

$(10m-189)(9-10)=(21-m)^2 \Rightarrow 189-10m=441-42m+m^2 \Rightarrow m^2-32m+252=0 \Rightarrow m=18,14$.

Roots of $x^2+10x+21=0$ are $-3,-7$. When $m=18$ roots of $x^2+9x+18=0$ are $-3,-6$.

In that case equation formed with -7 and -6 is $x^2+13x+42=0$ When $m=14$ roots of $x^2+9x+14=0$ are $-2,-7$.

In that case equation formed with -3 and -2 are $x^2+5x+6=0$.

316. Following condition for common roots, we have

$(-3+120)(10+3)=(3+36)^2 \Rightarrow 117*13=39^2$ which is true and thus equations have a common root.

Roots of $x^2-x-12=0$ are $4,-3$ and roots of $3x^2+10x+3=0$ are $-3,-\frac{1}{3}$ and thus common root is -3.

317. Condition for common root is given below:

$(p-q)(3q-2p)=(3-2)^2 \Rightarrow (2p-3q)(p-q)+1=0 \Rightarrow 2p^2+3q^2-5pq+1=0.$

318. The condition for common root is $(b-c)(a-b)=(a-c)^2$

$\Rightarrow ab-ac-b^2+bc=a^2+c^2-2ac \Rightarrow a^2+b^2+c^2-ab-ac-bc=0$

$\Rightarrow \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2=0 \Rightarrow a=b=c.$

319. Let α be the common root then

$\frac{\alpha^2}{pq_1-p_1q}=\frac{\alpha}{q-q_1}=\frac{1}{p_1-p}$. Clearly, the root is either $\frac{pq_1-p_1q}{q-q_1}$ or $\frac{q-q_1}{p_1-p}$.

320. Condition for having common root is:

$(-4b+3c)(-6a-2b)=(4a-2c)^2$. Solving this gives us required equation.

321. Condition for having a common root is:

$[(r-p)(q-r)-(p-q)^2][(p-q)(q-r)-(r-p)^2]=[(q-r)^2-(p-q)(r-p)]^2$, which is an equality and hence the equations have a common root.

322. Let α be a common root then

$\frac{\alpha^2}{ab^2-ac^2}=\frac{1}{b-c}=\frac{1}{ac-ab} \Rightarrow \alpha=-a(b+c)$ or $\alpha=-\frac{1}{a}$.

Let α,β be roots of first and α,γ be roots of the second equation. Then, $\alpha+\beta=-ab$ and $\alpha\beta=c$ also, $\alpha+\gamma=-ac$ and $\alpha\gamma=b$

$\Rightarrow 2\alpha+\beta+\gamma=-a(b+c)$ and $\alpha^2\beta\gamma=bc$

Equation formed by β and γ would be $x^2-(\beta+\gamma)x+\beta\gamma=0$.

For either values of α equation is $x^2-a(b+c)x+a^2bc=0$.

323. Let α is a common root then $x^2-px+q=0$ and $x^2-ax+b=0$. Let β be the second root of the first equationa then $\frac{1}{\beta}$ will be the second root of the second equation.

Clearly, $\alpha+\beta=p, \alpha\beta=q, \alpha+\frac{1}{\beta}=a, \frac{\alpha}{\beta}=b$.

$\therefore (q-b)^2 = (\alpha\beta - \frac{\alpha}{\beta})^2$,

$bq(p-a)^2 = \frac{\alpha}{\beta}(\alpha\beta)(\beta - \frac{1}{\beta})^2 = (\alpha\beta - \frac{\alpha}{\beta})^2$. Hence, proved.

324. It is a quadratic equation but satisfied by three values of $x = 1, 2, 3$ therefore it is an identity.

325. It is a quadratic equation but satisfied by three values of $x = a, b, c$ therefore it is an identity.

326. Let $x^5 = y$ then equation becomes $3y^2 - 2y - 8 = 0$.

Since it is satisfied by two distinct values and it is a quadratic equation therefore it is an equation.

327. $\frac{(x+2)^2-(x-2)^2}{x^2-4} = \frac{5}{6}$

$\Rightarrow \frac{8x}{x^2-4} = \frac{5}{6} \Rightarrow 5x^2 - 20 - 48x = 0 \Rightarrow x = 10, -\frac{2}{5}$.

328. Let $x = y^2 \Rightarrow \frac{2y+1}{3-y} = \frac{11-3y}{5y-9}$

$\Rightarrow 10y^2 - 13y - 9 = 33 - 20y + 3y^2 \Rightarrow 7y^2 + 7y - 42 = 0 \Rightarrow y = 2, -3$

$\Rightarrow x = 4, 9$ but $x = 9$ does not apply to the equation and is an impossible solution.

329. $(x+1)(x-3)(x+2)(x-4) = 336 \Rightarrow (x^2-2x-3)(x^2-2x-8) = 336$

Let $x^2 - 2x - 3 = y \Rightarrow y(y-5) = 336 \Rightarrow ey^2 - 5y - 336 = 0 \Rightarrow y = 21, -16$

$\Rightarrow x = -4, 6, 1 \pm 2\sqrt{3}i$.

330. Squaring $x + 1 + 2x - 5 + 2\sqrt{(x+1)(2x-5)} = 9 \Rightarrow 2\sqrt{(x+1)(2x-5)} = 13 - 3x$

Squaring again $4(x+1)(2x-5) = 9x^2 - 78x + 169 \Rightarrow x^2 - 66x + 189 = 0 \Rightarrow x = 3, 63$.

We see that $x = 63$ does not satisfy the equation hence the only solution is $x = 3$.

331. We have $2^{2x} + 2^{x+2} - 32 = 0 \Rightarrow (2^x - 4)(2^x + 8) = 0$. However, $2^x \neq 8 \Rightarrow 2^x = 4 \Rightarrow x = 2$.

332. Let the speed be x km/hour. Then, from the statement $\frac{800}{x} = \frac{800}{x+40} + \frac{2}{3}$

Solving we get $x = 200$ km/hour.

333. Let width be w meter. Thus, $(w+8)(w-2) = 119 \Rightarrow w^2 + 6w - 135 = 0 \Rightarrow w = 9, -15$ but width cannot be negative. Length is 11 m.

334. Equivalent equation is $-x^2 + 3x + 4 = 0$ and roots are $-1, 4$.

Since coefficient of x^2 is -ve the expression will be +ve if x lies between the root.

Therefore, for $-x^2 + 3x + 4 > 0$ the range is $]-1, 4[$.

335. $5x - 1 < (x+1)^2 \Rightarrow x^2 - 3x + 2 > 0.$

Roots of equivalent equation $x^2 - 3x + 2 = 0$ are $x = 2, 1.$

Since coefficient of x^2 is positive, x must lie outside the range of $[1, 2]$ for the expression to be positive.

Now considering, $(x+1)^2 < 7x - 3 \Rightarrow x^2 - 5x + 4 < 0$

Roots of the equivalent equation $x^2 - 5x + 4 = 0$ are $x = 1, 4$ and for expression to be negative x must lie inside the open interval $]1, 4[$.

Therefore, the only integral value satisfying the original expression is 3.

336. $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3 \Rightarrow \frac{2x^2+x-15}{2x^2+5x-12} > 0$

$2x^2 + x - 15 = 0$ has roots $x = -3, \frac{5}{2} \Rightarrow 2x^2 + 5x - 12 = 0$ has roots $x = -4, \frac{3}{2}$

Thus, the inequality will hold true for $x < -4$ and $-3 < x < \frac{3}{2}$ and $x > \frac{5}{2}$.

337. Let $y = \frac{x^2-3x+4}{x^2+3x+4} \Rightarrow (y-1)x^2 + 3(y+1)x + 4(y-1) = 0$

Since x is real, the discriminant will be greater that $0 \Rightarrow 9(y+1)^2 - 16(y-1)^2 \geq 0$

$-7y^2 + 50y - 7 \geq 0$. The roots are 7 and $\frac{1}{7}$

Since coefficient of y^2 is negative, for the expression to be positive y has to lie between the open interval formed by its roots i.e. $]\frac{1}{7}, 7[$

338. Let $y = \frac{x^2+34x-71}{x^2+2x-7} \Rightarrow (y-1)x^2 + 2(y-17)x + (71-y) = 0$

Since x is real, the discriminant will be greater that $0 \Rightarrow 4(y-17)^2 - 4(y-1)(71 - 7y) \geq 0$

$\Rightarrow y^2 - 14y + 45 \geq 0$. Its roots are 5 and 9

Since coefficient of y^2 is positive, therefore for the expression to be positive y has to lie outside the open interval formed by its roots. Thus, the expression has no value between 5 and 9.

339. Let $y = \frac{4x^2+36x+9}{12x^2+8x+1} \Rightarrow 4(3y-1)x^2 + 4(2y-9)x + y - 9 = 0.$

Since x is real, the discriminant will be greater that $0 \Rightarrow 16(2y-9)^2 - 16(3y-1)(y - 1) \geq 0 \Rightarrow y^2 - 8y + 72 \geq 0$

Corresponding equation is $y^2 - 8y + 72 = 0 \Rightarrow D = 64 - 288 = -224 < 0$

Since coefficient of y^2 is positive and discriminant is less than 0 therefore $y^2 - 8y + 72 \geq 0$ holds true for all value of y. Therefore, the expression can take any value.

340. Let $y = \frac{(x-a)(x-c)}{x-b} \Rightarrow x^2 - (a+c+y)x + ac + yb = 0$

Since x is real, the discriminant will be greater that 0

$\Rightarrow (a+c+y)^2 - 4(ac+yb) \geq 0 \Rightarrow y^2 + 2(a+c-2b)y + (a-c)^2 \geq 0.$

Corresponding equation is $y^2 + 2(a+c-2b)y + (a-c)^2 = 0$. Discriminant of above equation is $D = -16(a-b)(b-c)$

If $a > b > c$ then $D < 0$ and if $a < b < c$ then also $D < 0$.

Since coefficient of y^2 is positive and $D < 0$ the expression $y^2 + 2(a+c-2b)y + (a-c)^2 \geq 0$ is true for all real values of y.

Therefore, the given expression is capable of holding any value for the given conditions.

341. Given $x + y = k$ (say, a constant). Let $z = xy$, then $z = x(k-x) \Rightarrow x^2 - kx + z = 0$.

Since x is real, $D \geq 0$ for the above equation.

$k^2 - 4z \geq 0 \Rightarrow z \leq \frac{k^2}{4}$

Hence, the maximum value of $z = \frac{k^2}{4}$.

Thus, $x^2 - kx + \frac{k^2}{4} = 0 \Rightarrow \left(x - \frac{k}{2}\right)^2 = 0 \Rightarrow x = \frac{k}{2}$.

$\therefore y = \frac{k}{2}$ and thus xy is maximum when $x = y$.

342. Let $y = 3 - 6x - 8x^2 \Rightarrow 8x^2 + 6x + y - 3 = 0$. Since x is real, $D \geq 0$ for the this equation.

$\Rightarrow 36 - 32(y-3) \geq 0 \Rightarrow y \leq \frac{33}{8}$. Hence, maximum value of $y = \frac{33}{8}$

$\Rightarrow 64x^2 + 48x + 9 = 0 \Rightarrow (8x+3)^2 = 0 \Rightarrow x = -\frac{3}{8}$.

343. Let $y = \frac{12x}{4x^2+9} \Rightarrow 4yx^2 - 12x + 9y = 0$. Since x is real, $D \geq 0$ for the above equation.

$\Rightarrow 144 - 144y^2 \geq 0 \Rightarrow y^2 \leq 1 \Rightarrow -1 \leq y \leq 1 \Leftrightarrow |y| \leq 1 \Leftrightarrow \left|\frac{12x}{4x^2+9}\right| \leq 1$

Now, $\left|\frac{12x}{4x^2+9}\right| = 1 \Leftrightarrow 4|x|^2 - 12|x| + 9 = 0 \Rightarrow (2|x|-3)^2 = 0 \Rightarrow |x| = \frac{3}{2}$.

344. $x^2 + 9y^2 - 4x + 3 = 0$. Since x is real, $D \geq 0$ for the above equation.

$\Rightarrow (-4)^2 - 4(9y^2+3) \geq 0 \Rightarrow 9y^2 - 1 \leq 0 \Leftrightarrow y^2 \leq \frac{1}{9} \Rightarrow -\frac{1}{3} \leq y \leq \frac{1}{3}$

The given equation can also be written as $9y^2 + x^2 - 4x + 3 = 0$. Since y is real, $D \geq 0$ for the above equation.

$\Rightarrow -36(x^2-4x+3) \geq 0 \Rightarrow x^2-4x+3 \leq 0$

Since coefficient of x^2 is positive, it must lie between its root for the above expression to be negative. Therefore, x must lie between 1 and 3.

345. Given expression is $x^2-ax+1-2a^2>0$

Since x is real the discriminant of the corresponding equation has to be negative for it to be positive for all values of x.

$a^2-4(1-2a^2)<0 \Leftrightarrow 9a^2 \leq 4 \Rightarrow -\frac{2}{3}<a<\frac{2}{3}$.

346. Let α be a common factor, therefore it will satisfy both the equations.

$\alpha^2-11\alpha+a=0$ and $\alpha^2-14\alpha+2a=0$. By cross-multiplication

$\frac{\alpha^2}{-22a+14x}=\frac{\alpha}{a-2a}=\frac{1}{-14+11} \Rightarrow \frac{\alpha^2}{-8a}=\frac{\alpha}{-a}=-\frac{1}{3}$

From first two we have $\alpha=8$ and from last two we have $\alpha=\frac{a}{3} \therefore a=24$.

347. $y=mx$ is a factor of $ax^2+bxy+cy^2$ means $ax^2+bxy+cy^2$ will be zero when $y=mx$.

$ax^2+bx.mx+cm^2x^2=0 \Rightarrow cm^2+bm+a=0$. Similarly, $a_1m^2+b_1m+c_1=0$ since $my-x$ is a factor of $a_1x^2+b_1xy+c_1y^2$

Solving these two equations in m by cross-multiplication $\frac{m^2}{bc_1-ab_1}=\frac{m}{aa_1-cc_1}=\frac{1}{cb_1-ba_1}$

From first two we get, $m=\frac{bc_1-ab_1}{aa_1-cc_1}$, and from last two we get, $m=\frac{aa_1-cc_1}{cb_1-ba_1}$

Equating the two values of m obtained, we get $(bc_1-ab_1)(cb_1-ba_1)=(aa_1-cc_1)^2$.

348. We know that $ax^2+2hxy+by^2+2gx+2fy+c$ can be resolved into two linear factors if and only if

$abc+2fgh-af^2-bg^2-ch^2=0$ and $h^2-ab>0$. Given expression is $2x^2+mxy+3y^2-5y-2$

Here, $a=2, h=\frac{m}{2}, b=3, g=0, f=\frac{-5}{2}, c=-2 \Rightarrow h^2-ab=\frac{m^2}{4}-6>0 \Rightarrow m^2>24$

Applying the second condition, $-12-\frac{25}{2}+\frac{m^2}{2}=0 \Rightarrow m^2=49 \therefore m=\pm 7$.

349. Given expression is $ax^2+by^2+cz^2+2ayz+2bzx+2cxy$

$=z^2\left[a\left(\frac{x}{z}\right)^2+b\left(\frac{y}{z}\right)^2+c+2a\frac{y}{z}+2b\frac{x}{z}+2c\frac{xy}{z^2}\right]$

$=z^2(aX^2+bY^2+c+2aY+2bX+2cXY)$ where $X=\frac{x}{z}, Y=\frac{y}{z}$. Now this will resolve in linear factors if

$abc+2abc-a.a^2-b.b^2-c.c^2 \Longrightarrow a^3+b^3+c^3=3abc$.

350. Given expression is $2x^2 - y^2 - x + xy + 2y - 1$

Corresponding equation is $2x^2 - y^2 - x + xy + 2y - 1 = 0 \Rightarrow x = \frac{1-y\pm\sqrt{(1-y)^2+8(y^2-2y+1)}}{4} \Rightarrow x = 1-y, -\frac{1-y}{2}$.

Therefore, required linear factors are $x + y - 1$ and $2x - y + 1$.

351. Corresponding quadratic equation is $x^2 + 2(a+b+c)x + 3(ab+bc+ca) = 0$. It will be a perfect square if its discriminant is zero.

$\Rightarrow 4(a+b+c)^2 - 4.3(ab+bc+ca) = 0 \Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2 = 0 \Rightarrow a = b = c.$

352. Discriminant of the given equation is $D = 36 - 72 < 0$.

Now since coefficient of x^2 is less than zero the expression is always positive.

353. $8x - 15 - x^2 > 0 \Rightarrow x^2 - 8x + 15 < 0 \Rightarrow (x-3)(x-5) < 0$.

The above is true if x lies in the open interval $]3, 5[$.

354. $-x^2 + 5x - 4 > 0 \Rightarrow x^2 - 5x + 4 < 0 \Rightarrow (x-4)(x-1) < 0$.

The above is true if x lies in the open interval $]1, 4[$.

355. $x^2 + 6x - 27 > 0 \Rightarrow (x+9)(x-3) > 0$. This is true if $x < -9$ or $x > 3$.

356. $\frac{4x}{x^2+3} \leq 1 \Rightarrow x^2 + 3 \leq 4x \Rightarrow x^2 - 4x + 3 \leq 0$

$\Rightarrow (x-3)(x-1)le0$, This is true for closed interval $[1, 3]$.

357. $x^2 - 3x + 2 > 0 \Rightarrow (x-2)(x-1) > 0$. This is true for $x > 2$ or $x < 1$.

$x^2 - 3x - 4 \leq 0 \Rightarrow (x-4)(x+1) \leq 0$. This is true for $-1 \leq x \leq 4$.

Thus values of x which satisfy both are $-1 \leq x < 1$ and $2 < x \leq 4$.

358. Since roots of $ax^2 + bx + c$ are imaginary, therefore discriminant is negative. $\Rightarrow b^2 - 4ac < 0$.

Discriminant of $a^2x^2 + abx + ac$ is $D = a^2b^2 - 4a^3c = a^2(b^2 - 4ac) < 0$.

But coefficient of the expression is positive hence it will be always positive.

359. Let $y = \frac{x^2-2x+4}{x^2+2x+4} \Rightarrow (y-1)x^2 + 2(y+1)x + 4(y-1) = 0$

Since x is real discriminant will be greater or equal to zero.

$\Rightarrow 4(y+1)^2 - 16(y-1)^2 \geq 0 \Rightarrow y^2 + 2y + 1 - 4y^2 + 8y - 4 \geq 0 \Rightarrow -3y^2 + 10y - 3 \geq 0.$

Roots of corresponding equation are $\frac{1}{3}, 3$. Since coefficient of y^2 is negative, for above to be true y must lie between $\frac{1}{3}$ and 3.

360. Let $y = \frac{2x^2-3x+2}{2x^2+3x+2} \Rightarrow 2(y-1)x^2 + 3(y+1)x + 2(y-1) = 0$

Since x is real discriminant will be greater or equal to zero.

$\Rightarrow 9(y+1)^2 - 16(y-1)^2 \geq 0 \Rightarrow 9y^2 + 18y + 9 - 16y^2 + 32y - 16 \geq 0 \Rightarrow -7y^2 + 50y - 7 \geq 0$

Roots of the corresponding equation are $\frac{1}{7}, 7$. Since coefficients of y^2 is negative, for the above to be true y must lie between $\frac{1}{7}$ and 7.

361. Let $y = \frac{x^2-2x+p^2}{x^2+2x+p^2} \Rightarrow (y-1)x^2 + 2(y+1)x + (y-1)p^2 = 0.$

Since x is real, discriminant of above equation has to be greater or equal to zero.

$\Rightarrow 4(y+1)^2 - 4p^2(y-1)^2 \geq 0 \Rightarrow (1-p^2)y^2 + 2(1+p^2)y + 1 - p^2 \geq 0$

Since $p > 1$ coefficient of y^2 is negative and thus y must lie between its roots for the above to be true.

The roots are $y = \frac{-2(1+p^2) \pm \sqrt{4(1+p^2)^2 - 4(1-p^2)^2}}{2(1-p^2)}$

$y = \frac{p-1}{p+1}, \frac{p+1}{p-1}.$

362. Let $y = \frac{(x-1)(x+3)}{(x-2)(x+4)} \Rightarrow y = \frac{x^2+2x-3}{x^2+2x-8} \Rightarrow (y-1)x^2 + 2(y-1)x^2 + 3 - 8y = 0.$

Since x is real, discriminant must be greater than or equal to 0.

$4(y-1)^2 + 4(y-1)(8y-3) \geq 0 \Rightarrow y^2 - 2y + 1 + 8y^2 - 11y + 3 \geq 0 \Rightarrow 9y^2 - 13y + 4 \geq 0.$

For above to be true y must not lie between 1 and $\frac{4}{9}$.

363. Let $y = \frac{x+a}{x^2+bx+c^2} \Rightarrow yx^2 + (by-1)x - a + c^2y = 0.$

Since x is real, discriminant must be greater than or equal to 0.

$\Rightarrow (by-1)^2 - 4y(c^2y - a) \geq 0 \Rightarrow b^2y^2 - 2by + 1 + 4ay - 4c^2y^2 \geq 0 \Rightarrow (b^2 - 4c^2)y^2 + 2(2a-b)y + 1 \geq 0.$

Discriminant of corresponding equation is $D = 4(2a-b)^2 - 4(b^2 - 4c^2) = 4[4a^2 + b^2 - 4ab - b^2 + 4c^2] = 16(a^2 + c^2 - ab).$

Given $b^2 > 4c^2$ and $a^2 + c^2 > ab$ therefore $D < 0$ and coefficient of y^2 is negative. Therefore, y is capable of assuming any value.

364. Let $y = \frac{x^2-bc}{2x-b-c} \Rightarrow x^2 - 2yx + (b+c)y - bc = 0$

Since x is real, discriminant must be greater than or equal to 0.

$\Rightarrow 4y^2 - 4(b+c)y + 4bc \geq 0 \Rightarrow y^2 - (b+c)y + bc \geq 0.$

For above to be true y must not lie between b and c.

365. Given $x^2 - xy + y^2 - 4x - 4y + 16 = 0 \Rightarrow x^2 - (y+4)x + y^2 - 4y + 16 = 0$

Since x is real, discriminant has to be greater than or equal to 0.

$\Rightarrow (y+4)^2 - 4(y^2 - 4y + 16) \geq 0 \Rightarrow y^2 + 8y + 16 - 4y^2 + 16y - 64 \geq 0$

$\Rightarrow -3y^2 + 24y - 48 \geq 0 \Rightarrow y^2 - 8y + 16 \leq 0 \Rightarrow (y-4)^2 \leq 0$

The above inequality is only satisfied by $y = 4$. However, if $y = 4$ the given equation becomes

$x^2 - 8x + 16 = 0$ which is again only satisfied by $x = 4$.

366. Given $x^2 + 12xy + 4y^2 + 4x + 8y + 20 = 0 \Rightarrow x^2 + 4(1+3y)x + 4(y^2 + 2y + 5) = 0$

Since x is real, discriminant has to be greater than or equal to zero.

$\Rightarrow 16(1+3y)^2 - 16(y^2 + 2y + 5) \geq 0 \Rightarrow 1 + 6y + 9y^2 - y^2 - 2y - 5 \geq 0$

$\Rightarrow 8y^2 + 4y - 4 \geq 0 \Rightarrow 2y^2 + y - 1 \geq 0 \Rightarrow (2y-1)(y+1) \geq 0$

Therefore, y cannot lie between -1 and $\frac{1}{2}$. Rewriting the equation in terms of y

$4y^2 + 4(3x+2)y + x^2 + 4x + 20 = 0.$

Since x is real, discriminant has to be greater than or equal to zero.

$\Rightarrow (3x+2)^2 - x^2 - 4x - 20 \geq 0 \Rightarrow 8x^2 + 8x - 16 \geq 0 \Rightarrow x^2 + x - 2 \geq 0$

Therefore, x cannot lie between -2 and 1.

367. Let x be the length and y be the breadth then $x + 2y = 600$ and we have to maximize xy.

$xy = x\frac{600-x}{2} = z$ (say) $x^2 - 600x + 2z = 0.$

Since x is real, discriminant has to be greater than or equal to zero.

$\Rightarrow 360000 - 8z \geq 0 \Rightarrow z \leq 45000$. Thus, maximum area is 45000 mt. sq.

Substituting, $x^2 - 600x + 90000 = 0 \Rightarrow (x-300)^2 = 0 \Rightarrow x = 300 \Rightarrow y = 150.$

368. If $y - mx$ is a factor then equation reduces to $bm^2 + 2hm + a = 0$ and if $my + x$ is a factor then it reduces to $am^2 - 2hm + b = 0$. By cross-multiplication we have

$\frac{m^2}{-2h(a+b)} = \frac{m}{a^2-b^2} = \frac{1}{2h(a+b)}$. Thus, condition becomes $a + b = 0$ or $4h^2 + (a^2 - b^2) = 0.$

369. Roots of equation $P(x)Q(x) = 0$ will be the roots of equation $P(x) = 0$ i.e. $ax^2 + bx + c = 0$ and $Q(x) = -ax^2 + bx + c = 0$

Let D_1 and D_2 be the discriminants of two equations, then $D_1 + D_2 = b^2 - 4ac + b^2 + 4ac = 2b^2 > 0$.

Hence, $P(x)Q(x) = 0$ has at least two real roots.

370. Let D_1 be the discriminant of $bx^2 + (b-c)x + b - c - a = 0$ and D_2 be discriminant of $ax^2 + 2bx + b = 0$, then

$D_1 + D_2 = (b-c)^2 - 4b(b-c-a) + 4b^2 - 4ab = (b+c)^2 \geq 0$. Hence, if $D_2 < 0$, then $D_1 > 0$.

Therefore, roots of $bx^2 + (b-c)x + b - c - a = 0$ will be real if roots of $ax^2 + 2bx + b = 0$ are imaginary and vice versa.

371. Let $a = 2m+1, b = 2n+1, c = 2r+1$. Now $D = (2n+1)^2 - 4(2m+1)(2r+1)$

$=$ (an odd number) $-$ (an even number) $=$ an odd number.

If possible, let D be a perfect square then it has to be square of an odd number.

$\Rightarrow (2k+1)^2 = (2n+1)^2 - 4(2m+1)(2r+1) \Rightarrow (2m+1)(2r+1) = (n+k+1)(n-k)$.

If n and k are both odd or even then $n-k$ will be even or zero. However, if one is odd and one is even then $(n+k+1)$ will be even. So, R. H. S. is an even while L. H. S. is an odd number. Thus, D cannot be a perfect square. Hence, roots cannot be a rational numbers.

372. Let D_1 be discriminant of $ax^2 + 2bx + c = 0$ then $D_1 = 4b^2 - 4ac = 4k$, where $k = b^2 - ac$.

Let D_2 is discriminant of $(a+c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$

$\Rightarrow D_2 = 4(a+c)^2 b^2 - 4(a^2 + b^2 + k)(b^2 + c^2 + k) = -D1[4b^2 + (a-c)^2] \Rightarrow D_2 < 0 \because D_1 > 0$.

Therefore, roots of second equation are non-real complex numbers.

373. $D = 4[(C_r^n)^2 - C_{r-1}^n C_{r+1}^n] = 4(a-b)$, where $a = (C_r^n)^2, b = C_{r-1}^n C_{r+1}^n$

$\Rightarrow \frac{a}{b} = \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{n-r}\right) > 1 \Rightarrow a > b \Rightarrow D > 0$.

Thus, roots of given equation are real and distinct.

374. Let $y = e^{\sin x}$ then given equation becomes

$y - \frac{1}{y} - 4 = 0 \Rightarrow y = 2 \pm \sqrt{5} \therefore e^{\sin x} = 2 \pm \sqrt{5}$

$\sin x = \log_e(2 - \sqrt{5})$ is not defined.

$\sin x = \log_e(2 + \sqrt{5}) > 1$ is not possible. Hence, roots of given equation cannot be real.

375. Given equation is $az^2+bz+c+i=0$. $z=\frac{-b\pm\sqrt{b^2-4a(c+i)}}{2a}=\frac{-b\pm(p+iq)}{2a}$

where $\sqrt{b^2-4a(c+i)}=p+iq$. Now $b^2-4ac=p^2-q^2$ and $-4a=2qp$

Since z is purely imaginary $\frac{-b\pm p}{2a}=0\Rightarrow \pm p=b\Rightarrow -4a=2(\pm)q\Rightarrow q=\pm\frac{2a}{b}$

Then, $b^2-4ac=b^2-\frac{4a^2}{b^2}\Rightarrow c=\frac{a}{b^2}\Rightarrow a=b^2c$.

376. $D=a^2-4b$. Let a be an odd number then D is an odd number and a perfect square as roots are rational. Let $D=(2n+1)^2$, and $a=2m+1$ where $m,n\in I$.

Now roots $=\frac{-(2m+1)\pm(2n+1)}{2}=\frac{\text{an even no.}}{2}=$ an integer.

Similarly, it can be proven when a is an even no. then roots are integers.

377. Let α,β be integral roots of the given equation. $\alpha+\beta=-7$ and $\alpha\beta=14(q^2+1)$.

$\frac{\alpha\beta}{7}=2(q^2+1)=$ an integer.

$\therefore \alpha\beta$ is divisible by 7 and 7 is a prime number.

$\therefore$ at least one of α and β must be a multiple of 7.

Let $\alpha=7k$, where $k\in I\Rightarrow\beta=-7(k+1)$

Thus, $-\frac{2(q^2+1)}{7}=-7k(k+1)=$ an integer

Let $f(q)=q^2+1$ then it can be shown that $f(1),f(2),...,f(7)$ are not divisible by 7.

$f(q+7)=q^2+1+14q+49$ which is not divisible by 7 as q^2+1 is not divisible by 7.

Hence, α,β cannot be integers.

378. Given equation is $[a^3(b-c)+b^3(c-a)+c^3(a-b)]x^2-[a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)]x+abc[a^2(b-c)+b^2(c-a)+c^2(a-b)]=0$

But $a^3(b-c)+b^3(c-a)+c^3(a-b)=-(a-b)(b-c)(c-a)(a+b+c)$ and $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)=-(a-b)(b-c)(c-a)(ab+bc+ca)$ and $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a)$ the above equation becomes

$(a+b+c)x^2-(ab+bc+ca)x+abc=0$.

Roots are $\frac{(ab+bc+ca\pm\sqrt{(ab+bc+ca)^2-4abc(a+b+c)})}{2(a+b+c)}$, which will be equal if $D=0$.

If $\frac{1}{\sqrt{a}}\pm\frac{1}{\sqrt{b}}\pm\frac{1}{c}=0\Rightarrow\frac{\sqrt{bc}\pm\sqrt{ca}\pm\sqrt{ab}}{\sqrt{abc}}=0$

$\Rightarrow\sqrt{bc}\pm\sqrt{ca}\pm\sqrt{ab}=0$. Squaring

$bc+ca+ab\pm2\sqrt{abc}(\sqrt{a}\pm\sqrt{b}\pm\sqrt{c})=0\Rightarrow(bc+ca+ab)^2=4abc(a+b+c+\sqrt{bc}\pm\sqrt{ca}\pm\sqrt{ab})\Rightarrow D=0$ i.e. roots are equal.

379. Product of roots $= \frac{k+2}{k} = \frac{c}{a} \Rightarrow k = \frac{2a}{c-a}$

Sum of roots $= \frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a}$. Substituting for k

$\frac{c+a}{2a} + \frac{2c}{c+a} = -\frac{b}{a} \Rightarrow \frac{(a+c)^2+4ac}{2a(a+c)} = -\frac{b}{a}$

$\Rightarrow a(a+c)^2 + 4a^2c = -2abc - 2a^2b \Rightarrow (a+c)^2 + 4ac = -2bc - 2ab \Rightarrow (a+b+c)^2 = b^2 - 4ac.$

380. Given, $f(x) = ax^2 + bx + c$ and that α,β are the roots of the equation $px^2 + qx + r = 0$.

$\Rightarrow \alpha + \beta = -\frac{q}{p}$ and $\alpha\beta = \frac{r}{p}$.

Now $f(\alpha)f(\beta) = (a\alpha^2 + b\alpha + c)(a\beta^2 + b\beta + c)$

$= a^2\alpha^2\beta^2 + b^2\alpha\beta + c^2 + ab\alpha\beta(\alpha+\beta) + ac(\alpha^2+\beta^2) + bc(\alpha+\beta)$

$= a^2\frac{r^2}{p^2} + b^2\frac{r}{p} + c^2 - ab\frac{r}{p}\frac{q}{p} + ac\left(\frac{q^2}{p^2} - \frac{2r}{p}\right) - bc\frac{q}{p}$

$= \frac{1}{p^2}[a^2r^2 + b^2rp + c^2p^2 - abrq + acq^2 - 2acrp - bcqp] = \frac{1}{p^2}[(cp-ar)^2 + b^2rp - bcqp - abrq + acq^2]$

$= \frac{1}{p^2}[(cp-ar)^2 - (bp-aq)(cq-br)]$

Now since α, β are the roots of the equation $px^2 + qx + r = 0$

Therefore, if $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ have to have a common root then it has to be either α or β.

$f(\alpha) = 0$ or $f(\beta) = 0 \therefore f(\alpha)f(\beta) = 0 \Rightarrow (cp-ar)^2 - (bp-aq)(cq-br) = 0$

$\therefore bp - aq$, $cp - ar$, $cq - br$ are in G. P.

381. From the given equations it follows that q and r are roots of the equation

$a(p+x)^2 + 2bpx + c = 0 \Rightarrow ax^2 + 2(a+b)px + c = 0.$

Product of roots $qr = \frac{ap^2+c}{a} = p^2 + \frac{c}{a}$

382. Since α, β are the roots of the equation $x^2 - px - (p+c) = 0$

$\alpha + \beta = p$ and $\alpha + \beta = -(p+c)$. Now $(\alpha+1)(\beta+1) = -p - c + p + 1 = 1 - c$.

$\Rightarrow \frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$

$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)} = \frac{(\alpha+1)^2}{(\alpha+1)(\alpha-\beta)} + \frac{(\beta+1)^2}{(\beta+1)(\beta-\alpha)} = 1$. Hence, proved.

383. α, β are the roots of the equation $x^2 + px + q = 0$. $\therefore \alpha + \beta = -p$ and $\alpha\beta = q$

Since α, β are the roots of the equation $x^2 2n + p^n x^n + q^n = 0$.

Substituting it follows that α^n, β^n are the roots of the equation $y^2 + p^n y + q^n = 0$

$\therefore \alpha^n + \beta^n = (-p)^n$ and $\alpha^n\beta^n = q^n \Rightarrow (\alpha+\beta)^n = (-p)^n = p^n[\because$ n is even$]$.

Thus, $\alpha^n + \beta^n + (\alpha+\beta)^n = 0$

Dividing by β^n we have $\left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$

Dividing by α^n we have $\left(\frac{\beta}{\alpha}\right)^n + 1 + \left(\frac{\beta}{\alpha} + 1\right)^n = 0$

From last two equations it is evident that $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are roots of the equation $x^n + 1 + (x+1)^n = 0$.

384. Let α and β are the roots of the given equation.

Since roots are real and distinct $D > 0 \Rightarrow a^2 - 4b > 0 \Rightarrow b < \frac{a^2}{4}$

Again it is given that $|\alpha - \beta| < c \Rightarrow (\alpha - \beta)^2 < c^2$

$(\alpha+\beta)^2 - 4\alpha\beta < c^2 \Rightarrow a^2 - 4b < c^2 \Rightarrow 4b > a^2 - c^2 \Rightarrow \frac{a^2-c^2}{4} < b < \frac{a^2}{4}$.

385. Given, $ax^2 + bx + c - p = 0$ for two integral values of x say α and β.

Then, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c-p}{a}$

If possible, let $ax^2 + bx + c - 2p = 0$ for some integer k.

$ak^2 + bk + c - p = p \Rightarrow k^2 - (\alpha+\beta)k + \alpha\beta = \frac{p}{a} \Rightarrow (k-\alpha)(k-\beta) =$ an integer $= \frac{p}{a}$

But since p is prime this cannot hold true unless $a = p$ or $a = 1$

$a = p[\because a > 1] \Rightarrow (k-\alpha)(k-\beta) = 1$ which implies that $k - \alpha = k - \beta = 1$, which is not possible since $\alpha \neq \beta$

Thus, we have a contradiction. Hence, $ax^2 + bx + c \neq 2p$ for any integral value of x.

386. $\alpha + \beta = -p, \alpha\beta = q, \alpha^4 + \beta^4 = r, \alpha^4\beta^4 = s$

Let D be the discriminant of $x^2 - 4qx + 2q^2 - r = 0$ then

$D = 16q^2 - 4(2q^2 - r) = 8q^2 + 4r = 8\alpha^2\beta^2 + 4(\alpha^4 + \beta^4) = 4(\alpha^2 + \beta^2)^2$

$D \geq 0$ hence roots of the third equation are always real.

387. $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a} \Rightarrow \alpha_1 - \beta = -\frac{b_1}{a_1}$ and $-\alpha_1\beta = \frac{c_1}{a_1}$

$\Rightarrow \alpha + \alpha_1 = -\left(\frac{b}{a} + \frac{b_1}{a_1}\right)$.

Also, dividing $\alpha + \beta$ by $\alpha\beta$, $\frac{1}{\beta} + \frac{1}{\alpha} = -\frac{b}{c}$

Similarly, dividing $\alpha_1 - \beta$ by $-\alpha_1\beta$, $\frac{1}{\alpha_1} - \frac{1}{\beta} = -\frac{b_1}{c_1}$

Thus, $\frac{1}{\alpha} + \frac{1}{\alpha_1} = -\left(\frac{b}{c} + \frac{b_1}{c_1}\right)$

Equation whose roots are α and α_1 is

$x^2 - (\alpha + \alpha_1)x + \alpha\alpha_1 = 0 \Rightarrow \frac{x^2}{-(\alpha+\alpha_1)} + x - \frac{\alpha\alpha_1}{\alpha+\alpha_1} = 0$

$\frac{x^2}{\frac{b}{a}+\frac{b_1}{a_1}} + x + \frac{1}{\frac{b}{c}+\frac{b_1}{c_1}} = 0.$

388. Let α and β be roots of such quadratic equation given by $x^2 + px + q = 0$

$\Rightarrow \alpha + \beta = -p$ and $\alpha\beta = q$. Now quadratic equation whose roots are α^2 and β^2 is

$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0 \Rightarrow x^2 - (p^2 - 2q)x + q^2 = 0.$

But the equation remains unchanged, therefore,

$\frac{1}{1} = \frac{p}{p^2-2q} = \frac{q}{q^2} \Rightarrow q = q^2 \Rightarrow q(q-1) = 0 \Rightarrow q = 0, 1$

If $q = 0 \Rightarrow p = 0, -1$ and if $q = 1 \Rightarrow p = -2, 1$. Thus, four such quadratic equations are possible.

389. Given $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A. P. and a, b, c are in G. P.

Equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ will have a common root if

$\frac{2(bf-ec)}{cd-af} = \frac{cd-af}{2(ae-bd)} \Rightarrow 4(bf - ec)(ae - bd) = (cd - af)^2$

$4\left[\left(\frac{f}{c} - \frac{e}{b}\right)bc\right]\left[\left(\frac{e}{b} - \frac{d}{a}\right)ab\right] = \left(\frac{d}{a} - \frac{a}{f}\right)^2 a^2c^2$

$4k.k.b^2 = 4k^2ac$ where k is the c.d. of the A. P. i.e. $b^2 = ac$ which is true because a, b, c are in G. P.

390. Let α be the common root and β_1 another root of $x^2 + ax + 12 = 0$, β_2 be another root of $x^2 + bx + 15 = 0$ and β_3 be a root of $x^2 + (a + b)x + 36 = 0$.

$\Rightarrow \alpha + \beta_1 = -a$ and $\alpha\beta_1 = 12$, $\alpha + \beta_2 = -b$ and $\alpha\beta_2 = 15$, and $\alpha + \beta_3 = -(a + b)$ and $\alpha\beta_3 = 36$.

Thus, $2\alpha + \beta_1 + \beta_2 = \alpha + \beta_3 \Rightarrow \alpha = \beta_3 - \beta_1 - \beta_2$ and $\alpha(\beta_3 - \beta_1 - \beta_2) = 36 - 12 - 15 = 9$

$\Rightarrow \alpha^2 = 9 \Rightarrow \alpha = \pm 3$ but $\alpha > 0 \Rightarrow \alpha = 3 \Rightarrow \beta_1 = 4, \beta_2 = 5, \beta_3 = 12.$

391. Given $m(ax^2+2bx+c)+px^2+1qx+r=n(x+k)^2$. Equating coefficients for powers of x, we get

$ma+p=n, mb+q=nk, mc+r=nk^2 \Rightarrow m(ak-b)+pk-q=0 \Rightarrow m=-\frac{pk-q}{ak-b}$

$\Rightarrow m(bk-c)+qk-r=0 \Rightarrow m=-\frac{qk-r}{bk-c}$.

Equating values for m, $(ak-b)(qk-r)=(pk-q)(bk-c)$.

392. Given equation is $x^3-x^2+\beta x+\gamma=0$. Let it roots x_1, x_2, x_3 be $a-d, a, a+d$ respectively.

$\Rightarrow a-d+a+a+d=1 \Rightarrow a=\frac{1}{3} \Rightarrow (a-d)a+a(a+d)+(a-d)(a+d)=\beta \Rightarrow 3a^2-d^2=\beta \Rightarrow 1-3\beta=3d^2$

$(a-d)a(a+d)=\gamma \Rightarrow a(a^2-d^2)=\gamma \Rightarrow 1+27\gamma=9d^2$

Since d is real $\therefore 1-3\beta \geq 0 \Rightarrow \beta \leq \frac{1}{3}$ and $1+27\gamma \geq 0 \Rightarrow \gamma \geq -\frac{1}{27}$.

393. Let α be a common root, then

$\alpha^3+3p\alpha^2+3q\alpha+r=0 \ \ \ldots \ \ (1)$ and $\alpha^2+2p\alpha+q=0 \ \ \ldots \ \ (2)$

$(1)-\alpha(2)$ gives us $\Rightarrow p\alpha^2+2q\alpha+r=0 \ \ \ldots \ \ (3)$

By cross multiplication between (2) and (3)

$\frac{\alpha^2}{2(pr-q^2)}=\frac{\alpha}{pq-r}=\frac{1}{2(q-p^2)}$

Equating for values of α we get the desired condition.

394. Let α be a common root, then

$\alpha^3+2a\alpha^2+3b\alpha+c=0 \ \ \ldots \ \ (1)$ and $\alpha^3+a\alpha^2+2b\alpha=0 \ \ \ldots \ \ (2)$

Since $c \neq 0$, therefore $\alpha=0$ cannot be a common root. Therefore, from (2)

$\alpha^2+a\alpha+2b=0 \ \ \ldots \ \ (3)$

$(1)-\alpha(2) \Rightarrow a\alpha^2+b\alpha+c=0 \ \ \ldots \ \ $ x(4)

Solving (3) and (4) by cross-multiplication yields the desired result.

395. Given equation is $x^3+ax+b=0$ and α, β, γ be its real roots. Then we have

$\alpha+\beta+\gamma=0 \ \ \ldots \ \ (1)\ \alpha\beta+\beta\gamma+\alpha\gamma=a \ \ \ldots \ \ (2)\ \alpha\beta\gamma=-b$

Let $y=(\alpha-\beta)^2$, then $y=(\alpha+\beta)^2-4\alpha\beta \Rightarrow y=\gamma^2+\frac{4b}{\gamma} \Rightarrow \gamma^3-y\gamma+4b=0$.

Also, γ is a root of the original equation.

$\gamma^3 + a\gamma + b = 0 \Rightarrow (a+y)\gamma - 3b = 0 \Rightarrow \gamma = \frac{3b}{a+y}$

$\Rightarrow \frac{27b^3}{(a+y)^3} + a\left(\frac{3b}{a+y}\right) + b = 0 \Rightarrow y^3 + 6ay^2 + 9a^2y + 4a^3 + 27b^2 = 0$

We would have got same equation if we would have chosen $y = (\beta-\alpha)^2$ or $y = (\gamma-\alpha)^2$.

Hence, product of roots $-(4a^3+27b^2) = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 \geq 0 \therefore 4a^3+27b^2 \leq 0$.

396. α is a root of the equation $ax^2 + bx + c = 0 \therefore a\alpha^2 + b\alpha + c = 0$

Similarly, $-a\beta^2 + b\beta + c = 0$. Let $f(x) = \frac{a}{2}x^2 + bx + c = 0 \Rightarrow f(\alpha) = -\frac{a}{2}\alpha^2$,

and $f(\beta) = \frac{3}{2}\beta^2 \therefore f(\alpha)f(\beta) = -\frac{3}{4}a^2\alpha^2\beta^2 < 0 [\because \alpha,\beta \neq 0]$

$\therefore f(\alpha)$ and $f(\beta)$ have opposite signs. Therefore, $f(x)$ will have exactly one root between α and β.

397. Let $f(x) = ax^2 + bx + c = 0$. Since equation $ax^2 + bx + c = 0$ i.e. equation $f(x) = 0$ has no real root, therefore, $f(x)$ will have same sign for real values of x.

$\therefore f(1)f(0) > 0 \Rightarrow (a+b+c)c > 0$.

398. Let $f(x) = (x-a)(x-c) + \lambda(x-b)(x-d)$. Given $a > b > c > d$, now $f(b) = (b-a)(b-c) < 0$, and $f(d) = (d-a)(d-c) > 0$

Since $f(b)$ and $f(d)$ have opposite signs, therefore equation $f(x) = 0$ will have one real root between b and d.

Since one root is real and a, b, c, d, λ are all real the other root will also be real.

399. Let $f'(x) = ax^2 + bx + c$, then $f(x) = a\frac{x^3}{3} + b\frac{x^2}{2} + cx + k = \frac{2ax^3+3bx^2+4cx+6k}{6}$,

$\Rightarrow f(1) = \frac{2a+3b+6c+6k}{6} = k$. Again, $f(0) = k$

Thus, $f(0) = f(1)$ hence equation will have at least one root between 0 and 1 which implies that it will have a real root between 0 and 2.

400. Let $f(x) = \displaystyle\int (1+\cos^8 x)(ax^2+bx+c)\,dx$ then $f'(x) = (1+\cos^8 x)(ax^2+bx+c)$.

Given, $\displaystyle\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$.

$\Rightarrow f(1) - f(0) = f(2) - f(0) \Rightarrow f(1) = f(2)$.

Therefore, equation $f(x) = 0$ has at least one root between 1 and 2 which implies that $ax^2 + bx + c$ has a root between these two limits as $1 + \cos^8 x \neq 0$.

401. Given equation $f(x) - x = 0$ has non-real roots where $f(x) = ax^2 + bx + c$ is a continuous function.

$\therefore f(x) - x$ has same sign for all $x \in R$. Let $f(x) - x > 0 \ \forall \ x \in R$

$\Rightarrow f(f(x)) - f(x) > 0 \ \forall \ x \in R \Rightarrow f(f(x)) - x = f(f(x)) - f(x) + f(x) - x > 0 \ \forall \ x \in R$

Hence it has no real roots.

402. Let $f(x) = ax^2 - bx + c = 0$ and that α, β be its roots. Then, $f(x) = a(x-\alpha)(x-\beta)$.

Given $\alpha \neq \beta, 0 < \alpha < 1, 0 < \beta < 1$ and $a, b, c \in N$

Since quadratic equation has both roots between 0 and 1, therefore

$f(0)f(1) > 0$ but $f(0)f(1) = c(a - b + c) =$ an integer

Thus, $f(0)f(1) \geq 1 \Rightarrow a\alpha(1-\alpha)a\beta(1-\beta) = a^2\alpha\beta(1-\alpha)(1-\beta)$.

Let $y = \alpha(1-\alpha) \Rightarrow \alpha^2 - \alpha + y = 0$.

Since α is real $\therefore 1 - 4y \geq 0 \Rightarrow y \leq \frac{1}{4} \Rightarrow \alpha = \frac{1}{2}$ max value.

Similarly, maximum value of $\beta = \frac{1}{2}$.

Maximum value of $\therefore f(0)f(1) < \frac{a^2}{16} > 1 \Rightarrow a > 4 \Rightarrow a = 5$ [least integral value]

Since $ax^2 - bx + c = 0$ has real and distinct roots $\Rightarrow b^2 > 4ac$ [$\because a \geq 4, c \geq 1$]

$\Rightarrow b^2 \geq 20 \Rightarrow b \geq 5$.

403. Proceeding from previous question, $b^2 - 4ac > 0 \Rightarrow b^2 > 4.5.1[\because c \geq 1] \Rightarrow b = 5 \Rightarrow \log_5(abc) \geq 2$.

404. Given equation is $ax^2 + bx + 6 = 0$. Let $f(x) = ax^2 + bx + 6$

Since the equation has imaginary roots or real and equal roots, $f(0) = 6 > 0 \therefore f(x) \geq 0$ for all real x

$\Rightarrow f(3) \geq 0 \Rightarrow 9a + 3b + 6 \geq 0 \Rightarrow 3a + b \geq -2$ and hence least value is -2.

405. Let α, β, γ be the roots of the equation. Then,

$f(x) = 2x^3 - \frac{\alpha+\beta+\gamma}{2}x^2 + \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{2}x - \frac{\alpha\beta\gamma}{2} = 0$

Clearly, all roots have to be negative for signs to be satisfied as $a, b > 0$.

$f(0) = 4 > 0 \therefore f(1) > 0$ because sign of $f(x)$ will not change for all x.

$2 + a + b + 4 > 0 \Rightarrow a + b > -6$.

406. $f(x) = x^3 + 2x^2 + x + 5 = 0$ and $f'(x) = 3x^2 + 4x + 1$ which has roots -1 and $-\frac{1}{3}$.

$f(0) = 5$ and $f(x)$ is increasing in $(0, \infty)$ therefore it will have no root in $[0, \infty[$.

$f(-2) = 3 > 0$ and $f(-3) = -7 < 0$.

Since $f(-2)$ and $f(-3)$ are of opposite sign therefore equation $f(x) = 0$ will have one root between -2 and -3 and this will be only one root as $f(x)$ is increasing in $]-\infty, -1] \Rightarrow [\alpha] = -3$.

407. Given equation is $(x^2+2)^2 + 8x^2 = 6x(x^2+2)$. Let $y = x^2+2$ then above equation becomes $y^2 + 8x^2 = 6xy \Rightarrow y = 4x, 2x$.

If $y = 4x \Rightarrow x^2 - 4x + 2 = 0 \Rightarrow x = 2 \pm \sqrt{2}$.

If $y = 2x \Rightarrow x^2 - 2x + 2 = 0 \Rightarrow x = 1 \pm i$.

408. Given equation is $3x^3 = (x^2 + \sqrt{18}x + \sqrt{32})(x^2 - \sqrt{18}x - \sqrt{32}) - 4x^2 \Rightarrow 3x^3 = x^4 - (\sqrt{18}x + \sqrt{32})^2 - 4x^2$

$\Rightarrow x^2(3x+4) = x^4 - 2(3x+4)^2 \Rightarrow x^2y = x^4 - 2y^2$ where $y = 3x+4 \Rightarrow y = -x^2, \frac{x^2}{2}$.

If $y = -x^2 \Rightarrow x = \frac{-3\pm\sqrt{7}i}{2}$ and if $y = \frac{x^2}{2} \Rightarrow x = 3 \pm \sqrt{17}$.

409. Clearly, $(15+4\sqrt{14})^t(15-4\sqrt{14})^t = (225-224)^t = 1$. Let $(15+4\sqrt{14})^t = y$, then $(15-4\sqrt{14})^t = \frac{1}{y}$.

Substituting for the given equation

$y + \frac{1}{y} = 30 \Rightarrow y^2 - 30y + 1 = 0 \Rightarrow y = 15 \pm 4\sqrt{14}$

If $y = 15 + 4\sqrt{14} \Rightarrow t = 1$, then $x^2 - 2|x| = 1 \Rightarrow |x|^2 - 2|x| - 1 = 0$

$\Rightarrow |x| = 1 + \sqrt{2} \therefore x = \pm(1+\sqrt{2})$

If $y = 15 - 4\sqrt{14} \Rightarrow t = -1 \Rightarrow |x|^2 - 2|x| + 1 = 0 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$.

410. Given equation is $x^2 - 2a|x-a| - 3a^2 = 0$. When $a = 0$ equation becomes $x^2 = 0 \Rightarrow x = 0$

Let $a < 0$.

Case I: When $x < a$ then equation becomes

$x^2 + 2a(x-a) - 3a^2 = 0 \Rightarrow x^2 + 2ax - 5a^2 = 0 \Rightarrow x = -a \pm \sqrt{6}a$

Since $x < a$, $x = -a - \sqrt{6}a$ is not acceptable.

Case II: When $x > a$ the equation becomes

$x^2 - 2ax - a^2 = 0 \Rightarrow x = a \pm \sqrt{2}a$

Since $x > a$, $x = a + \sqrt{2}a$ is not acceptable.

Clearly, $x = a$ does not satisfy the equation.

411. $x^2 - x - 6 = 0 \Rightarrow x = -2, 3$

Case I: When $x < -2$ or $x > 3$ then $x^2 - x - 6 > 0$

Then equation becomes $x^2 - x - 6 = x + 2 \Rightarrow x^2 - 2x - 8 = 0$

$x = -2, 4$ but $x = -2$ is not acceptable as $x < -2$

Case II: When $-2 < x < 3$ $x^2 - x - 6 < 0$

Then equation becomes $-(x^2 - x - 6) = x + 2 \Rightarrow x^2 - 4 = 0 \Rightarrow x = 2$ because $x = -2$ is not acceptable.

Case III: Clearly $x = -2$ satisfies the equation by $x = 3$ does not.

412. $|x + 2| = 0 \Rightarrow x = -2$ and $|2^{x+1} - 1| = 0 \Rightarrow 2^{x+1} = 1 \Rightarrow x = -1$

Case I: When $x < -2$ then $x + 2 < 0$ and $2^{x+1} - 1 < 0$

Equation becomes $2^{-(x+2)} - [-(2^{x+1} - 1)] = 2^{x+1} + 1$

$\Rightarrow x = 3$

Case II: When $-2 < x < 1$ then $x + 2 > 0$ and $2^{x+1} - 1 < 0$

Equation becomes $2^{x+2} - [-(2^{x+1} - 1)] = 2^{x+1} + 1$

$\Rightarrow x = 1$

Case III: When $x > -1$ then $x + 2 > 0$ and $2^{x+1} - 1 > 0$

Equation becomes $2^{x+2} - (2^{x+1} - 1) = 2^{x+1} + 1$

$\Rightarrow x + 2 = x + 2$

which is true for all x but only values for $x > -1$ are acceptable.

Case IV: Clearly, $x = -2$ does not satisfy the equation but $x = -1$ satisfies it.

413. Given equation is $3^x + 4^x + 5^x = 6^x$. Then,

$\left(\frac{3}{6}\right)^x + \left(\frac{4}{6}\right)^x + \left(\frac{5}{6}\right)^x = 1$

Clearly, $x = 3$ satisfies the equation.

When $x > 3$, $\left(\frac{3}{6}\right)^x + \left(\frac{4}{6}\right)^x + \left(\frac{5}{6}\right)^x < 1$

When $x < 3$, $\left(\frac{3}{6}\right)^x + \left(\frac{4}{6}\right)^x + \left(\frac{5}{6}\right)^x > 1$

Therefore, $x = 3$ is the only solution.

414. Proceeding as previous problem $x = 2$ is the only solution.

415. $x = [x] + \{x\}$, given equation is $4\{x\} = x + [x] \Rightarrow \{x\} = \frac{2}{3}[x]$

$\because 0 < \{x\} < 1 \therefore 0 < \frac{2}{3}[x] < 1 \Rightarrow 0 < [x] < \frac{3}{2} \Rightarrow [x] = 1$

$\therefore \{x\} = \frac{2}{3} \Rightarrow x = \frac{5}{3}.$

416. Given, $[x]^2 = x(x - [x]) \Rightarrow [x]^2 = ([x] + \{x\})\{x\}[\because x = [x] + \{x\}]$

$y^2 = (y + z)z$, where $y = [x]$ and $z = \{x\} \Rightarrow z^2 + yz - y^2 = 0 \Rightarrow z = \frac{-y \pm \sqrt{5}y}{2}$

Since $0 < z < 1$ it implies that

if $z = -\frac{\sqrt{5}+1}{2}y$, then

$0 > y > -\frac{2}{\sqrt{5}+1} \Rightarrow -\frac{\sqrt{5}-1}{2} < y < 0$ is not possible as y is an integer.

If $z = \frac{\sqrt{5}-1}{2}y$ then $0 < y < \frac{2}{\sqrt{5}-1} \Rightarrow y = 1 \Rightarrow z = \frac{\sqrt{5}-1}{2}$ and $x = y + z = \frac{\sqrt{5}+1}{2}$.

417. Let $y = mx$ the equations become $x^3(1 - m^3) = 127$ and $x^3(m - m^2) = 42$.

Dividing we get $\frac{1-m^3}{m-m^2} = \frac{127}{42} \Rightarrow \frac{1+m+m^2}{m} = \frac{127}{42}[\because m = 1]$ does not satisfy the equations.

$\Rightarrow m = \frac{7}{6}, \frac{6}{7}$. Substituting we get $x = -6, y = -7$ and $x = 7, y = 6$.

418. Solving first two equations by cross-multiplication

$\frac{x}{7} = \frac{y}{7} = \frac{z}{7}$ or $x = y = z = k$.

Substituting in third equation $k = \pm\sqrt{7}$.

419. Let $x = u + v$ and $y = u - v$ then first equation becomes $(u + v)^4 + (u - v)^4 = 82$

$\Rightarrow u^4 + 6u^2v^2 + v^4 = 41$

Second equation becomes $2u = 4 \Rightarrow u = 2$. Substituting in this equation $v = \pm 5i, \pm 1$

$\therefore x = 2 \pm 5i, 3, 1$ and $y = 2 \mp 5i, 1, 3$.

420. Let $y = 2^x > 0$ then give equation becomes $\sqrt{a(y-2)+1} = 1 - y \Rightarrow y^2 - (a+2)y + 2a = 0$.

$y = 2, a$ but $y = 2$ does not satisfy the equation. When $y = a$ then $\sqrt{a(a-2)+1} = 1 - a \Rightarrow a \leq 1$

$\therefore 0 < a \leq 1[\because y > 0] \Rightarrow y = a \Rightarrow x = \log_2 a$, where $0 < a \leq 1$

When $a > 1$, given equation has no solution.

421. Given $(x-5)(x+m) = -2$.Since x and m are both integers, therefore, $x-5$ and $x+m$ are also integers.

So we have following combination of solutions:

$x-5=1$ and $x+m=2$ then $x=6, m=-8$

$x-5=2$ and $x+m=-1$ then $x=7, m=-8$

$x-5=-1$ and $x+m=2$ then $x=4, m=-2$

$x-5=-2$ and $x+m=1$ then $x=3, m=-2$

Thus, $m=-8, -2$.

422. Multiplying the equations we get $(xy)^{x+y} = (xy)^{2n} \therefore x+y=2n$ where $xy \neq 1$.

$\Rightarrow x^2 = y$ then $x + x^2 = 2n \Rightarrow x = \frac{-1\pm\sqrt{1+8n}}{2}$

But $x > 0 \therefore x = \frac{-1+\sqrt{1+8n}}{2} \Rightarrow y = x^2 = \frac{1+4n-\sqrt{1+8n}}{2}$.

423. Let $y = 12^{|x|}$, then given equation becomes $y^2 - 2y + a = 0 \Rightarrow y = 1 \pm \sqrt{1-a}$

$|x| = \log_{12}(1+\sqrt{1-a})$ as $y = 1-\sqrt{1-a}$ has to be rejected as $y > 1$.

But $\sqrt{1-a}$ has to be real $1-a \geq 0 \Rightarrow a \leq 1$

For $\log_{12}(1+\sqrt{1-a})$ to be defined $1+\sqrt{1-a} > 0 \therefore x = \pm\log_{12}(1+\sqrt{1-a})$.

424. Let $m = 2p+1$ and $n = 2q+1$ the $D = 4(2p+1)^2 - 8(2q+1) =$ an even no.

Let D be a perfect square then it has to be perfect square of an even no. Let that no. be $2r$ then

$4r^2 = 4(2p+1)^2 - 8(2q+1) \Rightarrow 2(2q+1) = (2p+1-r)(2p+1+r)$.

Clearly, if r is an even no. then L. H. S. is an even and R. H. S. is even no which is not possible.

Let r is an odd no. then R. H. S. is product of 2 even numbers. Let $2p+1-r = 2k$ and $2p+1+r = 2l$

$2(2q+1) = 4kl$ which is an odd no. $2q+1$ having equality to even no. $2kl$ which is again not possible. Thus, under the given conditions equation cannot have rational roots.

425. Equation representing points of local extrema is $f'(x) = 3ax^2 + 2bx + c = 0$.

Let one of these points is α and then second would be $-\alpha$.

Sum of these roots $= \alpha - \alpha = -\frac{2b}{3a} \Rightarrow b = 0$.

Product of roots $= -\alpha^2 = \frac{c}{3a}$ but since roots are opposite in equation it implies that a and c have opposite signs.

$\therefore b^2 - 4ac = -4ac > 0$ therefore roots of $ax^2 + bx + c$ will have real and distinct roots.

426. Given equation is $\frac{(x-a)(ax-1)}{x^2-1} = b$.

$ax^2 - (1+a^2)x + a = bx^2 - b \Rightarrow (a-b)x^2 - (1+a^2)x + a + b = 0.$

Discriminant $D^2 = (1+a^2)^2 - a^2 + b^2 = 1 + a^2 + a^4 + b^2 > 0[\because b \neq 0]$

Therefore, roots can never be equal.

427. Given equation is $C_r^n x^2 + 2C_{r+1}^n x + C_{r+2}^n = 0$. Let D be discriminant, then we have to prove that

$D = 4left(C_{r+1}^n) - 4(C_r^n.C_{r+2}^n) > 0$

$\Rightarrow \left[\frac{n!}{(r+1)!(n-r-r)!}\right]^2 - \frac{n!}{r!(n-r)!} \cdot \frac{n!}{(r+2)!(n-r-2)!} = \frac{n!^2}{r!(r+1)!(n-r-1)!(n-r-2)!}\left[\frac{1}{(r+1)(n-r-1)} - \frac{1}{(n-r)(r+2)}\right] > 0$

$\Rightarrow nr + 2n - r^2 - 2r - [nr + n - r^2 - r - r - 1] > 0 \Rightarrow n - 1 > 0.$

From given conditions minimum value of n is 4, hence above condition is true proving that roots are real.

428. $D = c^2(3a^2+b^2)^2 + 4abc^2(6a^2+ab-2b^2) = c^2(9a^4+b^4+6a^2b^2+4a^3b+4a^2b^2-8ab^3)$

$= c^2(3a^2 - b^2 + 4ab)^2$, which is a perfect square and hence roots are rational.

429. $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} + \frac{b}{\sqrt{ac}} = 0$

L.H.S. $= \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \frac{b}{\sqrt{ac}}$

$= \frac{\alpha+\beta}{\sqrt{\alpha\beta}} + \frac{b}{\sqrt{ac}} = \frac{-\frac{b}{a}}{\sqrt{\frac{c}{a}}} + \frac{b}{\sqrt{ac}} = 0.$

430. Let α be the root, then the second root would be α^3.

Product of roots $= \alpha^4 = a \Rightarrow \alpha = a^{\frac{1}{4}}$.

Sum of roots $= \alpha + \alpha^3 = -f(a) \Rightarrow f(a) = -a^{\frac{1}{4}} - a^{\frac{3}{4}}$.

Therefore, the general equation in x would be $f(x) = -x^{\frac{1}{4}} - x^{\frac{3}{4}}$.

431. Since α, β are roots of the equation $x^2 - px + q = 0$ therefore

$\alpha + \beta = p$ and $\alpha\beta = q$

$(\alpha^2-\beta^2)(\alpha^3-\beta^3)=(\alpha-\beta)^2(\alpha+\beta)[(\alpha^2+\beta^2)+\alpha\beta]=(p^2-4q)p(p^2+q)$, and

$\alpha^3\beta^2+\alpha^2\beta^3=\alpha^2\beta^2(\alpha+\beta)=pq^2$

Therefore, the equation would be

$x^2-p[(p^2-4q)(p^2+q)+q^2]x+p^2q^2(p^2-4q)(p^2+q)=0.$

432. $\alpha+\beta=b$ and $\alpha\beta=c$. Then proceeding like previous problem,

$(\alpha^2+\beta^2)(\alpha^3+beta^3)=[(\alpha+\beta)^2-2\alpha\beta][(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)]=(b^2-2c)(b^3-3bc)$, and

$\alpha^5\beta^3+\alpha^3\beta^5-2\alpha^4\beta^4=\alpha^3\beta^3(\alpha^2+\beta^2-2\alpha\beta)=c^3(b^2-4c).$

Therefore, the equation would be

$x^2-[(b^2-2c)(b^3-3bc)+c^3(b^2-4c)]x+(b^2-2c)(b^3-3bc)c^3(b^2-4c)=0.$

433. Let α,β be the roots then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

According to the question $\alpha+\beta=\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$\Rightarrow -\frac{b}{a}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}\Rightarrow -\frac{b}{a}=\frac{\frac{b^2}{a^2}}{\frac{c^2}{a^2}}-2\frac{1}{\frac{c}{a}}$

$\Rightarrow -\frac{b}{a}=\frac{b^2}{c^2}-2\frac{a}{c}\Rightarrow \frac{b^2}{ac}+\frac{bc}{a^2}=2.$

434. Given, $T=2\pi\sqrt{\frac{h^2+k^2}{gh}}$. Squaring, $h^2+k^2=\frac{T^2gh}{4\pi^2}$

$\Rightarrow h^2-\frac{T^2gh}{4\pi^2}+k^2=0$. Clearly, h_1 and h_2 are two possible roots of above equation, where

$h_1+h_2=\frac{T^2g}{4\pi^2}$ and $h_1h_2=k^2$.

435. Clearly, $\alpha_1+\alpha_2=-p$ and $\alpha_1\alpha_2=q$, $\beta_1+\beta2=-r$ and $\beta_1\beta_2=s$.

Solving the two equations in y and z by elimination we have

$\frac{\alpha_1}{\alpha_2}=\frac{\beta_1}{\beta_2}=k\Rightarrow\frac{p^2}{r^2}=\frac{(\alpha_1+\alpha_2)^2}{(\beta_1+\beta_2)^2}=\frac{\alpha_1^2(1+k^2)}{\beta_1(1+k^2)}=\frac{\frac{\alpha_1\alpha_2}{k}}{\frac{\beta_1\beta_2}{k}}=\frac{q}{s}.$

436. $-(1+\alpha\beta)=-(\frac{a+c}{a})$.

H. M. of α and $\beta=\frac{2\alpha\beta}{\alpha+\beta}=-\frac{2c}{b}$, but since a,b,c are in H. P. it becomes

$=-\frac{2c}{\frac{2ac}{a+c}}=-(\frac{a+c}{a})=-(1+\alpha\beta).$

437. Given equation is $x+1=\lambda x-\lambda^2x^2 \Rightarrow \lambda^2x^2+(1-\lambda)x+1=0$.

$\Rightarrow \alpha+\beta=\frac{\lambda-1}{\lambda^2}$ and $\alpha\beta=\frac{1}{\lambda^2}$.

Also given that, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=r-2$

$\Rightarrow \alpha^2+\beta^2=(r-2)\alpha\beta \Rightarrow (\alpha+\beta)^2=r\alpha\beta$

$\frac{(\lambda-1)^2}{\lambda^4}=\frac{r}{\lambda^2} \Rightarrow \lambda_1+\lambda_2=\frac{2}{1-r}$ and $\lambda_1\lambda_2=\frac{1}{1-r}$.

Now it is trivial to deduce the desired result.

438. Let α,β be roots of $ax^2+bx+c=0$ then

$\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

According to question, $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{m}{l}$ and $\frac{1}{\alpha\beta}=\frac{n}{l}$.

From product of roots, $\frac{c}{a}=\frac{l}{n}$ and from sum of roots $\frac{b}{c}=\frac{m}{l}$.

439. Let the roots are l, lm, lm^2, lm^3 which is an increasing G. P.

Sum of roots for first equation $=l(1+m)=3$

Sum of roots for second equation $=lm^2(1+m)=12 \Rightarrow m^2=4 \Rightarrow m=2$ because G. P. is increasing.

$\Rightarrow l=1$.

$\Rightarrow A=l^2m=2$ and $B=l^2m^5=32$.

440. For first equation, $p+q=2$ and $pq=A$. For second equation, $r+s=18$ and $rs=B$.

Let a be the first term and d be the common difference, then

$p=a-3d, q=a-d, r=a+d, s=a+3d$.

Substituting in sums we have $2a-4d=2$ and $2a+4d=18 \therefore a=5$ and $d=2$

$\therefore p=-1, q=3, r=7, s=11 \therefore A=-3$ and $B=77$.

441. $\alpha+\beta=-a$ and $\alpha\beta=-\frac{1}{2a^2}$. Now, $\alpha^4+\beta^4=((\alpha+\beta)^2-2\alpha\beta)^2-2\alpha^2\beta^2$

$=2+a^4+\frac{1}{2a^4}$.

Let $a^4+\frac{1}{2a^4}=y \Rightarrow 2a^8-2a^4y-1=0$.

Since a is real. $\therefore y^2-2\geq 0 \Rightarrow y\geq\sqrt{2}[\because a^4\geq 0] \Rightarrow \alpha^4+\beta^4\geq 2+\sqrt{2}$.

442. $\alpha+\beta=p$ and $\alpha\beta=q$.

$$\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}=\sqrt[4]{\left(\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}\right)^4}$$

$$=\sqrt[4]{\alpha+\beta+6\sqrt{\alpha\beta}+4\sqrt[4]{\alpha\beta(\alpha^2+\beta^2)}}=\sqrt[4]{p+6\sqrt{q}+4\sqrt[4]{q(p^2-2q)}}.$$

443. Let $\alpha, beta$ be roots of first equation and γ,δ be that of second equation.

$\alpha+\beta=\frac{b}{a},\alpha\beta=\frac{c}{a}$ and $\gamma+\delta=\frac{c}{b},\gamma\delta=\frac{a}{b}$

According to question, $\alpha-\beta=\gamma-\delta\Rightarrow(\alpha+\beta)^2-4\alpha\beta=(\gamma+\delta)^2-4\gamma\delta$

$\Rightarrow\frac{b^2}{a^2}-\frac{4c}{a}=\frac{c^2}{b^2}-\frac{4a}{b}\Rightarrow b^4-a^2c^2=4ab(bc-a^2).$

444. A cubic equation whose roots are α,β,γ is given by $f(x)=(x-\alpha)(x-\beta)(x-\gamma)$

$\therefore f'(x)=(x-\alpha)(x-\beta)+(x-\beta)(x-\gamma)+(x-\alpha)(x-\gamma)$

Now it is trivial to prove that a sign change occurs for the given limits for $f'(x)$ and thus a root lies in these limits.

445. Let $x_1,x_2,\ldots,x_n$ are the n roots of the given polynomial equation. If all the roots are equal then we will have the relationship

$(x_1-x_2)^2+(x_1-x_3)^2+\cdots+(x_1-x_n)^2+(x_2-x_3)^2+\cdots+(x_2-x_n)^2+\cdots+(x_{n-1}-x_n)^2>0$

$\Rightarrow(n-1)(x_1^2+x_2^2+\cdots+x_n^2)-2(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+x_2x_4+\cdots+x_2x_n+\cdots+x_{n-1}x_n)>0$

$\Rightarrow(n-1)(x_1^2+x_2^2+\cdots+x_n^2)+(2n-2)(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+x_2x_4+\cdots+x_2x_n+\cdots+x_{n-1}x_n)-2n(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+x_2x_4+\cdots+x_2x_n+\cdots+x_{n-1}x_n)>0$

$\Rightarrow(n-1)(x_1+x_2+\cdots+x_n)^2-2n(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+x_2x_4+\cdots+x_2x_n+\cdots+x_{n-1}x_n)>0$

Now from polynomial $x_1+x_2+\cdots+x_n=-a_1$ and $x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+x_2x_4+\cdots+x_2x_n+\cdots+x_{n-1}x_n=a_1$.

$\therefore(n-1)a_1^2-2na_2>0$. But it is given that $(n-1)a_1^2-2na_2<0$, hence all the roots cannot be equal.

446. Since $\alpha,\beta,\gamma,\delta$ are in A. P. let $\alpha=l-3m,\beta=l-m,\gamma=l+m,\delta=l+3m$ where l is the first term and m is the common difference of A. P.

$\alpha+\beta=-\frac{b}{a},\alpha\beta=\frac{c}{a}$ and $\gamma+\delta=-\frac{q}{p},\gamma\delta=\frac{r}{p}$

$\frac{D_1}{D_2}=\frac{b^2-4ac}{q^2-4pr}=\frac{\frac{b^2}{a^2}-\frac{4c}{a}}{\frac{a^2}{p^2}-\frac{4r}{p}}\frac{a^2}{p^2}=\frac{(\alpha-\beta)^2}{(\gamma-\delta)^2}\frac{a^2}{p^2}=\frac{4d^2}{4d^2}\frac{a^2}{p^2}.$

447. R.H.S. $= \frac{q^2-4pr}{p^2} = \frac{q^2}{p^2} - 4\frac{r}{p} = (\alpha+\beta+2h)^2 - 4(\alpha+h)(\beta+h)$

$= (\alpha+h-\beta-h)^2 = (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = \frac{b^2}{a^2} - 4\frac{c}{a} = \frac{b^2-4ac}{a^2} =$ L.H.S.

448. L.H.S. $= 2h = (\alpha+h+\beta+h) - (\alpha+\beta) = -\frac{q}{p} - (\frac{b}{a}) = \frac{b}{a} - \frac{q}{p} =$ R.H.S.

449. $\alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$ and $\alpha^4+\beta^4 = -\frac{m}{l}, \alpha^4\beta^4 = \frac{n}{l}$.

Discriminant of given quadratic equation, $D = 16a^2c^2l^2 - 4a2^l(2c^2l + a^2m) = 8a^2c^2l^2 - 4a^4lm$

$= 4a^4l^2\left(2\frac{c^2}{a^2} - \frac{m}{l}\right) = 4a^4l^2(2\alpha^2\beta^2 + \alpha^4 + \beta^4) = 2a^4l^2(\alpha^2+\beta^2)^2$.

Therefore, roots of the given equation can be computed which are found to be $(\alpha+\beta)^2, -(\alpha+\beta)^2$ which are equal and opposite in sign.

450. $\alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$ and $\gamma+\delta = -\frac{m}{l}, \gamma\delta = \frac{n}{l}$

Equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$ is

$x^2 - (\alpha\gamma+\beta\delta+\alpha\delta+\beta\gamma)x + (\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma) = 0$

$\Rightarrow x^{-}(\alpha+\beta)(\gamma+\delta)x + ((\alpha^2+\beta^2)\gamma\delta + (\gamma^2+\delta^2)\alpha\beta) = 0$

$\Rightarrow a^2l^2x^2 - ablmx + (b^2-2ac)ln + (m^2-2ln)ac = 0$.

451. Since p and q are roots of the equation $x^2+bx+c=0$ therefore $p+q=-b$ and $pq=c$

Equation whose roots are b and c is $x^2 - (b+c)x + bc \Longrightarrow x^2 + (p+q-pq)x - pq(p+q) = 0$.

452. p and q are roots of the equation $3x^2 - 5x - 2 = 0$.

$\Rightarrow p+q = \frac{5}{3}$ and $pq = -\frac{2}{3}$.

Equation whose roots are $3p-2q$ and $3q-2p$ is

$x^2 - (p+q)x - 6p^2 - 6q^2 + 13pq = 0 \Rightarrow 3x^2 - 5x - 100 = 0$.

453. Sum of roots $= 2\alpha = -p$ and product of roots $= \alpha^2 - \beta = q \Rightarrow \beta = \frac{p^2-4q}{4}$.

Equation whose roots are $\frac{1}{\alpha} \pm \frac{1}{\sqrt{\beta}}$ is $x^2 - \frac{2}{\alpha}x + \frac{1}{\alpha^2} - \frac{1}{\beta} = 0$

$\Rightarrow x^2 + \frac{2}{p}x + \frac{1}{p^2} - \frac{4}{p^2-4q} = 0 \Rightarrow (p^2-4q)(p^2x^2+4px) = 16q$.

454. Sum of roots is $\alpha^2\left(\frac{\alpha^2-\beta^2}{\beta}\right) + \beta^2\left(\frac{\beta^2-\alpha^2}{\alpha}\right)$

$= \frac{(\alpha^2-\beta^2)(\alpha^3-\beta^3)}{\alpha\beta} = \frac{(\alpha+\beta)(\alpha-\beta)^2(\alpha^2+\beta^2+\alpha\beta)}{\alpha\beta} = \frac{p}{q}(p^2-4q)(p^2-q)$.

Product of roots is $-\alpha\beta(\alpha^2-\beta^2)^2 = -q(\alpha-\beta)^2(\alpha+\beta)^2 = -p^2q(p^2-4q)$.

Hence the equation having these as roots is $qx^2 - p(p^2-q)(p^2-4q)x - p^2q^2(p^2-4q) = 0$.

455. Solving the system of equations, we have $u = -\frac{1}{3}, v = \frac{2}{3}$ and $w = \frac{5}{3}$.

Now, $(b-c)^2 + (c-a)^2 + (d-b)^2 = a^2 + 2b^2 + 2c^2 + d^2 - 2bc - 2ca - 2bd$, but because a, b, c, d are in G.P. therefore, $ad = bc, ca = b^2$ and $bd = c^2 \Rightarrow a^2 + 2b^2 + 2c^2 + d^2 - 2bc - 2ca - 2bd = (a-d)^2$.

Rewriting the first quadratic equaiton, $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$ becomes

$\Rightarrow -\frac{9}{10}x^2 + (a-d)^2x + 2 = 0 \Rightarrow 9x^2 - 10(a-d)^2x - 20 = 0$. Equation whose roots will be reciprocal of this equation will be $\frac{9}{x^2} - \frac{10(a-d)^2}{x} - 20 = 0 \Rightarrow 20x^2 + (a-d)^2x - 9 = 0$, which is what we had to prove.

456. Because $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of the equation $(\beta_1 - x)(\beta_2 - x)\ldots(\beta_n - x) + A = 0$, therefore

$(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)\ldots(\beta_n - \alpha_n) + A = 0.$

Therefore, equation having $\beta_1, \beta_2, \ldots, \beta_n$ as roots is

$(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n) + A = 0.$

457. Given $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of the equation $x^n + ax + b = 0$.

$\Rightarrow (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = x^n + nax - b$

$\Rightarrow \lim_{x\to\alpha_1}(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) = \frac{x^n+nax-b}{x-\alpha_1}$

Applying L'Hospital's rule, $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n) = nx^{n-1} + na = n(x^{n-1} + a)$.

458. We have $1+\alpha^2 = (\alpha+i)(\alpha-i)$ and so on for other terms of the first given root $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)$.

Let $P(x) = x^4 + qx^2 + rx + t$ then $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2) = P(i)P(-i) = (1-q+t+ri)(1-q+t-ri) = (1-q+t)^2 + r^2$.

Hence sum of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)$ and 1 is $(1-q+t)^2 + r^2 + 1$ and product is $(1-q+t)^2 + r^2$. Thus, we deduce the equation as

$x^2 - [(1-q+t)^2 + r^2 + 1]x + (1-q+t)^2 + r^2 = 0.$

459. Given α, β, γ are roots of $x^3 + px + q = 0$, so we have

$\alpha + \beta + \gamma = 0, \alpha\beta + \beta\gamma + \gamma\alpha = p, \alpha\beta\gamma = -q.$

Now sum of $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}, \frac{\gamma+1}{\gamma}$ is $\frac{3\alpha\beta\gamma+\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} = \frac{3q-p}{q}$.

Product of these roots taken two at a time is $\frac{3\alpha\beta\gamma+2(\alpha\beta+\beta\gamma+\alpha\gamma)+\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{3q-2p}{q}$

Product of all taken together is $\frac{\alpha\beta\gamma+\alpha\beta+\beta\gamma+\gamma\alpha+\alpha+\beta+\gamma+1}{\alpha\beta\gamma} = \frac{q-p-1}{q}$.

Thus the cubic equation having these roots is $x^3 - \frac{3q-p}{q}x^2 + \frac{3q-2p}{q}x - \frac{q-p-1}{q} = 0 \Rightarrow qx^3 + (p-3q)x^2 + (3q-2p)x + 1 + p - q = 0$.

460. Given equations are $ax^2+bx+c=0$ and $a_1x^2+b_1x+c_1=0$. Let α be the root which satisfies first equation and its reciprocal satisfies the second equation. Then,

$a\alpha^2+b\alpha+c=0$ and $\frac{a_1}{\alpha^2}+\frac{b_1}{\alpha}++c_1=0 \Rightarrow c_1\alpha^2+b_1\alpha+a_1=0$.

By cross multiplication $\alpha = \frac{cc_1-aa_1}{ab_1-bc_1} = \frac{ba_1-b_1c}{cc_1-aa_1} \Rightarrow (aa_1-cc_1)^2 = (bc_1-ab_1)(b_1c-a_1b)$.

461. Let $(\alpha,\beta),(\beta,\gamma),(\gamma,\alpha)$ be three pairs of roots which satisfy the given equation. Then, we have

$\alpha+\beta=-p, \beta+\gamma=-q, \alpha+\gamma=-r$, and hence, sum of all the common roots is obtained by adding these three equations

$\alpha+\beta+\gamma = -\frac{p+q+r}{2}$.

462. The second equation is $(2x\sin\theta-1)^2=0$ i.e. it has only one root, $x=\frac{1}{2\sin\theta}$. Since it has a common root with first equation and first equation has equal roots then that implies that first equation also has one root which is $\frac{1}{2\sin\theta}$.

Observing that coefficients in first equation are cyclic we deduce that $x=1$ will satisfy the equation. Hence, $\frac{1}{2\sin\theta}=1 \Rightarrow \sin\theta=\frac{1}{2}$.

$\Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}$, is the general solution of θ.

463. Let α is a root of $x^2-x+a=0$ then 2α will be a root of $x^2-x+3a=0$. Thus,

$\alpha^2-\alpha+a=0$ and $4\alpha^2-2\alpha+3a=0$. By cross-multiplication, we have

$\frac{\alpha^2}{-3a+2a} = \frac{\alpha}{3a-4a} = \frac{1}{-2+4} \Rightarrow a^2 = -2a \Rightarrow a = 0, -2$.

However, it is given that $a \neq 0, \therefore a = -2$.

464. If $(x_1,y_1),(x_2,y_2)$ are the two solutions, then y_1,y_2 are the two solutions of the quadratic in y. Then we will have two cases:

Case I: $x_1=y_1, x_2=y_2$. In this case the equation becomes $x^2+2lx+m=0$ therefore $a=2l, m=b$.

Case II: $x_1=y_2, x_2=y_1$. In this case $x_1y_1+l(x_1+y_1)+m=0$. Replacing y_1 with x_2, we get $b-al+m=0$.

465. Given that roots of the equation $10x^3 - cx^2 - 54x - 27 = 0$ are in H.P. Therefore if we replace x with $\frac{1}{x}$ then roots will be in A.P.

$\Rightarrow \frac{10}{x^3} - \frac{c^2}{x} - \frac{54}{x} - 27 = 0 \Rightarrow 27x^3 + 54x^2 + cx - 10 = 0.$

Let the roots are $a-d, a, a+d$, then sum of roots $3a = -\frac{54}{27} \Rightarrow a = -\frac{2}{3}$, which is a root of the equation. Substituting this in new equation we find $c = 9$.

466. Given that a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$ such that $c^2 = -ab$.

$\Rightarrow a + b + c = -p, ab + bc + ca = q$ and $abc = -r \Rightarrow c^3 = -abc = r$.

$pq = -(a+b+c)(ab+bc+ca) = -[a^2b + abc + ca^2 + ab^2 + b^2c + abc + abc + bc^2 + c^2a] = -(a^2b + abc + +ca^2 + ab^2 + b^2c + abc + abc - ab^2 - a^2b)$

$= -(3abc + a^2c + b^2) \therefore pq - 4r = -r - a^2c - b^2c \Rightarrow (pq-4r)^3 = -c^3(a^2+b^2+c^2).$

L.H.S. $= (p^2 - 2q)^3.r = -[(a+b+c)^2 - 2(ab+bc+ca)].c^3 = -c^3(a^2+b^2+c^2) =$ R.H.S.

467. If $\alpha + i\beta$ is one root of $x^3 + qx + r = 0$ then $\alpha - i\beta$ will be another root. Let γ be the third root.

Sum of roots $2\alpha + \gamma = 0 \Rightarrow \gamma = -2\alpha$. Since γ is a root of given equation, therefore

$(-2\alpha)^3 - 2q\alpha + r = 0$, and hence we have our equation is $x^3 + qx - r = 0$.

468. Clearly, $\alpha + \beta + \gamma = -\frac{1}{2}, \alpha\beta + \beta\gamma + \gamma\alpha = 0, \alpha\beta\gamma = 2$.

We have to find $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}$

$= \frac{1}{\alpha}(\beta+\gamma) + \frac{1}{\beta}(\gamma+\alpha) + \frac{1}{\gamma}(\alpha+\beta) = \frac{1}{\alpha}\left(-\frac{1}{2} - \alpha\right) + \frac{1}{\beta}\left(-\frac{1}{2} - \beta\right) + \frac{1}{\gamma}\left(-\frac{1}{2} - \gamma\right)$

$= -\frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3 = -\frac{1}{2}\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right) - 3 = -3.$

469. Given equations are $x^3 + px^2 + qx + r = 0$ and $x^3 + p'x^2 + q'x + r' = 0$. Let α, β are common roots. Then putting α and β in the equations and subtracting

$(p-p')\alpha^2 + (q-q')\alpha + (r-r') = 0$ and $(p-p')\beta^2 + (q-q')\beta + (r-r') = 0$.

Thus, the quadratic equation whose roots are α, β is $(p-p')x^2 + (q-q')x + (r-r') = 0$.

470. Let α, β, γ are the roots the given equation and are in G.P. Then, $\beta^2 = \alpha\gamma$ and also $\alpha\beta\gamma = -\frac{d}{a} \Rightarrow \beta = -\left(\frac{d}{a}\right)^{1/3}$.

Substituting the value of β thus obtained in the given equation

$a\left(-\frac{d}{a}\right)+3b\left(-\frac{d}{a}\right)^{2/3}+3c\left(-\frac{d}{a}\right)^{1/3}+d=0 \Rightarrow ac^3=b^3d$, which the needed condition.

471. Let α, β, γ are the roots of the equation $x^3-px^2+qx-r=0$, then

$\alpha+\beta+\gamma=p, \alpha\beta+\beta\gamma+\gamma\alpha=q, \alpha\beta\gamma=r.$

Mean of H.P. $=\beta=\frac{3\alpha\beta\gamma}{\alpha\beta+\beta\gamma+\gamma\alpha}=\frac{3r}{q}$. Substituting this in given equation

$\left(\frac{3r}{q}\right)^3-p\left(\frac{3r}{q}\right)^2+q\frac{3r}{q}-r=0 \Rightarrow 27r^3-9pqr^2+2rq^3=0 \Rightarrow 27r^2+2q^3=9pqr.$

472. Let α, β, γ be the roots of the given equation. Also given that $f(0)$ and $f(-1)$ are odd.

$f(0)=\text{odd} \Rightarrow d=\text{odd}, f(-1)=-1+b-c+d=\text{odd} \Rightarrow b-c=\text{odd}.$

Also, $\alpha\beta\gamma=-d=$ odd which implies α, β, γ are all odd. However,

$b-c=-[(\alpha+\beta+\gamma)-(\alpha\beta+\beta\gamma+\gamma\alpha)]=-[\text{odd}-\text{odd}]=\text{even}$

which contradicts the assumption that all roots are integers.

473. Let α, β, γ are roots of the equation $2x^3+ax^2+bx+4=0$, then

$\alpha+\beta+\gamma=-\frac{a}{2}, \alpha\beta+\beta\gamma+\gamma\alpha=\frac{b}{a}$ and $\alpha\beta\gamma=-2.$

Since all coefficients are positive hence all roots are negative. Let $\alpha=-p, \beta=-q$ and $\gamma=-r$, then

$p+q+r=\frac{a}{2}, pq+qr+rp=\frac{b}{2}$ and $pqr=2.$

Now A.M$\geq$ G.M. $\Rightarrow \frac{p+q+r}{3}\geq(pqr)^{\frac{1}{3}} \Rightarrow \frac{a}{6}\geq 2^{1/3}$

also, because A.M.$\geq$ G.M $\Rightarrow \frac{pq+qr+rp}{3}\geq(pqr)^{2/3} \Rightarrow b\geq 6.4^{1/3}$

Adding we arrive at the required inequality.

474. Given equations are $a_1x^3+b_1x^2+c_1x+d_1=0$ and $a_2x^3+b_2x^2+c_2x+d_2=0$. Let α be a common repeated root then

$a_1\alpha^3+b_1\alpha^2+c_1\alpha+d_1=0$ and $a_2\alpha^3+b_2\alpha^2+c_2\alpha+d_2=0$

Multiplying first equation by a_2 and second equation by a_1 and subtracting, we get

$(a_2b_2-a_1b_2)x^2+(a_2c_1-a_1c_2)x+(a_2d_1-a_1d_2)=0$

Also, the derivatives will be equal to zero because they have a common root i.e.

$3a_1x^2+2b_1x+c_1=0$ and $3a_2x^2+2b_2x+c_2=0$ and hence the condition is

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_1 & 2b_1 & c_1 \\ a_2b_1-a_1b_2 & a_2c_1-a_1c_2 & a_2d_1-a_1d_2 \end{vmatrix}=0$$

475. Given equations are $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^3 + b_2x^2 + c_2x + d_2 = 0$. Because cubic equation has a repeated root therefore its derivative will be equal to 0, and hence

$3a_2x^2 + 2b_2x + c_2 = 0$. Multiplying first equation by a_2x and second by a_1 and subtracting, we get

$(a_1b_2 - a_2b_1)x^2 + (a_1c_2 - a_2c_1)x + a_1d_2 = 0$ and thus from these three equations we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 3a_2 & 2b_1 & c_2 \\ a_1b_2 - a_2b_1 & a_1c_2 - a_2c_1 & a_1d_2 \end{vmatrix} = 0$$

476. Given that α, β, γ are roots of $x^3 - ax^2 + bx - c = 0$ then we have

$\alpha + \beta + \gamma = a, \alpha\beta + \beta\gamma + \gamma\alpha = b$ and $\alpha\beta\gamma = c$.

We know that if a, b, c are sides of a triangle and perimeter is $2s$ then area is given by $\sqrt{s(s-a)(s-b)(s-c)}$, therefore area of required triangle is

$$\Delta = \frac{1}{4}\sqrt{(\alpha+\beta+\gamma)(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)(\beta+\gamma-\alpha)}$$

$$= \frac{1}{4}\sqrt{a(\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha - \alpha^3 - \beta^3 - \gamma^3 - 2\alpha\beta\gamma)}$$

$$= \frac{1}{4}\sqrt{a[4(\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + 3\alpha\beta\gamma) -} \text{(square root continued)}$$

$$\sqrt{(\alpha^3 + \beta^3 + \gamma^3 + 3\alpha^2\beta + 3\alpha\beta^2 + 3\beta\gamma^2 + 3\beta^2\gamma + 3\alpha\gamma^2 + 3\alpha^2\gamma + 6\alpha\beta\gamma) - 8\alpha\beta\gamma]}$$

$$= \frac{1}{4}\sqrt{a[4(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - (\alpha+\beta+\gamma)^3 - 8\alpha\beta\gamma]}$$

$= \frac{1}{4}\sqrt{a(4ab - a^3 - 8c)}$, hence proved.

477. Given $a < b < c < d$ and $\mu(x-a)(x-c) + \lambda(x-b)(x-d) = 0$. Let $f(x) = \mu(x-a)(x-c) + \lambda(x-b)(x-d) = 0$

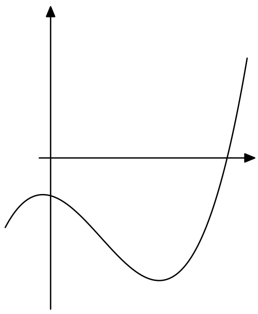
$f(a) = \lambda(a-b)(a-d)$, $f(c) = \lambda(c-b)(c-d) \Rightarrow f(a)f(c) < 0$ and similarly $f(b)f(d) < 0$. Thus the equation has one root between a and c and second root between b and d which implies that both the roots are real for real μ and λ.

478. Let $f(x) = 3x^5 - 5x^3 + 21x + 3\sin x + 4\cos x + 5 = 0$ then $f(\infty) = -\infty$ and $f(\infty) = \infty$.

$f'(x) = 15x^4 - 15x^2 + 21 + 3\cos x - 4\sin x = 15(x^4 - 2x^2 + 1 + x^2) + 6 + 3\cos x - 4\sin x > 0 \; \forall \; x \in (-\infty, \infty)$ which means $f(x)$ is increasing.

Thus, we see that $f(x)$ can have only one real root.

479. The plot is given below(not in linear scale):



$f'(x) = 3x^2 - 20x - 11 = 0 \Rightarrow x = \frac{10\pm\sqrt{133}}{3}$ which shows two points in the graph where tangent is parallel to x-axis. We see that after the higher value of this root the graph is increasing and cuts x-axis. So we substitute the increasing values of x to obtain the integral part of root. $x = \frac{10+\sqrt{133}}{3} \approx 7.16$. We find that $f(8) < f(9) < f(10) < f(11) < 0$ but $f(12) > 0$. So the root lies between 11 and 12, and hence the integral part is $[x] = 11$.

480. $f(x) = (x-m)(b_n x^n + \cdots + b_0) = (x-m)g(x)$ for some $b_0, \ldots, b_m \in \mathbb{Z}$. Then

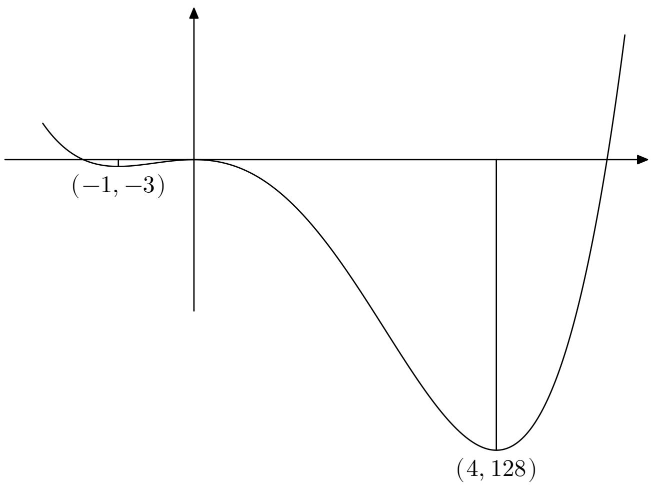
$f(0) = -m.g(0)$ and $f(1) = (1-m).g(1)$ but either $-m$ or $1-m$ is even. Observe that $f(0) = a_n$ and $f(1) = \sum_{i=0}^{n} a_i$.

481. Let $g(x) = e^x f(x)$ then $g''(x) = e^x[f(x) + 2f'(x) + f''(x)]$. $\therefore$ Roots of equation $f(x) + 2f'(x) + f''(x) = 0$ will be same as those of equation $g''(x) = 0$ as $e^x \neq 0$.

Also, since $e^x > 0$, therefore roots of the equation $f(x) = 0$ and $g(x) = 0$ will be same.

Clearly, $g(x) = 0$ will have $\alpha, beta, \gamma$ as roots and hence $g'(x) = 0$ will have roots a between α and $beta$ and a root b between β and γ. Hence equation $g''(x) = 0$ will have a root between a and b, which obviously lies between α and γ.

482. The plot is given below(not in linear scale):



Let $f(x) = x^4 - 4x^3 - 8x^2 \Rightarrow f'(x) = 4x^3 - 12x^2 - 16x = 4x(x-4)(x+1)$ so at $x = -1, 0, 4$ there will be tangents and the direction of $f(x)$ will change.

From the graph it is clear that for $f(x) + a = 0$ to have four real roots $0 \leq a \leq 3$.

483. Let α, β be two distinct roots of the given equation. Then $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$. Using A.M $\geq$ G.M. For $0 < \alpha, 1 - \alpha, \beta, 1 - \beta < 1$

So $\frac{1-\alpha+\alpha}{2} \geq \sqrt{\alpha(1-\alpha)} \Rightarrow \alpha(1-\alpha) \leq \frac{1}{4}$

Similarly $\beta(1-\beta) \leq \frac{1}{4} \Rightarrow \alpha\beta(1-\alpha)(1-\beta) < \frac{1}{16}$

$\Rightarrow \alpha\beta[1 - (\alpha+\beta) + \alpha\beta] < \frac{1}{16} \Rightarrow 16c(a - b + c) < a^2$

However, $\min[c(a-b+c)] = 1$ so $a^2 > 16$ Thus, $a_{\min} = 5$.

Now $2 < \alpha + \beta < 4 \Rightarrow 2a < b < 4a \Rightarrow b_{\min} = 11$.

484. Let $f(x) = (x-1)^5 + (x+2)^7 + (7x-5)^9 - 10$ then $f(-\infty) = -\infty$ and $f(\infty) = \infty$. $f'(x) = 5(x-1)^4 + 7(x+2)6 + 63(7x-5)^8 > 0$ which makes $f(x)$ and increasing function, which means it can cut x-axis only once; yielding only one root.

485. Given, $\sqrt{2(x+3)} - \sqrt{x+2} = 3$. Squaring $2x + 6 + x + 2 - 2\sqrt{2(x+3)(x+2)} = 9$.

Squaring again, $\Rightarrow 8(x+2)(x+3) = (1-3x)^2 \Rightarrow x^2 - 46x - 47 = 0 \Rightarrow x = 47, -1$.

Substituting these in the original equation, we quickly find that $x = 47$ is the actual root and $x = -1$ is the extraneous root. Hence, $\tan\theta = 47, \tan\phi = -1$, and hence

$\tan(\theta + \phi) = \frac{23}{24}$ and $\cot(\theta - \phi) = -\frac{23}{24}$.

486. **Case I:** When $x < -1$ then the equation becomes $-x - 1 + x - 3x + 3 + 2x - 4 = x + 2 \Rightarrow 2x = -4 \Rightarrow x = -2$.

Case II: When $-1 < x < 0$, then $x + 1 + x - 3x + 3 + 2x - 4 = x + 2 \Rightarrow x = x + 2$, which is not possible.

Case III: When $0 < x < 1$, then $x + 1 - x - 3x + 3 + 2x - 4 = x + 2 \Rightarrow -x = x + 2 \Rightarrow x = -1$, which is not possible.

Case IV: When $1 < x < 2$, then $x + 1 - x + 3x - 3 + 2x - 4 = x + 2 \Rightarrow 5x - 6 = x + 2 \Rightarrow x = 2$, which is not possible.

Case V: When $x \geq 2$, then $x + 1 - x + 3x - 3 - 2x + 4 = x + 2 \Rightarrow x + 2 = x + 2$, which is true.

Hence, the solution is $x = -2, x \geq 2$.

487. **Case I:** When $x < -1$, then $\frac{1}{2^{x+1}} - 2^x = -2^x + 1 + 1 \Rightarrow x = -2$.

Case II: When $-1 < x < 0$, then $2^{x+1} - 2^x = -\frac{1}{2^x} + 1 + 1 \Rightarrow 2^{2x+1} - 3.2^x + 1 = 0 \Rightarrow 2^x = 0, 2^x = \frac{1}{2}$, which is not possible.

Case III: When $x \geq 0$, $2^{x+1} - 2^x = 2^x - 1 + 1 \Rightarrow 0 = 0$.

Hence, the solution is $x = -2, x \geq 0$.

488. **Case I:** When $x < 0, y < 0$, then $x^2 - 2x + y = 1, x^2 - y = 1 \Rightarrow x = \frac{1-\sqrt{5}}{2}, y = \frac{1-\sqrt{5}}{2}$

Case II: When $x < 0, y > 0$, then $x^2 - 2x + y = 1, x^2 + y = 1 \Rightarrow -2x = 0, y = 1$

Case III: When $0 < x < 2, y < 0$, then $-x^2 + 2x + y = 1, x^2 - y = 1 \Rightarrow 2x = 2, y = 0$
Case IV: When $0 < x < 2, y > 0$, then $-x^2 + 2x + y = 1, x^2 + y = 1 \Rightarrow -2x^2 + 2x = 0, x = 0, 1, y = 1, 0$

Case V: When $x > 2, y < 0$, then $x^2 - 2x + y = 1, x^2 - y = 1 \Rightarrow 2x^2 - 2x = 2 \Rightarrow x = \frac{1\pm\sqrt{2}}{2} < 2$, which is not possible.

Case VI: When $x > 2, y > 0$, then $x^2 - 2x + y = 1, x^2 + y = 1 \Rightarrow x = 0, y = 1$, which is not possible.

Hence, the solution is $x = 0, y = 1, x = y = \frac{1-\sqrt{5}}{2}, x = 1, y = 0$.

489. Given equation is $|x^2 + 4x + 3| + 2x + 5 = 0 \Rightarrow |(x+1)(x+3)| + 2x + 5 = 0$.

Case I: When $x < -3$, then $x^2 + 4x + 3 + 2x + 5 = 0 \Rightarrow x^2 + 6x + 8 = 0 \Rightarrow x = \frac{-6\pm\sqrt{4}}{2}, \Rightarrow x = -4, -2$. But $x = -2$ is not possible.

Case II: When $-1 < x < -3$, then $-x^2 - 4x - 3 + 2x + 5 = 0 \Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = -1 \pm \sqrt{3}$. But $x = -1 + \sqrt{3}$ is not possible.

Case III: When $x > -1$, then $x^2 + 4x + 3 + 2x + 5 = 0 \Rightarrow x = -4, -2$, which is not possible.

Hence, the solution is $x = -4, -1 - \sqrt{3}$.

490. Given equation upon simplification is $x^4 + 6x^3 - 9x^2 - 162x - 243 = 0$ and $x \neq -3$.

Let us assume that $x^4 + 6x^3 - 9x^2 - 162x - 243 = (x^2 + ax + b)(x^2 + cx + d)$. Comparing coefficients,

$a + c = 6, b + d + ad = -9, ad + bc = -162, bd = -243$, which is four equations with four unknowns. Solving these, we have $a = -3, b = -9, c = 9, d = 27$, and hence, the solution is

$x = \frac{3\pm3\sqrt{5}}{2}, \frac{-9\pm3\sqrt{3}i}{2}$.

491. Given equation is $\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$. We observe that $[x]$ cannot be negative because that will make L.H.S. negative while R.H.S. is positive.

Case I: When $\{x\} \geq \frac{1}{2}$, then $2[x] = 2[x] + 1$. Putting $[x] = n$, where $n \in \mathbb{P}$.

Given equation is $\{x\} = \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3}$. Putting $x = 1, 2, 3, ...$ we observe that $\{x\}$ is not satisfied and the function is decreasing in nature.

Case II: When $\{x\} < \frac{1}{2}$, then $\{x\} = \frac{1}{n} + \frac{1}{2n} - \frac{1}{3}$.

$\Rightarrow \{x\} = \frac{6+3-2n}{6n}$, now we see that numerator becomes negative once $n \geq 5$, thus those values are ruled out. We see that $x = 2, 3, 4$ are the only values which satisfy the given conditions.

492. Let $k = \log_a x \log_{10} a \log_a 5 = \log_a 5^{\log_{10} x}$, then $a^k = 5^{\log_{10} x} = 5^l$(let $\log_{10} x = l$).

Let $m = \log_{10}\left(\frac{x}{10}\right) = \log_{10} x - 1 = l - 1$ and $n = \log_{100} x + \log_4 2 = \frac{1}{2}\log_{10} x + \frac{1}{2}\log_2 2 = \frac{l+1}{2}$.

$\therefore 9^n = 9^{\frac{l+1}{1}} = 3^{l+1} = 3.3^l$.

According to question $\frac{6}{5}.5^l - \frac{3^l}{3} = 3.3^l \Rightarrow 5^{l-2} = 3^{l-2}$, which is possible only if $l = 2 \Rightarrow x = 100$.

493. $5^{\frac{1}{x}} + 125 = 5^{\log_5 6 + 1 + \frac{1}{2x}} = 5^{\log_5 6}.5.5^{\frac{1}{2x}}$

$\Rightarrow 5^{\frac{1}{x}} + 125 = 6.5.5^{\frac{1}{x}} \Rightarrow k^2 + 125 = 30k$, where $k = 5^{\frac{1}{2x}}$

$\Rightarrow k = 5, 25 \Rightarrow x = \frac{1}{2}, \frac{1}{4}$.

494. Taking log of both sides with base x, we have

$\frac{2}{3}\left[(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right] = \frac{1}{2}\log_x 2$

$\Rightarrow \frac{2}{3}\left[(\log_2 x)^2 + \log_2 x - \frac{5}{4} = \frac{1}{2\log_2 x}\right]$ (Putting $\log_2 x = y$)

$\Rightarrow y^2 + y - \frac{5}{4} = \frac{3}{4y} \Rightarrow 4y^3 + 4y^2 - 5y - 3 = 0$.

Observing that sum of coefficients is zero, we quickly deduce that $y = 1$ is one of the solution. Thus, the above equation is reduced to

$4y^2 + 8y + 3 = 0 \Rightarrow y = -\frac{1}{2}, -\frac{3}{2}$.

And hence, $x = 2, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$.

495. Given $3x^2 = 8[x] - 1$. Let $[x] = 1$, then $x = \sqrt{\frac{7}{3}}$ and when $[x] = 2 \Rightarrow x = \sqrt{5}$. However, when $[x] = 3$, $x = \sqrt{\frac{23}{3}} < 3$, which is not possible. Further values are not possible because if we increase $[x]$ linearly then L.H.S. will increase exponentially.

Thus, two possible values are $\sqrt{\frac{7}{3}}$ and $\sqrt{5}$.

496. Let $y = t + \sqrt{t^2 - 1}$, then $\frac{1}{y} = t - \sqrt{t^2 - 1}$ and $y + \frac{1}{y} = 2t$

Thus, the given equation becomes $y^{x^2-2x} + \frac{1}{y^{x^2-2x}} = y + \frac{1}{y}$

Let $z = y^{x^2-2x}$, then given equation is $z - y + \frac{1}{z} - \frac{1}{y} = 0$

$\Rightarrow (z - y)\left(1 - \frac{1}{zy}\right) = 0 \Rightarrow z = y$ or $z = \frac{1}{y} \Rightarrow x = 1, 1 \pm \sqrt{2}$.

497. Multiplying first equation by 2 and subtracting, we get

$5y^2 + 10y - 15 = 0 \Rightarrow y^2 + 2y - 3 = 0 \Rightarrow y = -3, 1$. If $y = -3, -3x + 27 - x - 12 - 7 = 0 \Rightarrow -4x + 8 = 0 \Rightarrow x = 2$. If $y = 1, x + 3 - x + 4 - 7 = 0 \Rightarrow 0 = 0$ so all values of $x \in \mathbb{R}$ will satisfy the equation.

498. We have $2^{x-1}.27^{\overline{x+2}}^{x}$ = 3. Taking log with base 2, we have

$x - 1 + \frac{2x-2}{x+2}\log_2 3 = 0 \Rightarrow x - 1 - \frac{2x-2}{x+2} + \frac{2x-2}{x+2}(\log_2 3 + \log_2 2) = 0$

$\Rightarrow \frac{x^2-x}{x+2} + \frac{2x-2}{x+2}\log_2 6 = 0 \Rightarrow \frac{x-1}{x+2}(x + \log_2 6) = 0 \Rightarrow x = 1, -2\log_2 6$.

499. We have $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1} \Rightarrow 2^{2x} + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}}$

$\Rightarrow 2^{2x-1}.3 = 3^{x-\frac{1}{2}}.4 \Rightarrow 2^{2x-3} = 3^{x-\frac{3}{2}}$.

$x = \frac{3}{2}$ is a solution which satisfies both sides, and is the only solution.

500. We have $\log_{10}[98 + \sqrt{x^3 - x^2 - 12x + 36}] = 2$. Taking antilog,

$\sqrt{x^3 - x^2 - 12x + 36} = 2 \Rightarrow x^3 - x^2 - 12x + 32 = 0 \Rightarrow (x + 4)(x^2 - 5x + 8) = 0$.

We find that the only real solution is $x = -4$.

501. Given, $\log_{2x+3}(6x^2 + 23x + 21) = 4 - \log_{3x+7}(4x^2 + 12x + 9) \Rightarrow \log_{2x+3}(2x + 3)(3x + 7) = 4 - \log_{3x+7}(2x + 3)^2$

$\Rightarrow 1 + \log_{2x+3}(3x + 7) = 4 - 2\log_{3x+7}(2x + 3) \Rightarrow \log_{2x+3}(3x + 7) - \log_{3x+7}(2x + 3) = 3$

Let $\log_{2x+3}(3x + 7) = z$ then $\log_{3x+7}(2x + 3) = \frac{1}{z}$, and given equation becomes

$z + \frac{2}{z} = 3 \Rightarrow z = 1, 2 \Rightarrow 2x + 3 = 3x + 7 \Rightarrow x = -4$, which is not possible as $2x + 3 > 0$ and $3x + 7 = (2x + 3)^2 \Rightarrow 4x^2 + 9x + 2 = 0 \Rightarrow x = -\frac{1}{4}, -2$, but again $x = -2$ is not possible as it makes $2x + 3 < 0$.

Hence, the only possible solution is $x = -\frac{1}{4}$

502. Rewriting the given equation $y^4 - 2x^4 = 1402 \Rightarrow (y^2 + \sqrt{2}x^2)(y^2 - \sqrt{2}x^2) = 701 \times 2$

Suppose x, y are integers then $x^2, y^2 > 0$, which implies

$y^2 + \sqrt{2}x^2 = 701$ and $y^2 - \sqrt{2}x^2 = 2$. Adding, $2y^2 = 703$, which has no integral solution.

503. Given equation is $|x-1|^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$. Clearly, $x > 1$ for $\log_x 9$ to be defined. So the equation becomes

$(x-1)^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$, taking log of both sides

$(2\log_3 x - 4\log_x 3 - 7)[\log(x-1)] = 0$. So either

$2\log_3 x - 4\log_x 3 - 7 = 0$ or $\log(x-1) = 0 \Rightarrow x - 1 = 1 \Rightarrow x = 2$.

Let $\log_3 x = z$ then $\log_x 3 = \frac{1}{z}$, so we have

$2z^2 - 7z - 4 = 0 \Rightarrow z = 4, -\frac{1}{2}$ which gives us $x = 81, \frac{1}{\sqrt{3}}$ but $x > 1$ so $x = 81$ is the second solution.

504. One of the solutions is $\cos x = 1$ which will make exponent $\frac{1}{2}$ equalizing both sides. Thus, $x = 2n\pi$ is our first solution.

The second solution can be obtained by setting exponent to zero i.e. $\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2} = 0$ giving us $\sin x = 1, \frac{1}{2}$ but if $\sin x = 1$ then $\cos x = 0$, which makes th equation invalid.

Therefore, $\sin x = \frac{1}{2}$ is our second solution. Thus, $x = n\pi + (-1)^n\frac{\pi}{6}, n \in \mathbb{I}$.

505. We have the equation $(x+a)(x+1991) + 1 = 0 \Rightarrow (x+a)(x+1991) = -1$

Either $x + a = 1$ and $x + 1991 = -1 \Rightarrow a = 1993$ or $x + a = -1$ and $x + 1991 = 1 \Rightarrow a = 1989$.

506. Given equation is $2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7 \Rightarrow 2^{\sin^2 x} + \frac{10}{\sin^2 x} = 7$.

Let $2^{\sin^2 x} = y$, then the equation becomes $y + \frac{10}{y} = 7 \Rightarrow y^2 - 7y + 10 = 0 \Rightarrow y = 2, 5$.

Now $y = 5$ makes $\sin^2 x > 1$, which is not possible. If $y = 2 \Rightarrow 2^{\sin^2 x} = 2 \Rightarrow \sin x = \pm 1 \Rightarrow x = n\pi + (-1)^n\left(\pm\frac{\pi}{2}\right)$.

507. Given equation is $x + \log_{10}(1 + 2^x) = x\log_{10} 5 + \log_{10} 6 \Rightarrow x(1 - \log_{10} 5) + \log_{10}(1 + 2^x) = \log_{10} 6$

$\Rightarrow x(\log_{10} - \log_{10} 5) + \log_{10}(1 + 2^x) = \log_{10} 6 \Rightarrow x\log_{10} 2 + \log_{10}(1 + 2^x) = \log_{10} 6$

$\Rightarrow \log_{10} 2^x(1 + 2^x) = \log_{10} 6 \Rightarrow 2^x(1 + 2^x) = 6 \Rightarrow 2^x = 2, -3$ but for real values of x, $2^x \neq -3$, thus, $2^x = 2 \Rightarrow x = 1$.

508. Given equation is $\log_a(ax).\log_x(ax)+\log_{a^2}(a)=0 \Rightarrow (1+\log_a x)(1+\log_x a)+\frac{1}{2}=0$

$\Rightarrow 2(\log_a x)^2+5\log_a x+2=0 \Rightarrow \log_a x=-2,-\frac{1}{2} \Rightarrow x=\frac{1}{a^2},\frac{1}{\sqrt{a}}$.

509. Given equation is $\sqrt{11x-6}+\sqrt{x-1}=\sqrt{4x+5}$, squaring, we get

$11x-6+4x+5+2\sqrt{(11x-6)(x-1)}=4x+5 \Rightarrow \sqrt{(11x-6)(x-1)}=-4x+6$

Squaring again, $11x^2-17x+6=16x^2-48x+36 \Rightarrow 5x^2-31x+30=0 \Rightarrow x=\frac{6}{5},5$ but $x=5$ does not satisfy the given equation, and is result of squaring.

510. Given equation is $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$. Sqauring,

$3x^2-7x-30=(x-5)^2+2x^6-7x-5+2(x-5)\sqrt{2x^2-7x-5}$

$\Rightarrow (x-5)(5-\sqrt{2x^2-7x-5})=0$, so $x=5$ is one of the solutions. The other solution will be given by

$5=\sqrt{2x^2-7x-5}$, squaring again, $2x^2-7x-30=0 \Rightarrow x=6,-\frac{5}{2}$, but $x=-\frac{5}{2}$ does not satisfy the equation.

Hence, $x=5,6$ are the solutions.

511. Given euations are $y=2[x]+3$ and $y=3[x-2] \Rightarrow y=3[x]-6$. Solving yields $y=21,[x]=9$ giving $[x+y]=30$.

512. $\sum_{i=1}^{n}(x-a_i)^2=nx^2-2(a_1+a_2+\cdots+a_n)x+(a_1^2+a_2^2+\cdots+a_n^2)$, which is a quadratic equation in x and coefficient of x^2 is $n>0$, therefore, this quadratic equation will have least value at $x=\frac{a_1+a_2+\cdots+a_n}{n}$.

513. Let the quotient be $\frac{n}{n^2-1}$, $n\in\mathbb{N}$. According to question,

$\frac{n+2}{n^2-1+2}>\frac{1}{3} \Rightarrow n^2-3n-5<0 \Rightarrow \frac{3}{2}-\frac{\sqrt{29}}{2}<n<\frac{3}{2}+\frac{\sqrt{29}}{2}$.

Also, $0<\frac{n-3}{n^2-1-3}<\frac{1}{10} \Rightarrow 0<\frac{n-3}{n^2-4}<\frac{1}{10}$

Taking the first inequality, $\frac{n-3}{n^2-4}>0 \Rightarrow -2<n<2$ or $3<n<\infty$.

Taking the second inequality $\frac{n-3}{n^2-4}<\frac{1}{10} \Rightarrow \frac{n^2-10n+26}{10(4-n^2)}<0 \Rightarrow -n<-2$ or $n>2$.

Thus, we have $3<n<\frac{3}{2}+\frac{\sqrt{29}}{2} \Rightarrow n=4$ (since n is a natural number)

Thus, we deduce the quotient to be $\frac{4}{4^2-1}=\frac{4}{15}$.

514. Let $f(x) = ax^2 + bx + c$, then $g(x) = f(x) + f'(x) + f''(x) = ax^2 + bx + c + 2ax + b + 2a = ax^2 + (b + 2a)x + 2a + b + c$.

Given $ax^2 + bx + c > 0\ \forall\ x \in \mathbb{R} \therefore b^2 - 4ac < 0$ and $a > 0$.

Discriminant of $g(x)$, $D = (b + 2a)^2 - 4a(2a + b + c) = (b^2 - 4ac) - 4a^2 < 0$ and $a > 0$.

Thus, $g(x) > 0\ \forall\ x \in \mathbb{R}$.

515. From given equation it is clear that $f(x) \geq 0\ \forall\ x \in \mathbb{R}$ and

$f(x) = (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2) \geq 0\ \forall\ x \in \mathbb{R}$

$\therefore$ Discriminant of its corresponsing equation $D \leq 0$, because coefficient of x^2 is positive.

$\Rightarrow 4(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 - 4(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \leq 0$

$\Rightarrow (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$.

516. Given equation is $x(x+1)(x+m)(x+m+1) = m^2 \Rightarrow [x^2 + (m+1)x + m][x^2 + (m+1)x] = m^2$

$\Rightarrow y^2 + my - m^2 = 0$, where $y = x^2 + (m+1)x$. $\therefore y = \frac{-m \pm \sqrt{5}}{2}$

$\Rightarrow 2x^2 + 2(m+1)x - (\sqrt{5} - 1)m = 0$ and $2x^2 + 2(m+1)x + (\sqrt{5} + 1)m = 0$. Thus, given equation will have four real roots if these two equations have two real roots each.

$\therefore 4(m+1)^2 + 8(\sqrt{5} - 1)m > 0$ and $4(m+1)^2 - 8(\sqrt{5} + 1)m > 0$

$\Rightarrow m^2 + 2\sqrt{5}m + 1 > 0$ and $m^2 - 2\sqrt{5}m + 1 > 0$. Thus, $|m| > 2 + \sqrt{5}$ or $|m| < \sqrt{5} - 2$.

517. Given equation is $x^4 + (a-1)x^3 + x^2 + (a-1)x + 1 = 0 \Rightarrow \left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x} + (a-1)\left(x + \frac{1}{x}\right) + 1 = 0$

$\Rightarrow y^2 + (a-1)y - 1 = 0$, where $y = x + \frac{1}{x}$

$\therefore y = \frac{-(a-1) \pm \sqrt{(a-)^2 + 4}}{2} = -\frac{(a-1) \mp \sqrt{(a-1)^2 - 4}}{2}$

$\Rightarrow 2x^2 + [(a-1) - \sqrt{(a-1)^2 + 4}]x + 2 = 0$ and $2x^2 + [(a-1) + \sqrt{(a-1)^2 + 4}]x + 2 = 0$

Let α, β be roots of first and γ, δ be the roots of second, then

$\alpha + \beta = -\frac{(a-1) - \sqrt{(a-1)^2 + 4}}{2}$ and $\alpha\beta = 1$, $\gamma + \delta = -\frac{(a-1) + \sqrt{(a-1)^2 + 4}}{2}$ and $\gamma\delta = 1$

$\because \sqrt{(a-1)^2 + 4} > a - 1$, therefore, $\alpha + \beta > 0$ and $\alpha\beta > 0$, which means α, β are positive. Thus, the equation $2x^2 + [(a-1) + \sqrt{(a-1)^2 + 4}]x + 2 = 0$ must have two negative roots.

For both roots to be negative $D > 0 \Rightarrow [(a-1)+\sqrt{(a-1)^2+4}]^2 - 16 > 0$

$\Rightarrow a-1+\sqrt{(a-1)^2+4}-4 > 0[\because a-1+\sqrt{(a-1)^2+4}]+4 > 0$

$\Rightarrow \sqrt{(a-1)^2+4} > 5-a \Rightarrow a \geq 5$ or $(a-1)^2+4 > (5-a)^2$ where $a > 5$.

$\Rightarrow \frac{5}{2} < a < \infty$.

518. Given equation is $x^4+2ax^3+x^2+2ax+1=0 \Rightarrow x^2+\frac{1}{x^2}+2a\left(x+\frac{1}{x}\right)+1=0$

$\Rightarrow \left(x+\frac{1}{x}\right)^2 - 2.x.\frac{1}{x}+2a\left(x+\frac{1}{x}\right)+1=0 \Rightarrow y^2+2ay-1=0$, where $y = x+\frac{1}{x}$

$\Rightarrow y = -a \pm \sqrt{a^2+1}$. When $y = -a+\sqrt{a^2+1} = x+\frac{1}{x} \Rightarrow x^2+(a-\sqrt{a^2+1})x+1=0$, and, when $y = -a-\sqrt{a^2+1} = x+\frac{1}{x} \Rightarrow x^2+(a+\sqrt{a^2+1})x+1=0$.

Let α,β be roots of first equation and γ,δ be roots of second equation. Then,

$\alpha+\beta = \sqrt{a^2+1}-a$, $\alpha\beta = 1$ and $\gamma+\delta = -(a+\sqrt{a^2+1})$, $\gamma\delta = 1$.

Clearly α, β are both imaginary or positive so from question γ, δ both must be negative. $\Rightarrow D \geq 0$, which leads to

$(a+\sqrt{a^2+1})^2-4 > 0 \Rightarrow \sqrt{a^2+1} > 2-a \Rightarrow \frac{3}{4} < a < \infty$.

519. Given system of equations can be written as $ax_1^2+(b-1)x_1+c = x_2-x_1, ax_2^2+(b-1)x_1+c = x_3-x_2, \dots, ax_{n-1}^2+(b-1)x_{n-1}+c = x_n-x_{n-1}, ax_n^2+(b-1)x_n+c = x_1-x_n$

$\therefore f(x_1)+f(x_2)+\cdots+f(x_n) = 0 \quad (1)$

Case I: When $(b-1)^2-4ac < 0$.

In this case $f(x_1), f(x_2), \dots, f(x_n)$ will have same sign as that of $a \therefore f(x_1)+f(x_2)+\cdots+f(x_n) \neq 0$.

Hence, the given system of equations has no solution.

Case II: When $(b-1)^2-4ac = 0$.

In this case $f(x_1), f(x_2), \dots, f(x_n) \geq 0$ or $f(x_1), f(x_2), \dots, f(x_n) \leq 0$, From (1), $f(x_1)+f(x_2)+\cdots+f(x_n) = 0 \Rightarrow f(x_1) = f(x_2) = \cdots = f(x_n) = 0$

But $f(x_i) = 0 \Rightarrow ax_i^2+(b-1)x_i+c = 0 \Rightarrow x_i = \frac{1-b}{2a}$

$\therefore x_1 = x_2 = \cdots = x_n = \frac{1-b}{2a}$.

Case III: When $(b-1)^2-4ac > 0$.

Roots of equation $ax^2 + (b-1)x + c = 0$ are $\alpha,\beta = \frac{1-b\pm\sqrt{(1-b)^2-4ac}}{2a}$.

If $x_1, x_2, \ldots, x_n$ lie between α and β, then $f(x_1) + f(x_2) + \cdots + f(x_n) \neq 0$ (because it is < 0 or > 0 as $a > 0$ or $a < 0$)

If $x_1, x_2, \ldots, x_n$ lie in $(-\infty, \alpha)$ or (β, ∞) then also $f(x_1) + f(x_2) + \cdots + f(x_n) \neq 0$.

If all roots are either α or β then $f(x_1) + f(x_2) + \cdots + f(x_n) = 0$.

520. **Case I:** When $x > 1$. We will have $x^2 - \frac{3}{16} > 0 \Rightarrow x < -\frac{\sqrt{3}}{4}$ or $x > \frac{\sqrt{3}}{4}$, and $x^2 - \frac{3}{16} > x^4 \Rightarrow \frac{1}{4} < x^2 < \frac{3}{4}$.

Thus, we see that no value of x satisfies all these inequalities at the same time.

Case II: When $x < 1$. We will have $x^2 - \frac{3}{16} > 0$, which will impose same set of inequalities, and $x^2 - \frac{3}{16} < x^4 \Rightarrow x^2 < \frac{1}{4}$ or $x^2 > \frac{3}{4}$.

Thus, $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, 1\right)$ represents the set of solution.

521. Given $\log_{\frac{1}{2}} x^2 \geq \log_{\frac{1}{2}}(x+2) \Rightarrow x^2 \leq x + 2 \Rightarrow x^2 - x - 2 \leq 0 \Rightarrow -1 \leq x \leq 2, x \neq 0$. For lagrithm to be defined $x \neq 0$ and $x > -2$.

Also, $49x^2 - 4m^4 \leq 0 \Rightarrow -\frac{2}{7}m^2 \leq \frac{2}{7}m^2$.

According to question, $[-1, 2] \subseteq \left[-\frac{2}{7}m^2, \frac{2}{7}m^2\right]$

$\therefore -\frac{2}{7}m^2 \leq -1 \Rightarrow m^2 \geq \frac{7}{2}$ and $\frac{2}{7}m^2 \geq 2 \Rightarrow m^2 \geq 7$.

Thus, $-\infty < m \leq -\sqrt{7}$ or $\sqrt{7} \leq m < \infty$.

522. We have to find a for which $1 + \log_5(x^2+1) \geq \log_5(ax^2 + 4x + a)$ is valid $\forall\, x \in \mathbb{R}$.

$\Rightarrow \log_5 5 + \log_5(x^2+1) \geq \log_5(ax^2 + 4x + a) \Rightarrow 5(x^2+1) \geq ax^2 + 4x + a$

$\Rightarrow (5-a)x^2 - 4x + 5 - a \geq 0$

$D \leq 0 \Rightarrow 16 - 4(5-a^2) \leq 0 \Rightarrow a \leq 3$ or $a \geq 7$ and $5 - a > 0 \Rightarrow a < 5$. Combining $-\infty < a \leq 3$.

For $\log_5(ax^2 + 4x + a)$ to be defined $ax^2 + 4x + a > 0$ for all real x. So $D < 0 \Rightarrow 16 - 4a^2 < 0 \Rightarrow a < -2$ or $a > 2$ and $a > 0$. Combining $2 < a < \infty$.

Thus, common values are given by $2 < a \leq 3$.

523. $2x^2 + 2x + \frac{7}{2} > 0\ \forall\, x \in \mathbb{R}$ because discriminant of corresponding equation is less than 0 and coefficient of x^2 is greater than 0.

Thus, $\log_x\left(2x^2+2x+\frac{7}{2}\right)$ is defined $\forall\ x \in \mathbb{R}$.

For $\log_x a(x^2+1)$ to be defined $0 < a < \infty$.

Given equation is $1+\log_2\left(2x^2+2x+\frac{7}{2}\right) \geq \log_2(ax^2+a) \Rightarrow \log_2 2+\log_2\left(2x^2+2x+\frac{7}{2}\right) \geq \log_2(ax^2+a)$

$\Rightarrow \log_2 2\left(2x^2+2x+\frac{7}{2}\right) \geq \log_2(ax^2+a) \Rightarrow 4x^2+4x+7 \geq ax^2+a$

$\Rightarrow (4-a)x^2+4x+7-a \geq 0$. Let D be discriminant of corresponding equation, then

$D = 16-4(4-a)(7-a) = 4(4-a^2+11a-28) = -4(a-3)(a-8)$.

When $D > 0, a \neq 4, 3 < a < 8$

When $D = 0 \Rightarrow a = 3, 8$. When $a = 3$, the equation becomes $x^2+4x+4 \geq 0\ \forall\ x \in \mathbb{R}$.

When $a = 8$, the equation becomes $-(2x-1)^2 = 0$, when $x = \frac{1}{2}$.

When $a = 4$, the equation becomes $4x+3 \geq 0$ for infinitely many real values of x.

The equation will be satisfied for $a < 4$ and $D < 0 \Rightarrow (a-3)(a-8) > 0 \Rightarrow a < 3$ or $a > 8 \therefore -\infty < a < 3$.

Combining all these we get possible values of a by $-\infty < a \leq 8$.

524. Let $a-c = \alpha, b-c = \beta, c+x = u$, then for $\sqrt{a-c}$ and $\sqrt{b-c}$ to be real $\alpha,\beta \geq 0$. Also, as $x > -c \Rightarrow u > 0$.

Let xm $y = \frac{(a+x)(b+x)}{c+x} = \frac{(u+\alpha)(u+\beta)}{u} = \frac{u^2+(\alpha+\beta)u+\alpha\beta}{u} = u+\alpha+\beta+\frac{\alpha\beta}{u}$

$\Rightarrow u^2+(\alpha+\beta-y)+\alpha\beta = 0$, and because u is real.

$\therefore (\alpha+\beta-y)^2-4\alpha\beta \geq 0 \Rightarrow y^2-2(\alpha+\beta)y+(\alpha-\beta)^2 \geq 0$

Corresponding roots are $y = \frac{2(\alpha+\beta)\pm\sqrt{4(\alpha+\beta)^2-4(\alpha-\beta)^2}}{2} = \alpha+\beta \pm 2\sqrt{\alpha\beta}$

$= (\sqrt{\alpha}+\sqrt{\beta})^2, (\sqrt{\alpha}-\sqrt{\beta})^2$

But if $y \leq (\sqrt{\alpha}-\sqrt{\beta})^2 \Rightarrow y-(\alpha+\beta)+2\sqrt{\alpha\beta} \leq 0$ is not posssible, because $y-\alpha-\beta = u+\frac{\alpha\beta}{u} > 0$.

Thus, least values of y is $(\sqrt{\alpha}+\sqrt{\beta})^2 = (\sqrt{a-c}+\sqrt{b-c})^2$.

525. Let $y = 4(a-x)[x-a+\sqrt{a^2+b^2}] = 4z(-z+k)$, where $z = a-x$ and $k = \sqrt{a^2+b^2} \Rightarrow 4x^2-4kz+y = 0$

Because z is real, therefore, $D \geq 0 \Rightarrow 18k^2-16y \geq 0 \Rightarrow y \leq (a^2+b^2)$

$\Rightarrow y \ngtr a^2+b^2$.

526. Let $y = \frac{x^2+2x\cos 2\alpha+1}{x^2+2x\cos 2\beta+1} \Rightarrow (y-1)x^2 + 2(y\cos 2\beta - \cos 2\alpha) + y - 1 = 0$

Because x is real, therefore, $D \geq 0 \Rightarrow 4(y\cos 2\beta - \cos 2\alpha)^2 - 4(y-1)^2 \geq 0$

$(1 - \cos^2 2\beta)y^2 + 2(\cos 2\alpha \cos 2\beta - 1)y + 1 - \cos^2 2\alpha \leq 0 \Rightarrow \sin^2 2\beta y^2 + 2(\cos 2\alpha \cos 2\beta - 1)y + \sin^2 2\alpha \leq 0$

Roots of corresponding equation are $\frac{2(1-\cos 2\alpha \cos 2\beta) \pm 4\sin(\alpha-\beta)\sin(\alpha+\beta)}{2\sin^2 2\beta}$

$= \frac{\sin^2\alpha}{\sin^2\beta}, \frac{\cos^2\alpha}{\cos^2\beta}$, which are real and unequal and dicriminant is also greater than zero. Coefficient of y^2 is also greater than zero.

Thus, y does not lie between the roots.

527. Let $y = \frac{2a(x-1)\sin^2\alpha}{x^2-\sin^2\alpha} \Rightarrow yx^2 - 2a\sin^2\alpha x + (2a-y)\sin^2\alpha = 0.$

Because x is real, therefore, $D \geq 0 \Rightarrow 4a^2\sin^4\alpha - 4y(2a-y)\sin^2\alpha \geq 0 \Rightarrow a^2\sin^2\alpha - y(2a-y) \geq 0 \Rightarrow y^2 - 2ay + a^2\sin^2\alpha \geq 0.$

Roots of the corresponding equation are $y = 2a\sin^2\frac{\alpha}{2}, 2a\cos^2\frac{\alpha}{2}$.

Hence, y does not lie between these roots.

528. Let $y = \tan(x+\alpha)/\tan(x-\alpha) = \left(\frac{p+q}{1-pq}\right)\left(\frac{1+pq}{p-q}\right)$, where $p = \tan x$ and $q = \tan\alpha$.

$\Rightarrow y = \frac{qp^2+(1+q^2)p+q}{-qp^2+(1+q^2)p-q}$

$\Rightarrow q(y+1)p^2 + (1+q^2)(1-y)p + q(1+y) = 0$, but p is real, and hence $D \geq 0$.

$\Rightarrow (1+q^2)^2(1-y)^2 - 4q^2(1+y)^2 \geq 0 \Rightarrow (1-q^2)^2y^2 - 2[(1+q^2)^2 + 4q^2]y + (1-q^2)^2 \geq 0$

Disrciminant of corresponding equation is $64(1+q^2)^2q^2$ and roots are $\left(\frac{1-q}{1+q}\right)^2, \left(\frac{1+q}{1-q}\right)^2$

So roots are $\left(\frac{1-\tan\alpha}{1+\tan\alpha}\right)^2, \left(\frac{1+\tan\alpha}{1-\tan\alpha}\right)^2 = \tan^2\left(\frac{\pi}{4}-\alpha\right), \tan^2\left(\frac{\pi}{4}+\alpha\right)$.

Since roots are real and unequal and the coefficient of y^2 is greater than zero, and hence, y cannot lie between the given values.

529. Let $y = \frac{ax^2+3x-4}{3x-4x^2+a} \Rightarrow (4y+a)x^2 + 3(1-y)x - (ay+4) = 0.$

Since x is real, therefore, $D \geq 0 \Rightarrow 9(1-y)^2 + 4(4y+a)(ay+4) \geq 0 \Rightarrow (9+16a)y^2 + 2(2a^2+23)y + 9 + 16a \geq 0$

Discriminant of corresponding equation $D' = 4(2a^2+23)^2 - 4(9+16a)^2 = 16(a+4)^2(a-1)(a-7)$

If $1 < a < 7 \Rightarrow D' < 0$ and $9 + 16a > 0$, then $(9 + 16a)y^2 + 2(2a^2 + 23)y + 9 + 16a > 0 \ \forall \ y \in \mathbb{R}$.

Hence, given expression can assume any value if $1 < a < 7$.

530. Let $y = \frac{(ax-b)(dx-c)}{(bx-a)(cx-d)} = \frac{adx^2-(bd+ac)x+bc}{bcx^2-(ac+bd)x+ad}$

$\Rightarrow (bcy - ad)x^2 + (1 - y)(bd + ac) + ady - bc = 0.$

Because x is real, therefore, $D \geq 0$

$\Rightarrow (bd + ac)^2(1 - y)^2 - 4(bcy - ad)(ady - bc) \geq 0 \Rightarrow (bd - ac)^2 y^2 - 2[(bd + ac)^2 - 2(a^2d^2 + b^2c^2)]y + (bd - ac)^2 \geq 0$

Discriminant of corresponding equation $D' = -16(ad - bc)(a^2 - b^2)(c^2 - d^2)$

Because $a^2 - b^2$ and $c^2 - d^2$ are having same sign, therefore, $D' \leq 0$.

Hence, y can have any real value.

Answers of Chapter 5
Combinatorics

1. Given, $P_4^n = 360 \Rightarrow n(n-1)(n-2)(n-3) = 3 \times 4 \times 5 \times 6 \Rightarrow n = 6.$

2. Given, $P_3^n = 9240 \Rightarrow n(n-1)(n-2) = 20 \times 21 \times 22 \Rightarrow n = 22.$

3. Given, $P_r^{10} = 720 = 8 \times 9 \times 10 \Rightarrow r = 3.$

4. Given, $P_{n-1}^{2n+1} : P_n^{2n-1} = 3 : 5 \Rightarrow \frac{(2n+1)!}{(n+2)!} \cdot \frac{(2n-1)!}{(n--1)!} = \frac{3}{5} \Rightarrow \frac{(2n+1)2n}{n(n+1)(n+2)} = \frac{3}{5}$

   $\Rightarrow 3n^2 - 11n - 4 = 0 \Rightarrow n = 4, -\frac{1}{3}$, but n is an integer. Hence, $n = 4$.

5. Given, $P_4^n = 12 \times P_2^n \Rightarrow n(n-1)(n-2)(n-3) = 12 \times n(n-1) \Rightarrow n^2 - 5n - 6 = 0 \Rightarrow n = 6, -1.$

   But $n > 0 \Rightarrow n = 6$ is the only solution.

6. Given, $P_5^n = 20 \times P_3^n \Rightarrow (n-3)(n-4) = 20 \Rightarrow n^2 - 7n - 8 = 0 \Rightarrow n = 8, -1.$

   But $n > 0 \Rightarrow n = 8$ is the only solution.

7. Given, $P_4^n : P_4^{n+1} = 3 : 4 \Rightarrow \frac{n!}{(n-4)!} \cdot \frac{(n-3)!}{(n+1)!} = \frac{3}{4}$

   $\Rightarrow \frac{(n-3)}{n+1} = \frac{3}{4} \Rightarrow 4n - 12 = 3n + 3 \Rightarrow n = 15.$

8. Given $P_r^{20} = 6840 = 18 \times 19 \times 20 \Rightarrow r = 3.$

9. Given, $P_{k+1}^{k+5} = \frac{11(k-1)}{2} . P_k^{k+3} \Rightarrow (k+5)(k+4)(k+3) \cdots 6.5 = \frac{11(k-1)}{2} . (k+3)(k+2) \cdots 5.4$

   $\Rightarrow (k+5)(k+4) = 22k - 22 \Rightarrow k^2 - 13k + 42 = 0 \Rightarrow k = 6, 7.$

10. Given, $P_{r+1}^{22} : P_{r+2}^{20} = 11 : 52 \Rightarrow \frac{22!}{(21-r)!} \cdot \frac{(18-r)!}{20!} = \frac{11}{54}$

    $\Rightarrow \frac{22.21}{(21-r)(20-r)(19-r)} = \frac{11}{52} \Rightarrow (21-r)(20-r)(19-r) = 42.52 = 12.13.14 \Rightarrow r = 7.$

11. Given, $P_2^{m+n} = 90 \Rightarrow (m+n)(m+n-1) = 10.9 \Rightarrow m+n = 10$, and

    $P_2^{m-n} = 30 \Rightarrow (m-n)(m-n-1) = 6.5 \Rightarrow m-n = 6 \Rightarrow m = 8, n = 2.$

12. Given, $P_r^{12} = 11880 \Rightarrow \frac{12!}{(12-r)!} = 9 \times 10 \times 11 \times 12 \Rightarrow r = 4.$

13. Given, $P_{r+6}^{56} : P_{r+3}^{54} = 30800 : 1 \Rightarrow \frac{56!}{(50-r)!} \cdot \frac{(51-r)!}{54!} = 30800$

    $\Rightarrow 56 \times 55 \times (51-r) = 30800 \Rightarrow 51 - r = 10 \Rightarrow r = 41.$

14. $n.P_n^n = n.n! = (n+1-1).n! = (n+1)! - n!$. Similarly, $(n-1).P_{n-1}^{n-1} = n! - (n-1)!, \dots, 2.P_2^2 = 3! - 2!, 1.P_1^1 = 2! - 1!$.

    Adding these, we obtain L.H.S. $= (n+1)! - 1! = P_{n+1}^{n+1} - 1 =$ R.H.S.

15. Given, $C_{30}^n = C_4^n \Rightarrow \frac{n!}{30!(n-30)!} = \frac{n!}{4!(n-4)!}$

    Equating $n - 30 = 4$ and $n - 4 = 30$, we obtain $n = 34$ from both.

16. Given, $C_{12}^n = C_8^n \Rightarrow \frac{n!}{(n-12)!12!} = \frac{n!}{(n-8)!8!} \Rightarrow n - 12 = 8$ and $n - 8 = 12$. Thus, $n = 20$

    $C_{17}^{20} = \frac{20!}{17!3!} = \frac{20\times 19\times 18}{3\times 2} = 1140$, and $C_{20}^{22} = \frac{22!}{20!2!} = \frac{22\times 21}{2} = 231$.

17. Given, $C_r^{18} = C_{r+2}^{1}8 \Rightarrow \frac{18!}{(18-r)!r!} = \frac{18!}{(r+2)!(16-r)!} \Rightarrow 18 - r = r + 2 \Rightarrow r = 8$ and $r = 16 - r \Rightarrow r = 8$.

    $C_6^r = C_6^8 = \frac{8!}{6!2!} = 28$.

18. Given, $C_{n-4}^n = 15 \Rightarrow \frac{n!}{(n-4)!4!} = 15 \Rightarrow n(n-1)(n-2)(n-3) = 3\times 4\times 5\times 6 \Rightarrow n = 6$.

19. Given, $C_r^{15} : C_{r-1}^{1}5 = 11 : 5 \Rightarrow \frac{15!}{(15-r)!r!} \cdot \frac{(r-1)!(16-r!)}{15!} = \frac{11}{5} \Rightarrow \frac{16-r}{r} = \frac{11}{5} \Rightarrow r = 5$.

20. Given, $P_r^n = 2520 \Rightarrow \frac{n!}{(n-r)!} = 2520$ and $C_r^n = 21 \Rightarrow \frac{n!}{(n-r)!r!} = 21$

    $\Rightarrow \frac{2520}{r!} = 21 \Rightarrow r! = 120 \Rightarrow r = 5 \Rightarrow n(n-1)(n-2)(n-3)(n-4) = 2520 = 7\times 6\times 5\times 4\times 3 \Rightarrow n = 7$.

21. We know that $C_r^n = C_{n-r}^n \Rightarrow C_{13}^{20} = C_7^{20}$ and $C_{14}^{20} = C_6^{20}$.

    $\therefore C_{13}^{20} + C_{14}^{20} - C_6^{20} - C_7^{20} = 0$.

22. Given, $C_{r-1}^n = 36 \Rightarrow \frac{n!}{(n-r+1)(r-1)!} = 36$, $C_r^n = 84 \Rightarrow \frac{n!}{(n-r)!r!} = 84$, and $\frac{n!}{(n-r-1)!(r+1)!} = 126$.

    Dividing first two, $\frac{r}{n-r+1} = \frac{3}{7} \Rightarrow 3n = 10r - 3$, and dividing last two

    $\frac{r+1}{n-r} = \frac{2}{3} \Rightarrow 2n = 5r + 3$. Solving these two equations, we have $n = 9, r = 3$.

23. Thousand's place can be filled in 5 ways, hundred's place can be filled in 4 ways, ten's place can be filled in 3 ways and unit's place can be filled in 2 ways.

    Thus, total number of 4 digit numbers is $5\times 4\times 3\times 2 = 120$.

    Alternatively, it is $P_4^5 = 120$.

24. Hundred's place can be filled in 3 ways excluding $0, 2, 3$, ten's place can be filled in 5 ways and unit's place can be filled in 4 ways.

    Thus, no. of numbers between 400 and 1000 is $5\times 4\times 3 = 60$.

25. **Case I:** When the number is of three digits i.e. between 300 and 1000.

Hundred's place can be filled in 3 ways using $3, 4$ or 5, ten's place can be filled in 5 ways and unit's place can be filled in 4 ways.

Thus, total no. of three digit numbers is $5 \times 4 \times 3 = 60$.

Case II: When the number is of four digits i.e. between 1000 and 3000.

Thousand's place can be filled in 2 ways using 1 or 2. Three remaining places can be filled in P_3^5 ways i.e. 60 ways.

Therefore, total no. of four digit numbers is $2 \times 60 = 120$.

Thus, total no. of numbers between 300 and 3000 is $60 + 120 = 180$.

26. **Case I:** When 2 is at thousands place.

Hundred's placec can be filled in 4 ways using $3, 4, 5, 6$. Two remaining places can be filled in P_2^5 i.e. 20 ways. Number of numbers formed in this case is $4 \times 20 = 80$.

Case II: When thousands place is occupied by $3, 4, 5$ or 6.

We see that there are four ways to fill thousands place. Three remaining placed can be filled in P_3^6 i.e. 120 ways. Number of numbers formed in this case is $4 \times 120 = 480$.

Hence, total no. of numbers is $80 + 480 = 560$.

27. **Case I:** When the number is of one digit.

There will be four positive numbers excluding 0.

Case II: When the number is of two digits.

Ten's place can be filled in 4 ways using $1, 2, 3$ or 4. Unit's place can be filled in P_1^4 ways. Total no. of one digit numbers is $4 \times P_1^4 = 16$.

Case III: When the number is of three digits.

Hundred's place can be filled in 4 ways like previous case. Remaining two places can be filled in P_2^4 ways. Total no. of three digit numbers is $4 \times P_2^4 = 48$.

Case IV: When the number is four digits.

Thousand's place can be filled in 4 ways like previous case. Remaining three places can be filled in P_3^4 ways. Total no. of four digit numbers is $4 \times P_3^4 = 96$.

Case V: When the number is of five digits.

Ten thousand's place can be filled in 4 ways. Remaining four places can be filled in P_4^4 ways. Total no. of five digit numbers is $4 \times P_4^4 = 96$.

Thus, total no. of numbers formed is $4 + 16 + 48 + 96 + 96 = 260$.

28. Total no. of numbers will be $P^4_4 = 24$. Now since there are 4 digits and 24 numbers each no. will occur at each place for 6 times. Thus, sum of digits at each place would be $6(1+2+3+4) = 60$.

    Therefore, sum of all numbers $60(1+10+100+1000) = 66660$.

29. When any digit except 0 will occupy unit's place the thousand's place has to be occupied by the other two digits. Thus, total no. of such numbers is $3 \times 2 \times P^2_2 = 12$. Thus, 4 numbers for each of positive digits.

    When one of $1, 2, 3$ occupy thousand's place total no. of numbers is $3 \times P^3_3 = 18$. Thus, 6 numbers for each of the positive digits.

    Sum of digits at units, tens and thousands place will be $4(1+2+3) = 24$ and sum of digits at thousands place will be $6(1+2+3) = 36$.

    Thus, sum of numbers formed is $24(1+10+100) + 36 \times 1000 = 38{,}664$.

30. Each of the four digits $1, 2, 2, 3$ occurs at each place $\frac{P^3_3}{2!}$ i.e. 3 times. Thus, sum of digits at each place is $3(1+2+2+3) = 24$.

    Thus, sum of numbers formed $24(1+10+100+1000) = 26{,}664$.

31. Each friennd can be sent invitation by one servant. Since there are three servants each friend can receive an invitaion in three ways. Thus, total no. of ways of sending invitations is $3^6 = 729$.

32. Each prize can be given to any boy. Thus, each prize can be given in 7 ways, and hence, three prizes can be given in $7^3 = 543$ ways.

33. Each arm can occupy four positions, and thus, five arms can have $4^5 = 1024$ ways. But when all arms are in rest position no signal can be made. Hence, total no. of signal is $1024 - 1 = 1023$ ways.

34. Each ring of lock can have one of the ten letters, then three rings can have 10^3 combinations of the letters. However, one of the combinations will be a successful combination.

    Thus, total no. of possible unsuccessful attempts that can be made is $1000 - 1 = 999$.

35. We have to find numbers which are greater than 1000 but not greater than 4000 i.e. $1000 < x \leq 4000$ which is same as $1000 \leq x < 4000$.

    Now thousands place can be filled with $1, 2, 3$ i.e. in 3 ways. Hundreds, tens and units place can be filled in 5 ways each.

    Thus, total no. of numbers which can be formed is $3 \times 5^3 = 375$.

36. There are three groups. We can arrange three groups in 3! ways. 8 Indians can be arranged among themselves in 8! ways, 4 Ameriacans in 4! ways and 4 Englishmen in 4! ways.

    Thus, required answer is $3!\,8!\,4!\,4!$.

37. Total no. of volumes is $4 + 1 + 1 + 1 = 7$. We can arrange these volumes in 7! ways. 8 books volume can be arranged in 8! ways, volume having 5 books can be arranged in 5! ways and volume of 3 books can be arranged in 3! ways.

    Thus, required no. of arrangements is $7!\,8!\,5!\,3!$.

38. Taking all copies of the same book as one, we have 5 books, which can be arranged in 5! ways.

    All copies being identical can be arranged only in 1 way. Thus, required no. of arrangements is $5! = 120$.

39. The no. of permutations of the 10 papers without restriction is 10!.

    We find our no. of ways in which the best and worst paper come together then subtract from total no. of permutations to get the no. of permutations in which they never come together.

    Taking the best and the worst paper as one paper we have 9 papers, which can be arranged in 9! ways, but the two papers can be arranged among themselves in 2! ways. Thus, total no. of permutatiosn in which both the papers are toegther is $9!\,2!$.

    Thus, no. of permutations in which both are not together is $10! - 9!\,2! = 8.9!$.

40. Total no. ways in which all of them can be seated is $(5 + 3)! = 8!$. Taking all the girls as one total no. of persons is 6.

    The no. of ways in which these can be seated is 6!, but the 3 girls can be arranged in 3! ways. Thus, total no. of ways, when all three girls are together can be seate, is $6!\,3!$.

    Thus, total no. of ways in which all girls are not together is $8! - 6!\,3! = 36{,}000$.

41. Let us first position I.A. students. $*IA*IA*IA*IA*IA*IA*IA*$. The IA indicated the position where I.A. students sit and * indicated the positions where I.Sc. students can sit. We observe that there are 8 open places where I.Sc. students can sit.

    Now, 7 I.A. students can be seated in 7! ways and 8 I.Sc. students can be seated in P_5^8 ways.

    Thus, no. of required arrangements is $7!\,.\,\frac{8!}{3!}$.

42. Positioning the boys first, we have $*B*B*B*B*B*B*B*$, where Bs represents the 7 boys and $*$s represents the open positions for girls.

    7 boys can be arranged in 7! ways and 3 girls can be seated in P_3^8 ways. Thus, required no. of seating arrangememnts is $7!\,.\,\frac{8!}{5!} = 42.8!$.

43. **Case I:** When a boy sits at the first place. The possible arrangement in this case is $BGBGBGBG$, where B represents a boy and G represents a girl. Now, 4 boys and 4 girls can be arranged among themselves in 4! ways. Thus, no. of possible seating arrangement in this case is $4!\,4!$.

Case II: When a girl sits at the first place. Like previous case the possible no. of seating arrangements is same i.e. $4!\,4!$.

Thus, total no. of seating arrangements is $2.4!\,4! = 1152$.

44. Possible arrangements will have the form $BGBGBGB$, where B represents a boy, and G represents a girl. 4 boys can be seated in 4! ways and 3 girls can be seated in 3! ways.

    Thus, total no. of seating arrangements is $4!\,3!$.

45. There are 12 letters in the word civilization; out of which 4 are i's and other are different.

    Therefore, total no. of permutations is $\frac{12!}{4!}$, which included the word civilization itself.

46. There are 10 letters in the word university; out of which 4 are vowels, and i occurs twice. The consonants do not have repetition.

    Treating the 4 vowels as one letter, because they have to appear together, we have 7 letters. These 7 letters can be arranged in 7! ways. But the four vowels can be arranged among themselves in $\frac{4!}{2!}$ ways.

    Thus, total no.of words possible is $7!\,\frac{4!}{2!}$.

47. There are 8 letters in the word director; out of which 3 are vowels, and r occurs twice. Thus, total no. of words is $\frac{8!}{2!}$.

    When the vowels are together, taking them as one letter, we have 6 letters, which can be arranged in $\frac{6!}{2!}$, but the three vowels can be arranged in 3! ways among themselves, making the total no. of words in which vowels are together $3!\,\frac{6!}{2!}$.

    Thus, no. of words in which all three vowels are not together is $\frac{8!}{2!} - 3!\,\frac{6!}{2!}$.

48. There are 7 letters in the word welcome; out of which e occurs twice. Thus, total no. of words that can be formed is $\frac{7!}{2!}$.

    If 'o' comes at end then we will have 6 letters left giving us total no. of words as $\frac{6!}{2!}$.

49. There are 10 letters in the word California; out of which 5 are consonants without repetition and 5 vowels with a and i occurring twice.

    Thus, consonants can be arranged in 5! ways and vowels can be arranged in $\frac{5!}{2!2!}$ ways.

    Thus, total no. of words possible such that consoanats and vowels occupy their respective places is $\frac{5!5!}{2!2!}$.

50. There are 6 letters in the word pencil with two vowels and three even positions. Thus, vowels can be arranged in $P_2^3 = 6$ ways.

    Rest four positions can be filled in $4! = 24$ ways. Thus, total no. of words is $24 \times 6 = 144$.

51. From 5 letters $5! = 120$ words can be formed. Consider the form of word when no two vowels are together. $VCVCV$, where C represents consonants and V represents the vowels.

    Clearly, consonants can be arranged in $2!$ ways and vowels can be arranged in $P_3^3 = 3! = 6$ ways.

    Thus, no. of words where vowels are not together is $2 \times 6 = 12$.

52. There are seven digits given and we have to form numbers greater than one million, which implies all seven digits will have to used. Among the given digits 3 comes thrice and 2 comes twice. Thus, total no. of numbers which can be formed is $\frac{7!}{3!2!} = 420$.

    However, these numbers also contain the numbers where zero is the first digits making them less than one million. no. of such numbers is $\frac{6!}{3!2!} = 60$.

    Hence, no. of numbers greater than one million is $420 - 60 = 360$.

53. i. Total no. of persons is $5 + 4 = 9$. With no restirctions they can be seated at a round table in $(9 - 1)! = 8!$ ways.

    ii. Treating all British as a single person because they have to be together we have 6 persons which can be seated in $5!$ ways. But 4 Britishers can be arranged among themselves in $4!$ ways making the total no. of ways $5!\,4!$.

    iii. This is equal to $8! - 5!\,4!$ from previous parts.

    iv. First we seat the 5 Indians in $4!$ ways. Then that will leave 5 positions open for Britishers between Indians to sit, which gives us P_4^5 ways. Thus, total no. of ways in which no two Britishers are together is $4!\,5!$.

54. 5 Indians can be seated in a circle in $4!$ ways. We will have 5 positions between Indians in which we can seat 5 Britishers in $P_5^5 = 5!$ ways.

    Thus, total no. of required ways is $5!\,4!$.

55. Taking the two delegates who have to always sit together as a single person we have 19 persons which can be seated in $18!$ ways around a round table.

    However, the two delegates themselves can be arranged in $2!$ ways making the required no. of ways $18!\,2!$.

56. no. of four digit numbers which can be formed with $1, 2, 4, 5, 7$ i.e. 5 digits is $P_4^5 = 120$.

57. Units place cannot be filled with 0 so it can be filled in 4 ways using one of $1, 2, 3, 4$. Rest four positions can be filled in $P_4^4 = 4! = 24$ ways.

Thus, no. of 5 digit numbers is $4 \times 24 = 96$.

58. no. of given digits is 7 and we have to make numbers between 100 and 1000 i.e. three digit numbers. Since there is no zero in the given digits the required no. of numbers is $P_3^7 = 210$.

59. Units place be filled in 5 ways excluding 0 and two remaining places can be filled by remaining 5 digits in $P_2^5 = 20$ ways.

    Thus, total no. of required numbers is $5 \times 10 = 100$.

60. We have 10 digits. Units place can be filled in 9 ways excluding 0. Rest 8 places can be filled using remaining 9 digits in $P_8^9 = 9!$ ways.

    Thus, total no. of 9 digit numbers with no repetition is 9.9!.

61. Thousannds place can be filled in 5 ways excluding 0. Rest three places can be filled using remaining 5 digits in $P_3^5 = 60$ ways.

    Thus, no. of required numbers is $5 \times 60 = 300$.

62. Thousands place can be filled in 2 ways using either 5 or 9. Rest three places can be filled in 3! ways using remaining three digits.

    Thus, no. of required numbers is $2.3! = 12$.

63. **Case I:** When the number is of three digits.

    Hundreds place can be filled in 3 ways using $3, 4$ or 5. Remaining two places can be filled in $P_2^5 = 20$ ways using remaining 5 digits.

    Thus, no. of three digit numbers is $3 \times 20 = 60$.

    Case II: When the number is of four digits.

    Thousands place can be filled in 3 ways using $1, 2$ or 3. Remaining three place can be filled in $P_3^5 = 60$ ways using remaining 5 digits.

    Thus, no. of four digit numbers is $3 \times 60 = 180$.

    Thus, no. of required numbers is $60 + 180 = 240$.

64. Since the number has to be divisible by 5 the units place digit has to be either 0 or 5.

    Case I: When 0 is at units place. Rest three places can be filled in $P_3^4 = 24$ ways using remaining 4 digits.

    Thus, no. of four digit numbers in this case is 24.

    Case II: When 5 is at units place. Thousands place can be filled in 3 ways using $4, 6$ or 7. Remaining three places can be filled in $P_2^3 = 6$ ways using remaining 3 digits.

    Thus, no. of four digit numbers in this case is $3 \times 6 = 18$.

Hence, total no. of required numbers is $18 + 24 = 42$.

65. Since the number has to be even, therefore, units place can be filled by either 2 or 4 i.e. in 2 ways. Rest four places can be filled in $P_4^4 = 4! = 24$ ways.

Thus, total no. of 5 digit numbers is $2 \times 24 = 48$.

66. Since the no. has to be divisible by 5 units place can be occupied only by 0 and 5.

Case I: When the no. is of one digit. There are two such numbers 0 and 5.

Case II: When the no. is of two digits. If 0 occurs at units place then tens place can be filled in 9 ways giving us 9 numbers. However, when 5 occurs at units place then tens place can be filled in 8 ways giving us 8 numbers. Thus, total no. of two digits numbers is 17.

Caae III: When the no. is of three digits. If 0 occurs at units place then remaining two places can be filled in $P_2^9 = 72$ ways. If 5 is at units place then hundreds place can be filled in 8 ways excluding zero and tens place can be filled in 8 ways using remaining 8 digits. Thus, in this case otal no. of numbers is $72 + 8 \times 8 = 136$.

Thus, total no. of numbers is $2 + 17 + 136 = 155$.

67. Hundreds place can be filled in 5 ways excluding 0. Rest of two places can be filled in $P_2^5 = 20$ ways.

Thus, total no. of numbers is $5 \times 20 = 100$.

For odd numbers, units place can be filled in 2 ways using 5 or 7. Hundreds place can be filled in 4 ways excluding 0 and units place can also be filled in 4 ways using remaining digits.

Thus, total no. of odd numbers is $2 \times 4 \times 4 = 32$.

68. **Case I:** When the no. is of one digit. There are three such numbers $0, 2$ and 4.

Case II: When the no. is of two digits. When units place is occupied by 0, tens place can be filled in 4 ways, making no. of such numbers 4. If units place is occupied by 2 or 4 i.e. in two ways then tens place can be filled in 3 ways excluding 0, making no. of such numbers $2 \times 3 = 6$.

Thus, no. of two digit numbers is $4 + 6 = 10$.

Case III: When the no. is of three digits. When units place is occupied by 0, remaining two places can be filled in $P_2^4 = 12$ ways, making no. of such numbers 12. If units place is occupied by 2 or 4 i.e. in two ways then hundreds place can be filled in 3 ways excluding 0 and tens place can be filled in 3 ways using remaining three digits, making no. of such numbers $2 \times 3 \times 3 = 18$.

Thus, no. of three digit numbers is $12 + 18 = 30$.

Case IV: When the no. if of four digits. When units place is occupied by 0, remaining three places can be filled in $P_3^4 = 24$ ways, making no. of such numbers 24. Following similarly, when units place is occupied by 2 or 4, no. of such numbers is $2 \times 3 \times 3 \times 2 = 36$.

Thus, no. of four digit numbers is $24 + 36 = 60$.

Case V: When the no. is of five digits. In this case, units place must be occupied by 0 and not by 2 or 4. Then remaining 4 places can be filled in $P_4^4 = 24$ ways.

Thus, total no. of even numbers is $3 + 10 + 30 + 60 + 24 = 127$.

69. Once we fix 5 at tens place we have 5 open places and 5 different digits, which can be arranged in P_5^5 ways.

    Thus, no. of required numbers is 120.

70. We have 7 digits, and have to form four digit numbers. no. of such numbers possible is $P_4^7 = 840$.

    We have to find numbers greater than 3400. First we compute numbers between 3400 and 4000. The thousands place can be filled only by 3 and hundreds place can be filled by $4, 5, 6$ and 7 i.e. 4 ways. Remaining two positions can be filled in $P_2^5 = 20$ ways. Thus, no. of numbers between 3400 and 4000 is $4 \times 20 = 80$.

    Now we compute numbers greater than 4000. Thousands place can be filled by $4, 5, 6$ and 7 i.e. in 4 ways. Rest three places can be filled in $P_3^6 = 120$ ways. Thus, no. of such numbers is $4 \times 120 = 480$.

    Thus, no. of numbers greater than 3400 is $80 + 480 = 560$.

71. Since positions of 3 and 5 are fixed rest two positions can be filled with three remaining digits in $P_2^3 = 6$ ways. Thus, no. of such numbers is 6.

72. Thousands place can be filled in 5 ways excluding 0. Rremaining three places can be filled in $P_3^5 = 60$ ways using the five remaining digits. Thus, total no. of four digit numbers is $5 \times 60 = 300$.

    For numbers to be greater than 3000, thousands place has to be filled by $3, 4$ and 5 i.e. 3 ways. Remaining three places can be filled in $P_3^5 = 60$. Thus, no. of numbers greater than 3000 is $3 \times 60 = 180$.

73. **Case I:** When the no. is of one digit. Total no. of numbers possible in this case is 7 including 0.

    Case II: When the no. is of two digits. Tens place can be filled in 6 ways excluding 0 and units place can be filled in 6 ways with remaining digits.

    Thus, no. of two digit numbers is $6 \times 6 = 36$.

    Case III: When no. is of three digits. Following similarly the no. of numbers is $6 \times 6 \times 5 = 180$.

Case IV: When the no. is of four digits. Following similarly the no. of numbers is $6 \times 6 \times 5 \times 4 = 720$.

Case V: When the no. is of five digits. Following similarly the no. of numbers is $6 \times 6 \times 5 \times 4 \times 3 = 2160$.

Case VI: When the no. is of six digits. Following similarly the no. of numbers is $6 \times 6 \times 5 \times 4 \times 3 \times 2 = 4320$.

Case VII: When the no. is of seven digits. Following similarly the no. of numbers is $6 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 4320$.

Thus, total no. of numbers is $7 + 36 + 180 + 720 + 2160 + 4320 + 4320 = 11743$

74. We have 5 digits so when all of them are taken at a time then no. of possible numbers is $P_5^5 = 120$.

    Each digit will occupy each place for 24 numbers. Thus, sum of all numbers at any place is $24(1 + 3 + 5 + 7 + 9) = 600$. Therefore, sum of all such numbers is $600(1 + 10 + 100 + 1000 + 10000) = 6{,}666{,}600$.

75. We have 4 digits with 3 occurring twice. Thus, total no. of numbers is $\frac{P_4^4}{2!} = 12$. Now each of the digits will occur at each place $\frac{12}{4} = 3$ times.

    Thus, sum of digits at each place is $3(3 + 2 + 3 + 4) = 36$. Thus, sum of all possible numbers is $36(1 + 10 + 100 + 1000) = 39{,}996$.

76. Let us fix 2 at units place. Then, ten thousands place can be filled in 3 ways using 4, 6, 8 and remaining two places can be filled in $P_3^3 = 3!$ ways. Thus, total no. of numbers is $3 \times 6 = 18$.

    Number of numbers when 2 is at ten throusands place is $P_4^4 = 24$. Thus, each positive digit will occur at units, tens, hundreds and thousands place 18 times and at thousands place 24 times.

    Sum of the digits at units, tens, hundreds and thousands place will be each $18(2 + 4 + 6 + 8) = 360$ and sum of digits at ten thousands place is $24(2 + 4 + 6 + 8) = 480$.

    Thus, sum of all numbers will be $360(1 + 10 + 100 + 1000) + 480 \times 10000 = 5{,}199{,}960$.

77. Total no. of five digit numbers possible is $P_5^5 = 120$ where each digit will appear at each position $\frac{120}{5} = 24$ times.

    Thus, sum of digits at each place is $24(3 + 4 + 5 + 6 + 7) = 600$. Therefore, sum of all such numbers is $600(1 + 10 + 100 + 1000 + 10000) = 6{,}666{,}600$.

78. Let us fix 2 at units place. Then, thousands place can be filled in 2 ways using 3 or 5 and remaining two places can be filled in $P_2^2 = 2$ ways. Thus, total no. of numbers is $2 \times 2 = 4$.

Number of numbers when 2 is at thousands place is $P_3^3 = 6$. Thus, each positive digit will occur at units, tens, hundreds and thousands place 4 times and at thousands place 6 times.

Sum of digits at units, tens and hundreds place will eb each $4(2+3+5) = 40$ and sum of digits at thousands place will be $6(2+3+5) = 60$.

Thus, sum of all numbers will be $40(1+10+100) + 60 \times 1000 = 64{,}440$.

79. Each letter can be put in any one of the four letter boxes. Thus, 5 letters can be posted in 4^5 ways.

80. Each prize can be given in 5 ways. So three prizes can be given in 5^3 ways.

81. Each thing can be given in p ways to p person. Thus, n things can be given in p^n ways.

82. Each monkey can have a master in m ways. Thus, n monkeys can have a master in m^n ways.

83. First prize in mathematics and physics can be given in 10 ways and second prize in 9 ways. In chemistry, first prize can be given in 10 ways.

    Thus, total no. of ways is $10 \times 9 \times 10 \times 9 \times 10 = 81{,}000$.

84. The first animal can be picked in 3 ways with the possibility of it being a cow, a calf or a horse. Similarly, second animal can be picked in 3 ways. Proceeding this way all 12 animals for the stall can be picked in 3 ways.

    Thus, total no. of making the shipload is 3^{12}.

85. Each delegate can be put in a hotel in 6 ways. Therefore, 5 delegates can be put in 6^5 ways.

86. Ten thousands place can be filled in 4 ways exluding 0. Rest 4 places can be filled in 5 ways each. Thus, total no. of 5 digits numbers is $4 \times 5^4 = 2{,}500$.

87. Each ring can be put in a finger in 4 ways i.e. by putting it in any finger. Thus, 6 rings can be put in 4 fingers in 4^6 ways.

88. Thousands place can be filled in 3 ways using $3, 4$ or 5. Remaining places can be filled in 6^3 ways using any of the digits. But one of these numbers will be 3000 itself.

    Thus, no. of four digit numbers which can be made is $3 \times 6^3 - 1$.

89. When the number plate is of three digits, each place can be filled in 9 ways excluding zero. This gives us 9^3 number plates. Similalry, when the number plate is of four digits the no. of possible number plates is 9^4.

    Thus, total no. of number plates is $9^3 + 9^4 = 10 \times 9^3 = 7{,}290$.

90. Each question can be answered in 4 ways, therefore, 10 questions can be answered in 4^{10} ways.

Second part: First question can be answered in 4 ways. Now this choice won't be available for the second answer so there are 3 ways. Similarly, for third and so on. Thus, total no. of ways is 4×3^9.

91. Treating all volumes of a book as one book we have four books which can be arranged in 4! ways. However, books having 3 volumes can be arranged in 3! ways among themselves and similarly books having 2 volumes can be arranged in 2! ways among themselves.

    Thus, total no. of arranging given books is $4!\,3!\,3!\,2!\,2!$.

92. There are 14 books having different no. of copies. Treating all copies as one book we still have 14 books which can be arranged in 14! ways.

    Since copies are identical there is only one way to arrange them among themselves. Thus, total no. of arranging the given books is 14!.

93. Treating people of different nationalities as one person we have three persons, which can be arranged in 3! ways. Now 10 Indians can be arranged in 10! ways among themselves, 5 Americans can be arranged in 5! ways among themselves and 5 Britished can be arranged in 5! ways as well.

    Thus, total no. of ways of seating them is $3!\,10!\,5!\,5!$.

94. The pattern would be $GBGBGBGBGBGBG$ where B shows boys position and G indicates possible positions of girls. Boys can be arranged in 6! ways. For girls, there are 7 open positions and 4 girls can be seated in $P_4^7 = \frac{7!}{3!}$ ways.

    Thus, total no. of ways of seating them is $6!\,.\,\frac{7!}{31}$.

95. n books can be arranged in $n!$ ways. Now we will find the no. of arrangements when two given books which do not have to be together are together. Treating the two books as one book we have $n-1$ books which can be arraned in $(n-1)!$ ways. But the two books can be arranged in 2 ways among themselves, making the total no. of arrangements is $2.(n-1)!$.

    Thus, no. of arrangements when the two books are not together is $n! - 2.(n-1)! = (n-2)\,.(n-1)!$.

96. From previous problem, we find the answer to be $4.5! = 480$.

97. Following like previous problem, we find theh answer to be 480.

98. Following like previous problem on boys and girls we first seat the 15 I.Sc. students in 15! ways which gives us 16 open positions for $B.Sc.$ students, which can be seated in P_{12}^{16}.

    Thus, total no. of ways of seating the students is $15!\,.P_{12}^{16}$.

99. First we arrange black balls which will give us 20 positions in between them and on the edges for white balls. Since the balls are identical we can choose 18 positions out of 20 for white balls in $C_{18}^{20} = 190$ ways.

100. First we place p positive signs which will give us $p+1$ positions for negative signs between them and on the edges. Since signs are identical we can choose n positions out of $p+1$ in C_n^{p+1} ways.

101. m men can be seated in $m!$ ways which will have $m+1$ positions between them and on the edges for women so that no two women sit together. Now n women can be arranged in these $m+1$ positions in $P_n^{m+1} = \frac{(m+1)!}{(m-n+1)!}$ ways.

Thus, total no. of ways to seat them is $\frac{m!(m+1)!}{(m-n+1)!}$.

102. Following like previous problem, we have $m=5, n=3$, so the answer woulld be $\frac{5!6!}{3!}$.

103. We have 12 alphabets excluding c's out of which 5 are a's, 3 are b's, 1 d, 2 e's and 1 f, so these can be arranged in $\frac{12!}{5!3!2!}$ ways. Now these 12 alphabets will create 13 positions between them and on the edges which are to be filled by 3 c's in P_3^{13} ways.

Thus, total no. of arrangements is $\frac{12!}{5!3!2!} \times \frac{13!}{10!}$.

104. The word banana has 'a' repeating 3 times and 'n' repeating twice while total no. of alphabets is 6.

Hence, to no. of different permutations is $\frac{6!}{3!2!}$.

105. There are 13 alphabets in the word "circumference". 'c' comes thrice, 'r' comes twice, 'e' comes thrice and rest come once.

Thus, total no. of words that can be made is $\frac{13!}{3!3!2!}$.

106. Three copies of four books means 12 books with repetition of copies. Thus, total no. of arragements on the shelf is $\frac{12!}{3!3!3!3!}$.

107. There are 12 alphabets in the word "Independence". 'n' comes thrice, 'd' comes twice, 'e' comes four times, and rest come once.

Thus, total no. of words that can be made is $\frac{12!}{4!3!2!}$.

108. There are 8 alphabets in the word "Principal", of which, 'p' comes twice, 'i' comes twice and rest occur once. Treating all vowels as one alphabet we have 6 alphabets which can be arranged in $\frac{6!}{2!}$ ways.

However, the vowels themselves can be arranged among themselves in $\frac{3!}{2!}$ ways. Thus, total no. of words is $\frac{6!3!}{2!2!}$.

109. There are 11 alphabets in the word "Mathematics", of which, 'm' comes twice, 'a' comes twice, 't' comes twice and rest comes once. Thus, no. of words that can be formed is $\frac{11!}{2!2!2!}$.

Treating all vowels as one alphabet and all consonants as another we have two alphabets which can be arranged in 2! ways. But 4 vowels can be arranged in $\frac{4!}{2!}$ ways and 7 consonants can be arranged in $\frac{7!}{2!2!}$ ways.

Thus, total no. of such words is $\frac{2!7!4!}{2!2!2!}$.

110. There are 8 alphabets in the word "Director", of which, r comes twice and rest come once. Since the vowels have to come together, therefore we treat them as one alphabet making a total of 6 alphabets which can be arranged in $\frac{6!}{2!}$ ways.

However, the three vowels can be arranged in 3! ways among themselves making no. of such words $\frac{6!3!}{2!}$.

111. There are 8 alphabets in the word "Plantain", of which, 'a' and 'n' come twice and rest come once. Since the vowels have to come together, therefore we treat them as one alphabet making a total of 6 alphabets which can be arranged in $\frac{6!}{2!}$ ways.

However, the three vowels can be arraned in $\frac{3!}{2!}$ ways among themselves making no. of such words $\frac{6!3!}{2!2!}$.

112. There are 12 letters in the word "Intermediate", of which, 'e' comes thrice, 'i' and 't' comes twice and rest come once.

We first arrange vowels which can be done in $\frac{6!}{3!2!}$. Now because relative order does not change we have six positions for consonants giving us total no. of ways of arranging them as $\frac{6!}{2!}$.

Thus, total no. of such words is $\frac{6!6!}{3!2!2!}$.

113. There are 8 letters in the word "Parallel", of which, 'a' comes twice, 'l' comes thrice and rest comes once.

Total no. of arrangements is $\frac{8!}{3!2!}$. Treating all the ls as one letter we have 6 letters which can be arranged in $\frac{6!}{2!}$ ways in which all ls will be together.

Therefore, no. of words in which all ls are not together is $\frac{8!}{3!2!} - \frac{6!}{2!} = 3000$.

114. The parts are solved below:

i. Fixing 'D' at the first position; rest four positions can be filled in P_4^4 ways. Thus, no. of such words is $4! = 24$.

ii. Fixing 'I' at the end; rest four positions can be filled in P_4^4 ways. Thus, no. of such words is $4! = 24$.

iii. Fixing 'l' in the middle; rest four positions can be filled in P_4^4 ways. Thus, no. of such words is $4! = 24$.

iv. Fixing 'D' and 'I'; rest three positions can be fillled in P_3^3 ways. Thus, no. of such words is $3! = 6$.

115. There are 7 unique letter in the word "Violent" with 3 vowels. There are 4 odd places so three vowels can be arranged in $P_3^4 = 4!$ ways. Rest 4 consonants can be arrannged in $4! = 24$ ways. Thus, total no. of such words is $24 \times 24 = 576$.

116. There are 3 distinct consonants and 3 vowels, where 'o' repeats once in the word "Saloon". Since consonants and vowels have to occupy alternate place we will have two patterns. $VCVCVC$ and $CVCVCV$, where C represents consonants and V represents vowels.

Three consonants can be arranged in 3! arrangements and 3 vowels can be arranged in $\frac{3!}{2!}$ arrangement. Thus, total no. of arrangements is $3!\,3! = 36$.

117. There are 4 consonants and 3 vowels in the word "Article". Clearly, there are three even places which are to be occupied by vowels in 3! arrangements and consonants can be arranged in 4! arrangements for remaining 4 positions.

Thus, total no. of words is $4!\,3! = 144$.

118. Since the number has to be greater than 4 million and we are given 7 digits the ten millions place can be occupied by either 4 or 5 in 2 ways.

Remaning digits can be arranged in $\frac{6!}{2!2!} = 180$ arrangements as 2 and 3 repeat once. Thus, total no. of required numbers is $2 \times 180 = 360$.

119. In the given digits 2 comes thrice and 3 comes twicec so the no. of numbers is $\frac{7!}{3!2!} = 420$.

For odd numbers units place is to be occupied by 1, 3 or 5. When 1 or 5 occupy units place remaining positions can be filled in $\frac{6!}{3!2!} = 60$ ways making the number $2 \times 60 = 120$.

When one of the 3's occupy units place rest of the positions can eb filled in $\frac{6!}{3!} = 120$ ways. Thus, total no. of odd numbers is $120 + 120 = 240$.

120. There are four odd digits with both 1 and 3 repeating. The even no. 2 repeats once. In a 7 digits number there are four odd places which can be filled by odd numbers in $\frac{4!}{2!2!} = 6$ ways.

Even places can be filled by 2 and 4 can be filled in $\frac{3!}{2!} = 3$ ways. Thus, no. of required numbers is $6 \times 3 = 18$.

121. **Case I:** When the no. if is five digits.

When ten thousands place is occupied by $2, 3$ or 4 remaining four places can be filled in $\frac{P_4^5}{2!} = 60$ ways, making such numbers $60 \times 3 = 180$ in number.

When ten thousands place is occupied by 1 remaining four places can be filled in $P_4^5 = 120$ ways.

Thus, total no. of five digit numbers is $180 + 120 = 300$.

Case II: When the no. is of six digits.

When hundred thousands place is occupied by $2, 3$ or 4 remaining five places can be filled in $\frac{P_5^5}{2!} = 60$ ways, making such numbers $60 \times 3 = 180$ in number.

When hundred thousands place is occupied by 1 remaining four places can be filled in $P_5^5 = 120$ ways.

Thus, total no. of six digit numbers is $180 + 120 = 300$.

Thus, total no. of numbers is $300 + 300 = 600$.

122. When the digits are repeated thousands place can be filled in 5 ways excluding 0. Remaining 3 positions can be filled by 6 digits in 6^3 ways.

Thus, no. of such numbers is $5 \times 6^3 = 1080$.

To find the no. of numbers where at least one digit is repeated we find the no. of numbers where no digit is repeated and subtract it from previously obtained result.

For no repetition, thousands placec can be filled in 5 ways exluding 0. Remaning 3 places can be filled by 5 digits in $P_3^5 = 60$ ways.

Thus, no. of numbers without repetition is $60 \times 5 = 300$.

Thus, no. of numbers where at least one digit is repeated is $1080 - 300 = 780$.

123. There are a total of 9 flags, of which, 2 are red, 2 are blue and 5 are yellow. Thus, total no. of signals that can be made by using all of them at the same time is $\frac{9!}{2!2!5!}$.

124. When all are of same color P_1^6 signals can be made. When all are of two colors P_2^6 signals can be made and so on.

Thus, total no. of signals is $P_1^6 + P_2^6 + P_3^6 + P_4^6 + P_5^6 + P_6^6 = 1956$.

125. **Case I:** When 'e' is in first place. Remaining four places can be filled in 4! ways.

Case II: When 'e' is in second place. First place can be filled in 3 ways and remaining 3 places in 3! ways.

Case III: When 'e' is in third place. First two places in 3×2 ways and remaining two places in 2! ways.

Case IV: When 'e' is in fourth place. First three places in 3! ways and last place with 'i'.

Thus, total no. of words is $4! + 3 \times 3! + 6 \times 2! + 3! = 60$.

Second method: Total no. of words is 5!. In half of these 'e' will come before 'i' and in half of them after it. Thus, no. of words is $\frac{5!}{2} = 60$.

126. no. of ways in which 5 men can sit around a round table is $(5-1)! = 24$ arrangements.

127. When there is no restriction we have 10 girls and boys. Thus, total no. of arrangements would be 9!.

    When no girls are to sit together we first seat the boys in 4! arrangements giving us five open positions. These can be filled by 5 girls in 5! ways.

    Thus, total no. of seating arrangements is $4!\,5!$.

128. Treating all girls as a single girl we have 7 boys and girls which can be seated in 6! ways. But the 4 girls can be arranged in 4! ways among themselves.

    Thus, total no. of seating arrangements is $6!\,4!$.

129. The line can start with boys so we first seat the boys put the boys in 5! ways followed by girls in between boys in 5! ways. This can be repeated starting with girls in same manner.

    Thus, no. of lines that can be formed is $2.5!\,5!$.

    For a round table we have already solved previously giving us $4!\,5!$ no. of arrangements.

130. 6 boys can be seated first in 5! ways giving us 6 open places in which girls can be seated in P_5^6 ways. Thus, total no. of seating arrangements is $5!\,6!$.

131. Since in a necklace clockwise and anticlockwise does not matter, therefore, total no. of necklaces that can be made using 50 pearls is $\frac{49!}{2!}$.

132. Treating the two particular delegates as one delegate we have 19 delegates which can be seated in 18! ways. But the two delegates can be seated in 2! ways among themselves.

    Thus, total no. of seating arrangements is $18!\,2!$.

133. The question effectively asks for alternate seating arrangements among gentlemen and ladies. Thus, followin from problem solved previously total no. of seating arrangements would be $4!\,3!$.

134. 7 Englishmen can be seated in 6! ways giving us 7 open places which can be filled by 6 Indians in P_6^7 ways.

    Thus, total no. of seating arrangements is $6!\,7!$.

135. We know that if $C^n_x = C^n_y$ then either $x = y$ or $x + y = n$. Given, $C^{15}_{3r} = C^{15}_{r+3}$ therefore either $3r = r + 3$ or $3r + r + 3 = 15$.

However, $3r = r = 3 \Rightarrow r = \frac{3}{2}$, which is not possible, therefore, $3r + r + 3 = 15 \Rightarrow r = 3$ must be the case.

136. Given, $C^n_6 : C^{n-3}_3 \Rightarrow \frac{n!}{6!(n-6!)} \cdot \frac{3!(n-6)!}{(n-3)!} = \frac{33}{4}$

$\Rightarrow \frac{n!}{(n-3)!} \cdot \frac{3!}{6!} = \frac{n(n-1)(n-2)}{6.5.4} = \frac{33}{4} \Rightarrow n(n-1)(n-3) = 11.10.9 \Rightarrow n = 11.$

137. Given, $C^{47}_4 + \sum_{j=1}^{5} C^{52-j}_3$

$= C^{47}_4 + (C^{51}_3 + C^{50}_3 + C^{49}_3 + C^{48}_3 + C^{47}_3) = (C^{47}_4 + C^{47}_3) + (C^{51}_3 + C^{50}_3 + C^{49}_3 + C^{48}_3)$

$= C^4_4 8 + (C^{51}_3 + C^{50}_3 + C^{49}_3 + C^{48}_3 + C^{47}_3)\,[\because C^n_r + C^n_{r+1} = C^{n+1}_{r+1}]$

Repeating this we have the expression equal to C^{52}_4.

138. Let p be the product of r consecutive integers starting from n. Then, $p = n(n+1)(n+2)\cdots(n+r-1)$

$\Rightarrow \frac{p}{r!} = \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!} = \frac{1.2.3.....(n-1)n(n+1)(n+2)\cdots(n+r-1)}{1.2.3.....(n-1).r!}$

$= \frac{(n+r-1)!}{(n-1)!r!} = C^{n+r-1}_r$, which would be an integer, and hence, p is divisible by $r!$.

139. A triangle is formed with three vertices so the problem is essentially about choosing 3 out of m i.e. $C^m_3 = \frac{m(m-1)(m-2)}{6}$.

140. Number of children is 8. no. of children to be taken at a time is 3. Out of 8 children 3 can be selected in C^8_3 ways. Hence, the man has to go to zoo $C^8_3 = 56$ times.

Number of selection of 3 children out of 8 children including a particular child is $1 \times C^7_2 = 21$. Hence, a particular child will go 21 times to the zoo.

141. Let there be n students. no. of ways in which 2 students can be selected out of n is C^n_2 i.e. we have C^n_2 pairs.

But, for each pair of students no. of cards sent is 2. Thus, total no. of cards sent is $2.C^n_2 = n(n-1) = 600 \Rightarrow n = 25$ because $n \neq -24$.

Second method: Each student sends cards to $n - 1$ students. Thus, total no. of cards sent is $n(n-1) = 600 \Rightarrow n = 25$.

142. A polygon of m sides will have m vertices. When any two vertices of the polygon are joined, either a diagonal or a side is formed.

Total no. of selections of 2 points taken at a time from m points is C^m_2.

143. Total no. of persons is $6+4=10$. Total no. of selections of 5 persons out of 10 is C_5^{10}. Number of selections when no lady is taken is C_5^6.

Thus, no. of selections when at least one lady is present is $C_5^1 0 - C_5^6 = 252 - 6 = 246$.

144. (a) Total no. of selections of 3 points out of 10 points is $C_3^{10} = 120$. Number of selections of 3 points out of 4 collinear points is $C_3^4 = 4$.

Thus, no. of triangles formed is $120 - 4 = 116$.

(b) Total no. of selections of 2 points out of 10 points is $C_2^{10} = 45$. no. of selection of points when only one line is formed is $C_2^4 = 6$

Therefore, no. of straight lines formed is $45 - C_2^4 + 1 = 40$. (We take 1 line formed from four collinear points)

(c) Total no. of selections of 4 points out of 10 points is $C_4^{10} = 210$. no. of selection of points when no quadrilateral is formed is $C_3^4.C_1^6 + C_4^4.C_0^6 = 25$.

Thus, no. of quadrilaterals formed is $210 - 25 = 185$.

145. Zero or more oranges can be selected from 4 oranges in 5 ways because oranges are identical. Similalry, the no. of selection for apples would be 6 and for mangoes it would be 7.

Thus, no. of selections when all three types of fruits are selected from is $5 \times 6 \times 7 = 210$. But one of these selections will contain 0 fruits.

Thus, required no. of selections is 209.

146. no. of selections by which 1 or more green dye can be chosen is $C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 2^5 - 1$. no. of selections by which 1 or more blue dye can be chosen is $C_1^4 + C_2^4 + C_3^4 + C_4^4 = 2^4 - 1$. no. of selections by which 0 or more red dye can be chosen is $C_0^3 + C_1^3 + C_2^3 + C_3^3 = 2^3 = 8$.

Thus, required no. of selections is $21 \times 15 \times 8 = 3720$.

147. Factos of $216,000$ are 5 2s, 3 3s and 2 5s. Zero or more 2s can be selected in $5+1=6$ ways. Zero or more 3s can be selected in $3+1=4$ ways. Zero of more 5s can be selected in $2+1=3$ ways.

Thus, no. of divisors is $6 \times 4 \times 3 - 1 = 71$ because one of these would contain no factor. Adding 1 to the no. of divisors we have total no. of divisors as 72.

148. A student can fail in one , two, three, four or all of five subjects. Thus, no. of ways of failing is $C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 2^5 - 1 = 31$.

149. Each person can be given 4 things. no. of ways of giving 4 things out of 12 to the first person is C_4^{12}. Then, 8 things remain. no. of ways of giving 4 things out of 8 to the second person is C_4^8. Now third person can receive 4 things out of 4 in C_4^4 ways.

Thus, required no. of ways is $C_4^{12} \times C_4^8 \times C_4^4 = \frac{12!}{(4!)^3}$.

no. of ways in which 12 things can be divided equally among 3 sets is $\frac{12!}{(4!)^3.3}$.

150. There are 11 letters in the word "Examination" in which three occur in pairs i.e. 'A', 'N' and 'I'. The different letters are E, X, A, M, I, N, T, O i.e. 8.

Case I: When two pairs of identical letters are chosen.

The two pairs can be chosen from three in $C_2^3 = 3$ ways. These letters can be arranged among themselves in $\frac{4!}{2!2!} = 6$ ways. Thus, total no. of words formed is $3 \times 6 = 18$.

Case II: When one pair of identical letters is chosen and remaining two letters are different.

The pair of identical letters can be chosen in $C_1^3 = 3$ ways. The two different letters can be chosen in $C_2^7 = 21$ ways. These letters can be arranged in $\frac{4!}{2!}$ ways.

Thus, total no. of words formed is $3 \times C_2^7 \times \frac{4!}{2!} = 756$.

Case III: When all four letters are different.

no. of words that can be formed is $P_4^8 = 1680$.

Thus, total no. of words formed is $756 + 18 + 1680 = 2454$.

151. We need to select 4 vertices out of n of a polygon to form a quadrilateral. no. of selections of 4 points is C_4^n.

152. no. of ways of selecting 3 friends out of 7 is $C_3^7 = 35$. Thus, no. of parties that can be given is 35.

Suppose a particular friends is mandatory in a party then 2 other friends can be selected in C_2^6 ways. Thus, no. of parties a particular friend will attend is $C_2^6 = 15$.

153. If p things always occue then we have to select remaning $r - p$ things out of $n - p$ ways, which is C_{r-p}^{n-p}.

154. (a) If a particular member is always added then we have to choose 5 more from remaining 11, which is C_5^{11}.

(b) If a particular member is always excluded then we have to chhose 6 more from remaining 11, which is C_6^{11}.

155. (a) Total no. of ways of seating 6 students is $P_6^6 = 720$. Now we will put C and D together and subtract that from total no. of ways to find no. of ways of seating them when C and D are not together.

Treating C and D as one student we have 5 students which can be seated in $P_5^5 = 120$ ways. But these two can be arranged among themseleves in 2! ways making total no. of ways $120 \times 2 = 240$.

Thus, no. of ways of seating these 6 students together when C and D are not together is $720 - 240 = 480$.

(b) If C is always included then we need to select 3 more from remaining 5, which can be done in $C^5_3 = 10$ ways.

(c) Since E is always excluded we have only 5 students left. Thus, following previous part it can be done in $C^4_3 = 4$ ways.

156. Let there be n stations. To print a ticket we need a source station and a desination station. So different tickets which can be printed with n stations is C^n_2, which is 105 in our case.

$\therefore \frac{n!}{(n-2)!2!} = 105 \Rightarrow n(n-1) = 210 = 14.15 \Rightarrow n = 15.$

157. no. of ways to select 2 points to form a straight line out of 15 points is $C^{15}_2 = 105$. This will include 2 points out of 6 collinear points which will actually contain only 1 straight line out of it. So no. of ways to choose 2 points out of these 6 points is $C^6_2 = 15$. Thus, total no. of straight lines formed is $105 - 10 + 1 = 91$.

no. of ways of choosing 3 points out of 15 is $C^{15}_3 = 455$. We have to not consider cases when all three points aree selected from collinear points as those won't form a triangle. no. of selections of 3 points out of collinear points is $C^6_3 = 20$.

Thus, total no. of triangles formed is $455 - 20 = 435$.

158. no. of ways of choosing 4 points out of 10 is $C^{10}_4 = 210$. When 3 or 4 points are chosen from 5 collinear pooints the quadrilateral won't be formed. When we choose 3 points from collinear points we have $C^5_3 = 10$ ways, and 1 remaining point from 5 non-collinear points in 5 ways. Thus, total no. of such selections is $10 \times 5 = 50$.

When all four points are chosen from collinear points; this can be done in $C^5_4 = 5$ ways.

Thus, total no. of quadrilaterals formed is $210 - 50 - 5 = 155$.

159. There is a total of 12 points and we can choose 3 points from these in $C^{12}_3 = 220$ ways. However, these points must not come from points of same side.

Thus, no. of triangles formed is $220 - C^3_3 - C^4_3 - C^5_3 = 205$.

160. We need one goalkeeper in the team and two are available so goalkeeper can be chosen in 2 ways. Rest of 10 players can be chosen from remaining 12 players in $C^{12}_{10} = 66$ ways.

Thus, no. of ways in which a team of 11 out of 14 can be formed is $2 \times 66 = 122$.

161. 2 men can be chosen from 5 men in $C^5_2 = 10$ ways. Similarly, 2 women from 6 women can be chosen in $C^6_2 = 15$ ways.

Thus, total no. of ways of forming the committee is $10 \times 15 = 150$.

162. Since each boy is to receive one article at least one boy will receive 2 articles. These two articles can be given to one of the boys in C^8_2 ways. The second article can be given in C^7_1 ways and so on.

Since first article can be given to any of the seven boys the above result if multiplied by 7 will give us total no. of ways of distributing the articles.

Thus, total no. of ways is $7(C^8_2 + C^7_1 + C^6_1 + C^5_1 + C^4_1 + C^3_1 + C^2_1 + C^1_1)$.

163. **Case I:** When there are 3 ladies in the committee.

no. of ways of choosing 3 ways out of 4 ladies is C^4_3. Remaining 2 members can be selected out of 7 men is C^7_2 ways. Thus, no. of such committees is $C^4_3 \times C^7_2$.

Case II: When there are 4 ladies in the committee.

no. of ways of choosing 4 ways out of 4 ladies is C^4_4. Remaining 1 member can be selected out of 7 men is C^7_1 ways. Thus, no. of such committees is $C^4_4 \times C^7_1$.

Thus, total no. of committees is $84 + 7 = 91$.

164. There are three cases. Two questions from first group and four questions from second group, three questions from each group, and four questions from first group and two questions from second group.

This can be done in $C^5_2 \times C^5_4 + C^5_3 \times C^5_3 + C^5_4 \times C^5_2 = 50 + 100 + 50 = 200$.

165. 3 students can be chosen from 20 students in C^{20}_3 ways.

(a) When a particular professor is included the second professor for the committee out of remaining 9 professors can be included in C^9_1 ways.

Thus, total no. of such committees is $C^{20}_3 \times C^9_1$.

(b) When a particular profession is always excluded then two professors can be chosen from remaining 9 in C^9_2 ways.

Thus, total no. of such committees is $C^{20}_3 \times C^9_2$.

Thus, total no. of committees is $C^{20}_3 \times C^9_1 + C^{20}_3 \times C^9_2$.

166. The committee can comprise of 1, 2, 3, 4 or 5 girls, which can be selected out of 7 girls in $C^7_1, C^7_2, C^7_3, C^7_4$ or C^7_5 ways respectively.

Remaining 4, 3, 2, 1 boys can be selected out of 6 boys in C^6_4, C^4_3, C^4_2, C^4_1 ways respectively.

Thus, no. of ways in which committee can be formed is $C^7_1 \times C^6_4 + C^7_2 \times C^6_3 + C^7_3 \times C^6_2 + C^7_4 \times C^6_1 + C^7_5 \times C^6_0$.

167. (a) When there are no restrictions the committees can be formed by choosing 5 out of $6 + 4 = 10$ persons, which is $C^{10}_5 = 252$.

(b) When no lady is selected no. of ways to form committess is $C_5^6 = 6$. Thus, no. of committees when at least one lady is selected is $252 - 6 = 246$.

168. Total no. of committees would be C_5^{12}. no. of committees comprising only of men would be C_5^8.

Thus, no. of committees including at least one lady would be $C_5^{12} - C_5^8 = 736$.

169. Out of 6 hockey players $4, 5, 6$ hockey players can be selected in C_4^6, C_5^6, C_6^6 ways respectively. Remaining $8, 7, 6$ players can be chosen from remaining 9 players in C_8^9, C_7^9, C_6^9 ways respectively.

Thus, no. of ways in which players can be selected is $C_4^6 \times C_8^9 + C_5^6 \times C_7^9 + C_6^6 \times C_6^9 = 15 \times 9 + 6 \times 36 + 1 \times 84 = 435$.

170. Total no. of selections of 5 out of $7 + 4 = 11$ persons is C_5^{11}. When no ladies are selected, no. of ways of forming the boat party is C_5^7.

Thus, no. of ways of forming boat party when at least one lady is selected is $C_5^{11} - C_5^7 = 771$.

171. Since girls are not to be outnumbered we have to have $3, 4, 5$ or 6 girls out of 6 in the committee, which can be done in C_3^6, C_4^6, C_5^6 or C_6^6 ways respectively.

Remaining $3, 2, 1$ positions can be filled from 4 boys in C_3^4, C_2^4, C_1^4 ways respectively.

Thus, total no. of ways in which committee can be formed is $C_3^6 \times C_3^4 + C_4^6 \times C_2^4 + C_5^6 \times C_1^4 + C_6^6 = 20 \times 4 + 15 \times 6 + 6 \times 4 + 1 = 195$.

172. no. of relatives which can be invited is $5, 6, 7$ out of 8 relatives in C_5^8, C_6^8, C_7^8 ways. Remaining $2, 1$ friends can be chosen from remaining 4 friends which are no relatives in C_2^4, C_1^4 ways.

Thus, no. of ways in which invitations can be made is $C_5^8 \times C_2^4 + C_6^8 \times C_1^4 + C_7^8 = 56 \times 6 + 28 \times 4 + 8 = 336 + 112 + 8 = 456$.

173. The students can choose to answer the question paper in 4 ways. 5 questions from first paper and 2 from second paper, 2 questions from first paper and 5 questions from second paper, 4 questions from first paper and 3 from second paper, and 3 questions from first paper and 3 questions from second paper.

Because both papers contain 6 questions each the no. of ways for first and second method will be same and ways for third and fourth method will be same as well. So we can find no. of ways in two cases and multiply the sum by 2 to arrive at the answer.

Case I: When the student chooses first or second method.

5 questions can be chosen out of 6 in C_5^6 ways and 2 questions can be chosen out of 6 in C_2^6 ways.

Thus, no. of selections in this case is $C_5^6 \times C_2^6 = 6 \times 15 = 90$.

Case II: When the student chooses third or fourth method.

Following like previous case, no. of selections in this case is $C^6_4 \times C^6_3 = 15 \times 20 = 300$.

Thus, total no. of selections of questions is $2(90 + 300) = 780$.

174. We can choose 1 point out P and Q in C^2_1 and 2 from remaining other 8 points in C^8_2 ways, making no. of triangles $C^2_1 \times C^8_2 = 56$. Clearly, half of these would include P but exclude Q. Thus, 28 triangles will include P and exclude Q.

    In second case, both P and Q would be chosen in 1 way and 1 point from the other line would be chosen in $C^8_1 = 8$ ways. This gives us 8 triangles.

    Thus, total no. of triangles is $56 + 8 = 64$.

175. There can be two cases. First, when 1 vote is casted, and second, when 2 votes are casted.

    Case I: When 1 vote is casted.

    We can choose 1 from men or 1 from ladies. Thus, total no. of choices are $C^7_1 + C^3_1 = 7 + 3 = 10$.

    Case II: When 2 votes are casted.

    The two votes can be casted by making choices in three different manners. We can choose 2 men or 2 ladies or 1 man and 1 lady. Thus, total no. of choices are $C^7_2 + C^3_2 + C^7_1 \times C^3_1 = 21 + 3 + 21 = 45$.

    Thus, total no. of ways in which votes can be casted are $45 + 10 = 55$.

176. No. of ways of choosing boys are $C^{10}_3 = 120$. Let us first choose girls in an unrestricted manner. No. of ways of choosing 3 girls out of 7 are $C^7_3 = 35$. Now assume that the two particular girls, who cannot be together are always there in selection. Treating these two girls as one, and always selecting them gives us $C^5_1 = 5$ choices.

    However, these two girls cannot be in the same group, so total no. of choosing girls are $35 - 5 = 30$. And thus, no. of ways of forming the party is $120 \times 30 = 3600$.

177. Since there are three sets and we have to answer at least two questions from each set for a total of seven questions, so we will have to choose three questions from one of the sets.

    Thus, total no. of selecting questions is $C^4_3 \times C^5_2 \times C^6_2 + C^4_2 \times C^5_3 \times C^6_2 + C^4_2 \times C^5_2 \times C^6_3 = 600 + 900 + 1200 = 2700$.

178. From 5 apples we can choose one of $0, 1, 2, 3, 4, 5$ apples i.e. 6 selections. Similarly, for oranges no. of selections is 5, and for mangoes it is 4.

    Thus, total no. of selections are $6 \times 5 \times 4 = 120$. However, one of these selections will contains 0 fruits. Thus, the answer is $120 - 1 = 119$.

179. No. of ways to select red balls $C_1^4 + C_2^4 + C_3^4 + C_4^4 = 4 + 6 + 4 + 1 = 15$. No. of ways to select green balls $C_0^3 + C_1^3 + C_2^3 + C_3^3 = 8$.

Thus, total no. of selections are $15 \times 8 = 120$.

180. There are three bills. We can choose one, two or all them to form a sum. Thus, total no. of sums are $C_1^3 + C_2^3 + C_3^3 = 7$.

181. The boy can solve 1,2, 3, 4 or 5 questions from the paper. Thus, total no. of ways are $C_1^5 + C_2^5 + C_3^5 + C_4^5 + C_5^5 = 5 + 10 + 10 + 5 + 1 = 31$.

182. The voter can vote for one seat in C_1^6 ways, for two seats in C_2^6 ways, and for three seats in C_3^6 ways. Thus, total no. of ways in which the voter can vote are $C_1^6 + C_2^6 + C_3^6 = 41$.

183. Let there be n candidates out of which $n-1$ have to be elected. Total no. of ways in which this can be done are $C_1^n + C_2^n + \cdots + C_{n-1}^n = 30 \Rightarrow 2^n = 32 \Rightarrow n = 5$(Using binomial theorem).

184. 12 books are to be distributed equally among 4 person, so each will get 3 books. No. of ways to select 3 out of 12 are C_3^{12}, no. of selections for 3 out of remaining 9 are C_3^9 and so on.

Thus, total no. of ways of distributing the books are $C_3^{12} \times C_3^9 \times C_3^6 \times C_3^3 = \frac{12!}{(3!)^4}$.

185. no. of ways distributing n identical things among r people, where any person can get any no. of things is C_{r-1}^{n+r-1}. Therefore, required no. of ways are C_3^{13}.

186. 3 constants can be selected out of 10 consonants in C_3^{10} ways. 2 vowels out of 4 can be selected in C_2^4 ways. Now we have 5 alphabets which be arranged in 5! ways. Thus, total no. of words formed are $C_3^{10} \times C_2^4 \times 5!$.

187. We seat X, Y, Z on the side facing the window. Now from remaining 4 one has to sit on this side, which is C_1^4 ways of selection. From remaining 3 all 3 have to sit on the other side, which is C_3^3 ways of selection. Thus, total no. of selections are 4.

For each selection no. of arrnagements are $4! \times 3!$. Hence, total no. of ways of seating are $4! \times 3! \times 4 = 576$.

188. Six men have preferences. Suppose on one side we seat 4 men who wish to sit together then to fill remaining 4 positions we have to choose from 10 i.e. C_4^{10}. Now for the remaining 6 free seats we have 6 people, which can seated in 1 way. Both sides can be arranged in 8! ways.

Thus, no. of ways of seating them is $C_4^{10}.8!.8!$.

189. Since two women are to be seated on seats numbered 1 to 4, the no. of arrangements are $P_2^4 = 12$. Now, three men are to be seated on 5 remaining chairs; the no. of arrangements are $P_3^6 = 120$.

Thus, total no. of arrangements are $12 \times 120 = 1440$.

190. Consider $\frac{C_r^{2n}}{C_{r-1}^{2n}} = \frac{2n-r+1}{r}$, which has to be greater than 1 if C_r^{2n} has to be greatest. $\therefore 2n - r + 1 \geq r \Rightarrow r \leq n + \frac{1}{2}$.

Similarly, considering $\frac{C_r^{2n}}{C_{r+1}^{2n}}$, we find that $r \geq n - \frac{1}{2}$.

Combining the results $n - \frac{1}{2} \leq r \leq n + \frac{1}{2}$, but r has to be an integer. Thus, $r = n$.

191. In a seven digit number there are four odd places. There are two 1's, and two 3's, which can occupy these places. no. of such ways are $\frac{4!}{2!2!} = 6$.

For three even places, we have one 4, and two 2's. no. of ways in which three even places can be filled are $\frac{3!}{2!} = 3$.

Thus, total no. of required no. are $6 \times 3 = 18$.

192. There are 10 letters, and the words have five of these. no. of words where any letter can be repeated are 10^5. no. of letters where none of the letters are repeated are ${}^{10}P_5$.

Thus, no. of words where at least one letter is repeated are $10^5 - {}^{10}P_5$.

193. Ternary no. include 0, 1, 2. First we consider the case when the required seuqence begins with 210. We have six vacant places, which can be filled in $3^6 = 729$ ways. Similarly, no. of sequences which end with 210 will be 729.

However, there will be common sequences between these two which start with 210 and end with 210. no. of such sequences will be $3^3 = 27$ (Hint: we will have only three empty places).

Thus, no. of required numbers are $729 + 729 - 27 = 1431$.

194. A seven digit no. will be a no. ranging from 1,000,000 to 9,999,999. If units place is odd then sum of remaining six digits must be odd or if units place is even then sum of remaining six digits must be even to satisfy the condition given. Thus, half the no. will satisfy the given condition.

$\therefore$ Required number $= 9 \times 10 \times 10 \times 10 \times 10 \times 10 \times 5 = 45 \times 10^5$.

195. Treating 10 Indians as one person, we have $1 + 4 + 5 = 10$ persons. These can be seated in 10! ways. However, 10 Indians can be seated among themselves in 10! ways.

Thus, total no. of seating arrangements are $10!\,10!$.

196. Total no. of letters are 7; of which 2 are 'A', and 2 are 'R'. Total no. of arrangements when there is no restriction $= \frac{7!}{2!2!} = 1260$.

(a) Treating two 'R' as one i.e. we are considering words when both 'R's are together. no. of such words $= \frac{6!}{2!} = 360$. Thus, no. of words where 'R' are never together $= 1260 - 360 = 900$.

(b) Number of arrangements when two 'A's are together is 360 like previous case. Treating both 'A's and 'R''s as one i.e. when both are together, no. of words $= 5! = 120$.

Thus, required no. of words $= 360 - 120 = 240$.

(c) From (a) and (b) it follows that required number $= 900 - 240 = 660$.

197. no. of ways of dividing $m + n$ persons into two groups such that one has m persons, and the other has n persons is $C_m^{m+1}.C_n^n = \frac{(m+n)!}{m!n!}$.

Now, m persons can be seated around a round table in $(m-1)!$ ways; similarly n persons can be seated in $(n-1)!$ ways.

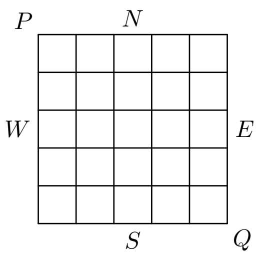
Thus, total no. of ways is $\frac{(m+n)!(m-1)!(n-1)!}{m!n!} = \frac{(m+n)!}{mn}$.

198. The signal can be made by using any no. of flags. Thus, required no. of signals is $P_1^5 + P_2^5 + P_3^5 + P_4^5 + P_5^5 = 325$.

199. There are 5 letters in the word 'Ought', which are all different. The alphabetical order of letters are G, H, O,T, U.

no. of words beginning with G, H, O are $4! \times 3 = 24 \times 3 = 72$. no. of words beginning with TG are $3! = 6$, and same for TH. no. of words beginning with TOG and TOH are $2! = 2$. TOUGH is the first word beginning with TOU. Thus, rank of the word 'TOUGH' in the dictionary will be $24 \times 3 + 6 \times 2 + 2 \times 2 + 1 = 89$.

200. Let the city be represented by a rectangle, whose sides are of length a and b North-South and East-West respectively.



Man has to go from P to Q. For this he has to travel a vertically downward and b horizontally Eastward. There are $m-1$ horizontal segments and $n-1$ vertical segments. Thus, from P to Q there are $m+n-2$ segments total. We have to choose $m-1$, and $n-1$ segments from these. This can be done in $\frac{(m+n-2)!}{(m-1)!(n-1)!}$ ways.

201. Let the n letters be denoted by $1, 2, 3, \ldots, n$. Let A_i denote the set of distribution of letters in envelopes so that only ith letter is put in the corresponding envelope. Then $n(A_i) = (n-1)!$, because rest $n-1$ letters can be put in $n-1$ envelopes in $(n-1)!$ ways.

Similarly, $n(A_i \cap A_j)$ i.e. putting two letters in correct envelopes is $(n-2)!$. Required number $= n(A_1' \cap A_2' \cap A_3' \cap \ldots \cap A_n') = n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)' = n! - n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)'$

$= n! - [\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \cdots + (-1)^n n(A_1 \cap A_2 \cap \ldots \cap A_n)]$

$= n! - [C_1^n(n-1)! - C_2^n(n-2)! \cdots] = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots\right].$

202. Number of non-congruent squares is 8 as they are of size $1 \times 1, 2 \times 2, 3 \times 3, \ldots, 8 \times 8$. Number of non-congruent rectanges, which are not squares $= C_2^8 = 28$.

Thus, required number $= 28 + 8 = 36$.

203. Let the three numbers selected from are a, b, c, which have to be in A.P. i.e. $a + c = 2b$. This implies that both a, c are either even or odd as sum has to be even.

Case I: When n is even.

Let $n = 2m$, then no. of odd and even numbers are same i.e. m. Thus, no. of ways in which a and c can be selected is $2 \times C_2^m = m(m-1) = \frac{1}{4}n(n-2)$.

Case II: When n is odd.

Let $n = 2m + 1$, then no. of odd numbers is $m + 1$, and no. of even numbers is m. Thus, a and c can be selected in $C_2^{m+1} + C_2^m = \frac{1}{4}(n-1)^2$ ways.

204. Since there are two packs of 52 cards, therefore, number of cards from same suit and denomination is 2 for each card.

no. of ways of selecting 26 cards out of 52 cards $= C_{26}^{52}$, however, each card can be selected in 2 ways. $\therefore$ Required numbers $= C_{26}^{52}.2^{26}$.

205. For n sides there will be n vertices. Selection of any 3 vertices will give us a triangle. no. of ways of selecting 3 vertices out of n vertices i.e. no. of triangles $= C_3^n = \frac{n(n-1)(n-2)}{6}$.

206. (a) If the n objects are $o_1, o_2, o_3, \ldots, o_n$, then possible solutions will be $o_1o_2o_3, o_2o_3o_4, o_3o_4o_5, \ldots, o_{n-2}o_{n-1}o_n$.

$\therefore$ Required number $= n - 2$.

(b) no. of ways to select 3 objetcs out of n objects without restriction $= C_3^n$. Thus, following from (a) required number $C_3^n - n + 2 = \frac{(n-3)(n^2-4)}{6}$.

207. Let a be the no. of stations before stop $1, b$ be the no. of stations before stop $2, c$ be the no. of stations before stop $3, d$ be the no. of stations before stop 4, and e be the no. of stations after stop 4. Then, $a + b + c + d + e = 8$, where $a \geq 0, b, c, d \geq 1, e \geq 0$.

Let $x = a, y = b - 1, z = c - 1, t = d - 1, w = e$, then $x + y + z + t + w = 5$.

$\therefore$ Required no. $=$ Number of non-negative integral solutions of the above equation $= C_r^{n+r-1} = C_5^9 = 126$.

208. no. of straight lines formed by given m points $= C_2^m = n$ (let). Total no. of points of intersections of these lines $= C_2^n$ under given conditions.

Consider a point A_1. No. of lines passing through $A_1 = m - 1$. No. of pair of lines intersecting at $A_1 = C_2^{m-1}$. Similarly, this will be the case for other points.

Hence, required no. of points of intersections $= C_2^n - m.C_2^{m-1} = \frac{m!}{8(m-4)!}$.

209. The word BAC cannot be spelled if the m selected coupons do not contain at least one of A, B or C.

no. of ways of selecting m coupon which are A or $B = 2^m$. This also includes when all m coupons are all A or all B. Similarly for B or C and for A and C. no. of ways of selecting m coupons where all are $A = 1$. Similarly for B and C.

Thus, required no. $= 2^m + 2^m + 2^m - 1 - 1 - 1 = 3(2^m - 1)$.

210. The staight cards can be $1-5, 2-6, 3-7, \dots, 6-10, 7-J, 8-Q, 9-K, 10-A$. Thus, we see there are 10 such straight hands. One card of any denomination can be picked from any of the suits in 4 ways. Thus, 5 cards of five different denominations can be selected from 4 suits in 4^5 ways.

Thus, number of ways of making selections $= 10 \times 4^5 = 10{,}240$.

If all cards are not from same suit then no. of ways of making selections $= 10 \times 4^5 - 10 \times 4 = 10{,}200$ because there aree 4 suits.

211. Let $A = a_1, a_2, \dots, a_n$. Consider element a_1. Either it is in P_1 or it is not. So total no. of ways for a_1 and $P_1 = 2$. No. of ways in which a_1 is in $P_1 = 1$, and same for not belonging i.e. in 1 way.

Total no. of ways for a_1, and m subsets $= 2^m$. No. of ways in which a_1 belongs to m subsets $= 1^m = 1$. No. of ways in which a_1 belongs to none of the subsets $= 1^m = 1$.

Thus, total no. of ways in which $a_1 \notin (P_1 \cap P_2 \cap \dots \cap P_m) = 2^m - 1$, $a_1 \notin (P_1 \cup P_2 \cup \dots \cup P_m) = 1^m$, and $a_1 \in (P_1 \cup P_2 \cup \dots \cup P_m) = 2^m - 1$.

i. Here exactly r elements of A belongs to $P_1 \cup P_2 \cup \dots \cup P_m$, and $n-r$ elements do not belong to $P_1 \cup P_2 \cup \dots \cup P_m$.

   $\therefore$ Required number $= C_r^n (2^m - 1)^r (1)^{n-r} = C_r^n (2^m - 1)^r$.

ii. Here exactly r elements of A belongs to $P_1 \cap P_2 \cap \dots \cap P_m$, and $n-r$ elements do not belong to $P_1 \cap P_2 \cap \dots \cap P_m$.

   $\therefore$ Required number $= C_r^n (2^m - 1)^{n-r} (1)^r = C_r^n (2^m - 1)^{n-r}$.

iii. Let $P_{m+1} = A - (P_1 \cup P_2 \cup \dots \cup P_m)$. Since $P_i \cap P_i = \phi, i \neq j$, where $i, j = 1, 2, \dots, m$.

   Each element of A should belong to exactly one of the $(m+1)$ subsets $P_1, P_2, \dots, P_m, P_{m+1}$. For one element there are $m+1$ ways so for n elements there are $(m+1)^n$ ways.

212. Given that number of boxes is $2m$, and number of identical balls is m. Number of ways to select m boxes out of $2m$ is C_m^{2m}. Because m balls are identical they can be arranged in $\frac{m!}{m!} = 1$ way.

$\therefore$ Required number $= C_m^{2m} = \frac{2m!}{m!m!}$.

We will make use of mathematical induction to show that $\frac{4^m}{2\sqrt{m}} \leq \frac{2m!}{m!m!} \leq \frac{4^m}{\sqrt{2m+1}}$.

Let $P(m) : \frac{4^m}{2\sqrt{m}} \leq \frac{2m!}{(m!)^2}$.

When $m = 1$, L.H.S. $= 2$, and R.H.S. $= 2$. The equality holds, and hence, $P(1)$ is true.

Let $P(k)$ be true $\Rightarrow \frac{4^k}{2\sqrt{k}} \leq \frac{2k!}{(k!)^2}$.

We have to prove that $P(k+1)$ is true i.e. $\frac{4^{k+1}}{2\sqrt{k+1}} \leq \frac{2(k+1)!}{[(k+1)!]^2} = \alpha$ (say).

Multiplying both sides of $P(k)$ with $\frac{(2k+1)(2k+2)}{(k+1)^2} = \frac{2(2k+1)}{k+1}$, we have

$\frac{(2k+2)!}{[(k+1)!]^2} \geq \frac{2(2k+1)4^k}{2\sqrt{k(k+1)}} = \frac{(2k+1)4^k}{\sqrt{k(k+1)}} = \beta$ (say).

Now $\frac{\beta}{\alpha} = \frac{(2k+1)4^k}{\sqrt{k(k+1)}} \cdot \frac{2\sqrt{k+1}}{4^{k+1}} = \frac{2k+1}{2\sqrt{k(k+1)}} = \frac{2k+1}{\sqrt{4k^2+4k}} = \frac{\sqrt{4k^2+4k+1}}{\sqrt{4k^2+4k}} > 1 \Rightarrow \beta > \alpha$.

Hence, $P(k+1)$ is true whenever $P(k)$ is true. Thus, $P(m)$ si true for all natural numbers m.

Let $Q(m) : \frac{2m!}{(m!)^2} \leq \frac{4^m}{\sqrt{2m+1}}$

When $m = 1$, L.H.S. $= 2$, and R.H.S. $= \frac{4}{\sqrt{3}}$, so L.H.S. $<$ R.H.S. making $Q(1)$ true.

Let $Q(k)$ be true i.e. $\frac{2k!}{(m!)^2} \leq \frac{4^k}{\sqrt{2k+1}}$

We have to prove that $Q(k+1)$ i.e. $\frac{(2k+2)!}{[(k+1)!]^2} \leq \frac{4^{k+1}}{\sqrt{2k+3}} = x$ (let).

Multiplying $Q(k)$ with $\frac{(2k+1)(2k+2)}{(rk+1)^2}$, we have $\frac{(2k+2)!}{[(k+1)!]^2} \leq \frac{4^{k+1}}{\sqrt{2k+1}} \cdot \frac{(2k+1)(2k+2)}{(k+1)^2} = \frac{4^k.2\sqrt{2k+1}}{k+1} = y$ (say).

$\therefore \frac{y}{x} = \frac{4^k.2\sqrt{2k+1}}{k+1} \cdot \frac{\sqrt{2k+3}}{4^{k+1}} = \frac{\sqrt{4k^2+8k+3}}{4k^2+8k+4} \Rightarrow y < x$.

Thus, $Q(k+1)$ is true, and hence $Q(m)$ is true for all m.

Hence, we have our proof using induction.

213. To form a parallelogram we need to select 2 lines from one set, and 2 from the other set. Thus, no. of parallelograms formed is $C_2^m \times C_2^n = \frac{1}{4}mn(m-1)(n-1)$.

214.

Number of ladies	Number of men	Number of committees
1	4	$C_1^4 C_4^6 = 60$

2	3	$(C_2^4 - C_0^2).C_3^6 = 100$
3	1	$(C_3^4 - C_1^2).C_2^6 = 30$
4	1	Not possible

215. **Case I:** 3 men from husband's side, and 3 ladies from wife's side. no. of ways to do this is $C_0^4 \times C_3^3 \times C_3^3 \times C_0^4 = 1$

Case II: 2 men, and 1 lady from husband's side, and 1 man and 2 ladies from wife's side. no. of ways to do this is $C_1^4 \times C_2^3 \times C_2^3 \times C_1^4 = 144$

Case III: 1 man, and 2 ladies from husband's side, and 2 men and 1 lady from wife's side. no. of ways to do this is $C_2^4 \times C_1^3 \times C_1^3 \times C_2^4 = 324$

Case IV: 3 ladies from husband's side, and 3 men from wife's side. no. of ways to do this is $C_3^4 \times C_0^3 \times C_0^3 \times C_3^4 = 16$

$\therefore$ Required number $= 1 + 144 + 324 + 16 = 485$.

216. For an intersection we need two lines such that they have one point on each of the given lines. Thus, total no. of ways to select these four points is $C_2^m \times C_2^n = \frac{1}{4}mn(m-1)(n-1)$.

217. Let y be the no. of children born after John and Mary marry. Then $x + x + 1 + y = 24 \Rightarrow 2x + y = 23$.

Let z be the no. of fights, then $z = C_1^x.C_1^y + C_1^x.C_1^{x+1} + C_1^y.C_1^{x+1} \Rightarrow z = xy + x(x+1) + y(x+1) = x(23-2x) + x^2 + x + (23-2x)(x+1) = -3x^2 + 45x + 23$

$\Rightarrow 3x^2 - 45x + z - 23 = 0$, now, because x is read $D \geq 0 \Rightarrow 45^2 - 12(z-23) \geq 0 \Rightarrow z \leq \frac{2301}{12} = 191.75$.

So, greatest value of z is 191.

218. Doing prime factorization, we have $2520 = 2^3 \times 3^2 \times 5 \times 7$. Each term of the product $(1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5)(1 + 7)$ is a divisor of 2520. Total no. of divisors is equal to total no. of terms in the product $= 48$. Sum of divisors $= (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5)(1 + 7) = 9360$.

219. There can be two sets of three positive integers whose sum is 5. These sets would be $\{1, 1, 3\}$ and $\{1, 2, 2\}$. Elements of both the set can be arranged in $\frac{3!}{2!}$ i.e. 3 ways.

$\therefore$ Required no. $= 3 \times C_3^5 \times C_1^2 \times C_1^1 + 3 \times C_2^5 \times C_2^3 \times C_1^1 = 150$.

220. Let $m = (n-1)!$, then $n! = mn$. Now $\frac{(n!)!}{(n!)^{(n-1)!}} = \frac{(mn)!}{(n!)^m}$, which is no. of ways of distributing mn things among m persons each having n things.

221. $\frac{(ab)!}{a!(b!)^a}$ is no. of ways of distributing ab different things in a sets each having b things, which is an integer.

222. Number of ways of distributing n identical objects in r groups, where each group can contain any number of objects, and the ordering matters $= P_r^{n+r-1} = P_6^{206}$.

223. **Method I:** Since each person has to get at least 3 things, if 3 persons get 3 things 4th can get at most 7 things. Thus,

Required numbers = coeff. of x^{16} in $(x^3 + x^4 + \cdots + x^7)^4$ = coeff. of c^{16} in $x^{12}(1 + x + \cdots + x^4)^4$ = coeff. of x^4 in $\left(\frac{1-x^5}{1-x}\right)^4 = C_4^7 = 35$.

Method II: Let the four persons be give a, b, c, d no. of things. Then, $a+b+c+d = 16$, where $a, b, c, d \geq 3$, then $w + x + y + z = 4, w, x, y, z \geq 0$, and $w = a - 3, x = b - 3, y = c - 3, z = d - 3$.

Required no. is solution of any of the above equations, which is number of ways of distributing 4 identical things among 4 persons, where each person can get any no. of things $= C_r^{n+r-1} = C_4^7 = 35$.

Method III: Sets of four posiitve integers each greater than or equal to 3 whose sum is 16 are $\{7, 3, 3, 3\}$, $\{6, 4, ,3, 3\}$, $\{5, 5, 3, 3\}$, $\{5, 4, 4, 3\}$, $\{4, 4, 4, 4\}$.

Elements of first set can be arranged in $\frac{4!}{3!} = 4$ ways. Elements of second set can be arranged in $\frac{4!}{2!} = 12$ ways. Elements of third set can be arranged in $\frac{4!}{2!2!} = 6$ ways. Elements of fourth set can be arranged in $\frac{4!}{2!} = 12$ ways. Elements of fifth set can be arranged in $\frac{4!}{4!} = 1$ way.

Required no. $= 4 + 12 + 6 + 12 + 1 = 35$.

224. Let no. of red, white, blue, and green balls be w, x, y and z respectively. From question, $w + x + y + z = 10$, where $w, x, y, z \geq 0$.

This is no. of ways of distributing 10 identical things among four persons where each can get any no. of things $= C_r^{n+r-1} = C_{10}^1 3 = 286$.

When the selections contain balls of each color the equation remains same, but $w, x, y, z \geq 1$. So $a + b + c + d = 6$, where, $a = w - 1, b = x - 1$ and so on.

In this case, the method is same but $n = 4, r = 6$, so the answer is $C_6^9 = 84$.

225. Let the questions contain $x_1, x_2, \ldots, x_8$ marks, then, from question $x_1 + x_2 + \cdots + x_8 = 30$, where $x - 1, x_2, \ldots, x_8 \geq 2$.

$\Rightarrow y_1 + y_2 + \cdots + y_8 = 14$, where $y_1 = x_1 - 2$, and so on.

Required number is no. of solutions of above equations $= C_8^{n+r-1} = C_{14}^{21} = 116{,}280$.

226. Total marks is $3 \times 50 + 100 = 250$, and the student must score 60% i.e. 150 marks.

Required number = coeff. of x^{150} in $(1 + x + \cdots + x^{50})^3(1 + x + \cdots + x^{100})$ = coeff. of x^{150} in $\left(\frac{1-x^{15}}{1-x}\right)^3 \frac{1-x^{101}}{1-x}$ = coeff. of x^{150} in $(1 - x^{51})^3(1 - x^{101})(1 - x)^{-4}$ = coeff. of x^{150} in $(1 - 3x^{51} + 3x^{102} - x^{101})(1 - x)^{-4}$ (leaving powers greater than 160)

$=$ coeff.of x^{150} in $(1-x)^{-4}-3$.coeff. of x^{99} in $(1-x)^{-4}+3$.coeff. of x^{48} in $(1-x)^{-4}-$ coeff. of x^{49} in $(1-x)^{-4}$

$= C_{150}^{153} - 3.C_{90}^{102} + 3.C_{48}^{51} - C_{49}^{52} = 110{,}556.$

227. Given, $x_1 + x_2 + \cdots + x_k = n$. Let $y_1 = x_1 - 1, y_2 = x_2 - 2, \ldots, y_k = x_k - k$, then $y_1 + y_2 + \cdots + y_k = n - (1 + 2 + \cdots + k) = n - \frac{k(k+1)}{2} = m$.

Then no. of solutions $= C_m^{m+k-1} = \frac{\left(n+k-\frac{k(k+1)}{2}-1\right)!}{\left(n-\frac{k(k+1)}{2}\right)!(k-1)!}$.

228. Given $x + y + z + w = 29$, where $x \geq 1, y \geq 2, \geq 3, w \geq 0$. Putting $p = x - 1, q = y - 2, r = z - 3$,

$p + q + r + w = 23$, where $p, q, , r, w \geq 0$.

Following like previous problems, required no. $= C_{23}^{26} = 2{,}600$.

229. Required number $=$ coeff. of x^{20} in $(1-x)^{-3}(1-x^4)^{-1} =$ coeff. of x^{20} in $(1 + C_1^3 x + C_2^4 x^2 + C_3^5 x^3 + \cdots + C_{20}^{22} x^{20} + \cdots)(1 + x^4 + x^8 + x^{12} + x^{16} + x^{20} + \cdots)$

$= 1 + C_4^6 + C_8^{10} + C_{12}^{14} + C_{16}^{18} + C_{20}^{22} = 536.$

230. From given equations we have $v + w = 15$, and $x + y + z = 3$.

Number of non-negative integral solution of these equations combined is $= C_5^7.C_{15}^{16} = 336$.

231. Given inequality is $3x + y + z \geq 30$. Let w is a non-negative integer such that $3x + y + z + w = 30$, where $x, y, z, w \geq 1$.

Let $a = x - 1, b = y - 1, c = z - 1, d = w$, then $3a + b + c + d = 25$, where $a, b, c, d \geq 0$.

Clearly, $0 \leq a \leq 8$. If $a = k$, then $b + c + d = 25 - 3k$.

no. of non-negative solutions of this equation is $C_{25-3k}^{27-3k} = C_2^{27-3k} = \frac{3}{2}(3k^2 - 53k + 234)$

$\therefore$ Required numbers $= \frac{3}{2}\sum_{k=0}^{k}(3k^2 - 53k + 234) = 1215.$

232. Given $a + b + c + d = 20$, where $a, b, c, d \geq 1$. Let us assume that $a < b < c < d$. Also let, $x = a, y = b - a, z = c - b, w = d - c \therefore a = x, b = y + x, c = x + y + z, d = x + y + z + w$

$\therefore 4x + 3y + 2z + w = 20. \therefore$ Sum of minimum values of $4x, 3y, 2z$ and $w = 4 + 3 + 2 + 1 = 10$.

Required number $=$ number of positive unequal integral solutions of above equation

$=$ coeff. of x^{10} in $(1-x^4)^{-1}(1-x^3)^{-1}(1-x^2)^{-1}(1-x)^{-1} =$ coeff. of x^{10} in $[(1+x^4+x^8)(1+x^3+x^6+x^9)(1+x^2+x^4+\cdots+x^{10})(1+x+x^2+\cdots+x^{10})]$(leaving terms greater than x^{10})

$=$ coeff. of x^{10} in $[(1+x^3+x^4+x^6+x^7+x^8+x^9+x^{10})(1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+5x^8+4x^9+6x^{10})] = 23$

But a, b, c, d can be arranged in 4! ways among themselves. Thus, total no. of unique solutions is $23 \times 4! = 552$.

233. Any no. between 1 and 1,000,000 must be of less than seven digits. Thus, $a_1+a_2+a_3+a_4+a_5+a_6 = 18$, where $a_1, a_2, \ldots, a_5 \in 0, 1, 2, \ldots, 9$, and the number is of the form $a_1a_2a_3a_4a_5a_6$.

$\therefore$ Required number $=$ coefficient of x^{18} in $(1+x+x^2+\cdots+x^9)^6 =$ coeff. of x^{18} in $\left(\frac{1-x^{10}}{1-x}\right)^6$

$=$ coeff. of x^{18} in $(1-x^{10})^6(1-x)^{-6} =$ coeff. of x^{18} in $(1-6x^{10})(1-x)^{-6}$(leaving out powers greater than x^{18})

$= C_{18}^{6+18-1} - 6.C_8^{6+8-1} = 25{,}927$.

234. Required number $=$ coefficient of x^n in $(1+x+x^2+\cdots+x^n)^2(1+x)^n =$ coeff. of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^2(1+x)^n =$ coeff. of x^n in $(1-2x^{n+1}+x^{2n+2})(1-x)^{-2}(1+x)^n =$ coeff. of x^n in $(1-x)^{-2}(1+x)^n$(leaving powers greater than n)

$=$ coeff. of x^n in $(1-x)^{-2}2-(1-x)^n =$ coeff. of x^n in $(1-x)^{-2}[2^n - C_1^n 2^{n-1}(1-x) + C_2^n 2^{n-2}(1-x)^2 - \cdots + (-1)^n C_n^n (1-x)^n] =$ coeff. of x^n in $[2^n(1-x)^{-2} - C_1^n 2^{n-1}(1-x)^{-1}]$(other terms will not contain x^n)

$= 2^n.C_n^{2+n-1} - C_1^n.2^{n-1}C_n^{1+n-1} = 2^{n-1}(2n+2)$.

235. no. of ways in which one crew out of 3 can be arranged on the steering is P_1^3.

Since 2 particular sailors are always to remain on bow side, therefore, 2 more sailors for bow side can be selected out of remaining 6 sailors in C_2^6, and 4 sailors for stroke side can be selected out of remaining 4 in C_4^4 ways.

Now 4 sailors on bow side can be arranged among themselves in 4! ways. Again 4 sailors on stroke side can be arranged among themselves in 4! ways.

$\therefore$ Required no. $= P_1^3.C_2^6.4!\,4! = 25{,}920$.

236. Total no. of letters is 11. E and N occurs thrice, D occurs twice, and rest occur once.

Case I: When three letters are identical, and remaining two are identical. We can select three E's and two N's or three E's and two D's or three N's and two E's or three N's and two D's. Thus, there are total 4 ways.

Case II: When three letters are identical, and remaining two are different. Letters selected can be three E's and two out of I, N, D, P, T or three N's and two out of I, E, D, P, T.

no. of ways to select is $1 \times C_2^5 + 1 \times C_2^5 = 20$.

Case III: When a two letters are identical, and there are two such letters, and the fifth letter is different. Letter selected can be two E's, two N's, and one out of I, D, P, T or two E's, two D's, and one out of I, N, P, T or two N's, two D's, and one out of I, E, P, T.

no. of ways to make these selections is $3 \times C_1^4 = 12$.

Case IV: When two letters are same, and remaining three are different. Letters selected can be two E's and three out of I, N, P, D, T or two N's and three out of I, E, P, D, T or two D's and three out of I, E, P, N, T.

no. of ways to make these selections is $3 \times C_3^5 = 30$.

Case V: When all five letters are different. no. of ways to make these selections is $C_5^6 = 6$.

Adding all these we get 72 as the answer.

Second Method: Previous method is direct, however, for bigger and more complex problems it becomes tedious.

Required number = coeff. of x^5 in $(1+x+x^2+x^3)^2(1+x+x^2)(1+x)^3$ = coeff. of x^5 in $(1+x^2+x^4+x^6+2x+2x^3+2x^3+2x^4+2x^5)(1+x+x^2)(1+x)^3$

= coeff. of x^5 in $(1+2x+3x^2+4x^3+3x^4+2x^5)(1+x+x^2)(1+3x+3x^2+x^3)$ = coeff. of x^5 in $(1+2x+3x^2+4x^3+3x^4+2x^5)(1+4x+7x^2+7x^3+4x^4+x^5)$ = $1+8+21+28+12+2 = 72$.

237. Here a occurs twice, l thrice, and p, r, e once.

no. of combinations = coeff. of x^4 in $(1+x+x^2+x^3)(1+x+x^2)(1+x)^3$ = coeff. of x^4 in $(1-x)^{-5}(1-x^2)^3(1-x^3)(1-x^4) = 22$

no. of permutations = coeff. of x^4 in $4!\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+x+\frac{x^2}{2!}\right)(1+x)^3$

$= 286$.

238. L.H.S. = $\sum_{n=1}^{n}(n^2+1)n! = (1^2+1).1! + (2^2+1).2! + \cdots + (n^2+1).n!$

We can expand this as follows:

$(1^2+1).1! = 2!$

$(2^2+1).2! = 3.3! - 2.2.2!$

$(3^2+1).3! = 4.4! - 2.3.3!$

$(4^2+1).4! = 5.5! - 2.4.4!$

...

$[(n-1)^2+1].(n-1)! = n.n! - 2.(n-1).(n-1)!$

$(n^2+1).n! = (n+1).(n+1)! - 2.n.n!$

Adding first two yields $3.3! - 3.2! = 2.3!$, adding this to third, we get $4.4! - 2.2.3! = 3.4!$, and so on. Thus, sum would be $n.(n+1)!$, which is easily verified.

239. Given, $\frac{P_4^{n+4}}{(n+2)!} - \frac{143}{4.n!} < 0 \Rightarrow \frac{(n+4)!}{n!(n+2)!} - \frac{143}{4.n!} < 0$

$\Rightarrow (n+4)(n+3) - \frac{143}{4} < 0 \Rightarrow 4n^2 + 28n - 95 < 0$, which is only true for $n = 1, 2$.

240. Given, $\frac{195}{4.n!} - \frac{(n+3)(n+2)(n+1)}{(n+1)!} > 0 \Rightarrow \frac{195}{4} - (n+3)(n+2).0$

$\Rightarrow 171 - 4n^2 - 20n > 0 \Rightarrow 4n^2 + 20n - 171 < 0$

Roots of corresponding quadratic equations are $\frac{-20\pm\sqrt{400+2736}}{8}$, which gives us $n = 1, 2, 3, 4$ as integral values for n to satisfy the inequality.

241. Given, ${}^{n-2}P_4 :{}^{n+2} C_8 = 16 : 57 \Rightarrow \frac{(n-2)!}{(n-6)!}.\frac{8!(n-6)!}{(n+2)!} = \frac{16}{57}$

$\Rightarrow \frac{8!}{(n+2)(n+1)n(n-1)} = \frac{16}{57}$. Solving this gives us $n = 19$.

242. Given, $P_r^n = P_{r+1}^n \Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \Rightarrow n - r = 1$,

and $C_r^n = C_{r-1}^n \Rightarrow \frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r+1)!}$

$\Rightarrow \frac{1}{r} = \frac{1}{n-r+1} \Rightarrow n - 2r + 1 = 0$. Solving these two equations, we have $n = 3, r = 2$.

243. Given, $P_{r-1}^n : P_r^n : P_{r+1}^n = a : b : c \Rightarrow \frac{1}{(n-r+1)(n-r)} : \frac{1}{(n-r)} : 1 = a : b : c$

Now, $b^2 = \frac{1}{(n-r)^2}$, and $a(b+c) = \frac{1}{(n-r+1)(n-r)}\left[\frac{1}{n-r}+1\right] = \frac{1}{(n-r)^2}$, and thus, $b^2 = a(b+c)$.

244. Given, $C_{r+1}^{n+1} : C_r^n : C_{r-1}^{n-1} = 11 : 6 : 3$.

Taking first two, we have $\frac{(n+1)!}{(n-r)!(r+1)!} : \frac{n!}{r!(n-r)!} \Rightarrow \frac{n+1}{r+1} = \frac{11}{6} \Rightarrow 6n - 11r = 5$

Taking last two, we have $\frac{n!}{r!(n-r)!}.\frac{(r-1)!(n-r)!}{(n-1)!} = \frac{n}{r} = \frac{6}{3} = 2 \Rightarrow n = 2r$

Solving these two obtained equations, we have $r = 5, n = 10$.

245. We have to prove that $\sum_{k=m}^{n} C_r^k = C_{r+1}^{n+1} - C_{r+1}^m$

L.H.S. $= C_r^m + C_r^{m+1} + \cdots + C_r^n$. Adding and subtracting C_{r+1}^m, we have

L.H.S. $= C_{r+1}^m + C_r^m + C_r^{m+1} + \cdots + C_r^n - C_{r+1}^m$

We know that $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$. Applying this repeatedly on the above expression we arrive at the desired result.

246. We have to prove that $C_r^n + 3.C_{r-1}^n + 3.C_{r-2}^n + C_{r-3}^n = C_r^{n+3}$

We will make use of the fact that $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$.

L.H.S. $= C_r^{n+1} + 2.C_{r-1}^n + 2.C_{r-2}^n + C_{r-2}^{n+1} = C_r^{n+1} + 2.C_{r-1}^{n+1} + C_{r-2}^{n+1}$

$= C_r^{n+2} + C_{r-1}^{n+2} = C_r^{n+3} =$ R.H.S.

247. We have to find r for which ${}^{18}C_{r-2} + 2.{}^{18}C_{r-1} +^{18} C_r \geq^{20} C_{13}$.

L.H.S. $= \left(C_{r-2}^{18} + C_{r-1}^{18}\right) + \left(C_{r-1}^{18} + C_r^{18}\right) = C_{r-1}^{19} + C_r^{19} = C_r^{20}$.

Comparing with R.H.S., clearly $r = 7, 8, 9, \ldots, 13$.

248. We have to prove that ${}^{4n}C_{2n} :^{2n} C_n = 1.3.5 \ldots (4n-1) : [1.3.5 \ldots (2n-1)]^2$.

L.H.S. $= \frac{C_{2n}^{4n}}{C_n^{2n}} = \frac{4n!}{2n!2n!} \cdot \frac{n!n!}{2n!} = \frac{4n.(4n-1)(4n-2)\ldots(2n+2)(2n+1)}{2n!2n!}.n!\,n!$

$= 1.3.5 \ldots (4n-1) : [1.3.5 \ldots (2n-1)]^2 \left[\because \frac{2n!}{n!} = 2n.(2n-1).(2n-2). \ldots (n+2)\,(n+1)\right]$

249. We have to find the positive integral values of x such that $C_4^{x-1} - C_3^{x-1} - \frac{5}{4}(x-2)(x-3) < 0$.

L.H.S. $= C_4^{x-1} - C_3^{x-1} - \frac{5}{2}.C_2^{x-2} = \frac{(x-2)!}{(x-5)!}\left[\frac{x-1}{4!} - \frac{x-1}{3!(x-4)} - \frac{5}{2.2!(x-4)}\right]$

$= \frac{(x-2)!}{(x-5)!3!}\left[\frac{x-1}{4} - \frac{x-1}{x-4} - \frac{15}{2(x-4)}\right]$

We know that factorials are always positive, hence the expression under brackets must be less than zero.

$\Rightarrow (x-1)(x-4) - 4(x-1) - 30 < 0, x - 4 > 0 \Rightarrow x^2 - 5x + 4 - 4x + 4 - 30 < 0 \Rightarrow x^2 - 9x - 22 < 0$. Roots of this equations are $\frac{9 \pm \sqrt{81+88}}{2} = -2, 11$ and x must lie between these two roots.

Thus, $x = 5, 6, 7, \ldots, 10$.

250. We have to prove that ${}^{2n}P_n = 2^n.1.3.5 \ldots (2n-1)$.

L.H.S. $= P_n^{2n} = \frac{2n!}{n!n!} = \frac{2n.(2n-1)(2n-2)\ldots.4.3.2.1}{n.(n-1)(n-2)\ldots.3.2.1} = 2^n.1.3.5. \ldots (2n-1) =$ R.H.S.

251. We have to prove that there cannot exist two positive integers n and r for which $C_r^n, C_{r+1}^n, C_{r+2}^n$ are in G.P.

For the terms to be in G.P. $\frac{C_r^n}{C_{r+1}^n} = \frac{C_{r+1}^n}{C_{r+2}^n}$

$\Rightarrow \frac{n!}{r!(n-r)!} \cdot \frac{(r+1)!(n-r-1)!}{n!} = \frac{n!}{(r+1)!(n-r-1)!} \cdot \frac{(r+2)!(n-r-2)!}{n!} \Rightarrow \frac{r+1}{n-r} = \frac{r+2}{n-r-1}$

$\Rightarrow nr + n - r^2 - r - r - 1 = nr + 2n - r^2 - 2r \Rightarrow -1 = n$, which is not possible.

252. We have to prove that there cannot exist two positive integers n and r for which $C^n_r, C^n_{r+1}, C^n_{r+2}, C^n_{r+3}$ are in A.P.

$\Rightarrow C^n_{r+1} - C^n_r = C^n_{r+2} - C^n_{r+1} = C^n_{r+3} - C^n_{r+2} \Rightarrow \frac{n!}{(r+1)!(n-r-1)!} - \frac{n!}{r!(n-r)!} = \frac{n!}{(r+2)!(n-r-2)!} - \frac{n!}{(r+1)!(n-r-1)!} = \frac{n!}{(r+3)!(n-r-3)!} - \frac{n!}{(r+2)!(n-r-2)!}$

Taking first two, $\frac{n!}{r!(n-r-1)!}\left[\frac{1}{(r+1)} - \frac{1}{n-r}\right] = \frac{n!}{(r+1)!(n-r-2)!}\left[\frac{1}{r+2} - \frac{1}{n-r-1}\right]$

$\Rightarrow \frac{1}{n-r-1}\frac{n-2r-1}{(r+1)(n-r)} = \frac{1}{r+1}\cdot\frac{n-2r-3}{(r+2)(n-r-1)} \Rightarrow nr - 2r^2 - r + 2n - 4r - 2 = n^2 - 2nr - 3n - nr + 2r^2 + 3r.$

Similarly, we can find another equation by considering second and the third term to get another equation in n and r. Solving these two we cannot find integral solutions in n and r.

253. For all positive integers we have to show that $2.6.10\ldots(4n-6)(4n-2) = (n+1)(n+2)\ldots(2n-2)2n$.

L.H.S. $= 2^n[1.3.5\ldots(2n-3)(2n-1)] = \frac{2^n[1.2.3.4.5\ldots(2n-3)(2n-1)2n]}{2.4.6\ldots.2n}$

$= \frac{2^n[1.2.3.4.5\ldots(2n-3)(2n-1)2n}{2^n[1.2.3\ldots n]} = \frac{2n!}{n!} = (n+1)(n+2)\ldots(2n-1)2n =$ R.H.S.

254. We have to show that ${}^{47}C_4 + \sum_{i=0}^{3}{}^{50-i}C_3 + \sum_{j=1}^{5}{}^{56-j}C_{53-j} = {}^{57}C_4$.

L.H.S. $= C^{47}_4 + C^{47}_3 + \sum_{i=0}^{2} C^{50-i}_3 + \sum_{j=1}^{5} C^{56-j}_{53-j}$

Now, we make use of the fact that $C^n_r + C^n_{r+1} = C^{n+1}_{r+1}$.

L.H.S. $= C^{48}_4 + C^{48}_3 + \sum_{i=0}^{1} C^{50-i}_3 + \sum_{j=1}^{5} C^{56-j}_{53-j}$

Proceeding this way we have, L.H.S. $= C^{51}_4 + \sum_{j=1}^{5} C^{56-j}_{53-j} = C^{51}_4 + C^{51}_{48} + \sum_{j=1}^{4} C^{56-j}_{53-j} = C^{51}_4 + +C^{51}_3 + \sum_{j=1}^{4} C^{56-j}_{53-j}$

Again repeating like earlier we find L.H.S. $= C^{57}_4 =$ R.H.S.

255. We have to show that ${}^nC_k + \sum_{j=0}^{m}{}^{n+j}C_{k-1} = {}^{n+m+1}C_k$.

L.H.S. $= C^n_k + C^n_{k-1} + \sum_{j=1}^{m} C^{n+j}_{k-1}$

Now, we make use of the fact that $C^n_r + C^n_{r+1} = C^{n+1}_{r+1}$.

$\therefore$ L.H.S. $= C^{n+1}_k + \sum_{j=1}^{m} C^{n+j}_{k-1} = C^{n+1}_k + C^{n+1}_{k-1} + \sum_{j=2}^{m} C^{n+j}_{k-1} = C^{n+2}_k + \sum_{j=2}^{m} C^{n+j}_{k-1}$

Repeating this we obtain, L.H.S. $= C^{n+m}_k + C^{n+m}_{k-1} = C^{n+m+1}_k =$ R.H.S.

256. We have to show that ${}^mC_1 + {}^{m+1}C_2 + \cdots + {}^{m+n-1}C_n = {}^nC_1 + {}^{n+1}C_2 + \cdots + {}^{n+m-1}C_m$.

Adding $1 = C^m_0$ to L.H.S., we have $C^m_0 + C^m_1 + C^{m+1}_2 + C^{m+2}_3 + \cdots + C^{m+n-1}_n$

Now, we make use of the fact that $C^n_r + C^n_{r+1} = C^{n+1}_{r+1}$.

$\therefore$ L.H.S. $= C^{m+1}_1 + C^{m+1}_2 + C^{m+2}_3 + \cdots + C^{m+n-1}_n = C^{m+1}_2 + C^{m+2}_3 + \cdots + C^{m+n-1}_n$

Repeating, we obtain L.H.S. $= C^{m+n}_n = C^{m+n}_m$

Proceeding similarly for R.H.S. we obtain it as $C^{m+n}_m =$ L.H.S.

257. For the number to be divisible by 25, the last two digits have to be 25, 50 or 75.

Case I: When last two digits are 25 or 75.

Ten thousand's place can be filled in 5 ways. Thousand's place can be filled in 5 ways, and so on.

Thus, total no. of numbers $2 \times 5 \times 5 \times 4 = 200$

Case II: When last two digits are 50.

Ten thousand's place can be filled in 6, and so on. Thus, no. of numbers is $6 \times 5 \times 4 = 120$.

Thus, total no. of numbers divisible by 25 is 320.

258. A no. is divisible by 4 if last two digits of the no. are divisible by 4. Thus, the last two digits can be 12, 24, 32, 52.

We fix last two digits. Ten thousand's place can be filled in 3 ways, thousand's place can be filled in 2 ways, hundred's place can be filled in 1 way. Thus, total no. of numbers is $4 \times 3 \times 2 = 24$.

259. For the no. to be divisible by 3 the sum of digits must be divisible by 3. We take two cases when the number contains 0, and when the number does not contain 0.

Case I: These can be a combination of 123, 135, 234, 345. Thousand's place can be filled in 3 ways, hundred's place can be filled in 3 ways, ten'place can be filled in 2 ways.

Thus, no. of numbers is $4 \times 3 \times 3 \times 2 = 72$.

Case II: These can be a combination of 1245, which can have $4! = 24$ combinations.

Thus, total no. of such numbers is 96.

For the no. to be divisible by 6 the sum of digits must be divisible by 3 and last digit must be even.

Proceeding similarly, we obtain 52 as the no. of numbers, which are divisible by 6.

260. Let us assume that the three 3's are different. Then, Thousand's, and hundred's place can be filled in 3 ways, ten's place can be filled in 2 ways, and unit's place can be filled in 1 way.

Thus, no. of numbers $= 3 \times 3 \times 2 = 18$. However, 3's are same so no. of numbers is $18/2 = 9$.

Out of 9 numbers at unit's, ten's, and hundred's place 3 will come four times, 1 will come twice, and 0 will come thrice.

On thousand's place 3 will come six times, and 1 will come thrice.

Sum of digits at unit's place $= 4 * 3 + 2 * 1 = 14$

Sum of digits at ten's place $= 4 * 3 + 2 * 1 = 14$

Sum of digits at hundred's place $= 4 * 3 + 2 * 1 = 14$

Sum of digits at thousand's place $= 6 * 3 + 3 * 1 = 21$

Thus, sum of numbers is 22,554.

261. Note that repetition is allowed here. Total no. of ways of taking 1 thing at a time out of n things $= n$. Total no. of ways of taking 2 thing at a time out of n things $= n^2$. Total no. of ways of taking 3 thing at a time out of n things $= n^3$ and so on. Total no. of ways of taking r thing at a time out of n things $= n^r$

Total no. of ways $= n + n^2 + n^3 + \cdots + n^r = \frac{n(n^r-1)}{n-1}$.

262. Smallest seven digit numbers is 1,000,000. Largest seven digit no. is 9,999,999. Total no. of seven digit numbers is 9,000,000. Half of these will have even sum, which is 4,500,000.

263. Ways of choosing k numbers out of $r(r \leq n) = r^k$. However, $(r-1)^k$ will not have r as maximum. $\therefore$ Required answer $= r^k - (r-1)^k$.

264. No. of ways of filling most significant positions is 9. No. of ways of filling next position is 8, because consecutive digits cannot be same. This is true for all remaining positions.

Thus, required number $= 9.8^{n-1}$.

265. No. of ways to fill first position is 26, because it has to be an alphabet. No. of ways to fill next five positions is 36.

However, the identifier can be of up to six characters. Thus, total no. of identifiers $= 26 + 26.36 + 26.36^2 + \cdots + 26.36^5 = 26.\frac{36^6-1}{35}$.

266. First we compute total no. of five digit numbers. No. of ways to fill ten thousand's position is 9. No. of ways to fill rest of positions is 10. $\therefore$ Total no. of give digit numbers is 9×10^4.

However, these numbers will include numbers without repetition. So we compute numbers without repetition. No. of numbers without repetition is $9 \times 9 \times 8 \times 7 \times 6 =$ 27,216.

Thus, no. of numbers with repetition is $90{,}000 - 27{,}216 = 62{,}784$.

267. Total no. of numbers between 2×10^4, and 6×10^4 is 4×10^4. Half of these would have sum of digits as even, which is 20,000.

268. (a) Treating A_1, and A_2 as one entity, total no. of ways of arranging them is 9!. However, A_1, and A_2 can be arranged among themselves in 2! ways. Thus, total no. of ways in which A_1, and A_2 are next to each other is $9!\,2!$.

(b) Total no. of permutations is 10!. In half of these A_1 will be above A_2. Thus, required numbers is $\frac{10!}{2!}$.

269. Since no man can sit between two women, therefore all men have to sit together. Treating all men as one man, no. of ways to seat them together is $(n+1)!$. However, m men can be arranged in $m!$ ways among themselves.

Thus, total no. of ways of seating them together is $(n+1)!\,m!$.

270. There are two I's, two T's, three E's, and rest of the characters come once each. Because vowels cannot come between two consonants, the vowels have to come together. Treating all the vowels as one character, total no. of characters is 7.

No. of ways to arrange them is $\frac{7!}{2!}$. Six vowels can be arranged among themselves in $\frac{6!}{2!3!}$.

Thus, desired no. of words is $\frac{7!6!}{2!2!3!} = 151{,}200$.

271. Total no. of arrangements $= \frac{18!}{5!6!7!}$. Treating all balls of same color as one ball so that they stay together, total no. of arrangements is 3!.

Thus, required numbers $= \frac{18!}{5!6!7!} - 3!$.

272. No. of ways of seating men together $= 7!$. No. of ways of seating women together $= 3!$. No. of ways of seating two men together $= 2!$.

No. of arrangements when three ladies, and two men are together $= 7!\,3!\,2!$.

Treating all ladies as one we have 8 people, and ladies can be seated in 3! ways among themselves.

No. of arrangements with ladies together, and no condition on seating men $= 8!\,3!$.

$\therefore$ Desired no. of arrangements $= 8!\,3! - 7!\,3!\,2!$.

273. Total no. of permutations $= n!$. Treating p things as one thing we have $n - p + 1$ things. No. of arrangements when p things are together is $(n-p+1)!$. p things can be arranged among themselves in $p!$ ways.

$\therefore$ No. of ways in which p things are together $= n! - (n-p+1)!\,p!$.

274. Consider -+-+-+-+-+-+-+-, where '+' denotes the positions of +, and '-' denotes the positions of $-$. There are 7 positions for $-$, which have to be filled by 4.

Thus, required no. of arrangements $= C^7_4 = 35$.

275. Let gentlemen be 'G', and ladies be 'L'. They can be seated as GLGGLGLG.

Gentlemen can exchange places in 5! ways, and ladies can exchange places in 3! ways. So total no. of ways $= 5!\,3! = 720$.

276. There are three S', two C's, and one U and E each.

(a) Treating two C's as one character. SXSXSXS can be a way where S is position of S and X is position of other characters.

No. of ways to fill S $= C^4_3 = 4$

However, rest of 3 characters can be arranged in 3! ways. Thus, total no. of ways $= 4.3! = 24$.

(b) Total no. of permutations of letters(T) $= \frac{7!}{2!3!}$

With two C together(A) $= \frac{6!}{2!}$

With three S together(B) $= \frac{6!}{2!} - \frac{5!}{2!}$

With both S and C together $= 5! - 4!$

$\therefore$ Desired answer $= T - A - B + C = 96$.

277. No. of words beginning with E $= 5!$

No. of words beginning with H $= 5!$

No. of words beginning with ME $= 4!$

No. of words beginning with MH $= 4!$

No. of words begining with MOE $= 3!$

No. of words begining with MOH $= 3!$

No. of words begining with MOR $= 3!$

No. of words beginning with MOTE $= 2!$

There are two words which begin with MOTH and MOTHER is first of them.

$\therefore$ Rank of MOTHER $= 309$.

278. There are 7 destinations, and Delhi is the final destination. Thus, there are 8 places, where passengers can go to. Let the intermediate stations be $S_1, S_2, \dots, S_7$.

People starting at Kolkata will have 8 destinations. People starting at S_1 will have 7 destinations, and so on.

Thus, total no. of possible tickets $= 8 + 7 + \dots + 1 = 36$.

Thus, total no. of sets possible is C_5^{36}.

279. There are 10 destinations, and London is the final destination. Thus, there are 10 places, where passengers can go to. Let the intermediate stations be $S_1, S_2, \dots, S_9$.

People starting at Cambridge will have 10 destinations. People starting at S_1 will have 9 destinations, and so on.

No. of selections of two out of ten is $C_2^{10} = 45$.

Thus, no. of sets of tickets is C_6^{45}.

280. A day can be either clear or overcast. Thus, we have two possibilities. Total no. of possibilities for 7 days $= 2^7 = 128$.

281. No. of ways of selecting 1 book $= C_1^{2n+1}$

No. of ways of selecting 2 books $= C_2^{2n+1}$

...

No. of ways of selecting n book $= C_n^{2n+1}$

Let $S = C_1^{2n+1} + C_2^{2n+1} + \dots + C_n^{2n+1} = 63$

We know that, $C_0^{2n+1} + C_1^{2n+1} + C_2^{2n+1} + \dots + C_n^{2n+1} + \dots + C_{2n+1}^{2n+1} = 2^{2n+1}$

We also know that $C_r^n = C_{n-r}^n \Rightarrow 1 + 2S + 1 = 2^{2n+1} \Rightarrow 1 + S = 2^{2n} = 64 \Rightarrow n = 3$.

282. k flowers can be chosen from the first bag in C_k^k ways.

k flowers can be chosen from the second bag in C_k^{k+1} ways.

...

k flowers can be chosen from the mth bag in C_k^{k+m-1} ways.

Total no. of ways is $C_k^k + C_k^{k+1} + \dots + C_k^{k+m-1} = S(\text{let})$.

We know that, $C_r^n = C_{n-r}^n$, applying this, we have

$S = C_0^k + C_1^{k+1} + \dots + C_{m-1}^{k+m-1} = C_0^{k+1} + C_1^{k+1} + \dots + C_{m-1}^{k+m-1}$

Now repeatedly applying $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$

$\Rightarrow S = C_{m-1}^{k+m} = C_{k+1}^{k+m}$.

283. No. of ways of selecting 11 persons out of 50 $= C^{50}_{11}$.

Treating A, B, C as one person, no. of ways of choosing 11 when all three are part of the committee $= C^{47}_{8}$.

Thus, desired answer $= C^{50}_{11} - C^{47}_{8}$.

284. Let S_1, S_2, S_3 be the three intermediate stations where the train stops. P, a, S_1, b, S_2, c, S_3, d, Q be the stations. Let a, b, c, d be the no. off stations between P and S_1, S_1 and S_2, S_2 and S_3, and S_3 and Q

$a + b + c + d + S_1 + S_2 + S_3 = m$, where $a \geq 0, b \geq 1, c \geq 1, d \geq 0, S_1 = S_2 = S_3 = 1$

Thus, the equation i.e. no. of stations $= m - 2$ when no two of the stations where train stops are consecutive.

$\therefore$ No. of ways to choose 3 out of $m - 2$ is C^{m-2}_{3}.

285. Let $A = \{a_1, a_2, \ldots, a_n\}$. For each element a_i there are four possibilities: (i) $a_i \in P$ and $a_i \in Q$, (ii) $a_i \,/\!\in P$ and $a_i \in Q$, (iii) $a_i \in P$ and $a_i \,/\!\in Q$, and (iv) $a_i \,/\!\in P$ and $a_i \,/\!\in Q$

(a) For $P \cap Q$ to contain exactly two elements, we have to choose 2 elements out of n i.e. we have C^n_2 ways. Remaining $n - 2$ elements can choose one of three states i.e. 3^{n-2} ways.

Thus, no. of ways in which $P \cap Q$ will contain exactly 2 elements is $C^n_2.3^{n-2}$.

(b) Since $P \cap Q = \phi$, all n elements can choose one of three states. Thus, total no. of ways is 3^n.

286. Let $A = \{a_1, a_2, \ldots, a_n\}$. For element a_1, and one subset P_1 there are two possibilities, either $a_1 \in P_1$ or $a_1 \,/\!\in P_1$.

Total no. of ways for one element a_1 of A, and one subset $P_1 = 2$.

No. of ways in which a_1 does not belong to $P_1 = 1$.

Total no. of ways for a_1, and m subsets $= 2^m$.

Total no. of ways for a_1 to belong to m subsets $= 1^m$

Total no. of ways for a_1 does not belong to m subsets $= 1^m$

$\therefore a_1 \in (P_1 \cap P_2 \cap \ldots \cap P_m) = 1^m$, $a_1 \,/\!\in (P_1 \cap P_2 \cap \ldots \cap P_m) = 2^m - 1^m$, and $a_1 \in (P_1 \cup P_2 \cup \ldots \cup P_m) = 2^m - 1^m$

i. We have to choose one element out of n i.e. we have n ways. Remaining $n - 1$ elements are in $P_1 \cup P_2 \cup \ldots \cup P_m$ i.e $(2^m - 1^m)^{n-1}$. Thus, total no. of ways is $n(2^m - 1^m)^{n-1}$.

ii. All n elements are in $P_1 \cup P_2 \cup \ldots \cup P_m$ i.e. no. of ways is $(2^m - 1^m)^n$.

iii. All n elements are not in $P_1 \cap P_2 \cap \ldots \cap P_m$ i.e. no. of ways is $(2^m - 1^m)^n$.

287. No. of possible choices are $(3{,}1{,}1),(1{,}3{,}1),(1{,}1{,}3),(2{,}2{,}1),(2{,}1{,}2),(1{,}2{,}2)$ where each number represents no. of choices from a paper.

For $(3, 1, 1)$ no. of choices $= C_3^5 \times C_1^5 \times C_1^5 = 250$.

For three such sets no. of choices $= 750$.

For $(2, 1, 2)$ no. of choices $= C_2^5 \times C_2^5 \times C_1^5 = 500$.

For three such sets no. of choices $= 1{,}500$.

$\therefore$ Total no. of choices in which the questions can answered $= 2{,}250$.

288. The product will be divisible by 3 if one of the numbers is divisible by 3.

Case I: When one of the numbers is divisible by 3.

Total no. of ways $= 33 \times 67 = 2{,}211$

Case II: When both the numbers are divisible by 3.

Total no. of ways $= C_2^{33} = 528$.

Thus, total no. of ways of selecting two numbers is $2{,}211 + 528 = 2{,}739$.

289. No. of ways of choosing two husbands $= C_2^5 = 10$. After selecting two husbands, we have 3 wives to choose from. No. of ways of choosing two wives out of three is $C_2^3 = 3$.

However, wives can be part of either side, thus, total no. of ways $= 10 \times 3 \times 2 = 60$.

290. The line which is parallel to nn concurrent line has to be part of all triangles. Also, the line which is parallel to it will be part of no triangle. Thus, total no. of possible triangles $= C_2^{n-1}$.

291. Total no. of points of intersection $= C_2^n = m$(say). If these points are not collinear then total no. of triangles formed $= C_3^m$.

One line will have $n-1$ collinear points. These lines will not form any triangle among themselves. Thus, no. of ways to select 3 out of them C_2^{n-1}.

However, there are n such lines, therefore, no. of triangles, which will not be formed is $n.C_2^{n-1}$.

Thus, required answer $= C_3^m - n.C_3^{n-1}$.

292. There can be $3, 4$ or 5 bowlers in the team.

Thus, total no. of ways of selecting team $= C_3^5 \times C_8^{10} + C_4^5 \times C_7^{10} + C_5^5 \times C_6^{10}$.

293. From each bag $1, 2, 3, \ldots, m$ balls can be selected.

No. of ways selecting 1 ball from both the bags $= C_1^m \times C_1^m = (C_1^m)^2$.

No. of ways selecting 2 ball from both the bags $= C_2^m \times C_2^m = (C_2^m)^2$.

...

No. of ways selecting m ball from both the bags $= C_m^m \times C_m^m = (C_m^m)^2$.

$\therefore$ Total no. of ways $= (C_1^m)^2 + (C_2^m)^2 + (C_3^m)^2 + \cdots + (C_m^m)^2 = C_m^{2m} - 1$.

294. There can be $5, 6, 7$ or 8 men in the committee. So no. of ways to form the committee is $C_5^8 \times C_7^9 + C_6^8 \times C_6^9 + C_7^8 \times C_5^9 + C_8^8 \times C_4^9$.

For women to be in majority there must be at least 7 women, which means 5 men, and only one such committee is possible. No. of such committees is ${}^8_5 \times C_7^9$.

For men to be in majority there must be at least 7 men. No. of such committees is $C_7^8 \times C_5^9 + C_8^8 \times C_4^9$.

295. Let the distance between lines be 1 unit. For squares with side 1 unit: Along m horizontal lines $m-1$ squares can be formed, and along n vertical lines $n-1$ squares can be formed. Thus, total no. of such squares is $(m-1)(n-1)$.

For squares of 2 units, no. of such squares is $(m-2)(n-2)$.

Since $m < n$, total no. of squares is $= \sum_{i=1}^{m-1}(m-i)(n-i) = \sum_{i=1}^{m-1}[mn-(m+n)i+i^2]$

$= mn(m-1) - (m+n)\frac{m(m-1)}{2} + \frac{m(m-1)(2m-1)}{6} = \frac{1}{6}m(m-1)[6n-3(m+n)+2m-1] = \frac{1}{6}m(m-1)(2n-m-1)$.

296. This problem is same as previous problem, and has same answer.

297. Total no. of ways of dividing $3n$ elements in three groups which contain equal no. of elements $= \frac{3n!}{3!(n!)^3}$.

298. No. of ways in which 50 different things can be divided 5 sets three of them having 12 things and two of them having 7 things each $= \frac{50!}{(12!)^3(7!)^2 3!2!}$.

299. No. of ways of distributing n things in groups such that they contain $a, b, c, \ldots, k$ things is $\frac{n!}{a!b!c!\ldots k!}$, which is an integer.

300. $\frac{(n^2!)}{(n!)^{n+1}}$ can be rewritten as $\frac{(n^2)!}{n!(n!)^n}$, which is distributing n^2 different things in n groups such that each group contains n things.

301. No. of ways of dividing n different things a groups each containing b things $= \frac{n!}{(b!)^a a!}$, which is an integer. Thus, $(n-1)!$ is divisible by both a and b.

302. No. of ways dividing kn different things in k groups each containing n things is $\frac{(kn)!}{k!(n!)^k}$, which is an integer.

303. Let $r = 20, n = 5$, then required number $= C_r^{n+r-1} = C_{20}^{24}$.

304. C_r^{n+r-1}.

305. Given, $x, y, z \geq 1$, and $x + y + z = n$. Let $a = x - 1, b = y - 1, c = z - 1$, then $a + b + c = n - 3$, where $a, b, c \geq 0$.

No. of solutions of the equation $a + b + c = n - 3$ is $C_n^{n+3-1} = C_n^{n+2} = C_2^{n+2}$.

306. $x + y + z = 0\ \forall x, y, z \geq -5$ can be rewritten as $a + b + c = 15\ \forall a, b, c \geq 0$. Thus, no. of solutions of the above equation $C_2^{15+3-1} = 136$.

307. Required answer is coeff. of x^{3n} in $\left(1 + x + x^2 + \cdots + x^n\right)^3\left(1 + x + x^2 + \cdots + x^{2n}\right)$

$= \left(1 - x^{n+1}\right)^3\left(1 - x^{2n+1}\right)(1 - x)^{-4} = \left(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3}\right)\left(1 - x^{2n+1}\right)\left(1 + C_1^4 x + C_2^5 x^2 + \cdots + C_3^{n+3} + \cdots\infty\right)$

$= C_3^{3n+3} - 3.C_3^{2n+2} + 3.C_3^{n+1} - C_3^{n+2} = \frac{n+1}{6}.(5n^2 + 10n + 6)$.

308. Here $n = 10, r = 3$ so no. of non-negative solutions is $C_2^9 = 36$.

309. Clearly, $0 \leq x \leq 8$. If $x = k$, then $y + z = 24 - 3k$.

No. non-negative integral solutions of the above equation $= C_{24-3k}^{2+24-3k-1} = C_1^{25-3k} = 25 - 3k$.

$\therefore$ Required number $= \sum_{k=0}^{8}(25 - 3k) = 225 - 108 = 117$

310. Given, $w + x + y + z = 29$, where $x \geq 1, y \geq 2, z \geq 3, w \geq 0$. Let $a = x - 1, b = y - 2, c = z - 3$

$\Rightarrow w + a + b + c = 23$. Hence, total no. of solutions $= C_{4-1}^{23+4-1} = 2{,}600$.

311. No. of non-negative solutions of the equation $a + b + c + d = 20$ is $C_{4-1}^{20+4-1} = C_3^{23} = 1{,}771$.

312. Possible equations are $x_1 + x_2 + \cdots + x_k = 0, 1, 2, 3, \ldots, n$, which has no. of solutions $1 = C_0^{k-1} = C_0^k, k = C_1^k, C_2^{k+1} \ldots, C_n^{n+k-1}$.

Now, applying $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$, repeatedly, we have C_n^{n+k} as the answer.

313. $x + y = \frac{10-z}{2} \therefore x + y \leq 10$. Let $x + y + t = 10$, then $x, y, t \geq 0$

$\therefore$ Required numbers $= C_{10}^{12} = C_2^{12} = 66$.

314. **Case I:** We consider set of 2 numbers. Let the numbers be a and b.

Given, $\frac{a+b}{2} = 60 \Rightarrow a + b = 120$. Both a and b cannot be 60, because 60 cannot be used twice. Let $0 \leq a \leq 59$, and $61 \leq b \leq 120$.

Total no. of ways in which a can be chosen $= 60$. Value of b depends on chosen value of a. Thus, total no. of ways of choosing two numbers is 60.

Case II: We consider set of 3 numbers. Let the numbers be a, b and c. Given, $\frac{a+b+c}{3} = 60 \Rightarrow a+b+c = 180$

Case a: Let $0 \le a \le 59, 0 \le b \le 59$, and $c \ge 60$.

a can be chosen in 60 ways, and b can be chosen in 59 remaining ways. Value of c will depend on values of a and b. No. of ways of choosing $a, b, c = 60 \times 59 = 3{,}540$.

Case b: $a = 60, b+c = 120$. Like **case I**, the no. of ways to select a, b, c is 60.

Case c: $61 \le a \le 90, 61 \le b \le 90$ and $c < 60$.

a and b can be selected in $30 \times 29 = 870$ ways. c depends on a, and b.

$\therefore$ Total no. of ways of making selection is 4,530.

315. There are 11 letters, 'S' comes twice, and others once. Since we have to select 'T', we have 4 letters to select.

Case I: 2 S, and 2 others. No. of ways to select is $C^2_2.C^8_2 = 28$.

Case II: No. of ways of selecting 1 S, and 3 others is $C^8_3 = 56$.

Case III: No. of ways of selecting no S is $C^8_4 = 70$.

Thus, total no. of ways of making the required word is 154.

316. There are 3 O's, 2 P's, and R's, one each of T, I and N.

Case I: When all four letters are distinct. We have four letters to be selected out of six. So no. of selections is C^6_4, and no. of arrangements is P^6_4.

Case II: With one letter repeated. P, R, and O are repeated, and we can choose one, which can be done in C^3_1, other letters can be selected in C^5_2 ways. Thus, no. of selections is 30. Total no. of such words is $30 \times \frac{4!}{2!} = 360$.

Case III: Two letter repeated. No. of ways to select 2 out of P, R, and O is C^3_2. These letters can be arranged in $\frac{4!}{2!2!} = 6$. Therefore, total no. of arrangements is $3 \times 6 = 18$.

Case IV: 3 O's at a time. No. of ways of selecting remaining one letter is C^5_1. No. of arrangements is $\frac{4!}{3!}$. Total no. of such words is $5 \times 4 = 20$.

Now total can be computed trivially.

317. Three odd places in the five letter word can be filled by five non-repeated letters C, E, H, I, and S. No. of permutations is $P^5_3 = 60$. For two even places we have six repeating letters, two each of M, A, and T. This leads to two cases:

Case I: Both are different. This can be done in $P^3_2 = 6$ ways.

Case II: Both are same. This can be done in $C^3_1 = 3$ ways.

Thus, even places can be filled in 9 ways. And thus, total no. of words that can be formed is $60 \times 9 = 540$.

318. No. of selections of three letters from Box 1 is $C^6_3 = 20$. No. of selections of two letters from Box 2 is $C^4_2 = 6$. Total no. of selections is $20 \times 6 = 120$.

No. of ways to arrange five letters is $5! = 120$. Thus, total no. of codewords that can be formed is $120 \times 120 = 14{,}400$.

319. No. of ways of arranging r out of m on one side is P^m_r. Similarly, no. of ways of arranging s out of m on the other side is P^m_s. No. of ways of arranging remaining is $(2m - r - s)!$.

Thus, total no. of ways of seating them is $P^m_r P^m_s.(2m - r - s)!$.

320. No. of ways of selecting 4 or 6 out of 10 is $C^{10}_6 = C^{10}_4 = 210$.

6 people can be seated around a round table in $5! = 120$ ways. Similarly, 4 people can be seated around a round table in $3! = 6$ ways.

Thus, total no. of ways of seating people is $210 \times 120 \times 6 = 151{,}200$.

321. There are a total of $2n + m + 1$ seats, $2n$ grandchildren occupy $2n$ seats at the end in $2n!$ ways. Since the grandfather does not wish to have a grandchild at either end so he can occupy one of $m - 1$ seats in the middle. m sons and daughters can occupy m seats in $m!$ ways.

Thus, total no. of ways is $(2n)!\,m!(m - 1)$.

322. Let $S_1, S_2, S_3, \ldots, S_{2n}$ be the seats. The table has n seats on two sides i.e. $2n$ seats. The master, and the mistress always sit opposite to each other. Let the two guests, which must not be placed to each other are called X and Y.

There are 4 special positions for X. These are S_1, S_n, S_{n+1}, and S_{2n}. When X seats at these positions, then remaining $2n - 2$ guests can be seated in $(2n - 2)!$ ways. Hence total no. of arrangements is $4(2n - 2)(2n - 2)!$ because Y can sit at $2n - 2$ positions.

For other positions of X there will be $2n - 3$ positions for Y. Now total no. of ways in this case will be $(2n - 3)(2n - 4)(2n - 2)!$ [$2n - 4$ positions for X, and $(2n - 2)!$ no. of arrangements for remaining guests].

Thus, total no. of ways is $(2n - 2)![4(2n - 2) + (2n - 3)(2n - 4)] = (2n - 2)!(4n^2 - 6n + 4)$.

323. We have to choose n objects out of $3n$ because n like objects are always there in the selection. No. of ways of making these selections is C^{3n}_n.

Total no. of arrangements is $C^{3n}_n \times \frac{2n!}{n!} = \frac{3n!}{(n!)^2}$.

324. There are four pairs whose sum is 9, and then there is 9. We know that for a number to be divisible by 9, the sum of digits of that number must be divisible by 9. So we can pick three pairs out of these four, and one nine to construct a 7-digit number.

No. of ways of making this selection is $C^4_3 \times C^1_1 = 4$. No. of ways to arrange these 7 digits is 7!. Thus, no. of such numbers is $4 \times 7!$.

325. Following like previous problems we have 4 such pairs. The 9th digit can be 9 or 0.

Case I: When 9th digit is 0.

No. of ways of making selection is $C^4_4 \times C^1_1 = 1$. The 0 cannot be most significant digit. No. of ways of filling most significant position is 8. No. of ways to fill remaining positions is 8!. Thus, total no. of numbers is $8 \times 8!$.

Case II: When 9th digit is 9. Total no. of possible numbers in this case is 9!.

Thus, total no. of numbers is $17 \times 8!$.

326. For the product of five consecutive digits to be divisible by 7, the five digits must contain 7. Thus, 7 is fixed. No. of ways of arranging remaining 8 digits is 8!, which is the answer.

327. Total no. of outcomes is $6^3 = 216$. Total no. of outcomes in which no 5 comes is $5^3 = 125$. Thus, no. of outcomes in which at least one 5 shows is 91.

328. No. of outcomes of a single throw of a dice is 6. No. of outcomes of n dice throw is 6^n. Now, we consider the cases where only even no. are the outcome. No. of even numbers is $2, 4, 6$. Thus, no. of even only outcomes is 3^n.

Thus, no. of outcomes where at least one odd outcome has come is $6^n - 3^n$.

329. A no. from 1 to 1000 will have three digits. Thus, 5 can come once, twice or thrice. There is only one no. 555 where it comes thrice. So we have to calculate the other two cases.

Case I: When 5 comes once in the number.

We can choose one place out of three in C^3_1 ways, and remaining two places in 9^2 ways. Thus, no. of times 5 will be written is $3 \times 9^2 = 243$.

Case II: When 5 comes twice in the number.

We can choose two places out of three in C^3_2 ways, and remaining one place can be filled in 9 ways. Thus, 5 will be written $2 \times 3 \times 9 = 54$ times.

Thus, total no. of times 5 will be written is $243 + 54 + 3 = 300$.

330. For prime number p, and k the highest power of p that can divide the $n!$ is given by $\sum \left\lfloor \frac{n}{p^k} \right\rfloor$.

Given $n = 33$, and $p = 2$. Thus, required no. $= \left\lfloor \frac{33}{2} \right\rfloor + \left\lfloor \frac{33}{2^2} \right\rfloor + \left\lfloor \frac{33}{2^3} \right\rfloor + \left\lfloor \frac{33}{2^4} \right\rfloor + \left\lfloor \frac{33}{2^5} \right\rfloor = 31$.

331. Since $\frac{1}{3} = 0.33\ldots$ so up to $\frac{1}{3} + \frac{33}{50}$ the terms will be 0. So $E = 17$. Then we proceed like previous problem, required answer is $\left\lfloor \frac{17}{2} \right\rfloor + \left\lfloor \frac{17}{2^2} \right\rfloor + \left\lfloor \frac{17}{2^3} \right\rfloor + \left\lfloor \frac{17}{2^4} \right\rfloor = 8 + 4 + 2 + 1 = 15$.

332. The set of unit's place is $\{0, 1, 2, \ldots, 9\}$, the set of ten's place is $\{1, 4, 9\}$, and the set of hundred's place is $\{1, 2, \ldots, 9\}$.

There will be a total of $9 \times 3 \times 10 = 270$. For each possible choice of ten's place there will be 90 numbers, for each choice of unit's place there will be 27 number, and for each choice of hundred's place there will be 30 numbers.

Thus, sum of these numbers is $27 \times 100 \times (1 + 2 + \cdots + 9) + 90 \times 10 \times (1 + 4 + 9) + 30 \times (1 + 2 + \cdots + 9) = 135{,}450$.

333. In the first round numbers marked will be $1, 16, 31, \ldots, 991$. The second round will begin with $991 + 15 - 1000 = 6$. So the numbers marked in the second round will be $6, 21, 36, \ldots, 996$. The third round will begin with $996 + 15 - 1000 = 11$. So numbers marked will be $11, 26, 41, \ldots, 986$. In the next round next number marked will be $986 + 15 - 1000 = 1$, and the cycle will repeat.

Thus, marked numbers are $1, 6, 11, 16, 21, \ldots$ i.e. numbers which leave remainder 1 when divided by 5. So no. of marked numbers is $\frac{1000}{5} = 200$. So no. of marked numbers is 800.

334. Partitioning S according to given condition we have following for A, $\{1, 2, 4, 8\}, \{3, 6\}, \{5, 10\}, \{7\}, \{9\}$. Now for each of these we have $n + 1$ choices if n is the no. of elements because once we include a no. the larger one must be included. Thus, total no. of subsets is $5.3.3.2.2 = 180$.

335. Each card can be dealt to two persons in two ways. Thus, n cards can be dealt in 2^n ways. However, in two of the ways each person will receive no card. Thus, total no. of ways of dealing cards in required manner is $2^n - 2 = 2(2^{n-1} - 1)$.

336. We can choose 0 A to 4 A i.e. in 5 ways. Similarly, we can choose B in 4 ways, C, D, E in 2 ways. Thus, total no. of ways is $5 \times 4 \times 2^3 = 160$.

However, in one of these ways no letter is selected. Thus, required answer is $160 - 1 = 159$.

337. We can have a factor by picking zero 'a', and up to seven 'a' i.e. in 8 ways. Similarly, b can be chosen in 5 ways, and so on.

Thus, total no. of ways is $8 \times 5 \times 4 \times 2^3 = 1{,}280$. However, in one way all are selected, and in another none(factor is 1 in this case). Thus, required answer is $1{,}280 - 2 = 1{,}278$.

338. Following like previous problem, b_i can be chosen in $p_i + 1$ ways. Thus, total no. of positive divisors is $\prod_{i=1}^{n} (p_i + 1)$.

339. First square can be selected in 64 ways. Once we have selected a square we cannot choose from same row and column. Thus, the second square can be chosen from remaining seven rows and seven columns i.e. from 49 squares.

Thus, total no. of ways chosing squares is $64 \times 49 = 3{,}136$. However, in choosing these squares there are duplicates in the sense that squares are chosen twice. Thus, required answer is $\frac{3{,}136}{2} = 1{,}568$.

340. Let $A = \{1, 2, 3, \ldots, 2n-1, 2n\}$. If we pick 1 as one of the elements of the pair then the second can be obtained in $2n-1$ ways. Similarly, if we pick 2 as one of the elements of the pair then second can be obtained in $2n-3$ ways because 2 elements have been already chosen for first pair, and so on.

Thus, number of pairings is $(2n-1)(2n-3)\cdots 3.1$.

341. Let A be a onsisting of 5 concyclic points, and B be the set consisting of 7 points.

Case I: Circle passes through 3 points of B. Number of circles is C_3^7.

Case II: Circle passes through 2 points of B, and 1 point of A. Number of circles is $C_2^7.C_1^5$.

Case III: Circle passes through 1 point of B, and 2 points of A. Number of circles is $C_1^7.C_2^5$.

Case IV: Circle passes through no points of B. Number of circles is 1.

Thus, total no. of circles is 211.

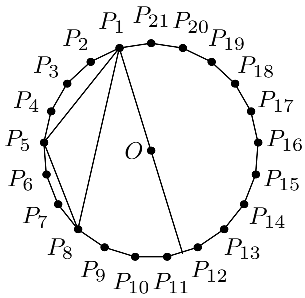
342. From 37 lines we get no. of intersections as C_2^{37}. However, for 13 lines we get one point A instead of C_2^{13}, and similarly for B, we get one point instead of C_2^{11}.

Hence, total no. of points of intersection is $C_2^{37} - C_2^{13} - C_2^{11} + 2 = 535$.

343. If we pick 2 points from m points then third can be picked in n ways giving us $C_2^m \times n$ triangles. If we pick 2 points from n points then third can be picked in m ways giving us $C_2^n \times m$ triangles.

Thus, total no. of triangles is $n.C_2^m + m.C_2^n = \frac{m^2n-mn+mn^2-mn}{2} = \frac{mn(m+n-2)}{2}$.

344. Since this is a regular polygon with odd no. of vertices no two of the vertices are places diagonally opposite, so there are no right-angled triangles. Let A be the no. of acute angled triangles, and O be the no. of obtuse angled triangles. To form a triangle we need to choose 3 vertices out of 21 which can be done in $C_3^{21} = 1330$ ways. Since the triangles are either actue or obtuse, we have $A + O = 1330$. Draw a diameter through point $P-1$. Now we consider point only on one side of diameter including P_1. So we need to select 2 out of 10. All these will be obtuse-angled triangles. No. of such triangles on both side is $21 * C_2^{10} = 945$. Thus, no. of acute-angled triangels is $A = 1330 - 945 = 385$.



A triangle $P_iP_jP_k$ is equilateral if these points are equispaced. Out of 21 points we have 7 such triplets. Thus, we have only 7 equilateral triangles.

Consider the diameter P_1OB, where B is the point, where P_1O meets the polygon. If we have an isosceles triangle at P_1 as its vertex then P_1B is the altitude, and the base is bisected by P_1B. This means that other two vertices are equally spaces from B. Clearly, there are 10 such pairs of vertices giving us 10 isosceles triangles for each vertes. But one of these is equilateral so we have 9 such triangles. For 21 points we will have 189 such isosceles triangles. Since we can consider equilateral triangles as isosceles triangles, total no. of isosceles triangles is $189 + 7 = 196$.

345. $S \cup T$ has r elements. This means that out of n elements r elements are present in either of S and T or in both. Thus, each element has 3 choices. No. of ways to select elements for S and $T = 3^r$.

Remaining $n - r$ elements have only 1 choice, and that is to be not in either of S and T. Total no. of ways of selecting $n - r$ elements is $1^{n-r} = 1$.

The r elements can be selected from n elements in C^n_r ways.

Thus, total no. of ways is $C^n_r.3^r$.

346. Given, $S = \overline{T}$. This means an element is present in either P or Q. Thus, for each element we have 2 choices. Hence, total no. of ways to select is 2^n.

347. Let $U = \{a_1, a_2, \dots, a_n\}$ be a set of n elements. Let S be the set of all subsets and B be the set of all binary sequences of n elements.

Let $A \in S$. Let $f : S \to B$ be a function that associates a binary sequence with B as follows: $a_i \in B$, iff ith term of sequence is 1.

For example, subset $\{a_2, a_4, \dots, a_{n-1}\}$ corresponds to binary sequence $010101 \dots 10$.

Observe that for every subset B, there is a binary sequence of n terms, and for every binary sequence of n terms as stated above, there is a subset B of U. Therefore, f is a bijection between S and B. Hence, no. of subsets = no. of binary sequences = 2^n.

348. Assign h for horizontal movement and v for vertical movement of 1 unit. One such path would be $h, h, h, v, v, v, h, h, v, h$.

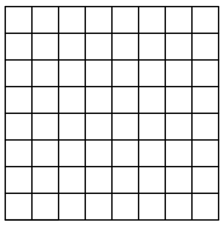
Note that there are 7 horizontal, and 6 vertical line segments, of one unit each, from A to B.

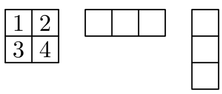
Since for every path between A, and B, there is a sequence of $7h$'s and $6v$'s, and for every sequence we have corresponding one path made up of horizontal and vertical lines. Therefore, there is a bijection between the set of all paths from A to B, and the set of all sequences of $7h$'s, and $6v$'s. Thus, number of paths from A to B = Number of sequences = Number of ways to select 7 places to put h out of 13 different places $= C_7^{13} = \frac{13!}{7!6!}$.

Number of path through C = Number of paths from A to C $\times$ Number of paths from C to B = Number of ways to select 4 placces to put h out of 8 different places $\times$ Number of ways to select 3 places to put 3 out of 5 dfferent places $= C_4^8 \times C_3^5$.

Similarly, number of paths from D to $B = \frac{4!}{2!2!}$. Number of paths including $CD = \frac{8!}{4!4!} \cdot \frac{4!}{2!2!}$.

349. A square chessboard is an 8×8 board as shown.





For two sides to be common first we have L shaped possibilities. Consider the 2×2 square shown. In this case if we take 1, 2, 4 or 2, 4, 3 or 4, 3, 1 or 3, 1, 2 then we will have one square whose two sides are common with others i.e. we will have 4 such squares. Now if we fit this square block along two rows we will have 7 such blocks, and then 7 along columns. Thus, total no. of required squares would be $4 \times 7 \times = 196$.

Now, the other possibility is a continuous block of three squares either horizontally or vertically as shown. We will get one required square in the middle of the blocks. Along the rows and columns we can fit 6 such blocks. Thus, no. of required squares $= 2 \times 6 \times 8 = 96$.

Thus, total no. of required squares is 292.

350. We will prove this by negation. Since we have to make 10 exact predictions this implies that we have to make 10 wrong predictions. No. of ways to select 10 matches out of 20 is C_{10}^{20}. Now we can make mistake in 2 ways. Thus, total no. of ways of making mistakes is 2^{10}.

Thus, total no. of ways of making 10 exact correct predictions is $C_{10}^{20}.2^{10}$.

351. A forecast for a match can be done in 3 ways. So for five matches total no. of forecasts is $3^5 = 243$. Following like previous problem total no. of making 0, 1, 2, 3, 4 and 5 errors is $C_0^5.2^0, C_1^5.2, C_2^5.2^2, C_3^5.2^3, C_4^5.2^4$ and $C_5^5.2^5$.

352. The votes has to vote for at least 1 candidate and at most $n-1$ candidates. Here, n is the number of candidates. For each candidate, the votes has 2 options: either vote for the candidate or not vote for the candidate. So number of ways $= 2^n$. But we have to subtract 2 for two cases: one when the voter does not vote for any candidate, and one when the voter votes for every candidate.

Hence, $2^n - 2 = 62 \Rightarrow n = 6$.

353. Every lamp can have two states: on or off. Thus, total no. of states is $2^{10} = 1024$. However, in one of these cases all the lapms will be off, and the hall will not be illuminated.

Thus, no. of ways of illuminating the hall is $= 1024 - 1 = 1013$.

354. We observe that last when India wins the last match the series is over. So first case is when India wins all 5 matches, and there is only one way this can happen. Second case is when India loses one match. In this case we exclude last match to get total no. of matches as 5. Now we can choose to make India win remaining 4 in $C_5^4 = 1$ way. Similarly, for remaining cases it would be $C_4^6, C_4^7, C - 4^8$.

Adding these, we get 126 as desired answer.

355. For each book 0 or more copies can be selected. So there are $p + 1$ ways for each book. Thus, for n books we have $p + 1$^n ways. But in one of these ways no book is selected.

Thus, total no. of selecting books is $(p + 1)^n - 1$.

356. We have to have at least 2 students and at most we can have $n - 2$ students in the team.

Thus, total no. of ways of selecting them is $C_2^n + C_3^n + C_4^n + \cdots + C_{n-2}^n$.

We know that $C_0^n + C_1^n + C_2^n + \cdots + C_n^n = 2^n$, and thus, $C_2^n + C_3^n + C_4^n + \cdots + C_{n-2}^n = 2^n - C_0^n - C_1^n - C_{n-1}^n - C_n^n = 2^n - 2n - 2$.

357. No. of elements in $a_2, a_3, \ldots, a_{n+1} = n$. We can choose $a_1 0$ or more times up to m times. Thus, no. of ways of choosing a_1 is $m + 1$.

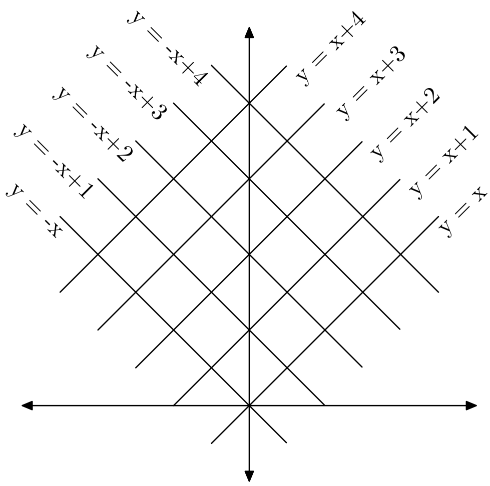
We can pick a combination of a_1 with all others in 0 to n numbers, giving us a total of 2^n.

Thus, total no. of prime factors is $(m + 1)\, 2^n$. However, one of these factors is 1. Thus, required no. of factors is $(m + 1)\, 2^n - 1$.

358. In a polygon 4 points make 2 sides and produce one point of intersection of 2 diagonals. Thus, no. of points is $C_4^n = 70 \Rightarrow n = 8$.

Thus, no. of diagonals is $C_2^8 - 8 = 20$(8 is subtracted because that is the no. of sides formed when two points are joined).

359. The plot of given lines is shown below:



If you observe carefully, you will find that side of one square in the diagram is $\sqrt{0.5^2+0.5^2}=\frac{1}{\sqrt{2}}$. So the diagonal would be 1. However, we want squares with diagonal 2. So we have to combine four squares into 1. No. of such squares is 9.

360. Total no. of triangles formed is $C_3^8 = 56$. When two sides are common with octagon the no. of triangles formed will be 8 as you can find by choosing any two adjacent sides of octagon and then shifting them around octagon. When we pick one side with octagon we cannot choose the third vertex of triangle with adjacent sides, thus, we will have only four points to choose from, and there are 8 sides. So no. of such triangles is $8\times 4 = 32$.

Thus, required no. of triangles is $56-8-32 = 16$.

361. An intersection point requires four points on the circle. Thus, total no. of such points is $C_4^9 = 126$.

362. For each element of set A we can have n elements in set B. Thus, total no. of functions would be n^m.

363. The set is $A=\{2,3,5,7,11,13,17,19,23,29\}$ i.e. having 10 elements. We can form a rational number of the form $\frac{p}{q}$. Thus, we can choose these numbers in C_2^{10} ways. But the rational number can be also of the form $\frac{q}{p}$. Thus, no. of rational numbers is $C_2^{10}\times 2 = 90$. One of the rational numbers will be 1 when $p=q$.

Thus, total no. of rational numbers is 91.

364. In the subset we put a_3 so now we have to choose 2 more elements out of remaining $n-1$ elements of the given set.

This can be done in C_2^{n-1} ways.

365. Clearly, no. of m element subsets is C_m^n. Now if we follow previous problem then no. of subsets containing a_4 is C_{m-1}^{n-1}.

Given, $C_m^n = k.C_{m-1}^{n-1} \Rightarrow \frac{n!}{m!(n-m)!} = k.\frac{(n-1)!}{(n-1)!(m-n)!} \Rightarrow n = mk$.

366. Let no. of subsets is $C_0^{2n+1} + C - 1^{2n+1} + C_2^{2n+2} + \cdots + C_n^{2n+1} = N$.

From binomial theorem we know that $C_0^{2n+1} + C - 1^{2n+1} + C_2^{2n+2} + \cdots + C_{2n+}^{2n+1} = 2^{2n+1}$. We also know that $C_r^n = C_{n-r}^n$.

$\therefore 2N = 2^{2n+1} \Rightarrow N = 2^{2n}$.

367. Total no. of subsets is 2^n. Half of these will contain even no. of elements i.e. 2^{n-1}.

368. When $x = n+1$; x, y can be chosen in n^2 ways. When $z = n$; x, y can be chosen in $(n-1)^2$ ways and so on.

Thus, total no. of ways is $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

369. This is same as selecting 6 numbers from 1 to 9 (0 is excluded because if zero is selected, that makes it the least and should be paced at the first place that makes it a 5 digit number) i.e. $C_6^9 = 84$.

370. If $y = n$ then x can take $n-1$ values from 1 to $n-1$ and z can take $n+1$ values from 0 to $n+1$. y can vary between 2 to 9.

Hence, required answer is $\sum_2^9 (n-)(n+1) = 276$.

371. $x_1 < x_2 \leq x_3 < x_4 < x_5 \leq x_6$ can be broken into four cases. These are $x_1 < x_2 < x_3 < x_4 < x_5 < x_6, x_1 < x_2 = x_3 < x_4 < x_5 < x_6, x_1 < x_2 < x_3 < x_4 < x_5 = x_6$, and $x_1 < x_2 = x_3 < x_4 < x_5 = x_6$.

Thus, required no. of numbers is $C_6^9 + C_5^9 + C_5^9 + C_4^9 = C_6^{11}$.

372. First we consider numbers starting with 12. Rest of the 4 positions can be filled in $C_4^7 = 35$ ways. Next we consider 13. In this case we can fill remaining places in $C_4^6 = 15$ ways. Next we consider 14, which gives us $C_4^5 = 5$ numbers. Next we choose 15, which gives us $C_4^4 = 1$ number. Next we have 23, which gives us $C_4^6 = 15$. Till now we have 71 numbers.

So 72nd number will be 245678 and the sum of the digits is 32.

373. For a radical center 3 circles are required. The total no. of radical centers is C_3^n. The total no. of radical axes is C_2^n.

Thus, $C_2^n = C_3^n \Rightarrow n = 5$.

374. No. of ways = No. of one-to-one mappings + No. of mappings in which 0 and 1 map to the same element and 2 maps to a different element + 0 maps to a different element and 1 and 2 maps to same element + No. of constant functions

$= C_3^8 + C_2^8 + C_2^8 + C_1^8 = C_3^{10}$.

375. We can choose 0 to n objects from n identical objects in 1 way. However, we can choose from 0 to n objects from $2n$ different objects in $C_0^{2n}, C_1^{2n}, \ldots, C_n^{2n}$.

We know that $C_0^{2n} + C_1^{2n} + \cdots + C_n^{2n} + C_{n+1} + \cdots + C_{2n}^{2n} = 2^{2n}$

$\Rightarrow C_0^{2n} + C_1^{2n} + \cdots + C_n^{2n} = 2^{2n-1} + C_n^{2n}$.

376. Let the no. of players be n. For $n-2$ participants no. of games would be C_2^{n-2}.

The two players played 3 games each, therefore, $C_2^{n-2} + 6 = 84 \Rightarrow \frac{(n-2)(n-3)}{2} = 78 \Rightarrow n^2 - 5n - 150 = 0 \Rightarrow n = 15$

377. For 1 digit no. we have $C - 1^9$ numbers. For 2 digits, we have C_2^9 and so on. So the total no. of numbers is $2^9 - 1$.

378. There is a total of 8! ways possible. For each legal way we can reorder within each column in 3!, 2! and 3! ways to obtain any of the 8! orderings (legal or illegal). Thus, the number of ways to shoot the targets is $\frac{8!}{3!3!2!} = 560$.

379. Consider a vertex A. Suppose A is not a vertex of hexagon. Arrange nine balls in the form of a circle of which one is A. We choose six out of nine gaps in which to place a green ball(a vertex of hexagon) to ensure that no two green balls are adjacent(this ensures that no side of hexagon coincides with quindecagon). Now we label the vertics in alphabetical order as we go clockwise around the circle. There are C_6^9 ways to do this.

Now suppose A is a vertex of hexagon. We color it green. We arrange nine blue balls around A for form a circle which includes this ball. Doing this creates 10 gaps in which to place green balls but two of them are adjacent to A, leaving 8 gaps. We can choose 5 out of the remaining 8 in C_5^8 ways.

Thus, total no. of hexagons possible is $C_6^9 + C_5^8$.

380. Required no. of triplets = No. of triplets without repetition − No. of triplets with repetition $= n^3 - n(n-1)(n-2) = 3n^2 - 2n$.

381. Maximum can be chosen in m ways, and minimum can be chosen in $m-1$ ways or vice-versa. Then we havev $n-5$ no. remaining of which $m-2$ are to be chosen with repetition, which can be done in $(n-5)^{m-2}$.

Thus, required answer is $m(m-1)(n-5)^{m-2}$.

382. First rook can be places in 8^2 ways. Then second can be places in 7^2 and so on. However, the rooks are identical so no. of ways is $\frac{(8!)^2}{8!} = 40{,}320$.

383. First position can be filled in 3 ways, and rest in 2. Thus, no. of words is $3.2^6 = 192$.

384. Since we have to find numbers, which are not equivalent we need to consider combinations and not permutations because permutations will give equivalent numbers.

When all digits are different; no. of numbers is C_5^{10}. When 3 digits are different, and 2 digits are identical; no. of numbers is $C_1^{10} \times C_3^9$. When 2 digits are different, and 3 digits are identical; no. of number is $C_1^{10} \times C_2^9$. Similarly, when 4 digits are identical, and fifth is different; no. of numbers is $C_1^{10} \times C_1^9$. When 2 pairs of 2 digits are identical, and fifth is different; no. of numbers is $C_1^{10} \times C_1^9 \times C_1^8$. When 3 digits are identical, and 2 digits are also identical but a different digit; no. of numbers is $C_1^{10} \times C_1^9$.

Adding these we get desired answer.

385. Let the n objects be $o_1, o_2, o_3, \dots, o_n$.

Total no. of ways of selecting three objects so that no two of them are consecutive = Total no. of ways of selecting three objects − No. of ways of selecting three consecutive objects − No. of ways of selecting three objects in which two are consecutive and one is separate

Total no. of ways of selecting 3 objects out of n is C_3^n.

The three consecutive objects can be selected in the pattern: $o_1o_2o_3, o_2o_3o_4, \dots, o_{n-1}o_no_1, o_no_1o_2$. So no. of ways in which three consecutive objects can be selected is n.

Now we select two consecutive objects and one separated. We can pick o_1o_2 and third can be o_4 to o_{n-1}. Now there can be n such pairs. So no. of ways is $n(n-4)$.

Thus, required no. is $C_3^n - n - n(n-4) = \frac{n}{6}(n-4)(n-5)$.

386. Let us seat the 18 persons in 17! ways. Now we seat the two brothers around a person in 2! ways. Now there are 18 persons around which the two brothers can be seated. Thus, total no. of ways is 2.18!.

387. A cube can be rotated into $6 \times 4 = 24$ configurations (i.e. the one face can be any one of the 6, and then there are 4 ways to rotate it that keeps that face red)

So number of ways of painting the cube is $\frac{6!}{24} = 30$.

388. n things can be arranged in circular fashion in $(n-1)!$ ways. However, r things are alike so like linear permutations we have required answer as $\frac{(n-1)!}{r!}$.

389. 52 cards among 4 players is a distribution problem. The answer according to formula is $\frac{52!}{(13!)^4}$.

52 cards in 4 groups is a division problem. The answer according to formula is $\frac{52!}{(13!)^4 4!}$.

390. There are two mutually exclusive cases. In first case, 2 children get none, one child gets three and remaining get one each. In second case, 2 chidlren get none, two get two toys each and remaining get one each.

In first case, no. of ways is $\frac{10!}{2!3!7!}.10!$, and in second case it is $\frac{10!}{2!(2!)^2 2!6!}10!$.

391. There are three possible distributions. When divided in $4, 2, 1$ way; the no. of ways is $\frac{7!}{4!2!1!}.3! = 630$. When divided in $2, 2, 3$ way; the no. of ways is $\frac{7!}{2!2!3!} \cdot \frac{1}{2!}.3! = 630$. When divided in $3, 3, 1$ way; the no. of ways is $\frac{7!}{(3!)^2} \cdot \frac{1}{2!}.3! = 420$.

Thus, total no. of ways is 1680.

392. No. of ways of distributing 15 things in groups of $8, 4$ and 3 is $\frac{15!}{8!4!3!}$.

393. Required no. of ways according to formula is $\frac{8!}{3!3!2!2!2!}.2! = \frac{8!}{(3!)^2.2!}$.

394. $3n$ things can be distributed among 3 persons in $\frac{3n!}{(n!)^3} = k$ ways. These can be put in 3 groups in $\frac{3n!}{(n!)^3.3!} = \frac{k}{3!}$ ways.

395. No. of ways of giving all prizes to one person is C_1^m. Total no. of ways giving prizes is m^n. Thus, required answer is $m^n - m = m^{n-1}(m-1)$.

396. Clearly, one child out of $n-1$ will get 2 toys. 1 child can be left out in n ways. The extra toy can be chosen in n ways. Remaining $n-1$ toys can be distributed among $n-1$ children in $(n-1)!$ ways. However, if the child getting 2 toys gets toy A as the 'extra' toy and toy B as the 'ordinary' toy, this is the same as if this child gets toy B as the extra toy and toy A as the ordinary toy. So we have counted $2x$ as many combinations as we need.

So number of ways is $n \times n \times (n-1) \times \frac{(n-1)!}{2} = n!\,C_2^n$.

397. Any of the x_is can have a value of 0 through 8 to satisfy the sum i.e. if one of these have the value of 8 remaining four will have a value of 0. Thus, it is an arrangement of 12 objects, 8 of which are of one type, and 4 of which are of another type.

Total no. of such arrangements is $\frac{12!}{8!4!} = 495$.

398. Following like previous problem x_is can occupy value from 1 to 4 i.e. if one of these have the value of 4 remaining four will have value of 1. Thus, it is an arrangement of 7 objects, 4 of which are of one type, and 3 of another type.

Total no. of such arrangements is $\frac{7!}{4!3!} = 35$.

399. Let $x_1 = -2 + x_1'$, $x_2 = 1 + x_2'$, $x_3 = 2 + x_3'$. Then, we can rewrite the given equation as $x_1' + x_2' + x_3' + x_4 = 13$, where $x_1', x_2', x_3', x_4 \geq 0$.

Thus, no. of integral solutions is $\frac{16!}{13!3!} = 560$.

400. Let $u = x - 1, v = y - 2, w = z - 3$, then $u, v, w \geq 0$.

Thus, we have $u+v+w+t = 23$. Thus, total no. of solutions is $C_{4-1}^{23+4-1} = C_3^{26} = 2600$.

401. We can write the given equations as $x_1 + x_2 + x_3 = 5$, and $x_4 + x_5 = 15$.

The no. of solutions of the given system of equations is combinaiton of the solutions of these two equations, which is $C_1^{16} \times C_2^7 = 336$.

402. Let $x-1, x_2, x_3, x_4$ be the no. of red, black, white, and yellow balls selected by the child respectively. No. of ways to select 4 balls is equal to number of integeral solutions of $x_1+x_2+x_3+x_4=4$. Clearly, $\max(x_i)=5-i$.

Number of ways to select balls is coeff. of x^4 in $(1+x+x^2+x^3+x^4)(1+x+x^2+x^3)(1+x+x^2)(1+x)$ = coeff. of x^4 in $(1-x^5)(1-x^4)(1-x^3)(1-x^2)(1-x)^{-4}$

= coeff. of x^4 in $(1-x^2-x^3-x^4)(1-x)^{-4}$ = coeff. of x^4 in $(1-x)^{-4}$ − coeff. of x^2 in $(1-x)^{-4}$ − coeff. of x in $(1-x)^{-4}$ − constant term in $(1-x)^{-4}$

$= C_4^7 - C_2^5 - C_1^4 - C_0^3 = 20.$

403. If the student gets 60% marks in two papers then he needs to score only 30% in the third to fulfill the criterion of 150 marks.

Thus, the equaiton becomes $x_1+x_2+x_3=150$, where $60 \le x_1 \le 100$, marks scored in first paper, $60 \le x_2 \le 100$, marks scored in second paper, and $0 \le x_3 \le 30$, marks scored in third paper.

Thus, no. of ways of scoring 150 marks is coeff. of x^{150} in $(x^{60}+x^{61}+\cdots+x^{100})^2(1+x+x^2+\cdots+x^{30})$

= coeff. of x^{30} in $(1+x+x^2+\cdots+x^{40})^2(1+x+x^2+\cdots+x^{30})$ = coeff. of x^{30} in $(1-x)^{-3} = C_{3-1}^{30+3-1} = C_2^{32}$.

However, the two papers can be selected in C_2^3 ways. Thus, reuqired answer is $C_2^3 \times C_2^{32}$.

404. The equation is $x_1+x_2+\cdots+x_8=30$, where x_i is the marks allotted in the ith paper. Also, from question $2 \le x_i \le 16$.

No. of ways is coeff. of x^{30} in $(x^2+x^3+\cdots+x^{16})^8$ = Coeff. of x^{14} in $\left(\frac{1-x^{15}}{1-x}\right)^8$

= coeff. of x^{14} in $(1-x)^{-8} = C_{14}^{21} = 116{,}280$.

405. Let x_1, x_2, x_3, and x_4 be the marks obtained in paper $1, 2, 3$, and 4 respectively. Given that the cadidate needs $3n$ marks.

Thus, $x_1+x_2+x_3+x_4=3n$. Thus, no. of ways to get $3n$ marks is

= coeff. of x^{3n} in $(1+x+x^2+\cdots+x^n)^3(1+x+x^2+\cdots+x^{2n})$

= coeff. of x^{3n} in $(1-x^{n+1})^3(1-x^{2n+1})(1-x)^{-4}$

= coeff. of x^{3n} in $(1-x)^{-4}$ − 3. coeff. of x^{2n-1} in $(1-x)^{-4}$ − coeff. of x^{n-1} in $(1-x)^{-4}$ + 3. coeff. of x^{n-2} in $(1-x)^{-4}$

$= C_3^{3n+3} - 3C_3^{2n+2} - C_3^{n+2} + 3C_3^{n+1} = \frac{(n+1)(5n^2+10n+6)}{2}$.

406. Let $x_1, x_2, x_3, \ldots, x_7$ be the scores scored in seven shots. Clearly, x_i is an integer between 0 and 5, where $1 \leq i \leq 7$, and $i \in \mathbb{N}$. Thus,

$x_1 + x_2 + \cdots + x_7 = 30$, and hence we have following as answer

Coeff. of x^{30} in $(1 + x + x^2 + \cdots + x^5)^7 = 420$.

407. Number of non-negative integral solutions of the given equation = Coeff. of x^{20} in $(1-x)^{-3}(1-x^4)^{-1} = 536$.

408. No. of ways of distributing n things among r people, where anyone can get any number of thigs is C_{r-1}^{n+r-1}.

Hence, the answer is $C_3^{13} = 286$.

409. Given, $x + y + z = 100$, where x, y, z are positive integers i.e. $x, y, z \geq 1$.

Let $a = x - 1, b = y - 1, c = z - 1$, then $a + b + c = 97$. Now following like previous problem, no. of solutions is $C_{3-1}^{97+3-1} = 4851$.

410. Let $a = x_1 + 5, b = x_2 + 5, c = x_3 + 5$, then $a + b + c = 15$. No. of solutions of this solution is $C_{3-1}^{15+3-1} = C_2^{17} = 136$.

411. Let $a = x + 1, b = y + 1, c = z + 1, d = t + 1 \Rightarrow a + b + c + d = 24$, where $a, b, c, d \geq 0$. No. of solutions of this is $C_{4-1}^{24+4-1} = C_3^{27}$.

412. Let $x = a + 3, y = b - 1$, then $x, y \geq 0$. Thus, $x + y + c + d + e = 24$, and hence, no. of solutions of this equation is C_4^{28}.

413. Let d be the c.d. of the A.P., then $a = b - d, c = b + d \Rightarrow b = 7 \Rightarrow a + c = 14$, where $a, b \geq 1$.

Let $x = a - 1, y = c - 1 \Rightarrow x + y = 12$, where $x, y \geq 0$.

No. of solutions of this equation is $C_{2-1}^{12+2-1} = C_1^{13} = 13$.

414. Let $a = 2k + 1, b = 2l + 1, c = 2m + 1, d = 2n + 1$, where $k, l, m, n \in \mathbb{N}$. Then, $k + l + m + n = 8$.

No. of solutions of this equation is C_3^{11}.

415. Given, $x + y + 3z = 33 \Rightarrow x = y = 33 - 3z$.

If $z = 0, x + y = 33$, then no. of solutions is $C_{2-1}^{33+2-1} = C_1^{34} = 34$.

If $z = 1, x + y = 30$, then no. of solutions is $C_{2-1}^{30+2-1} = C_1^{31} = 31$.

If $z = 2, x + y = 27$, then no. of solutions is $C_{2-1}^{27+2-1} = C_1^{28} = 28$.

...

If $z = 11, x + y = 0$, then no. of solutions is $C_{2-1}^{2-1} = C_1^1 = 1$.

Thus, total no. of solutions is $1 + 4 + 7 + \cdots + 28 + 31 + 34 = 210$.

416. Let $a = x - 1, b = y - 1, c = z - 1$, then $3a + b + c = 22 \Rightarrow b + z = 22 - 3a$.

If $a = 0, b + c = 22$, then no. of solutions is $C^{22+2-1}_{2-1} = C^{23}_{1}$.

If $a = 1, b + c = 19$, then no. of solutions is $C^{19+2-1}_{2-1} = C^{20}_{1}$.

If $a = 2, b + c = 16$, then no. of solutions is $C^{16+2-1}_{2-1} = C^{17}_{1}$.

...

If $a = 7, y + z = 1$, then no. of solutions is $C^{1+2-1}_{2-1} = C^{2}_{1}$.

Thus, total no. of solutions is $2 + 5 + 8 + \cdots + 23 = 100$.

417. Given, $a+b+c \leq 8$, where $a, b, c \geq 1$. Let $x = a+1, y = b+1, z = c+1 \Rightarrow x+y+z \leq 5$, where $x, y, z \geq 0$.

No. of solutions of this equation is $\sum_{i=0}^{5} C^{i+3-1}_{3-1} = C^7_2 + C^6_2 + C^5_2 + C^4_2 + C^3_2 + C^2_2 = 21 + 15 + 10 + 6 + 3 + 1 = 56$.

418. Given, $x + y + z + w \leq 7$. No. of solutions of this equation is $\sum_{i=0}^{7} C^{i+4-1}_{4-1}$

$= C^3_3 + C^4_3 + C^5_3 + C^6_3 + C^7_3 + C^8_3 + C^9_3 + C^{10}_3 = 1+4+10+20+35+56+84+120 = 330$.

419. Given, $x + y + z = 100$. Since we have to find non-negative even integral we assume $x = 2a, y = 2b, z = 2c$, where $a, b, c \in \mathbb{N}$.

Therefore, $a + b + c = 50$. No. of solutions of this equation is C^{52}_2.

420. Proceeding like previous to previous problem, no. of solutions of this equation is $\sum_{i=0}^{2} 3C^{i+2}_3 = C^{27}_4$.

421. We have $15 < a + b + c \leq 20$, where $a, b, c > 0$. Let $x = a - 1, y = b - 1, z = c - 1$, therefore, $18 < x + y + z \leq 23$, where $x, y, z \geq 0$.

No. of solutions of this equation is $\sum_{i=18}^{23} C^{i+2}_2 = 685$.

422. Given, $(a + b + c)(p + q + r + s) = 21$. 21 can be factored in 4 ways. $(1, 21)$ and $(3, 7)$. So we have four combinations. $a + b + c = 1, p + q + r + s = 21$ is first pair; $a+b+c = 21, p+q+r+s = 1$ is second pair; $a+b+c = 3, p+q+r+s = 7$ is third pair; and $a + b + c = 7, p + q + r + s = 3$ is final and fourth pair.

Now, the no. of solutions of individual equations when multiplied with its counterpart will give the soluion, which is $C^5_2.C^{10}_3 + C^9_2.C^6_3 + C^{23}_2.C^4_1 + C^{24}_3.C^3_1$.

423. Clearly, $x + y + z = n$, and the no. of solutions in this case is $C^{n+2}_2 = \frac{(n+1)(n+2)}{2}$.

424. Let A, B, C donate a, b, c coins. Then $a+b+c=10$, where $a, b, c>0$, and $a \leq 6, b \leq 7, c \leq 8$.

Thus, required no. of ways = coeff. of x^{10} in $(x+x^2+\cdots+x^6)(x+x^2+\cdots+x^7)(x+x^2+\cdots+x^8)$

$=$ coeff. of x^7 in $(1+x+\cdots+x^5)(1+x+\cdots+x^6)(1+x+\cdots+x^7)$

$=$ coeff. of x^7 in $(1-x^6)(1-x^7)(1-x^8)(1-x)^{-3}=$ coeff. of x^7 in $(1-x^6-x^7)(1+C_1^3x+C_2^4x^2+C_2^5x^3+\cdots+C_2^9x^7)$

$=C_2^9-1=35$.

If $a, b, c \geq 0$ the solution is coeff. of x^{10} in $(1+x+\cdots+x^6)(1+x+\cdots+x^7)(1+x+\cdots+x^8)$

$=$ coeff. of x^{10} in $(1-x^7)(1-x^8)(1-x^9)(1-x)^{-3}=$ coeff. of x^{10} in $(1-x^7-x^8-x^9)(1+C_1^3x+C_2^4x^2+C_2^5x^3+\cdots+C_1^{12}x^{10})$

$=C_2^{12}-C_3^5-C_2^4-C_1^3=47$.

Similarly, coeff. of x^15 will give us C_2^8 as the answer.

425. Total marks is 250. The student must score 60% i.e. 150 marks.

Thus, required answers = coeff. of x^{150} in $(1+x+\cdots+x^{50})^3(1+x+\cdots+x^{100})$

$=$ coeff. of x^{150} in $(1-x^{51})^3(1-x^{101})(1-x)^{-4}=$ coeff. of x^{150} in $(1-x)^{-4}$ - 3. coeff. of x^{99} in $(1-x)^{-4}+$ 3. coeff. of x^{48} in $(1-x)^{-4}-$ coeff. of x^{49} in $(1-x)^{-4}$

$=C_{150}^{153}-3.C_{99}^{102}+3.C_{48}^{51}-C_{49}^{52}=110{,}551$.

426. Clearly, $210 \leq x+y+z \leq 300$, where x, y, z are marks scored in Physics, Chemistry, and Mathematics.

If $x+y+z=210$, then no. of solutions is C_2^{211} and so on.

So the answer is $\sum_{i=211}^{302} C_2^i$.

427. Given, $a+b+c+d=6$, where a, b, c, d are the values of the up face of the four dices.

Clearly, $1 \leq a, b, c, d \leq 6$. Let $x=a-1, y=b-1, z=c-1, t=d-1$, then

$x+y+z+t=2$, where $x, y, z, t \geq 0$.

No. of solutions of the above equation is $C_3^{4+2-1}=10$.

428. Let the digits be a, b, c, d, e, f, then clearly $a, b, c, d, e, f \geq 0$. From quesiton

$a+b+c+d+e+f=5$, which has $C_5^{10}=252$ solutions. However, this also includes numbers from 1 to 99. Then, $a+b=5$, then solution of this euqation is $C_1^6=6$.

Thus, required answer is 246.

429. Let the scores be a, b, c in three throws of the dice. According to question, $a+b+c=14$, where $1 \leq a, b, c \leq 6$.

Let $x=a-1, y=b-1, z=c-1$, then $x+y+z=11$.

Thus, required answer is coeff. of $x^1 1$ in $(1+x+x^2+\cdots+x^5)^3=(1-x^6)^3(1-x)^{-3}$

$=C_2^{13}-3.C_2^7=15$.

430. The possible triplets are $(1, 3, 10)$, $(1, 2, 15)$, $(2, 3, 5)$, $(1, 5, 6)$, $(1, 1, 30)$. Taking permutations, we have $3!+3!+3!+3!+\frac{3!}{2!}=27$.

431. The factors are $2, 3, 7, 5, 5$. Now, $2, 3, 5$ can belong to one of a, b, c, d, e, thus we have 5 ways to put these. Both the fives can belong to one of five variables in 5 ways, and to two of five variabbles in $C_2^5=10$ ways.

Thus, total no. of ways of distributing the factors is $5 \times 5 \times 5 \times (5+10)=1875$.

432. For $y=1$, there is only one solution. For $y=2, 3, 5$ there is only one factor, which can be put in 3 ways to one of x_1, x_2, x_3. For $y=6, 10, 15$, there are two factors, which can be put in 3×3 ways to the three variables. For $y=30$ there are three factors, which can be put in 3^3 ways to three variables.

Adding these, we get 64 as desired answer.

433. The factors x_i can be positive or negative, which gives us a factor of 2^{10} for number of solutions.

$1080000=2^6 \times 3^3 \times 5^4$. Let the powers of 2 among x_is be a_i, then $\sum\limits_{i=0}^{10} a_i=6$, which gives us $C_9^{15}=C_6^{15}$ solutions.

Let the powers of 3 among x_is be b_i, then $\sum\limits_{i=0}^{10} b_i=3$, which gives us $C_9^{12}=C_3^{12}$ solutions.

Similarly, for powers of 5, we have C_4^{13} solutions.

Thus, total no. of solutions is $2^{10}C_6^{15}C_3^{12}C_4^{13}$.

434. Let r be the number of zeros in $x_1, x_2, \ldots, x_{10}$, where $0 \leq r \leq 9$. There are C_r^{10} ways to choose zeros. Thus, the number of non-zero x_i is $10-r$. When a positive integer a is given, there are two non-zero x_i, which satisfy $|x_i|=a$(one a and another $-a$).

The no. of solutions of $a_1+a_2+\ldots+a_{10}=100$, where $a_i>0$ is given by C_{9-r}^{99}.

Thus, the answer is $\sum_{r=0}^{9} 2^{10-r}C_r^{10}C_{9-r}^{99}$.

435. Let r_n denote the number of regions made by n lines. Then, $r_0=1, r_1=2, r_2=4$. Consider that there are $n-1$ lines none of which are horizontal. Now we draw a

horizontal line. The $n-1$ lines will cut the horizontal line in n parts which will divide the old regions in two parts generating n new regions.

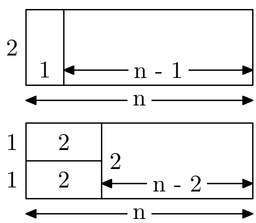
Thus, we can write $r_n = r_{n-1} + n \Rightarrow r_n - r_1 = \sum\limits_{n=2}^{n} n \Rightarrow r_n = 1 + \frac{n(n+1)}{2}$.

436. Let r_n denote the number of regions made by n circles. Then, $r_0 = 1, r_1 = 2, r_2 = 4, r_3 = 8$. Let $n-1$ circles create r_{n-1} regions.

$n$the circle will intersect with these $n-1$ circles at 2 points i.e. a total of $2(n-1)$ points dividing nth circle in $2(n-1)$ arcs. Each arc will fall in some old region and divide those regions in 2 parts, and thus generate $2(n-1)$ new regions.

$\Rightarrow r_n = r_{n-1} + 2(n-1)$, where $n \geq 2 \Rightarrow r_n = r_1 + \sum\limits_{i=2}^{n} 2(i-1) \Rightarrow r_n = n^2 - n + 2$.

437. The diagram is given below:



Let no. of ways be w_n, then $w_0 = 1, w_1 = 1, w_2 = 2$. Let $n \geq 2$. We divide the board in two parts A and B depending upon the domino placed at first place.

A : Those perfect covers in which there is a vertical domino at first place as shown in figure.

B : Those perfect covers in which there are two horizontal dominos at first place as shown in figure.

Now perfect covers in A = perfect covers in $2(n-1)$ board. $\Rightarrow |A| = w_{n-1}$. Similarly, $|B| = w_{n-2}$.

$\Rightarrow w_n = w_{n-1} + w_{n-2}$, which is Fibonacci sequence.

Now we will find the general solution using analysis which is a bit advanced, undergraduate math.

A power series of type $\sum a_n x^n$ has a radius of convergence $R \geq 0$ such that the series is convergent if $|x| < R$, and divergent if $|x| > R$. If two power series $\sum a_n x^n$ and $\sum b_n x^n$ are such that they are equal for all values of x in some interval $(-R, R)$ then $a_n = b_n$.

Let a_n denote the $n$the Fibonacci number and consider the power series $F(x) = \sum\limits_{n=0}^{\infty} a_n x^n$. If this series converges in certain intereval $(-R, R)$ then we have following relations for $|x| < R$:

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$xF(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$$

$$x^2 F(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$$

Now this gives us $F(x)-xF(x)-x^2F(x)=a_0+(a_1-a_0)x \Rightarrow F(x)=\frac{1}{1-x-x^2}$

Roots of the equation $x^2-x-1=0$ are $\alpha,\beta=\frac{1\pm\sqrt{5}}{2}$, and hence

$$F(x)=\frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha x}-\frac{\beta}{1-\beta x}\right)$$

We also make use of another power series $\frac{x}{1-x}=1+x+x^2+x^3+\cdots$ for $|x|<1$

and we see that $a_n=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$ and hence we have general solution for Fibonacci series.

438. Let m_n be the minimum no. of moves that will transfer from one peg to the other peg under given restrictions. By basic enumeration, we see that $m_1=1$ and $m_2=3$.

We first transfer $n-1$ smaller disks to peg 3 in m_{n-1} moves. Then move the largest disk to peg 2, and finally transfer $n-1$ disks from peg 3 to peg 2 using same m_{n-1} moves.

Thus, $m_n=2m_{n-1}+1 \Rightarrow m_n+1=2(m_{n-1}+1)=2^{n-1}(a_1+1)=2^n \Rightarrow m_n=2^n-1$.

439. Using the formulal for derangements $D_5=5!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right]=44$.

440. Let A_k be the set of integers from 1 to 1000, which are divisible by k. We have to find $n(A_2\cup A_3\cup A_5)$.

$n(A_2)=\left\lfloor\frac{1000}{2}\right\rfloor=500$, $n(A_3)=\left\lfloor\frac{1000}{3}\right\rfloor=333$, $n(A_5)=\left\lfloor\frac{1000}{5}\right\rfloor=200$.

$n(A_2\cap A_3)=\left\lfloor\frac{1000}{6}\right\rfloor=166$. Similarly, $n(A_3\cap A_5)=66$, $n(A_1\cap A_5)=100$, $n(A_2\cap A_3\cap A_5)=33$.

Hence, $n(A_2\cup A_3\cup A_5)=500+333+200-166-66-100+33=734$.

Hence, numbers not divisible by given numbers is $1000-734=266$.

441. There are seven persons altogether. No. of ways of arranging them around the table is 6!.

$n(A)=$ when two Americans are together $=5!\,2!=240$, $n(B)=$ when two Britishers are together $=240$.

$\Rightarrow n(A\cap B)=n(A)+n(B)-n(A\cap B)=384$

Hence $n(\bar{A}\cap\bar{B})=$ Total $-n(A\cup B)=336$.

442. Let C denote the selection of cards in which clubs are absent; D denote the selection of cards in which diamonds are absent; H denote the selection of cards in which hearts are absent; and, S denote the selection of cards in which spades are absent. Then,

$|C|=|D|=|H|=|S|=C_5^{39}$, $|C\cap D|=\ldots=C_5^{26}$, $|C\cap D\cap H|=\ldots=C_5^{13}$ and $|C\cap D\cap H\cap S|=0$.

$\therefore |C \cup D \cup H \cup S| = 4.C_5^{39} - 6.C_5^{26} + 4.C_5^{13}$.

Hence, required answer is $C_5^{52} - 4.C_5^{39} + 6.C_5^{26} - 4.C_5^{13}$.

443. Total no. of distributions when there are no conditions is 4^6. No. of distributions when one box is empty is $C_1^4.3^6$; no. of distributions when two boxes are empty is $C_2^4.2^6$; and, no. of distributions when three boxes are empty is $C_3^4.1^6$.

Thus, required no. is $4^6 - C_1^4.3^6 + C_2^4.2^6 - C_3^4 = 2260$.

Similarly, when exactly one box is empty is $C_1^4(3^6 - C_1^3.2^6 + C_2^3) = 2160$.

444. Total no. of ways without any condition is $10^2 = 100$. Let $b - a \geq 6 \Rightarrow 1 \leq a < b \leq 10 \Rightarrow 1 \leq a < b - 5 \leq 5 \Rightarrow C_2^5 = 10$.

Similarly, $a - b \geq 6$ will give 10 ways. Thus, required answer is $100 - 10 - 10 = 80$.

445. Let H, I and D represent the set of people who read The Hindu, Indian Express and Deccan Herald, respectively.

According to question, $|H \cap I \cap D| \leq 150$, $|H| = 70$, $|I| = 80$, $|D| = 50$, $|H \cap I| = 30$, $|H \cap D| = 20$, $|I \cap D| = 25$.

$|H \cap I \cap D| = |H| + |I| + |D| - |H \cap I| - |I \cap D| - |H \cap D| + |H \cap I \cap D|$

In order to maximize $|H \cap I \cap D|$ we have to maximize $|H \cup I \cup D|$ i.e. 150.

Hence, $|H \cap I \cap D| = 25$.

446. Let $n(S)$ be 100. $\therefore n(S) \geq n(E \cup H) = n(E) + n(H) - n(E \cap H) \Rightarrow n(E \cap H) \geq 45$.

Similarly, $n(S) \geq n(L \cup A) = n(L) + n(A) - n(L \cap A) \Rightarrow n(L \cap A) \geq 65$.

$n(S) = 100 \geq n[(E \cap H) \cup (L \cap A)] \Rightarrow n(E \cap H \cap L \cap A) \geq 10$.

That is there is at least 10% of the people must have lost all four.

447. $A \cap B \subset A \Rightarrow n(A \cap B) \leq n(A) \Rightarrow d \leq a$. Also, $n(A \cap B) \leq B \Rightarrow n(A \cap B) \leq n(B) \Rightarrow d \leq b$.

$n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow c + d = a + b$.

448. When there is no restriction and repetition allowed, each of n digits can be filled in 3 ways using one of $1, 2$ or 3.

Thus, no. of ways is 3^n.

The number of n-digit numbers all of whose digits are $1, 2$ or 3 is 3^n. The number of n-digit all of whose digits are 1 and 2, each occurring at least once is $2^n - 2$.

Thus, total no. of digits is $3^n - 3(2^n - 2) - 3 = 3^n - 3.2^n + 3$.

449. $10^{60} = 2^{60} \times 5^{60}$, therefore, $n(A) = 61^2$. $20^{50} = 2^{100} \times 5^{50}$, therefore, $n(B) = 101 \times 51$, and similarly, $n(C) = 41^3$.

Clearly, $n(A \cap B) = 61 \times 51$, where 61 is the no. of ways of selecting 2 and 51 is the no. of ways of selecting 5. Similarly, $n(B \cap C) = n(A \cap C) = 41^2$ and $n(A \cap B \cap C) = 41^2$.

$\therefore n(A \cap B \cap C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C) = 73{,}001$.

450. First we assign 3 for x_1, and 7 for x_3. Now the problem remains of assigning remaining 18 among x_1, x_2, and x_3 such that x_1 has at most 6, x_2 at most 8, and x_3 at most 10.

The number of ways of distributing 18 to these three is $C_{3-1}^{18+3-1} = C_2^{20} = 190$.

Let d_1 be the set of distributions, where x_1 gets at least 7; d_2 be the set of distributions, where x_2 gets at least 9, and d_3 be the set of distributions, where x_3 gets at least 11.

$|d_1| = C_{3-1}^{18-7+3-1} = C_2^{13} = 78$, $|d_2| = C_{3-1}^{18-9+3-1} = C_2^{11} = 55$, and $|d_3| = C_{3-1}^{18-11+3-1} = C_2^9 = 36$.

$|d_1 \cap d_2| = C_{3-1}^{18-7-9+3-1} = 6$, $|d_2 \cap d_3| = C_2^{18-9-11+3-1} = C_2^0 = 0$, and $|d_1 \cap d_3| = 1$.

Also, $|d_1 \cap d_2 \cap d_3| = 0 \Rightarrow |d_1 \cup d_2 \cup d_3| = 162$.

So required no. of solutions is $190 - 162 = 28$.

451. Let A_i be the set of days on which ith friend is present, where $i = 1, 2, 3, \dots, 6$.

Given, $n(A_i) = 7$, and $n(A_i') = 7$. Also given, $|A_i \cap A_j| = 5$, $|A_i \cap A_j \cap A_k| = 4$, $|A_i \cap A_j \cap A_k \cap A_l| = 3$, $|A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 2$, and $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 1$, where i, j, k, l, m are distinct between 1 and 6.

$\therefore |A_1 \cup A_2 \cup \dots \cup A_6| = 13$. This is total no. of dinners, where at least one friend is present.

Total no. of dinners is $|A_i| + |A_i'| = 14$. Therefore, total no. of dinners, where I was along is 1.

452. Let M_r denote the set of morning rainy days, and A_r be the set of afternoon rainy days. Then M_r' will represet clear mornings, and A_r' will represnet clear afternoons.

Given, $M_r \cap A_r = \phi$, $M_r' = 6$, $A_r' = 5$, and $M_r \cup A_r = 7$. Also, given that M_r and A_r are disjoint sets, and $n(M_r) = d - 6$, $n(A_r) = d - 5$.

Applying PIE, we get $n(M_r \cup A_r) = n(M_r) + n(A_r) - n(M_r \cap A_r) \Rightarrow d = 9$.

453. This is a straight problem of derangements. Applying the formula, we get

$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right]$.

454. We can choose one number in $C_1^5 = 5$ ways. Rest 4 numbers are a case of derangements, which is $D_4 = 9$.

Thus, total no. of ways is 45.

455. First part is direct application of derangements. Using derangement formula, we have

$D_5 = 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right] = 44.$

No. of ways to place 2 balls in correct box = Total no. of ways − No. of ways to place all balls in wrong boxes − No. of ways to place 1 ball in correct box

$= 5! - 44 -$ No. of ways to place 1 ball in correct box

To put one ball in correct box we can choose the ball in C_1^5 ways, and then apply derangement formula for remaining 4 i.e. $D_4 = 9$. So no. of ways to put one ball correctly is same as previous problem i.e. $9 \times C_1^5 = 45$.

Thus, required answer is $120 - 44 - 45 = 31$.

456. First part is a direct derangement question. $D_6 = 6!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{1}{6!}\right] = 265.$

No. of ways to place at least 4 letters correctly = 4 letters placed correctly +5 letters placed correctly +6 letters placed correctly $= C_4^6 \times 1 + 0 + 1 = 16$.

No. of ways to place at most 3 letters correctly = 6 letters placed correctly + 5 letters placed correctly +4 letters placed correctly +3 letters placed correctly

$= 1 + 0 + C_4^6 + C_3^6 .3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 56.$

457. Let A denote the set of no. divisible by 2, B by 3, and C by 7.

$n(A) = 50, n(B) = 33, n(C) = 14, n(A \cap B) = 16, n(A \cap C) = 7, n(B \cap C) = 4, n(A \cap B \cap C) = 2.$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) = 50 + 33 + 14 - 16 - 7 - 4 + 2 = 72.$

$\therefore |\overline{A \cup B \cup C}| = 100 - 72 = 28.$

458. Let A denote the set of no. divisible by 2, B by 3, and C by 5.

$n(A) = 250, n(B) = 166, n(C) = 100, n(A \cap B) = 83, n(A \cap C) = 50, n(B \cap C) = 33, n(A \cap B \cap C) = 16.$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) = 250 + 166 + 100 - 83 - 50 - 33 + 16 = 366$

$\therefore |\overline{A \cup B \cup C}| = 500 - 366 = 134.$

459. When least element is 3, remaining 2 can be chosen from 7 in C_2^7 ways. When 7 is greatest then remaining 2 can be chosen from 6 in C_2^6 ways. No. of subsets satisfying both conditions is 3 because elements remaining are 4, 5, 6.

Then from PIE, the answer is $C_2^6 + C_2^7 - 3 = 33$.

460. Optional digits are $3, 4, 5, 6$. Number of n-digit numbers made from $2, 3, 4, 5, 6, 7$ is 6^n.

Number of n-digit numbers containing $3, 4, 5, 6, 7$ is 5^n, and similarly for $2, 3, 4, 5, 6$.

Number of n-digit numbers containing only $3, 4, 5, 6$ is 4^n.

Thus, required answer is $6^n - 2 \times 5^n - 4^n$.

461. If the integer is of one digit then there are no repeated digits. If the integer is of two digits then we can fill first position in 9 ways excluding 0, and second position in 9 ways excluding first digit; this gives us 81 integers. Similarly, for 3 digit integer we have 648 digits wiithout repetition.

Thus, we have 738 integers without repetition.

462. No. of perfect squares is 10^4. No. of perfect cubes is 464. No. of fifth powers is 39. Squares, which are also cubes is 21. Fifth powers, which are also squares is 6. Fifth powers, which are also cubes is 2(excluding 1^5, which is already accounted for).

Thus, no. of such powers is $10^4 + 464 + 39 - 21 - 6 - 2 = 10474$.

Thus, required answer is $10^8 - 10474 = 99{,}989{,}526$

463. Zero letters can be chosen in 1 way. One letter can be chosen in 26 ways. Two letters can be chosen in 26^2, and three in 26^3 ways. Similarly, digits can be chosen in $(1 + 10 + 10^2 + 10^3 + 10^4)$ ways.

Thus, no. of licenses which can be produced is $(1+26+26^2+26^3).(1+10+10^2+10^3+10^4)$. However, one of these contain neither a digits nor a letter i.e. it is empty, which is not allowed. Hence, the answer is $(1 + 26 + 26^2 + 26^3).(1 + 10 + 10^2 + 10^3 + 10^4) - 1$.

If 85 combinations of letters is not allowed then the answer would be $(1 + 26 + 26^2 + 26^3 - 85).(1 + 10 + 10^2 + 10^3 + 10^4)$.

464. MAT caannot be spelled if only one of the letters or two of them is chosen. No. of ways of chossing one letter is $C^3_1 = 3$.

No. of ways in which any two can be chosen is $C^3_2(2^k - 2)$. Thus, $3 + C^3_2(2^k - 2) = 93 \Rightarrow K = 5$.

465. $10^{40} = 2^{40}.5^{40}$. Thus, it has $41 \times 41 - 1 = 1680$ factors. Factors of $20^{30} = 2^{60}.5^{30}$ not dividing 10^{40} have the form 2^m5^n with $41 \leq m \leq 60$, and $0 \leq n \leq 30$ i.e. no. of such numbers is $20.31 = 620$.

466. Total no. of permutations is $7! = 5040$. Let A be the set of words in which '*beg*' appear. Treating it as one letter we have $n(A) = 5! = 120$, and we have same for '*cad*' i.e. $n(B) = 120$, where B is the set of words where '*cad*' appears.

However, both can also come together, which is $n(A \cap B) = 3! = 6$ by treating both of them as one letter.

Thus, $n(A \cup B) = 120 + 120 - 6 = 234$, and hence, $n(\overline{A \cup B}) = 5040 - 234 = 4806$.

467. Total no. of permutations is $\frac{9!}{2!} = 181{,}440$.

Let A represent the set of words in which 'HIN' comes together. Treating 'HIN' as one letter, we have $n(A) = 7!$ permutations. Similarly, if we 'DUS' treat as one, and B is the set in which it comes together,we have $n(B) = \frac{7!}{2!}$ permutations as N comes twice. If C is the corresponding set for 'TAN' then $n(C) = 7!$.

$\therefore n(A \cap B) = n(A \cap C) = n(B \cap C) = 5! = 120, n(A \cap B \cap C) = 3! = 6.$

$\Rightarrow n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) = 5040 + 5040 + 2520 - 120 - 120 - 120 + 6.$

Hence, $n(\overline{A \cup B \cup C}) = 181440 - (5040 + 5040 + 2520 - 120 - 120 - 120 + 6) = 169{,}194$.

468. Total no. of permutatiosn is $N = \frac{8!}{2!2!2!2!} = 2520$.

Let α represent the sets, where one of the letters come toegther then $n(\alpha) = \frac{7!}{2!2!2!} = 630$.

Let $\alpha\beta$ represent the sets, where two of the letters come together then $n(\alpha\beta) = \frac{6!}{2!2!} = 180$.

Similarly, $n(\alpha\beta\gamma) = \frac{5!}{2!} = 60$, and $n(\alpha\beta\gamma\delta) = 4! = 24$.

Applying PIE, we get the desired result as $N - 4n(\alpha) + 6(\alpha\beta) - 4(\alpha\beta\gamma) + n(\alpha\beta\gamma\delta) = 864$.

469. Since we have to find non-negative values, it implies that values can be zero also.

No. of ways of distributing 15 among three variables is $C_2^{17} = 136$.

Let d_1 be the no. of distributions, where x_1 gets at least 6; d_2 be the set of distributions, where x_2 gets at least 7; and d_3 be the set of distributions, where x_3 gets at least 8.

$|d_1| = C_{3-1}^{15-6+3-1} = C_2^{11}$, $|d_2| = C_2^{10}$, and $|d_3| = C_2^{9}$.

$|d_1 \cap d_2| = C_{3-1}^{15-6-7+3-1} = C_2^4$, $|d_1 \cap d_3| = C_2^3$, and $|_.2 \cap d_3| = C_2^2$.

Also, $|d_1 \cap d_2 \cap d_3| = 0 \Rightarrow |d_1 \cup d_2 \cup d_3| = 55 + 45 + 36 - 6 - 3 - 1 = 126$.

$\Rightarrow |\overline{d_1 \cup d_2 \cup d_3}| = 136 - 126 = 10$.

470. No. of years divisible by 4 between 1000 and 3000(both inclusive) is 501. No. of years divisible by 100 between 100 between given range is 21, and no. of years divisible by 400 is 5.

Thus applying PIE, we have $501 - 21 + 5 = 485$ as the answer.

471. Let S denote all the onto functions. Total no. of ways of mapping elements is $|S| = 3^6$. For $i = 1, 2, 3$ let $S_i \subset S$ denote the functions that have i not in their image.

So the answer is $|S| - |S_1 \cup S_2 \cup S_3| = 3^6 - |S_1 \cup S_2 \cup S_3|$. Now $|S_1 \cup S_2 \cup S_3|$ can be found using PIE, to be equal to $C_1^3|S_1| - C_2^3|S_1 \cap S_2| = 3.2^6 - 3.1^6$.

Thus, answer is $3^6 - 3.2^6 + 3$.

472. There are two possible ways to construct these 6-digit numbers.

First, three digts of one kind, and remaining three of different kind. No. of such numbers is $C_1^{10}.C_3^9.\frac{6!}{3!} = 100{,}800$.

Second, two digits of one kind and remaining two of different kind. No. of such numbers is $C_2^{10}.C_2^8.\frac{6!}{2!2!} = 226{,}800$.

However, one-tenth of these numbers will start with 0. So the answer is $\frac{9(100800+226800)}{10} = 294{,}840$.

473. R.H.S. $= (n-1)(D_{n_1} + D_{n-2}) = (n-1)(n-2)!\Big[(n-1)\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-1}}{(n-1)!}\Big) + \Big(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-2}}{(n-2)!}\Big)\Big]$

$= (n-1)!\Big[n - 1 + 1 - \frac{1}{1!}(n-1+1) + \frac{1}{2!}(n-1+1) - \cdots + \frac{(-1)^{n-2}}{(n-2)!}(n-1+1) + \frac{(-1)^{n-1}(n-1)}{(n-1)!}\Big]$

$= n!\Big[1 - \frac{1}{1!} + \frac{1}{21} - \cdots + \frac{(-1)^n}{n!}\Big] = D_n =$ L.H.S.

474. $\lim\limits_{x\to\infty} \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots$.

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, and hence,

$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots =$ L.H.S.

475. We can pick a glove either left or right in 5 ways, and a person in 5 ways as well. Thus, no. of ways to give a glove is 25. Once we have given a glove our choice for the matching pair is only one because one person has to get a correct pair of gloves.

Now we have to distribute remaining gloves in such a way that no one gets a correct pair. We pick all left gloves and distribute them in 4! ways. Now distributing the right gloves is a derangement problem, which is equal to $4!\Big[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\Big] = 9$.

Thus, total no. of ways is $25 \times 4! \times 9 = 5400$.

476. L.H.S. $= \sum\limits_{r=1}^{n} r!\,r = \sum\limits_{r=1}^{n} [(r+1)! - r!] = 2! - 1! + 3! - 2! + \cdots + (n+1)! - n! = (n+1)! - 1 =$ R.H.S.

477. When all matches are correct the condition is satisfied, which can be done in 1 way. There is no way to answer with 5 correct matches and 1 incorrect match because if 5 are correct then 6th is automatically correct.

When 4 are correct, and 2 are incorrect. 4 can be selected in C^6_4 ways. 2 incorrect matches can be done in 1 way. Thus, total is $C^6_4 = 15$.

Finally, we can have 3 correct, and 3 incorrect matches. This can be done in $C^6_3 \times 2 = 40$ ways.

Thus, total no. of ways to pass is 56.

478. Let $R(n)$ be the number of regions formed by n parabolas. When there are $n-1$ parabolas, nth parabola will be intersected at $2(n-1)$ points, so it will be divided into $2n-1$ arcs. Each arc adds a region. So $R(n) = R(n-1) + 2n - 1$. We have base case as $R(1) = 2$.

    Hence, $R(n) = n^2 + 1 \Rightarrow R(10) = 101$.

479. Let there be n staircases. We can reach nth stair from $(n-1)$th or $(n-2)$th stair. Thus, $f(n) = f(n-1) + f(n-2)$.

    Our base case is $f(1) = 1, f(2) = 2$, and hence $f(12) = 233$.

480. Like previous problem the recurrence relation is same, and hence, $f(10) = 144$.

481. Every paving of $1 \times n$ is paving of $1 \times (n-1)$ with 1×1 at the end or $1 \times (n-2)$ with 1×2 at the end or $1 \times (n-3)$ with 1×3 at the end.

    Hence, the recurrence relation is $f(n) = f(n-1) + f(n-2) + f(n-3)\, \forall n \geq 3$.

    The base cases are $f(0) = 1, f(1) = 1, f(2) = 2$, and hence, $f(7) = 44$.

482. For first part, the no. of ways are $3^5 = 243$. For second part, the no. of ways are $r^n - C^r_1(r-1)^n + C^r_2(r-2)^n - \cdots$

    Putting $n = 5$, and $r = 3$ in the above equation, we get

    No. of ways $= 3^5 - C^3_1 2^5 + C^3_1 .1^5 = 150$.

483. The terms of the exxpansion $(a+b+c+d)^{24}$ will have the form $k.a^{e_1}b^{e_2}c^{e_3}d^{e_4}$, where k is a constant, and $e_i, 0 \leq e_i \leq 24$ are integers such that $e_1 + e_2 + e_3 + e_4 = 24$.

    The problem is equivalent to ditributing 24 identical balls in 4 distinguishable boxes, where empty boxes are allowed. The no. is $C^{24+4-1}_{24} = C^{27}_{24}$.

484. The number of ways of dividing n identical balls into r boxes so that no box remains empty is $C^{n-1}_{r-1} = C^4_2 = 6$.

485. No. of ways of distributing balls is equal to no. of integral solutions of the equation $x_1 + x_2 + x_3 = 10$, where $2 \leq x_i \leq 4, i = 1, 2, 3$.

    No. of ways is coeff. of x^{10} in $(x^2 + x^3 + x^4)^3 =$ coeff. of x^4 in $(1-x^3)^3(1-x)^{-3} = C^{4+3-1}_4 - 3.C^{3+1-1}_1 = 6$.

486. Let the boys get $a, a+b, a+b+c$ toys respectively such that $a, b, c \geq 1$. Thus, we have $3a + 2b + c = 14$.

Therefore, no. of solutions = coeff. of x^{14} in $(x^3+x^6+x^9+\cdots)(x^2+x^4+\cdots)(x+x^2+x^3+\cdots)$

$=$ coeff. of x^8 in $(1+x^2+x^3+x^4+x^5+2x^6+x^7+2x^8)(1+x+x^2+\cdots+x^8)=10$.

Since these numbers can be assigned in 3! ways, so, total no. of ways is 60.

487. Required answer is $S(5,1)+S(5,2)+S(5,3).S(5,1)=1, S(5,2)=2^{5-1}-1=15$.

$S(5,3)=S(4,2)+3S(4,3)=25$. So the answer is 41.

488. The answer is $P_3(6n)+P_2(6n)+P_1(6n)$.

We know that $P_k(b)-P_k(n-k)=P_{k-1}(n-1) \Rightarrow P_3(6n)-P_3(6n-3)=P_2(6n-1)=\left\lfloor\frac{6n-1}{2}\right\rfloor=3n-1$

and $P_3(6n-3)-P_3(6n-6)=P_2(6n-4)=\left\lfloor\frac{6n-4}{2}\right\rfloor=3n-2$

$\Rightarrow P_3(6n)-P_3(6(n-1))=3(2n-1) \Rightarrow P_3(6n)-P_3(6)=3(n^2-1)$

As $P_3(6)=3 \Rightarrow P_3(6n)=3n^2 \Rightarrow P_2(6n)=3n$ and $P_1(6n)=1$.

Hence, answer is $3n^2+3n+1$.

489. Total no. of ways is 2^n. However, in 2 of the ways all the balls go in each box but the question says that boxes cannot remain empty.

Thus, answer is 2^n-2.

490. $S(n,2)=2^{n-1}-1$.

491. **Case I:** When no box is empty. No. of ways of distributing the balls is $C_4^9=126$.

Case II: When one box is empty. No. of ways = Selection of one box $\times$ Distribution of 10 balls into 4 boxes $=C_1^5\times C_3^9=420$.

Case III: When two boxes are empty. No. of ways = (Selection of two boxes $-$ Adjacent boxes)*times* Distribution of 10 balls into 3 boxes $=(C_2^5-4)\times C_2^9=216$.

Case IV: When three boxes are empty. There is only one way to select three boxes when adjacent boxes are not empty. No. of ways is $1\times C_1^9=9$.

Thus, total no. of ways is $126+420+216+9=771$.

492. The answer is equivalent to integral solution of $x_1+x_2+x_3+x_4+x_5+x_6=12$ such that $1\le x_i\le 3, i=1,2,3,4,5,6$.

No. of integral solutions is coeff. of x^{12} in $(x+x^2+x^3)^6=$ coeff. of x^6 in $(1+x+x^2)^6$

$=$ coeff. of x^6 in $(1-x^3)^6(1-x)^{-6}=$ coeff. of x^6 in $(1-6x^3+15x^6-\ldots)(1+6x+C_2^7x^2+C_3^8x^3+C_4^9x^4+C_5^{10}x^5+C_6^{11}x^6+\cdots)$

$=C_6^{11}-6.C_3^8+3=141$.

493. Following like previous problem, maximum a person can get is 10 rupees when five others get the minimum of 4 rupees.

Thus, the answer is coeff. of x^{30} in $(x^4+x^5+x^6+x^7+x^8+x^9+x^{10})^6$ = coeff. of x^6 in $(1-x^7)^6(1-x)^{-6} = C_6^{11} = 462$.

494. Consider following: $rr|rr|rr|r|r$, where r represents the rings and $|$ represents the empty space. We have 8 rings(different) and 4 empty space(same), which can be arranged in $\frac{12!}{4!}$ ways.

495. This is equivalent to no. of integeral solutions of the equation $x_1+x_2+x_3+x_5 = 15$ such that $x_i \geq 2, i = 1, 2, 3, 4, 5$.

$\Rightarrow y_1+y_2+y_3+y_4+y_5 = 5$, where $y_i \geq 0$.

No. of integral solution of this equation is C_4^9.

496. No. of ways of distributing n identical objects into r different groups is C_{r-1}^{n+r-1}.

So no. of ways to distribute 3 blue balls is $C_3^6 = 20$. No. of ways to distribute 4 red balls is $C_3^7 = 35$. No. of ways to distribure 2 red balls is $C_3^5 = 10$.

Thus, total no. of ways is 7000.

497. First we give Manya 2 candy bars, which will leave us with 13 of those. No. of ways to distribute 13 candy bars among 4 people is $C_3^{13} = 560$.

These include the cases where Tanya gets more than 5. Let us calculate the cases where Tanya gets at least 6. In this case we give 8 candy bars to Tanya and Manya, which will leave us with 7 of those. No. of ways to distribute 7 candy bars is $C_3^{10} = 120$.

Thus, required answer is $560 - 120 = 440$.

498. Total no. of such sequences is 4^n. No. of sequences where 2 does not appear is 3^n and same is the case when 3 dpes not appear. However, these include cases where both do not appear and those are 2^n.

Hence, the answer is $4^n - 2.3^n + 2^n$.

499. This is the case of putting n distinguishable balls into r boxes, which is $3^{15} - C_1^3 2^{15} + C_2^3.1^{15}$.

He can choose in two ways $3, 1, 1$ and $2, 21$, which are $C_3^5 \times C_1^5 \times C_1^5 = 250$ and 500. However, for the first case there are 3 ways to choose the box from which 3 balls can be chosen so it becomes 750 and same for second case which is 1500.

Thus, total no. of ways is 2250.

500. No. of ways of selecting boxes will remain constant 3!.

Case I: The balls are distributed in $5 \sim 2 \sim 2$. No. ways of selecting balls is $C_5^9 \times C_2^4 \times C_2^2 = 378$.

Case II: The balls are distributed in $4 \sim 2 \sim 3$. No. of ways is $C_4^9 \times C_2^5 \times C_3^3 = 1260$.

Case III: The balls are distributed in $3 \sim 3 \sim 3$. No. of ways is $C_3^6 \times C_3^6 \times C_3^3 = 280$.

Thus, total no. of ways is $(378 + 1260 + 280)6 = 11508$.

501. 5 balls can be selected out of 12 in C_5^{12} ways. Remaining balls can be put into two boxes in 2^7 ways.

Thus, total no. of ways is $C_5^{12}.2^7$.

502. Let the daughters get x, y and z coins. Then, $x + y + z = 101$ and $x \leq y + z = 101 - x \Rightarrow x \leq 50$.

Thus, $x, y, z \leq 50$ because any daughter can have this larger share. This is equivalent to coeff. of x^{101} in $(1 + x + \cdots + x^{50})^3$

$= C_2^{103} - 3.C_{50}^{52}$.

503. Let the two sets be A and B. 2 cannot be in the same set as 1 and 4. Because if 2 and 1 are in the same set then the difference $2 - 1 = 1$ violates the given condition. Same is true for 2 and 4. Let 2 be in set A and $1, 4$ in set B.

Clearly, 3 has to be in set A as $4 - 3 = 1$. So the sets now are $A = \{2, 3, \_\}$ and $B = \{1, 4, \_\}$. Now we cannot put 5 in either of these sets, and hence, the result.

504. We divide the square into 9 unit squares. Out of the 10 points distributed in the big square at least one of the small squares must have at least two points by the Pigeonhole principle. Since these two points lie in the same unit square maximum distance between them can be $\sqrt{2}$ units.

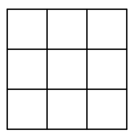
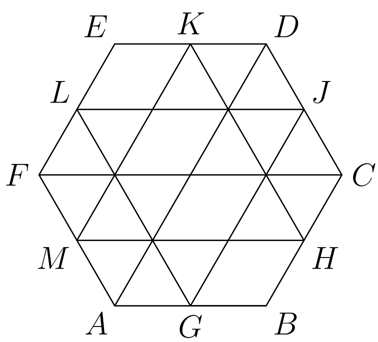


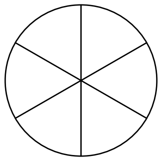
Figure 5.1

505. Let $ABCDEF$ represet the hexagon and G, H, I, J, K, L be midpoints of the sides, each side being 2cm. Join midpoints and sides as shown which will yield 24 equilateral triangles, each of side $1cm$.



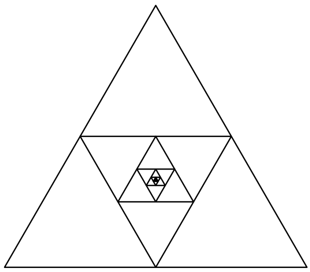
We have to place 25 points in these 24 triangles so as to violate the required condition, which is not possible. Hence, the result.

506. Draw the circle and divide it into six equal parts as shown.



We have to place 7 points in these 6 equal parts so as to violate the required condition, which is not possible. Hence, the result.

507. Draw the equilateral triangle and join the mid-points of every side.



We have initial triangle of side 1 cm. Upon joining mid-points we get 4 equilateral triangles of side $\frac{1}{2}$ cm futher reduced by same factor on further joining of mid-points of reulting triangles.

Now as we can see that we cannot place $5 = 4^1 + 1$ points without violating the required condition it propagates to smaller triangles as well. Hence, the result.

508. There are 34 integers in the progression. We consider the pairs whose sum is 104 i.e. $(4, 100)$, $(7, 97)$, $(10, 94)$, ..., $(49, 55)$. Thus we have 16 pairs. In these pairings we leave out 1 and 52 as they do not have a correpoding pairing element.

Thus, we can pick 1 element out of 16 pairs along with 1 and 52, which will not have a sum of 104. However, the moment we pick 19th number it will form at least one pair with one of the 16 chosen numbers from pairs. Hence, the result is proven.

509. Since $n(X) = 10$, the number of non-empty, proper subsets of X is $2^{10} - 2 = 1022$.

The sum of the elements of the proper subsets of X can possibly range from 1 to $\sum_{i=1}^{9}(90 + i)$ i.e. 1 to 855.

Thus, the 1022 subsets can have sums from 1 to 855. Thus, by Piegonhole principle, at least two subsets B and C will have the same sum. If B and C are not disjoint, then let

$X = B - (B \cap C)$ and $Y = C - (B \cap C)$.

Clearly, X and Y are disjoint, non-empty, and have the same sum of their elements.

We define $s(A) =$ sum of the elements A. We have B and C not necessarily disjoint such that $s(B) = s(C)$.

Clealrly, $s(X) = s(B) - s(B \cap C)$ and $s(Y) = s(C) - s(B \cap C)$. However, $s(B) = s(C)$, and hence $s(X) = s(Y)$.

510. In base-10 or decimal number system we have 10 digits; 0 through 9. For 3 numbers of 4-digits not to have common numbers we need 12 digits. Thus, by Pigeonhole principle at least two must have a common digit.

Similarly, in base-7 we have 7 digits(not a correct word for base-7 symbols, possible word is *heptit*); 0 through 7. For 2 numbers of 4-digit we need at least 8-digits for no repetition which is not possible as we have only 7 available for use. Hence, the result.

511. To represent 3 numbers of k-digits such that there is no common digit between them and without repetition we need $3k$ digits. However, we have only $2k$ available to us. Now $k \geq 1$ so $3k - 2k \geq 1$, and hence, the result.

Similarly, for $3k + 1$ digit numbers will have $3k + 3$ digits in all, but we have only $2k + 1$ base i.e. $2k + 1$ different digits. And hence, the result.

512. Pairs which have the sum $2n + 2nd$ are $[a, a + 2nd], [a + d, a + (2n - 1)d], \ldots, [a + (n - 1)d, a + (n + 1)d]$.

Now since, we cannot have sum equal to $2a + 2nd$ so we drop one element from each pair leaving us n terms and we also include $a + nd$ to this giving us $n + 1$ terms, giving us the required result for part one.

In the second part of the problem $S = \{a, a + d, \cdots + a + (2n + 1)d\}$. Pairs which give $2a + (2n + 1)d$ as sum are $[a, a + (2n + 1)d], [a + d, a + 2nd], \ldots, [a + nd, a + (n + 1)d]$. Thus, we have $n + 1$ pairs. If we pick one from each pair we get $n + 1$ terms of A, which satisfy required condition.

513. Let $a = \tan A$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The reason for choosing this is because in the given range the tan function is strictly increasing and covers the complete $\mathbb{R}$.

We divide the interval into four equal intervals, each of length $\frac{\pi}{4}$. Now using Pigeonhole principle we find that at least two of the values lie in one interval such that it falls in one of these. Let us call these x and y then $0 < \tan^{-1} x - \tan^{-1} y < \frac{\pi}{4} \Rightarrow 0 < \frac{x-y}{1+xy} < 1$.

514. Let the two numbers be x and y. According to question $x^2 - y^2 = 100k$, such that $k \in \mathbb{N}$ and $x, y \in \{0, 1, 2, \ldots, 100\}$.

Clearly, $(x + y)(x - y) = 100k$. Since numbers are between 0 and 100 so the sum is going to be divisible by 100. Let us pick these pairs $(0, 0), (1, 99), \ldots, (49, 51)$, which is 50 pairs. So we can pick one of these 50 and the extra 50, which is not part of any pair.

The moment we choose 52nd number it has to be part of a pair, whose other member is already chosen. Hence, the result.

515. **Case I:** When each person has at least one acquaintance. In this case a person can have $1, 2, 3, \ldots, 6$ acquaintances, but we have 7 people. Hence, the result from Pigeonhole principle.

 Case II: When a person can have zero acquaintances. Assume that one person has zero acquaintances. So effectively we can remove him from group. Thus, we are left with 6 and like case I, from Pigeonhole principle we have the same result.

516. This problem is similar to the square problem of 3 units, which we solves earlier and can be solved similarly.

517. If we try to add smallest differences 11 times then we have $(1+2+3+4)\times 11+5\times 6 = 140$ The difference 5 occurs only six times because we have takes 44 differences already. Thus, if we pick 51st number then one of the differences, which we have already taken will reappear making it 12th one from Pigeonhole principle. Hence, result.

518. Let the sticks be $s_1, s_2, \ldots, s_{10}$. According to question, $1 < s_1 \leq s_2 \leq s_3 \leq \cdots s_{10} < 55$.

 Let us assume that triangles are not possible, then $2+2 \leq s_1 + s_2 \leq s_3 \Rightarrow s_3 \geq 4$. Similarly, we find $s_4, s_5, \ldots, S_{10}$. We find that $s_{10} > 55$, which is a contradiction. Hence, the result.

519. The sums are that of three row, columns, and two diagonals. Minimum value of these sums is $1+1+1=3$ and maximum is $3+3+3=9$. Thus, we have only 7 values for 8 sums.

 Hence, from Pigeonhole principle at least two sums must be equal.

520. Consider a lattice point (x, y, z). Classify it according to the parity (oddness or evenness) of its entries. There are exactly eight different ordered triples of even and odd. Thus two of the nine given points must have the same parity. Suppose that (x, y, z) and (a, b, c) are two points such that each coordinate has the same parity. Then $\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2}$ are all integers and are the coordinates of the midpoint of the line joining (x, y, z) and xm (a, b, c). This midpoint is a lattice point on the interior of the line segment joining (x, y, z) and (a, b, c).

521. Let us assume that first two numbers are 1 and 2. Then, we apply the rule $x \to x-1$, which gives next numbers as $7, 21, 107, 427, 1707$. If this is the case then triplets cannot found but such is not the case, and hence, the result.

522. We draw a ring of given radii around every points. These rings can be contained within a circle of radius 19 and area $19^2\pi$. The sum of all the rings is $650.(3^2-2^2)\pi$, which is more than $9.19^2\pi$.

 Hence, from Pigeonhole principle, the result.

523. Proceeding like previous problem we put the cyclinder around all of 14001 marbles, giving us a total area of $14001 \times \frac{25}{4}\pi$.

These would be contained in a rectangle of 125×155 which is more than 12 times the area covered by cyclinders. Hence, the result.

524. Consider any 12 points on the line. Since we have only 11 colors, at least two points must be of the same color, whose distance apart is integral.

525. mod 11 will divide the difference into 11 equivalence classes. Two number are of same equialence $a \equiv b = m \Rightarrow m|(a-n)$. So when we will choose 12 numbers two of them will fall in the same equivalence class.

In other words, the remainder can be $0, 1, 2, 3, \ldots, 10$. Out of these only 10 remainders will satisfy the criterion, which we can get by using 11 numbers. 12th will always fall in one of these 11 classes, and thus, difference will be divisible by 11.

Answers of Chapter 6
Mathematical Induction

1. Let $P(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   Putting $n = 1$, we see $P(1) = 1 = \frac{1.2.3}{6} = 1$. So $P(1)$ is true.

   Let it be true for $n = k$. Now for $n = k + 1$,

   $P(k+1) = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

   $= \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = P(k+1)$.

   Thus, by mathematical induction, the result.

2. Let $P(n) = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

   $P(1) = \frac{1}{1.2} = \frac{1}{1+1} = P(1)$, which is true for $n = 1$. Let it be true for $n = k$

   $\Rightarrow P(k) = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$.

   Adding $\frac{1}{(k+1)(k+2)}$, on both sides, we get

   $P(k+1) = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

   $= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2} = P(k+1)$.

   Hence, by mathematical induction, the result.

3. Let $P(n) = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

   $P(1) = 1^3 = 1 = \left(\frac{1.2}{2}\right)^3 = 1$, which is true for $n = 1$. Let it be true for $n = k$.

   $\Rightarrow P(k) = 1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$

   Adding $(k+1)^3$ to both sides, we get

   $P(k+1) = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$

   $= \frac{(k+1)^2[k^2+4k+4]}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2 = P(k+1)$.

   Hence, by mathematical induction, the result.

4. Let $P(n)=\frac{1}{a+d}+\frac{a}{(a+d)(a+2d)}+\cdots+\frac{a}{[a+(n-1)d](a+nd)}=\frac{n}{a+nd}$

$P(1)=\frac{1}{a+d}$ =R.H.S., which is true for $n=1$. Let it be true for $n=k$

$\Rightarrow P(k)=\frac{1}{a+d}+\frac{a}{(a+d)(a+2d)}+\cdots+\frac{a}{[a+(k-1)d](a+kd)}=\frac{k}{a+kd}$.

Adding $\frac{a}{[a+kd][a+(k+1)d]}$ to both sides, we get

$P(k+1)=\frac{1}{a+d}+\frac{a}{(a+d)(a+2d)}+\cdots+\frac{a}{[a+(k-1)d](a+kd)}+\frac{a}{[a+kd][a+(k+1)d]}$

$=\frac{k}{a+kd}+\frac{a}{[a+kd][a+(k+1)d]}=\frac{ka+k(k+1)d+a}{[a+kd][a+(k+1)d]}=\frac{(k+1)(a+kd)}{[a+kd][a+(k+1)d]}=\frac{k+1}{a+(k+1)d}=P(k+1)$

Hence, by mathematical induction, the result.

5. Let $P(n)=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}\forall n\in\mathbb{N}$

$P(1)=\frac{1}{1.2.3}=\frac{1.4}{4.2.3}=\frac{1}{2.3}$, which is true for $n=1$. Let it be true for $n=k$.

$\Rightarrow P(k)=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\cdots+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$

Adding $\frac{1}{(k+1)(k+2)(k+3)}$ to both sides, we get

$P(k+1)=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\cdots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k++3)}=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$

$=\frac{(k+1)(k+4)}{4(k+2)(k++3)}=P(k+1)$.

Hence, by mathematical induction, the result.

6. Let $P(n)=1.3+2.3^2+3.3^3+\cdots+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$

$P(1)=1.3=3=\frac{(2-1).3^2+3}{4}=3$, which is true for $n=1$. Let it be true for $n=k$.

Adding $(k+1)3^{k+1}$ to both sides, we get

$\Rightarrow P(k+1)=1.3+2.3^2+3.3^3+\cdots+k.3^k+(k+1).3^{k+1}=\frac{(2k-1)3^{k+1}+3}{4}$

$+(k+1).3^{k+1}=\frac{(2k-1).3^{k+1}+3+(4k+4).3^{k+1}}{4}$

$=\frac{(6k+3).3^{k+1}}{4}=\frac{[2(k+2)-3].3^{k+1}}{4}=P(k+1)$.

Hence, by mathematical induction, the result.

7. Let $P(n) = 1 + 4 + 7 + \cdots + 3n - 2 = \frac{n(3n-1)}{2}$.

$P(1) = 1 = \frac{1.(3-1)}{2} = 1$, which is true for $n = 1$. Let it be true for $n = k$.

$P(k) = 1 = 4 + 7 + \cdots + 3k - 2 = \frac{k(3k-1)}{2}$

Adding $3k + 1$ to both sides, we get

$\Rightarrow P(k+1) = 1 + 4 + 7 + \cdots + 3k - 2 + 3k + 1 = \frac{k(3k-1)}{2} + 3k + 1$

$= \frac{3k^2-k+6k+2}{2} = \frac{3k^2+5k+2}{2} = \frac{(k+1)(3k+2)}{2} = \frac{(k+1)[3(k+1)-1]}{2} = P(k+1)$.

Hence, by mathematical induction, the result.

8. Let $P(n) = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

$P(1) = 1^2 = 1 = \frac{(12-1)(2+1)}{3} = 1$, which is true for $n = 1$. Let it be true for $n = k$

$P(k) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$.

Adding $(2k+1)^2$ to both sides, we get

$\Rightarrow P(k+1) = 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2 = \frac{k(4k^2-1)}{3} + (2k+1)^2$

$= \frac{4k^3-k+12k^2+12k+3}{3} = \frac{4k^3+122k^2+11k+3}{3} = \frac{(k+1)(2k+1)(2k+3)}{3} = P(k+1)$.

Hence, by mathematical induction, the result.

9. Let $P(n) = 1 - 3^2 + 5^2 - 7^2 + \cdots + (4n-3)^2 - (4n-1)^2 = -8n^2$

$P(1) = 1 - 3^2 = -8 = -8.1^2$, which is true for $n = 1$. Let it be true for $n = k$.

$P(k) = 1 - 3^2 + 5^2 - 7^2 + \cdots + (4k-3)^2 - (4k-1)^2 = -8k^2$

Adding $(4k+1)^2 - (4k+2)^2$ to both sides, we get

$P(k+1) = 1 - 3^2 + 5^2 - 7^2 + \cdots + (4k-3)^2 - (4k-1)^2 + (4k+1)^2 - (4k-3)^2$

$= -8k^2 + (4k+1)^2 - (4k+3)^2 = -8k^2 - 16k - 8 = -8(k+1)^2$.

Hence, by mathematical induction, the result.

10. Let $P(n) = 3.6 + 6.9 + 9.12 + \cdots + 3n(3n+3) = 3n(n+1)(n+2)$.

$P(1) = 3.6 = 3.1(1+1)(1+2) = 3.6$, which is true for $n = 1$. Let it be true for $n = k$

$P(k) = 3.6 + 6.9 + 9.12 + \cdots + 3k(3k+3) = 3k(k+1)(k+2)$

Adding $3(k+1)[3(k+1)+3]$ to both sides, we get

$P(k+1) = 3.6 + 6.9 + 9.12 + \cdots + 3k(3k+3) + 3(k+1)[3(k+1)+3]$

$= 3k(k+1)(k+2) + 3(k+1).3(k+2) = 3(k+1)(k+2)(k+3) = P(k+1).$

Hence, by mathematical induction, the result.

11. We have to prove that $1^3 = 1, 2^3 = 3+5, 3^3 = 7+9+11, 4^3 = 13+15+17+19$.

First term contains one term, second terms contains two terms and so on. Hence, kth term will contain k terms.

Sum of no. of terms till kth term is $1+2+\cdots+k = \frac{k(k+1)}{2}$.

And, hence $(k+1)$th term will begin with $1 + \left(\frac{k(k+1)}{2}\right)2 = k^2+k+1$ and will contain $k+1$ terms with a c.d. of 2. Let it be true for $P(k)$ i.e. $t_k = k^3$.

Thus, $t_{k+1} = \frac{k+1}{2}[2(k^2+k+1) + (k+1-1)2] = \frac{k+1}{2}[2k^2+4k+2] = (k+1)^3$.

Hence, by mathematical induction, the result.

12. Let $P(n) = \sum\limits_{r=1}^{n} r.C_r^n = n.2^{n-1}$.

$P(1) = 1.C_0^1 = 1 = 1.2^{1-1} = 1$. Hence, it is true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow C_1^k + 2.C_2^k + \cdots + k.C_k^k = k.2^{k-1}$

$P(k+1) = C_1^{k+1} + 2.C_2^{k+1} + \cdots + (k+1).C_{k+1}^{k+1}$

$= (C_0^k + C_1^k) + 2(C_1^k + C_2^k) + \cdots + (k+1)(C_k^k + 0)$

$= (C_0^k + 2C_1^k + \cdots + (k+1)C_k^k) + (C_1^k + 2.C_2^k + \cdots + k.C_k^k)$

$= 2^k + k.2^{k-1} + k.2^{k-1} = (k+1).2^k = P(k+1).$

Hence, by mathematical induction, the result.

13. Let $P(n) = \sum\limits_{r=1}^{n} r(2r+1) = \frac{n(n+1)(4n+5)}{6}$.

$P(1) = 1.(2+1) = 3 = \frac{1.2.9}{6} = 3$. Hence, it it true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow \sum\limits_{r=1}^{k} r(2r+1) = \frac{k(k+1)(4k+5)}{6}$.

$P(k+1) = \sum\limits_{r=1}^{k+1} r(2r+1) = \frac{k(k+1)(4k+5)}{6} + (k+1)(2k+3)$

$= \frac{(k+1)}{6}\left[\frac{4k^2+5k+12k+18}{}\right] = \frac{(k+1)(k+2)[4(k+1)+5]}{6}$.

Hence, by mathematical induction, the result.

14. Let $P(n) = 1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

$P(1) = 1.2.3 = 6 = \frac{1.2.3.4}{4} = 6$, which is true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow 1.2.3 + 2.3.4 + 3.4.5 + \cdots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$

Adding $(k+1)(k+2)(k+3)$ to both sides, we get

$P(k+1) = 1.2.3 + 2.3.4 + 3.4.5 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$

$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$.

Hence, by mathematical induction, the result.

15. Let $P(n) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

$P(1) = \frac{1}{4} = \frac{1}{3+1} = \frac{1}{4}$, which is true for $n = 1$. Let it be true for $n = k$

$\Rightarrow P(k) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

Adding $\frac{1}{(3k+1)(3k+4)}$ to both sides, we get

$P(k+1) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \cdots + \frac{1}{(3n-2)(3n+1)} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = P(k+1)$

Hence, by mathematical induction, the result.

16. Let $P(n) = 7 + 77 + 777 + \cdots + \underbrace{7\ldots77}_{n \text{ digits}} = \frac{7}{81}(10^{n+1} - 9n - 10)$

$P(1) = 7 = \frac{7}{81}(10^2 - 9 - 10) = 7$, which is true for $n = 1$. Let it be true for $n = k$

$\Rightarrow 7 + 77 + 777 + \cdots + \underbrace{7\ldots77}_{k \text{ digits}} = \frac{7}{81}(10^{k+1} - 9k - 10)$

Adding $\underbrace{7\ldots77}_{k+1\text{~digits}}$ to both sides, we get

$P(k+1) = 7 + 77 + 777 + \cdots + \underbrace{7\ldots77}_{k \text{ digits}} + \underbrace{7\ldots77}_{k+1 \text{ digits}}$

$= \frac{7}{81}(10^{k+1} - 9k - 10) + \underbrace{7\ldots77}_{k+1 \text{ digits}} = \frac{7}{81}(10^{k+1}(10^{k+1} - 9k - 10)) + \frac{7}{9}(10^{k+1} - 1)$

$= \frac{7}{9}\left[\frac{10^{k+1}-9k-10+9.10^{k+1}-9}{9}\right] = \frac{7}{81}[10^{k+2} - 9(k+1) - 10]$.

Hence, by mathematical induction, the result.

17. Let $P(n)=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+n}=\frac{2n}{n+1}$

$P(1)=1=\frac{2.1}{1+1}=1$, which is true for $n=1$. Let it be true for $n=k$

$\Rightarrow P(k)=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+k}=\frac{2k}{k+1}$

Adding $\frac{1}{1+2+3+\cdots+(k+1)}$ to both sides,we get

$P(k+1)=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+n}+\frac{1}{1+2+3+\cdots+(k+1)}=\frac{2k}{k+1}+\frac{1}{1+2+3\cdots+(k+1)}$

$=\frac{2k}{k+1}+\frac{2}{(k+1)(k+2)}=\frac{2}{k+1}.\frac{k^2+2k+1}{k+2}=\frac{2(k+1)}{k+2}.$

Hence, by mathematical induction, the result.

18. Let $P(n)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2n+2}$

$P(1)=1-\frac{1}{2^2}=\frac{3}{4}=\frac{1+2}{2.1+2}=\frac{3}{4}$, which is true for $n=1$. Let it be true for $n=k$

$\Rightarrow P(k)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+2}$

Multiplying both sides with $1-\frac{1}{(k+2)^2}$, we get

$P(k+1)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)=\frac{k+2}{2k+2}\left(1-\frac{1}{(k+2)^2}\right)$

$=\frac{k+2}{2k+2}.\frac{k^2+4k+3}{(k+2)^2}=\frac{k+3}{2k+4}.$

Hence, by mathematical induction, the result.

19. Let $P(n)=1.3+2.3^2+\cdots+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}.$

$P(1)=3=\frac{(2.1-1).3^{1+1}+3}{4}=\frac{12}{4}=3$, which is true for $n=1$. Let it be true for $n=k$

$\Rightarrow P(k)=1.3+2.3^2+\cdots+k.3^k=\frac{(2k-1)3^{k+1}+3}{4}$

Adding $(k+1).3^{k+1}$, to both sides, we get

$P(k+1)=1.3+2.3^2+\cdots+k.3^k+(k+1).3^{k+1}=\frac{(2k-1)3^{k+1}+3}{4}+(k+1).3^{k+1}$

$=\frac{(2k-1).3^{k+1}+3+(4k+4).3^{k+1}}{4}=\frac{(6k+3).3^{k+1}=3}{4}=\frac{[2(k+1)-1].3^{k+2}+3}{4}$

Hence, by mathematical induction, the result.

20. Let $P(n) = \cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \sin\frac{n\alpha}{2}\csc\frac{\alpha}{2}\cos\frac{(n+1)\alpha}{2}$.

$P(1) = \cos\alpha = \sin\frac{\alpha}{2}\csc\frac{\alpha}{2}\cos\frac{1+1}{2}\alpha = \cos\alpha$, which is true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow P(k) = \cos\alpha + \cos 2\alpha + \cdots + \cos k\alpha = \sin\frac{k\alpha}{2}\csc\frac{\alpha}{2}\cos\frac{(k+1)\alpha}{2}$

Adding $\cos(k+1)\alpha$, to both sides, we get

$P(k+1) == \cos\alpha + \cos 2\alpha + \cdots + \cos k\alpha + \cos(k+1)\alpha = \sin\frac{k\alpha}{2}\csc\frac{\alpha}{2}\cos\frac{(k+1)\alpha}{2} + \cos(k+1)\alpha$

$= \frac{1}{2}\csc\frac{\alpha}{2}\left[2\sin\frac{k\alpha}{2}\cos\frac{(k+1)\alpha}{2} + 2\sin\frac{\alpha}{2}\cos(k+1)\alpha\right]$

$= \frac{1}{2}\csc\frac{\alpha}{2}\left[\sin\frac{(2k+1)\alpha}{2} - \sin\frac{\alpha}{2} + \sin\frac{(2k+3)\alpha}{2} - \sin\frac{(2k+1)\alpha}{2}\right]$

$= \csc\frac{\alpha}{2}\cos\frac{(k+2)\alpha}{2}\sin\frac{(k+1)\alpha}{2}$

Hence, by mathematical induction, the result.

21. Let $P(n) = \tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \cdots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$P(1) = \tan\alpha = \cot\alpha - 2\cot 2\alpha = \frac{1}{\tan\alpha} - \frac{2}{\tan 2\alpha} = \frac{1}{\tan\alpha} - \frac{1-\tan^2\alpha}{\tan\alpha} = \tan\alpha$,

which is true for $n = 1$. Let it be true for $n = k$

$\Rightarrow P(k) = \tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \cdots + 2^{k-1}\tan 2^{k-1}\alpha = \cot\alpha - 2^k\cot 2^k\alpha$

Adding $2^k\tan 2^k\alpha$, to both sides, we get

$P(k+1) = \tan\alpha + 2\tan 2\alpha + 2^2\tan 2^2\alpha + \cdots + 2^{k-1}\tan 2^{k-1}\alpha + 2^k\tan 2^k\alpha$

$= \cot\alpha - 2^k\cot 2^k\alpha + 2^k\tan 2^k\alpha = \cot\alpha - 2^k\left(\cot 2^k\alpha - \tan 2^k\alpha\right)$

$= \cot\alpha - 2^{k+1}\left(\frac{1-\tan 2^{k+1}\alpha}{2\tan 2^k\alpha}\right) = \cot\alpha - 2^{k+1}\cot 2^{k+1}\alpha$.

Hence, by mathematical induction, the result.

22. Let $P(n) = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \cdots + \tan^{-1}\frac{1}{n^2+n+1} = \tan^{-1}\frac{n}{n+2}$

$P(1) = \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{1+2} = \tan^{-1}\frac{1}{3}$, which is true for n = 1. Let it be true for $n = k$

$\Rightarrow P(k) = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \cdots + \tan^{-1}\frac{1}{k^2+k+1} = \tan^{-1}\frac{k}{k+2}$

Adding $\tan^{-1}\frac{1}{k^2+3k+3}$, to both sides, we get

$P(k+1) = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \cdots + \tan^{-1}\frac{1}{k^2+k+1} + \tan^{-1}\frac{1}{k^2+3k+3}$

$= \tan^{-1}\frac{k}{k+2} + \tan^{-1}\frac{1}{k^2+3k+3} = \tan^{-1}\frac{\frac{k}{k+2}+\frac{1}{k^2+3k+3}}{1-\frac{k}{k+2}.\frac{1}{k^2+3k+3}} = \tan^{-1}\frac{k^3+3k^2+3k+k+2}{k^3+5k^2+9k+6-k}$

$= \tan^{-1}\frac{k^3+3k^2+4k+2}{k^3+5k^2+8k+6} = \tan^{-1}\frac{k+1}{k+3}.$

Hence, by mathematical induction, the result.

23. $u_3 = u_2 + u_1$. Substituting $n = 3$ in the given formula $u_3 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right]$

$= \frac{1}{\sqrt{5}}\left[\frac{1+3\sqrt{5}+15+5\sqrt{5}}{8} - \frac{1-3\sqrt{5}+15-5\sqrt{5}}{8}\right] = \frac{1}{\sqrt{5}}\frac{16\sqrt{5}}{8}2 = u_1 + u_2.$

Thus, the relation holds for $n = 3$. Similarly, we can prove that it holds for $m = 1, 2$. Let it hold for $n = k$ and $k + 1$.

$\Rightarrow u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$ and $u_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right].$

$u_{k+2} = u_k + u_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(1 + \frac{1+\sqrt{5}}{2}\right) - \left(\frac{1+\sqrt{5}}{2}\right)^k\left(1 - \frac{1-\sqrt{5}}{2}\right)\right]$

$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}^k}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^2\right] = u_{k+2}.$

Hence, by mathematical induction, the result.

24. Let $P(n) = p^{n+1} + (p+1)^{2n-1}$.

$P(1) = p^2 + p + 1$, which is divisible by $p^2 + p + 1$, and hence, our statement is true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow P(k) = p^{k+1} + (p+1)^{2k-1}$ is divisible by $p^2 + p + 1$ i.e. $p^{k+1} + (p+1)^{2k+1} = (p^2 + p + 1)Q(p)$, where $Q(p)$, is a polynomial of p.

$P(k+1) = p^{k+2} + (p+1)^{2k+1} = p.p^{k+1} + (p+1)^2(p+1)^{2k-1}$

$\therefore P(k) = (p^2 + p + 1)^2 Q(k)$, making it divisible by $p^2 + p + 1$.

Hence, by mathematical induction, the result.

25. Let $P(n) = 2^n > 2n + 1$, where $n > 2$

$P(3) = 2^3 = 8 > 2.3 + 1 = 7$, hence, it is true for $n = 3$. Let it be true for $n = k$.

$\Rightarrow P(k) = 2^k > 2k + 1$, where $k > 2$.

$P(k+1) = 2^{k+1} = 2.2^k = 4k + 2 = 2k + 2k + 2 \because k \geq 3 \therefore 2k + 2 > 3$, making our statement true for $n = k + 1$.

hence, by mathematical induction, the result.

26. Let $P(n) = 2^n > n^3$, where $n \geq 10$.

$P(10) = 2^{10} = 1024 > 10^3 = 1000$, hence, it is true for $n = 3$. Let it be true for $n = k$.

$\Rightarrow P(k) = 2^k > k^3$

$P(k+1) = 2^{k+1} > 2.k^3 > (k+1)^3 \Rightarrow k^3 - 3k^2 - 3k - 1 > 0 \Rightarrow (k-1)^3 - 6k > 0$

Let $k = 10 + a$, where $a \geq 0$, hence, $(9+a)^3 - 60 - 6a = 669 + 183a + 27a^2 + a^3 > 0$

hence, by mathematical induction, the result.

27. Given, $n > 1$, so we start with $n = 2$. $\Rightarrow \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} > 2\tan\alpha \because 1 - \tan^2\alpha < 1$,

which is true for $n = 2$. Let the statement for $n = k$

$\Rightarrow \tan k\alpha > k\tan\alpha$. For $n = k+1$

$\tan(k+1)\alpha = \frac{\tan\alpha+\tan k\alpha}{1-\tan\alpha.\tan k\alpha} > \frac{k\tan\alpha+\tan\alpha}{1-\tan k\alpha\tan\alpha} > (k+1)\tan\alpha \because 1 - \tan\alpha\tan k\alpha < 1$

Hence, by mathematical induction, the result.

28. Let $P(n) = n^4 < 10^n \; \forall \; n \geq 2$

For $n = 1$, $P(2) = 2^4 < 10^2 \Rightarrow 16 < 100$, which is true for $n = 2$. Let $P(k)$ be true i.e. $k^4 < 10^k$.

We have to prove that $P(k+1)$ is true i.e. $(k+1)^4 < 10^{k+1}$.

Clearly, $10^{k+1} > 10k^4$. Now, $\frac{10k^4}{(k+1)^4} = 10\left(\frac{k}{k+1}\right)^4$

$\because k \geq 2 \Rightarrow \left(\frac{k}{k+1}\right)^4 \geq \frac{2^4}{3^4} \Rightarrow 10\left(\frac{k}{k+1}\right)^4 \geq 10.\frac{16}{81} > 1$.

Thus, $10^{k+1} > (k+1)^4$. Hence, by mathematical induction, the result.

29. Let $P(n) = 1^3 + 3^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$.

$P(1) = 1^3 = 1 = 1^2(2.1^1 - 1) = 1$, which is true for $n = 1$. Let it be true for $n = k$.

$P(k) = 1^3 + 3^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1)$

Adding $(2k+1)^3$, to both sides, we get

$P(k+1) = 1^3 + 3^3 + \cdots + (2k-1)^3 + (2k+1)^3 = k^2(2k^2 - 1) + (2k+1)^3$

$= 2k^4 - k^2 + 8k^2 + 12k^2 + 6k + 1 = (k+1)^2[2(k+1)^2 - 1]$.

Hence, by mathematical induction, the result.

30. Let $P(n) = 3.2^2 + 3^3.2^3 + \cdots + 3^n.2^{n+1} = \frac{12}{5}(6^n - 1)$.

$P(1) = 3.2^2 = 12 = \frac{12}{5}(6^1 - 1) = 12$, which is true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow P(k) = 3.2^2 + 3^3.2^3 + \cdots + 3^k.2^{k+1} = \frac{12}{5}(6^k - 1).$

Adding $3^{k+1}.2^{k+2}$, to both sides, we get

$P(k+1) = 3.2^2 + 3^3.2^3 + \cdots + 3^k.2^{k+1} + 3^{k+1}.2^{k+2} = \frac{12}{5}(6^k - 1) + 3^{k+1}.2^{k+2}$

$= \frac{12}{5}(6^k - 1) + 2.6^{k+1} = \frac{(2.6^{k+1} - 12 + 10.6^{k+1})}{5} = \frac{12}{5}(6^{k+1} - 6).$

Hence, by mathematical induction, the result.

31. Let $P(n) = \frac{1}{1.4} + \frac{1}{4.7} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$

$P(1) = \frac{1}{4} = \frac{1}{3.1+1} = \frac{1}{4}$, which is true for $n = 1$. Let it be true for $n = k$.

$\Rightarrow P(k) = \frac{1}{1.4} + \frac{1}{4.7} + \cdots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

Adding $\frac{1}{(3k+1)(3k+4)}$, to both sides we get

$P(k+1) = \frac{1}{1.4} + \frac{1}{4.7} + \cdots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)} = \frac{k+1}{3(k+1)+1}.$

Hence, by mathematical induction, the result.

32. Let $P(n) = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

Clearly, it is true for $n = 1$. Let it be true for $n = k$, i.e.

$P(k) = (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta.$

$P(k+1) = (\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= [\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos\theta\sin k\theta + \cos k\theta\sin\theta)]$

$= \cos(k+1)\theta + i\sin(k+1)\theta.$

Hence, by mathematical induction, the result.

33. Let $P(n) = \cos\theta.\cos 2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}.$

$P(1) = \cos\theta = \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$, which is true for $n = 1$.

Let it be true for $n = k$.

$\Rightarrow P(k) = \cos\theta.\cos 2\theta \ldots \cos 2^{k-1}\theta = \frac{\sin 2^k\theta}{2^k\sin\theta}$

Multiplying both sides with $cos 2^k\theta$, we get

$P(k+1) = \cos\theta.\cos 2\theta \ldots \cos 2^k\theta = \frac{\sin 2^k\theta}{2^k\sin\theta}.\cos 2^k\theta = \frac{2\sin 2^k\theta\cos 2^k\theta}{2^{k+1}\sin\theta} = \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta}$

Hence, by mathematical induction, the result.

34. Let $P(n) = \sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\frac{n+1}{2}\alpha$

$P(1) = \sin\alpha = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\frac{1+1}{2}\alpha = \sin\alpha$, which is true for $n = 1$.

Let it be true for $n = 1$

$\Rightarrow P(k) = \sin\alpha + \sin 2\alpha + \cdots + \sin k\alpha = \frac{\sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\frac{k+1}{2}\alpha$

Adding $\sin(k+1)\alpha$, to both sides, we get

$P(k+1) = \sin\alpha + \sin 2\alpha + \cdots + \sin k\alpha + \sin(k+1)\alpha = \frac{\sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\frac{k+1}{2}\alpha + \sin(k+1)\alpha$

$= \frac{\sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\frac{k+1}{2} + 2\sin\frac{k+1}{2}\alpha\cos\frac{k+1}{2}\alpha$

$= \sin\frac{k+1}{2}\alpha\left[\frac{\sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}} + 2\cos\frac{k+1}{2}\alpha\right] = \sin\frac{k+1}{2}\alpha\left[\frac{\sin\frac{k\alpha}{2} + 2\cos\frac{k+1}{2}\alpha\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\right]$

$= \sin\frac{k+1}{2}\alpha\left[\frac{\sin\frac{k\alpha}{2} + \sin\frac{k+2}{2}\alpha - \sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}}\right]$

$= \frac{\sin\frac{(k+1)\alpha}{2}}{\sin\frac{\alpha}{2}} \sin\frac{k+1}{2}\alpha$

Hence, by mathematical induction, the result.

35. Given, $a_1 = 1$ and $a_{n+1} = \frac{a_n}{n+1}$, $n \geq 1$. Clearly, $a_2 = \frac{a_1}{1+1} = \frac{1}{2} = \frac{1}{2!}$, which means it holds true for $n = 1$.

Let it be true for $n = k \Rightarrow a_{k+1} = \frac{1}{k!}$.

$P(k+1) = \frac{a_k}{k+1} = \frac{1}{k!(k+1)} = \frac{1}{(k+1)!}$.

Hence, by mathematical induction, the result.

36. Given, $a_1 = 1, a_2 = 5$ and $a_{n+2} = 5a_{n+1} - 6a_n$, $n \geq 1$

$\Rightarrow a_3 = 5a_2 - 6a_1 = 25 - 6 = 19 = 3^3 - 2^3 = 19$, which is true for $n = 3$.

Let it be true for $n = k$ and $n = k + 1$

$\Rightarrow a_k = 3^k - 2^k$ and $a_{k+1} = 3^{k+1} - 2^{k+1}$

$a_{k+2} = 5a_{k+1} - 6a_k = 5.3^{k+1} - 5.2^{k+1} - 6.3^k + 6.2^k = 9.3^k - 4.2^k = 3^{k+2} - 2^{k+2}$.

Hence, by mathematical induction, the result.

37. Given, $u_0 = 2, u_1 = 3$ and $u_{n+1} = 3u_n - 2u_{n-1}$ and $u_n = 2^n + 1$.

$u_2 = 3u_1 - 2u_0 = 3.3 - 2.2 = 5 = 2^2 + 1$, which is true for $n = 2$.

Let it be true for $n = k$ and $n = k + 1$

$\Rightarrow u_k = 2^k - 1$ and $u_{k+1} = 2^{k+1} - 1$

$u_{k+2} = 3.u_{k+1} - 2.u_k = 3.2^{k+1} - 3 - 2.2^k + 2 = 4.2^k + 1 = 2^{k+2} + 1.$

Hence, by mathematical induction, the result.

38. Given, $a_0 = 0, a_1 = 1$ and $a_{n+1} = 3a_n - 2a_{n-1}$, and $a_n = 2^n - 1$.

$a_2 = 3a_1 - 2a_0 = 3 = 2^2 - 1$, which is true for $n = 2$.

Let it be true for $n = k$ and $n = k + 1$

$\Rightarrow a_k = 2^k - 1$ and $a_{k+1} = 2^{k+1} - 1$

$\Rightarrow a_{k+2} = 3.2^{k+1} - 3 - 2.2^k + 2 = 4.2^k - 1 = 2^{k+2} - 1.$

Hence, by mathematical induction, the result.

39. Given, $A_1 = \cos\theta$, $A_2 = \cos 2\theta$, and for every natural number $m > 2$, $A_m = 2A_{m-1}\cos\theta - A_{m-2}$.

$A_3 = 2A_2\cos\theta - A_1 = 2\cos\theta\cos 2\theta - \cos\theta = \cos 3\theta + \cos\theta - \cos\theta = \cos 3\theta$, which is true for $n = 3$.

Let it be true for $n = k$ and $n = k + 1$

$\Rightarrow A_k = \cos k\theta$ and $A_{k+1} = \cos(k+1)\theta$

$A_{k+2} = 2\cos(k+1)\theta\cos\theta - \cos k\theta = \cos(k+2)\theta + \cos k\theta - \cos k\theta = \cos(k+2)\theta$

Hence, by mathematical induction, the result.

40. Let $P(n) = (2\cos\theta - 1)(2\cos 2\theta - 1)\cdots(2\cos 2^{n-1}\theta - 1) = \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1}$

$P(1) = 2\cos\theta - 1 = \frac{2\cos 2\theta + 1}{2\cos\theta + 1} = \frac{4\cos^2\theta - 1}{2\cos\theta + 1} = 2\cos\theta - 1$, which is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = (2\cos\theta - 1)(2\cos 2\theta - 1)\cdots(2\cos 2^{m-1}\theta - 1) = \frac{2\cos 2^m\theta + 1}{2\cos\theta + 1}$

Multiplying both sides by $2\cos 2^m\theta - 1$, we get

$P(m+1) = (2\cos\theta - 1)(2\cos 2\theta - 1)\cdots(2\cos 2^{m-1}\theta - 1)(2\cos 2^m\theta - 1) = \frac{2\cos 2^m\theta + 1}{2\cos\theta + 1}(2\cos 2^m\theta - 1)$

$= \frac{4\cos 2^m\theta - 1}{2\cos\theta} = \frac{2\cos 2^{m+1}\theta + 1}{2\cos\theta}$

Hence, by mathematical induction, the result.

41. Let $P(n) = \tan^{-1}\frac{x}{1.2+x^2} + \tan^{-1}\frac{x}{2.3+x^2} + \cdots + \tan^{-1}\frac{x}{n(n+1)+x^2} = \tan^{-1}x - \tan^{-1}\frac{x}{n+1}$, $x \in \mathbb{R}$.

$P(1) = \tan^{-1}\frac{x}{1.2+x^2} = \tan^{-1}x - \tan^{-1}\frac{x}{1+1} = \tan^{-1}\frac{2x-x}{2+x^2} = \tan^{-1}\frac{x}{1.2+x^2}$, which is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(n) = \tan^{-1}\frac{x}{1.2+x^2} + \tan^{-1}\frac{x}{2.3+x^2} + \cdots + \tan^{-1}\frac{x}{m(m+1)+x^2} = \tan^{-1}x - \tan^{-1}\frac{x}{m+1}$

Adding $\tan^{-1}\frac{x}{(m+1)(m+2)+x^2}$, to both sides, we get

$P(m+1) = \tan^{-1}\frac{x}{1.2+x^2} + \tan^{-1}\frac{x}{2.3+x^2} + \cdots + \tan^{-1}\frac{x}{m(m+1)+x^2} + \tan^{-1}\frac{x}{(m+1)(m+2)+x^2} = \tan^{-1}x - \tan^{-1}\frac{x}{m+1} + \tan^{-1}\frac{x}{(m+1)(m+2)+x^2}$

$= \tan^{-1}x - \tan^{-1}\frac{x}{m+1} + \tan^{-1}\frac{x}{m+1} - \tan^{-1}\frac{x}{m+2} = \tan^{-1}x - \tan^{-1}\frac{x}{m+2}$.

Hence, by mathematical induction, the result.

42. Let $P(n) = 3 + 33 + \cdots + \underbrace{33\ldots3}_{n\text{ digits}} = \frac{10^{n+1}-9n-10}{27}$.

$P(1) = 3 = \frac{10^2-9-19}{27} = \frac{81}{27} = 3$, which is true for $n = 1$.

Let it be true for $n = m$

$P(n) = 3 + 33 + \cdots + \underbrace{33\ldots3}_{m\text{ digits}} = \frac{10^{m+1}-9n-10}{27}$

Adding $\underbrace{3\ldots33}_{m+1\text{ digits}}$, to both sides, we get

$P(m+1) = 3 + 33 + \cdots + \underbrace{33\ldots3}_{m\text{ digits}} + \underbrace{33\ldots3}_{m+1\text{ digits}} = \frac{10^{m+1}-9n-10}{27} + \underbrace{33\ldots3}_{m+1\text{ digits}}$

$= \frac{10^{m+1}-9m-10}{27} + +\frac{3}{9}(10^{m+1}-1) = \frac{10^{m+1}-9m-10+9.10^{m+1}-9}{27}$

$= \frac{10^{m+2}-9m-19}{243} = \frac{10^{m+2}-9(m+1)-10}{27}$.

Hence, by mathematical induction, the result.

43. Let $P(n) = \int_0^\pi \frac{\sin(2n+1)x}{\sin x}dx = \pi$.

$P(1) = \int_0^\pi \frac{\sin 3x}{\sin x}dx = \int_0^\pi \frac{3\sin x - 4\sin^3 x}{\sin x}dx$

$= 3\int_0^\pi dx - 4\int_0^\pi \sin^2 x dx = [3x]_0^\pi - 2\int_0^\pi (1-\cos 2x)\,dx$

$= 3\pi - [2x]_0^\pi + [\sin 2x]_0^\pi = \pi + 0 = \pi$, which is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = \int_0^\pi \frac{\sin(2m+1)x}{\sin x} dx = \pi$

Now $P(m+1) - P(m) = \int_0^\pi \frac{\sin(2m+3)x - \sin(2m+1)d}{\sin x} dx = \int_0^\pi \frac{2\cos(2m+2)x \sin x}{\sin x} dx$

$= \int_0^\pi \cos(2m+2)x dx = 0 \Rightarrow P(m+1) = P(m) = \pi$.

Hence, by mathematical induction, the result.

44. Let $P(n) = \int_0^\pi \frac{\sin^2 nx}{\sin^2 x} dx = n\pi$.

$P(1) = \int_0 \frac{\sin^2 x}{\sin^2 x} dx = [x]_0^\pi = \pi$, which is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = \int_0^\pi \frac{\sin^2 mx}{\sin^2 x} dx = m\pi$

$P(m+1) - P(m) = \int_0^\pi \frac{\sin^2(m+1)x - \sin^2 mx}{\sin^2 x} dx = \int_0^\pi \frac{\cos 2mx - \cos(2m+2)x}{\sin^2 x} dx$

$= \int_0^\pi \frac{2\sin(2m+1)\sin x}{\sin^2 x} dx = \pi$(we have proved this in previous problem)

$\Rightarrow P(m+1) = (m+1)\pi$(because $P(m) = m\pi$).

Hence, by mathematical induction, the result.

45. Let $P(n) = \tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \cdots + \tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1}(n+1) - \frac{\pi}{4}$.

$P(1) = \tan^{-1} \frac{1}{1+2} = \tan^{-1} 2 - \tan^{-1} 1 = \tan^{-1}(1+1) - \frac{\pi}{4}$, which is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = \tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \cdots + \tan^{-1} \frac{1}{1+m+m^2} = \tan^{-1}(m+1) - \frac{\pi}{4}$.

Adding $\tan^{-1} \frac{1}{1+(m+1)+(m+1)^2}$, to both sides, we get

$P(m+1) = \tan^{-1} \frac{1}{1+1+1^2} + \tan^{-1} \frac{1}{1+2+2^2} + \cdots + \tan^{-1} \frac{1}{1+m+m^2} + \tan^{-1} \frac{1}{1+(m+1)+(m+1)^2} = \tan^{-1}(m+1) - \frac{\pi}{4} + \tan^{-1} \frac{1}{1+(m+1)+(m+1)^2}$

$= \tan^{-1}(m+1) - \frac{\pi}{4} + \tan^{-1}(m+2) - \tan^{-1}(m+1) = \tan^{-1}(m+1) - \frac{\pi}{4}$.

Hence, by mathematical induction, the result.

46. Let $P(n) = n(n+1)(n+5)$. $P(1) = 1.2.6 = 12$, which is divisible by 6 i.e. the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = m(m+1)(m+5) = m^3 + 6m^2 + 5m = 6k$,where $k \in \mathbb{N}$.

$P(m+1) = (m+1)(m+2)(m+6) = m^3+9m^2+15m+12 = 6k+3m^2+15m+12 = 6k+3(m+1)(m+4)$

Clearly, $3(m+1)(m+4) = 6q$, where $q \in \mathbb{N}$.

Hence, by mathematical induction, the result.

47. Let $P(n) = n^3 + (n+1)^3 + (n+2)^3 .P(1) = 1^3 + 2^3 + 3^3 = 36$, which is divisible by 9 i.e. the statement is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = m^3 + (m+1)^3 + (m+2)^3 = 9k$, where $k \in \mathbb{N}$.

$P(m+1) = (m+1)^3 + (m+2)^3 + (m+3)^3 = 9k + 9m^2 + 27m + 27 = 9(k + m^2 + 3m + 3)$.

Hence, by mathematical induction, the result.

48. Let $P(n) = n(n^2 + 20)$, where n is even where $n \in \mathbb{P}$.

$P(2) = 2.24 = 48$, which is divisible by 48 i.e. the statement is true for $n = 2$.

Let it be true for $n = 2m$

$\Rightarrow P(2m) = 2m(4m^2 + 20) = 8m(m^2 + 5) = 48k$, where $k \in \mathbb{P}$.

$P(2m+2) = (2m+2)\left[4m^2 + 8m + 24\right] = 8(m+1)(m^2 + 2m + 6) = 8(m^3 + 2m^2 + 6m + m^2 + 2m + 6)$

$= 8(m^3 + 3m^2 + 8m + 6) = 8(48k + 3m^2 + 3m + 6) = 24(16k + m^2 + m + 2)$.

Now, we have to prove that $m^2 + m + 2$ is divisible by 2.

Let $Q(m) = m^2 + m + 2$. $Q(1) = 4$, which is divisible by 2. Let it be true for $m = t$.

$\Rightarrow Q(t) = t^2 + t + 2 = 2x$, where $x \in \mathbb{Z}$.

$Q(t+1) = t^2 + 2t + t + 4 = 2x + 2t + 2$, which is divisible by 2.

Hence, by mathematical induction, the result.

49. Let $P(n) = 4^n - 3n - 1$. $P(1) = 4 - 3 - 1 = 0$, which is divisible by 9, so it is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = 4^m - 3m - 1 = 9k$, where $k \in \mathbb{N}$.

$P(m+1) = 4^{m+1} - 3(m+1) - 1 = 4.4^m - 3m - 4 = 4(9k + 3m + 1) - 3m - 4 = 36k + 9m$, which is divisible by 9.

Hence, by mathematical induction, the result.

50. Let $P(n) = 3^{2n} - 1$. $P(1) = 3^2 - 1 = 8$, which is divisible by 8. So the statement is true for $n = 1$.

    Let it be true for $n = m$

    $\Rightarrow P(m) = 3^{2m} - 1 = 8k$, where $k \in \mathbb{N}$.

    $P(m+1) = 3^{2m+2} - 1 = 9.3^{2m} - 1 = 9(8k+1) - 1 = 72k + 8$, which is divisible by 8.

    Hence, by mathematical induction, the result.

51. Let $P(n) = 5.2^{3n-2} + 3^{3n-1}$. $P(1) = 5.2 + 3^2 = 19$, which is divisible by 19, so the statement is true for $n = 1$.

    Let it be true for $n = m$

    $\Rightarrow P(m) = 5.2^{3m-2} + 3^{3m-1} = 19k$, where $k \in \mathbb{N}$

    $P(m+1) = 5.2^{3m+1} + 3^{3m+2} = 40.2^{3m-2} + 27.3^{3m-1} = 8(5.2^{3m-2} + 3^{3m-1}) + 19.3^{3m-1} = 8.19k + 19.3^{3m-1}$,

    which is divisible by 19.

    Hence, by mathematical induction, the result.

52. Let $P(n) = 7^{2n} + 2^{3n-3}.3^{n-1}$. $P(1) = 7^2 + 2^0 3^0 = 50$, which is divisible by 25, so the statement is true for $n = 1$.

    Let it be true for $n = m$

    $\Rightarrow P(m) = 7^{2m} + 2^{3m-3}.3^{m-1} = 25k$, where $k \in \mathbb{N}$.

    $P(m+1) = 7^{2m+2} + 2^{3m}.3^m = 49.7^{2m} + 24.2^{3m-3}.3^{m-1} = 25.7^{2m} + 24.25k$, which is divisible by 25.

    Hence, by mathematical induction, the result.

53. Let $P(n) = 10^n + 3.4^{n+2} + 5$. $P(1) = 10 + 3.4^3 + 5 = 207$, which is divisible by 9, so the statement is true for $n = 1$.

    Let it be true for $n = m$

    $\Rightarrow P(m) = 10^m + 3.4^{m+2} + 5 = 9k$, where $k \in \mathbb{N}$.

    $P(m+1) = 10^{m+1} + 3.4^{m+3} + 5 = 10.10^m + 12.4^{m+2} + 5 = 10.9k + 9.10^m + 9.4^{m+2}$, which is divisible by 9.

    Hence, by mathematical induction, the result.

54. Let $P(n) = 3^{4n+2} + 5^{2n+1}$. $P(1) = 3^6 + 5^3 = 729 + 125 = 854$, which is divisible by 14, so the statement is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = 3^{4m+2} + 5^{2m+1} = 14k$, where $k \in \mathbb{N}$.

$P(m+1) = 3^{4m+6} + 5^{2m+3} = 81.3^{4m+2} + 25.5^{2m+1} = 25.14k + 56.3^{4m+2}$, which is divisible by 14.

Hence, by mathematical induction, the result.

55. Let $P(n) = 3^{2n+2} - 8n - 9$. $P(1) = 3^4 - 8.1 - 9 = 64$, which is divisible by 64, so the statement is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = 3^{2m+2} - 8m - 9 = 9^{m+1} - 8m - 9 = 9.9^m - 8m - 9 = 64k$, where $k \in \mathbb{N}$.

$P(m+1) = 3^{2m+4} - 8(m+1) - 9 = 9^{m+2} - 8m - 17 = 81.9^m - 8m - 17 = 64k + 72.9^m - 8 = 64k + 8(9^{m+1} - 1)$.

Now we will prove that $9^{m+1} - 1$ is divisible by 8. Let $Q(n) = 9^{n+1} - 1$. $Q(1) = 80$, which is divisible by 8.

Let it be true for $n = r$.

$Q(r) = 9^{r+1} - 1 = 8s$, where $s \in \mathbb{N}$.

$Q(r+1) = 9.9^{r+1} - 1 = 8.8^{r+1} + 8s$, which is divisible by 8.

Hence, by mathematical induction, the result.

56. Let $P(n) = n^7 - n$. $P(1) = 1^7 - 1 = 0$, which is divisible by 7, so the statement is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = m^7 - m = 7k$, where $k \in \mathbb{N}$.

$P(m+1) = (m+1)^7 - (m+1) = m^7 + C^7_1 m^6 + C^7_2 m^5 + C^7_3 m^4 + C^7_4 m^3 + C^7_5 m^2 + C^7_6 m + 1 - m - 1$

$= m^7 - m + C^7_1 m^6 + C^7_2 m^5 + C^7_3 m^4 + C^7_4 m^3 + C^7_5 m^2 + +C^7_6 m = 7k + 7s$, where $s \in \mathbb{N}$, which is divisible by 7.

Hence, by mathematical induction, the result.

57. Let $P(n) = 11^{n+2} + 12^{2n+1}$. $P(1) = 11^3 + 12^3 = 1331 + 1728 = 3059$, which is divisible by 133, so the statement is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = 11^{m+2} + 12^{2m+1} = 133k$, where $k \in \mathbb{N}$.

$P(m+1) = 11.11^{m+2} + 144.12^{2m+1} = 11.133k + 133.12^{2m+1}$, which is divisible by 133.

Hence, by mathematical induction, the result.

58. Let $P(n) = 10^{2n-1} + 1$. $P(1) = 10^{2-1} + 1 = 11$, which is divisible by 11, so the statement is true for $n = 1$.

Let it be true for $n = m$

$\Rightarrow P(m) = 10^{2m-1} + 1 = 11k$, where $k \in \mathbb{N}$.

$P(m+1) = 10^{2m+1} + 1 = 100.10^{2m-1} + 1 = 99.10^{2m-1} + 11k$, which is divisible by 11.

Hence, by mathematical induction, the result.

59. Let $P(n) = 7^n - 3^n$. $P(1) = 7 - 3 = 4$, which is divisible by 4, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 7^m - 3^m = 4k$, where $k \in \mathbb{N}$.

$P(m+1) = 7^{m+1} - 3^{m+1} = 7.7^m - 3.3^m = 4.7^m + 3.4k$, which is divisible by 4.

Hence, by mathematical induction, the result.

60. Let $P(n) = 2.7^n + 3.5^n - 5$. $P(1) = 2.7 + 3.5 - 5 = 24$, which is divisible by 24, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$P(m) = 2.7^m + 3.5^m - 5 = 24k$, where $k \in \mathbb{N}$.

$P(m+1) = 2.7^{m+1} + 3.5^{m+1} - 5 = 14.7^m + 15.5^m - 5 = 4.7^m + 5.24k + 20 = 4(7^m + 5) + 120k$.

Now we will prove that $7^m + 5$ is divisible by 6.

$Q(n) = 7^n + 5$. $Q(1) = 7 + 5 = 12$, which is divisible by 6, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow Q(m) = 7^m + 5 = 6s$, where $s \in \mathbb{N}$.

$Q(m+1) = 7^{m+1} + 5 = 6.7^m + 6s$, which is divisible by 6.

Hence, by mathematical induction, the result.

61. Let $P(n) = 3^{2n} - 1$. $P(1) = 3^2 - 1 = 8$, which is divisible by 8, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 3^{2m} - 1 = 8k$, where $k \in \mathbb{N}$

$P(m+1) = 3^{2m+2} - 1 = 9.3^{2m} - 1 = 8.3^{2m} + 8k$, which is divisible by 8.

Hence, by mathematical induction, the result.

62. Let $P(n) = 10^n + 3.4^{n+2} + 5$. $P(1) = 10 + 3.4^3 + 5 = 297$, which is divisible by 9, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 10^m + 3.4^{m+2} + 5 = 9k$, where $k \in \mathbb{N}$.

$P(m+1) = 10^{m+1} + 3.4^{m+3} + 5 = 9.10^m + 9.4^{m+3} + 9k$, which is divisible by 9.

Hence, by mathematical induction, the result.

63. Let $P(n) = 5^{2n+1} + 2^{n+4} + 2^{n+1}$. $P(1) = 5^3 + 2^5 + 2^2 = 125 + 32 + 4 = 161$, which is divisible by 23, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 5^{2m+1} + 2^{m+4} + 2^{m+1} = 23k$, where $k \in \mathbb{N}$.

$P(m+1) = 5^{2m+3} + 2^{m+5} + 2^{m+2} = 25.5^{2m+1} + 2.2^{m+4} + 2.2^{m+1} = 23.5^{2m+1} + 2.23k$, which is divisible by 23.

Hence, by mathematical induction, the result.

64. Let $P(n) = 7^{2n} - 1$. $P(1) = 7^2 - 1 = 48$, which is divisible by 8, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 7^{2m} - 1 = 8k$, where $k \in \mathbb{N}$.

$P(m+1) = 7^{2m+2} - 1 = 49.7^{2m} - 1 = 48.7^{2m} + 48k$, which is divisible by 8.

Hence, by mathematical induction, the result.

65. Let $P(n) = 3^{2n+2} - 8n - 9$. $P(1) = 3^4 - 8 - 9 = 64$, which is divisible by 8, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 3^{2m+2} - 8m - 9 = 8k$, where $k \in \mathbb{N}$.

$P(m+1) = 3^{2m+4} - 8(m+1) - 9 = 8.3^{2m+2} + 8k - 8$, which is divisible by 8.

Hence, by mathematical induction, the result.

66. Let $P(n) = 41^n - 14^n$. $P(1) = 41 - 14 = 27$, which is divisible by 27, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 41^m - 14^m = 27k$, where $k \in \mathbb{N}$.

$P(m+1) = 41^{m+1} - 14^{m+1} = 27.41^m + 27k$, which is divisible by 27.

Hence, by mathematical induction, the result.

67. Let $P(n) = 15^{2n-1} + 1$. $P(1) = 15 + 1 = 16$, which is divisible by 16, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 15^{2m-1} + 1 = 16k$, where $k \in \mathbb{N}$.

$P(m+1) = 15^{2m+1} + 1 = 225.15^{2m-1} + 1 = 224.15^{2m-1} + 16$, which is divisible by 16.

Hence, by mathematical induction, the result.

68. Let $P(n) = 5^{2n+1} + 3^{n+2}.2^{n-1}$. $P(1) = 5^3 + 3^3.2^0 = 125 + 27 = 152$, which is divisible by 19, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 5^{2m+1} + 3^{m+2}.2^{m-1} = 19k$, where $k \in \mathbb{N}$.

$P(m+1) = 5^{2m+3} + 3^{m+3}.2^m = 25.5^{2m+1} + 6.3^{m+2}.2^{m-1} = 19.5^{2m-1} + 6.19k$, which is divisible by 19.

Hence, by mathematical induction, the result.

69. Let $P(n) = 10^n + 3.4^{n+2} + 5$. $P(1) = 10 + 3.4^3 + 5 = 10 + 192 + 5 = 207$, which is divisible by 9, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 10^m + 3.4^{m+2} + 5 = 9k$, where $k \in \mathbb{N}$.

$P(m+1) = 10^{m+1} + 3.4^{m+3} + 5 = 10.10^m + 12.4^{m+2} + 5 = 6.10^m + 4.9k - 15 = 3(2.10^m - 5) + 4.9k$.

Now we will prove that $2.10^m - 5$ is divisible by 3.

Let $Q(n) = 2.10^n - 5$. $Q(1) = 20 - 5 = 15$, which is divisible by 3, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow Q(m) = 2.10^m - 5 = 3s$, where $s \in \mathbb{N}$.

$Q(m+1) = 2.10^{m+1} - 5 = 20.10^m - 5 = 18.10^m + 3s$, which is divisible by 3.

Hence, by mathematical induction, the result.

70. Let $P(n) = 9^n - 8n - 1$. $P(1) = 9 - 8 - 1 = 0$, which is divisible by 64, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 9^m - 8m - 1 = 64k$, where $k \in \mathbb{N}$.

$P(m+1) = 9^{m+1} - 8(m+1) - 1 = 9.9^m - 8m - 9 = 9(9^m - 8m - 1) + 64m = 9.64k + 64m$, which is divisible by 64.

Hence, by mathematical induction, the result.

71. Let $P(n) = n^3 + 3n^2 + 5n + 3$. $P(1) = 1 + 3 + 5 + 3 = 12$, which is divisible by 3, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = m^3 + 3m^2 + 5m + 3 = 3k$, where $k \in \mathbb{N}$.

$P(m+1) = (m+1)^3 + 3(m+1)^2 + 5(m+1) + 3 = m^3 + 3m^2 + 5m + 3 + 3m^2 + 3m + 1 + 6m + 3m + 5 = 3k + 3m^2 + 9m + 6$, which is divisible by 3.

Hence, by mathematical induction, the result.

72. $P(n) = (n+1)(n+2)(n+3)(n+4)(n+5)$. $P(1) = 2.3.4.5.6 = 720$, which is divisible by 120, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = (m+1)(m+2)(m+3)(m+4)(m+5) = 120k$, where $k \in \mathbb{N}$.

$P(m+1) - P(m) = 5(m+2)(m+3)(m+4)(m+5)$

Among the four consecutive numbers $(m+2)(m+3)(m+4)(m+5)$, there has to be at least one multiples of $2, 3$ and 4 each. Thus, $P(m+1)$ is divisible by 120.

Hence, by mathematical induction, the result.

73. Let $P(n) = n^5 - n$. $P(1) = 1 - 1 = 0$, which is divisible by 5, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = m^5 - m = 5k$, where $k \in \mathbb{N}$.

$P(m+1) = (m+1)^5 - (m+1) = m^5 - m + C_1^{m4} + C_2^5 m^3 + C_3^5 m^2 + C_4^5 m$, which is divisible by 5.

Hence, by mathematical induction, the result.

74. Let $P(n) = (1+x)^n - nx - 1$. $P(1) = 1 + x - x - 1 = 0$, which is divisible by x^2, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = (1+x)^m - mx - 1 = Q(x)x^2$, where $Q(x)$ is a polynomial in x.

$P(m+1) = (1+x)^{m+1} - (m+1)x - 1 = (1+x)(1+x)^m - mx - x - 1 = Q(x)x^2 + x(1+x)^m - x$

$= Q(x)x^2 + x[C_0^m + C_1^m x + \cdots + C_m^m x^m] - x$, which is divisible by x^2.

Hence, by mathematical induction, the result.

75. Let $P(n) = n(n^2-1)$. $P(1) = 0$, which is divisible by 24, so the statement is true for $n = 1$.

Llet it be true for $n = m$, where $m = 2k+1, \forall k \in N$.

$\Rightarrow P(m) = (2k+1)(4k^2+4k) = 4k(2k+1)(k+1) = 4k(2k^2+3k+1) = 24s$, where $s \in \mathbb{N}$.

$P(m+2) = (2k+3)(4k^2+12k+8) = 8k^3+36k^2+52k+24 = 24s+24k^2+48k+24$, which is divisible by 24.

Hence, by mathematical induction, the result.

76. Let $P(n) = n(n^2+20)$. $P(2) = 2.24 = 48$, which is divisible by 48, so the statement is true for $n = 1$.

Let it be true for $n = 2m$, where $m \in \mathbb{N}$.

$\Rightarrow P(2m) = 8m(m^2+5) = 48s$, where $s \in \mathbb{N}$.

$P(2m+2) = 8(m+1)(m^2+2m+6) = 8(m^3+3m^2+8m+6) = 48s + 24m^2 + 24m + 48$.

Now we will prove that $Q(n) = n^2 + n = n(n+1)$ is divisible by 2. We can prove this by induction or by just simple observation product of two consecutive integers is always divisible by 2.

Hence, by mathematical induction, the result.

77. Let $P(n) = 2^{2n} + 1$ and $Q(n) = 2^{2n} - 1$. $P(1) = 2^2 + 1 = 5$ and $Q(2) = 2^4 - 1 = 15$. Both are divisible by 5. So statements are true for $n = 1, 2$ respectively.

Let they are true for $n = 2m, 2m+1$ where $\in \mathbb{N}$.

$\Rightarrow P(2m+1) = 2^{4m+2} + 1 = 5k$ and $Q(2m) = 2^{4m} - 1 = 5l$, where $l, m \in \mathbb{N}$.

$P(2m+3) = 2^{4m+6} + 1 = 15.2^{4m+2} + 5k$, which is divisible by 5.

$Q(2m+2) = 2^{4m+4} - 1 = 15.2^{4m} + 5l$, which is also divisible by 5.

Hence, by mathematical induction, the result.

78. Let $P(n) = 5^{2n} + 1$. $P(1) = 5^2 + 1 = 26$, which is divisible by 13, so the statement is true for $n = 1$.

Let it be true for $n = 2m + 1$, where $m \in \mathbb{N}$.

$\Rightarrow P(2m + 1) = 5^{4m+2} + 1 = 13k$, where $k \in \mathbb{N}$.

$P(2m + 3) = 5^{4m+6} + 1 = 625.5^{4m+2} + 1 = 624.5^{4m+2} + 13k$, which is divisible by 13.

Hence, by mathematical induction, the result.

$5^{99} = 5.5^{98} = 5.5^{2.49} = 5(5^{98} + 1) - 5$. This will leave the remainder $13 - 5 = 8$, when divided by 13.

79. Let $P(n) = 4.6^n + 5^{n+1}$. $P(1) = 4.6 + 5^2 = 49$, which leaves remainder 9 when divided by 20, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 4.6^m + 5^{m+1} = 20k + 9$, where $k \in \mathbb{N}$.

$P(m + 1) = 4.6^{m+1} + 5^{m+2} = 24.6^m + 5.5^{m+1} = 4.6^m + 5.20k + 45$.

Now we will prove that $4.6^m + 45$ will leave remainder 9 when divided by 20.

Let $Q(n) = 4.6^n + 45$. $Q(1) = 49$, which leaves remainder 9 when divided by 20, so the statement is true for $n = 1$.

Let it be true for $n = r$.

$Q(r) = 4.6^m + 45 = 20s + 9$, where $s \in \mathbb{N}$.

$Q(r+1) = 4.6^{m+1} + 45 = 24.6^m + 45 = 20.6^m + 20s + 9$, which will leave remainder 9, when divided by 20.

Hence, by mathematical induction, the result.

80. Let $P(n) = 3^n + 8^n$. $P(1) = 3 + 8 = 11$, which is not divisiible by 8, so the statement is true for $n = 1$. Quick observation tells us that 3^m will be odd, while 8^n will be even so sum would be odd, which will not be divisible by 8. We will prove this by induction.

Let it be true for $n = m$.

$\Rightarrow P(m) = 3^m + 8^m = 8k + s$, where k,$s \in \mathbb{N}$ such that $s \in \{1,2,3, \dots, 7\}$.

$P(m + 1) = 3.3^m + 8.8^m = 8k + s + 5.8^m$, which will leave remainder s when divided by 8.

Hence, by mathematical induction, the result.

81. Let $P(n) = 2^{2^n} + 1$. If this has last digit as 7 then it will leave remainder 7 when divided by 10. $P(2) = 2^4 + 1 = 17$, which leaves remainder 7, so the statement is true for $n = 2$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 2^{2^m} + 1 = 10k + 7$, where $k \in \mathbb{N}$.

$P(m+1) = 2^{2^{m+1}} + 1 = 2^{2.2^m} + 1 = (10k+6)^2 + 1 = 100k^2 + 120k + 37$, which leaves remainder 7, when divided by 10.

Hence, by mathematical induction, the result.

82. Let $P(n) = \frac{n^3}{3} + n^2 + \frac{5}{3}n + 1$. $P(1) = \frac{1}{3} + 1 + \frac{5}{3} + 1 = 4$, which is a natural number, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = \frac{m^3}{3} + m^2 + \frac{5m}{3} + 1 = k$, where $k \in \mathbb{N}$.

$\Rightarrow P(m+1) = \frac{(m+1)^3}{3} + (m+1)^2 + \frac{5m+5}{3} + 1 = \frac{m^3}{3} + m^2 + \frac{5m}{3} + 1 + m^2 + m + \frac{1}{3} + 2m + 1 + \frac{5}{3} + 1$

$= k + m^2 + 3m + 4$, which is a ntural number.

Hence, by mathematical induction, the result.

83. Let $P(n) = x^n + y^n$. $P(1) = x + y$, which is divisible by $x + y$, so the statement is true for $n = 1$. Similarly, $x^3 + y^3$ is divisible by $x + y$.

Let it be true for $n = 2m - 1, 2m + 1$, where $m \in \mathbb{N}$

$\Rightarrow P(2m-1) = x^{2m-1} + y^{2m-1} = f(x, y)(x + y)$, where $f(x, y)$ is a polynomial in x, y.

$\Rightarrow P(2m+1) = x^{2m+1} + y^{2m+1} = g(x, y)(x + y)$, where $g(x, y)$ is a polynomial in x, y.

$P(2m+3) = x^{2m+3} + y^{2m+3} = (x^2 + y^2)(x^{2m+1} + y^{2m+1}) - x^2y^2(x^{2m-1} + y^{2m-1}) = (x^2+y^2)(x+y)g(x, y) - x^2y^2(x+y)f(x, y)$, which is divisible by $x + y$.

Hence, by mathematical induction, the result.

84. Let $P(n) = x^n - y^n$. $P(1) = x - y$, which is divisible by $x - y$, so the statement is true for $n = 1$. Similarly, it is true for $n = 2$.

Let it be true for $n = m, m - 1$.

$P(m-1) = x^{m-1} - y^{m-1} = f(x, y)(x - y)$, where $f(x, y)$ is a polynomial in x and y.

$\Rightarrow P(m) = x^m - y^m = g(x, y)(x - y)$, where $g(x, y)$ is a polynomial in x and y.

$P(m+1) = x^{m+1} - y^{m+1} = (x^m - y^m)(x+y) - xy(x^{m-1} - y^{m-1})$, which is divisible by $x - y$.

Hence, by mathematical induction, the result.

85. Let $P(n) = x(x^{n-1} - na^{n-1}) + a^n(n-1)$. $P(2) = x(x-2a) + a^2 = (x-a)^2$, which is divisible by $(x-a)^2$, so the statement is true for $n = 2$.

Let it be true for $n = m$

$\Rightarrow P(m) = x(x^{m-1} - ma^{m-1}) + a^m(m-1) = f(x,y)(x-a)^2$, where $f(x,y)$ is a polynomial in x, y.

$P(m+1) = x(x^m - (m+1)a^m) + ma^{m+1} = x.f(x,y)(x-a)^2 + ma^{m-1}(x-a)^2$, which is divisible by $(x-a)^2$.

Hence, by mathematical induction, the result.

86. Let $P(n) = \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$. $P(1) = \frac{1}{5} + \frac{1}{3} + \frac{7}{15} = 1$, which is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = \frac{m^5}{5} + \frac{m^3}{3} + \frac{7m}{15} = k$, where $k \in \mathbb{N}$.

$P(m+1) = \frac{(m+1)^5}{5} + \frac{(m+1)^3}{3} + \frac{7m+7}{15} = \frac{m^5}{5} + \frac{m^3}{3} + \frac{7m}{15} + m^4 + 2m^3 + 2m^2 + m + m^2 + m + \frac{1}{5} + \frac{1}{3} + \frac{1}{15} = k + 1 + m^4 + 2m^3 + 3m^2 + 2m$, which is a natural number.

Hence, by mathematical induction, the result.

87. Let $P(n) = \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$. $P(1) = \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = 1$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = \frac{m^7}{7} + \frac{m^5}{5} + \frac{2m^3}{3} - \frac{m}{105} = k$, where $k \in \mathbb{N}$.

$P(m+1) = \frac{(m+1)^7}{7} + \frac{(m+1)^5}{5} + \frac{2(m+1)^3}{3} - \frac{m+1}{105} = \frac{m^7}{7} + \frac{m^5}{5} + \frac{2m^3}{3} - \frac{m}{105} + \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} + \frac{C^7_1m^6 + C^7_2m^5 + C^7_3m^4 + C^7_4m^3 + C^7_5m^2 + C^7_6m}{7} + \frac{C^5_1m^4 + C^5_2m^3 + C^5_3m^2 + C^5_4m}{5} + \frac{2C^3_1m^2 + 2C^3_2m}{3}$

$= k + 1 + \frac{C^7_1m^6 + C^7_2m^5 + C^7_3m^4 + C^7_4m^3 + C^7_5m^2 + C^7_6m}{7} + \frac{C^5_1m^4 + C^5_2m^3 + C^5_3m^2 + C^5_4m}{5} + \frac{2C^3_1m^2 + 2C^3_2m}{3}$, which is a natural numebr.

Hence, by mathematical induction, the result.

88. Let $P(n) = 2^n > n^2, n \geq 5$. $P(5) = 32 > 25$, so the statement is true for $n = 5$.

Let it be true for $n = m$, where $m \in \mathbb{N}$, and $m \geq 5$.

$\Rightarrow P(m) = 2^m > m^2$.

$P(m+1) = 2^{m+1} > (m+1)^2 \Rightarrow 2.2^m > m^2 + 2m + 1$.

Now, $2m^2 - (m^2 + 2m + 1) = m^2 - 2m - 1 = (m-1)^2 - 2$. Let $k = 5 + a$, where $a \geq 0$.

$(4+a)^2 - 2 = a^2 + 8a + 14 \geq 0$

$\Rightarrow 2m^2 > (m+1)^2 \Rightarrow 2(m+1)^2 > (m+1)^2$.

Hence, by mathematical induction, the result.

89. Let $P(n) = 1 + 2 + \cdots + n \leq \frac{1}{8}(2n+1)^2$. $P(1) = 1 \leq \frac{9}{8}$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 1 + 2 + \cdots + m \leq \frac{1}{8}(2m+1)^2$

$P(m+1) = 1 + 2 + \cdots + m + (m+1) \leq \frac{1}{8}(2m+1)^2 + m + 1 = \frac{1}{8}(2m+3)^2$, which is true.

Hence, by mathematical induction, the result.

90. Let $P(n) = n^n < (n!)^2$, $n > 2$. $P(3) = 3^3 < (3!)^2 = 27 < 36$, so the statement is true for $n = 3$.

Let it be true for $n = m$, where $m \in \mathbb{N}$, and $m > 2$.

$\Rightarrow P(m) = m^m < (m!)^2$.

$P(m+1) = (m+1)^{m+1} < [(m+1)!]^2$. Dividing $P(m+1)$ by $P(m)$, we get

$\left(\frac{(m+1)^{m+1}}{m^m}\right) < \left(\frac{(m+1)!}{m!}\right)^2 = \frac{m}{(m+1)^2}\left(\frac{m+1}{m}\right)^{m+1} < 1 \Rightarrow (m+1)^{m+1} < [(m+1)!]^2$.

Hence, by mathematical induction, the result.

91. Let $P(n) = n! > 2^n$, $n > 3$. $P(4) = 24 > 16$, so the statement is true for $n = 4$.

Let it be true for $n = m$, sich that $m > 3$, and $m \in \mathbb{N}$.

$\Rightarrow P(m) = m! > 2^m$.

$P(m+1) = (m+1)! > 2^{m+1}$. Dividing $P(m+1)$ by $P(m)$, we get

$m + 1 > 2$, which is true.

Hence, by mathematical induction, the result.

92. Let $P(n) = n! < \left(\frac{n+1}{2}\right)^n$, $n > 1$. $P(2) = 2! < \left(\frac{3}{2}\right)^2 = \frac{9}{4}$, so the statement is true for $n = 2$.

Let it be true for $n = m$.

Also, let $F(m) = \left(\frac{m+1}{2}\right)^2$ and $G(m) = m!$.

So $F(m) > G(m)$. $\frac{F(m+1)}{F(m)} \cdot \frac{G(m)}{G(m+1)} = \frac{1}{2} \cdot \frac{(m+2)^{m+1}}{(m+1)^m} \cdot \frac{m!}{(m+1)!}$

$= \frac{1}{2\left(\frac{n+2}{n+1}\right)}^{n+1} > 1 \Rightarrow F(m+1) > G(m+1).$

Hence, by mathematical induction, the result.

93. Let $P(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}$, $n > 1$. $P(2) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{13}{24}$, so the statement is true for $n = 2$.

Let it be true for $n = m$.

$\Rightarrow P(m) = \frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3} + \cdots + \frac{1}{2m} > \frac{13}{24}.$

Adding $\frac{1}{2m+2} + \frac{1}{2m+1} - \frac{1}{m+1}$ to both sides

$P(m+1) = \frac{1}{m+2} + \frac{1}{m+3} + \cdots + \frac{1}{2m} + \frac{1}{2m+1} + \frac{1}{2m+2} > \frac{13}{24}.$

Now $\frac{1}{2m+2} + \frac{1}{2m+1} - \frac{1}{m+1} = \frac{2m+1+2m+2-4m-2}{(2m+1)(2m+2)} = \frac{1}{(2m+1)(2m+2)} > 0.$

Thus, $P(m+1)$ is also true.

Hence, by mathematical induction, the result.

94. Let $P(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1 \forall n \in \mathbb{N}$. $P(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 2$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{3m+1} > 1.$

Adding $\frac{1}{3m+2} + \frac{1}{3m+3} + \frac{1}{3m+4} - \frac{1}{m+1}$, to L.H.S., we get

$P(m+1) = \frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3} + \frac{1}{3m+2} + \frac{1}{3m+3} + \frac{1}{3m+4} - \frac{1}{m+1}$

Now, $\frac{1}{3m+2} + \frac{1}{3m+3} + \frac{1}{3m+4} - \frac{1}{m+1}$

$= \frac{(3m+4)(3m+3)+(3m+2)(2m+4)+(3m+2)(3m+3)-3(3m+2)(3m+4)}{(3m+2)(3m+3)(3m+4)} > 0$

Thus, $P(m+1) > 1$.

Hence, by mathematical induction, the result.

95. Let $P(n) = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$. $P(2) = \frac{5}{4} < 2 - \frac{1}{2} = \frac{3}{2}$, so the statement is true for $n = 2$.

Let it be true for $n = m$.

$P(m) = 1 + \frac{1}{4} + \cdots + \frac{1}{m^2} < 2 - \frac{1}{m}.$

$P(m+1) = 1 + \frac{1}{4} + \cdots + \frac{1}{(m+1)^2} < 2 - \frac{1}{m} + \frac{1}{(m+1)^2}.$

Now $\frac{1}{(m+1)^2} - \frac{1}{m} = \frac{m-m^2-2m-1}{m(m+1)^2} < \frac{1}{m+1}$.

Thus, $P(m+1) < 2 - \frac{1}{m+1}$.

Hence, by mathematical induction, the result.

96. Let $P(n) = (2n+7) < (n+3)^2$. $P(1) = 8 < 16$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 2m + 7 < (m+3)^2$.

$P(m+1) = 2m + 9 < (m+4)^2$. Subtracting $P(m+1) - P(m)$, we get

$2 < 2m + 7$, which is true for $m \in \mathbb{N}$.

Hence, by mathematical induction, the result.

97. Let $P(n) = 2^n > n \forall n \in \mathbb{N}$. $P(1) = 2 > 1$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 2^m > m$.

$P(m+1) = 2^{m+1} > m + 1$. Dividing $P(m+1)$ by $P(m)$, we get

$2 > \frac{m+1}{m}$, which is true for $m \in \mathbb{N}$.

Thus, $P(m+1)$ is true if $P(m)$ is true.

Hence, by mathematical induction, the result.

98. Let $P(n) = 1 + 2 + 3 + \cdots + n < \frac{(2n+1)^2}{8} \Rightarrow \frac{n(n+1)}{2} < \frac{(2n+1)^2}{8} \Rightarrow n(n+1) < \frac{(2n+1)^2}{4}$. $P(1) = 2 < \frac{9}{4}$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$P(m) = m(m+1) < \frac{(2m+1)^2}{4}$, where $m \in \mathbb{N}$.

Adding $m + 1$ to both sides we get

$P(m+1) = 1 + 2 + \cdots + m + m + 1 < \frac{(2m+1)^2}{8} + (m+1)$

$= \frac{4m^2+4m+1+8m+8}{8} = \frac{(2m+3)^2}{8}$.

Hence, by mathematical induction, the result.

99. Let $P(n) = 1^2 + 2^2 + \cdots + n^2 > \frac{n^3}{3}$. $P(1) = 1 > \frac{1}{3}$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) = 1^2 + 2^2 + \cdots + m^2 > \frac{m^3}{3}$, where $m \in \mathbb{N}$.

Adding $(m+1)^2$ to both sides

$P(m+1) = 1^2+2^2+\cdots+m^2+(m+1)^2 > \frac{m^3}{3}+(m+1)^2 = \frac{m^3+3m^2+6m+3}{3} > \frac{(m+1)^3}{3}$.

Hence, by mathematical induction, the result.

100. Let $P(n) = 2^n > n^2$, where $n \geq 5$. $P(5) = 2^5 > 5^2 = 32 > 25$, so the statement is true for $n = 1$.

Let it be true for $n = m$, where $m \geq 5$.

$P(m) = 2^m > m^2$.

$P(m+1) = 2^{m+1} > (m+1)^2 = 2.m^2 > (m+1)^2 \Rightarrow m^2 - 2m - 1 > 0$, which is true for $m > 5$.

Hence, by mathematical induction, the result.

101. Let $P(n) = \frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$, where $n > 1$. $P(2) = 12 > \frac{16}{3}$, so the statement is true for $n = 2$.

Let it be true for $n = m$.

$\Rightarrow P(m) = \frac{2m!}{(m!)^2} > \frac{4^m}{m+1}$.

$P(m+1) = \frac{(2m+2)!}{[(m+1)!]^2} > \frac{4^{m+1}}{m+2} = \frac{(2m+2)(2m+1)}{(m+1)^2}.\frac{4^m}{m+1} > \frac{4^{m+1}}{m+2}$

$= \frac{2(2m+1)}{(m+1)} > \frac{4}{m+2}$, which is true for $m > 1$.

It is trivial to prove that $\frac{(2m+1)(m+2)}{m+1} > 2$ as $\frac{(2m+1)(m+2)}{m+1} = \frac{(\sqrt{2m}+\sqrt{2})^2+m}{m+1}$.

Hence, by mathematical induction, the result.

102. Let $P(n) = (1+x)^n > 1 + nx$. $P(2) = 1 + 2x + x^2 > 1 + 2x$, which is true for $x > -1$.

Let it be true for $n = m$

$\Rightarrow P(m) = (1+x)^m > 1 + mx$

$P(m+1) = (1+x)^{m+1} > 1+(m+1)x = (1+x)(1+mx) > 1+(m+1)x = mx^2 > 0$, which is true for $n > 1$, $x > -1$.

Hence, by mathematical induction, the result.

103. $t_n = 2t_{n-1} + 2t_{n-2}$. $t_3 = 8 + 2 = 10$, and $t_n = \frac{1}{2}\left[(1+\sqrt{3})^3 + (1-\sqrt{3})^3\right]$

$= \frac{1}{2}\left[1 + 3\sqrt{3} + 9 + 3\sqrt{3} + 1 - 3\sqrt{3} + 9 - 3\sqrt{3}\right] = 10$, so the statement is true for $n = 3$.

Let it be true for $n = m, m+1$.

$t_m = \frac{1}{2}[(1+\sqrt{3})^m + (1-\sqrt{3})^m]$, $t_{m+1} = \frac{1}{2}[(1+\sqrt{3})^{m+1} + (1-\sqrt{3})^{m+1}]$.

$t_{m+2} = 2t_{m+1} + 2t_m = (1+\sqrt{3})^m[1+1+\sqrt{3}] + (1-\sqrt{3})^m[1+1-\sqrt{3}]$

$= \frac{1}{2}(1+\sqrt{3})^m(4+2\sqrt{3}) + \frac{1}{2}(1-\sqrt{3})^m(4-2\sqrt{3}) = \frac{1}{2}[(1+\sqrt{3})^{m+2} + (1-\sqrt{3})^{m+2}]$

$(\because 4+2\sqrt{3} = 1+\sqrt{3}^2 + 2\sqrt{3})$

Hence, by mathematical induction, the result.

104. Let $P(n) \equiv x^n + y^n = a^n + b^n$, whihc is true for $n = 1, 2$ as given. Let it be true for $n = m, m+1$. Also, from these statements $xy = ab$

$\Rightarrow P(m) \equiv x^m + y^m = a^m + b^m$, $P(m+1) \equiv x^{m+1} + y^{m+1} = a^{m+1} + b^{m+1}$.

$P(m+2) \equiv x^{m+2} + y^{m+2} = x(a^{m+1} + b^{m+1} - y^{m+1}) + y(a^{m+1} + b^{m+1} - x^{m+1})$

$= (a^{m+1} + b^{m+1})(x+y) - xy(x^m + y^m) = (a^{m+1} + b^{m+1})(a+b) - ab(a^m + b^m) = a^{m+2} + b^{m+2}$.

Hence, by mathematical induction, the result.

105. Let $P(n) \equiv \frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$. $P(1) \equiv \frac{1}{2} \leq \frac{1}{\sqrt{4}}$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) \equiv \frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2m-1}{2m} \leq \frac{1}{\sqrt{3m+1}}$.

Multiplying both sides with $\frac{2m+1}{2m+2}$, we get

$P(m+1) \equiv \frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2m+1}{2m+2} \leq \frac{2m+1}{2m+2} \cdot \frac{1}{\sqrt{3m+1}} = \frac{2m+1}{2m+2} \cdot \frac{1}{\sqrt{3m+1}}$.

$\frac{2n+1}{2n+2}\frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{3n+4}} \Rightarrow 12n^2 + 19n \leq 24n^2 + 20n$.

Hence, by mathematical induction, the result.

106. Let $P(n) \equiv \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \frac{25}{36}$. $P(1) \equiv \frac{1}{2} < \frac{25}{36}$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) \equiv \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} < \frac{25}{36}$.

Adding $\frac{1}{2m+1} + \frac{1}{2m+2} - \frac{1}{m+1}$, to both sides

$P(m+1) \equiv \frac{1}{m+2} + \frac{1}{m+3} + \cdots + \frac{1}{2m+1} + \frac{1}{2m+2} < \frac{25}{36}\frac{1}{2m+1} + \frac{1}{2m+2} - \frac{1}{m+1}$

$= \frac{25}{36} + \frac{1}{(2n+1)(2n+2)} < \frac{25}{36} + \frac{1}{4n(n-1)} < \frac{25}{36} - \frac{1}{4n}$.

Hence, by mathematical induction, the result.

107. Let $P(n) \equiv \sqrt{a+\sqrt{a+\sqrt{a+\cdots n \sim \text{terms}}}} \leq \frac{1+\sqrt{4a+1}}{2}$. $P(1) \equiv \sqrt{a} \leq \frac{1+\sqrt{4a+1}}{2}$, so the statement is true for $n=1$.

Let it be true for $n=m$.

$\Rightarrow P(m) \equiv \sqrt{a+\sqrt{a+\sqrt{a+\cdots m \sim \text{terms}}}} \leq \frac{1+\sqrt{4a+1}}{2}$

$P(m+1) \equiv \sqrt{a+\sqrt{a+\sqrt{a+\cdots(m+1) \sim \text{terms}}}} \leq \frac{1+\sqrt{4a+1}}{2} \equiv \sqrt{a+\frac{1+\sqrt{4a+1}}{2}} \leq \frac{1+\sqrt{4a+1}}{2}$

$\Rightarrow \frac{2a+\sqrt{4a+1}}{2} < \frac{(1+\sqrt{4a+1})^2}{4}$, which is true.

Hence, by mathematical induction, the result.

108. Let $P(n) \equiv \sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n}}}} < 3$, where $n \geq 2, n \in \mathbb{N}$. $P(2) \equiv \sqrt{2} < 3$, so the statement is true for $n=2$.

Let it be true for $n=m$.

$P(m) = \sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n}}}} < 3$. Taking log of both sides, we get

$\Rightarrow \frac{1}{2}\log 2 + \frac{1}{4}\log 3 + \frac{1}{8}\log 4 + \cdots + \frac{1}{2^{m-1}}\log m < \log 3 - \frac{1}{2^{n-2}}\log m$

Now it is trivial to show it for $n=m+1$.

Hence, by mathematical induction, the result.

109. We have $x^{3n} = a_n x + b_n + c_n x^{-1} \forall n \in \mathbb{N}$. For $n=1$, $x^3 = a_1 x + b_1 + c_1 x^{-1} = (a_0+b_0)x + (a_0+b_0+c_0) + (a_0+c_0)x^{-1} = x+1$, so the statement is true for $n=1$.

Let it be true for $n=m$.

$\Rightarrow x^{3m} = a_m x + b_m + c_m x^{-1}$.

$x^{3m+3} = (a_m x + b_m + c_m x^{-1})x^3 = (a_m x + b_m + c_m x^{-1})(x+1) [\because x^3 = x+1]$

$= a_m x + a_m x^2 + b_m + b_m x + c_m x^{-1} + c_m = x(a_m+b_m) + a_m x^{-1}x^3 + b_m + c_m + c_m x^{-1}$

$= x(a_m+b_m) + a_m x^{-1}(1+x) + b_m + c_m x^{-1} + c_m = x(a_m+b_m) + a_m x^{-1} + a_m x + b_m + c_m + c_m x^{-1}$

$= a_{m+1}x + b_{m+1} + c_{m+1}x^{-}1$.

Hence, by mathematical induction, the result.

110. Let $P_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$. $P_1 = 6$, which is divisible by 2, so the statement is true for $n=1$.

Let it be true for $n=m$.

$\Rightarrow P_m = (3+\sqrt{5})^m + (3-\sqrt{5})^m = 2k$, where $k \in \mathbb{N}$.

We observe that $r_1 r_2 = 6$ and $r_1 r_2 = 4$, where $r_1 = 3+\sqrt{5}$ and $r_2 = 3-\sqrt{5}$, where r_1 and r_2 are roots of the equation $r^2 - 6r + 4 = 0$.

Thus, P_m satisfied the recurrence relation $P_{m+2} - 6P_{m+1} + 4P_m = 0$.

Hence, by mathematical induction, the result.

111. Let $P(n) \equiv x_1^2 + 3x_2^2 + 5x_3^2 + \cdots + (2n-1)x_n^2 \leq (x_1 + x_2 + \cdots + x_n)^2$, where $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$.

$P(1) \equiv x_1^2 + 3x_2^2 \leq x_1^2 + x_2^2 + 2x_1x_2 \Rightarrow x_2 \leq x_1$, so the statement is true for $n = 1$. Let it be true for $n = m$.

$\Rightarrow P(m) \equiv x_1^2 + 3x_2^2 + 5x_3^2 + \cdots + (2m-1)x_m^2 \leq (x_1 + x_2 + \cdots + x_m)^2$.

$P(m+1) \equiv x_1^2 + 3x_2^2 + 5x_3^2 + \cdots + (2m-1)^2 x_m^2 + (2m+1)x_{m+1}^2 \leq (x_1 + x_2 + \cdots + x_{m+1})^2 = (x_1 + x_2 + \cdots + x_m)^2 + 2x_{m+1}(x_1 + x_2 + \cdots + x_m) + x_{m+1}^2$

$\Rightarrow mx_{m+1} \leq x_1 + x_2 + \cdots + x_m$, which is true.

Hence, by mathematical induction, the result.

112. Let $P(n) \equiv |\sin(x_1 + x_2 + \cdots + x_n)| \leq |\sin x_1| + |\sin x_2| + \cdots + |\sin x_n|$, where $x_1, x_2, \ldots, x_n \in [0, \pi]$.

$P(1) \equiv |\sin x_1| \leq |\sin x_1|$, so the statement is true for $n = 1$.

Let it be true for $n = m$.

$\Rightarrow P(m) \equiv |\sin(x_1 + x_2 + \cdots + x_m)| \leq |\sin x_1| + |\sin x_2| + \cdots + |\sin x_m|$.

$P(m+1) \equiv |\sin(x_1 + x_2 + \cdots + x_m + x_{m+1})| \leq |\sin x_1| + |\sin x_2| + \cdots + |\sin x_m| + |\sin x_{m+1}|$

$= |\sin(x_1 + x_2 + \cdots + x_m)\cos x_{m+1} + \cos(x_1 + x_2 + \cdots + x_m)\sin x_{m+1}| \leq |\sin x_1| + |\sin x_2| + \cdots + |\sin x_m| + |\sin x_{m+1}|$

$\leq |\sin(x_1 + x_2 + \cdots + x_m)| + |\sin x_{m+1}| \leq |\sin x_1| + |\sin x_2| + \cdots + |\sin x_m| + |\sin x_{m+1}|$, which is true.

Hence, by mathematical induction, the result.

113. Let $P(n) \equiv \tan x_1 - \tan x_2 + \cdots + (-1)^n \tan x_n \geq \tan(x_1 - x_2 + \cdots + (-1)^n x_n)$, where $\frac{\pi}{2} > x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$.

For $n = 1$, $\tan x_1 = \tan x_1$, so the statement is true for $n = 1$. For $n = 2$, $\tan x_1 - \tan x_2 \geq \tan(x_1 - x_2) = \frac{\tan x_1 - \tan x_2}{1 + \tan x_1 \tan x_2} \Rightarrow 1 + \tan x_1 \tan x_2 \geq 1$, so the statement is also true for $n = 2$.

Let it be true for $n = m$.

$\Rightarrow \tan x_1 - \tan x_2 + \cdots + (-1)^m \tan x_m \geq \tan(x_1 - x_2 + \cdots + (-1)^m x_m)$

$P(m+1) \equiv \tan x_1 - \tan x_2 + \cdots + (-1)^{m+1} \tan x_{m+1} \geq \tan(x_1 - x_2 + \cdots + (-1)^n \tan_{m+1}) = \frac{\tan(x_1 - x_2 + \cdots + (-1)^m x_m) - (-1)^{m+1} \tan x_{m+1}}{1 + \tan(x_1 - x_2 + \cdots + (-1)^m x_m) \tan x_{m+1}} \leq \tan(x_1 - x_2 + \cdots + (-1)^m x_m)$

Hence, by mathematical induction, the result.

114. Let $P(n) \equiv a_1^r - a_2^r + \cdots + (-1)^n a_n^r \geq (a_1 - a_2 + \cdots + (-1)^n a_n)^r$, where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0, r \geq 1$.

For $n = 1$, $a_1^r = a_1^r$, so the statement is true for $n = 1$. For $n = 2$, $r = 2$, $a_1^2 - a_2^2 \geq a_1^2 + a_2^2 - 2a_1a_2 \Rightarrow 2a_1a_2 - 2a_2^2 \geq 0$, so the statement is also true for $n = 2, r = 2$.

Let it be true for $n = m$.

$\Rightarrow P(m) \equiv a_1^r - a_2^r + \cdots + (-1)^m a_m^r \geq (a_1 - a_2 + \cdots + (-1)^m a_m)^r$

$P(m+1) \equiv a_1^r - a_2^r + \cdots + (-1)^m a_m^r + (-1)^{m+1} a_{m+1} \geq (a_1 - a_2 + \cdots + (-1)^m a_m + (-1)^{m+1} a_{m+1})^r$.

This can be proven by mathematical induction by varying r trivially.

Hence, by mathematical induction, the result.

Answers of Chapter 7

Binomials, Multinomials and Expansions

1. Using binomial theorem, $\left(x+\frac{1}{x}\right)^5 = C_0^5x^5 + C_1^5x^4.\frac{1}{x} + C_2^5x.\frac{1}{x^2} + C_3^5x^2.\frac{1}{x^3} + C_4^5x.\frac{1}{x^4} + C_5^5.\frac{1}{x^5}$

   $= C_0x^5 + C_1^5x^3 + C_2^5x + C_3^5.\frac{1}{x} + C_4^5.\frac{1}{x^3} + C_5^5.\frac{1}{x^5}$.

2. $(10.1)^5 = (10+0.1)^5$, so we proceed like previous problem to get

   $(10.1)^5 = C_0^5 10000 + C_1^5 1000 + C_2^5 10 + C_3^5\frac{1}{10} + C_4^5\frac{1}{1000} + C_5^5\frac{1}{100000}$

   $= 100000 + 5000 + 100 + 1 + .005 + .00001 = 15101.00501$.

3. $(x+\sqrt{x-1})^6 + (x-\sqrt{x-1})^6 = C_0^6x^6 + C_1^6x^5\sqrt{x-1} + C_2^6x^4\sqrt{(x-1)^2} + C_3^6x^3\sqrt{(x-1)^3} + C_4^6x^2\sqrt{(x-1)^4} + C_5^6x\sqrt{(x-1)^5} + C_6^6\sqrt{(x-1)^6} + C_0^6x^6 - C_1^6x^5\sqrt{x-1} + C_2^6x^4\sqrt{(x-1)^2} - C_3^6x^3\sqrt{(x-1)^3} + C_4^6x^2\sqrt{(x-1)^4} - C_5^6x\sqrt{(x-1)^5} + C_6^6\sqrt{(x-1)^6}$

   $= 2x^6 + 30x^4(x-1) + 30x^2(x-1)^2 + 2(x-1)^3$.

4. Consider the expansion of $(x+a)^n$ and $(x-a)^n$. The sum of real terms will be A and the sum of imaginary terms will be B.

   $(x+a)^n = C_0^nx^n + C_1^nx^{n-1}.a + C_2^nx^{n-2}a^2 + \cdots + C_n^na^n = A+B$, and $(x-a)^n = C_0^nx^n - C_1^nx^{n-1}.a + C_2^nx^{n-2}a^2 + \cdots + C_n^n(-a)^n = A-B$

   Multiplying, we get

   $(x^2-a^2)^n = A^2 - B^2$.

5. Let $(7+4\sqrt{3})^n = \alpha + \beta$, where α is a positive integer and β is a proper fraction.

   Cleaerly, $0 < 7-4\sqrt{3} < 1\left[\because 7-4\sqrt{3} = \frac{49-48}{7+4\sqrt{3}} = \frac{1}{7+4\sqrt{3}}\right]$

   $\therefore 0 < (7-4\sqrt{3})^n < 1 = \beta_1$(let), then $0 < \beta_1 < 1$.

   $\alpha + \beta + \beta_1 = 2[7^n + C_2^n 7^{n-2}.48 + \cdots] =$ an even number.

   $\Rightarrow \beta + \beta_1 =$ an even number $-\alpha =$ an integer.

   $\because 0 < \beta < 1$ and $0 < \beta_1 < 1 \therefore 0 < \beta + \beta_1 < 2$. Thus, $\beta + \beta_1 = 1$.

   $\therefore \alpha + 1 =$ an even number $\Rightarrow \alpha =$ an odd number.

6. Proceeding from previous problem, $(\alpha+\beta)(1-\beta) = (\alpha+\beta)\beta_1 = (7+4\sqrt{3})^n(7-4\sqrt{3})^n = 1$.

7. $t_r = C_r^{10} y^{10-r}.\left(\frac{c^3}{y^2}\right)^r$. We have to find coefficient of $\frac{1}{y^2}$, hence, $10-r-2r=-2 \Rightarrow r=4$.

   Thus, coefficient is $C_4^{10}.c^{12}$.

8. We have to find coefficient of x^9 in $(1+3x+3x^2+x^3)^{15} = (1+x)^{45}$. Therefore, coefficient is C_9^{45}.

9. We have to find term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$. The general term will be $t_r = C_{r-1}^9.\left(\frac{3}{2}x^2\right)^{9-r+1}\left(-\frac{1}{3x}\right)^{r-1}$.

   $\Rightarrow 21-3r=0 \Rightarrow r=7$. So coefficient is $(-1)^6.C_6^9\left(\frac{3}{2}\right)^{10-7}.\frac{1}{3^6} = \frac{7}{18}$.

10. $(1+x)^m\left(1+\frac{1}{x}\right)^n = x^{-n}(1+x)^{m+n}$. We have to find term independent of x in the expansion, which is coefficient of x^n in $(1+x)^{m+n}$.

    Coeff. is $C_n^{m+n} = \frac{(m+n)!}{m!n!}$.

11. Coeff. of x^{-1} in $(1+3x^2+x^4)\left(1+\frac{1}{x}\right)^8$ = coeff. of x^{-1} in $\left(1+\frac{1}{x}\right)^8$ + coeff. of x^{-1} in $3x^2\left(1+\frac{1}{x}\right)^8$ + coeff. of x^{-1} in $x^4\left(1+\frac{1}{x}\right)^8$

    = coeff. of x^{-1} in $\left(1+\frac{1}{x}\right)^8$ + coeff. of x^{-3} in $3\left(1+\frac{1}{x}\right)^8$ + coeff. of x^{-5} in $\left(1+\frac{1}{x}\right)^8$

    General term is given by $t_r = C_{r-1}^n\left(\frac{1}{x}\right)^{r-1} = C_{r-1}^n x^{1-r}$.

    When $r-1=1 \Rightarrow r=2$, we have coeff. as C_1^8. When $r-1=3 \Rightarrow r=4$, we have coeff. as C_3^8, and similarly coeff. of x^{-5} is C_5^8.

    Thus, required coeff. of x^{-1} is $C_1^8 + 3.C_3^8 + C_5^8 = 232$.

12. rth term in the expansion of $(1-x)^n$ is $C_{r-1}^{2n-1}(-1)^{r-1}x^{r-1}$, so $(r+1)$th term will have the term x^r.

    $\Rightarrow a_{r-1} = (-1)^{r-1}C_{r-1}^{2n-1}$ and $a_{2n-r} = (-1)^{2n-r}C_{2n-r}^{2n-1}$.

    We know that $C_r^n = C_{n-r}^n$ and $(-1)^{2n}=1$. Hence, $a_{2n-r} = (-1)^{-r}C_{r-}^{2n-1}$.

    Thus, $a_{r-1} + a_{2n-r} = 0$.

13. Let the rth term be independent of x. $t_r = C_{r-1}^{10}(\sqrt{x})^{10-r+1}\left(\frac{k}{x^2}\right)^{r-1} = C_{r-1}^{10}x^{\frac{11-r}{2}-2r+2}k^{r-1}$.

    Since the term is independent of $x \Rightarrow 15-5r=0 \Rightarrow r=3$.

So the term is $C^{10}_2 k^2 = 405 \Rightarrow k = \pm 3$.

14. kth term in the expansion is given by $t_k = C^{n-3}_{k-1} x^{n-3-k+1}(x^{-2})^{k-1}$

$= C^{n-3}_{k-1} x^{n-3k}$. Let this term contain $x^{2r} \Rightarrow 2r = n - 3k \Rightarrow k = \frac{n-2r}{3}$.

Since $n - 2r$ is not a multiple of 3, k cannot be an integer. So no term will contain x^{2r}.

15. Let rth term be independent of x. $t_r = C^n_{r-1}(x^a)^{n-r+1}(x^{-b})^{r-1}$.

This will be independent of x if $an - ar + a - br + b = 0 \Rightarrow an = (a+b)(r-1) \Rightarrow r = 1 + \frac{an}{a+b}$.

Clearly, r will be an integer only if an is a multiple of $a + b$.

16. $\left(x + \frac{1}{x}\right)^7 = C^7_0 x^7 + C^7_1 x^6 . \frac{1}{x} + C^7_2 x^5 . \frac{1}{x^2} + C^7_3 x^4 . \frac{1}{x^3} + C^7_4 x^3 . \frac{1}{x^4} + C^7_5 x^2 . \frac{1}{x^5} + C^7_6 x . \frac{1}{x^6} + C^7_7 . \frac{1}{x^7}$

$= C^7_0 x^7 + C^7_1 x^5 + C^7_2 x^3 + C^7_3 x + C^7_4 . \frac{1}{x} + C^7_5 . \frac{1}{x^3} + C^7_6 . \frac{1}{x^5} + C^7_7 . \frac{1}{x^7}$

$= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$.

17. $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6 = C^6_0\left(\frac{2x}{3}\right)^6 + C^6_1\left(\frac{2x}{3}\right)^5 . \left(-\frac{3}{2x}\right) + C^6_2\left(\frac{2x}{3}\right)^4\left(-\frac{3}{2x}\right)^2 + C^6_3\left(\frac{2x}{3}\right)^3\left(-\frac{3}{2x}\right)^3 + C^6_4\left(\frac{2x}{3}\right)^2\left(-\frac{3}{2x}\right)^4 + C^6_5\left(\frac{2x}{3}\right)\left(-\frac{3}{2x}\right)^5 + C^6_6\left(-\frac{3}{2x}\right)^6$

$= \frac{64}{729}x^6 - \frac{32}{2x}x^4 + \frac{20}{3}x^2 - 20 + \frac{135}{4}x^2 - \frac{243}{8}x^4 + \frac{729}{64}x^6$.

18. Given, $(1 + ax)^n = 1 + 8x + 24x^2 + \cdots \Rightarrow 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \cdots = 1 + 8x + 24x^2 + \cdots$

Comparing coefficients of powers of x

$an = 8, \frac{n(n-1)}{2}a^2 = 24 \Rightarrow 32n^2 - 32n = 24n^2 \Rightarrow n = 4 \Rightarrow a = 2$.

19. 7th term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$ is $C^9_6\left(\frac{4x}{5}\right)^3\left(-\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$.

20. $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = C^6_0 2^3 + C^6_1 4\sqrt{2} + C^6_2 2^2 + C^6_3 2\sqrt{2} + C^6_4 2 + C^6_5\sqrt{2} + C^6_6 + C^6_0 2^3 - C^6_1 4\sqrt{2} + C^6_2 2^2 - C^6_3 2\sqrt{2} + C^6_4 2 - C^6_5\sqrt{2} + C^6_6$

$= 2.C^6_0 2^3 + 2.C^6_2 2^2 + 2.C^6_4 2 + 2.C^6_6 = 198$.

21. According to questions $(x + a)^n = A + B$, because A is the sum of odd terms and B is the sum of even terms. From Binomial theorem $(x - a)^n = A - B$.

$\Rightarrow (x + a)^{2n} = A^2 + B^2 + 2AB$ and $(x - a)^{2n} = A^2 + B^2 - 2AB$.

Subtracting, we get $(x + a)^{2n} - (x - a)^{2n} = 4AB$.

22. Let $(5+2\sqrt{6})^n = \alpha + \beta$, where α is a positive integer and β is a proper fraction.

Cleaerly, $0 < 5 - 2\sqrt{6} < 1\left[\because 5 - 2\sqrt{6} = \frac{25-24}{5+2\sqrt{6}} = \frac{1}{5+2\sqrt{6}}\right]$

$\therefore 0 < (5-2\sqrt{6})^n < 1 = \beta_1(\text{let})$, then $0 < \beta_1 < 1$.

$\alpha + \beta + \beta_1 = 2[5^n + C_2^n 5^{n-2}.24 + \cdots] =$ an even number.

$\Rightarrow \beta + \beta_1 =$ an even number $-\alpha =$ an integer.

$\because 0 < \beta < 1$ and $0 < \beta_1 < 1 \therefore 0 < \beta + \beta_1 < 2$. Thus, $\beta + \beta_1 = 1$.

$\therefore \alpha + 1 =$ an even number $\Rightarrow \alpha =$ an odd number.

23. Proceeding from previous problem, $(\alpha+\beta)(1-\beta) = (\alpha+\beta)\beta_1 = (3+\sqrt{8})^n(3-\sqrt{8})^n = 1$.

24. Let rth term contains the term x. Then $t_r = C_{r-1}^9(2x)^{10-r}\left(-\frac{3}{x}\right)^{r-1}$

Since the term contains x, therefore $10 - r - r + 1 = 1 \Rightarrow r = 5$.

Thus, coefficient is $C_4^9 2^5.3^4 = 2592.C_4^9$.

25. Let rth term contain x^7 in the expansion of $(3x^2 + (5x)^{-1})^{11}$. $t_r = C_{r-1}^{11}(3x^2)^{12-r}.(5x^{-1})^{r-1}$.

Since the term contains x^7, therefore $24 - 2r - r + 1 = 7 \Rightarrow r = 6$.

Thus, coefficient is $C_5^{11}.\frac{3^6}{5^5}$.

26. Let rth term contain x^9 in the expansion of $(2x^2 - x^{-1})^{20}$. Then $t_r = C_{r-1}^{20}(2x^2)^{21-r}(-x^{-1})^{r-1}$.

Since the term contains x^9, therefore $42 - 2r - r + 1 = 9$, which does not yield an integral value for r. Therefore, coefficient is 0.

27. Let rth term contain x^{24} in the expansion of $(x^2 + 3ax^{-1})^{15}$. Then $t_r = C_{r-1}^{15}(x^2)^{16-r}(3ax^{-1})^{r-1}$.

Since the term contains x^{24}, therefore $32 - 2r - r + 1 = 24 \Rightarrow r = 3$.

Therefore, the coefficient is $C_2^{16} 9a^2$.

28. Let rth term contain x^9 in the expansion of $(x^2 - (3x)^{-1})^9$. Then $t_r = C_{r-1}^9(x^2)^{10-r}\left(-\frac{1}{3x}\right)^{r-1}$.

Since the term contains x^9, therefore $20 - 2r - r + 1 = 9 \Rightarrow r = 4$.

Therefore, the coefficient is $C_3^9.\frac{-1}{3^3} = -\frac{28}{9}$.

29. Let rth term contain x^{-7} in the expansion of $\left(2x - \frac{1}{3x^2}\right)^{11}$. Then $t_r = C^{11}_{r-1}(2x)^{12-r}\left(-\frac{1}{3x^2}\right)^{r-1}$.

    Since the term contains x^{-7}, therefore, $12 - r - 2r + 2 = -7 \Rightarrow r = 7$.

    Therefore, coefficient is $C^{11}_6 \frac{2^5}{3^6}$.

30. Let rth term contain x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$. $t_r = C^{11}_{r-1}(ax^2)^{12-r}.\left(\frac{1}{bx}\right)^{r-1}$.

    Since the term contains x^7, therefore $24 - 2r - r + 1 = 7 \Rightarrow r = 6$.

    Therefore, coefficient is $C^{11}_5 a^6 b^{-5}$. Let sth term contain x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$. Then $t_r = C^{11}_{r-1}(ax)^{12-r}\left(-\frac{1}{bx^2}\right)^{r-1}$.

    Since the term contains x^{-7}, therefore $12 - r - 2r + 2 = -7 \Rightarrow r = 7$.

    Therefore, the coefficient is $C^{11}_6 a^5 b^{-6}$.

    Since the coefficients are equal $ab = 1[\because C^{11}_5 = C^{11}_6]$.

31. Let rth term contain x^p in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$. Then $t_r = C^{2n}_{r-1}(x^2)^{2n+1-r}\frac{1}{x^{r-1}}$.

    Since the term contains x^p, therefore $4n + 2 - 2r - r + 1 = p \Rightarrow r = \frac{4n-p}{3} + 1$.

    Therefore, the coefficient is $\frac{2n!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$.

32. The problems are solved below:

    i. Let rth term be indnependent of x in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$. Then $t_r = C^{2n}_{r-1}x^{2n+1-r}.\frac{1}{x^{r-1}}$

       Since the term is independent of x, therefore $2n + 1 - r - r + 1 = 0 \Rightarrow r = n + 1$.

       Therefore, the coefficient is $C^{2n}_n = \frac{(2n)!}{n!n!}$.

    ii. Let rth term be independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^{15}$. Then $t_r = C^{15}_{r-1}(2x^2)^{16-r}\left(\frac{1}{x}\right)^{r-1}$.

       Since the term is independent of x, therefore $32 - 2r - r + 1 = 0 \Rightarrow r = 11$.

       Therefore, the coefficient is $C^{15}_{10}.2^5 = 32.C^{15}_{10}$.

iii. Let rth term be independent of x in the expansion of $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$. Then $t_r = C^{10}_{r-1}\left(\sqrt{\frac{x}{3}}\right)^{11-r}\left(\frac{3}{2x^2}\right)^{r-1}$.

Since the term is independent of x, therefore $\frac{11-r}{2} - 2r + 2 = 0 \Rightarrow r = 3$.

Therefore, the coefficient is $C^{10}_2.\left(\frac{1}{3^4}\right).\left(\frac{3^2}{2^2}\right) = \frac{5}{4}$.

iv. Let rth term be independent of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$. Then $t_r = C^{12}_{r-1}(2x^2)^{13-r}\left(\frac{1}{x}\right)^{r-1}$.

Since the term is independent of x, therefore $26 - 2r - r + 1 = 0 \Rightarrow r = 9$.

Therefore, the coefficient is $C^{12}_8.2^4 = 7920$.

v. Let rth term be independent of x in the expansion of $\left(2x^2 - \frac{3}{x^3}\right)^{25}$. Then $t_r = C^{25}_{r-1}(2x^2)^{26-r}\left(-\frac{3}{x^3}\right)^{r-1}$.

Since the term is independent of x, therefore $52 - 2r - 3r + 3 = 0 \Rightarrow r = 11$.

Therefore, the coefficient is $C^{25}_{10}.2^{15}3^{10}$.

vi. Let rth term be independent of x in the expansion of $\left(x^3 - \frac{3}{x^2}\right)^{15}$. Then $t_r = C^{15}_{r-1}(x^3)^{16-r}\left(-\frac{3}{x^2}\right)^{r-1}$.

Since the term is independent of x, therefore $48 - 3r - 2r + 2 = 0 \Rightarrow r = 10$.

Therefore, the coefficient is $C^{15}_9(-3)^9 = -3^9.C^{15}_0$.

vii. Let rth term be independent of x in the expansion of $\left(x^2 - \frac{3}{x^3}\right)^{10}$. Then $t_r = C^{10}_{r-1}(x^2)^{11-r}\left(-\frac{3}{x^3}\right)^{r-1}$.

Since the term is independent of x, therefore $22 - r - 3r + 3 = 0 \Rightarrow r = 5$.

Therefore, the coefficient is $C^{10}_4.(-3)^4 = 3^4.C^{10}_4$.

viii. Let rth term be independent of x in the expansion of $\left(\frac{1}{2}x^{1/3} + x^{-1/3}\right)^8$. Then $t_r = C^8_{r-1}\left(\frac{1}{2}x^{1/3}\right)^{9-r}(x^{-1/3})^{r-1}$.

Since the term is independent of x, therefore $9 - r - r + 1 = 0 \Rightarrow r = 5$.

Therefore, the coefficient is $C^8_4.\frac{1}{2^4} = \frac{35}{8}$.

33. Let rth term be independent of x in the expansion of $\left(x+\frac{1}{x^2}\right)^n$. Then $t_r = C^n_{r-1}x^{n+1-r}\frac{1}{(x^2)^{r-1}}$.

Since the term is independent of x, therefore $n+1-r-2r+2=0 \Rightarrow r=\frac{n}{3}+1$.

Therefore, the coefficient is $C^n_{\frac{n}{3}} = \frac{n!}{(\frac{n}{3})!(\frac{2n}{3})!}$.

34. First we will find coefficient of x^m and then of x^n in the expansion of $(1+x)^{m+n}$. Let pth term contain x^m.

Then $t_P = C^{m+n}_{p-1}x^{p-1}$. Since it contains x^m, therefore $p=m+1$.

Thus coefficient is C^{m+n}_m. Similarly, we find the coefficient of term containing x^n as C^{m+n}_n. We know that $C^n_r = C^n_{n-r}$.

Therefore, $C^{m+n}_m = C^{m+n}_n$. Hence, proved.

35. 4th term in the expansion of $\left(px+\frac{1}{x}\right)^n$ is given by $t_4 = C^n_3(px)^{n+1-4}\frac{1}{x^3}$.

Since the term is independent of x, therefore $n-3-3=0 \Rightarrow n=6$.

So the term is $C^6_3p^3 = \frac{5}{2} \Rightarrow p=\frac{1}{2}$.

36. There are 13 terms in the expansion of $\left(x-\frac{1}{2x}\right)^{12}$. So 7th term will be the middle term.

$t_7 = C^{12}_6.x^6\left(-\frac{1}{2x}\right)^6 = C^{12}_6.\frac{1}{2^6} = \frac{231}{16}$.

37. There are 8 terms in the expansion of $\left(2x^2-\frac{1}{x}\right)^7$. These are 4th and 5th terms.

$t_4 = C^7_3(2x^2)^4\left(-\frac{1}{x}\right)^3 = -560x^5$, $t_5 = C^7_4(2x^2)^3\left(-\frac{1}{x}\right)^4 = 280x^2$.

38. There are $2n+1$ terms in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$. So the middle term is $(n+1)$th term.

$t_{n+1} = C^{2n}_n x^{2n-n}.\frac{1}{x^n} = C^{2n}_n = \frac{2n!}{n!n!} = \frac{1.3.5...(2n-1).2^n}{n!}$.

39. There are $2n+1$ terms in the expansion of $(1+x)^{2n}$. So the middle term is $(n+1)$th term.

$t_{n+1} = C^{2n}_n x^n$. So the ccoefficient is C^{2n}_n.

There are $2n$ temrs in the expansion of $(1+x)^{2n-1}$. So the middle terms are nth and $(n+1)$th terms.

Coefficients are C^{2n-1}_{n-1} and C^{2n-1}_n.

Clearly, $C^{2n-1}_{n-1} + C^{2n-1}_n = C^{2n}_n$. Hence, proved.

40. The solutions are given below:

i. There will be 21 terms in the expansions of $\left(\frac{2x}{3}-\frac{3y}{2}\right)^{20}$. So 11th term will be the middle term.

$t_{11}=C^{20}_{10}\left(\frac{2x}{3}\right)^{21-10}.\left(-\frac{3y}{2}\right)^{10}=C^{20}_{10}x^{10}y^{10}.$

ii. There will be 7 terms in the expansions of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^{6}$. So 4th term will be the middle term.

$t_4=C^6_3\left(\frac{2x}{3}\right)^3\left(-\frac{3}{2x}\right)^3=-20.$

iii. There will be 8 terms in the expansion of $\left(\frac{x}{y}-\frac{y}{x}\right)^7$. So 4th and 5th termss will be the middle terms.

$t_4=C^7_3.\left(\frac{x}{y}\right)^4\left(-\frac{y}{x}\right)^3=-\frac{35x}{y}$, $t_5=C^7_4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right)^3=\frac{35y}{x}.$

iv. The middle term of the expansion $(1+x)^{2n}$ will be the $(n+1)$th term.

$t_{n+1}=C^{2n}_nx^n=\frac{2n!}{n!n!}x^n.$

v. $(1-2x+x^2)^n=(1-x)^{2n}$ so $(n+1)$th term will be the middle term.

$t_{n+1}=C^{2n}_n(-x^n)=(-1)^n\frac{2n!}{n!n!}x^n$

41. The general rth term will be given by $t_r=C^{2n+1}_{r-1}\left(\frac{x}{y}\right)^{2n+1+1-r}.\left(\frac{y}{x}\right)^{r-1}.$

Since it will $2n+2$ terms there will be two middle terms. $(n+1)$th and $(n+2)$th terms will be middle terms. Since the powers of x and y are symmetric if any term has to be free of x and y then it has to be middle terms.

$t_{n+1}=C^{2n+1}_n\left(\frac{x}{y}\right)^{2n+1+1-n-1}\left(\frac{y}{x}\right)^n=C^{2n+1}_n\frac{x}{y}$

$t_{n+2}=C^{2n+1}_{n+1}\frac{y}{x}$. Both of these terms are not free of x and y.

We also prove that no term is free of x and y by considering general term. Since the term has to be independent of x and y, we consider the general term.

$2n+2-r--r+1==\Rightarrow r=\frac{2n+1}{2}$, which cannot be an integer. So no terms is free of both x and y.

42. There will be $2n+1$ terms in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$. So the middle term would be $(n+1)$th term.

$t_{n+1}=C^{2n}_nx^{2n+1-n-1}\frac{-1^n}{x^n}=(-1)^n\frac{2n!}{n!n!}=\frac{1.3.5...(2n-1)}{n!}.(-2)^n.$

43. $t_{2r+1} = C^{43}_{2r}x^{2r}$ and $t_{r+2} = C^{43}_{r+1}x^{r+1}$.

Given that coefficients are equal. $\therefore C^{43}_{2r} = C^{4}_{r+1}3 \Rightarrow 2r + r + 1 = 43 \Rightarrow r = 14$.

44. Coefficient of rth term in the expansion of $(1+x)^{20}$ is $C^{2}_{r-1}0$, and the coefficient of $(r+4)$th term is C^{20}_{r+3}.

Clearly, for coefficients to be equal $r - 1 + r + 3 = 20 \Rightarrow r = 9$.

45. Following like previous problem, $r - 3 + 2r + 3 = 18 \Rightarrow r = 6$.

46. Following like previous problem, $2r + 4 + r - 7 = 39 \Rightarrow r = 14$. So $C^{r}_{12} = 91$.

47. Following like previous problem, $3r - 1 + r + 1 = 2n \Rightarrow r = \frac{n}{2}$.

48. Following like previous problem, $p + p + 2 = 2n \Rightarrow p = n - 1$.

49. Coefficient of $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is C^{n+1}_{r}. Coefficients of rth and $(r+1)$th terms in the expansion of $(1+x)^n$ are C^{n}_{r-1} and C^{n}_{r} respectively.

Clearly, $C^{n}_{r-1} + C^{n}_{r} = C^{n+1}_{r}$. Hence, proved.

50. Since we have to find numerically greatets term we can replaced $-$ sign with $+$. Let rth term be the greatest term in the expansion of $\left(7 + \frac{10}{3}\right)^{11}$. $t_r = C^{11}_{r-1}7^{12-r}\left(\frac{10}{3}\right)^{r-1}$. We consider $(r+1)$th term as well. $t_{r+1} = C^{11}_{r}7^{11-r}\left(\frac{10}{3}\right)^{r}$

$\frac{t_r}{t_{r+1}} = \frac{21r}{(12-r)\,10} \geq 1 \Rightarrow r \geq 3\frac{27}{31}$.

Replacing r with $r-1$, $\frac{t_{r-1}}{t_r} = \frac{21r-21}{130-10r} \Rightarrow r \leq 4\frac{27}{31} \therefore r = 4$.

So the greatest term will be $C^{11}_{3}7^{8}\frac{10^3}{3^3} = \frac{440}{9}7^{8}5^{3}$.

51. In any binomial expansion, the middle terms have the greatest coefficient. Therefore, $(n+1)$th term will have greatest coefficient.

$t_n = C^{2n}_{n-1}x^{n-1}$, $t_{n+1} = C^{2n}_{n}x^{n}$, $t_{n+2} = C^{2n}_{n+1}x^{n+1}$

$\therefore \frac{t_{n+1}}{t_{n+2}} = \frac{n+1}{n}.\frac{1}{x}$. Since t_{n+1} is the greatest term $\frac{t_{n+1}}{t_{n+2}} > 1 \Rightarrow x < \frac{n+1}{n}$.

Similarly, considering t_n and t_{n+1}, $x > \frac{n}{n+1}$.

52. The greatest terms are calculated below:

   i. $\left(2 + \frac{9}{5}\right)^{10}$ will have 6th term as the middle term, which will be greatest.

   $t_6 = C^{10}_{5}.2^{5}.\left(\frac{9}{5}\right)^{5} = C^{10}_{5}\left(\frac{18}{5}\right)^{5}$.

   ii. For $(4-2)^7$ let t_r is the greatest term. Then $\frac{t_r}{t_{r+1}} > 1$ and $\frac{t_r}{t_{r-1}} > 1$. Substituting and evaluating, we find $r = 3$.

$t_3 = C^7_2.4^5.2^2 = 86016.$

iii. For $(5+2)^{10}$ let t_r is the greatest term. Then $\frac{t_r}{t_{r+1}} > 1$ and $\frac{t_r}{t_{r-1}} > 1$. Substituting and evaluating, we find $r = 4$.

$t_4 = C^{13}_3 5^{10} 2^3.$

53. In any binomial expansion, the middle terms have the greatest coefficient. Therefore, $(15+1)$th term will have greatest coefficient.

$t_{15} = C^{30}_{14} x^{14}$, $t_{16} = C^{30}_{15} x^{15}$, $t_{17} = C^{30}_{16} x^{16}$

$\therefore \frac{t_{16}}{t_{17}} = \frac{16}{15} \cdot \frac{1}{x}$. Since t_{16} is the greatest term $\frac{t_{16}}{t_{17}} > 1 \Rightarrow x < \frac{16}{15}$.

Similarly, considering t_{15} and t_{16}, $x > \frac{15}{16}$.

54. Given, $6^{2n} - 35n - 1 = 36^n - 35n - 1 = (1+35)^n - 35n - 1 = 35^2[C^n_2 + 35.C^n_3 + \cdots + 35^{n-2}]$

$= 1225[C^n_2 + 35.C^n_3 + \cdots + 35^{n-2}] = 1225\times$ a positive integer if $n \geq 2$.

If $n = 1$, given expression becomes 0. Hence, for all positive integral values of n, $6^{2n} - 35n - 1$ is divisible by 1225.

55. $2^{4n} - 2^n(7n+1) = 16^n - 2^n(7n+1) = (2+14)^n - 2^n(7n+1) = 14^2[C^n_2.2^{n-2} + C^n_3.2^{n-3}.14 + \cdots + 14^{n-2}]$, which is divisible by 196 for all positive values of n. If $n = 1$, given expression becomes 0, which is also divisible by 196.

56. $3^{4n+1} + 16n - 3 = 3(3^{4n} - 1) + 16n = 3[81^n - 1] + 16n = 3[(1+80)^n - 1] + 16n$

$= 3[80n + C^n_2 80^2 + C^n_3 80^3 + \cdots + 80^n] + 16n = 256[n + 75(C^n_2 + C^n_3.80 + \cdots + 80^{n-2})]$,

which is divisible by 256 for all $n \in \mathbb{N}$.

57. The problems are solved below:

i. $4^n - 3n - 1 = (1+3)^n - 3n - 1 = C^n_2 3^2 + C^n_3 3^3 + \cdots + 3^n$

$= 9[C^n_2 + C^n_3.3 + \cdots + 3^{n-2}]$,

which is divisible by 9 for $n \geq 2$. When $n = 1$, the given expression becomes 0, and hence divisible by 9. Thus, given expression is divisible by 9 for all $n \in \mathbb{P}$.

ii. $2^{5n} - 31n - 1 = (1+31)^n - 31n - 1 = C^n_2 31^2 + C^n_3 31^3 + \cdots + 31^n$

$= 961[C^n_2 + C^n_3.31 + \cdots + 31^{n-2}]$,

which is divisible by 961 for $n \geq 2$. When $n = 1$, the given expression becomes 0, and hence divisible by 961. Thus, given expression is divisible by 961 for all $n \in \mathbb{P}$.

iii. $3^{2n+2} - 8n - 9 = 9(1+8)^n - 8n - 9 = 9[1 + 8n + C_2^n.8^2 + C_3^n.8^3 + \cdots + 8^n] - 8n - 9 = 64[n + 9(C_2^n + C_3^n 8 + \cdots + 8^{n-2})]$,

which is divisible by 64 for $n \geq 2$.

iv. $2^{5n+5} - 31n - 32 = 32(1+31)^n - 31n - 32 = 32[1 + 31n + C_2^n.31^2 + C_3^n.31^3 + \cdots + 31^n] - 31n - 32 = 961[n + 32(C_2^n + C_3^n 31 + \cdots + 31^{n-2})]$,

which is divisible by 961 for $n > 1$.

v. $3^{2n} - 1 + 24n - 32n^2 = (1+8)^n - 1 + 24n - 32n^2 = 1 + 8n + 32n^2 - 32n + 8^3(C_3^n + C_4^n.8 + \cdots + 8^{n-3}) - 1 + 24n - 32n^2 = 8^3(C_3^n + C_4^n.8 + \cdots + 8^{n-3})$,

which is divisible by 512 for $n > 2$.

58. Let the three consecutive coefficients in the expansion of $(1+x)^n$ be the rth, $(r+1)$th and $(r+2)$th, which are given to be $165, 330$ and 462 respectively.

$\therefore C_{r-1}^n = 165 \Rightarrow \frac{n!}{(r-1)!(n-r+1)!} = 165$

$C_r^n = 330 \Rightarrow \frac{n!}{r!(n-r)!} = 330$, and $C_{r+1}^n = \frac{n!}{(r+1)!(n-r-1)!} = 462$.

From first two, we have $\frac{r}{n-r+1} = \frac{1}{2} \Rightarrow 3r = n+1$.

From last two, we have $\frac{r+1}{n-r} = \frac{5}{7} \Rightarrow 12r = 5n - 7$

Thus, $n = 11, r = 4$. So positions of coefficients are the 4th, 5th and 6th respectively.

59. Let a_1, a_2, a_3 and a_4 be the coefficients of the rth, $(r+1)$th, $(r+2)$th and $(r+3)$th terms respectively in the expansion of $(1+x)^n$.

$\therefore a_1 = C_{r-1}^n, a_2 = C_r^n, a_3 = C_{r+1}^n$, and $a_4 = C_{r+2}^n$.

$\frac{a_2}{a_1} = \frac{n-r+1}{r} \Rightarrow \frac{a_1+a_2}{a_1} = \frac{n+1}{r} \Rightarrow \frac{a_1}{a_1+a_2} = \frac{r}{n+1}$.

Similarly, $\frac{a_2}{a_2+a_3} = \frac{r+1}{n+1}$, and $\frac{a_3}{a_3+a_4} = \frac{r+2}{n+1}$.

Clearly, $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$.

60. 2nd terms $= C_1^n x^{n-1} y = 240$, 3rd term $= C_2^n x^{n-2} y^2 = 720$, and 4th term $= C_3^n x^{n-3} y^3 = 1080$.

From first two, we have $\frac{240}{720} = \frac{2}{n-1} \cdot \frac{x}{y}$.

From last two, we have $\frac{720}{1080} = \frac{3}{n-2} \cdot \frac{x}{y}$

From these two equations $\frac{1}{2} = \frac{2(n-2)}{3(n-1)} \Rightarrow n = 5$.

$\Rightarrow y = \frac{3x}{2}$.

$\Rightarrow 240 = C_1^n x^{n-1} y \Rightarrow x^5 = 32 \Rightarrow x = 2 \Rightarrow y = 3$.

61. Let the index of the power be n. And let a, b, c be the rth, $(r+1)$th, $(r+2)$th coefficients respectively in the expansion of $(1+x)^n$.

$a = C_{r-1}^n, b = C_r^n$, and $c = C_{r+1}^n$.

$\frac{a}{b} = \frac{r}{n-r+1} \Rightarrow an + a = r(a+b), \frac{b}{c} = \frac{r+1}{n-r} \Rightarrow bn - br = cr + c \Rightarrow bn - c = r(b+c)$

$\Rightarrow n = \frac{2ac+ab+bc}{b^2-ac}$.

62. The coefficient of 14th, 15th and 16th terms in the expansion of $(1+x)^n$ will be C_{13}^n, C_{14}^n and C_{15}^n respectively. Given that these are in A.P. $\Rightarrow 2C_{14}^n = C_{13}^n + C_{15}^n$.

$\Rightarrow 2.\frac{n!}{14!(n-14)!} = \frac{n!}{13!(n-13)!} + \frac{n!}{15!(n-15)!}$

$\Rightarrow 2.15(n-13) = 15.14 + (n-13)(n-14) \Rightarrow n = 23, 34$.

63. Let the three consecutive terms are rth, $(r+1)$th and $(r+2)$th. Then, $C_{r-1}^n = 56, C_r^n = 70, C_{r+1} = 56$.

From first two, we have $\frac{r}{n-r+1} = \frac{4}{5}$, and from last two, we have $\frac{r+1}{n-r} = \frac{5}{4}$.

Solving these gives us $n = 8, r = 4$.

64. Let the three consecutive terms are rth, $(r+1)$th and $(r+2)$th. Then, $C_{r-1}^n = 220, C_r^n = 495, C_{r+1} = 792$.

From first two, we have $\frac{r}{n-r+1} = \frac{4}{9}$, and from last two, we have $\frac{r+1}{n-r} = \frac{5}{8}$.

Solving these gives us $n = 12$.

65. $t_3 = C_2^n a^{n-2} x^2 = 84, t_4 = C_3^n a^{n-3} x^3 = 280$, and $t_5 = C_4^n a^{n-4} x^4 = 560$.

From first two, we have $\frac{3}{n-2} \cdot \frac{a}{x} = \frac{3}{10}$, and from last two we have $\frac{4}{n-3} \cdot \frac{a}{x} = \frac{1}{2}$.

$\Rightarrow \frac{3(n-3)}{4(n-2)} = \frac{3}{5} \Rightarrow n = 7, \Rightarrow x = 2, a = 1$.

66. $t_6 = C_5^n x^{n-5} y^5 = 112, t_6 = C_6^n x^{n-6} y^6 = 7$, and $t_8 = C_7^n x^{n-7} y^7 = \frac{1}{4}$.

From first two, we have $\frac{6}{n-5} \cdot \frac{x}{y} = 16$, and from last two we have $\frac{7}{n-6} \cdot \frac{x}{y} = 28$

$\Rightarrow \frac{6(n-6)}{7(n-5)} = \frac{4}{7} \Rightarrow n = 7, x = 4, y = \frac{1}{2}$.

67. Let the binomial expansion be $(x+y)^n$. $a = C_5^n x^{n-5} y^5, b = C_6^n x^{n-6} y^6, c = C_7^n x^{n-7} x^7$, and $d = C_8^n x^{n-8} y^8$.

From first two, we have $\frac{b}{a}=\frac{n-5}{6}.\frac{y}{x}$, from second and third, we have $\frac{c}{b}=\frac{n-6}{7}.\frac{y}{x}$, and from kast two we gace $\frac{d}{c}=\frac{n-7}{8}.\frac{y}{x}$.

Now from first two we have, $\frac{b^2}{ac}=\frac{7(n-5)}{6(n-6)}$ and $\frac{c^2}{bd}=\frac{8(n-6)}{7(n-7)}$

Subtracting 1 from both of these, we have $\frac{b^2-ac}{ac}=\frac{7(n-5)-6(n-6)}{6(n-6)}$, and $\frac{c^2-bd}{bd}=\frac{8(n-6)-7(n-7)}{7(n-7)}$

Dividing, we get $\frac{b^2-ac}{c^2-bd}=\frac{4a}{3c}$.

68. (a) Let a, b, c and d be the rth, $(r+1)$th, $(r+2)$th, and $(r+3)$th term of the binomial expansion $(x+y)^n$.

$a=C^n_{r-1}, b=C^n_r, c=C^n_{r+1}$, and $d=C^n_{r+2}$.

$\frac{b}{a}=\frac{n-r+1}{r}. \Rightarrow \frac{a+b}{a}=\frac{n+1}{r}, \frac{c}{b}=\frac{n-r}{r+1}\Rightarrow\frac{c+b}{b}=\frac{n+1}{r+1}$

$\frac{d}{c}=\frac{n-r-1}{r+2}\Rightarrow\frac{c+d}{c}=\frac{n+1}{r+2}$. Clearly, $\frac{a+b}{a},\frac{b+c}{b},\frac{c+d}{c}$ are in H.P.

(b) $(bc+ad)(b-c)=\frac{(n!)^3}{[(r-1)!]^3[(n-r-2)!]^3}\left(\frac{1}{r(n-r)(n-r-1)}\cdot\frac{1}{r(r+1)(n-r-1)}-\frac{1}{(n-r+1)(n-r)(n-r-1)}\cdot\frac{1}{r(r+1)(r+2)}\right)\left(\frac{1}{r(n-r)(n-r-1)}-\frac{1}{r(r+1)(n-r-1)}\right)$

Now it is trivial to prove that $(bc+ad)(b-c)=2(ac^2-b^2d)$.

69. The coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are C^n_4, C^n_5 and C^n_6. Given that these are in A.P., therefore

$2C^n_5=C^n_4+C^n_6\Rightarrow\frac{2}{5(n-5)}=\frac{1}{(n-4)(n-5)}+\frac{1}{5.6}\Rightarrow n=7,14.$

70. The coefficients of second, third and fourth terms in the expansion of $(1+x)^{2n}$ are in A.P.

$\Rightarrow 2C^{2n}_2=C^{2n}_1+C^{2n}_3\Rightarrow\frac{2}{2(2n-2)}=\frac{1}{(2n-1)(2n-2)}+\frac{1}{2.3}$

$\Rightarrow 2n^2-9n+7=0.$

71. The coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are in A.P.

$\Rightarrow 2C^n_r=C^n_{r-1}+C^n_{r+1}\Rightarrow\frac{2}{r(n-r)}=\frac{1}{(n-r)(n-r+1)}+\frac{1}{r(r+1)}\Rightarrow n^2-n(4r+1)+4r^2-2=0.$

72. Let the coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^n$ are in the ratio of 182 : 84 : 30.

$t_r=C^n_{r-1}, t_{r+1}=C^n_r$ and $t_{r+2}=C^n_{r+1}$

From first two, we have $\frac{r}{n-r+1}=\frac{13}{6}$, and from last two we have $\frac{r+1}{n-r}=\frac{14}{5}$

From these two equations we have $n=18$.

73. Given series is $C_1+2.C_2+3.C_3+\cdots+n.C_n$. Its rth term $t_r=r.C^n_r=n.C^{n-1}_{r-1}[\because r.C^n_r=n.C^{n-1}_{r-1}]$

Now $C_1+2.C_2+3.C_3+\cdots+n.C_n=\sum_{r=1}^{n}r.C^n_r=\sum_{r=1}^{n}n.C^{n-1}_{r-1}$

$=n[C^{n-1}_0+C^{n-1}_1+C^{n-1}_2+\cdots+C^{n-1}_{n-1}]=n(1+1)^{n-1}=n.2^{n-1}$.

Calculus Method: $(1+x)^n=C_0+C_1.x+C_2.x^2+\cdots+C_n.x^n$

Differentiating w.r.t. x, we get $n(1+x)^{n-1}=C_1+2C_2.x+\cdots+nC_n.x^{n-1}$

Putting $x=1$, we get $n.2^{n-1}=C_1+2.C_2+3.C_3+\cdots+n.C_n$.

74. Given series is $C_0+2.C_1+3.C_2+\cdots+(n+1).C_n$.

Its rth term is $t_r=r.C^n_{r-1}=(r-1).C^n_{r-1}+C^n_{r-1}=n.C^{n-1}_{r-2}+C^n_{r-1}[\because(r-1).C^n_{r-1}=n.C^{n-1}_{r-2}]$

Now, $C_0+2.C_1+3.C_2+\cdots+(n+1).C_n=\sum_{r=1}^{n+1}t_r=\sum_{r=1}^{n+1}n.C^{n-1}_{r-2}+\sum_{r=1}^{n+1}C^n_{r-1}$

$=n[C^{n-1}_0+C^{n-1}_1+\cdots+C^{n-1}_{n-1}]+(C^n_0+C^n_1+\cdots+C^n_n)$

$=n.2^{n-1}+2^n=2^{n-1}(n+2)$.

Calculus Method: $(1+x)^n=C_0+C_1.x+C_2.x^2+\cdots+C_n.x^n$

Multiplying with x, we get $x(1+x)^n=C_0.x+C_1.x^2+C_2.x^3+\cdots+C_n.x^{n+1}$

Differentiating w.r.t. x, we get $(1+x)^n+nx(1+x)^{n-1}=C_0+2C_1.x+3C_2.x^2+\cdots+(n+1)C_n.x^n$

Putting $x=1$, we get $C_0+2.C_1+3.C_2+\cdots+(n+1).C_n=2^{n-1}(n+2)$.

75. Given series is $C_0+3.C_1+5.C_2+\cdots+(2n+1).C_n$.

Its rth term is $t_r=(2r-1)C_{r-1}=[2(r-1)+1]C_{r-1}=2(r-1)C_{r-1}+C_{r-1}$

$=2.nC^{n-1}_{r-2}+C_{r-1}[\because(r-1)C_{r-2}=n.C^{n-1}_{r-2}]$

Now, $C_0+3.C_1+5.C_2+\cdots+(2n+1).C_n=\sum_{r=1}^{n+1}t_r=2n\sum_{r=1}^{n+1}C^{n-1}_{r-2}+\sum_{r=1}^{n+1}C_{r-1}$

$=2n(C^{n-1}_0+C^{n-1}_1+\cdots+C^{n-1}_{n-1})+(C_0+C_1+\cdots+C_n)$

$=2n.2^{n-1}+2^n=2^n(n+1)$.

Calculus Method: $(1+x)^n = C_0 + C_1.x + C_2.x^2 + \cdots + C_n.x^n$

Putting $x = x^2$ and multiplying with x, we get

$x(1+x^2)^n = C_0.x + C_1.x^3 + C_2.x^5 + \cdots + C_n.x^{2n+1}$

Differentiating both sides w.r.t. x, we get

$(1+x^2)^n + 2x^2.n(1+x^2)^{n-1} = C_0 + C_1.3x^2 + C_2.5x^4 + \cdots + C_n.(2n+1)\,x^{2n+1}$

Putting $x = 1$, we get

$C_0 + 3.C_1 + 5.C_2 + \cdots + (2n+1)\,.C_n = 2n.2^{n-1} + 2^n = 2^n(n+1).$

76. We have to prove that $C_1 - 2.C_2 + 3.C_3 - 4.C_4 + \cdots + (-1)^{n-1}\,n.C_n = 0.$

rth term $t_r = (-1)^{r-1}\,r.C_r^n = (-1)^{r-1}\,.nC_{r-1}^{n-1}$

$\sum\limits_{r=1}^{n} t_r = \sum\limits_{r=1}^{n}(-1)^{r-1}\,n.C_{r-1} = n.\sum\limits_{r=1}^{n}(-1)^{r-1}\,C_{r-1}$

$= n(C_0^{n-1} - C_1^{n-1} + C_2^{n-1} - C_3^{n-1} + \cdots + (-1)^{n-1}\,C_{n-1}^{n-1})$

$= n(1-1)^{n-1} = 0.$

Calculus Method: $(1+x)^n = C_0 + C_1.x + C_2.x^2 + \cdots + C_n.x^n.$

Differentiating both sides w.r.t. x, we get

$n(1+x)^{n-1} = C_1 + 2x.C_2 + 3x^2C_3 + \cdots + nx^{n-1}C_n$

Putting $x = -1$, we get

$n(1-1)^{n-1} = C_1 - 2C_2 + 3C_3 - \cdots + (-1)^{n-1}\,.nC_n = 0.$

77. Given series is $C_0 + \frac{C_1}{2} + \frac{C_3}{3} + \cdots + \frac{C_n}{n+1}.$

Its rth term is $t_r = \frac{C_{r-1}^n}{r}.$

$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \sum\limits_{r=1}^{n+1}\frac{C_{r-1}^n}{r} = \sum\limits_{r=1}^{n+1}\frac{C_r^{n+1}}{n+1}\left[\because \frac{C_{r-1}^n}{r} = \frac{C_r^{n+1}}{n+1}\right]$

$= \frac{1}{n+1}(C_1^{n+1} + C_2^{n+2} + \cdots + C_{n+1}^{n+1}) = \frac{2^{n+1}-1}{n+1}\,[\because$ we add and subtract $C_0^{n+1}\,]$.

Calculus Method: $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Integrating within limits 0 and 1, we have

$\left[\frac{(1+x)^n}{n}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_3\frac{x^3}{3} + \cdots + C_n\frac{x^{n+1}}{n+1}\right]_0^1$

$\Rightarrow \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}.$

78. Given series is $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$.

Its rth term $t_r = (-1)^r \frac{C^n_{r-1}}{r}$

Now $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots + (-1)^n \frac{C_n}{n+1} = \sum_{r=1}^{n+1} (-1)^{r-1} \frac{C^n_{r-1}}{r}$

$= \sum_{r=1}^{n+1} (-1)^{r-1} \frac{C^{n+1}_r}{n+1} \left[\because \frac{C^n_{r-1}}{r} = \frac{C^{n+1}_r}{n+1}\right]$

$= \frac{1}{n+1} [C^{n+1}_1 - C^{n+1}_2 + C^{n+1}_3 - \cdots + (-1)^n C^{n+1}_{n+1}]$

$= \frac{1}{n+1} [-(1-1)^{n+1} + C^{n+1}_0] = \frac{1}{n+1}$.

Calculus Method: $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$

Integrating within limits 0 and -1, we have

$\left[\frac{(1+x)^n}{n}\right]_0^{-1} = \left[C_0 x + C_1 \frac{x^2}{2} + C_3 \frac{x^3}{3} + \cdots + C_n \frac{x^{n+1}}{n+1}\right]_0^{-1}$

$\Rightarrow -\frac{1}{n+1} = -\left[C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots + (-1)^n \frac{C_n}{n+1}\right]$

$\Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$.

79. Given series is $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots$

$t_r = \frac{C^n_{2r-1}}{2r} = \frac{C^{n+1}_{2r}}{n+1}$

Now, $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots = \frac{1}{n+1} \sum_{r=1} C^{n+1}_{2r} = \frac{1}{n+1} [C^{n+1}_2 + C^{n+1}_4 + C^{n+1}_6 + \cdots]$

$= \frac{2^n - 1}{n+1} [\because C^n_0 + C^n_2 + C^n_4 + \cdots = 2^{n-1}]$.

Calculus Method: Adding the results of last two problems, we have

$2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots\right] = \frac{2^{n+1} - 1 - 1}{n+1} = \frac{2(2^n - 1)}{n+1}$

$\Rightarrow \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots = \frac{2^n - 1}{n+1}$.

80. Given series is $2.C_0 + 2^2 . \frac{C_1}{2} + 2^3 . \frac{C_2}{3} + \cdots + 2^{n+1} . \frac{C_n}{n+1}$

$t_r = 2^r . \frac{C^n_{r-1}}{r} = 2^r . \frac{C^{n+1}_r}{n+1}$.

Now, $2.C_0 + 2^2 . \frac{C_1}{2} + 2^3 . \frac{C_2}{3} + \cdots + 2^{n+1} . \frac{C_n}{n+1} = \frac{1}{n+1} \sum_{r=1}^{n+1} C^{n+1}_r . 2^r$

$= \frac{1}{n+1}[C_1^{n+1}.2 + C_2^{n+1}.2^2 + C_3^{n+1}.2^3 + \cdots + C_{n+1}^{n+1}.2^{n+1}]$

$= \frac{1}{n+1}[(1+2)^{n+1} - 1] = \frac{3^{n+1}-1}{n+1}.$

Calculus Method: $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Integrating within the limits of 0 and 2, we have

$\left[C_0x + C_1.\frac{x^2}{2} + C_2.\frac{x^3}{3} + \cdots + C_n..\frac{x^{n+1}}{n+1}\right]_0^2 = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^2$

$\Rightarrow C_0.2 + \frac{C_1}{2}.2^2 + \frac{C_2}{3}.2^3 + \cdots + \frac{C_n}{n+1}.2^{n+1} = \frac{3^{n+1}-1}{n+1}.$

81. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiplying these two, we have $(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \cdots + C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n)$

Coeff. of x^{n+r} on R.H.S. $= C_0c_r + C_1C_{r+1} + \cdots + C_{n-r}C_n$, and on L.H.S. $= C_{n+r}^{2n} = \frac{(2n)!}{(n+r)!(n-r)!}$.

82. $(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \cdots + C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n)$

Equating the coeff. of x^n, we have

$C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = C_n^{2n} = \frac{2n!}{n!n!}.$

83. $t_r = r.\frac{C_r^n}{C_{r-1}^n} = n - r + 1$

L.H.S. $= t_1 + t_2 + t_3 + \cdots + t_n = n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$.

84. $[(1+x)^n]^2 = (C_0 + C_1.x + C_2.x^2 + \cdots + C_n.x^n)^2$, and $(1+x)^{2n} = (C_0^{2n} + C_1^{2n}.x + C_2^{2n}.x^2 + \cdots + C_{2n}^{2n}.x^{2n})$

Putting $x = 1$, it is trivial to see that

$(1 + C_1 + C_2 + \cdots + C_n)^2 = 1 + C_1^{2n} + C_2^{2n} + \cdots + C_{2n}^{2n}.$

85. $[(1+x)^n]^5 = (C_0 + C_1.x + C_2.x^2 + \cdots + C_n.x^n)^5$, and $(1+x)^{5n} = (C_0^{5n} + C_1^{5n}.x + C_2^{5n}.x^2 + \cdots + C_{5n}^{5n}.x^{5n})$

Putting $x = 1$, it is trivial to see that

$(1 + C_1 + C_2 + \cdots + C_n)^5 = 1 + C_1^{5n} + C_2^{5n} + \cdots + C_{5n}^{5n}.$

86. We have to prove that $C_0 + 5.C_1 + 9.C_2 + \cdots + (4n+1).C_n = (2n+1)2^n$.

L.H.S. $= 4.0.C_0 + 4.1.C_1 + 4.2.C_2 + \cdots + 4nC_n + C_0 + C_1 + C_2 + \cdots + C_n$

We know that $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$.

$C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n$. Its rth term $t_r = r.C^n_r = n.C^{n-1}_{r-1}[\because r.C^n_r = n.C^{n-1}_{r-1}]$

Now $C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n = \sum_{r=1}^{n} r.C^n_r = \sum_{r=1}^{n} n.C^{n-1}_{r-1}$

$= n[C^{n-1}_0 + C^{n-1}_1 + C^{n-1}_2 + \cdots + C^{n-1}_{n-1}] = n(1+1)^{n-1} = n.2^{n-1}$

Multiplying with 4, we have

$4.0.C_0 + 4.1.C_1 + 4.2.C_2 + \cdots + 4nC_n = 2n.2^n$.

Thus, $C_0 + 5.C_1 + 9.C_2 + \cdots + (4n+1).C_n = (2n+1)2^n$.

87. We have to prove that $1 - (1+x)C_1 + (1+2x)C_2 - (1+3x)C_3 + \cdots = 0$.

$\Rightarrow 1 - 1 + 1 - 1 + \cdots - x[C_1 - 2.C_2 + 3.C_3 - \cdots] = 0$

That is we have to prove that $C_1 - 2.C_2 + 3.C_3 - \cdots = 0$, which has been proven earlier.

88. We have to prove that $3.C_1 + 7.C_2 + \cdots + (4n-1).C_n = (2n-1)2^n$.

L.H.S. $= 4.0.C_0 + 4.1.C_1 + 4.2.C_2 + \cdots + 4nC_n - [C_0 + C_1 + C_2 + \cdots + C_n]$

We know that $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$.

$C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n$. Its rth term $t_r = r.C^n_r = n.C^{n-1}_{r-1}[\because r.C^n_r = n.C^{n-1}_{r-1}]$

Now $C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n = \sum_{r=1}^{n} r.C^n_r = \sum_{r=1}^{n} n.C^{n-1}_{r-1}$

$= n[C^{n-1}_0 + C^{n-1}_1 + C^{n-1}_2 + \cdots + C^{n-1}_{n-1}] = n(1+1)^{n-1} = n.2^{n-1}$

Multiplying with 4, we have

$4.0.C_0 + 4.1.C_1 + 4.2.C_2 + \cdots + 4nC_n = 2n.2^n$.

Thus, $3.C_1 + 7.C_2 + \cdots + (4n-1).C_n = (2n-1)2^n$.

89. We have to prove that $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \cdots = \frac{2^n}{n+1}$.

$t_r = \frac{C^n_{2r-2}}{2r-1} = \frac{C^{n+1}_{2r-1}}{n+1}$

Now $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \cdots = \frac{2^n}{n+1} = \frac{1}{n+1}[C^{n+1}_0 + C^{n+1}_3 + C^{n+1}_5 + \cdots]$

$= \frac{2^n}{n+1}$.

90. We know that $(1+x)^n = C^n_0 + C^n_1 x + C^n_2 x^2 + \cdots + C^n_n x^n$ and $(1+x)^{n+1} = C^{n+1}_0 + C^{n+1}_1 x + C^{n+2}_2 x^2 + \cdots + C^{n+1}_{n+1} x^{n+1}$.

Multiplying equations above and equating coeff. of x^{n+1}, we get

$C_0^nC_1^{n+1} + C_1^nC_2^{n+1} + \cdots + C_n^nC_{n+1}^{n+1} = C_{n+1}^{2n+1} = \frac{(2n+1)!}{(n+1)!n!}$.

91. Given series is $C_0 - 2.C_1 + 3.C_2 - \cdots + (-1)^n(n+1)C_n = 0$.

$t_r = (-1)^{r-1}.rC_{r-1}^n = (-1)^r(r-1)C_{r-1}^n + (-1)^{r-1}C_{r-1}^n = (-1)^{r-1}.nC_{r-2}^{n-1} + (-1)^{r-1}C_{r-1}^n$.

$C_0 - 2.C_1 + 3.C_2 - \cdots + (-1)^n(n+1)C_n = \sum_{r=1}^{n+1}(-1)^{r-1}.nC_{r-2}^{n-1} + \sum_{r=1}^{n+1}(-1)^{r-1}C_{r-1}^n$

$= -n(C_0^{n-1} - C_1^{n-1} + \cdots + (-1)^{n-1}C_{n-}^{n-1}) + (C_0^n - C_1^n + C_2^n + \cdots + (-1)^{n-1}C_n^n)$

$= -n(1-1)^{n-1} + (1-1)^n = 0$.

Calculus Method: Given $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Multiplying with x, and differentiating w.r.t. x, we get

$1.(1+x)^n + x.n(1+x)^{n-1} = C_0 + C_1.2x + C_2.3x^2 + \cdots + (n+1)C_n.x^n$

Putting $x = -1$, we get

$C_0 - 2.C_1 + 3.C_2 - \cdots + (-1)^n(n+1)C_n = 0$.

92. Given series is $C_0 - 3.C_1 + 5.C_2 - \cdots + (-1)^n(2n+1)C_n = 0$.

$t_r = (-1)^{r-1}(2r-1)C_{r-1} = (-1)^{r-1}[2(r-1)+1]C_{r-1} = 2(-1)^{r-1}C_{r-1}^n + (-1)^{r-1}C_{r-1}^n = 2(-1)^{r-1}.nC_{r-2}^{n-1} + (-1)^{r-1}.C_{r-1}^n$

$C_0 - 3.C_1 + 5.C_2 - \cdots + (-1)^n(2n+1)C_n = 2n\sum_{r=1}^{n+1}(-1)^{r-1}C_{r-2}^{n-1} + \sum_{r=1}^{n+1}C_{r-1}^n$

$= -2n[C_0^{n-1} - C_1^{n-1} + \cdots + (-1)^nC_{n-1}^{n-1}] + [C_0 - C_1 + C_2 - \cdots + (-1)^nC_n]$

$= -2n(1-1)^{n-} + (1-1)^n = 0$.

Calculus Method: We know that $(1+x^2)^n = C_0 + C_1x^2 + C_2x^4 + \cdots + C_nx^{2n}$

Multiplying both sides with x, and differentiating w.r.t. x, we get

$C_0 + 3C_1x^2 + 5.C_2x^4 + \cdots + (2n+1)C_nx^{2n} = (1+x^2)^n + nx.2x(1+x^2)^n$

Putting $x = -1$, we have

$C_0 - 3.C_1 + 5.C_2 - \cdots + (-1)^n(2n+1)C_n = 0$.

93. Given series is $a - (a-1)C_1 + (a-2)C_2 - (a-3)C_3 + \cdots + (-1)^n(a-n)C_n = 0$.

$t_r = (-1)^{r-1}[a+1-r]C_{r-1}^n = (-1)^{r-1}[a-(r-1)]C_{r-1} = a(-1)^{r-1}C_{r-1} - (-1)^{r-1}n.C_{r-2}^{n-1}$.

$a-(a-1)C_1+(a-2)C_2-(a-3)C_3+\cdots+(-1)^n(a-n)C_n = a\sum_{r=1}^{n+1}(-1)^{r-1}C_{r-1}-n\sum_{r=1}^{n+1}.C_{r-2}^{n+1}$

$= a[C_0-C_1+C_2-\cdots+(-1)^nC_n]-n[C_0^{n-1}-C_1^{n-1}+C_2^{n-1}-\cdots+(-1)^nC_{n-1}^{n-1}]$

$= a(1-1)^n-n(1-1)^{n-1}=0.$

Calculus Method: L.H.S. $= a-(a-1)C_1+(a-2)C_2-(a-3)C_3+\cdots+(-1)^n(a-n)C_n$

$= a[C_0-C_1+C_2-\cdots+(-1)^nC_n]+1[C_1-2.C_2+3.C_3-\cdots+(-1)^n(-n)C_n]$

We have shown both the series in brackets equal to zero earlier.

94. Given series is $1^2.C_1+2^2.C_2+3^2C_3+\cdots+n^2.C_n=n(n+1)2^{n-2}$.

$t_r=r^2.C_r=r.n.C_{r-1}^{n-1}=n.[r-1+1]C_{r-1}^{n-1}=n(n-1).C_{r-2}^{n-2}+n.C_{r-2}^{n-2}.$

$1^2.C_1+2^2.C_2+3^2C_3+\cdots+n^2.C_n=n(n-1)[C_0^{n-2}+C_1^{n-2}+\cdots+C_{n-2}^{n-2}]+n[C_0^{n-1}+C_1^{n-1}+C_2^{n-2}+\cdots+C_{n-1}^{n-1}]$

$= n(n-1).2^{n-2}+n.2^{n-1}=2^{n-2}[n(n-1)+2n]=n(n+1).2^{n-2}.$

Calculus Method: $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$

Differentiating w.r.t. x, we have

$n(1+x)^{n-1}=C_1+2.C_1x+3.C_2x^2+\cdots+n.C_nx^{n-1}$

Multiplying both sides w.r.t. x, and differentiating again, we have

$n[(1+x)^{n-1}+(n-1)x(1+x)^{n-2}]=1^2.C_1+2^2.C_1x+3^2C_2x^2+\cdots+n^2.C_nx^{n-1}$

Putting $x=1$, we arrive at

$1^2.C_1+2^2.C_2+3^2C_3+\cdots+n^2.C_n=n[2^{n-1}+(n-1)2^{n-2}]=n(n+1)2^{n-2}.$

95. Given series is $C_0-2^2.C_1+3^2.C_2-\cdots+(-1)^n(n+1)^2C_n=0$, $n>2$.

$t_{r+1}=(-1)^rr^2C_r=(-1)^r[r^2+2r+1]C_r$

$\sum_{r=0}^{n}(-1)^r.C_r^n=C_0-C_1+C_2-\cdots+(-1)^n.C_n=0$

$\sum_{r=0}^{n}(-1)^r.rC_r = \sum_{r=0}^{n}(-1)^r.nC_{r-1}^{n-1} = -n[C_0^{n-1}-C_1^{n-1}+C-2^{n-1}-\cdots+(-1)^{n-1}C_{n-1}^{n-1}]$

$= -n(1-1)^{n-1}=0.$

$$\sum_{r=0}^{n}(-1)^r.r^2C_r^n = \sum_{r=0}^{n}(-1)^r.r.nC_{r-1}^{n-1} = n\sum_{r=0}^{n}(-1)^r(r-1)C_{r-1}^{n-1} + n\sum_{r=0}^{n}C_{r-1}^{n-1}$$

$$= n(n-1)\sum_{r=0}^{n}(-1)^r.C_{r-2}^{n-2} + n.0 = 0.$$

Thus, $\sum_{r=0}^{n}t_r = 0$.

96. Given series is $C_0.abc - C_1(a-1)(b-1)(c-1) + C_2(a-2)(b-2)(c-2) - \cdots + (-1)^n.C_n(a-n)(b-n)(c-n) = 0$.

$t_{r+1} = (-1)^r[abc + (a+b+c)r^2 - (ab+bc+ca)r - r^3]C_r^n$.

$$\sum_{r=0}^{n}(-1)^r.C_r^n = C_0 - C_1 + C_2 - \cdots + (-1)^n.C_n = 0$$

$$\sum_{r=0}^{n}(-1)^r.rC_r = \sum_{r=0}^{n}(-1)^r.nC_{r-1}^{n-1} = -n[C_0^{n-1} - C_1^{n-1} + C - 2^{n-1} - \cdots + (-1)^{n-1}C_{n-1}^{n-1}]$$

$$= -n(1-1)^{n-1} = 0.$$

$$\sum_{r=0}^{n}(-1)^r.r^2C_r^n = \sum_{r=0}^{n}(-1)^r.r.nC_{r-1}^{n-1} = n\sum_{r=0}^{n}(-1)^r(r-1)C_{r-1}^{n-1} + n\sum_{r=0}^{n}C_{r-1}^{n-1}$$

$$= n(n-1)\sum_{r=0}^{n}(-1)^r.C_{r-2}^{n-2} + n.0 = 0.$$

Similarly, $\sum_{r=0}^{n}(-1)^r.r^3.C_r^n = 0$. And thus $\sum_{r=0}^{n}t_r = 0$.

97. $t_{r+1} = r^2.C_r p^r q^{n-r} = r.n.C_{r-1}^{n-1}p^r q^{n-r} = n.(r-1+1)C_{r-1}^{n-1}p^r q^{n-r}$

$= n[(n-1)C_{r-2}^{n-2} + C_{r-1}^{n-1}]p^r q^{n-r} = n(n-1)p^2C_{r-2}^{n-2}p^{r-2}q^{n-2-(r-2)} + np^{n-1}.C_{r-1}^{n-1}p^{r-1}q^{n-1-(r-1)}$

$$\therefore \sum_{r=0}^{n}t_{r+1} = n(n-1)p^2(p+q)^{n-2} + np(p+q)^{n-1} = n(n-1)p^2 + np = n^2p^2 + npq.$$

98. $(1+x)^{10} = C_0 + C_1x + C_2x^2 + \cdots + C_{10}x^{10}$

Integrating between limits of 0 and 2, we have

$$\left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \cdots + C_{10}\frac{x^{11}}{11}\right]_0^2 = \left[\frac{(1+x)^{11}}{11}\right]_0^2$$

$$\Rightarrow 2.C_0 + \frac{2^2}{2}.C_1 + \frac{2^3}{3}.C_2 + \cdots + \frac{2^{11}}{11}.C_{11} = \frac{3^{11}-1}{11}.$$

99. Given series is $\frac{2^2}{1.2}C_0+\frac{2^3}{2.3}C_2+\frac{2^4}{3.4}C_2+\cdots+\frac{2^{n+2}}{(n+1)(n+2)}C_n=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$.

$t_r=\frac{2^{r+1}}{r(r+1)}C^n_{r-1}=\frac{2^{r+1}}{r+1}.\frac{C^{n+1}_r}{n+1}=\frac{2^{r+1}}{r(r+1)}.\frac{C^{n+1}_r}{r+1}=\frac{2^{r+1}}{n+1}\frac{C^{n+2}_{r+1}}{n+2}$

$\Rightarrow\frac{2^2}{1.2}C_0+\frac{2^3}{2.3}C_2+\frac{2^4}{3.4}C_2+\cdots+\frac{2^{n+2}}{(n+1)(n+2)}C_n=\frac{1}{(n+1)(n+2)}\sum_{r=1}^{n+1}2^{r+1}.C^{n+2}_{r+1}$

$=\frac{1}{(n+1)(n+2)}[2^2.C^{n+2}_2+2^3.C^{n+2}_3+\cdots+2^{n+2}.C^{n+2}_{n+2}]$

$\frac{1}{(n+1)(n+2)}[(1+2)^{n+2}-C^{n+2}_0-C^{n+2}_1]$

$=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$.

Calculus Method: $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$.

Integrating within the limits of 0 and x, we have

$\left[C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+\cdots+C_n\frac{x^{n+1}}{n+1}\right]_0^x=\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x$

$\Rightarrow\left[C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+\cdots+C_n\frac{x^{n+1}}{n+1}\right]_0^x=\frac{(1+x)^{n+1}}{n+1}-\frac{1}{n+1}$

Integrating again within the limits of 0 and 2, we arrive at

$\frac{2^2}{1.2}C_0+\frac{2^3}{2.3}C_2+\frac{2^4}{3.4}C_2+\cdots+\frac{2^{n+2}}{(n+1)(n+2)}C_n=\left[\frac{(1+x)^{n+2}}{(n+1)(n+2)}-\frac{x}{n+1}\right]_0^2$

$=\frac{3^{n+2}}{(n+1)(n+2)}-\frac{2}{n+1}-\frac{1}{(n+2)(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$.

100. Let $S_n=C_1-\frac{C_2}{2}+\frac{C_3}{3}-\cdots+\frac{(-1)^nC_n}{n}=n-\frac{1}{2}.\frac{n(n-1)}{2!}+\frac{1}{3}.\frac{n(n-1)(n-2)}{3!}-\cdots$

$=(n-1+1)-\frac{1}{2}.\frac{(n-1)(n-2+2)}{2!}+\frac{1}{3}.\frac{(n-1)(n-2)(n-3+3)}{3!}-\cdots$

$=\left[(n-1)-\frac{1}{2}.\frac{(n-1)(n-2)}{2!}+\frac{1}{3}.\frac{(n-1)(n-2)(n-3)}{3!}-\cdots\right]+\left[1-\frac{n-1}{2!}+\frac{(n-1)(n-2)}{3!}-\cdots\right]$

$=S_{n-1}+\frac{1}{n}[C_1-C_2+C_3-\cdots]=S_{n-1}+\frac{1}{n}\Rightarrow S_n-S_{n-1}=\frac{1}{n}$

And thus, $S_{n-1}-S_{n-2}=\frac{1}{n-1},S_{n-2}-S_{n-3}=\frac{1}{n-2},\ldots,S_2-S_1=\frac{1}{2},S_1=1$.

Adding, we get $S_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

Calculus Method: $1-C_1x+C_2x^2-C_3x^3+\cdots+(-1)^nC_nx^n=(1-x)^n$

$\Rightarrow C_1-C_2x+C_3x^2-\cdots+(-1)^{n-1}C_nx^{n-1}=\frac{1-(1-x)^n}{x}$

Integrating between the limits of 0 and 1, we arrive at

$$\left[C_1 x - C_2.\frac{x^2}{2} + C_3.\frac{x^3}{3} - \cdots + (-1)^{n-1} C_n.\frac{x^n}{n}\right]_0^1 = \int_0^1 \frac{1-(1-x)^n}{x}\,dx$$

Now $\int_0^1 \frac{1-(1-x)^n}{x}\,dx = \int_0^1 \frac{1-z^n}{1-z}\,dz$, where $z = 1-x$

$$= \int_0^1 (1 + z + z^2 + \cdots + z^n)\,dz = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

101. $(1-x)^n = C_0 - C_1 x + C_2 x^2 - \cdots + (-1)^n C_n x^n$

Substituting $x = x^4$, we have

$(1-x^4)^n = C_0 - C_1 x^4 + C_2 x^8 - \cdots + (-1)^n C_n x^{4n}$

Integrating within the limits of 0 and 1, we deduce

$$\left[C_0 x - C_1 \frac{x^5}{5} + C_2.\frac{x^9}{9} - \cdots + (-1)^n C_n \frac{x^{4n+1}}{4n+1}\right] = \int_0^1 (1-x^4)^n\,dx$$

Now we will evaluate the R.H.S. Let $I_n = \int_0^n (1-x^4)^n\,dx = [x(1-x^4)^n]_0^1 - \int_0^1 x.n(1-x^4)^{n-1}.(-4x^3)\,dx$

$$= -4n \int_0^1 (1-x^4)^{n-1}(1-x^4-1)\,dx = -4nI_n + 4nI_{n-1} \Rightarrow \frac{I_n}{I_{n-1}} = \frac{4n}{4n+1}$$

Now, $\frac{I_n}{I_0} = \frac{I_n}{I_{n-1}}.\frac{I_{n-1}}{I_{n-1}}\cdots\frac{I_3}{I_2}.\frac{I_2}{I_1}.\frac{I_1}{I_0}$

$$= \frac{4n}{4n+1}.\frac{4n-4}{4n-3}.\frac{4n-8}{4n-7}\cdots\frac{4}{5} = \frac{4^n.n!}{5.9.\cdots(4n+1)}.$$

Aliter: Putting $x^2 = \sin\theta, 2xdx = \cos\theta d\theta \therefore dx = \frac{\cos\theta d\theta}{\sqrt{\sin\theta}}$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{2}$.

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\cos^{2n+1}\theta}{2\sqrt{\sin\theta}}\,d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n}\theta.\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$$

$$= [\cos^{2n}\theta.\sqrt{\sin\theta}]_0^{\frac{\pi}{2}} - 2n\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta(-\sin\theta)\sqrt{\sin\theta}\,d\theta$$

$$= 0 + 2n\int_0^{\frac{\pi}{2}} \frac{\cos^{2n-1}\theta.\sin^2\theta}{\sqrt{\sin\theta}}\,d\theta = 4n\int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta(1-\cos^2\theta).\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$$

$$= 4n\int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta.\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta - 4n\int_0^{\frac{\pi}{2}} \cos^{2n}\theta.\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$$

$$= 4I_{n-1} - 4nI_n \Rightarrow \frac{I_n}{I_{n-1}} = \frac{4n}{4n+1}$$

And now we can proceed like earlier.

102. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + C_nx^n$

Multiply both sides with x^{n-1} to get

$x^{n-1}(1-x)^n = C_0x^{n-1} - C_1x^n + C_2x^{n+1} - \cdots + (-1)^n C_nx^{2n-1}$

Integrate both sides between limits 0 and 1 to get

$$\int_0^1 x^{n-1}(1-x)^n\,dx = \left[C_0.\frac{x^n}{n} - C_1.\frac{x^{n+1}}{n+1} + C_2.\frac{x^{n+2}}{n+2} - \cdots + C_n\frac{x^{2n}}{2n}\right]_0^1$$

$$\Rightarrow \frac{C_0}{n} - \frac{C_1}{n+1} + \frac{C_2}{n+2} - \cdots + (-1)^n\frac{C_n}{2n} = \int_0^1 x^{n-1}(1-x)^n\,dx$$

Now we evaluate $\int_0^1 x^{n-1}(1-x)^n\,dx$.

Let $I_{n-1,n} = \int_0^1 x^{n-1}(1-x)^n\,dx = \left[x^{n-1}.\frac{(1-x)^{n+1}}{-(n+1)}\right]_0^1 - \int_0^1 (n-1)x^{n-2}\frac{(1-x)^{n+1}}{-((n+1)}\,dx$

$$= 0 + \frac{n-1}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1}\,dx = \frac{n-1}{n+1}I_{n-2,n+1}$$

$$= \frac{n-1}{n+1}\cdot\frac{n-2}{n+2}I_{n-3,n+2} = \frac{n-1}{n+1}\cdot\frac{n-2}{n+2}\cdot\frac{n-3}{n+3}I_{n-4,n+3} = \cdots = \frac{n-1}{n+1}\cdot\frac{n-2}{n+2}\cdot\frac{n-3}{n+3}\cdots\frac{1}{2n-1}I_{0,2n-1}$$

$$= \frac{n-1}{n+1}\cdot\frac{n-2}{n+2}\cdot\frac{n-3}{n+3}\cdots\frac{1}{2n-1}int_0^1x^0(1-x)^{2n-1}\,dx = \frac{n-1}{n+1}\cdot\frac{n-2}{n+2}\cdot\frac{n-3}{n+3}\cdots\frac{1}{2n-1}\cdot\frac{1}{2n}$$

$$= \frac{n!(n-1)!}{2n!}.$$

103. L.H.S. $= \left(\frac{C_0}{n} - \frac{C_0}{n+1}\right) - \left(\frac{C_1}{n+1} - \frac{C_1}{n+2}\right) + \cdots + (-1)^n\left(\frac{C_n}{2n} - \frac{C_n}{2n+1}\right)$

$$= \frac{C_0}{n} - \frac{C_1}{n+1} + \frac{C_2}{n+2} - \cdots + (-1)^n\frac{C_n}{2n} - \left[\frac{C_0}{n+1} - \frac{C_1}{n+2} + \cdots + (-1)^n\frac{C_n}{2n+2}\right]$$

Now the problem is similar to previous one and we proceed similarly to arrive at the answer.

104. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$

Multiply both sides by x^{jk-1} to get

$C_0x^{k-1} - C_{-1}\,x^k + C_2x^{k+1} - \cdots + (-1)^n C_nx^{n+k-1} = x^{k-1}(1-x)^n$

Integrate both sides with limts 0 and 1 to get

$$\left[C_0\frac{x^k}{k} - C_1\frac{x^{k+1}}{k+1} + C_2\frac{x^{k+2}}{k+2} - \cdots + (-1)^n C_n\frac{x^{n+k}}{n+k}\right]_0^1 = \int_0^1 x^{k-1}(1-x)^n\,dx$$

$$\Rightarrow \frac{C_0}{k} - \frac{C_1}{k+1} + \frac{C_2}{k+2} - \cdots + (-1)^n \frac{C_n}{k+n} = \int_0^1 x^{k-1}(1-x)^n\, dx.$$

Now we evaluate $\int_0^1 x^{k-1}(1-x)^n\, dx$.

$$I_{k-1,n} = \int_0^1 x^{k-1}(1-x)^n\, dx = \left[x^{k-1}.\frac{(1-x)^{n+1}}{-(n+1)}\right]_0^1 + \frac{k-1}{n+1}\int_0^1 x^{k-2}(1-x)^{n+1}\, dx$$

$$= 0 + \frac{k-1}{n+1} I_{k-2,n+1}.$$

Now we proceed like previous problem we obtain the answer as $\frac{n!}{k(k+1)(k+2)\cdots(n+k)}$.

105. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$ and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiply both of above equation we arrive at

$(C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n) = (1-x^2)^n$

Coeff. of x^n on L.H.S. $= C_0^2 - C_1^2 + C_2^2 - \cdots + (-1)^n C_n^2$

R.H.S. $= C_0 - C_1x^2 + C_2x^4 + \cdots$

We observe that power of x are even on right hand side. So if n is odd then coeff. is 0.

If n is even then coeff. of x^n on R.H.S. $= (-1)^{\frac{n}{2}} C_{n/2} = (-1)^{\frac{n}{2}} \frac{n!}{\left(\frac{n!}{2}\right)^2}$.

106. $(1+x)^n = C_0^n + C_1^n + C_2^nx^2 + \cdots + C_{r-1}^nx^{r-1} + C_r^nx^r + \cdots + C_n^nx^n$ and $(1+x)^m = C_0^m + C_1^m + C_2^mx^2 + \cdots + C_{r-1}^mx^{r-1} + C_r^mx^r + \cdots + C_m^nx^m$.

Multiply these two and equating the coeff. of x^r we get

$C_r^mC_0^n + C_{r-1}^mC_1^n + C_{r-2}^mC_2^n + \cdots + C_0^mC_r^n = C_r^{m+n}$.

107. $(1-x)^{2n} = C_0^{2n} - C_1^{2n}x + C_2^{2n}x^2 - \cdots + (-1)^{2n} C_{2n}^2 n$ and $(x+1)^{2n} = C_0^{2n}x^{2n} + C_1^{2n}x^{2n-1} + \cdots + C_{2n}^{2n}$.

We multiply these two and equate the coeff. of x^{2n} to arrive at

$(C_0^{2n})^2 - (C_1^{2n})^2 + \cdots + (-1)^{2n}(C_{2n}^{2n})^2 = (-1)^n C_n^{2n}$.

108. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Differentiating both sides w.r.t. x we deduce

$n(1+x)^{n-1} = C_1 + 2.C_2x + 3.C_3x^2 + \cdots + n.C_nx^{n-1}$

Also, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiply last two equations to arrive at

$n(1+x)^{2n-1} = (C_1 + 2.C_2x + 3.C_3x^2 + \cdots + n.C_nx^{n-1})(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n)$

Equating coeff. of x^{n-1}, we get

$C_1^2 + 2.C_2^2 + 3.C_3^2 + \cdots + n.C_n^2 = n.C_{n-1}^{2n-1} = \frac{(2n-1)!}{[(n-1)!]^2}$.

109. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Integrating within limits 0 and x, we get

$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + C_1.\frac{x^2}{2} + C_2.\frac{x^3}{3} + \cdots + C_n.\frac{x^{n+1}}{n+1}\right]_0^x$

$\Rightarrow \frac{(1+x)^{n+1}-1}{n+1} = C_0x + C_1.\frac{x^2}{2} + C_2.\frac{x^3}{3} + \cdots + C_n.\frac{x^{n+1}}{n+1}$

Also, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiplying last two equations and equating coeff. of x^{n+1} we get desired reqult.

110. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$

Multiply with x to get

$x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - \cdots + (-1)^n C_nx^{n+1}$

Differentiating w.r.t. x gives us

$(1-x)^n - nx(1-x)^{n-1} = C_0 - 2.C_1x + 3.C_2x^2 - \cdots + (-1)^n.(n+1).C_nx^n$

We multiply again with x and differentiate again w.r.t. x to get

$(1-x)^n - nx(1-x)^{n-1} - 2nx(1-x)^{n-1} + n(n-1)x^2(1-x)^{n-2} = C_0 - 2^2.C_1x + 3^2.C_2x^2 - \cdots + (-1)^n.(n+1)^2.C_nx^n$

Putting $x = 1$ gives us

$C_0 - 2^2C_1 + 3^2C_2 - \cdots + (-1)^n(n+1)^2C_n = 0.$

111. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$

Integrating within the limits 0 and x gives us

$-\frac{(1-x)^{n+1}}{n+1} + \frac{1}{n+1} = C_0x - C_1.\frac{x^2}{2} + C_2.\frac{x^3}{3} - \cdots + (-1)^n C_n.\frac{x^{n+1}}{n+1}$

Integrating again within the limits 0 and 1 we arrive at

$\left[\frac{(1-x)^{n+2}}{(n+1)(n+1)} + \frac{x}{n+1}\right]_0^1 = \frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n\frac{C_n}{(n+1)(n+2)}$

$\frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n\frac{C_n}{(n+1)(n+2)} = \frac{1}{n+2}$.

112. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$

Multiplying with x gives us

$x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - \cdots + (-1)^n C_nx^{n+1}$

Integrating within limits 0 and 1 yields

$$\left[-\frac{x(1-x)^{n+1}}{n+1}\right]_0^1 + \int_0^1 \frac{(1-x)^{n+1}}{n+1}\,dx = \frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n \frac{C_n}{(n+1)(n+2)}$$

$$\Rightarrow \left[-\frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1 = \frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n \frac{C_n}{(n+1)(n+2)}$$

$$\Rightarrow \frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n \frac{C_n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}.$$

113. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$

Multiplying with x^2 gives us

$x^2(1-x)^n = C_0x^2 - C_1x^3 + C_2x^4 - \cdots + (-1)^n C_nx^{n+2}$

Integrating within limits 0 and 1 yields

$$\left[C_0.\frac{x^3}{3} - C_1.\frac{x^4}{4} + C_2.\frac{x^5}{5} - \cdots + (-1)^n C_n.\frac{x^{n+3}}{n+3}\right]_0^1 = \int_0^1 x^2(1-x)^n\,dx$$

Now we evaluate $\displaystyle\int_0^1 x^2(1-x)^n\,dx$

$$\int_0^1 x^2(1-x)^n\,dx = \left[-\frac{x^2(1-x)^{n+1}}{n+1}\right]_0^1 + \int_0^1 \frac{2x(1-x)^{n+1}}{(n+1)}\,dx$$

$$= 0 - \left[\frac{2x(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1 + \frac{2}{(n+1)(n+2)}\int_0^1 (1-x)^{n+2}\,dx$$

$$= -0 - \frac{2}{(n+1)(n+2)}\left[\frac{(1-x)^{n+3}}{n+3}\right]_0^1 = \frac{2}{(n+1)(n+2)(n+3)}.$$

114. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Integrating within limits 0 and 3 we get the desired result.

115. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^n C_nx^n$

Multiply with x to get

$x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - \cdots + (-1)^n C_nx^{n+1}$

Differentiating w.r.t. x leads us to

$$[(1-x)^n - nx(1-x)^{n-1}] = C_0 - 2.C_1x + 3.C_2x^2 - \cdots + (-1)^n.(n+1).C_nx^n$$

Also, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiplying last two equations gives us

$(1-x^2)^n - nx(1+x)(1-x^2)^{n-1} = (C_0 - 2.C_1x + 3.C_2x^2 - \cdots + (-1)^n.(n+1).C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n)$

Equating the coeff. of x^n gives us

$C_0^2 - 2.C_1^2 + 3.C_2^2 - \cdots + (-1)^n(n+1)C_n^2 = (-1)^{\frac{1}{2}}C_{\frac{n}{2}}.nC_{\frac{n}{2}-1}^{n-1}$

$= (-1)^{\frac{n}{2}}\left[C_{\frac{n}{2}} + n.C_{\frac{n}{2}-1}^{n-1}\right]$

$\Rightarrow 2.\frac{\left(\frac{n!}{2}\right)^2}{n!}[C_0^2 - 2.C_1^2 + 3.C_2^2 - \cdots + (-1)^n.(n+1)C_n^2] = (-1)^{n/2}(2+n).$

116. $2\sum_{0\le i\le n}\sum_{i<j\le n} C_iC_j = (C_0 + C_1 + C_2 + \cdots + C_n)^2 - (C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2) = (2^n)^2 - C_n^{2n} = 2^{2n} - \frac{2n!}{(n!)^2}$

$\Rightarrow \sum_{0\le i\le n}\sum_{i<j\le n} C_iC_j = 2^{2n-1} - \frac{2n!}{2(n!)^2}.$

117. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, $(1+y)^n = C_0 + C_1y + C_2y^2 + \cdots + C_ny^n$, and $(x+y)^n = C_0x^n + C_1x^{n-1}y + C_2x^{n-2}y^2 + \cdots + C_ny^n$

We multiply all three and equate the coeff. of x^ny^n which is equal to $C_0^3 + C_1^3 + C_2^3 + \cdots + C_n^3$.

118. Let $(1+x-3x^2)^{2163} = a_0 + a_1x + a_2x^2 + \cdots + a_{6489}x^{6489}$.

Putting $x = 1$, we get sum of coefficients as

$a_0 + a_1 + a_2 + \cdots + a_{6489} = (1+1-3)^{2163} = (-1)^{2163} = -1.$

119. Putting $x = 1, \omega, \omega^2$ in the given equation, and adding we get the desired result.

120. $(r+1)$th term in the given expression is given by $t_{r+1} = C_r^{10}2^{\frac{10-r}{2}}3^{\frac{r}{5}}$.

For rational terms r has to a multiple of 5 for 3 and $\frac{10-r}{2}$ has to be a multiple of 2. In the given series for the first condition $r = 0, 5, 10$, and for the second condition $r = 0, 2, 4, 6, 8, 10$.

So common values of r are 0 and 10.

$\therefore$ Sum of rational terms $= t_1 + t_{11} = C_0(\sqrt{2})^{10} + C_{10}^{10}(3^{1/5})^{10} = 41.$

121. $\frac{2^{4n}}{15} = \frac{16^n}{15} = \frac{(1+15)^n}{15} = \frac{1+C_1.15+C_2.15^2+\cdots+C_n.15^n}{15}.$

It is clear from above that the fractional part would be $\frac{1}{15}$.

122. Let $(\sqrt{3}+1)^{2n} = p + f$, where p is the integral part and $0 < f < 1$.

$(\sqrt{3}+1)^{2n} = [(\sqrt{3}+1)^2]^n = (4+2\sqrt{3})^n = 2^n(2+\sqrt{3})^n$

Thus, $p + f = 2^n(2+\sqrt{3})^n$. Also, $0 < \sqrt{3-1} < 1 \therefore 0 < (\sqrt{3}-1)^{2n} < 1$

Let $f_1 = (\sqrt{3}-1)^{2n} = 2^n(2-\sqrt{3})^n$

$p + f + f_1 = 2^n.2[2^n + C_2.2^n(\sqrt{3})^2 + \cdots] = 2^{n+1}$. an integer = an integer.

$f + f_1 =$ even number $-p =$ an integer. Also, $0 < f + f_1 < 2$ and thus, $f + f_1 = 1$.

Hence, integer just above $(\sqrt{3}+1)^{2n}$ i.e. $p + 1$ is divisible by 2^{n+1}.

123. Clearly $0 < f < 1$. $R = (5\sqrt{5}+11)^{2n+1}$

Let $f' = (5\sqrt{5}-11)^{2n+1}$. $\because 0 < 5\sqrt{5}-11 < 1 \therefore 0 < (5\sqrt{5}-11)^{2n+1} < 1$.

$R - f' = 2[C_1^{2n+1}.(5\sqrt{5})^{2n}.11 + C_3^{2n+1}(5\sqrt{5})^{2n-2}.11^3 + \cdots] =$ an even number

$f - f' =$ an even number $-[R] =$ an integer. But $-1 < f - f' < 1$.

Thus, $f - f' = 0 \Rightarrow f = f'. \Rightarrow Rf' = ((5\sqrt{5}+11)^{2n+1})((5\sqrt{5}-11)^{2n+1}) = 4^{2n+1} = Rf[\because f = f']$.

124. $101^{50} - 99^{50} = (100+1)^{50} - (100-1)^{50}$

$= 2[C_1^{50}.100^{49} + C_3^{50}100^{47} + \cdots + C_{49}^{50}100]$

$= 100^{50} + 2[C_3^{50}100^{47} + \cdots + C_{49}^{50}100]$

$\therefore 101^{50} - 99^{50} > 100^{50} \Rightarrow 101^{50} > 100^{50} + 99^{50}$.

125. $t_1 = \sum\limits_{r=0}^{n} C_r\left(\frac{1}{2}\right)^r = \left(1-\frac{1}{2}\right)^r = \frac{1}{2^n}$, $t_2 = \sum\limits_{r=0}^{n} C_r\left(\frac{3}{4}\right)^r = \left(1-\frac{3}{4}\right)^r = \frac{1}{2^{2n}}$, ... and so on.

Therefore, required sum $= \frac{1}{2^n} + \frac{1}{2^{2n}} + \cdots + \frac{1}{2^{mn}} = \frac{1}{2^n}\left[\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right] = \frac{1-\frac{1}{2^{mn}}}{2^n-1}$.

126. $32^{32} = (2+3\times 10)^{32} = 2^{32} + 10k$, where $k \in \mathbb{N}$. Therefore last digit in 32^{32} is same as last digit in 2^{32}.

$2^{32} = (2^5)^6.2^2 = 32^6.4 = (2+10)^6.4 = 4.(2^6 + 10r)$, where $r \in \mathbb{N}$.

Therefore, last digit in 2^{32} is same as last digit in $4.2^6 =$ last digit in $16 = 6$.

127. Let $n = 2m$, where $m \in \mathbb{P}$, then $k = 3n$.

Now, L.H.S. $= \sum\limits_{r=1}^{3m}(-3)^{r-1}C_{2r-1}^{6m} = C_1^{6m} - C_3^{6m}.3 + C_5^{6m}.3^2 - \cdots + (-1)^{3m-1}C_{6m-1}^{6m}3^{3m-1}$

$= \frac{1}{\sqrt{3}}[C_1^{6m}\sqrt{3} - C_3^{6m}(\sqrt{3})^3 + C_5^{6m}(\sqrt{3})^5 - \cdots + (-1)^{3m-1}C_{6m-1}^{6m}(\sqrt{3})^{6m-1}]$

We observe that $(-1)^{3m-1} = i^{6m-2} = -i.i^{6m-1}$.

Now $(1+\sqrt{3}i)^{6m} = 1 + C_1^{6m}\sqrt{3}i + C_2^{6m}(\sqrt{3}i)^2 + C_3^{6m}(\sqrt{3}i)^3 + \cdots + C_{6m-1}^{6m}(\sqrt{3}i)^{6m-1} + C_{6m}^{6m}(\sqrt{3}i)^{6m}$

$= [1 - C_2^{6m}.3 + C_4^{6m}.3^2 - \cdots] + i[C_1^{6m}\sqrt{3} - C_3^{6m}(\sqrt{3})^3 + \cdots + C_{6m-1}^{6m}.i^{6m-2}(\sqrt{3})^{6m-1}]$

However, $(1+\sqrt{3}i)^{6m} = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{6m} = 2^{6m}$.

Equating coeff. of imaginary parts yields

$C_1^{6m}\sqrt{3} - C_3^{6m}(\sqrt{3})^3 + \cdots + C_{6m-1}^{6m}.i^{6m-2}(\sqrt{3})^{6m-1} = 0.$

128. $(a+x)^n = a^n + C_1^n a^{n-1}x + C_2^n a^{n-2}x^2 + \cdots + x^n$

Thus, $t_0 = a^n, t_1 = C_1^n a^{n-1}x, t_2 = C_2^n a^{n-2}x^2, \ldots$

Now we replace x with ix to get

$(a+ix)^n = a^n + C_1^n a^{n-1}ix - C_2^n a^{n-2}x^2 + \cdots + (ix)^n$

$= (a^n - C_2^n a^{n-2}x^2 + \cdots) + i(C_1^n a^{n-1}x - C_3^n a^{n-3}x^3 + \cdots)$

$= (t_0 - t_2 + t_4 - \cdots) + i(t_1 - t_3 + t_5 - \cdots)$

Taking modulus and squaring yields

$(a^2+x^2)^n = (t_0 - t_2 + t_4 - \cdots)^2 + i(t_1 - t_3 + t_5 - \cdots)^2.$

129. Given, $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$.

Putting $x = 1$ yields

$a_0 + a_1 + a_2 + \cdots + a_{2n} = 3^n.$

130. Putting $x = -1$ yields

$a_0 - a_1 + a_2 - \cdots + a_{2n} = 1.$

131. Putting $x = 1, \omega, \omega^2$ yields

$a_0 + a_1 + a_2 + \cdots + a_{2n} = 3^n,$

$a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{2n}\omega^{2n} = 0$, and

$a_0 + a_1\omega^2 + a_2\omega^4 + \cdots + a_{2n}\omega^{4n} = 0.$

Adding we get $3(a_0 + a_3 + a_6 + \cdots) = 3^n$

$\Rightarrow a_0 + a_3 + a_6 + \cdots = 3^{n-1}.$

132. $S_n = 1 + q + q^2 + \cdots + q^n = \frac{1-q^{n+1}}{1-q}$ and $S'_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \cdots + \left(\frac{q+1}{2}\right)^n = \frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\frac{q+1}{2}} = \frac{2^{n+1}-(q+1)^{n+1}}{(1-q).2^n}$.

Now $C_1^{n+1} + C_2^{n+1}.S_1 + C_3^{n+1}.S_2 + \cdots + C_{n+1}^{n+1}.S_n = C_1^{n+1}\left(\frac{1-q}{1-q}\right) + C_2^{n+1}\left(\frac{1-q^2}{1-q}\right) + C_3^{n+1}\left(\frac{1-q^3}{1-q}\right) + \cdots + C_{n+1}^{n+1}\left(\frac{1-q^{n+1}}{1-q}\right)$

$= \frac{1}{1-q}[(C_1^{n+1} + C_2^{n+1} + C_3^{n+1} + \cdots + C_{n+1}^{n+1}) - (C_1^{n+1}q + C_2^{n+1}q^2 + C_3^{n+1}q^3 + \cdots + C_{n+1}^{n+1}q^{n+1})]$

$= \frac{1}{1-q}[2^{n+1} - 1 - \{(1+q)^{n+1} - 1\}] = \frac{1}{1-q}[2^{n+1} - (1+q)^{n+1}] = 2^n S'_n.$

133. $(r+1)$th term in the given expansion is given by $t_{r+1} = C_r^{1000} 9^{\frac{1000-r}{4}}.8^{\frac{r}{6}}$, where $r = 0, 1, 2, \ldots, 1000$.

For rational terms r has to be a multile of $6 = 0, 6, 12, 18, \ldots, 996$ and $1000 - r =$ a multiple of $4 = 0, 4, 8, 12, \ldots, 1000$.

From both of these the common values are multiple of 12, which is L.C.M. of 4 and 6. Thus, sum of rational terms would be 84.

134. $(r+1)$th term in the given expansion is given by $t_{r+1} = C_r^{15} 2^{\frac{15-r}{3}} 3^{\frac{r}{5}}$, where $r = 0, 1, 2, \ldots, 15$.

For rational terms r has to be a multiple of $5 = 0, 5, 10, 15$ and $15 - r =$ a multiple of $3 = 0, 3, 6, 9, 12, 15$.

The common values will depend on the L.C.M. of 3 and 5 which is 15. So there are two terms which will satisfy the criteria; first terms and last term.

Sum would be $= C_0^{15} 2^5 + C_{15}^{15} 3^3 = 32 + 27 = 59$.

135. t_3 in the expansion of $(x + x\log_{10} x)^5$ is $C_2^5 x^3.x^2(\log_{10} x)^2 = 1{,}000{,}000$

$\Rightarrow x^5(\log_{10} x)^2 = 100{,}000 \Rightarrow x = 10$.

136. Replacing $x - \frac{1}{x} = y$ we have $(1+y)^3 = 1 + 3y + 3y^2 + y^3$. Substituting back we get

$\left(x + 1 - \frac{1}{x}\right)^3 = 1 + 3\left(x - \frac{1}{x}\right) + 3\left(x^2 + \frac{1}{x^2} - 2\right) + \left(x - \frac{1}{x}\right)^3 = 1 + 3x - \frac{3}{x} + 3x^2 + \frac{3}{x^2} - 6 + x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$.

137. It is given that coefficients of second, third and fourth terms are the first, third and fifth terms of an A.P. i.e. they are in A.P.

$\therefore 2.C_2^m = C_1^m + C_3^m \Rightarrow 2.\frac{m(m-1)}{2} = m + \frac{m(m-1)(m-2)}{6}$

$\Rightarrow m - 1 = 1 + \frac{(m-1)(m-2)}{6} \Rightarrow m^2 - 9m + 14 = 0$

$\Rightarrow m = 2, 7$ but it cannot be 2 as we have more than 3 terms, so $m = 7$.

$t_6 = C_5^7[2^{\log(10-3^x)+(x-2)\log 3}] = 21 \Rightarrow 2^{\log(10-3^x)+\log 3^{x-2}} = 1 = 2^0$

$\Rightarrow \log(10-3^x) + \log 3^{x-2} = 0 \Rightarrow (10-3^x)3^{x-2} = 1$

$\Rightarrow 3^{2x} - 10.3^x + 9 = 0 \Rightarrow 3^x = 9, 1 \Rightarrow x = 2, 0.$

138. Sixth term of the expansion $\left[2^{\log_2\sqrt{9^{x-1}+7}} + \frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$ is given by $t_6 = C_5^7(\sqrt{9^{x-1}+7})^2 . \frac{1}{(3^{x-1}+1)} = 84$

$\Rightarrow \frac{9^{x-1}+7}{3^{x-1}+1} = 4 \Rightarrow x = 1, 2.$

139. We have to prove that $\frac{1}{(81)^n} - \frac{10}{(81)^n}.C_1^{2n} + \frac{10^2}{(81)^n}.C_2^{2n} - \frac{10^3}{(81)^n}.C_3^{2n} + \cdots + \frac{10^{2n}}{(81)^n} = 1.$

$\Rightarrow \frac{1}{81^n}[C_0^{2n} - C_1^{2n}10 + C_2^{2n}10^2 - C_3^{2n}10^3 + \cdots + C_{2n}^{2n}10^{2n}] = 1$

$\Rightarrow \frac{1}{81^n}(1-10)^{2n} = \frac{1}{81^n}.(-9)^{2n} = 1.$

140. We know that $C_r = C_{n-r}$. Thus we can rewrite the given series as

$\lim_{n\to\infty} = C_0 - C_1.\frac{2}{3} + C_2.\left(\frac{2}{3}\right)^2 - C_3.\left(\frac{2}{3}\right)^3 + \cdots + (-1)^n C_n.\left(\frac{2}{3}\right)^n$

$= \left(1 - \frac{2}{3}\right)^n = \frac{1}{3^n} = 0.$

141. Given, $E = (6\sqrt{6}+14)^{2n+1} = [E] + F$. Let $F' = (6\sqrt{6}-14)^{2n+1} = \frac{20^{2n+1}}{(6\sqrt{6}+14)^{2n+1}}$

$E - F' = C_1^{2n+1}(6\sqrt{6})^{2n}.14 + C_3^{2n+1}(6\sqrt{6})^{2n-2}.14^3 + ... =$ an even number.

$F - F' =$ an even number $-[E] =$ an integer. But $0 < F < 1$ and $0 < F' < 1 \Rightarrow -1 < F - F' < 1$

$\Rightarrow F - F' = 0 \Rightarrow F = F'.$

$\therefore EF = EF' = 20^{2n+1}.$

142. $(17)^{256} = (289)^{128} = (290-1)^{128} = C_0.290^{128} - C_1.290^{127} + \cdots + (-1)^{128}C_{128}$

Clearly, all the terms except last term would be multiple of 10. So at unit's place last term will occur which is 1.

Similarly for ten's place only second last term will matter which is $-C_{127}.290 = -128 \times 290 = -37120$. This is a negative term and the highest term is positive and multiple of 100 so the tens place unit will be $10 - 2 = 8$.

Similarly for hundred's place third last term will matter which is $C_{126}.290^2 = 683564800$ and thus the digit will be 6 after adjusting with second last term.

143. We have to prove that for $n \geq 3$, $n^{n+1} > (n+1)^n$, $forall\ n \in \mathbb{P}$.

Rewriting $\Rightarrow n > \left(1 + \frac{1}{n}\right)^n$.

Now $\left(1+\frac{1}{n}\right)^n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \frac{n(n-1)(n-2)\cdots[n-(n-1)]}{n!}\cdot\frac{1}{n^n}$

$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

$= 1 + 2\left[1 - \frac{1}{2^n}\right] = 3 - \frac{1}{2^{n-1}} < 3$.

Now we have been given that $n \geq 3$ and hence $n^{n+1} > (n+1)^n$.

144. We have to prove that $2 < \left(1+\frac{1}{n}\right)^n < 3\ \forall\ n \in \mathbb{N}$.

Proceeding like previous problem we obtain $2 \leq \left(1+\frac{1}{n}\right)^n < 3$.

145. We have to prove that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.

Rewriting as $(1992^{1998} - 1955^{1998}) - (1938^{1998} - 1901^{1998})$. We know that $a^n - b^n$ is divisible by $a - b$. Thus given expression is divisible by 37.

Rewriting again as $(1992^{1998} - 1938^{1998}) - (1955^{1998} - 1901^{1998})$, which is divisible by 54.

Since 37 is a prime number there will be no common factor with 54. Hence, given expression is divisible by $37 \times 54 = 1998$.

146. $53^{53} = (50+3)^{53} = 50k + 3^{53}$ and $33^{33} = (30+3)^{33} = 30k + 3^{33}$.

So if the difference $3^{53} - 3^{33}$ is divisible by 10 then our proof will be complete.

We observe the powers of 3. $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, \ldots$. Thus we see that it repeats in the fashion of $3, 9, 7, 1, 3, 9, 7, 1, \ldots$ as far as last digits are concerned.

$53 = 4 * 13 + 1$ and $33 = 4 * 8 + 1$ so the last digits of both will be same i.e. 9. Hence, the given difference is divisible by 10.

147. $(1+x)^{m+1} = C_0^{m+1} + C_1^{m+1}x + C_2^{m+1}x^2 + \cdots + C_{m+1}^{m+1}x^{m+1}$

$\Rightarrow [(1+x)^{m+1} - 1 - x^{m+1}] = C_1^{m+1}x + C_2^{m+1}x^2 + \cdots + C_m^{m+1}x^{m+1}$.

Putting $x = 1, 2, 3, \ldots, n$ and adding we get the desired result.

148. $\sum_{i=1}^{k}\sum_{k=1}^{n} C_k^n C_i^k = \sum_{k=1}^{n}(C_k^n C_1^k + C_k^n C_2^k + C_k^n C_3^k + \cdots + C_k^n C_k^k)$

$= (C_1^n C_1^1 + C_2^n C_1^2 + C_3^n C_1^3 + \cdots + C_n^n C_1^n) + (C_2^n C_2^2 + C_3^n C_2^3 + \cdots + C_n^n C_2^n) + (C_3^n C_3^3 + \cdots + C_n^n C_n^n C_3^n) + \cdots + C_n^n C_n^n$

$= C_1^n(C_1^1 + C_0^1) - C_1^n C_0^1 + C_2^n(C_0^2 + C_1^2 + C_2^2) - C_2^n C_0^2 + C_3^n(C_0^3 + C_1^3 + C_2^3 + C_3^3) - C_3^n C_0^3 + \cdots$

$= C_1^n 2 + C_2^n 2^2 + C_3^n 2^3 + \cdots + C_n^n 2^n - 1 - [C_0^n + C_1^n + C_2^n + \cdots + C_n^n - 1] = (1+2)^n - (1+1)^n = 3^n - 2^n.$

149. We have to prove that $\sum_{r=0}^{n}(-1)^r.{}^nC_r \frac{1+r\log_e 10}{(1+\log_e 10^n)^r} = 0$. Note that this equality will hold for positive n but not for $n=0$ for which L.H.S. is equal to 1.

$\sum_{r=0}^{n}(-1)^r.{}^nC_r \frac{1+r\log_e 10}{(1+\log_e 10^n)^r} = \sum_{r=0}^{n}(-1)^r C_r^n \frac{1}{(1+\log 10^n)^r} + \sum_{r=0}^{n}(-1)^r C_{r-1}^{n-1}\frac{n\log 10}{(1+\log 10^n)^r}$

$= 1 + \sum_{r=1}^{n-1}(C_r^{n-1} + C_{r-1}^{n-1})\frac{1}{(1+\log 10^n)^r} + (-1)^n\frac{1}{(1+\log 10^n)^r} - \sum_{r=0}^{n-1}(-1)^r C_r^{n-1}\frac{\log 10^n}{(1+\log 10^n)^{r+1}}$

$= 1 + \sum_{r=1}^{n-1}(-1)^r C_r^{n-1}\frac{1}{(1+\log 10^n)^r} - \sum_{r=0}^{n-2}(-1)^r C_r^{n-1}\frac{1}{(1+\log 10^n)^{r+1}} + (-1)^n\frac{1}{(1+\log 10^n)^r} - \sum_{r=0}^{n-1}(-1)^r C_r^{n-1}\frac{\log 10^n}{(1+\log 10^n)^{r+1}}$

$= 1 + \sum_{r=1}^{n-1}(-1)^r C_r^{n-1}\frac{1}{(1+\log 10^n)^r} + (-1)^n\frac{1}{(1+\log 10^n)^r} - \sum_{r=0}^{n-2}(-1)^r C_r^{n-1}\frac{1}{(1+\log 10^n)^r} + (-1)^n\frac{\log 10^n}{(1+\log 10^n)^n}$

$= 1 + (-1)^{n-1}\frac{1}{(1+\log 10^n)^{n-1}} + (-1)^n\frac{1}{(1+\log 10^n)^{n-1}} - 1 + (-1)^n\frac{\log 10}{(1+\log 10^n)^n} = 0.$

150. $32^{32} = 2^{160} = (3-1)^{160} = 3m+1$, where $m \in \mathbb{N}$.

$32^{32^{32}} = 32^{3m+1} = 2^{3(5m+1)}.2^2 = 23(8^{5m}) = 32(1+7)^{5m}$

$= 32(1+7k), k \in \mathbb{N} = 4 + 28 + 7(32k) = 4 + 7r$, where $r \in \mathbb{N}$.

Thus, remainder would be 4.

151. Let $t = x-3$, then $x-2 = 1+t$. Now $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$

$\Rightarrow \sum_{r=0}^{2n} a_r(1+t)^r = \sum_{r=0}^{2n} b_r t^r$

$\Rightarrow a_0 + a_1(1+t) + \cdots + a_{n-1}(1+t)^{n-1} + 1.(1+t)^n + 1.(1+t)^{n+1} + \cdots + 1.(1+t)^{2n} = b_0 + b_1 t + b_2 t^2 + \cdots + b_{2n} t^{2n}$

Equating the coefficients of t^n gives us

$b_n = C_n^n + C_n^{n+1} + C_n^{n+2} + \cdots + C_n^{2n} = C_{n+1}^{n+2} + C_n^{n+2} + \cdots + C_n^{2n} [\because C_n^n = C_{n+1}^{n+1} = 1]$

...

$= C_{n+1}^{2n+1}$.

152. Given, x^{50} in $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + 1001x^{1000}$.

Rewriting, $(1+x)^{1000}\left[1 + 2\frac{x}{1+x} + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right]$

$= (1+x)^{1000}[1 + 2k + 3k^2 + \cdots + 1001k^{1000}]$, where $k = \frac{x}{1+x}$

$$\begin{aligned} S &= 1 + 2k + 3k^2 + \cdots + 1001k^{1000} \\ kS &= \qquad k + 2k^2 + \cdots + 1000k^{1000} + 1001k^{1001} \end{aligned}$$

Subtracting $(1-k)S = 1 + k + k^2 + \cdots + k^{1000} - 1001k^{1001} = \frac{1-k^{1001}}{1-k} - 1001k^{1001}$

$S = (1+x)^2\left[1 - \frac{x^{1001}}{(1+x)^{1001}}\right] - 1001.\frac{x^{1001}}{(1+x)^{1000}}$

So given expression becomes $(1+x)^{1002} - x^{1001}(1+x) - 1001x^{1001}$, and hence, coefficient of x^{50} is C_{50}^{1002}.

153. L.H.S. = coeff. of x^n in $(1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \cdots + (1+x)^{n+k}$

Now, $(1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \cdots + (1+x)^{n+k} = (1+x)^n\left[\frac{(1+x)^{k+1}-1}{x}\right]$

$= \frac{1}{x}(1+x)^{n+k+1} - \frac{1}{x}(1+x)^n$

Equating coefficient of x^n gives us

$C_n^n + C_n^{n+1} + C_n^{n+2} + \cdots + C_n^{n+1} = C_{n+1}^{n+k+1}$(there is no power of x^n in second term).

154. Let $S = x + 2x^2 + 3x^3 + \cdots + nx^n$ then $xS = x^2 + 2x^3 + \cdots + (n-1)x^n + nx^{n+1}$

Subtracting yields $(1-x)S = x + x^2 + x^3 + \cdots + x^n - nx^{n+1} \Rightarrow S = x\frac{(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}$

$\Rightarrow (1 + x + 2x^2 + 3x^3 + \cdots + nx^n)^2 = \left[1 + x\frac{(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}\right]^2$

Coefficient of x^n = coefficient of x^n in $\left[1 + \frac{x}{(1-x)^2}\right]^2$(leaving terms containing powers of x greater than n)

$=$ coefficient of x^n in $2x(1-x)^{-2} + x^2(1-x)^{-4} = 2.C_{n-1}^n + C_{n-2}^{n+1} = 2.C_1^n + C_3^{n+1}$

$= \frac{n(n^2+11)}{6}$.

155. $1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n = \frac{(1+x)^{n+1}-1}{x}$.

Clearly, coefficient of x^k in $\frac{(1+x)^{n+1}-1}{x}$ is equal to the coefficient of x^{k+1} in $(1+x)^{n+1} = C^{n+1}_{k+1}$.

156. $(x+1)^n + (x+1)^{n-1}(x+2) + (x+1)^{n-2}(x+2)^2 + \cdots + (x+2)^n$ is a G.P. with first term $(x+1)^n$ and common ratio $\frac{x+2}{x+1}$.

Thus, sum of the series is $\frac{(x+1)^n\left[\frac{(x+2)^n}{(x+1)^n}-1\right]}{\frac{x+2}{x+1}-1} = \frac{(x+2)^n-(x+1)^n}{\frac{1}{x+1}} = (x+1)[(x+2)^n - (x+1)^n]$.

Thus, coefficient of x^3 in $(x+1)[(x+2)^n - (x+1)^n]$ is equal to coefficient of x^2 in $(x+2)^n - (x+1)^n +$ coefficient of x^3 in $[(x+2)^n - (x+1)^n]$

$= C^n_{n-2}.2^{n-2} - C^n_{n-2} + C^n_{n-3}.2^{n-3} - C^n_{n-3} = C^{n+1}_3(2^{n-2}-1)$.

157. $\left(\frac{a+1}{a^{2/3}-a^{1/3}+1} - \frac{a-1}{a-a^{1/2}}\right)^{10} = \left(a^{1/3} + 1 - \frac{\sqrt{a}+1}{\sqrt{a}}\right)^{10} = (a^{1/3} - a^{-1/2})^{10}$.

Let $(r+1)$th term be independent of a, then $t_{r+1} = C^{10}_r a^{(10-r)/3}(-a)^{-r/2}$.

For this term to be independent of a, $\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 5r = 0 \Rightarrow r = 4$.

So 5th term is independent of a. $t_5 = C^{10}_4 = 210$.

158. Coefficient of x^2 in $\left(x+\frac{1}{x}\right)^{10}(1-x+2x^2) =$ coefficient of x^2 in $\left(x+\frac{1}{x}\right)^{10}$ − coefficient of x in $\left(x+\frac{1}{x}\right)^{10} + 2.$ coefficient of term independeng of x in $\left(x+\frac{1}{x}\right)^{10}$.

Consider $(r+1)$th term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$.

$t_{r+1} = C^{10}_r.x^{10-r}.x^{-r}$ so power of x is $10-2r$.

If $10-2r = 2 \Rightarrow r = 4$; for $10-2r = 1$ we do not have an integral r; and for $10-2r = 0 \Rightarrow r = 5$.

So final answer is $C^{10}_4 + 2.C^{10}_5 = 714$.

159. $(1+x-2x)^6 = (1-x)^6(1+2x)^6 = (1-6x+15x^2-20x^3+15x^4-6x^5+x^6)(1+12x+60x^2+160x^3+140x^4+192x^5+64x^6)$

Hence, coefficient of x^4 is $240 - 6\times 160 + 15\times 60 - 20\times 12 + 15 = -45$.

160. We have to find the term independent of x in $(1+x+2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

We consider $(r+1)$th term in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

$t_{r+1}=C_r^9\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$

So the power of x would be $18-3r$. This term is multiplied with $1+x+2x^3$ so for 1 the value of $18-3r=0$ for the term to be independent of x i.e. $r=6$. For x there will be no such term because $18-3r=-1$ does not give an integral r. For $2x^3$ the term would be $18-3r=-3 \Rightarrow r=7$.

So the final term would be $C_6^9\left(\frac{3}{2}x^2\right)^3\left(-\frac{1}{3x}\right)^6+2x^3.C_7^9\left(\frac{3}{2}x^2\right)^2\left(-\frac{1}{3x}\right)^7$

$=\frac{17}{54}$.

161. We have to find the term independent of x in $\left(x^2+\frac{1}{x^3}\right)^7(2-x)^{10}$.

The terms in $(2-x)^{10}$ will have terms in which powers of x will vary from 0 to 10 increasing by 1.

We consider $(r+1)$th term in $\left(x^2+\frac{1}{x^3}\right)^7$.

$t_{r+1}=C_r^7(x^2)^{7-r}.\left(\frac{1}{x^3}\right)^r$. So the power of x would be $14-5r$. We vary r which gives us $14, 9, 4, -1, -6, -11$ for $r=0,1,2,3,4,5$ and so on. Out of these only powers of -1 and -6 can be neutralized by the second expansion. Thus, $r=3,4$.

Corresponding terms in the expansion of $(2-x)^{10}$ are $-C_1^{10}2^9x$ and $C_6^{10}2^4x^6$.

Corresponding terms in the expansion of $\left(x^2+\frac{1}{x^3}\right)^7$ are $C_3^7x^{-1}$ and $C_4^7x^{-6}$. Now it is trivial to obtain the final answer.

162. We have to find the term independent of x in $(1+x+x^{-2}+x^{-3})^{10}$. Rewriting

$(1+x+x^{-2}+x^{-3})^{10}=\frac{1}{x^{30}}(x^3+x^4+x+1)=\frac{1}{x^{30}}[(1+x)(1+x^3)]^{10}$.

Powers of x in first expansion are $0,1,2,3,\ldots,10$ and in the second expansion are $0,3,6,9,\ldots,30$. We need to add powers such that the sum is 30. Such powers are $(0,30),(3,27),(6,24),(9,21)$.

Thus, the term is $C_0^{10}C_{10}^{10}+C_3^{10}C_9^{10}+C_6^{10}C_8^{10}+C_9^{10}C_7^{10}$.

163. Powers of x would be $1,2,3$ for a_1,a_2,a_3 respectively as we observe from the given series.

Now we consider $(r+1)$th term of $(1+x^2)^2(1+x)^n=(1+2x^2+x^4)(1+x)^n$.

Coeff. of term containing x would be $c_1=C_1^n$. Coeff. of term containing x^2 would be $c_2=C_2^n+2.C_0^n$ and coeff. of term containing x^3 would ve $c_3=C_3^n+2.C_1^n$.

Given that c_1, c_2, c_3 are in A.P. $\Rightarrow C_3^n + 3C_1^n = 2C_2^n + 4$

$\Rightarrow \frac{n(n-1)(n-2)}{3!} + 3n = n(n-1) + 4 \Rightarrow n = 2, 3, 4.$

164. We have to show that $C_1^m + C_2^{m+1} + C_3^{m+2} + \cdots + C_n^{m+n-1} = C_1^n + C_2^{n+1} + C_3^{n+2} + \cdots + C_n^{m+n-1}$.

L.H.S. $= C_1^m + C_2^{m+1} + C_3^{m+2} + \cdots + C_n^{m+n-1}$

Adding and subtracting C_0^m and applying $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$ repeatedly

$\Rightarrow C_0^m + C_1^m + C_2^{m+1} + C_3^{m+2} + \cdots + C_n^{m+n-1}$

$= C_1^{m+1} + C_2^{m+1} + C_3^{m+2} + \cdots + C_n^{m+n-1} = C_2^{m+2} + C_3^{m+2} + \cdots + C_n^{m+n-1}$

...

$= C_{n-1}^{m+n-1} + C_n^{m+n-1} = C_n^{m+n}$

Doing similar steps, R.H.S. $= C_m^{m+n}$ and thus, L.H.S. = R.H.S.

165. We have $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$.

(a) Putting $x = \frac{1}{x}$, we have $\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \sum_{r=0}^{n} a_r \frac{1}{x^r}$

$\Rightarrow (1 + x + x^2)^n = \sum_{r=0}^{n} a_r x^{2n-r}$.

Putting $r = 2n - r$ we see that $a_r = a_{2n-r}$.

(b) Putting $x = 1$ gives us $3^n = a_0 + a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n + a_{n+1} + \cdots + a_{2n}$

Using the result obtained in first section we see that $a_0 = a_{2n}, a_1 = a_{2n-1}, \ldots, a_{n-1} = a_{n+1}$, and thus,

$2(a_0 + a_1 + \cdots + a_{n_1}) + a_n = 3^n \Rightarrow a_0 + a_1 + \cdots + a_{n_1} = \frac{1}{2}(3^n - a_n)$

(c) Given, $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$

Difference w.r.t. x yields $n(1 + 2x)(1 + x + x^2)^{n-1} = \sum_{r=0}^{2n} r a_r x^{r-1}$

Multiplying both sides by $(1 + x + x^2)$ yields

$n(1 + 2x)(1 + x + x^2)^n = n(1 + 2x) \sum_{r=0}^{2n} a_r x^r = (1 + x + x^2) \sum_{r=0}^{2n} r a_r x^{r-1}$

Equating coefficients of $x^r(0 < r < 2n)$ gives us

$na_r + 2na_{r-1} = (r+1)a_{r+1} + ra_r + (r-1)a_{r-1} \Rightarrow (r+1)a_{r+1} = (n-r)a_r + (2n-r+1)a_{r-1}$.

166. Given, $(1-x^3)^n = \sum_{r=0}^{n} a_r.x^r.(1-x)^{3n-2r}$.

Rewriting $\frac{(1-x^3)^n}{(1-x)^{3n}} = \sum_{r=0}^{n} a_r.\frac{x^r}{(1-x)^{2r}}$

$\Rightarrow \left(\frac{1+x+x^2}{(1-x)^2}\right)^n = \sum_{r=0}^{n} a_r\alpha^2$, where $\alpha = \frac{x}{(1-x)^2}$

$\Rightarrow (1+3\alpha)^n = \sum_{r=0}^{n} a_r\alpha^r$

Equating the coefficient of α^r yields $a_r = C_r^n 3^r$.

167. No. of terms in the expansion of $(1+x)^{2n}$ is $2n+1$, so middle term would be $n+1$. Coefficient of this term would be C_n^{2n}.

Coefficient of x^n in the expansion of $(1+x)^{2n-1}$ is C_n^{2n-1}.

$2.C_n^{2n-1} = 2\frac{(2n-1)!}{n!(n-1)!} = \frac{2n.(2n-1)!}{n!n!} = \frac{(2n)!}{n!n!} = C_n^{2n}$.

168. The middle term has the greatest coefficient. In C_r^{200} the middle term would be 101st term. Coefficient of 101st term is C_{100}^{200}.

169. Let r be the no. of people needed for making maximum no. of committees. So no. of committees is C_r^{20}. Since the middle term has the largest coefficient so 11th term willl have largest no. of committees. Thus, no. persons chosen should be 10.

170. Let there be a and b permutations then $a+b=2n$. Thus, no. of permutations is C_a^{2n} and C_{2n-a}^{2n} which are greatest when $a=b$ because those will be middle terms.

171. We consider the general $(r+1)$th term's coefficient, which is, $C_r^{3}{}^{7-r}2^r$.

This ciefficient shoul be equal to $(r+2)$th term's coefficient, which is, $C_{r+1}^{7}3^{6-r}2^{r+1}$.

Equating and solving gives us $r=3$, so 4th and 5th terms have equal coefficients.

172. Let $(1+5x^2-7x^3)^{2000} = a_0 + a_1x + a_2x^2 + \cdots + a_{6000}^{6000}$.

Putting $x=-1$, we get

$a_0 + a_1 + a_2 + \cdots + a_{6000} = (1+5-7)^{2000} = 1$.

173. Given that sum of the binomial coefficients of the expansion $\left(3^{-\frac{x}{4}} + 3^{\frac{5x}{4}}\right)^n$ is 64.

$(1+1)^n = 64 \Rightarrow n = 6$[We put $3^{-\frac{x}{4}} = 3^{\frac{5x}{4}} = 1$]

Here middle term will be greatest term, which is 4th term.

According to question $t_4 = (n-1) + t_3 = 5 + t_3$.

$\Rightarrow C_3^6\left(3^{-\frac{x}{4}}\right)^3\left(3^{\frac{5x}{4}}\right)^3 = 5 + C_2^6\left(3^{-\frac{x}{4}}\right)^4\left(3^{\frac{5x}{4}}\right)^2$

$\Rightarrow 20.3^{3x} = 5 + 15.3^{\frac{3x}{2}} \Rightarrow 20y^2 - 15y - 5 = 0$, where $y = 3^{\frac{3x}{2}}$

$4y^2 - 3y - 1 = 0 \Rightarrow y = 1, -\frac{1}{4} \Rightarrow y = 1[\because y = 3^{\frac{3x}{2}} > 0]$

$\Rightarrow x = 0 \Rightarrow [\alpha] = 0 \Rightarrow 0 \leq \alpha < 1.$

174. Let $(5p - 4q)^n = a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + \cdots + a_nq^n$

Putting $p = 1, q = 1$ gives us

$a_0 + a_1 + a_2 + \cdots + a_n = (5-4)^n = 1.$

175. Let $(1 - 3x + x^3)^{201}.(1 + 5x - 5x^2)^{503} = a_0 + a_1x + a_2x^2 + \cdots + a_{2112}x^{3520}$

Putting $x = 1$ gives us

$a_0 + a_1 + a_2 + \cdots + a_{2112} = (-1)^{201}.1^{503} = -1.$

176. Putting $x = 1$ in $(tx^2 - 2x + 1)^n$ we have sum of coefficients as $(t-1)^n$. Similarly putting $x = 1, y = 1$ in $(x - ty)^n$ we have sum of coefficients as $(1-t)^n$.

Given that sum of coefficients is equal so $(t-1)^n = (-1)^n(t-1)^n$.

Consider n to be odd; then $(-1)^n = -1$ so there is no way these will be equal if $t-1$ is not zero. So the only possible value of $(t-1)^n = 0$, which gives $t = 1$.

Similarly, if n is even; then $(t-1)^n = (1-t)^n$ for all values of n.

177. Given, $(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$

Putting $x = 1$ gives us $a_0 + a_1 + a_2 + a_3 + \cdots + a_n = 2^n$

Putting $x = i$ gives us $(1+i)^n = a_0 + a_1i - a_2 - a_3i + \cdots + a_ni^n$

Taking modulus and squaring yields

$2^n = (a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots)^2.$

Q.E.D.

178. Let rth term be the greatest term, which is given by $t_r = \sqrt{3}\left[C_{r-1}^{20}\left(\frac{1}{\sqrt{3}}\right)^{r-1}\right]$. Similarly $t_{r+1} = \sqrt{3}\left[C_r^{20}\left(\frac{1}{\sqrt{3}}\right)^r\right]$

$\frac{t_r}{t_{r+1}} = \frac{C^2_{r-1}0}{C^{20}_r}\sqrt{3} = \frac{r}{21-r}\sqrt{3} \geq 1$

$\Rightarrow \sqrt{3} rgeq 21 - r \Rightarrow r \geq= 7.69$(approximately).

Similarly $\frac{t_r}{t_{r-1}} \geq 1 \Rightarrow r \leq 8.5$(approximately).

The only integer between these limits is 8. Hence, $t_8 = \frac{25840}{9}$ is the greatest term.

179. Let t_r represent the rth term of the expansion of $(x+a)^{15}$.

Now, $t_{11} = C^{15}_{10}x^5a^{10}, t_8 = C^{15}_7 x^8 a^7$, and $t_{12} = C^{15}_{11}x^4a^{11}$

Given, that t_{11} is G.M. of t_8 and t_{12}, thus

$\left(C^{15}_{10}x^5a^{10}\right)^2 = C^{15}_7 x^8 a^7 . C^{15}_{11}x^4a^{11}$

Solving this gives us, $\frac{x}{a} = \sqrt{\frac{77}{75}}$.

Let t_r be the greatest term. Then $\frac{t_r}{t_{r+1}} = \frac{r}{16-r} . \frac{x}{a} \geq 1$

$\Rightarrow r \geq 7.947 \Rightarrow r = 8$.

$\Rightarrow t_8 = \frac{15!}{7!8!}\left(\frac{77}{75}\right)^4 a^{15}$.

180. $t_{n+1} = C^{2n}_n x^n, t_{n+2} = C^{2n}_{n+1}x^{n+1}$, and $t_n = C^{2n}_{n-1}x^{n-1}$.

$\frac{t_{n+1}}{t_{n+2}} = \frac{n+1}{n} . \frac{1}{x}$, and $\frac{n+1}{n}x$.

Since t_{n+1} is the only greatest term popssible here.

$\Rightarrow \frac{t_{n+1}}{t_{n+2}} > 1 \Rightarrow x < \frac{n+1}{n}$, and similarly $\frac{t_{n+1}}{t_n} > 1 \Rightarrow x > \frac{n}{n+1}$.

Thus, $x \in \left(\frac{n}{n+1}, \frac{n+1}{n}\right)$, which is given as $x \in \left(\frac{10}{11}, \frac{11}{10}\right)$, and thus $n = 10$.

Given that t_4 in the expansion of $\left(kx + \frac{1}{x}\right)^m$ is $\frac{n}{4}$.

$t_4 = \frac{n}{4} = \frac{5}{2} = C^m_3(kx)^{m-3} . \frac{1}{x^3} \Rightarrow C^m_3 k^{m-3}x^{m-6} = \frac{5}{2}$

Since the term is independent of $x \Rightarrow m = 6 \Rightarrow C^6_3 k^3 = \frac{5}{2} \Rightarrow k = \frac{1}{2} \Rightarrow mk = 3$.

181. $t_4 = C^{10}_3 2^7\left(\frac{3}{8}x\right)^3, t_3 = C^{10}_2 2^8\left(\frac{3}{8}x\right)^2$, and $t_5 = C^{10}_4 2^6\left(\frac{3}{8}x\right)^4$.

Given that 4th term has greatest numerical value, so $\frac{t_4}{t_3} > 1 \Rightarrow \frac{2!8!}{3!7!}\frac{3}{16}x > 1 \Rightarrow \frac{x}{2} > 1$, and

$\frac{t_4}{t_5} > 1 \Rightarrow \frac{4!6!}{3!7!}\frac{16}{3x} > 1 \Rightarrow x < \frac{64}{21}$

$\Rightarrow 2 < x < \frac{64}{21}$.

182. Let the binomial expansion be $(x+y)^n$ and a, b, and c be the coefficients of rth, $(r+1)$th, and $(r+2)$th terms respectively. Then,

$a = C^n_{r-1}, b = C^n_r, c = C^n_{r+1}$.

Discriminant of the quadratic equation $ax^2+2bx+c=0$ is $D = 4b^2-4ac = 4b^2\big(1-\frac{a}{b}\cdot\frac{c}{b}\big)$

$= 4(C^n_r)^2\Big(1-\frac{C^n_{r-1}}{C^n_r}\cdot\frac{C^n_{r+1}}{C^n_r}\Big) = 4(C^n_r)^2\big(1-\frac{r}{n-r+1}\cdot\frac{n-r}{r+1}\big)$

$= 4(C^n_r)^2\cdot\frac{n+1}{(n-r+1)(r+1)} > 0$.

Hence, roots of the quadratic equation are real and unequal.

183. $9^n+7 = (1+8)^n+7 = C^n_0 + C^n_1.8 + C^n_2.8^2 + \cdots + C^n_n.8^n + 7$

$= 8\big(1 + C^n_1 + C^n_2.8 + C^n_3.8^2 + \cdots + C^n_n.8^{n-1}\big)$

$= 8$.an integer.

Thus, 9^n+7 is divisible by 8.

184. For $n=1, 3^{2n+1}+2^{n+2} = 3^3+2^3 = 27+8 = 35$, which is divisible by 7.

Let it be true for $n=m$ i.e. $3^{2m+1}+2^{m+2} = 7k$, where $k \in \mathbb{N}$.

For $n=m+1, 9.3^{2m+1}+2.2^{m+2} = 7.3^{2m+1}+2(3^{2m+1}+2^{m+2}) = 7.3^{2m+1}+7k$, which is divisible by 7.

Q.E.D.

185. Let $C^n_{r-1}, C^n_r, C^n_{r+1}$ be in G.P. $\therefore \frac{C^n_r}{C^n_{r-1}} = \frac{C^n_{r+1}}{C^n_r}$

$\Rightarrow \frac{n-r+1}{r} = \frac{n-(r+1)+1}{r+1} \Rightarrow n = -1$, which is not possible.

Let $C^n_{r-1}, C^n_r, C^n_{r+1}$ be in H.P. $\Rightarrow \frac{2}{C^n_r} = \frac{1}{C^n_{r-1}} + \frac{1}{C^n_{r+1}}$

$\Rightarrow 2 = \frac{n-r+1}{r} + \frac{r+1}{n-r} \Rightarrow (n-2r)^2+n = 0$, whihc is not possible.

186. $a_n =$ coefficient of x^n in $(1+x+x^2)^n =$ coefficient of y^n in $(1+y+y^2)^n$.

Given, $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n}$.

Putting $x = -\frac{1}{x}$, we get $(x^2-x+1)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \cdots + a_{2n}$

$\therefore a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2 =$ coefficient of x^{2n} in $(1+x+x^2)^n(x^2-x+1) = (x^4+x^2+1)$

$=$ coefficient of y^n in $(1+y+y^2)^n$, where $y = x^2$

$= a_n$.

187. We have $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n}$ and we have proved that

$a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2 = a_n.$

Putting $x = \frac{1}{x}$, we have

$(x^2+x+1) = a_0x^{2n} + a_1x^{2n-1} + \cdots + a_{2n}.$

Equating the coefficients of same powers of x gives us $a_0 = a_{2n}, a_1 = a_{2n-1}, \cdots$. This also follows from earlier result where we proved $a_r = a_{2n-r}$.

Thus, $a_0^2 - a_1^2 + a_2^2 - \cdots + (-1)^{n-1}a_{n-1}^2 + (-1)^n a_n^2 + (-1)^{n+1}a_{n+1}^2 + \cdots + a_{2n}^2 = 0$

$\Rightarrow a_0^2 - a_1^2 + a_2^2 - \cdots + (-1)^n a_{n-1}^2 = \frac{1}{2}a_n[1-(-1)^n a_n].$

Q.E.D.

188. Let $E = \sum\limits_{0\le i<j}\sum\limits_{0\le j\le n}(C_i+C_j)^2$, where $i = 0, 1, 2, \ldots, (n-1)$, and $j = 1, 2, 3, \ldots, n$ and $i < j$.

$E = n(C_0^2 + C_1^2 + \cdots + C_n^2) + 2\sum\limits_{0\le i<j}\sum\limits_{0\le j\le n} C_iC_j$

$= n.C_n^{2n} + [(C_0 + C_1 + C_2 + \ldots + C_n)^2 - (C_0^2 + C_1^2 + \cdots + C_n^2)]$

$= n.C_n^{2n} + (2^n)^2 - C_n^{2n} = (n-1)C_n^{2n} + 2^{2n}.$

189. Let $E = \sum\limits_{0\le i<j}\sum\limits_{0\le j\le n}(i+j)C_iC_j$, where $i = 0, 1, 2, \ldots, (n-1)$, and $j = 1, 2, 3, \ldots, n$ and $i < j$. Clearly, $n-i = n, (n-1), (n-2), \ldots, 3, 2, 1$ and $n-j = n-1, n-2, n-3, \ldots, 2, 1, 0$.

Thus, we see that $n-j$ behaves as i and $n-i$ behaves as j.

$\Rightarrow E = \sum\limits_{0\le i<j}\sum\limits_{0\le j\le n}(n-j+n-i)C_{n-j}C_{n-i} = \sum\limits_{0\le i<j}\sum\limits_{0\le j\le n}[2n-(i+j)]C_iC_j[\because C_r = C_{n-r}]$

$= 2n.\sum\limits_{0\le i<j}\sum\limits_{0\le j\le n}C_iC_j - E \Rightarrow 2E = 2n\sum\limits_{0\le i<j}\sum\limits_{0\le j\le n}C_iC_j$

$E = \frac{n}{2}[(C_0 + C_1 + \cdots + C_n)^2 - (C_0^2 + C_1^2 + \cdots + C_n^2)] = \frac{n}{2}(2^{2n} - C_n^{2n}).$

190. L.H.S. $= \frac{n!}{(m+n)!}\left[\frac{(m+n)!}{m!n!}C_0 + \frac{n(m+n)!}{(m+1)!n!}C_1 + \cdots + \frac{n!(m+n)!}{(m+n)!n!}C_n\right]$

$= \frac{n!}{(m+n)!}[C_n^{m+n}C_0^n + C_{n-1}^{m+n}C_1^n + \cdots + C_0^{m+n}C_n^n]$

We know that $(1+x)^{m+n} = C_0^{m+n} + C_1^{m+n}x + C_2^{m+n}x^2 + \cdots + C_{m+n}^{m+n}x^{m+n}$

and $(1+x)^n = C_0^n + C_1^nx + C_2^nx^2 + \cdots + C_n^nx^n$

Coefficient of x^n in $(1+x)^{m+n}(1+x)^n = C_n^{m+n}C_0^n + C_{n-1}^{m+n}C_1^n + \cdots + C_0^{m+n}C_n^n = C_n^{m+2n}$

$\therefore$ L.H.S. $= \frac{n!}{(m+n)!}C_n^{m+2n} = \frac{(m+2n)!}{(m+n)!(m+n)!} =$ R.H.S.

191. rth factor of $(C_0+C_1)(C_1+C_2)(C_2+C_3)\cdots(C_{n-1}+C_n)$ is given by

$t_r = C_{r-1} + C_r = C_r^{n+1} = \frac{n+1}{r}.C_{r-1}.$

Now, $(C_0+C_1)(C_1+C_2)(C_2+C_3)\cdots(C_{n-1}+C_n) = t_1.t_2.t_3\ldots t_n$

$= \left(\frac{n+1}{1}C_0\right)\left(\frac{n+1}{2}C_1\right)\left(\frac{n+1}{3}C_2\right)\cdots\left(\frac{n+1}{n}C_{n-1}\right)$

$= \frac{(n+1)^n}{n!}C_1C_2\ldots C_n.$

192. L.H.S. $= \frac{1}{n!}\left[\frac{n!}{1!(n-1)!} + \frac{n!}{3!(n-3)!} + \frac{n!}{5!(n-5)!} + \cdots + \frac{n!}{(n-1)!1!}\right]$

$= \frac{1}{n!}(C_1 + C_3 + C_5 + \cdots + C_{n-1}) = \frac{2^{n-1}}{n!}.$

193. R.H.S. $= \sum_{r=0}^{n}(-1)^r\frac{C_r^n}{C_r^{r+3}} = \sum_{r=0}^{n}(-1)^r\frac{n!}{r!(n-r)!}\cdot\frac{r!3!}{(r+3)!}$

$= 3!\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!(r+3)!} = \frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^r\frac{(n+3)!}{(r+3)!(n-r)!}$

$= \frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^r C_{r+3}^{n+3}$

$= \frac{3!}{(n+1)(n+2)(n+3)}[C_3^{n+3} - C_4^{n+3} + C_5^{n+3} - \cdots + (-1)^n C_{r+3}^{n+3}]$

$\because C_0^{n+3} - C_1^{n+3} + C_2^{n+3} - C_3^{n+3} + \cdots + (-1)^{n+3}C_{n+3}^{n+3} = (1-1)^{n+3} = 0$

$\Rightarrow C_3^{n+3} - C_4^{n+3} + C_5^{n+3} - \cdots + (-1)^n C_{r+3}^{n+3} = C_0^{n+3} - C_1^{n+3} + C_2^{n+3}.$

$\Rightarrow \frac{3!}{(n+1)(n+2)(n+3)}[C_3^{n+3} - C_4^{n+3} + C_5^{n+3} - \cdots + (-1)^n C_{r+3}^{n+3}] = \frac{3!}{(n+1)(n+2)(n+3)}\left[1 - (n+3) + \frac{(n+3)(n+2)}{2}\right] = \frac{3!}{2(n+3)}.$

194. $C_0^n = C_0^{n-1}, -C_1^n = -C_0^{n-1} - C_1^{n-1}, C_2^n = C_1^{n-1} + C_2^{n-1}, \ldots, (-1)^{m-1}C_{m-1}^n = (-1)^{m-1}C_{m-2}^{n-1} + (-1)^{m-1}C_{m-1}^{n-1}$

Adding gives us

$C_0 - C_1 + C_2 - \cdots + (-1)^{m-1}C_{m-1} = (-1)^{m-1}C_{m-1}^{n-1} = (-1)^{m-1}\frac{(n-1)!}{(m-1)!(n-m)!} = (-1)^{m-1}\frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}.$

195. Let d be the common divisor of $C_1^{2n}, C_3^{2n}, C_5^{2n}, \dots, C_{2n-1}^{2n}$.

We know that $C_1^{2n} + C_3^{2n} + C_5^{2n} + \dots + C_{2n-1}^{2n} = 2^{2n-1}$.

Thus, because d is a common divisor so it will have a form of 2^k because it has to divide 2^{2n-1}. Thus, $0 < k \leq 2n-1$.

Let $n = 2^m.r$, where r is an odd positive integer. $\Rightarrow 2n = 2^{m+1}.r \Rightarrow C_1^{2n} = 2n = 2^{m+1}.r$

Thus, common divisor $\leq 2^{m+1}$. We claim that 2^{m+1} divides all of $C_1^{2n}, C_3^{2n}, C_5^{2n}, \dots, C_{2n-1}^{2n}$.

For odd positive integer p, $C_p^{2n} = \frac{2n}{p} C_{p-1}^{2n-1} = \frac{2^{m+1}r}{p} C_{p-1}^{2n-1} = 2^{m+1}\left(\frac{r.C_{p-1}^{2n-1}}{p}\right)$.

196. $2\sum_{r=0}^{n} C_r^n . \sin rx \cos(n-r)x = (C_0^n \sin 0x \cos nx + C_n^n \sin nx \cos 0x) + (C_1^n \sin x \cos(n-1)x + C_{n-1}^n \sin(n-1)x \cos x) + \dots + (C_n^n \sin nx \cos 0x + C_0^n \sin 0x \cos nx)$

$= (C_0 + C_1 + C_2 + \dots + C_n) \sin nx = 2^n \sin nx$

$\Rightarrow \sum_{r=0}^{n} C_r^n . \sin rx \cos(n-r)x = 2^{n-1} \sin nx$.

197. We have proven earlier that $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n.2^{n-1}$.

Rewriting $a.C_0 + (a-b).C_1 + (a-2b).C_2 + \dots + (a-nb).C_n = a(C_0 + C_1 + C_2 + \dots + C_n) - b(C_1 + 2C_2 + 3C_3 + \dots + nC_n)$

$= a.2^n - bn2^{n-1} = 2^{n-1}(2a - nb)$.

198. Given, $a^2.C_0 - (a-1)^2.C_1 + (a-2)^2.C_2 - \dots + (-1)^n(a-n)^2.C_n = 0$, $n > 3$.

$= a^2[C_0 - C_1 + C_2 - \dots + (-1)^n C_n] + 2a[C_1 - 2C_2 + 3C_3 - \dots + (-1)^n C_n] - [C_1 - 2^2C_2 + 3^2C_3 - \dots]$

All three series' have been proven equal to zero earlier, and thus, sum is zero.

199. Given that $a_0, a_1, a_2, \dots, a_n$ form an A.P. Let d be the common difference of this A.P. Then

$a_1 = a_0 + d, a_2 = a_0 + 2d, \dots, a_n = a_0 + nd$.

We have to prove that $a_0 - a_1.C_1 + a_2C_2 - \dots + (-1)^n a_nC_n = 0$

L.H.S. $= a_0 - (a_0 + d)C_1 + (a_0 + 2d)C_2 - \dots + (-1)^n(a_0 + nd)C_n$

$= a_0(C_0 - C_1 + C_2 - \dots + (-1)^n C_n) - d(C_1 - 2C_2 + \dots - (-1)^n nC_n)$

We have proven the two series in question equal to be zero. Q.E.D.

200. We have to prove that $\sum_{r=0}^{n}(-1)^r(a-r)(b-r)C_r = 0$.

$$\sum_{r=0}^{n}(-1)^r(a-r)(b-r)C_r=\sum_{r=0}^{n}(-1)^r[ab-(a+b)r+r^2]C_r$$

This will lead to three series $[C_0-C_1+C_2-\cdots+(-1)^nC_n]$, $[C_1-2C_2-\cdots+(-1)^nnC_n]$ and $[C_1-2^2C_2+3^2C_3-\cdots+(-1)^nn^2C_n]$.

We have proven the three series in question equal to be zero. Q.E.D.

201. We have to prove that $\sum_{r=0}^{n}(-1)^r(a-r)(b-r)(c-r)C_r=0$.

$\Rightarrow \sum_{r=0}^{n}(-1)^r(a-r)(b-r)(c-r)C_r=0=\sum_{r=0}^{n}(-1)^r[abc-(ab+bc+ca)r+(a^2+b^2+c^2)r^2-r^3]C_r=0$.

Out of these four series first three have been proven to be equal to zero. Now we will prove that $\sum_{r=0}^{n}(-1)^r r^3C_r=0$.

Consider $(1-x)^n=C_0-C_1x+C_2x^2-\cdots+(-1)^nC_nx^n$

Differentiating w.r.t. x gives us

$-n(1-x)^{n-1}=-C_1+2C_2x-3C_3x^2+\cdots+(-1)^nnC_nx^{n-1}$

Now we multiply with x and differentiate again to get

$-n(1-x)^{n-1}+n(n-1)x(1-x)^{n-2}=-C_1+2^2C_2x-3^2C_3x^2+\cdots+(-1)^nn^2C_nx^{n-1}$

Repeating previous step and putting $x=1$ gives us

$-C_1+2^3C_2-3^3C_3+\cdots+(-1)^nn^3C_n=0$. Q.E.D.

202. We have to prove that $\frac{C_0}{2^n}+\frac{2.C_1}{2^n}+\cdots+\frac{(n+1)C_n}{2^n}=16$.

Earlier we have proven that $C_0+2C_1+3C_2+\cdots+(n+1)C_n=(n+2)2^{n-1}$

Substituting this result gives us

$\frac{n+2}{2}=16\Rightarrow n=30$.

203. Clearly, $a_2=a_1+d, a_3=a_1+2d,\ldots,a_{n+1}=a_1+nd$, where d is the common difference of the A.P.

We have to prove that $\sum_{k=0}^{n}a_{k+1}C_k=2^{n-1}(a_1+a_{n+1})$.

L.H.S. $=a_1C_0+a_2C_1+a_3C_2+\cdots+a_{n+1}C_n=a_1(C_0+C_1+C_2+\cdots+C_n)+d(C_1+2C_2+\cdots+nC_n)=a_1.2^n+d.n.2^{n-1}$

$=2^{n-1}(2a_1+nd)=2^{n-1}(a_1+a_{n+1})=$ R.H.S.

204. $S = a + (a+d)C_1 + (a+2d)C_2 + \cdots + (a+nd)C_n = a(C_0 + C_1 + C_2 + \cdots + C_n) + d(C_1 + 2C_2 + \cdots + nC_n) = a.2^n + d.n.2^{n-1} = 2^{n-1}[2a + nd] = 2^n \frac{2a+nd}{2} = 2^n . \frac{s}{n+1}$

$\Rightarrow (n+1)S = 2^n.s$

205. Given that $(1 + x + x^2 + \cdots + x^p)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{np}x^{np}$

Differentiating w.r.t. x and putting $x = 1$ gives us

$n(1 + 2 + 3 + \cdots + p)(p+1)^{n-1} = a_1 + 2a_2 + 3a_3 + \cdots + npa_{np}$

$\Rightarrow \frac{1}{2}np(p+1)^n = a_1 + 2a_2 + 3a_3 + \cdots + npa_{np}$.

206. We have to prove that $\sum_{k=0}^{15} \frac{C_k^{15}}{(k+1)(k+2)} = \frac{2^{17}-18}{16.17}$.

Consider $(1+x)^{15} = C_0^{15} + C_1^{15}x + C_2^{15}x^2 + \cdots + C_{15}^{15}x^{15}$

Integrating w.r.t. x with limits 0 and x gives us

$\left[\frac{(1+x)^{16}}{16}\right]_0^x = \left[C_0^{15} + \frac{C_1^{15}}{2}x^2 + \cdots + \frac{C_{15}^{15}x^{16}}{16}\right]_0^x$

$\Rightarrow \frac{(1+x)^{16}}{16} - \frac{1}{16} = C_0^{15} + \frac{C_1^{15}}{2}x^2 + \cdots + \frac{C_{15}^{15}x^{16}}{16}$

Integrating again w.r.t. x with limits 0 and 1 we get

$\sum_{k=0}^{15} \frac{C_k^{15}}{(k+1)(k+2)} = \left[\frac{(1+x)^{17}}{16.17} - \frac{x}{16}\right]_0^1$

$= \frac{2^{17}}{16.17} - \frac{1}{16.17} - \frac{1}{16} = \frac{2^{17}-18}{16.17}$.

Q.E.D.

207. $(1-x)^n = C_0 - C_1x + C_2x^2 - \cdots + (-1)^nC_nx^n$

Substituting $x = x^3$, we have

$(1-x^3)^n = C_0 - C_1x^3 + C_2x^6 - \cdots + (-1)^nC_nx^{3n}$

Integrating within the limits of 0 and 1, we deduce

$\left[C_0x - C_1\frac{x^4}{4} + C_2.\frac{x^7}{7} - \cdots + (-1)^nC_n\frac{x^{3n+1}}{3n+1}\right] = \int_0^1 (1-x^3)^n\,dx$

Now we will evaluate the R.H.S. Let $I_n = \int_0^n (1-x^3)^n\,dx = [x(1-x^3)^n]_0^1 - \int_0^1 x.n(1-x^3)^{n-1}.(-3x^2)\,dx$

$= -3n\int_0^1 (1-x^3)^{n-1}(1-x^3-1)\,dx = -3nI_n + 3nI_{n-1} \Rightarrow \frac{I_n}{I_{n-1}} = \frac{3n}{3n+1}$

Now, $\frac{I_n}{I_0} = \frac{I_n}{I_{n-1}} \cdot \frac{I_{n-1}}{I_{n-1}} \cdots \frac{I_3}{I_2} \cdot \frac{I_2}{I_1} \cdot \frac{I_1}{I_0}$

$= \frac{3n}{3n+1} \cdot \frac{3n-3}{3n-2} \cdot \frac{3n-6}{3n-5} \cdots \frac{3}{4} = \frac{3^n.n!}{4.7.\cdots(3n+1)}$.

208. We have proven earlier that $\frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \cdots + (-1)^n \frac{C_n}{(n+1)(n+2)} = \frac{1}{n+2}$.

Thus, $\sum_{r=0}^{n} \frac{(-1)^r C_r}{(r+1)(r+2)} = \frac{1}{n+2}$.

209. We have to prove that $\sum_{r=0}^{n} \frac{C_r.3^{r+3}}{(r+1)(r+2)(r+3)} = \frac{4^{n+3}-1-\frac{3}{2}(n+3)(3n+8)}{(n+1)(n+2)(n+3)}$.

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$

Integrating within limits 0 and x gives us

$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0 x + \frac{C_1}{2} x^2 + \frac{C_2}{3} x^3 + \cdots + \frac{C_n}{n+1} x^{n+1}$

Integrating again with limits 0 and x gives us

$\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{C_0}{2} x^2 + \frac{C_1}{2.3} x^3 + \frac{C_2}{3.4} x^4 + \cdots + \frac{C_n}{(n+1)(n+2)} x^{n+2}$

Integrating again with limits 0 and 3 gives us

$$\frac{4^{n+3}}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)} - \frac{3}{(n+1)(n+2)} - \frac{9}{2(n+1)} = \sum_{r=0}^{n} \frac{C_r.3^{r+3}}{(r+1)(r+2)(r+3)}$$

$\sum_{r=0}^{n} \frac{C_r.3^{r+3}}{(r+1)(r+2)(r+3)} = \frac{4^{n+3}-1-\frac{3}{2}(n+3)(3n+8)}{(n+1)(n+2)(n+3)}$.

210. $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$

Integrating within limits 0 and x gives us

$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0 x + \frac{C_1}{2} x^2 + \frac{C_2}{3} x^3 + \cdots + \frac{C_n}{n+1} x^{n+1}$

Multiplying with x and differentiang w.r.t. x gives us

$\frac{(1+x)^{n+1} + (n+1)x(1+x)^n}{n+1} - \frac{1}{n+1} = 2C_0 x + \frac{3}{2} C_1 x^2 + \frac{4}{3} C_2 x^3 + \cdots + \frac{n+2}{n+1} C_n x^{n+1}$

Putting $x = 1$ gives us

$\sum_{r=0}^{n} \frac{r+2}{r+1} C_r = \frac{2^n(n+3)-1}{n+1}$.

211. Proceeding like previous to previous problem and adjusting last step with integration between 0 and x and integrating once more for the fourth time and putting $x = 3$ we get the desired result.

212. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiplying and equating the power of x^{n-3} gives us

$C_0C_3 + C_1C_4 + C_2C_5 + \cdots + C_{n-3}C_n =$ coefficient of x^{n-3} in $(1+x)^{2n}$

$\Rightarrow \sum_{r=0}^{n-3} C_rC_{r+3} = \frac{(2n)!}{(n+3)!(n-3)!}$.

213. We have to find $\sum_{0\leq i<j} \sum_{0\leq j\leq n} C_iC_j$.

$\left(\sum C_k\right)^2 = \sum C_k^2 + 2\sum_{0\leq i<j} \sum_{0\leq j\leq n} C_iC_j$.

We know that $(\sum C_k)^2 = (2^n)^2 = 2^{2n}$ and $\sum C_k^2 = \frac{2n!}{n!n!}$

$\therefore \sum_{0\leq i<j} \sum_{0\leq j\leq n} C_iC_j = 2^{2n-1} - \frac{(2n-1)!}{n!(n-1)!}$.

214. Given that $S_n = C_0C_1 + C_1C_2 + \cdots + C_{n-1}C_n$.

$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$

Multiplying and equating power of x^{n-1} gives us

$S_n = C_0C_1 + C_1C_2 + \cdots + C_{n-1}C_n =$ coefficient of x^{n-1} in $(1+x)^{2n} = C_{n-1}^{2n}$

$\Rightarrow \frac{S_{n+1}}{S_n} = \frac{(2n+2)!}{n!(n+2)!} \cdot \frac{(n-1)!(n+1)!}{(2n)!} = \frac{(2n+2)(2n+1)}{n(n+2)} = \frac{15}{4} \Rightarrow n = 2, 4$.

215. $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$.

Multiplying first equation with x and differentiating first equation w.r.t. x gives us

$(1+x)^n + nx(1+x)^{n-1} = C_0 + 2C_1x + 3C_2x^2 + \cdots + (n+1)C_nx^n$.

Multiplying this with second equation and then equating coefficient of x^n gives us

$C_0^2 + 2.C_1^2 + 3.C_2^2 + \cdots + (n+1)C_n^2 =$ coefficient of x^n in $(1+x)^{2n} + nx(1+x)^{2n-1}$

$= C_n^{2n} + n.C_{n-1}^{2n-1} = \frac{(n+2)(2n-1)!}{n!(n-1)!}$.

216. We have $x^n(2+x)^n = (2x+x^2)^n = [(x+1)^2 - 1]^n$

$\Rightarrow x^n(2^n + C_1^n.2^{n-1} + \cdots + C_n^nx^n) = C_0^n(x+1)^{2n} - C_1^n(x+1)^{2n-2} + C_2^n(x+1)^{2n-4} - \cdots$

Equating coefficients of x^n gives us

$C_0.C_n^{2n} - C_1.C_n^{2n-2} + C_2.C_n^{2n-4} - \cdots = 2^n.$

217. We have to find $\sum_{0\le i\le j}\sum_{0\le j\le n}(i+j)(C_i+C_j+C_iC_j)$.

Earlier we have proven that $\sum_{0\le i\le j}\sum_{0\le j\le n}(i+j)(C_iC_j) = \frac{n}{2}\left(2^{2n1} - \frac{2n!}{(n!)^2}\right)$

Now $\sum_{0\le i\le j}\sum_{0\le j\le n}(i+j)(C_i+C_j)$ we proceed similarly to get

$\sum_{0\le i\le j}\sum_{0\le j\le n}(i+j)(C_i+C_j) = \sum_{0\le i\le j}\sum_{0\le j\le n}(2n-i-j)(C_i+C_j) = E$

$\Rightarrow 2E = \sum_{0\le i\le j}\sum_{0\le j\le n}2n(C_i+C_j) = 2n\sum_{0\le j\le n}[(C_0+C_j)+(C_1+C_j)+\cdots+(C_{j-1}+C_j)]$

$= 2n[(2C_0+C_1+C_2+\cdots+C_n)+(C_0+2C_1+C_2+\cdots+C_n)+\cdots+(C_0+C_1+\cdots+2C_{n-1})+C_n]$

$= 2n^2[C_0+C_1+C_2+\cdots+C_n] \Rightarrow E = n^2.2^n$

218. Given that $(1+x+x^2)^n = a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$.

Putting $x=-x$ gives us $(1-x+x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \cdots + a_{2n}$

Multiplying and equating coefficients of x^{2n-2r} gives us

$a_0a_{2r} - a_1a_{2r+1} + a_2a_{2r+2} - \cdots + a_{2n-2r}a_{2n} =$ coefficient of x^{2n-2r} in $(x^4+x^2+1)^n$

Putting $x^2 = y$ it is coefficient of y^{n-2} in $(1+y+y^2) = a_{n-r}$.

Earlier we have proven that for the given series $a_r = a_{2n-r} \Rightarrow a_{n-r} = a_{n+r}$.

219. $\frac{P_{n+1}}{P_n} = \frac{C_0^{n+1}.C_1^{n+1}\ldots C_{n+1}^{n+1}}{C_0^n.C_1^n\ldots C_n^n}$

$= \frac{C_0^{n+1}}{C_0^n}.\frac{C_1^{n+1}}{C_1^n}.\cdots\frac{C_n^{n+1}}{C_n^n}.C_{n+1}^{n+1}$

$= \frac{(n+1)^n}{n!}.$

220. We have to prove that $\sum_{r=1}^{n} r^3\left(\frac{C_r}{C_{r-1}}\right)^2 = \frac{1}{12}n(n+1)^2(n+2)$.

$= \sum_{r=1}^{n} r(n+1-r)^2\,[\because \frac{C_r}{C_{r-1}} = \frac{(n+1-r)}{r}]$

which is now a trivial matter of applying summation rules.

221. We know that $(1+x)^n = C_0+C_1x+C_2x^2+\cdots+C_n$. Putting $x=1,-1$ gives us

$C_0+C_1+C_2+\cdots+C_n = 2^n$, and $C_0-C_1+C_2-\cdots+(-1)^nC_n = 0$

Adding and subtracting previous two equations gives us

$C_0 + C_2 + C_4 + \cdots = 2^{n-1}$, and $C_1 + C_3 + C_5 + \cdots = 2^{n-1}$.

Putting $x = i$ gives us

$(C_0 - C_2 + C_4 - C_6 + \cdots) + i(C_1 - C_3 + C_5 - C_7 + \cdots) = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

Equating imaginary parts gives us

$C_1 - C_3 + C_5 - C_7 + \cdots = 2^{n/2}\sin\frac{n\pi}{4}$

$\Rightarrow C_3 + C_7 + C_{11} + \ldots = \frac{1}{2}\left[2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right]$.

222. We have $(1 + x + x^2)^{20} = a_0 + a_1x + a_2x^2 + \cdots = a_{40}x^{40}$.

Putting $x = 1$ gives us

$a_0 + a_1 + a_2 + \cdots + a_{40} = 4^{20} = 2^{40}$

Putting $x = -1$ gives us

$a_0 - a_1 + a_2 - a_3 + \cdots + a_{40} = 2^{20}$

Adding gives us

$a_0 + a_2 + a_4 + \cdots + a_{40} = 2^{39} + 2^{19}$

Clearly, $a_{40} = 2^{20} \Rightarrow a_0 + a_2 + a_4 + \cdots + a_{38} = 2^{39} + 2^{19} - 2^{20} = 2^{19}(2^{20} - 1)$

223. We have $(1 + x + x^2)^{20} = a_0 + a_1x + a_2x^2 + \cdots = a_{40}x^{40}$.

Putting $x = 1$ gives us

$a_0 + a_1 + a_2 + \cdots + a_{40} = 4^{20} = 2^{40}$

Putting $x = -1$ gives us

$a_0 - a_1 + a_2 - a_3 + \cdots + a_{40} = 2^{20}$

Subtracting gives us

$a_1 + a_3 + a_5 + \cdots + a_{39} = 2^{40} - 2^{20}$

Now $a_{39} =$ coefficient of x^{39} in $(1 + x + 2x^2)^{20}$

$=$ coefficient of x^{39} in $[1 + x(1 + 2x)]^{20} = C_{20}^{20}x^{20}(1 + 2x)^{20}$

$=$ coefficient of x^{19} in $(1 + 2x)^{20} = 20 \times 2^{19}$

$\Rightarrow a_1 + a_3 + a_5 + \cdots + a_{37} = 2^{19}(2^{20} - 21)$.

224. $1 - C_1x + C_2x^2 - C_3x^3 + \cdots + (-1)^n C_nx^n = (1-x)^n$

$\Rightarrow C_1 - C_2x + C_3x^2 - \cdots + (-1)^{n-1} C_nx^{n-1} = \frac{1-(1-x)^n}{x}$

Integrating between the limits of 0 and 1, we arrive at

$$\left[C_1x - C_2.\frac{x^2}{2} + C_3.\frac{x^3}{3} - \cdots + (-1)^{n-1} C_n.\frac{x^n}{n}\right]_0^1 = \int_0^1 \frac{1-(1-x)^n}{x}dx$$

Now $\int_0^1 \frac{1-(1-x)^n}{x}dx = \int_0^1 \frac{1-z^n}{1-z}dz$, where $z = 1-x$

$= \int_0^1 (1 + z + z^2 + \cdots + z^n)\,dz = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$

Now $\frac{1}{n(n-1)} + \frac{1}{n} = \frac{n}{n(n-1)}, \frac{2}{(n-1)(n-2)} + \frac{1}{n-1} = \frac{n}{(n-1)(n-2)}$ and so on till $\frac{n-2}{2.3} + \frac{1}{3} = \frac{n}{2.3}$(we will have 1 and $\frac{1}{2}$ left)

and $\frac{n}{n(n-1)} = n\left[\frac{1}{n-1} - \frac{1}{n}\right], \frac{n}{(n-1)(n-2)} = n\left[\frac{1}{n-2} - \frac{1}{n-1}\right]$ and so on.

So sum would be $\frac{n}{2} - \frac{n}{n} + 1 + \frac{1}{2} = \frac{n+1}{2}.$

225. We have to prove that $\sum_{0\le i<j}\sum_{0\le j\le n} \frac{i}{C_i} + \frac{j}{C_j} = \frac{n^2}{2}\sum_{r=0}^{n}\frac{1}{C_r}$ $[0 \le i \le j \le n]$.

Let $S = \sum_{0\le i<j}\sum_{0\le j\le n} \frac{i}{C_i} + \frac{j}{C_j}$

Replacing i by $n-i$ and j by $n-j$

$$S = \sum_{0\le i<j}\sum_{0\le j\le n} \frac{n-i}{C_{n-i}} + \frac{n-j}{C_{n-j}} = \sum_{0\le i<j}\sum_{0\le j\le n} \frac{n-i}{C_i} + \frac{n-j}{C_j}$$

$$2S = n\sum_{0\le i<j}\sum_{0\le j\le n}\left(\frac{1}{C_i} + \frac{1}{C_j}\right)$$

$$S = \frac{n}{2}\sum_{r=0}^{n-1}\frac{n-r}{C_r} + \sum_{r=1}^{n}\frac{r}{C_r} = \frac{n}{2}\sum_{r=0}^{n}\frac{n}{C_r} = \frac{n^2}{2}\sum_{r=0}^{n}\frac{1}{C_r}.$$

226. $\sum_{0\le i<j}\sum_{0\le j\le n} i.j.C_i.C_j = \sum_{0\le i<j}\sum_{0\le j\le n}\left(i.\frac{n}{i}.C_{i-1}^{n-1}\right)\left(j.\frac{n}{j}.C_{j-1}^{n-1}\right)$

$= \frac{n^2}{2}[2^{2n-1} - C_{n-1}^{2n-1}]$ (using result obtained earlier)

$= n^2\left(2^{2n-3} - \frac{1}{2}C_{n-1}^{2n-1}\right).$

227. $t_r = (-1)^{r-1}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}\right).C_r^n.$

Now $(-1)^r(1+x+x^2+\cdots+x^{r-1}).C_r^n = \frac{(1-x^r)}{1-x}(-1)^{r-1}.C_r^n$

$= \frac{1}{1-x}\sum_{r=1}^{n}(-1)^{r-1}C_r - (-1)^{r-1}x^r.C_r^n = \frac{1}{1-x}[(-1)\{(1-1)^n-1\}+(1-x)^n-1] = (1-x)^{n-1}$

$\Rightarrow \sum_{r=1}^{n}(-1)^{r-1}(1+x+x^2+\cdots+x^{r-1}).C_r^n = (1-x)^{n-1}$

Integrating w.r.t. x between limits 0 and 1 gives us

$\sum_{r-1}^{n}(-1)^{r-1}.C_r^n\int_0^1(1+x+x^2+\cdots+x^{r-1})\,dx = \left[-\frac{(1-x)^n}{n}\right]_0^1$

$\Rightarrow \sum_{r=1}^{n}t_r = \frac{1}{n}.$

228. The general term of the expansion of $(1+2x+3x^2)^4\frac{4!}{\alpha_1!\alpha_2!\alpha_3!}1^{\alpha_1}(2x)^{\alpha_2}(3x^2)^{\alpha_3}$

$= \frac{4!}{\alpha_1!\alpha_2!\alpha_3!}(2x)^{\alpha_2}(3x^2)^{\alpha_3}$, where $\alpha_1, \alpha_2, \alpha_3$ are non-negative integers satisfying the condition

$\alpha_1+\alpha_2+\alpha_3 = 4.$

For coefficient of x^5, $\alpha_2+2\alpha_3 = 5.$

Thus, if $\alpha_2 = 1$, then $\alpha_3 = 2$ making $\alpha_1 = 1$ and if $\alpha_2 = 3$, then $]alpha_3 = 1$ making $\alpha_1 = 0$.

Thus, required coefficient of $x^5 = \frac{4!}{1!1!2!}2.3^2 + \frac{4!}{0!3!1!}2^3.3 = 312.$

229. General term in the expansion of $(2x-3y+4z)^9$ is $\frac{9!}{a!b!c!}(2x)^a(-3y)^b(4z)^c$.

Coefficient of $x^3y^4z^2$ means $a=3, b=4$, and $c=2$.

Thus, coefficient is $\frac{9!}{3!4!2!}2^3.(-3)^4.4^2 = 13063600.$

230. General term in the expansion of $(2x-3y+4z)^{100}$ is $\frac{9!}{a!b!c!}(2x)^a(-3y)^b(4z)^c$

where $a+b+c = 100$ is the required condition.

No. of terms in the expansion of $(2x-3y+4z)^{100}$ is = no. of non-negative integral solutions of the above condition

$= C_r^{n+r-1} = C_{100}^{102} = 5151.$

231. General term in the expansion of $(1+x+x^2)^3$ is $\frac{3!}{a!b!c!}1^a.x^b.x^{2c}$ where $a+b+c=3$.

For coefficient of $x^4, b+2c=4$. If $b=2 \Rightarrow c=1$ making $a=0$ and if $b=0 \Rightarrow c=2$ amking $a=1$.

Thus, coefficient of x^4 is $\frac{3!}{2!}+\frac{3!}{2!}=6$.

232. General term in the expansion of $(7+x+x^2+x^3+x^4+x^5)^3$ is $\frac{3!}{a!b!c!d!e!f!}7^a x^b x^{2c} x^{3d} x^{4e} x^{5f}$, where $a+b+c+d+e+f=3$ and $b+2c+3d+4e+5f=10$

If $f=2$ and $a=1$ making $b=c=d=e=0$ is one solution. If $f=1$ then $b=1, e=1$ is another solution making $a=c=d=0$; $c=1, d=1$ is another solution making $a=b=e=0$. If $c=1$ and $e=2$ then $a=b=d=f=0$ and $d=2, e=1$ then $a=b=c=f=0$ are two other solutions.

Thus coefficient is $\frac{3!}{2!}7+3!+3!+\frac{3!}{2!}+\frac{3!}{2!}=39$.

233. General term in the expansion of $(1+3x-2x^3)^{10}$ is $\frac{10!}{a!b!c!}1^a.(3x)^b(-2x^3)^c$, where $a+b+c=10$.

For coefficient of x^7 we have $b+3c=7$. If $b=1$ then $c=2$ making $a=7$. If $b=4$ then $c=1$ making $a=5$ If $b=7$ then $c=0$ making $a=3$.

Thus, coefficient is $\frac{10!}{7!1!2!}3^1.(-2)^2+\frac{10!}{5!4!1!}3^4.(-2)^1+\frac{10!}{3!7!}3^7=62640$.

234. General term in the expansion of $(xy+yz+zx)^6$ is $\frac{6!}{a!b!c!}(xy)^a.(yz)^b.(zx)^c$, where $a+b+c=6$.

For coefficient of $x^3y^4z^5$ we have $a+c=3, a+b=4, b+c=5$.

$\Rightarrow b-c=1, c-a=1, b-a=2$. If $a=1, \Rightarrow b=3, c=2$ is one solution.

Thus, coefficient is $\frac{6!}{1!3!2!}=60$.

235. The coefficient will be greatest when maximum no. of terms will have equal power which is 4 for three terms and 3 for 4th term.

The greatest coefficient is $\frac{15!}{3!(4!)^3}$.

236. Following like problem solved earlier no. of terms is $C_{100}^{100+5-1}=C_4^{104}$.

237. We know that if n is a negative integer or a fraction and $|x|<1$ then

$(1+x)^n=1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ to ∞

Putting $x=-x$ and $n=-2$ gives us

$(1-x)^{-2}=1+\frac{-2}{1!}(-x)+\frac{-2(-2-1)}{2!}x^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3+\cdots$ to ∞

$=1+2x+3x^2+4x^3+\cdots$ to ∞.

238. $\frac{a+bx}{(1-x)^2} = (a+bx)(1-x)^{-2} = (a+bx)(1+2x+3x^2+4x^3+\cdots+nx^{n-1}+(n+1)x^n+\cdots)$

Coefficient of $x^n = nb + (n+1)a = (a+b)n + a$. Given that $(a+b)n + a = 2n+1$.

Equating the coefficient we have $a+b=2, a=1 \Rightarrow b=1$.

Substituting back gives us $\frac{1+x}{(1-x)^2} = (1+x)(1+2x+3x^2+4x^3+\cdots)$

$= 1 + 3x + 5x + 7x^3 + \cdots$

Putting $x = \frac{1}{2}$ gives us $1 + 3.\frac{1}{2} + 5.\frac{1}{2^2} + 7.\frac{1}{2^3} + \cdots$.

239. $(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots$ to ∞.

Comparing first two terms we have $nx = \frac{1}{3}, \frac{n(n-1)x^2}{2!} = \frac{1.3}{3.6}$

$\Rightarrow \frac{n^2x^2}{n(n-1)} \cdot \frac{2!}{x^2} = \frac{1}{9} \cdot \frac{3.6}{1.3}$

$\Rightarrow \frac{2n}{n-1} = \frac{2}{3} \Rightarrow n = -\frac{1}{2} \Rightarrow x = -\frac{2}{3}$.

Hence, sum of the given series is $(1+x)^n = \left(1-\frac{2}{3}\right)^{-\frac{1}{2}} = \sqrt{3}$.

240. We know that if n is a negative integer or a fraction and $|x| < 1$ then

$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$ to ∞

Putting $x = -x$ and $n = -1$ gives us

$(1-x)^{-1} = 1 + \frac{-1}{1!}(-x) + \frac{-1(-1-1)}{2!}x^2 + \frac{-1(-1-1)(-1-2)}{3!}(-x)^3 + \cdots$ to ∞

$= 1 + x + x^2 + x^3 + \cdots$ to ∞.

241. Following like previous problem; we put $n = -1$ to get

$(1+x)^{-1} = 1 + \frac{-1}{1!}x + \frac{-1(-1-1)}{2!}x^2 + \frac{-1(-1-1)(-1-2)}{3!}x^3 + \cdots$ to ∞

$= 1 - x + x^2 - x^3 + \cdots$ to ∞.

242. Following like previous problem; we put $n = -2$ to get

$(1+x)^{-2} = 1 + \frac{-2}{1!}x + \frac{-2(-2-1)}{2!}x^2 + \frac{-2(-2-1)(-2-2)}{3!}x^3 + \cdots$ to ∞

$= 1 - 2x + 3x^2 - 4x^3 + \cdots$ to ∞.

243. Following like previous problem; we put $n = -3$ and $x = -x$ to get

$(1-x)^{-3} = 1 + \frac{-3}{1!}(-x) + \frac{-3(-3-1)}{2!}(-x)^2 + \frac{-3(-3-1)(-3-2)}{3!}(-x)^3 + \cdots$ to ∞

$= 1 + 3x + 6x^2 + 10x^3 + \cdots$ to ∞.

244. Following like previous problem; we put $n = -3$ to get

$(1+x)^{-3} = 1 + \frac{-3}{1!}x + \frac{-3(-3-1)}{2!}x^2 + \frac{-3(-3-1)(-3-2)}{3!}x^3 + \cdots$ to ∞

$= 1 - 3x + 6x^2 - 10x^3 + \cdots$ to ∞.

245. Following like previous problem; we put $n = -\frac{1}{5}$ to get

$(1+x)^{-\frac{1}{5}} = 1 + -\frac{1}{5.1!}x + \frac{-1}{5}.\left(-\frac{1}{5}-1\right).\frac{1}{2!}x^2 + \frac{-1}{5}.\left(-\frac{1}{5}-1\right).\left(-\frac{1}{5}-2\right).\frac{1}{3!}x^3 + \cdots$ to ∞

$= 1 - \frac{x}{5} + \frac{3x^2}{25} - \frac{11x^3}{125} + \cdots$ to ∞.

246. $\left(\frac{2x}{3} - \frac{3}{2x}\right)^{-3/2} = \left(\frac{3}{2x}\right)^{\frac{3}{2}}\left(1 - \frac{9}{4x^2}\right)^{-\frac{3}{2}}$

$= \left(\frac{3}{2x}\right)^{\frac{3}{2}}\left[1 + \frac{-3}{2}.\left(-\frac{9}{4x^2}\right) + \frac{-3}{2}.\left(-\frac{3}{2} - 1\right).\frac{1}{2!}.\left(-\frac{9}{4x^2}\right)^2 + \frac{-3}{2}\left(-\frac{3}{2} - 1\right)\left(-\frac{3}{2} - 2\right).\frac{1}{3!}.\left(-\frac{9}{4x^2}\right)^3 + \cdots\right]$

$= \left(\frac{3}{2x}\right)^{\frac{3}{2}}\left[1 + \frac{3}{2}.\frac{9}{4x^2} + \frac{3.5}{4.2!}\frac{81}{16x^4} + \frac{3.5.7}{8.3!}.\frac{729}{64x^6} + \cdots\right]$.

247. $\left(1 - \frac{x}{2}\right)^{-2} = 1 + \frac{-2}{1!}. - \frac{x}{2} + \frac{-2(-2-1)}{2!}\left(-\frac{x}{2}\right)^2 + \cdots$

$= 1 + x + \frac{3}{4}x^2$ are the first three terms.

248. Coefficient of x^6 is $-\frac{5}{2}.\left(-\frac{5}{2}-1\right)\left(-\frac{5}{2}-2\right)\left(-\frac{5}{2}-3\right)\left(-\frac{5}{2}-4\right).\frac{1}{5!}(-2)^5 = \frac{15015}{16}$.

249. $t_{r+1} = -\frac{1}{2}.\left(-\frac{1}{2}-1\right).\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-r+1\right).\frac{1}{r!}.(-2x)^r$

$= \frac{1.3.5\cdots(2r-1)}{r!}x^r$ and the coefficient is $\frac{1.3.5.\cdots(2r-1)}{r!}$.

250. $(1 + 2x + 3x^2 + 4x^3 + \ldots \text{ to } \infty)^{3/2} = (1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$ to ∞.

251. $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \cdots = 1 + \frac{1}{4} + \frac{1.3}{4.8} + \cdots$ to ∞

Comparing the series gives us

$nx = \frac{1}{4}$ and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.8}$

$\Rightarrow \frac{n^2x^2}{n(n-1)}.\frac{2!}{x^2} = \frac{1}{16}.\frac{4.8}{1.3} = \frac{2}{3}$

$\Rightarrow \frac{2n}{n-1} = \frac{2}{3} \Rightarrow 6n = 2n - 2 \Rightarrow n = -\frac{1}{2}$

$\Rightarrow x = -\frac{1}{2} \Rightarrow (1+x)^n = \left(1-\frac{1}{2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}.$

252. Similar to previous problem $nx = \frac{2}{6}$ and $\frac{n(n-1)}{2!}x^2 = \frac{2.5}{6.12}$

$\Rightarrow \frac{n^2x^2}{n(n-1)} \cdot \frac{2!}{x^2} = \frac{1}{9} \cdot \frac{6.12}{2.5} = \frac{4}{5}$

$\Rightarrow \frac{2n}{n-1} = \frac{4}{5} \Rightarrow 10n = 4n - 4 \Rightarrow n = -\frac{2}{3}$

$x = \frac{1}{3} \cdot -\frac{3}{2} = -\frac{1}{2}$

$(1+x)^n = \left(1-\frac{1}{2}\right)^{-\frac{2}{3}} = 4^{1/3}.$

253. GIven $y = x - x^2 + x^3 - x^4 + \cdots$ to $\infty \Rightarrow 1 - y = 1 - x + x^2 - x^3 + \cdots$ to ∞

$\Rightarrow 1 - y = (1+x)^{-1} = \frac{1}{1+x} \Rightarrow 1 + x = (1-y)^{-1}$

$\Rightarrow x = y + y^2 + y^3 + \cdots$ to ∞.

254. $\frac{1}{1+x+x^2} = \frac{1-x}{1-x^3} = (1-x)(1-x^3)^{-1}.$

Now is it trivial to prove the required condition.

255. $t_n = \frac{2n}{(n+1)!} = \frac{2n+1}{(2n+1)!} - \frac{1}{(2n+1)!} = \frac{1}{2n!} - \frac{1}{(2n+1)!}$

$\Rightarrow t_1 = \frac{1}{2!} - \frac{1}{3!}, t_r = \frac{1}{4!} - \frac{1}{5!}$

Sum of given series is $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

$= e^{-1} = \frac{1}{e}.$

256. $t_n = \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$

$t_1 0 + \frac{1}{0!} = 1, t_2 = \frac{1}{0!} + \frac{1}{1!}, t_3 = \frac{1}{1!} + \frac{1}{2!}$

Sum of given series is $\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) = e + e = 2e.$

257. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$

$\log(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

$= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

258. $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \log(1+x) \Rightarrow 1 + x = e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$\Rightarrow x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

259. Since α, β are the roots of the equation $ax^2 + bx + c = 0 \therefore \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

$\log(a - bc + cx^2) = \log\left[a\left(1 - \frac{b}{a}x + \frac{c}{a}x^2\right)\right] = \log[a(1 + (\alpha+\beta)x + \alpha\beta x^2)] = \log[a(1 + \alpha x + \beta x)] = \log a + \log(1 + \alpha x) + \log(1 + \beta x)$

$= \log a + \left[\alpha x - \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^2}{3} - \cdots\right] + \left[\beta x - \frac{(\beta x)^2}{2} + \frac{(\beta x)^3}{3} - \cdots\right]$

$= \log a + (\alpha + \beta)x - \frac{(\alpha^2+\beta^2)}{2}x^2 + \ldots$ to ∞

260. $t_n = \frac{n}{(2n+1)!} = \frac{1}{2}\left[\frac{2n}{(2n+1)!}\right]$

We have proven earlier that if $t_n = \frac{2n}{(2n+1)!}$ then sum of the series is $\frac{1}{e}$, thus, in this case sum of the series would be $\frac{1}{2e}$.

261. $t_n = \frac{2n-1}{2n!} = \frac{1}{(2n-1)!} - \frac{1}{2n!}$

$t_1 = \frac{1}{1!} - \frac{1}{2!}, t_2 = \frac{1}{3!} - \frac{1}{4!}, t_3 = \frac{1}{5!} - \frac{1}{6!}$

Sum $= 1 - e^{-1}$

262. $t_n = \frac{1+2+\cdots+n}{(n+1)!} = \frac{n}{2.n!} = \frac{1}{2} \cdot \frac{1}{(n-1)!}$

$\therefore$ Sum $= \frac{e}{2}$

263. $t_n = \frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{n^2-1+1}{(n-1)!} = \frac{n+1}{(n-2)!} + \frac{1}{(n-1)!}$

Now $\frac{n+1}{(n-2)!} = \frac{n-2+3}{(n-2)!} = \frac{1}{(n-3)!} + \frac{3}{(n-2)!}$

$\therefore$ Sum $= 5e$.

264. $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots$

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Thus, $1 - \log 2 = \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots$

265. $\log(1+x) - \log(x-1) = \log(1+x) - \log x + \log x - \log(x-1) = \log(1+\frac{1}{x}) - \log(1-\frac{1}{x})$

$= 2\left[\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots\right]$

266. $2\log x - \log(x+1) - \log(x-1) = -[\log(1+\frac{1}{x}) - \log(1-\frac{1}{x})]$

$= \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^5} + \cdots$ to ∞.

267. $\log[(1+x)^{1+x}\log(1-x)^{1-x}] = (1+x)\log(1+x) + (1-x)\log(1-x)$

$= (1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + (1-x)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)$

$= -2\left(\frac{x^2}{2} + \frac{x^4}{4}\right) + 2x\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$

$= 2\left(\frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^4}{5.6} + \cdots\right).$

Answers of Chapter 8
Determinants

1. Let $\Delta = \begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix}$

$\Delta = \begin{vmatrix} 1 & 4 & 0 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix} [R_1 \to R_1 - R_2] = \begin{vmatrix} 1 & 4 & 0 \\ 0 & -7 & 7 \\ 0 & -16 & 5 \end{vmatrix} [R_2 \to R_2 - 3R_1 \text{ and } R_3 \to R_3 - 5R_1]$

$= 1(-35+112) = 77.$

2. Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(b+c-a-b) = (a-b)(b-c)(c-a).$

3. Let $a = 2, b = 3, c = 4$ then $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

We can solve this like previous problem which gives us $\Delta = (2-3)(3-4)(4-2) = 2.$

4. Let $\Delta = \begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -5 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix} [R_1 \to R_1 - R_2]$

$= \begin{vmatrix} 1 & 4 & -5 \\ 0 & -7 & 22 \\ 0 & -31 & 46 \end{vmatrix} [R_2 \to R_2 - 3R_1; R_3 \to R_3 - 5R_1]$

$= (-322+682) = 360.$

5. Let $\Delta = \begin{vmatrix} 18 & 1 & 17 \\ 22 & 3 & 19 \\ 26 & 5 & 21 \end{vmatrix} = \begin{vmatrix} 18 & 1 & 17 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{vmatrix} [R_3 \to R_3 - R_2; R_2 \to R_2 - R_1]$

$= 0$(bceuase two rows are identical).

6. Let $\Delta = \begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix} [R_1 \to R_2 - R_1]$

$= \begin{vmatrix} 1 & 4 & 0 \\ 0 & -7 & 7 \\ 0 & -16 & 5 \end{vmatrix} [R_2 \to R_2 - 3R_1; R_3 \to R_3 - R_1]$

$= (-35 + 112) = 77.$

7. $\Delta = \begin{vmatrix} 1 & 4 & 9 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{vmatrix} [R_3 \to R_3 - R_2; R_2 \to R_2 - R_1] = \begin{vmatrix} 1 & 4 & 9 \\ 3 & 5 & 7 \\ 2 & 2 & 2 \end{vmatrix} [R_3 \to R_3 - R_2]$

$= \begin{vmatrix} 1 & 3 & 5 \\ 3 & 2 & 2 \\ 2 & 0 & 0 \end{vmatrix} [C_3 \to C_3 - C_2; C_2 \to C_2 - C_1] = 2(6 - 10) = -8.$

8. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$

$= (a+b+c)[(c-b)(b-c) - (a-b)(a-c)] = (a+b+c)(ab+bc+ca-a^2-b^2-c^2) = -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

Given that a, b, c are positive so $a + b + c > 0$ and since a, b, c are unequal so $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$. Thus, $\Delta < 0$.

9. Let $\Delta = \begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = \begin{vmatrix} a+b+c & a+b & a \\ b+c+a & b+c & b \\ a+b+c & c+a & c \end{vmatrix} [C_1 \to C_1 + C_3]$

$= (a+b+c) \begin{vmatrix} 1 & a+b & a \\ 1 & b+c & b \\ 1 & c+a & c \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a+b & a \\ 0 & c-a & b-a \\ 0 & c-b & c-a \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$

$= (a+b+c)[(c-a)^2 - (c-b)(b-a)] = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3 + b^3 + c^3 - 3abc.$

10. $\Delta = \begin{vmatrix} 1+a_1+a_2+a_3 & a_2 & a_3 \\ 1+a_1+a_2+a_3 & 1+a_2 & a_3 \\ 1+a_1+a_2+a_3 & a_2 & 1+a_3 \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$= (1 + a_1 + a_2 + a_3) \begin{vmatrix} 1 & a_2 & a_3 \\ 1 & 1+a_2 & a_3 \\ 1 & a_2 & 1+a_3 \end{vmatrix} = (1 + a_1 + a_2 + a_3) \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a_2 & 1+a_3 \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$$= (1 + a_1 + a_2 + a_3)\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} = 1 + a_1 + a_2 + a_3.$$

11. $\Delta = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & b+c+a & -(a+b+c) \\ 1 & a & c+a+2b \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$$

$= 2(a+b+c)^3\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a & c+a+2b \end{vmatrix}$(taking $a+b+c$ common from first and second row)

$$= 2(a+b+c)^3[-1.-1-1.0] = 2(a+b+c)^3.$$

12. $\Delta = \begin{vmatrix} 2a & 2b & a-b-c \\ 2b & 2c & b-c-a \\ 2c & 2a & c-a-b \end{vmatrix} [C_1 \to C_1 + C_2; C_2 \to C_2 - C_3]$

$$= 4\begin{vmatrix} a & b & a-b-c \\ b & c & b-c-a \\ c & a & c-a-b \end{vmatrix}$$

$$= 4\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 4(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 4(a^3+b^3+c^3-3abc).$$

13. $\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & \\ 2c & 2c & c-a-b \end{vmatrix} [R_1 \to R_1 + R_2 + R_3]$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} [C_1 \to C_2 - C_1; C_3 \to C_3 - C_1]$$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} = (a+b+c)^3.$$

14. $\Delta = \frac{1}{x.y.z}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}[C_1 \to xC_1; C_2 \to yC_2; C_3 \to zC_3]$

$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$(exchanging rows twice)

$= \begin{vmatrix} 1 & 0 & \\ x^2 & y^2 - x^2 & z^2 - x^2 \\ x^3 & y^3 - x^3 & z^3 - x^3 \end{vmatrix}[C_2 \to C_2 - C_1; C_3 \to C_3 - C_1]$

$= \begin{vmatrix} (y-x)(y+x) & (z-x)(z+x) \\ (y-x)(y^2+xy+x^2) & (z-x)(z^2+zx+x^2) \end{vmatrix}$

$= (y-x)(z-x)\begin{vmatrix} y+x & z+x \\ y^2+xy+y^2 & z^2+zx+x^2 \end{vmatrix}$

$= (y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+yx+x^2 & (z^2-y^2)+zx-zy \end{vmatrix}[C_2 \to C_2 - C_1]$

$= (y-x)(z-x)(z-y)\begin{vmatrix} y+z & 1 \\ y^2+xy+x^2 & x+y+z \end{vmatrix}$

$= (y-x)(z-x)(z-y)[(y+x)(x+y+z) - y^2 - xy - x^2] = (x-y)(y-z)(z-x)(xy+yz+zx)$.

15. $\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}[C_1 \to aC_1; C_2 \to bC_2; C_3 \to cC_3]$

$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$

$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}[C_1 \to C_1 + C_2 + C_3]$

$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$

$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}[R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$

$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1+a^2+b^2+c^2.$

16. $\Delta = a_1a_2a_3\begin{vmatrix} \frac{1}{a_1}+1 & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{a_1} & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix}[C_1 \to \frac{C_1}{a_1}; C_2 \to \frac{C_2}{a_2}; C_3 \to \frac{C_3}{a_3}]$

$= a_1a_2a_2\begin{vmatrix} 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} & \frac{1}{a_2} & \frac{1}{a_3} \\ 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3} & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix}[C_1 \to C_1+C_2+C_3]$

$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)\begin{vmatrix} 1 & \frac{1}{a_2} & \frac{1}{a_3} \\ 1 & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ 1 & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix}$

$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)\begin{vmatrix} 1 & \frac{1}{a_2} & \frac{1}{a_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}[R_2 \to R_2-R_1; R_3 \to R_3-R_1]$

$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right).$

17. $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} + \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

$= xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$(by exchanging two columns)

$= (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (1+xyz)(x-y)(y-z)(z-x)$(from the value of a circular determinant)

$\Delta = 0 \Rightarrow 1+xyz = 0[\because x \neq y; y \neq z; z \neq x] \Rightarrow xyz = -1.$

18. $\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ a & c & a+b \end{vmatrix}[R_1 \to R_1-R_2-R_3]$

$= \frac{1}{c}\begin{vmatrix} 0 & -2c & -2b \\ 0 & c(c+a-b) & b(c-a-b) \\ c & c & a+b \end{vmatrix}[R_2 \to cR_2-bR_3]$

$= \frac{1}{c}c(-2bc)[c-a-b-(c+a-b)] = -2bc.-2a = 4abc.$

19. $$\Delta = \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ b^2-(c+a)^2 & (c+a)^2-b^2 & b^2 \\ 0 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix} [C_1 \to C_1 - C_2; C_2 \to C_2 - C_3]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b+c-a & c+a-b & b^2 \\ 0 & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b+c-a & c+a-b & b^2 \\ 2a-2b & -2a & 2ab \end{vmatrix} [R_3 \to R_3 - R_1 - R_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} [C_1 \to C_1 + C_2]$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(c+a) & b^2 \\ 0 & 0 & 2ab \end{vmatrix} [C_1 \to aC_1 + C_3; C_2 \to bC_2 + C_3]$$

$$= \frac{(a+b+c)^2}{ab}.ab.2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2ab(a+b+c)^2[(b+c)(c+a) - ab] = 2abc(a+b+c)^3.$$

20. $$\Delta = \begin{vmatrix} 15-x & 1 & 10 \\ -4-2x & 0 & 6 \\ -8 & 0 & 3 \end{vmatrix} = 0 [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$$

$$\Rightarrow (12 + 6x - 48) = 0 \Rightarrow x = 6.$$

21. $$\Delta = \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix} = 0 [C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow \begin{vmatrix} -x & c & b \\ -x & b-x & a \\ -x & a & c-x \end{vmatrix} = 0 [\because a+b+c=0]$$

$$\Rightarrow (-x) \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & c & b \\ 0 & b-c-x & a-b \\ 0 & a-c & c-b-x \end{vmatrix} = 0 [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$$

$$\Rightarrow x[(b-c-x)(c-b-x) - (a-c)(a-b)] = 0 \Rightarrow x(x^2 - a^2 - b^2 - c^2 + ab + bc + ca) = 0$$

$\Rightarrow x=0; x^2=(a^2+b^2+c^2)-\frac{1}{2}[(a+b+c)^2-(a^2+b^2+c^2)]=\frac{3}{2}(a^2+b^2+c^2)[\because a+b+c=0] \Rightarrow x=\pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$.

22. $D_1=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}=\begin{vmatrix} a & b & c \\ tx & ty & tc \\ g & h & k \end{vmatrix}$

$=t\begin{vmatrix} a & b & c \\ x & y & z \\ g & h & k \end{vmatrix}=t\begin{vmatrix} a & x & g \\ b & y & h \\ c & z & k \end{vmatrix}$(changing rows into corresponding columns)

$=-t\begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix}=-tD_2$.

23. $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix}=\frac{1}{a^2b^2c^2}\begin{vmatrix} a^3 & a^2bc & a^3bc \\ b^3 & ab^2c & ab^3c \\ c^3 & abc^2 & abc^3 \end{vmatrix}[R_1\to a^2R_1; R_2\to b^2R_2; R_3\to c^2R_3]$

$=\frac{abc.abc}{a^2b^2c^2}\begin{vmatrix} a^3 & a & a^2 \\ b^3 & b & b^2 \\ c^3 & c & c^2 \end{vmatrix}$

$=\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$(exchanging columns twice).

24. Let x be the first term and y the common ratio of the G.P. then

$a=xy^{p-1}\Rightarrow \log a=\log x+(p-1)\log y; b=xy^{q-1}\Rightarrow \log b=\log x+(q-1)\log y; c=xy^{r-1}\Rightarrow \log c=\log x+(r-1)\log(r-1)$

$\Rightarrow \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}=\begin{vmatrix} \log x+(p-1)\log y & p & 1 \\ \log x+(q-1)\log y & q & 1 \\ \log x+(r-1)\log y & r & 1 \end{vmatrix}$

$=\begin{vmatrix} (p-1)\log y & p & 1 \\ (q-1)\log y & q & 1 \\ (r-1)\log y & r & 1 \end{vmatrix}[C_1\to C_1-\log x.C_3]$

$=\log y\begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}=\log y\begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix}=0[C_1\to C_1+C_3]$.

25. $\Delta=\begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix}[C_2\to C_2-C_1; C_3\to C_3-C_1]$

$=xy$.

26. $\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} [C_1 \to C_1 - C_2; C_2 \to C_2 - C_3]$

$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$

$= (a-b)(b-c)[b^2+bc+c^2-a^2-ab-b^2] = (a-b)(b-c)[(c-a)(a+b+c)].$

27. $\Delta = \begin{vmatrix} 0 & b-a & b^2-a^2 \\ 0 & c-b & b^2-c^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$= (b-a)(c-b)\begin{vmatrix} 0 & 1 & b+a \\ 0 & 1 & c+b \\ 1 & a+b & a^2+b^2 \end{vmatrix}$

$= (b-a)(c-b)(c+b-b-a) = (a-b)(b-c)(c-a).$

28. $\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2+ca-bc \\ 0 & b-c & b^2-c^2+ab-ca \\ 1 & c & c^2-ab \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & a+b+c \\ 1 & c & c^2-ab \end{vmatrix}$

$= 0$(two rows are identical).

29. $\Delta = \frac{1}{abc}\begin{vmatrix} a & abc & abc(b+c) \\ b & abc & abc(c+a) \\ c & abc & abd(a+b) \end{vmatrix} [R_1 \to aR_1; R_2 \to bR_2; R_3 \to cR_3]$

$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = abc\begin{vmatrix} a-b & 0 & b-a \\ b-c & 0 & c-b \\ c & 1 & a+b \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3] = 0.$

30. Check previous problem.

31. Let x be the first term and d be the common difference of the corresponding A.P. Then4

$\frac{1}{a} = x + (p-1)d; \frac{1}{b} = x + (q-1)d; \frac{1}{c} = x + (r-1)d$

$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} = (p-q)d; \frac{c-b}{bc} = (q-r)d.$

$\Delta = \begin{vmatrix} c(b-a) & p-q & 0 \\ a(c-b) & q-r & 0 \\ ab & r & 1 \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$$= \begin{vmatrix} c(b-a) & \frac{b-a}{abd} & 0 \\ a(c-b) & \frac{c-b}{bcd} & 0 \\ ab & r & 1 \end{vmatrix} = (b-a)(c-b)\begin{vmatrix} c & \frac{1}{abd} & 0 \\ a & \frac{1}{bcd} & 0 \\ ab & r & 1 \end{vmatrix} = 0$$

32. $\Delta = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 1 & -4 \\ -2 & 4 & 0 \end{vmatrix}$(putting $x = 0$) $= \begin{vmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ -2 & 4 & 12 \end{vmatrix} [C_3 \to C_3 + 3C_2]$

$= 10 = t$

33. $\Delta = abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

We have evaluated this new determinant equal to $(a-b)(b-c)(c-a)$ earlier and thus our required result is proven.

34. We can write $b = a + d$ and $c = a + 2d$, where d is the common difference of the A.P. Then,

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ 1 & 1 & d \\ 1 & 1 & d \end{vmatrix} [R_3 \to R_3 - R_2; R_2 \to R_2 - R_1]$$

$= 0$(because two rows are equal).

35. $\Delta = \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} [R_1 \to R_1 + R_2 + R_3]$

$= 0$(because $1 + \omega + \omega^2 = 0$).

36. $\Delta = k\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix} [R_3 \to R_3 - R_2; R_2 \to R_2 - R_1]$

$= k.$

37. $\Delta = \begin{vmatrix} a^2+b^2+c^2+x & b^2 & c^2 \\ a^2+b^2+c^2+x & b^2+x & c^2 \\ a^2+b^2+c^2+x & b^2 & c^2+x \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$$= (a^2+b^2+c^2+x)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+x & c^2 \\ 1 & b^2 & c^2+x \end{vmatrix} = \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} [R_3 \to R_3 - R_1; R_2 \to R_2 - R_1]$$

$= (a^2+b^2+c^2+x)x^2.$

38. $\Delta = \begin{vmatrix} a-b & b-c & a^2-b^2 \\ b-c & c-b & b^2-c^2 \\ c & a+b & c^2 \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & a-c \\ 1 & -1 & b+c \\ c & a+b & c^2 \end{vmatrix}[R_1 \to R_1 - R_2] = -(a-b)(b-c)(c-a)(a+b+c).$$

39. $$\Delta = \begin{vmatrix} a+b+c & a-b & a \\ b+c+a & b-c & b \\ c+a+b & c-a & c \end{vmatrix}[C_1 \to C_1 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 0 & a+c-2b & a-b \\ 0 & b+a-2c & b-c \\ 1 & c-a & c \end{vmatrix}[R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$$

$$= (a+b+c)(ab+bc+ca-a^2-b^2-c^2) = 3abc - a^3 - b^3 - c^3.$$

40. $$\Delta = 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 1 & c+a & a+b \\ 1 & a+b & b+c \end{vmatrix}[C_1 \to C_1 + C_2 + C_3]$$

$$= 2(a+b+c)\begin{vmatrix} 0 & b-a & c-b \\ 0 & c-b & a-c \\ 1 & a+b & a+c \end{vmatrix}[R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$$

$$= 2(a+b+c)(ab+bc+ca-a^2-b^2-c^2) = -2(a^3+b^3+c^3-3abc).$$

41. $$\Delta = \begin{vmatrix} a-b & b-c & x+c \\ a-b & b-c & x+b \\ a-b & b-c & x+c \end{vmatrix}[C_1 \to C_1 - C_2; C_2 \to C_2 - C_3]$$

$= 0$(because two columns are equal).

42. Multiplying each row with -1 gives us

$$\Delta = -\begin{vmatrix} 0 & q-p & r-p \\ p-q & 0 & r-q \\ p-r & q-r & 0 \end{vmatrix}$$

Changing rows into corresponding columns

$$\Delta = -\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = -\Delta \Rightarrow \Delta = 0.$$

43. Let L.H.S. $= \Delta = (3a+3b)\begin{vmatrix} 1 & a+b & a+2b \\ 1 & a & a+b \\ 1a+2b & a & \end{vmatrix}[C_1 \to C_1 + C_2 + C_3]$

$$= (3a+3b)\begin{vmatrix} 0 & b & b \\ 0 & -2b & b \\ 1a+2b & a & \end{vmatrix}[R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$$

$$= 9b^2(a+b) = \text{R.H.S.}$$

44. $\Delta = \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+c & c \end{vmatrix} [C_1 \to aC_1 + bC_2 + cC_3]$

$= \frac{a^2+b^2+c^2}{a}\begin{vmatrix} 0 & -c & a+b \\ 0 & -a & -a \\ 1 & b+a & \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$= \frac{a^2+b^2+c^2}{a}(ac + a^2 + ab) = (a^2+b^2+c^2)(a+b+c).$

Thus, given determinant has the same sign as $a+b+c$ because $a^2+b^2+c^2$ is always positive for real values of a, b, c.

45. $\Delta = \frac{1}{ab}\begin{vmatrix} 0 & a(bc'+b'c) - b(ca'+c'a) & ab'c' - bc'a' \\ 0 & b(ca'+c'a) - c(ab'-ab) & bc'a' - ca'b' \\ ab & ab'+a'b & a'b' \end{vmatrix} [R_1 \to aR_1 - bR_2; R_2 \to bR_2 - cR_3]$

$= \frac{ab}{ab}(ab' - a'b)(b'c - bc')(a'c - c'a) = (ab' - a'b)(b'c - bc')(a'c - c'a).$

46. $\Delta = \frac{1}{abc}\begin{vmatrix} a(b^2+c^2) & a^2b & a^2c \\ ab^2 & b(c^2+a^2) & b^2c \\ ac^2 & c^2b & c(a^2+b^2) \end{vmatrix} [R_1 \to aR_1; R_2 \to bR_2; R_3 \to cR_3]$

$= \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

$= \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ 0 & c^4 + c^2a^2 - b^2c^2 & b^2c^2 - a^2b^2 - b^4 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} [R_1 \to R_1 - R_2 - R_3; R_2 \to c^2R_2 - b^2R_3]$

$= 4a^2b^2c^2.$

47. $\Delta = \frac{1}{a^2b^2c^2}\begin{vmatrix} (\alpha+\gamma)^2 & \beta^2 & \beta^2 \\ \gamma^2 & (\alpha+\beta)^2 & \gamma^2 \\ \alpha^2 & \alpha^2 & (\beta+\gamma)^2 \end{vmatrix} [C_1 \to a^2C_1; C_2 \to b^2C_2; C_3 \to c^2C_3$ and then applying $ab = \alpha, bc = \beta, ca = \gamma]$

We have evaluated this determinant earlier to be equal to $2\alpha\beta\gamma(\alpha+\beta+\gamma)^3$ and $\alpha\beta\gamma = a^2b^2c^2$.

Thus, $\Delta = 2(ab+bc+ca)^3$.

48. $\Delta = \frac{1}{abc}\begin{vmatrix} c(a+b)^2 & c^2a & bc^2 \\ ca^2 & a(b+c)^2 & a^2b \\ b^2c & ab^2 & b(c+a)^2 \end{vmatrix} [R_1 \to cR_1; R_2 \to aR_2; R_3 \to bR_3]$

$$= \begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

We have evaluated this determinant to be equal to $2abc(a+b+c)^3$ earlier.

$\therefore \Delta = 2abc(a+b+c)^3$.

49. $\Delta = \frac{1}{abc} \begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix} [R_1 \to cR_1; R_2 \to aR_2; R_3 \to bR_3]$

We have evaluated this determinant to be equal to $4a^2b^2c^2$.

$\therefore \Delta = 4abc$.

50. $\Delta = \begin{vmatrix} 0 & 0 & x-a \\ a & a & a \\ b & x & b \end{vmatrix} [R_1 \to R_1 - R_2]$

$= (x-a)(ax-ab) = 0 \Rightarrow x = a, b$.

51. $\Delta = \begin{vmatrix} x & 2 & 1 \\ 6 & x+4 & -x \\ 7 & 8 & x \end{vmatrix} [C_3 \to C_3 - C_2] = \begin{vmatrix} x & 2 & 1 \\ 13 & x+12 & 0 \\ 7 & 8 & x \end{vmatrix} [R_2 \to R_2 + R_3]$

$= \begin{vmatrix} 0 & 0 & 1 \\ 13 & x+12 & 0 \\ 7-x^2 & 8-2x & x \end{vmatrix} [C_1 \to C_1 - xC_3; C_2 \to C_2 - 2C_3]$

$= x^3 + 12x^2 - 33x + 20 = 0$

We observe that sum of coefficients is zero so 1 would be a factor. The other factor is $x^2 + 13x - 20 = 0$; whose roots are $\frac{-13 \pm \sqrt{249}}{2}$.

52. $\Delta = \begin{vmatrix} x-12 & 2-3x & 0 \\ 4 & x & 1 \\ x-20 & 2-5x & 0 \end{vmatrix} [R_1 \to R_1 - 3R_2; R_3 \to R_3 - 5R_2]$

$= (x-20)(3x-2) - (x-12)(5x-2) \Rightarrow x = \pm 2\sqrt{2}$.

53. $\Delta = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$= (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1] = (x+a+b+c)x^2$

$\Rightarrow x = -(a+b+c), 0$.

54. $\Delta = \begin{vmatrix} x+10 & 5 & 2 \\ x+10 & 7+x & 6 \\ x+10 & 5 & 3+x \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$= \begin{vmatrix} 0 & 0 & -x-1 \\ x+14 & x+7 & 6 \\ x+10 & 5 & 3+x \end{vmatrix} [R_1 \to R_1 - R_3] = (x+1)(x^2+12x)$

$\Rightarrow x = 0, -1, -12.$

55. $\Delta = \begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix} + \begin{vmatrix} b & b+c & c+a \\ c & c+a & a+b \\ a & a+b & b+c \end{vmatrix}$

$= \begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ a & c & a+b \\ b & a & b+c \end{vmatrix} + \begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix} [=0] + \begin{vmatrix} b & c & c+a \\ a & c & a+b \\ a & b & b+c \end{vmatrix}$

$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} [=0] + \begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix} [=0] + \begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix} [=0] + \begin{vmatrix} b & c & c \\ c & a & a \\ a & b & b \end{vmatrix} [=0] + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$

$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$(exchanging columns of second determinant twice).

56. This problem is similar to previous problem and can be solved similarly.

57. This problem is similar to previous problem and can be solved similarly.

58. $\Delta = \begin{vmatrix} -2a & a^2+1 & a \\ -2b & b^2+1 & b \\ -2c & c^2+1 & c \end{vmatrix} [C_1 \to C_1 - C_2]$

Taking out -2 from C_1 makes C_1 and C_3 equal, and thus, $\Delta = 0$.

59. Multiplying all rows by -1 and changing rows intocorresponding columns we observe that $\Delta = -\Delta \Rightarrow \Delta = 0$.

60. $\Delta = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} [R_1 \to aR_1; R_2 \to bR_2; R_3 \to cR_3]$

$= \frac{abc}{abc}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$(exchanging columns twice).

61. $\Delta = \frac{1}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ xyz & yzx & zxy \end{vmatrix} [C_1 \to xC_1; C_2 \to yC_2; C_3 \to zC_3]$

$$= \frac{xyz}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}.$$

62. Changing rows twice gives us third determinant from first determinant.

$$\Delta = -\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix}[R_1 \leftrightarrow R_2] = \begin{vmatrix} y & x & z \\ b & a & c \\ q & p & r \end{vmatrix}[C_1 \leftrightarrow C_2]$$

$$= \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}\text{(changing rows into corresponding columns).}$$

63. $\Delta = m!(m+1)!(m+2)!\begin{vmatrix} 1 & m+1 & (m+1)(m+2) \\ 1 & m+2 & (m+2)(m+3) \\ 1 & m+3 & (m+3)(m+4) \end{vmatrix}$

$$= m!(m+1)!(m+2)!\begin{vmatrix} 1 & m+1 & (m+1)^2 \\ 1 & m+2 & (m+2)^2 \\ 1 & m+3 & (m+3)^2 \end{vmatrix}[C_3 \to C_3 - C_2]$$

Using the result obtained earlier

$= m!(m+1)!(m+2)!(-1).(-1).2.$

$\frac{\Delta}{(m!)^3} = 2m^3 + 8m^2 + 10m + 4$, and thus, divisibility condition is fulfilled.

64. $\Delta = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5 \neq 0.$

$\Delta_1 = \begin{vmatrix} 4 & 1 \\ 9 & -3 \end{vmatrix} = -21, \Delta_2 = \begin{vmatrix} 1 & 4 \\ 2 & 9 \end{vmatrix} = 1$

By Cramer's rule, $x = \frac{\Delta_1}{\Delta} = \frac{21}{5}, y = \frac{\Delta_2}{\Delta} = -\frac{1}{5}.$

65. $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1) + 1(1-1) + 3(-1-1) = -2.$

$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 9(1+1) + 1(6-2) + 3(-6-2) = -2$

$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 2(6-2) - 9(1-1) + 3(2-6) = -4.$

$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2(2+6) + 1(2-6) + 9(-1-1) = -6$

By Cramer's rule, $x = \frac{\Delta_1}{\Delta} = 1, y = \frac{\Delta_2}{\Delta} = 2, z = \frac{\Delta_3}{\Delta} = 3.$

66. $\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \Delta_1 = \begin{vmatrix} 6 & 3 \\ 10 & 6 \end{vmatrix} = 6 \neq 0.$

Hence, given system of equations is inconsistent and has no solution.

67. $\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 3 & 4 & 1 \\ 5 & 4 & 1 \end{vmatrix} [C_1 \to C_1 + C_3; C_2 \to C_2 + C_3]$

$= -1(12 - 20) = 8 \neq 0.$

Hence, given system of equations is consistent and has unique solution.

68. $\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0, \; \Delta_1 = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0, \; \Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0.$

Hence, given system of equations is consistent and has infinite number of solutions.

69. $\Delta = \begin{vmatrix} 2 & 1 & 13 \\ 6 & 3 & 18 \\ 1 & -1 & -3 \end{vmatrix} = 2(-9 + 18) - 1(-18 - 18) + 13(-6 - 3) = -63 \neq 0.$

Thus, we have case of inconsistent solutions.

70. $\Delta = \begin{vmatrix} 1 & 1 & -6 \\ 3 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 1(-2 - 2) - 1(6 + 2) - 6(-3 + 1) = 0$

Hence, given system of equations has non-trivial solution.

71. $\Delta = \begin{vmatrix} 1 & 1 & -k \\ 3 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 0 \text{(for non-trivial solution)} \Rightarrow k = 6.$

Solving the system of equation gives us $z = \frac{x}{2}$ and $y = 2x$. Thus, solution is given by $x = t, y = 2t, z = \frac{t}{2}$, where t is an arbitrary number.

72. $\Delta = \begin{vmatrix} 1 & -2 \\ 7 & 6 \end{vmatrix} = 6 + 14 = 20, \Delta_1 = \begin{vmatrix} 0 & -2 \\ 40 & 6 \end{vmatrix} = 80$

$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 7 & 40 \end{vmatrix} = 40 \Rightarrow x = \frac{\Delta_1}{\Delta} = 4, y = \frac{\Delta_2}{\Delta} = 2.$

73. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & -3 \\ -1 & 0 & 1 \end{vmatrix} = 1(2) - 1(3 - 3) + 1(2) = 4$

$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 0 & 2 & -3 \\ 2 & 0 & 1 \end{vmatrix} = 9(2) - 1(6) + 1(-4) = 8$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 3 & 0 & -3 \\ -1 & 2 & 1 \end{vmatrix} = 1(6) - 9(3-3) + 1(6) = 12$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 3 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 1(4) - 1(6) + 9(2) = 16$$

$\Rightarrow x = \frac{\Delta_1}{\Delta} = 2, y = \frac{\Delta_2}{\Delta} = 3, z = \frac{\Delta_3}{\Delta} = 4.$

74. $\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & -5 \\ 3 & -4 & 2 \end{vmatrix} = 1(6-20) + 1(4+15) + 1(-17) = 12$

$$\Delta_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 3 & -5 \\ -1 & -4 & 2 \end{vmatrix} = -1(9) + 1(25) = 16$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 7 & -5 \\ 3 & -1 & 2 \end{vmatrix} = 1(9) + 1(-23) = -14$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 7 \\ 3 & -4 & -1 \end{vmatrix} = 1(25) + 1(-23) = 2$$

$\Rightarrow x = \frac{4}{3}, y = -\frac{7}{6}, z = \frac{1}{6}.$

75. $\Delta = \begin{vmatrix} 2 & 3 & -3 \\ 5 & -2 & 2 \\ 1 & 7 & -5 \end{vmatrix} = 2(-4) - 3(-25-2) - 3(37) = -38$

$$\Delta_1 = \begin{vmatrix} 0 & 3 & -3 \\ 19 & -2 & 2 \\ 5 & 7 & -5 \end{vmatrix} = -3(-105) - 3(143) = -114$$

$$\Delta_2 = \begin{vmatrix} 2 & 0 & -3 \\ 5 & 19 & 2 \\ 1 & 5 & -5 \end{vmatrix} = 2(-105) - 3(6) = -228$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 0 \\ 5 & -2 & 19 \\ 1 & 7 & 5 \end{vmatrix} = 2(-143) - 3(6) = -304$$

$\Rightarrow x = 3, y = 6, z = 8.$

76. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (k-b)(b-c)(c-k)$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = -(a-k)(k-c)(c-a)$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (a-b)(b-k)(k-a)$$

$\Rightarrow x = \frac{(k-b)(k-c)}{(a-b)(a-c)}, y = \frac{(k-a)(k-c)}{(b-a)(b-c)}, z = \frac{(k-a)(k-b)}{(c-a)(c-b)}.$

77. $\Delta = \begin{vmatrix} 3 & 9 \\ 9 & 27 \end{vmatrix} = 0, \Delta_1 = \begin{vmatrix} 5 & 9 \\ 10 & 27 \end{vmatrix} \neq 0.$

Hence, the given system of equations is inconsistent and has no solution.

78. $\Delta = \begin{vmatrix} 5 & -3 \\ 1 & 1 \end{vmatrix} \neq 0, \Delta_1 = \begin{vmatrix} 3 & -3 \\ 7 & 1 \end{vmatrix} \neq 0, \Delta_2 = \begin{vmatrix} 5 & 3 \\ 1 & 7 \end{vmatrix} \neq 0.$

Hence, the given system of equations has unique solution.

79. $\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0, \Delta_1 = \begin{vmatrix} 5 & 2 \\ 15 & 2 \end{vmatrix} = 0, \Delta_2 = \begin{vmatrix} 1 & 5 \\ 3 & 15 \end{vmatrix} = 0.$

Hence, the given system of equations has infinite solutions.

80. $\Delta = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 5 \\ 1 & 4 & -2 \end{vmatrix} = 2(-22) - 3(-11) + 1(11) = 0.$

$$\Delta_1 = \begin{vmatrix} 5 & 3 & 1 \\ 7 & 1 & 5 \\ 3 & 4 & -2 \end{vmatrix} = 5(-22) - 3(-29) + 1(25) = 2.$$

Hence, the given system of equations is inconsistent and has no solution.

81. $\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & 6 \\ 2 & 7 & 4 \end{vmatrix} = 1(-26) - 1(12) - 1(34) = -72$

$$\Delta_1 = \begin{vmatrix} -2 & 1 & -1 \\ 26 & 4 & 6 \\ 31 & 7 & 4 \end{vmatrix} = -2(-26) - 1(-82) - 1(38) \neq 0$$

$$\Delta_2 = \begin{vmatrix} 1 & -2 & -1 \\ 6 & 26 & 6 \\ 2 & 31 & 4 \end{vmatrix} = 1(-82) + 2(12) - 1(134) \neq 0$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -2 \\ 6 & 4 & 26 \\ 2 & 7 & 31 \end{vmatrix} = 1(-38) - 1(134) - 2(34) \neq 0$$

Hence, the given system of equations has unique solution.

82. $\Delta = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 1(-4k+6) - k(-8) + 3(9-2k) = 0 \Rightarrow k = \frac{33}{2}.$

Solving the system of equations for this value of k gives us $2x + 15y = 0$ and $x + 5z = 0$.

Therefore, $x = t, y = \frac{-2t}{15}, z = \frac{-t}{5}$, where t is an arbitrary number.

83. $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$

$= (a+b+c)[(c-b)(b-c) - (a-b)(a-c)] = (a+b+c)(ab+bc+ca-a^2-b^2-c^2) = -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

Since a, b, c are different Δ will acquire value zero only if $a+b+c=0$ for non-trivial solution.

84. $\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 3(2\lambda + 15) + 1(\lambda + 18) + 4(-7) = 0$

$\Rightarrow \lambda = -5.$

85. Let $A28 = A \times 100 + 2 \times 10 + 8 = pk$, $3B9 = 3 \times 100 + B \times 10 + 9 = qk62C = 6 \times 100 + 2 \times 10 + C = rk$ where p, q, r are integers.

$\Delta = \begin{vmatrix} A & 3 & 6 \\ pk & qk & rk \\ 2 & B & 2 \end{vmatrix} [R_2 \to R_2 + 10R_3 + 100R_1]$

$= k \begin{vmatrix} A & 3 & 6 \\ p & q & r \\ 2 & B & 2 \end{vmatrix}$, which is divisible by k.

86. $\Delta = \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix} = \frac{xyz}{2.6} \begin{vmatrix} 1 & x-1 & (x-1)(x-2) \\ 1 & y-1 & (y-1)(y-2) \\ 1 & z-1 & (z-1)(z-2) \end{vmatrix}$

$= \frac{xyz}{12}\begin{vmatrix} 1 & x-1 & (x-1)^2 \\ 1 & y-1 & (y-1)^2 \\ 1 & z-1 & (z-1)^2 \end{vmatrix} [C_3 \to C_3 + C_2] = \frac{xyz}{12}(x-y)(y-z)(z-x)[\because \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)]$.

87. $\Delta = \begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} [R_1 \to R_1 - R - 2; R_2 \to R_2 - R_3] = 0$

$\Rightarrow (p-a)[r(q-b) - b(c-r)] - (b-q)[0 - a(c-r)] = r(p-a)(q-b) + b(p-a)(r-c) + a(q-b)(r-c) = 0$

$\Rightarrow \frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0 \Rightarrow \frac{r}{r-c} + \left(\frac{b}{q-b} + 1\right) + \left(\frac{a}{p-a} + 1\right) = 0 + 1 + 1$

$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

88. $\Delta = \begin{vmatrix} x(x-2a) & x(2b-x) & 0 \\ 0 & -(x-2b) & x(ac-x) \\ a^2 & b^2 & (x-c)^2 \end{vmatrix} [R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$= x^2\begin{vmatrix} x-2a & -(x-2b) & 0 \\ 0 & x-2b & -(x-2c) \\ a^2 & b^2 & (x-c)^2 \end{vmatrix}$

$= x^2(x-2a)(x-2b)(x-2c)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a^2}{x-2a} & \frac{b^2}{x-2b} & x + \frac{c^2}{x-2c} \end{vmatrix}$

$= x^2(x-2a)(x-2b)(x-2c)\left(x + \frac{a^2}{x-2a} + \frac{b^2}{x-2b} + \frac{c^2}{x-2c}\right)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ \frac{a^2}{x-2a} & \frac{b^2}{x-ac} & 1 \end{vmatrix} [C_3 \to C_1 + C_2 + C_3]$

$= x^2(x-2a)(x-2b)(x-2c)\left(x + \frac{a^2}{x-2a} + \frac{b^2}{x-2b} + \frac{c^2}{x-2c}\right)$.

89. $\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+3d & a+2d \end{vmatrix}$

$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)(a+3d) & 2d & a+d \\ (a+3d)(a+4d) & 2d & a+2d \end{vmatrix} [C_2 \to C_2 - C_3]$

$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)2d & 0 & d \\ (a+3d)2d & 0 & d \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_2]$

$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} . -2d[2d^2(a+2d-a-3d)]$

$= \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

90. $\Delta = \frac{1}{(a+x)(b+c)(c+x)(a+y)(b+y)(c+y)(a+z)(b+z)(c+z)}\Delta_1,$

where $\Delta_1 = \begin{vmatrix} (b+x)(c+x) & (b+y)(c+y) & (b+z)(c+z) \\ (c+x)(a+x) & (c+y)(a+y) & (c+z)(a+z) \\ (a+x)(b+x) & (a+y)(b+y) & (a+z)(b+z) \end{vmatrix}$

$\Delta_1 = \begin{vmatrix} (b+x)(c+x) & (b+y)(c+y) & (b+z)(c+z) \\ (c+x)(a-b) & (c+y)(a-b) & (c-z)(a-b) \\ (b+x)(a-c) & (b+y)(a-c) & (b+z)(a-c) \end{vmatrix} [R_2 \to R_2 - R-1; R_3 \to R_3 - R_1]$

$= (a-b)(a-c)\begin{vmatrix} (b+x)(c+x) & (b+y)(c+y) & (b+z)(c+z) \\ c+x & c+y & c+z \\ b+x & b+y & b+z \end{vmatrix}$

$= (a-b)(a-c)\begin{vmatrix} x(c+x) & y(c+y) & z(c+z) \\ c+x & c+y & c+z \\ b-c & b-c & b-c \end{vmatrix} [R_1 \to R_1 - bR_2; R_3 \to R_3 - R_2]$

$= (a-b)(b-c)(a-c)\begin{vmatrix} (x-z)(c+x+z) & (y-z)(c+y+z) & z(c+z) \\ x-z & y-z & c+z \\ 0 & 0 & 1 \end{vmatrix} [C_1 \to C_1 - C_2; C_2 \to C_2 - C_3]$

$= (a-b)(b-c)(a-c)(x-z)(y-z)\begin{vmatrix} c+x+z & c+y+z & z(c+z) \\ 1 & 1 & c+z \\ 0 & 0 & 1 \end{vmatrix}$

$= (a-b)(b-c)(a-c)(x-z)(y-z)[c+x+z-c-y-z] = (a-b)(b-c)(a-c)(x-z)(y-z)(x-y)$

$\Rightarrow \Delta = \frac{(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)}{(a+x)(b+x)(c+x)(b+x)(b+y)(b+z)(c+x)(c+y)(c+z)}.$

91. Let $\alpha = s-a, \beta = s-b, \gamma = s-c$, then

$\beta+\gamma = 2s-(b+c) = a, \gamma+\alpha = b, \alpha+\beta = c, \alpha+\beta+\gamma = 3s-(a+b+c) = 3s-2s = s$

$\Delta = \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma+\alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2 \end{vmatrix}$

Follwing like problem solved earlier

$= 2\alpha\beta\gamma(\alpha+\beta+\gamma)^3 = 2(s-a)(s-b)(s-c)s^3.$

92. $\Delta = \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} x & ay+bx & cx+az \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix} [C_1 \to aC_1+bC_2+cC_3]$

$= \frac{1}{ax}(a^2+b^2+c^2)\begin{vmatrix} x^2+y^2+z^2 & b(x^2+y^2+z^2) & c(x^2+y^2+z^2) \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix} [R_1 \to xR_1 + yR_2+zR_3]$

$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}\begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$

$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix} [R_2 \to R_2-yR_1; R_3 \to R_3-zR_1]$

$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}[(cz+ax)(ax+by)-bcyz]$

$= (a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz)$

93. $\Delta = \begin{vmatrix} 2+4\sin 4\theta & \sin^2\theta & 4\sin\theta \\ 2+4\sin 4\theta & 1+\sin^2\theta & 4\sin\theta \\ 2+4\sin 4\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0[C_1 \to C_1+C_2+C_3]$

$= (2+4\sin 4\theta)\begin{vmatrix} 1 & \sin^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0[R_2 \to R_2-R_1; R_3 \to R_3-R_1]$

$\Rightarrow 2(2+4\sin 4\theta) = 0$

$\sin 4\theta = -\frac{1}{2} \Rightarrow 4\theta = \frac{7\pi}{6}, \frac{11\pi}{6} \Rightarrow \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$

94. $\Delta = \frac{1}{abc}\begin{vmatrix} a[a^2+(b^2+c^2)\cos\phi] & ba^2[1-\cos\phi] & ca^2(1-cos\phi) \\ ab^2(1-\cos\phi) & b[b^2+(c^2+a^2)\cos\phi] & cb^2(1-\cos\phi) \\ ac^2(1-\cos\phi) & bc^2(1-\cos\phi) & c[c^2+(a^2+b^2)\cos\phi] \end{vmatrix} [R_1 \to aR_1+bR_2+cR_3]$

$= \begin{vmatrix} a^2+(b^2+c^2)\cos\phi & a^2(1-\cos\phi) & a^2(1-\cos\phi) \\ b^2(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & b^2(1-\cos\phi) \\ c^2(1-\cos\phi) & c^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$

$= (a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & b^2(1-\cos\phi) \\ c^2(1-\cos\phi) & c^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix} [R_1 \to R_1 + R_2+R_3]$

Performing $C_1 \to C_1-C_2; C_2 \to C_2-C_3$, we get

$$= (a^2+b^2+c^2)\begin{vmatrix} 0 & 0 & 1 \\ -(a^2+b^2+c^2)\cos\phi & (a^2+b^2+c^2)\cos\phi & b^2(1-\cos\phi) \\ 0 & -(a^2+b^2+c^2)\cos\phi & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a^2+b^2+c^2)^2\cos^2\phi = \cos^2\phi$$

95. $\Delta = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b & -abc & bc^2+abc \\ a^c+abc & b^2c+abc & -abc \end{vmatrix} [R_1 \to aR_1; R-2 \to bR_2; R_3 \to cR_3]$

$$= \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & ab \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & ab \end{vmatrix} [R_1 \to R_1+R_2+R_3]$$

$= (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ca) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix} [C_2 \to C_2 - C_1; C_3 \to C_3 - C_1]$

$$= (ab+bc+ca)^3$$

96. Given $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{vu'-uv'}{v^2} \Rightarrow v^2\frac{dy}{dx} = vu' - uv'$

$$= v^3\frac{dy}{dx} = v^2u' - uvv'$$

Again differentiating w.r.t. x, we get

$$v^3\frac{d^2y}{dx^2} + 3v^2v'\frac{dy}{dx} = 2vv'u' + v^2u'' - uvv'' - (uv'+u'v)v'$$

$$v^3\frac{dy^2}{dx^2} = -2u'vv' + 2uv^{2\prime} + v^2u'' - uvv'' = \Delta$$

97. $\Delta = \begin{vmatrix} x & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$

$$= \begin{vmatrix} x & x & x \\ 0 & a & 0 \\ 0 & 0 & a^2 \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1] + (x+a)(x+a^2) - x^2$$

$$= xa^3 + x(a+a^2) + a^3 = a^3\left[1 + x\left(1 + \frac{1}{a} + \frac{1}{a^2}\right)\right]$$

$$= a^3\left[1 + \frac{x(a^3-1)}{a^2(a-1)}\right]$$

98. L.H.S. $= pa(qra^2 - p^2bc) - qb(q^2ca - prb^2) + rc(pqc^2 - r^2ab) = pqra^3 - abcp^3 - abcq^3 + pqrb^3 + pqrc^3 - abcr^3$

$= pqr(a^3+b^3+c^3) - abc(p^3+q^3+r^3) = pqr(a^3+b^3+c^3-3abc) - abc(p^3+q^3+r^3-3pqr)$

$= pqr(a^3+b^3+c^3-3abc) - 0[\because p+q+r=0]$

R.H.S. $= pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

$= pqr(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}[C_1 \to C_1 + C_2 + C_3]$

$= pqr(a+b+c)\begin{vmatrix} o & b-a & c-b \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix}[R_1 \to R_1 - R_2; R_2 \to R_2 - R_3]$

$= pqr(a^3+b^3+c^3-3abc) =$ L.H.S

99. R.H.S. $= \frac{1}{abc}\begin{vmatrix} a & abc & a(b+c) \\ b & abc & b(c+a) \\ c & abc & c(a+b) \end{vmatrix}[R_1 \to aR_1; R_2 \to bR_2; R_3 \to cR_3]$

$= -\frac{abc}{abc}\begin{vmatrix} 1 & a & ab+ac \\ 1 & b & bc+ba \\ 1 & c & ca+cb \end{vmatrix}$ Taking abc out and then applying $C_1 \leftrightarrow C_2$

$= -\begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}[C_3 \to C_3 - (ab+bc+ca)C_1]$

$= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}[R_1 \to aR_1; R_2 \to bR_2; R_3 \to cR_3]$

$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}[C_2 \leftrightarrow C_3; C_1 \leftrightarrow C_2]$

100. $\Delta = \begin{vmatrix} x^2 & x+1 & x-2 \\ 2x^2 & 3x & 3x-3 \\ x^2 & 2x-1 & 2x-1 \end{vmatrix} + \begin{vmatrix} x & x+1 & x-2 \\ 3x-1 & 3x & 3x-3 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

$= \begin{vmatrix} 2x^2 & 3x & 3x-3 \\ 2x^2 & 3x & 3x-3 \\ x^2 & 2x-1 & 2x-1 \end{vmatrix}[R_1 \to R_1 + R_3] + \begin{vmatrix} 2 & 3 & x-2 \\ 2 & 3 & 3x-3 \\ 4 & 0 & 2x-1 \end{vmatrix}[C_1 \to C_1 - C_3; C_2 \to C_2 - C_3]$

$$= 0 + \begin{vmatrix} 2 & 3 & x \\ 2 & 3 & 3x \\ 4 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 2 & 3 & -2 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$$

$= xA + B$, where $A = \begin{vmatrix} 2 & 3 & 1 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} 2 & 3 & -2 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$ which are determinants of 3rd order independent of x.

101. $\sum\limits_{r=1}^{n} D_r = D_1 + D_2 + \cdots + D_n$

$$= \begin{vmatrix} \sum\limits_{r=1}^{n} r & x & \frac{n(n+1)}{2} \\ \sum\limits_{r=1}^{n}(2r-1) & y & n^2 \\ \sum\limits_{r=1}^{n}(3r-2) & z & \frac{n(3n-1)}{2} \end{vmatrix} = \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

$= 0$ because first and third columns are identical.

102. $\Delta = \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$

$$\overline{\Delta} = \begin{vmatrix} -5 & 3-5i & \frac{3}{2}+4i \\ 3+5i & 8 & 4-5i \\ \frac{3}{2}-4i & 4+5i & 9 \end{vmatrix}$$

Exchanging rows and columns

$\overline{\Delta} = \Delta. \therefore \Delta$ is purely real.

103. Putting $b = -c$, we have

$$\Delta = \begin{vmatrix} -2a & a-c & a+c \\ -c+a & 2c & 0 \\ c+a & 0 & -2c \end{vmatrix}$$

$$= \begin{vmatrix} c-a & a-c & a-c \\ a-c & 2c & 0 \\ c+a & 0 & -2c \end{vmatrix} [R_1 \to R_1 + R_3]$$

$$= \begin{vmatrix} c-a & 0 & 0 \\ a-c & a+c & a-c \\ c+a & a+c & a-c \end{vmatrix} [C_2 \to C_2 + C_1; C_3 \to C_3 + C_1]$$

$$= (c-a)[(a^2-c^2) - (a^2-c^2)] = 0$$

Hence, $b+c$ is a factor of Δ. Similarly it can be proven that $a+b$ and $c+a$ are factors of Δ.

We see that, upon expansion of determinant, each term of the L.H.S. and R.H.S. is a homogeneous expression in a,b,c of 3rd degree.

Let $\begin{vmatrix} -2a & a+b & b+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(b+c)(c+a)(a+b)$, where k is independent of a,b,c

Putting $a=0, b=1, c=1$ we get

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = 2k$$

$k = 4$.

Thus, we have proven the required condition.

104. $F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(x) \\ g_1(a) & g_2(x) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(x) \\ g_1'(a) & g_2'(x) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(x) \\ g_1(a) & g_2(x) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$

$= 0+0+0$

$\because f_r(a) = g_r(a) = h_r(a)$, $r = 1,2,3$ in the first determinant last two, in the second determinant first and third, in the third determinant first two, rows are identical. Therefore, all determinants are zero.

105. Since $f(x) = 0$ is a quadratic equation with repeated root α, $\therefore f(x) = a_r(x-\alpha)^2$, where a_r is a constant.

Clearly $\Delta(x)$ is a polynomial of degree having a maximum value of 5.

$$\Delta(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$\Delta(\alpha) = 0[\because R_1$ and R_2 are identical].

$\Delta'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0[\because R_1$ and R_3 are identical].

Thus, we can say that $\Delta(x) = 0$ has two roots equal to α.

$\Rightarrow \Delta(x) = (x-\alpha)^2 g(x)$, where $g(x)$ is a polynomial of degree 3 at most.

$\Delta(x) = a(x-\alpha)^2 \frac{g(x)}{a} = a(x-\alpha)^2 .h(x)$, where $h(x) = \frac{g(x)}{a_1}$.

Thus, $\Delta(x) = f(x) .h(x)$, where $h(x)$ is a polynomial in x. Hence, $\Delta(x)$ is divisible by $f(x)$.

106. Let Δ be the determinant. Then,

$$\frac{d\Delta}{d\theta}=\begin{vmatrix}-\sin(\theta+\alpha) & -\sin(\theta+\beta) & -\sin(\theta+\gamma)\\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma)\\ \sin(\beta+\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta)\end{vmatrix}+\begin{vmatrix}\cos(\theta+\alpha) & cos(\theta+\beta) & \cos(\theta+\gamma)\\ \cos(\theta+\alpha) & cos(\theta+\beta) & \cos(\theta+\gamma)\\ \sin(\beta+\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta)\end{vmatrix}+$$
$$\begin{vmatrix}\cos(\theta+\alpha) & cos(\theta+\beta) & \cos(\theta+\gamma)\\ \cos(\theta+\alpha) & cos(\theta+\beta) & \cos(\theta+\gamma)\\ 0 & 0 & 0\end{vmatrix}$$

$$=0+0+0$$

Thus, Δ is a constant, which will be independent of θ.

107. $$\Delta=\begin{vmatrix}f & g & h\\ xf'+f & xg'+g & xh'+h\\ x^2f''+4xf'+2f & x^2g''+4xg'+2g & x^2h''+4xh'+2h\end{vmatrix}$$

$$=\begin{vmatrix}f & g & h\\ xf' & xg' & xh'\\ x^2f''+4xf' & x^2g''+2xg' & x^2h''+2xh'\end{vmatrix}[R_2\to R_2-R_1;\, R_3\to R_3-2R_1]$$

$$=\begin{vmatrix}f & g & h\\ xf' & xg' & xh'\\ x^2f'' & x^2g'' & x^2h''\end{vmatrix}[R_3\to R_3-4R_2]$$

$$=x^3\begin{vmatrix}f & g & h\\ f' & g' & h'\\ f'' & g'' & h''\end{vmatrix}$$

$$\Delta'=\begin{vmatrix}f' & g' & h'\\ f' & g' & h'\\ x^3f'' & x^3g'' & x^3h''\end{vmatrix}+\begin{vmatrix}f & g & h\\ f'' & g'' & h''\\ x^3f'' & x^3g'' & x^3h''\end{vmatrix}+\begin{vmatrix}f & g & h\\ f' & g' & h'\\ (x^2f'')' & (x^2g'')' & (x^2h'')'\end{vmatrix}$$

$=0+0+\begin{vmatrix}f & g & h\\ f' & g' & h'\\ (x^2f'')' & (x^2g'')' & (x^2h'')'\end{vmatrix}$ because two rows of first two determinants are equal.

108. $$\frac{d^n\{f(x)\}}{dx^n}=\begin{vmatrix}\frac{d^nx^n}{dx^n} & \frac{d^n\sin x}{dx^n} & \frac{d^n\cos x}{dx^n}\\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2}\\ a & a^2 & a^2\end{vmatrix}$$

$y=x^n,\ y_1=\frac{dy}{dx}=nx^{n-1},\ y_2=\frac{d^2y}{dy^2}=n(n-1)x^{n-1},\ \ldots\ y_n=n(n-1)\ldots 3.2.1=n!$

$y=\sin x,\ y_1=\cos x=\sin\left(\frac{\pi}{2}+x\right),\ y_2=\cos\left(\frac{\pi}{2}+x\right)=\sin\left(\frac{\pi}{2}+\frac{\pi}{2}+x\right)$

Proceeding in the same way $y_n=\sin\left(\frac{n\pi}{2}+x\right)$

Now $y = \cos x, y_1 = -\sin x = \cos\left(\frac{\pi}{2} + x\right), y_2 = -sin\left(\frac{\pi}{2} + x\right) = \cos\left(2\frac{\pi}{2} + x\right)$

Proceeding in the same way $y_n = \cos\left(n\frac{\pi}{2} + x\right)$

$$\frac{d^n\{f(x)\}}{dx^n} = \begin{vmatrix} n! & \sin\left(\frac{n\pi}{2} + x\right) & \cos\left(\frac{n\pi}{2} + x\right) \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$$

$$f^n(0) = \begin{vmatrix} n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} = 0 \text{ because first two rows are identical.}$$

109. $$\Delta = \begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos A\cos Q + \sin A\sin Q & \cos A\cos R + \sin A\sin R \\ \cos B\cos P + \sin B\sin P & \cos B\cos Q + \sin B\sin Q & \cos B\cos R + \sin B\sin R \\ \cos C\cos P + \sin C\sin P & \cos C\cos Q + \sin C\sin Q & \cos C\cos R + \sin C\sin R \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} + \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$= 0 + 0 = 0.$

As an alternative we can expand the determinant along first column and then split the addition taking $\cos P$ and $\sin P$ common and repeat it for others to get the desired result.

110. We know that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

$$\Rightarrow (a^3 + b^3 + c^3 - 3abc)^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} [C_1 \to -C_1; C_2 \leftrightarrow C_3] = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2bc - b^2 & a^2 \\ b^2 & a^2 & 2bc - c^2 \end{vmatrix}$$

111. L.H.S. $= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$

$= 0.0 = 0.$

112. $\Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -(15 + 2m)$

Case I: When $\Delta = 0, m = \frac{-15}{2}$

$\Delta_1 = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m \neq 0$

Hence, given system of equation has no solution when $m = \frac{-15}{2}$

Case II: When $m \neq \frac{-15}{2}$

$\Delta_2 = \begin{vmatrix} 2 & m \\ 2 & 20 \end{vmatrix} = 2(30 - m)$

$x = \frac{\Delta_1}{\Delta} = \frac{25m}{15+2m} > 0[\because x > 0]$

$\Rightarrow -\infty < m < \frac{-15}{2}$ or $0 < m < \infty$

$y = \frac{\Delta_2}{\Delta} = \frac{2(m-30)}{15+2m} > 0[\because y > 0]$

$\Rightarrow -\infty < m < \frac{-15}{2}$ or $30 < m < \infty$

Combining both we get, $-\infty < m < \frac{-15}{2}$ or $30 < m < \infty$.

113. $\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 7(\lambda + 5)$

Case I: When $\lambda \neq 5 \Rightarrow \Delta \neq 0$ which means the system of equations has unique solution.

Case II: When $\lambda = -5 \Rightarrow \Delta = 0$

Also, $\Delta_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ 3 & 5 & -5 \end{vmatrix} = 0$, $\Delta_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0$

$\Delta_3 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$

Since all the determinants are zero, in this case we have infinite solutions for given system of equations.

Putting the value of λ the set of equation becomes

$3x - y + 4z = 3; x + 2y - 3z = -2; 6x + 5y - 5z = -3$

From first two equations we get, $z = \frac{4-7x}{5}$

Substituting this in first we get $y = \frac{1-13x}{5}$

Thus the set of solutions is $x = t, y = \frac{1-13t}{5}, z = \frac{4-7t}{5}$.

114. $\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (p-2)(q-3)$

$\Delta_1 = \begin{vmatrix} 8 & p & 8 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = (p-2)(4q-15)$

$\Delta_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & 1 \\ 1 & 4 & 3 \end{vmatrix} = 0$

$\Delta_3 = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = p-2$

Case I: When $\Delta \neq 0$ i.e. $p \neq 2, q \neq 3$, given system of equations has unique solution.

Case II: When $\Delta = 0, p = 2$, or $q = 3$

When $p = 2 \Rightarrow \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

Thus, given system of equations has inifinite solutions.

When $q = 3 \Rightarrow \Delta_1 \neq 0$

Thus, given system of equations has no solutions.

115. For non-trivial solution

$\Delta = 0$ or $\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & \cos\alpha \end{vmatrix} = 0$

$\Rightarrow \lambda = \sin 2\alpha + \cos 2\alpha$

If $\lambda = 1, \sin 2\alpha + \cos 2\alpha = 1$

$\Rightarrow \sin 2\alpha = 1 - \cos 2\alpha = 2\sin^2\alpha$

$\Rightarrow 2\sin\alpha(\cos\alpha - \sin\alpha) = 0$

$\therefore \sin\alpha = 0$ or $\tan\alpha = 1$

$\therefore \alpha = n\pi$ or $\alpha = n\pi + \frac{\pi}{4}, n \in I$.

116. $\Delta = (a+b+c)\begin{vmatrix} 1 & b+c & a^2 \\ 1 & c+a & b^2 \\ 1 & a+b & c^2 \end{vmatrix} [C_1 \to C_1 + C_2]$

$= (a+b+c)\begin{vmatrix} 1 & b+a & a^2 \\ 0 & a-b & b^2-a^2 \\ 0 & a-c & c^2-a^2 \end{vmatrix} [R_3 \to R_3 - R_1; R_2 \to R_2 - R_1]$

$= (a+b+c)[(a-b)(c^2-a^2)-(a-c)(b^2-a^2)] = (a+b+c)(a-b)(c-a)(c+a-b-a)$

$= -(a+b+c)(a-b)(b-c)(c-a).$

117. $\Delta = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}-\sqrt{6} & 5-2\sqrt{10} & 0 \\ 3-\sqrt{15} & \sqrt{15}-10 & 0 \end{vmatrix} [R_2 \to R_2 - \sqrt{2}R_1;\, R_3 \to R_3 - \sqrt{5}R_1]$

$= 15\sqrt{2} - 25\sqrt{3}.$

118. $\Delta = \begin{vmatrix} x & x(x^2+1) & x \\ y & y(y^2+1) & y \\ z & z(z^2+1) & z \end{vmatrix} + \begin{vmatrix} x & x(x^2+1) & 1 \\ y & y(y^2+1) & 1 \\ z & z(z^2+1) & 1 \end{vmatrix}$

Observe that first and third columns of first determinant are identical.

$\Rightarrow \Delta = \begin{vmatrix} x & x(x^2+1) & 1 \\ y & y(y^2+1) & 1 \\ z & z(z^2+1) & 1 \end{vmatrix}$

$= \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix} + \begin{vmatrix} x & x^3 & x \\ y & y^3 & y \\ z & z^3 & z \end{vmatrix}$

Again second and third columns are identical in second determinant.

$\Delta = \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$

$= (x-y)(y-z)(z-x)(x+y+z).$

119. Let a and d be the first term and common difference of corresponding A.P.

$\frac{1}{x} = a + (l-1)d, \frac{1}{y} = a + (2m-1)d, \frac{1}{z} = a + (3n-1)d$

$\Delta = \frac{1}{xyz}\begin{vmatrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ l & 2m & 3n \\ 1 & 1 & 1 \end{vmatrix}$

$= \frac{1}{xyz}\begin{vmatrix} a+(l-1)d & a+(2m-1)d & a+(3n-1)d \\ l & 2m & 3n \\ 1 & 1 & 1 \end{vmatrix}$

$= \frac{1}{xyz}\begin{vmatrix} ld-d & 2md-2d & 3nd-3d \\ l & 2m & 3n \\ 1 & 1 & 1 \end{vmatrix} [R_1 \to R_1 - aR_3]$

$= \frac{1}{xyz}\begin{vmatrix} 0 & 0 & 0 \\ l & 2m & 3n \\ 1 & 1 & 1 \end{vmatrix} [R_1 \to R_1 - (R_2 - 1)d]$

$= 0.$

120. $\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2 - a^2 & b^3 - a^3 \\ 0 & c^2 - a^2 & c^3 - a^3 \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$

$= (b^2 - a^2)(c^3 - a^3) - (c^2 - a^2)(b^3 - a^3)$

$= (b - a)(c - a)[(b + a)(c^2 + ac + a^2) - (c + a)(b^2 + ab + a^2)]$

$= (b - a)(c - a)(bc^2 + abc + a^2b + ac^2 + a^2c + a^3 - b^2c - abc - a^2c - ab^2 - a^2b - a^3)$

$= (b - a)(c - a)(bc^2 + ac^2 - b^2c - ab^2) = (b - a)(c - a)[bc(c - b) + a(c^2 - b^2)]$

$= (b - a)(c - a)(c - b)(ab + bc + ca)$

We know that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - a)(c - a)(c - b)$

Hence, L.H.S. = R.H.S.

121. $\Delta = \begin{vmatrix} b^2 + c^2 & a^2 & bc \\ c^2 + a^2 & b^2 & ca \\ a^2 + b^2 & c^2 & ab \end{vmatrix} [C_1 \to -2C_3]$

$= (a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} [C_1 \to C_1 + C_2 + C_3]$

We know that $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

$\Delta = (a^2 + b^2 + c^2)(a + b + c)(a - b)(b - c)(c - a).$

122. Perform $C_2 \to C_2 - C_1$, take out -1 common then $C_1 \to C_1 + C_2 + 2C_3$, take out $x^2 + y^2 + z^2$ commong then $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$ will give two zeroes in first column which upon expansion gives the result.

123. Let $a_1b_1c_1 = 100 \times a_1 + 10 \times b_1 + c_1 = pk$, where $p \in I$

$a_2b_2c_2 = 100 \times a_2 + 10 \times b_2 + c_2 = qk$, where $q \in I$

$a_3b_3c_3 = 100 \times a_3 + 10 \times b_3 + c_3 = rk$, where $r \in I$

$$\Delta = \begin{vmatrix} a_1 & b_1 & pk \\ a_2 & b_2 & qk \\ a_3 & b_3 & rk \end{vmatrix} [C_3 \to 100C_1 + 10C_2 + C_3]$$

Thus, given determinant is divisible by k.

124. $$\Delta = \begin{vmatrix} a_1 & a_1x + b_1 & c_1 \\ a_2 & a_2x + b_2 & c_2 \\ a_3 & a_3x + b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x + b_1 & c_1 \\ b_2x & a_2x + b_2 & c_2 \\ b_3x & a_3x + b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1x + b_1 & c_1 \\ a_2 & a_2x + b_2 & c_2 \\ a_3 & a_3x + b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x + b_1 & c_1 \\ b_2 & a_2x + b_2 & c_2 \\ b_3 & a_3x + b_3 & c_3 \end{vmatrix}$$

$$= x\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} + x\begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

Clearly, first and last determinants are zero as they have identical columns.

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} + x^2\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

Exchanging first two columns of second determinant

$$= (1 - x^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

125. $$\Delta = abc\begin{vmatrix} \frac{1}{a} + 1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

$$= abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix} [C_2 \to C_2 - C_1; C_3 \to C_3 - C_1]$$

$$= abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)$$

Now since a, b, c are roots of $px^3 + qx^2 + rx + s = 0$

$\therefore px^3 + qx^2 + rx + s = (x - a)(x - b)(x - c)$

Comparing coefficients, $a+b+c=\frac{-q}{p}$

$ab+bc+ca=\frac{r}{p}$; $abc=\frac{-s}{p}$

Thus, $abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)=\frac{r-s}{p}$.

126. $\Delta=\begin{vmatrix}1 & a & a^4\\ 0 & b-a & b^4-a^4\\ 0 & c-a & c^4-a^4\end{vmatrix}[R_2\to R_2-R_1; R_3\to R_3-R-1]$

$=(b-a)(c^4-a^4)-(c-a)(b^4-a^4)$

$=(b-a)(c-a)[(c+a)(c^2+a^2)-(b+a)(b^2+a^2)]>0\ \forall\ a<b<c$

127. $\Delta=\begin{vmatrix}a & a^3 & a^4\\ b & b^3 & b^4\\ c & c^3 & c^4\end{vmatrix}-\begin{vmatrix}a & a^3 & 1\\ b & b^3 & 1\\ c & c^3 & 1\end{vmatrix}$

$=abc\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}-\begin{vmatrix}a & a^3 & 1\\ b & b^3 & 1\\ c & c^3 & 1\end{vmatrix}$

$=abc\begin{vmatrix}0 & a^2-b^2 & a^3-b^3\\ 0 & b^2-c^2 & b^3-c^3\\ 1 & c^2 & c^3\end{vmatrix}-\begin{vmatrix}0 & a-b & a^3-b^3\\ 0 & b-c & b^3-c^3\\ 1 & c & c^3\end{vmatrix}[R_1\to R_1-R_2; R_2\to R_2-R_3]$

$\Rightarrow abc[(a^2-b^2)(b^3-c^3)-(b^2-c^2)(a^3-b^3)]-[(a-b)(b^3-c^3)-(b-c)(a^3-b^3)]=0$

$abc(a-b)(b-c)[(a+b)(b^2+bc+c^2)-(b+c)(a^2+ab+b^2)]=(a-b)(b-c)(b^2+bc+c^2-a^2-ab-b^2)$

Becasue a,b,c are distinct we $a-b\neq 0; b-c\neq 0; c-a\neq 0$

$abc(a+b)(b^2+bc+c^2)-(b+c)(a^2+ab+b^2)=(b^2+bc+c^2-a^2-ab-b^2)$

$abc(ab^2+abc+ac^2+b^3+b^2c+bc^2-a^2b-ab^2-b^3-a^2c-abc-b^2c)=bc+c^2-a^2-ab$

$abc(ac^2+bc^2-a^2b-a^2c)=b(c-a)+(c^2-a^2)$

$abc[ac(c-a)+b(c^2-a^2)]=(c-a)(a+b+c)$

$abc(ab+bc+ca)=a+b+c$

128. Taking b_1,b_2,b_3 common from columns and multiplying rows with them, we get

$\Delta=\begin{vmatrix}x_1+a_1b_1 & a_1b_1 & a_1b_1\\ a_2b_2 & x_2+a_2b_2 & a_2b_2\\ a_3b_3 & a_3b_3 & x+a_3b_3\end{vmatrix}$

Taking x_1,x_2,x_3 common from rows

$$= x_1x_2x_3\begin{vmatrix} 1+\frac{a_1b_1}{x_1} & \frac{a_1b_1}{x_1} & \frac{a_1b_1}{x_1} \\ \frac{a_2b_2}{x_2} & 1+\frac{a_2b_2}{x_2} & \frac{a_2b_2}{x_2} \\ \frac{a_3b_3}{x_3} & \frac{a_3b_3}{x_3} & 1+\frac{a_3b_3}{x_3} \end{vmatrix}$$

$$= x_1x_2x_3\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{c_3}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{a_2b_2}{x_2} & 1+\frac{a_2b_2}{x_2} & \frac{a_2b_2}{x_2} \\ \frac{a_3b_3}{x_3} & \frac{a_3b_3}{x_3} & 1+\frac{a_3b_3}{x_3} \end{vmatrix} [R_1 \to R_1+R_2+R_3]$$

$$= x_1x_2x_3\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{c_3}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{a_2b_2}{x_2} & 1 & 0 \\ \frac{a_3b_3}{x_3} & 0 & 1 \end{vmatrix} [C_2 \to C_2-C_1; C_3 \to C_3-C_1]$$

$$= x_1x_2x_3\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{c_3}\right).$$

129. This problem is similar to problem 90 and can be solved similarly.

130. $\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$

$$= abc\begin{vmatrix} -2a & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix} [R_1 \to R_1-R_2-R_3]$$

$$= 4a^2b^2c^2.$$

131. $\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} [C_1 \to C_1-bC_3; C_2 \to C_2+aC_3]$

$$= (1+a^2+b^2)^2(1-a^2-b^2+2a^2)+2b^2(1+a^2+b^2)^2$$

$$= (1+a^2+b^2)^3.$$

132. We know that $P = \frac{a+b+c}{a}$; $A = \sqrt{s(s-a)(s-b)(s-c)}$

After that this problem is same as 91, and we just need to substitute for the values of A and P.

133. Taking a, b, c common from rows and multiplying with columns gives is the same determinant as in problem 88 and can be solved in same fashion.

134. $\Delta = \begin{vmatrix} x^3 & 6x^2a+2a^3 & (x-a)^3 \\ y^3 & 6y^2a+2a^3 & (y-a)^3 \\ z^3 & 6z^2a+2a^3 & (z-a)^3 \end{vmatrix} [C_2 \to C_2-C_3]$

$$= 2\begin{vmatrix} x^3 & 3x^2a + a^3 & (x-a)^3 \\ y^3 & 3y^2a + a^3 & (y-a)^3 \\ z^3 & 3z^2a + a^3 & (z-a)^3 \end{vmatrix}$$

$$= 2\begin{vmatrix} x^3 & 3x^2a + a^3 & 3xa^2 \\ y^3 & 3y^2a + a^3 & 3ya^2 \\ z^3 & 3z^2a + a^3 & 3za^2 \end{vmatrix} [C_3 \to C_3 - C_1 - C_2]$$

$$= 2a^3\begin{vmatrix} x^3 & 3x^2 + a^2 & 3x \\ y^3 - x^3 & 3(y^2 - x^2) & 3(y-x) \\ z^3 - x^3 & 3(z^2 - x^2) & 3(z-x) \end{vmatrix} [R_2 \to R_2 - R_1; R_3 \to R_2 - R_1]$$

Now we can take $y - x$ and $z - x$ common followed by expasion so that desired condition can be proven easily.

135. $$\Delta = \begin{vmatrix} 1-x & a & 0 \\ a & a^2 - x & x \\ a^2 & a^3 & -x \end{vmatrix} [C_3 \to C_3 - aC_2]$$

$$= x\begin{vmatrix} 1-x & a & 0 \\ a + a^2 & a^2 - x + a^3 & 0 \\ a^2 & a^3 & -1 \end{vmatrix} [R_2 \to R_2 + R_3]$$

$$= x[a(a + a^2) - (1-x)(a^2 + a^3 - x)]$$

$$= x(a^2 + a^3 - a^2 - a^3 + x + xa^2 + xa^3 - x^2)$$

$$= x^2(1 + a^2 + a^3) - x^3.$$

136. $y_1 = p\cos px, y_2 = -p^2 \sin px, y_3 = -p^3 \cos px, y_4 = p^4 \sin px$

$y_5 = p^5 \cos px, y_6 = -p^6 \sin px, y_7 = -p^7 \cos px, y_8 = p^8 \sin px$

$$\Delta = -p^6\begin{vmatrix} \sin px & p\cos px & -p^2 \sin px \\ -p^3 \cos px & p^4 \sin px & p^5 \cos px \\ \sin px & p\cos px & -p^2 \sin px \end{vmatrix}$$

Clearly first and last rows are identical.

$\Delta = 0.$

137. $$\Delta = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix} [C_1 \to C_1 - \sin\theta C_3; C_2 \to C_2 + \cos\theta C_3]$$

$$= \cos^2\theta + \sin^2\theta = 1.$$

138. $$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & 0 \\ -\sin\alpha & \cos\alpha\cos\beta & 0 \\ 0 & -\sin\beta & \frac{1}{\cos\beta} \end{vmatrix} [C_3 \to C_3 - \tan\beta C_2]$$

$= \frac{1}{\cos\beta}[\cos\beta(\cos^2\alpha + \sin^2\alpha)] = 1.$

139. Multiplying columns with a, b, c, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+x) & ab^2 & ac^2 \\ a^2b & b(b^2+x) & bc^2 \\ a^2c & b^2c & c(c^2+x) \end{vmatrix}$$

Now taking out a, b, c from rows, we have

$$= \begin{vmatrix} a^2+x & b^2 & c^2 \\ a^2 & b^2+x & c^2 \\ a^2 & b^2 & c^2+x \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 & -x \\ 0 & x & -x \\ a^2 & b^2 & c^2+x \end{vmatrix} [R_1 \to R_1 - R_3; R_2 \to R_2 - R_3]$$

$$= \begin{vmatrix} x & 0 & 0 \\ 0 & x & -x \\ a^2 & b^2 & a^2+c^2+x \end{vmatrix} [C_3 \to C_3 + C_1]$$

$\Rightarrow x^2(a^2+b^2+c^2+x) = 0$

$\Rightarrow x = 0, -(a^2+b^2+c^2).$

140. By observation if $x = n-1$ then first two columns are same. Similarly, if $x = n$ then first column is equal to sum of two other columns. Thus, $x = n, n-1$ are two possible solutions.

If we take $\frac{x!}{r!(x-r)!}$ common from first column and similarly for second and third, then we get a quadratic equation which will have two roots and we have found both of them.

141. $$\Delta = \frac{1}{a^2}\begin{vmatrix} u+a^2x & aw'-bu & av'-cu \\ w'+abx & av-bw' & au'-cw' \\ v'+acx & au'-bv' & aw-cv' \end{vmatrix} [C_2 \to aC_2 - bC_1; C_3 \to aC_3 - cC_1]$$

$$\Rightarrow x = -\begin{vmatrix} u & aw'-bu & av'-cu \\ w' & av-bw' & au'-cw' \\ v' & au'-bv' & aw-cv' \end{vmatrix} \div \begin{vmatrix} a^2 & aw'-bu & av'-cu \\ ab & av-bw' & au'-cw' \\ ac & au'-bv' & aw-cv' \end{vmatrix}.$$

142. We know that value of the determinant in denominator is $(a-b)(b-c)(c-a) = k$ (say)

$$f(a,b,c) = \begin{vmatrix} f(a)-f(b) & f(b)-f(c) & f(c) \\ 0 & 0 & 1 \\ a-b & b-c & c \end{vmatrix} [C_1 \to C_1 - C_2; C_2 \to C_2 - C_3] \div k$$

$= -(b-c)(f(a)-f(b)) - (a-b)(f(b)-f(c)) \div k$

$= -(a-b)(b-c)\left[\frac{f(a)-f(b)}{a-b} - \frac{f(b)-f(c)}{(b-c)}\right] \div k$

$= -(a-b)(b-c)(f(a,b) - f(b,c)) \div k$

$= (a-b)(b-c)(c-a)\frac{f(b,c)-f(a,b)}{c-a} = (a-b)(b-c)(c-a)f(a,b,c) \div (a-b)(b-c)(c-a)$

$= f(a,b,c)$.

143. Becasue A, B, C are angles of a triangle. $A + B + C = \pi$

Also, $e^{i\pi} = \cos\pi + i\sin\pi = -1$

Taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2, R_3, we get

$$\Delta = e^{i(A+B+C)}\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$= -\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

Taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2, C_3, we get

$$= \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4, \text{ which is purely real.}$$

144. $\Delta = \begin{vmatrix} 1 & \sin A\cos A & \cos^2 A \\ 1 & \sin B\cos B & \cos^2 B \\ 1 & \sin C\cos C & \cos^2 C \end{vmatrix} [C_1 \to C_1 + C_3]$

Performing $R_3 \to R_3 - R_1$; $R_2 \to R_2 - R_1$, we get

$= \sin(A-B)\sin(B-C)\sin(C-A) \geq 0$.

Now it is trivial to prove the second part.

145. Performing $C_1 \to C_1 - aC_2$; $C_2 \to C_2 - aC_3]$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cos nx - a\cos(n+1)x & \cos(n+1)x - a\cos(n+2)x & \cos(n+2)x \\ \sin nx - a\sin(n+1)x & \sin(n+1)x - a\sin(n+2)x & \sin(n+2)x \end{vmatrix}$$

$= \sin(n+1)x\cos nx - a\sin(n+1)x\cos(n+1)x - a\sin(n+2)x\cos nx + a^2\sin(n+2)x\cos(n+1)x - \sin nx\cos(n+1)x + a\sin nx\cos(n+2)x + a\sin(n+1)x\cos(n+1)x - a^2\sin(n+1)x\cos(n+2)x$

$= \sin(n+1-n)x - a\sin(n+2-n)x + a^2\sin(n+2-n-1)x$

$= \sin x - a\sin 2x + a^2 \sin x$

$= (a^2 - 2a\cos x + 1)\sin x.$

146. $\Delta = \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 2 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix} [C_1 \to C_1 + C_2]$

$= \begin{vmatrix} 0 & -1 & 0 \\ 2 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix} [R_1 \to R_1 - R_2]$

$= 2 - 4\sin 2x$

The above expression has maximum value for $0 < x < \frac{\pi}{2}$ when $x = \frac{\pi}{4}$.

147. Expanding the determinant we get $\Delta = -1 + 2\cos A\cos B\cos C + \cos^2 A + \cos^2 B + \cos^2 C$

Consider the expression $2(\cos^2 A + \cos^2 B + \cos^2 C)$

$= 1 + \cos 2A + 1 + \cos 2B + 1 + \cos 2C = 2 + 2\cos(A+B)\cos(A-B) + 2\cos^2 C$

$= 2 + 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C = 2 - 2\cos C[\cos(A-B) - \cos C]$

$= 2 - 2\cos C[\cos(A-B) 0 \cos(\pi - (A+B))] = 2 - 4\cos A\cos B\cos C$

Thus, $\Delta = 0$.

148. Since A, B, C are angles of an isosceles triangle, let $A = B$

Thus, first two columns become equal leading determinant to be zero.

149. $\Delta = \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$

$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log x \end{vmatrix}$

$= 0$ because all three rows are identical.

150. $\Delta = \begin{vmatrix} a^{2x} + a^{-2x} + 2 & a^{2x} + a^{-2x} - 2 & 1 \\ b^{2x} + b^{-2x} + 2 & b^{2x} + b^{-2x} - 2 & 1 \\ c^{2x} + c^{-2x} + 2 & c^{2x} + c^{-2x} - 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} a^{2x} + a^{-2x} & a^{2x} + a^{-2x} & 1 \\ b^{2x} + b^{-2x} & b^{2x} + b^{-2x} & 1 \\ c^{2x} + c^{-2x} & c^{2x} + c^{-2x} & 1 \end{vmatrix} [C_1 \to C_1 - 2C_3; C_2 \to C_2 + 2C_3]$

$= 0$ because first two columns are identical.

151. Considering first determinant only:

$$\Delta = \begin{vmatrix} 115 & 114 & 103 \\ 108 & 106 & 111 \\ 113 & 116 & 104 \end{vmatrix} [C_1 \leftrightarrow C_2; C_2 \leftrightarrow C_3]$$

Performing $R_1 \leftrightarrow R_3$

$$\Delta = -\begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$$

Thus, given condition is satisfied.

152. $$\sum_{n=1}^{N} U_n = \begin{vmatrix} \sum_{n=1}^{N} n & 1 & 5 \\ \sum_{n=1}^{N} n^2 & 2N+1 & 2N+1 \\ \sum_{n=1}^{N} n^3 & 3N^2 & 3N \end{vmatrix}$$

$$= \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left\{\frac{N(N+1)}{2}\right\}^2 & 3N^2 & 3N \end{vmatrix}$$

Taking $\frac{N(N+1)}{2}$ common from first column and then performing $C_1 \to C_1 - \frac{1}{6}(C_2 + C_3)$

$$= \frac{N(N+1)}{2} \begin{vmatrix} 0 & 1 & 5 \\ 0 & 2N+1 & 2N+1 \\ 0 & 3N^2 & 3N \end{vmatrix}$$

Since first column has only 0 as element, therefore, the sum of determinants is zero.

153. $\because A, B, C$ are angles of a triangle, therefore $A + B + C = \pi$; $\sin(A + B + C) = 0$; $\cos(A + B) = -\cos C$

$$\therefore \Delta = \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan A & 0 \end{vmatrix}$$

Changing rows into corresponding columns

$$= \begin{vmatrix} 0 & -\sin B & -\cos C \\ \sin B & 0 & -\tan A \\ \cos C & \tan A & 0 \end{vmatrix}$$

Taking -1 common from second and third columns, we have

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ \sin B & 0 & \tan A \\ \cos C & -\tan A & 0 \end{vmatrix} = -\Delta$$

$\Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0.$

154. Taking $b-a$ common from first and third columns

$$\Delta = (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

$$= (b-a)^2 \begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix} [C_1 \to C_1 - C_3]$$

Since the first two columns are same; the determinant is zero.

155. We can rewrite it as $\displaystyle\sum_{j=0}^{n-1} \Delta_j = \begin{vmatrix} \displaystyle\sum_{j=0}^{n-1} j & n & 6 \\ \displaystyle\sum_{j=0}^{n-1} j^2 & 2n^2 & 4n-2 \\ \displaystyle\sum_{j=0}^{n-1} j^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left\{\frac{n(n-1)}{2}\right\}^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n(n-1)}{2} \begin{vmatrix} 1 & n & 6 \\ \frac{2n-1}{3} & 2n^2 & 4n-2 \\ \frac{n(n-1)}{2} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n(n-1)}{2} \begin{vmatrix} 0 & n & 6 \\ 0 & 2n^2 & 4n-2 \\ 0 & 3n^3 & 3n^2-3n \end{vmatrix} [C_1 \to C_1 - \frac{C_3}{6}]$$

Since first column is entirely made up of zeros the value of determinant is zero, which is a constant as desired.

156. $\displaystyle\sum_{r=0}^{m} (2r-1) = \frac{1}{2}(m+1)(2m-1-1) = m^2-1$

$$\sum_{r=0}^{m} ({}^nC_r) = 2^m$$

$$\sum_{r=0}^{m} 1 = m+1$$

Thus, first two rows of determinant become zero leading the desired sum to be 0.

157. $\Delta = \begin{vmatrix} C_r^x & C_{r+1}^{x+1} & C_{r+2}^{x+1} \\ C_r^y & C_{r+1}^{y+1} & C_{r+2}^{y+1} \\ C_r^z & C_{r+1}^{z+1} & C_{r+2}^{z+1} \end{vmatrix} [C_3 \to C_3 + C_2; C_2 \to C_2 + C_1]$

Performing $C_3 \to C_3 + C_2$ we get the determinant on R.H.S.

158. $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} r & n+1 & 1 \\ \sum_{r=1}^{n} r^2 & 2n-1 & \frac{2n+1}{3} \\ \sum_{r=1}^{n} r^3 & 3n+2 & \frac{n(n+1)}{2} \end{vmatrix}$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & n+1 & 1 \\ \frac{n(n+1)(2n+1)}{6} & 2n-1 & \frac{2n+1}{3} \\ \left\{\frac{n(n+1)}{2}\right\}^2 & 3n+2 & \frac{n(n+1)}{2} \end{vmatrix}$$

If we take $\frac{n(n+1)}{2}$ common from first column then first and third column become same. Thus, $\sum_{r=1}^{n} \Delta_r = 0$.

159. $\sum_{r=1}^{n} 2^{r-1} = 1 + 2 + ... + 2^{n-1} = \frac{2^n-1}{2-1} = 2^n - 1$

$$\sum_{r=1}^{n} 2.3^{r-1} = 2.\frac{3^n-1}{3-1} = 3^n - 1$$

$$\sum_{r=1}^{n} 4.5^{r-1} = 4.\frac{35^n-1}{5-1} = 5^n - 1$$

Thus, we see that first row and third rows are equal leading the sum of the determinants to zero.

160. $\Delta = \begin{vmatrix} 2x-1 & 2x-3 & x^2-4x+4 \\ 2x-3 & 2x-5 & x^2-6x+9 \\ 2x-5 & 2x-7 & x^2-8x+16 \end{vmatrix} [C_1 \to C_1 - C_1; C_2 \to C_2 - C_3]$

$$= \begin{vmatrix} 2x-1 & 2x-3 & x^2 \\ 2x-3 & 2x-5 & x^2 \\ 2x-5 & 2x-7 & x^2 \end{vmatrix} + \begin{vmatrix} 2x-1 & 2x-3 & -4x \\ 2x-3 & 2x-5 & -6x \\ 2x-5 & 2x-7 & -8x \end{vmatrix} + \begin{vmatrix} 2x-1 & 2x-3 & 4 \\ 2x-3 & 2x-5 & 9 \\ 2x-5 & 2x-7 & 16 \end{vmatrix}$$

Clearly, if we perform $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$ will make R_1 and R_3 same in the first determinant.

This is also true for second determinant.

$$= \begin{vmatrix} 2 & 2 & -5 \\ 2 & 2 & -7 \\ 2x-5 & 2x-7 & 16 \end{vmatrix}$$

Clearly, the determinant is independent of x.

161. $$\Delta = \begin{vmatrix} 2 & 1+i & 3 \\ 1-i & 0 & 2+i \\ 3 & 2-i & 1 \end{vmatrix}$$

Taking complex conjugate and exchanging rows into corresponding columns

$$\overline{\Delta} = \begin{vmatrix} 2 & 1+i & 3 \\ 1-i & 0 & 2+i \\ 3 & 2-i & 1 \end{vmatrix} = \Delta$$

Since $\overline{\Delta} = \Delta$, the determinant is purely real.

162. $$\Delta = \begin{vmatrix} x-3 & 2x & 2 \\ 3x+2 & x & 1 \\ 5x+1 & 5x & 5 \end{vmatrix} + \begin{vmatrix} x-3 & 1 & 2 \\ 3x+2 & 2 & 1 \\ 5x+1 & 4 & 5 \end{vmatrix}$$

If we take out x common from second column of first determinant then second and third columns are same, making it zero. Now expandng second determinant

$$= \begin{vmatrix} x & 1 & 2 \\ 3x & 2 & 1 \\ 5x & 4 & 5 \end{vmatrix} + \text{a determinant of constants(say } k)$$

$$= x\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} [C_1 \to C_1 - C_2] + k$$

$$= x\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 4 \end{vmatrix} [R_3 \to R_3 - R_2] + k$$

$$= x\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{vmatrix} [C_3 \to C_3 - 2C_1] + k$$

$= k$.

163. $$\Delta = \begin{vmatrix} a^n - x & a^{n+1} - x & a^{n+2} - x \\ a^{n+3} - a^n & a^{n+4} - a^{n+1} & a^{n+5} - a^{n+2} \\ a^{n+6} - a^{n+3} & a^{n+7} - a^{n+4} & a^{n+8} - a^{n+5} \end{vmatrix} [R_2 \to R_2 - R_1;\, R_3 \to R_3 - R_2]$$

$$= a^{n(n+3)}\begin{vmatrix} a^n - x & a^{n+1} - x & a^{n+2} - x \\ a^3 - 1 & a^4 - a & a^5 - a^2 \\ a^3 - 1 & a^4 - a & a^5 - a^2 \end{vmatrix} = 0$$

Since second and third rows are same, the edterminant is zero.

164. $\Delta = \sum_{r=2}^{n}(-2)^r \begin{vmatrix} C_r^{n-2} & C_{r-1}^{n-2} & C_r^{n-2} \\ 0 & 1 & 1 \\ 0 & -1 & 9 \end{vmatrix} [C_1 \to C_1 + 2C_2 + C_3]$

$= \sum_{r=0}^{n}(-2)^r C_r^n - (C_0^n - 2C_1^n)$

$= 2n - 1 + (-1)^n.$

165. Performing $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$ and then taking out abc out from first two columns,

$$\Delta = abc\begin{vmatrix} bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \\ ab & 1 & c(a+b) \end{vmatrix}$$

Performing $C_3 \to C_3 + C_1$ and then taking $ab + bc + ca$ out

$$= abc(ab + bc + ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$

Since last two columns are same, the determinant is zero.

166. Putting $b = c$, we see that the determinant reduces to 0. Similarly, $c = a$ or $a = b$ or $a = d$ or $b = d$ or $c = d$ also reduces the determinant to zero.

We also see that the degree of polynomial of the determinant is six, and thus,

$$\begin{vmatrix} b+c-a-d & bc-ad & bc(a+d)-ad(b+d) \\ c+a-b-d & ca-bd & ca(b+d)-bd(c+a) \\ a+b-c-d & ab-cd & ab(c+d)-cd(a+b) \end{vmatrix} = k(b-c)(c-a)(a-b)(a-d)(b-d)(c-d)$$

Putting $a = 0, b = 1, c = 2, d = 3$ we evaluate $k = -2$, and thus, we have proven the desired equality.

167. Putting $b = c$, we see that the determinant reduces to zero. Similarly, $c = a$ or $a = b$ also reduced the determinant to zero. Also, putting $a = -b - c$ or $ab = -bc - ca$ makes the determinant zero.

We also see that the degree of polynomial of the determinant is six, and thus,

$$\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca + ab - bc & bc + ab - ca & bc + ca - ab \\ (a+b)(a+c) & (b+c)(b+a) & (c+a)(c+b) \end{vmatrix} = k(b-c)(c-a)(a-b)(a+b+c)(ab+bc+ca)$$

Putting $a = 0, b = 1, c = 2$ we evaluate $k = 3$.

168. Putting $l = m$ we see that the determinant reduces to zero. Similarly $l = n, n = p, m = n, m = p, n = p$ also reduce the determinant to zero.

We also see that the degree of polynomial of the determinant is six, and thus,

$\begin{vmatrix} 1 & (m+n-l-p)^2 & (m+n-l-p)^4 \\ 1 & (n+l-m-p)^2 & (n+l-m-p)^4 \\ 1 & (l+m-n-p)^2 & (l+m-n-p)^4 \end{vmatrix} = k(l-m)(l-n)(l-p)(m-n)(m-p)(n-p)$

Putting $l = 0, m = 1, n = 2, p = 3$ we find that $k = 64$, and hence we prove the required equality.

169. $\frac{d}{dt}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_2 & v_2 & w_2 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_3 & v_3 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$

First two determinants are zero because two rows are identical. Hence,

$\frac{d}{dt}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$.

170. Given $Y = sX \Rightarrow Y_1 = s_1X + sX_1 \Rightarrow Y_2 = s_2X + s_1X_1 + s_1X_1 + sX_2 = sX_2 + 2s_1X_1 + s_2X$, and similarly $Z = tX \Rightarrow Z_1 = t_1X + tX_1 \Rightarrow Z_2 = tX_2 + 2t_1X_1 + t_2X$.

L.H.S. $= \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} X & sX & tX \\ X_1 & s_1X + sX_1 & t_1X + tX_1 \\ X_2 & s_2X + 2s_1X_1 + sX_2 & t_2X + 2t_1X_1 + tX_2 \end{vmatrix}$

$= X\begin{vmatrix} 1 & s & t \\ X_1 & s_1X + sX_1 & t_1X + tX_1 \\ X_2 & s_2X + 2s_1X_1 + sX_2 & t_2X + 2t_1X_1 + tX_2 \end{vmatrix}$

$= X\begin{vmatrix} 1 & s & t \\ 0 & s_1X & t_1X \\ 0 & s_2X + 2s_1X_1 & t_2X + 2t_1x_1 \end{vmatrix}$ $[R_2 \to R_2 - R_1X_1; R_3 \to R_3 - R_1X_2]$

$= X^3\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$.

171. Let $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(\alpha) & g(\alpha) & h(\alpha) \\ f(\beta) & g(\beta) & h(\beta) \end{vmatrix}$

Clearly, $F(\alpha) = 0$ because first two rows become equal. $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(\alpha) & g(\alpha) & h(\alpha) \\ f(\beta) & g(\beta) & h(\beta) \end{vmatrix}$

If $F'(x) = 0$ then $F'(\alpha) = 0$ making α a repeated root.

$F(\beta) = 0$ because first and last rows are identical. Thus, $(x-\alpha)^2(x-\beta)$ is a factor of $F(x)$ so the required condition is $\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(\alpha) & g(\alpha) & h(\alpha) \\ f(\beta) & g(\beta) & h(\beta) \end{vmatrix} = 0$

172. We see that $\frac{d\Delta}{dx} = 0$ and hence it is a constant, independent of x.

173. Applying $C_1 \to C_1 - 2\sin x C_3$; $C_2 \to C_2 + 2\cos x C_3$

$$\Delta = \begin{vmatrix} 2 & 0 & -\sin x \\ 0 & 2 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix} = 2\sin^2 x + 2\cos^2 x = 2.$$

$\Rightarrow f'(x) = 0.$

$$\therefore \int_0^{\frac{\pi}{2}} [f(x) + f'(x)]\,dx = \int_0^{\frac{\pi}{2}} 2dx = \pi.$$

174. L.H.S. $= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} = 0.$

175. Let $\overrightarrow{V_r} = l_r\vec{\imath} + m_r\vec{\jmath} + n_r\vec{k}$. Then for $r = 1$ let $\overrightarrow{V_1} = \pm\vec{\imath}$, for $r = 2$, $\overrightarrow{V_2} = \pm\vec{\jmath}$, and $r = 3$, $\overrightarrow{V_3} = \pm\vec{k}$.

Thus, we get determinant as $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ or $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$

Hence, $\Delta = \pm 1$.

176. Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Cofactor matrix $C = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

$\text{adj}(A) = C^T$ but $A(\text{adj}(A)) = |A|I \Rightarrow |A||\text{adj}(A)| = |A|^3 \Rightarrow |\text{adj}(A)| = |A|^2$

$\therefore |C| = |A|^2 \Rightarrow \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

We can proceed similarly for second level of cofactors whose determinant will be Δ^4.

Thus, $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \Delta^6.$

177. Using Cramer's rule, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = 34 - 20 - 24 = -20$

$$\Delta_x = \begin{vmatrix} 17 & 4 & 1 \\ 2 & 2 & 9 \\ 1 & 6 & 3 \end{vmatrix} = 204 - 302 + 78 = -20$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 17 & 1 \\ 3 & 2 & 9 \end{vmatrix} = 151 - 90 - 141 = -80$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 17 \\ 3 & 2 & 2 \end{vmatrix} = -26 + 94 - 48 = 20$$

$\Rightarrow x = \frac{\Delta_x}{\Delta} = 1, y = \frac{\Delta_y}{\Delta} = 4, z = \frac{\Delta_z}{\Delta} = -1.$

178. $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a).$

Similarly, $\Delta_x = dbc(d-b)(b-c)(c-d)$, $\Delta_y = acd(a-d)(d-c)(c-a)$, and $\Delta_z = abd(a-b)(b-d)(d-a)$

$\Rightarrow x = \frac{d(d-b)(c-d)}{a(a-b)(c-a)}, y = \frac{d(a-d)(d-c)}{b(a-b)(b-c)}, z = \frac{d(b-d)(d-a)}{c(b-c)(c-a)}.$

If only two of a, b, c are zero the given system of equations has no solution. If $a = b = c$ and any of a, b, c is zero then the system of equations has infinite number of solutions.

179. Given $f(1) = 0 \Rightarrow a + b + c = 0$; $f(2) = -2 \Rightarrow 4a + 2b + c = -2$; $f(3) = -6 \Rightarrow 9a + 3b + c = -6$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = -1 + 5 - 6 = -2$$

$$\Delta_a = \begin{vmatrix} 0 & 1 & 1 \\ -2 & 2 & 1 \\ -6 & 3 & 1 \end{vmatrix} = -4 + 6 = 2$$

$$\Delta_b = \begin{vmatrix} 1 & 0 & 1 \\ 4 & -2 & 1 \\ 9 & -6 & 1 \end{vmatrix} = 4 - 6 = -2$$

$$\Delta_c = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & -2 \\ 9 & 3 & -6 \end{vmatrix} = 0$$

$\Rightarrow a = -1, b = 1, c = 0.$

180. $f(0)=6 \Rightarrow c=6$, $f(2)=11 \Rightarrow 4a+2b+c=11$, $f(-3)=6 \Rightarrow 9a-3b+c=6$.

$$\Delta=\begin{vmatrix} 0 & 0 & 1 \\ 42 & 1 & \\ 9 & -3 & 1 \end{vmatrix}=-30$$

$$\Delta_a=\begin{vmatrix} 6 & 0 & 1 \\ 11 & 2 & 1 \\ 0 & 6 & 1 \end{vmatrix}=-15$$

$$\Delta_b=\begin{vmatrix} 0 & 6 & 1 \\ 4 & 11 & 1 \\ 9 & 6 & 1 \end{vmatrix}=-45$$

$\Rightarrow a=\frac{1}{2}, b=\frac{3}{2}, c=6 \Rightarrow f(1)=a+b+c=8.$

181. $$\Delta=\begin{vmatrix} -a & b+c & b+c \\ c+a & -b & c+a \\ a+b & a+b & -c \end{vmatrix}=\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & -b & c+a \\ a+b & a+b & -c \end{vmatrix}[R_1 \to R_1+R_2+R_3]$$

$$=(a+b+c)\begin{vmatrix} 0 & 1 & 1 \\ 0 & -b & c+a \\ a+b+c & a+b & -c \end{vmatrix}[C_1 \to C_1-C_3]$$

$$=(a+b+c)^3$$

$$\Delta_x=\begin{vmatrix} b-c & b+c & b+c \\ c-a & -b & c+a \\ a-b & a+b & -a \end{vmatrix}=\begin{vmatrix} b-c & 0 & b+c \\ c-a & -(a+b+c) & c+a \\ a-b & a+b+c & -c \end{vmatrix}[C_2 \to C_2-C_3]$$

$$=(a+b+c)\begin{vmatrix} b-c & 0 & b+c \\ c-b & 0 & a \\ a-b & 1 & -c \end{vmatrix}[R_2 \to R_2+R_3]$$

$$=(c-b)(a+b+c)^2$$

$\Rightarrow x=\frac{c-b}{a+b+c}$. Since the given system of equations is cyclic, therefore, $y=\frac{a-c}{a+b+c}$, $z=\frac{b-a}{a+b+c}$.

182. $$\Delta=\begin{vmatrix} 7 & -7 & 5 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{vmatrix}=-70+35+35=0$$

$$\Delta_x=\begin{vmatrix} 3 & -7 & 5 \\ 7 & 1 & 5 \\ 5 & 3 & 5 \end{vmatrix}=-30-70+80=-20$$

Therefore, the system of equations is inconsistent and has no solution.

183. For system of equations to be consistent $\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 1+k & 2+k & 8 \\ 1 & -(1+k) & -(2+k) \end{vmatrix} = 0$

$\Rightarrow 3k^2 + 2k - 5 = 0 \Rightarrow k = 1, -\frac{5}{3}$.

184. $\Delta = \begin{vmatrix} (k+1)^3 & (k+2)^2 & (k+3)^3 \\ k+1 & k+2 & k+3 \\ 1 & 1 & 1 \end{vmatrix}$

$\Rightarrow k = -2$.

Answers of Chapter 9
Matrices

1. The matrix will be any one of the following type $1 \times 12, 12 \times 1, 2 \times 6, 6 \times 2, 3 \times 4, 4 \times 3$. So the answer is 6.

2. $a_{11} = 2.1 - 3.1 = -1, a_{12} = 2.1 - 3.2 = -4, a_{13} = 2.1 - 3.3 = -7$

   $a_{21} = 2.2 - 3.1 = 1, a_{22} = 2.2 - 3.2 = -2, a_{23} = 2.2 - 3.3 = -5$

   $\therefore A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \end{bmatrix}$.

3. $A + B = \begin{bmatrix} a - a & b + b \\ -b - b & a - a \end{bmatrix} = \begin{bmatrix} 0 & 2b \\ -2b & 0 \end{bmatrix}$.

4. $2X = (2X + Y) - Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$

   $\Rightarrow X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

5. $x^2 - 4x = -3 \Rightarrow x = 1, 3$. $x^2 = 1 \Rightarrow x = \pm 1$. $x^2 = -x + 2 \Rightarrow x = -2, 1$. $x^3 = 1 \Rightarrow x = 1, \omega, \omega^2$.

   Common value of x is 1.

6. $x + 3 = 0 \Rightarrow x = -3$. $2y + x = -7 \Rightarrow 2y = -4 \Rightarrow y = -2$. $z - 1 = 3 \Rightarrow z = 4$. $4a - 6 = 2a \Rightarrow 2a = 6 \Rightarrow a = 3$.

7. $4A - 3B = 4\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ -4 & 0 & 8 \\ 4 & -12 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 15 & 18 \\ -3 & 0 & 3 \\ 6 & 3 & 6 \end{bmatrix}$

   $= \begin{bmatrix} -8 & -7 & -6 \\ -1 & 0 & 5 \\ -2 & -15 & -2 \end{bmatrix}$.

8. A is a 2×3 matrix and B is a 3×2 matrix. AB is defined and will be a 2×2 matrix.

   $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 - 8 + 6 & 3 - 10 + 3 \\ -8 + 8 + 10 & -12 + 10 + 5 \end{bmatrix}$

   $= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$.

   BA is also defined and will be a 3×3 matrix.

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}.$$

Clearly, $AB \neq BA$.

9. Because associative law holds for matrix multiplication, therefore, $A(BC) = (AB)C$.

$$BC = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+gz \\ hx+by+fz \quad gx+fy+zc \end{bmatrix}$$

$$ABC = [xyz] \begin{bmatrix} ax+by+gz \\ hx+by+fz \\ gx+fy+zc \end{bmatrix}$$

$= x(ax+by+gz) + y(hx+by+fz) + z(gx+fy+zc) = ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz.$

10. $A' = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 5 & 4 \\ 3 & 0 & 3 \end{bmatrix}$

Let B be the matrix whose elements are cofactors of the corresponding elements of the matrix A. Then $B = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$

$$\therefore \text{adj}(A) = B' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}.$$

11. Let B be the matrix whose elements are cofactors of the corresponding elements of A. Then

$$B = \begin{bmatrix} -1 & 8 & 5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} \therefore \text{adj}(A) = B' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2 \therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

12. Let B be the matrix whose elements are cofactors of the corresponding elements of A. Then

$$B = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix} \therefore \text{adj}(A) = B' = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 21 \therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}.$$

$$A^{-1}A = \frac{1}{21}\begin{bmatrix} 2+6+13 & 4+9-13 & 10+3-13 \\ -3+12-9 & -6+18+9 & -15+6+9 \\ 5-6+1 & 10-9-1 & 25-3-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

13. $A^2 = A.A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$A^2 - 4A - 5I = O \Rightarrow A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = A^{-1}O = O$$

$$(A^{-1}A)A - 4(A^{-1}A) - 5A^{-1}I = O \Rightarrow IA - 4I - 5A^{-1} = O$$

$$\Rightarrow 5A^{-1} = A - 4I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

14. Let $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$.

Then the matrix equation of the gives system of equation becomes $AX = B$.

Now $A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -22 \neq 0$

Hence, A is non-singular. Therefore, the given system of equations will have the unique solution given by $X = A^{-1}B$.

Let C be the matrix whose elements are cofactors of the corresponding elements of A, then

$$C = \begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix} \therefore \operatorname{adj}(A) = C' = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$A^{1-} = \frac{\operatorname{adj}(A)}{|A|} = -\frac{1}{22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \therefore x = 1, y = 2, z = 5.$$

15. $AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

Given system of equations in matrix form is $BX = C$, where $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$

We have $BX = C$. Multiplying both sides with B^{-1}, $B^{-1}BX = B^{-1}C \Rightarrow IX = X = B^{-1}C$.

However, $AB = 4I_3 \Rightarrow \frac{A}{4}B = I_3 \Rightarrow B^{-1} = \frac{A}{4} \Rightarrow X = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\therefore x = 2, y = 1, z = -1.$

16. $x + y = 3, x - y = 7 \Rightarrow x = 5, y = -2.$

17. $x - y = -1, 2x - y = 0 \Rightarrow x = 1, y = 2$. $2x + x_1 = 5 \Rightarrow x_1 = 3, 3x + y_1 = 13 \Rightarrow y_1 = 10$

So P is $(1, 2)$ and Q is $(3, 10)$. $PQ = \sqrt{(1-3)^2 + (2-10)^2} = \sqrt{68} = 2\sqrt{17}$.

18. $2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$

$2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$.

19. $C = B - A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

20. $X = 2A + 3B - C = \begin{bmatrix} 4 & 6 & 8 \\ -6 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 9 & -12 & -15 \\ 3 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 5 & -1 & 2 \\ 7 & 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 8 & -5 & -9 \\ -10 & 6 & 4 \end{bmatrix}$.

21. $A - 2B + 3C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 10 & 12 \\ -2 & 0 & 2 \\ 4 & 2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 6 & 3 \\ -3 & 6 & 9 \\ -3 & -6 & 6 \end{bmatrix}$

$= \begin{bmatrix} -10 & -14 & -6 \\ -2 & 6 & 9 \\ -6 & -11 & 3 \end{bmatrix}$.

22. $P(x).P(y) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} . \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$

$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - cos x \sin y & -\sin x \sin y + \cos x \cos y \end{bmatrix} = \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix}$.

Similarly, it can be proven to be equal to $P(y).P(x)$.

23. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$= \begin{bmatrix} 1*1+0*0+0*a & 1*0+0*1+0*b & 1*0+0*0+0*-1 \\ 0*1+1*0+0*a & 0*0+1*1+0*b & 0*0+1*0+0*-1 \\ a*1+b*0+-1*a & a*0+b*1+-1*b & a*0+b*0+-1*-1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$.

24. $A^2 = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$

$= \begin{bmatrix} -1*-1+1*3+-1*5 & -1*1+1*-3+-1*-5 & -1*-1+1*3+-1*5 \\ 3*-1+-3*3+3*5 & 3*1+-3*-3+3*-5 & 3*-1+-3*3+3*5 \\ 5*-1+-5*3+5*5 & 5*1+-5*-3+5*-5 & 5*-1+-5*3+5*5 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = A$

$B^2 = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 0*0+4*1+3*-1 & 0*4+4*-3+3*4 & 0*3+4*-3+3*4 \\ 1*0+-3*1+-3*-1 & 1*4+-3*-3+-3*4 & 1*3+-3*-3+-3*4 \\ -1*0+4*1+4*-1 & -1*4+4*-3+4*4 & -1*3+4*-3+4*4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore A^2B^2 = A$

25. Given, $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 2*1+3*-1+4*0 & 2*3+3*2+4*0 & 2*0+3*1+4*2 \\ 1*1+2*-1+3*0 & 1*3+2*2+3*0 & 1*0+2*1+3*2 \\ -1*1+1*-1+2*0 & -1*3+1*2+2*0 & -1*0+1*1+2*2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1*2+3*1+0*-1 & 1*3+3*2+0*1 & 1*4+3*3+0*2 \\ -1*2+2*1+1*-1 & -1*3+2*2+1*1 & -1*4+2*3+1*2 \\ 0*2+0*1+2*-1 & 0*3+0*2+2*1 & 0*4+0*3+2*2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}.$$

Clearly, $AB \neq BA$.

26. Let $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

$$AB = \begin{bmatrix} 0*a^2+c*ab+-b*ac & 0*ab+c*b^2+-b*bc & 0*ac+c*bc+-b*c^2 \\ -c*a^2+0*ab+a*ac & -c*ab+0*b^2+a*bc & -c*ac+0*bc+a*c^2 \\ b*a^2+-a*ab+0*ac & b*ab+-a*b^2+0*bc & b*ac+-a*bc+0*c^2 \end{bmatrix}$$

= a zero matrix.

27. Given $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \therefore A^2 = \begin{bmatrix} 3*3+-5*-4 & 3*-5+-5*2 \\ -4*3+2*-4 & -4*-5+2*2 \end{bmatrix}$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$$

$\therefore A^2 - 5A - 14I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

28. Given $A=\begin{bmatrix}2&3\\1&2\end{bmatrix}\therefore A^2=\begin{bmatrix}2*2+3*1&2*3+3*2\\1*2+2*1&1*3+2*2\end{bmatrix}$

$=\begin{bmatrix}7&12\\4&7\end{bmatrix}$

$\therefore A^3=\begin{bmatrix}7*7+12*4&7*12+12*7\\4*7+7*4&4*12+7*7\end{bmatrix}=\begin{bmatrix}97&168\\56&97\end{bmatrix}$

Clearly, $A^3-4A^2+A=\begin{bmatrix}71&123\\41&71\end{bmatrix}$, which can be easily shown to be an orthogonal matrix.

29. $A^2=\begin{bmatrix}0.8*0.8+0.6*-0.6&0.8*0.6+0.6*0.8\\-0.6*0.8+0.8*-0.6&-0.6*0.6+0.8*0.8\end{bmatrix}$

Similarly we proceed for A^3 which turns out to be $\begin{bmatrix}-0.352&0.936\\-0.936&-0.352\end{bmatrix}$.

30. $f(A)=A^2-5A+7I$, where $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$

$A^2=\begin{bmatrix}3*3+1*-1&3*1+1*2\\-1*3+2*-1&-1*1+2*2\end{bmatrix}=\begin{bmatrix}8&5\\-5&3\end{bmatrix}$

Thus, $A^2-5A+7I$ is a 2×2 zero matrix, which is trivial to prove.

31. $AB=\begin{bmatrix}\cos\theta*\cos\phi+\sin\theta*\sin\phi&\cos\theta*\sin\phi+\sin\theta*\cos\phi\\\sin\theta*\cos\phi+\cos\theta*\sin\phi&\sin\theta*\sin\phi+\cos\theta*\cos\phi\end{bmatrix}$

$=\begin{bmatrix}\cos(\theta-\phi)&\sin(\theta+\phi)\\\sin(\theta+\phi)&\cos(\theta-\phi)\end{bmatrix}$

Similarly $BA=\begin{bmatrix}\cos(\theta-\phi)&\sin(\theta+\phi)\\\sin(\theta+\phi)&\cos(\theta-\phi)\end{bmatrix}$

Thus, $AB=BA$.

32. $f(A)=A^2-5A+6$, $A^2=\begin{bmatrix}2*2+0*2+1*1&2*0+0*1+1*-1&2*1+0*3+1*0\\2*2+1*2+3*1&2*0+1*1+3*-1&2*1+1*3+3*0\\1*2+-1*2+0*1&1*0+-1*1+0*-1&1*1+-1*3+0*0\end{bmatrix}$

$=\begin{bmatrix}5&-1&2\\9&-2&5\\0&-1&-2\end{bmatrix}$

Thus, $A^2-5A+6=\begin{bmatrix}1&-1&-3\\-1&-1&-10\\-5&4&4\end{bmatrix}$.

33. Given $A=\begin{bmatrix}5&3\\12&7\end{bmatrix}\therefore A^2=\begin{bmatrix}5*5+3*125*3+3*7&\\12*5+7*12&12*3+7*7\end{bmatrix}$

$$= \begin{bmatrix} 61 & 36 \\ 144 & 85 \end{bmatrix}$$

$\therefore A^2 - 12A - I = 0$

34. We have $\left(\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} + \begin{bmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 1+\omega & \omega+\omega^2 & \omega^2+1 \\ \omega+\omega^2 & \omega^2+1 & 1+\omega \\ \omega^2+\omega & 1+\omega^2 & \omega+1 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$

$$= \begin{bmatrix} 1+\omega+\omega^2+\omega^3+\omega^4+\omega^2 \\ \omega+\omega^2+\omega^3+\omega+\omega^4+\omega^3 \\ \omega^2+\omega+\omega+\omega^3+\omega^3+\omega^2 \end{bmatrix} = 2 \begin{bmatrix} 1+\omega+\omega^2 \\ 1+\omega+\omega^2 \\ 1+\omega+\omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

35. $I + A = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$

$$(I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \tan\frac{\alpha}{2}.\sin\alpha & \tan\frac{\alpha}{2}\cos\alpha - \sin\alpha \\ \sin\alpha - \tan\frac{\alpha}{2}\cos\alpha & \tan\frac{\alpha}{2}\sin\alpha + \cos\alpha \end{bmatrix}$$

Substituting $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ we get the desired result.

36. If we multiply two matrices on left-hand side and compare the terms with right-hand side then we will get four equations in x, y, z and u. We also have four unknowns, which is a solvable system of linear equations. The solution is left as an exercise.

37. We have $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3+5x+3 & 2+x+2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$$

$1 + 6 + 5x + 4x + x^2 = 0 \Rightarrow x^2 + 9x + 7 = 0 \Rightarrow x = \frac{-9 \pm \sqrt{53}}{2}$.

38. Product is $\begin{bmatrix} \cos^2\theta\cos^2\phi + \sin\theta\cos\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \sin\theta\cos\theta\sin^2\phi \\ \cos^2\theta\sin\theta\cos\theta + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \sin\phi\sin\theta\cos(\theta-\phi) \\ \cos\phi\sin\theta\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

Clearly the above matrix is a zero matrix of the difference of angles is an odd multiple of $\frac{\pi}{2}$.

39. This we will prove by mathematical induction. We have $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta + \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

Thus, the result is true for $n = 2$. Let it be true for $n = k$ i.e. $A^k = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$.

$$A^{k+1} = \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \sin k\theta\cos\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

$= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$, which is true for $n = k+1$.

Thus, we have proven the required result by mathematical induction.

40. We have $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We will prove this by mathematical induction like last problem.

$A^2 = \begin{bmatrix} 3*3-4*1 & 3*-4-4*-1 \\ 1*3-1*1 & 1*-4-1*-1 \end{bmatrix} = \begin{bmatrix} 1+2*2 & -4*2 \\ 2 & 1-2*2 \end{bmatrix}$, which is true for $n = 2$.

Let it be true for $n = k$ i.e. $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} (1+2k)*3-4k*1 & (1+2k)*-4-4k*-1 \\ k*3+(1-2k)*1 & k*-4+(1-2k)*-1 \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$, which is true for $n = k+1$.

Thus, we have proven the required result by mathematical induction.

41. $(aI + bA)^1 = aI + 1a^{1-1}bA = aI + bA$, thus the statement holds true for $n = 1$.

Let it be true for $n = k$ i.e. $(aI + bA)^k = a^kI + ka^{k-1}bA$

For $n = k+1$, $(aI + bA)^{k+1} = (a^kI + ka^{k-1}bA)(aI + bA) = a^{k+1}I + ka^{k-1}abA + a^kbA + ka^{k-1}bA * bA$

However, $A^2 = 0$, so$a^{k+1}I + ka^kbA + a^kbA = a^{k+1}I + (k+1)a^kbA$.

Thus, the statement is true for $n = k+1$, and, hence proved.

42. We know that for matrix multiplication it is not neccessary that $AB = BA$. However, $(A+B)(A-B) = A^2 - AB + BA - B^2$, which will be equal to $A^2 - B^2$ if $AB = BA$.

43. We can represent quanitity bought using a row matrix, for example, $Q = \begin{bmatrix} 8 & 10 & 4 \end{bmatrix}$ and rate as $R = \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$.

Total cost would be product of these two matrices i.e. $8*18+10*9+4*6 = 144+90+24 = 258$.

44. Let the amount invested in first fund is USD x, and in second fund USD $30000-x$. Then we can represent interest as

$$\begin{bmatrix} x & 30000-x \end{bmatrix} . \begin{bmatrix} 0.05 \\ 0.07 \end{bmatrix} = 2000$$

$\Rightarrow 0.05x + 2100 - 0.07x = 2000 \Rightarrow 0.02x = 100 \Rightarrow x = 5000$. Thus, amount to be invested in first fund is USD 5000, and in second fund USD 25000 should be invested.

45. The store owner has 240 shirts, 180 trousers and 300 pair of socks, which can be represented by a row matrix, $I = \begin{bmatrix} 240 & 180 & 300 \end{bmatrix}$ for example. The respective costs can be represented by a column matrix, $R = \begin{bmatrix} 50 \\ 90 \\ 12 \end{bmatrix}$.

Thus, according to question, total amount receieved would be $IR = 240*50+180*90+300*12 = \$24600$.

46. There are 120 physics books, 96 chemistry books, and 60 mathematics books. These can be represented by a row matric,$B = \begin{bmatrix} 120 & 96 & 60 \end{bmatrix}$, for example. The respective costs can be represented by a column matrix, $R = \begin{bmatrix} 8.3 \\ 3.45 \\ 4.5 \end{bmatrix}$.

Thus, total amount received by the store owner upon sell of all the books will be $BR = 120*8.3+96*3.45+60*4.5 = 1597.2$

47. Given, $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$. Therefore, $A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$AA' = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha. -\sin\alpha + \sin\alpha.\cos\alpha \\ -\sin\alpha.\cos\alpha + \cos\alpha.\sin\alpha & -\sin\alpha. -\sin\alpha + \cos\alpha.\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A'A = \begin{bmatrix} -\sin\alpha. -\sin\alpha + \cos\alpha.\cos\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus, $AA' = A'A = I_2$.

48. We know that for a symmetric matrix its trnaspose is equal to original matrix i.e. if A is the matrix then $A = A^T$, while for a skew-symmetric matrix its transpose is equal to neation of original matrix i.e. $A = -A^T$.

Thus, any square matrix can be represented as sum of its symmetric matrix(say S) and skew-symmetric matrix(say K) as $S = \frac{1}{2}(A + A^T)$ and $K = \frac{1}{2}(A - A^T)$.

Thus, $S = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$

and $K = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right) = \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & 2 & 0 \end{bmatrix}$.

49. We know that a matrix is orthogonal if its product with its transpose is yields an identity matrix. We have already shown this in second last problem for the given matrix.

50. The transpose of given matrix is equal to the original i.e. $A = A'$.

$$AA' = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} . \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1*-1+2*2+2*2 & -1*2+2*-1+2*2 & -1*2+2*2+2*-1 \\ 2*-1-1*2+2*2 & 2*2-1*-1+2*2 & 2*2-1*2+2*-1 \\ 2*-1+2*2-1*2 & 2*2+2*-1+-1*2 & 2*2+2*2-1*-1 \end{bmatrix} = I_3.$$

Thus, given matrix is orthogonal.

51. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$. Let B represent the matrix of cofactors of A, then

$$B = \begin{bmatrix} 3*4-3*2 & 3*2-4*2 & 3*2-3*3 \\ 3*3-4*2 & 4*1-3*3 & 3*2-3*1 \\ 2*2-3*3 & 3*2-2*1 & 3*1-2*2 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & 1 \end{bmatrix}.$$

Thus, $adj(A) = B' = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$.

52. First we find the adjoint of the matrix for which we need to find cofactors. Let C_{ij} represent the cofactors, then,

$C_{11} = \cos\alpha*1-0*0 = \cos\alpha$, $C_{12} = -(\sin\alpha*1-0*0) = -\sin\alpha$, $C_{13} = \sin\alpha*0-\cos\alpha*0 = 0$, $C_{21} = -(\sin\alpha*1-0*0)$, $C_{22} = \cos\alpha*1-0*0 = \cos\alpha$, $C_{23} = -(\cos\alpha*0-0*0) = 0, C_{31} = \cos\alpha*0-0*0 = 0, C_{32} = -\cos\alpha*0-0*0 = 0, C_{33} = \cos\alpha*1-0*0 = \cos\alpha$

Hence, $adj(A) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix}$

$$\therefore A*adj(A) = \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & 0 & 0 \\ 0 & \cos^2\alpha+\sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Now, $|A| = \cos\alpha(\cos\alpha*1-0*0)-(-\sin\alpha)(\sin\alpha*1-0*0)+0 = 1$

$|A|I = I$. Hence proven.

53. We know that $A(adj\ A) = |A|I$. Here $|A| = 1(3*10-2*0)-(-1)(2*10-18*0)+1(2*2-18*3) = 0$. Hence, $A(adj\ A) = 0$.

54. Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$|A| = 1(4*4-3*3)-3(1.*4-4*1)+3(1*3-4*1) = 1$

Matrix of cofactos is $\begin{bmatrix} (4*4-3*3) & (1*4-3*1) & (1*3-4*1) \\ (3*4-3*3) & (1*4-3*1) & (1*3-3*1) \\ (3*3-4*3) & (1*3-3*1) & (1*4-3*1) \end{bmatrix}$

Hence, $adj(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Because $|A| = 1$, therefore, $A^{-1} = adj(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

55. $|A| = 2(2*2-3*-2)-(-3)(2*2-3*3)+3(2*-2-2*3) = -25$

Matrix of cofactors is $\begin{bmatrix} (2*2-3*-2) & -(2*2-3*3) & (2*-2-2*3) \\ -(-3*2-3*-2) & (2*2-3*3) & -(2*-2-(-3)*3) \\ (-3*3-2*3) & -(2*3-2*3) & (2*2-(-3)*2) \end{bmatrix} =$
$\begin{bmatrix} 10 & 5 & -10 \\ 0 & -5 & -5 \\ -15 & 0 & 10 \end{bmatrix}$

$adj(A) = \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{5}\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$.

56. The inverse can be calculated like previous problems and is equal to $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

57. In previous problem we have formula for inverse of a 2×2 matrix.

$AB = \begin{bmatrix} 14 & 5 \\ 16 & 0 \end{bmatrix}$.

Hence, $(AB)^{-1} = \frac{1}{-80}\begin{bmatrix} 0 & -5 \\ -16 & -14 \end{bmatrix}$.

$A^{-1} = \frac{1}{-4}\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}$ $B^{-1} = \frac{1}{20}\begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$

$A^{-1}B^{-1} = \frac{1}{-80}\begin{bmatrix} 0 & -5 \\ -16 & -14 \end{bmatrix}$.

Hence, $(AB)^{-1} = A^{-1}B^{-1}$.

58. Given $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ so $A^{-1} = \cos^2 x \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix}$

$\therefore A' A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} . \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix}$

$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

59. (Hint: $(AB)^{-1} = B^{-1}A^{-1}$)

$A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ (from formula obtained in third last problem).

$B^{-1} = \begin{bmatrix} -\frac{9}{2} & -\frac{7}{2} \\ 4 & -3 \end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix} -47 & \frac{39}{2} \\ 41 & -17 \end{bmatrix}$.

60. We can write the given system of equations as $AX = B$, where $A = \begin{bmatrix} 3 & -2 \\ 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

$|A| = 3 * 3 - (-2) * 5 = 19 \Rightarrow A^{-1} = \frac{1}{19}\begin{bmatrix} 3 & 2 \\ -5 & 3 \end{bmatrix}$

$X = A^{-1}B$. We multiply the matrices and compare elements to get x and y as $\frac{23}{19}$ and $-\frac{32}{19}$ respectively.

61. Proceeding like previous problem we represent the given system of equations as $AX = B$, where $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$|A| = 2(2 * 2 - 3 * -2) - (-3)(2 * 2 - 3 * 3) + 3(2 * -2 - 2 * 3) = -25$

Matrix of cofactors is $\begin{bmatrix} 10 & 5 & -10 \\ 0 & -5 & -5 \\ -15 & 0 & 10 \end{bmatrix}$.

$$\Rightarrow adj(A) = \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}.$$

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$X = A^{-1}B$. We multiply the matrices and compare elements to get solution as $x = \frac{7}{5}$, $y = \frac{1}{5}$ and $z = -\frac{2}{5}$.

62. We represent the gives system of equations as $AX = B$, where $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 10 \end{bmatrix}$

$|A| = 0$, which means either it is inconsistent or has infinitely many solutions. However, we observe that $(2x + 3y = 5)\,(3 * 2x = 6x, 3 * 3y = 9y, 3 * 5 = 15$, and $3 * 10 = 30)$. Thus, the given system of equations is inconsistent

Answers of Chapter 10
Inequalities

1. We have $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$. It should be noted that equality holds if and only if $a = b$.

2. Similar to previous problem we have $a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \geq 0$. Similarly, equality holds if and only if $a = b$.

3. Squaring $\frac{a^2+b^2}{2} \geq \frac{a^2+b^2+2ab}{4} \Rightarrow \frac{(a-b)^2}{4} \geq 0$, which is true. Similar to previous problems equality holds if and only if $a = b$.

4. $\frac{a+b}{2} \geq \frac{2ab}{a+b} \Rightarrow (a-b)^2 \geq 0$, which is same as first problem.

5. $a + b - 1 - ab = (a-1)(1-b)$. We have $b < 1 < a$ making $(a-1)(1-b) > 0$.

6. $a^2 + b^2 - c^2 - (a+b-c)^2 = (a^2 - c^2) - [(a+b-c)^2 - b^2] = 2(a-c)(c-b) > 0$.

7. Multiplying both sides of the inequality by ab, where $ab > 0$, gives us $a^2 + b^2 \geq 2ab$.

8. Dividing both sides of the inequality $a^2 + b^2 \geq -2ab$ by ab, where $ab < 0$, gives us the required inequality.

9. We have $x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n$, which gives us $nx_1 \leq x_1 + x_2 + \cdots + x_n \leq nx_n \Rightarrow x_1 \leq \frac{x_1+x_2+\cdots+x_n}{n} \leq x_n$.

10. We are given that $\frac{x_1}{y_1} \leq \cdots \leq \frac{x_n}{y_n}$, from which we deduce that $y_i\frac{x_1}{y_1} \leq x_i \leq y_i\frac{x_n}{y_n}, i = 1, \ldots, n$. Adding all these inequalities leads to $\frac{x_1}{y_1}(y_1 + \cdots + y_n) \leq x_1 + \cdots + x_n \leq \frac{x_n}{y_n}(y_1 + \cdots + y_n)$. Hence, it follows that $\frac{x_1}{y_1} \leq \frac{x_1+\cdots+x_n}{y_1+\cdots+y_n} \leq x_n$.

11. Given that $x_1 \leq x_i \leq x_n, i = 1, \ldots, n$. Multiplying these inequalities $x_1^n \leq x_1 \cdots x_n \leq x_n^n \Rightarrow x_1 \leq (x_1 \cdots x_n)^{\frac{1}{n}} \leq x_n$.

12. If $a_1 + \cdots + a_n \geq 0$, then $|a_1 + \cdots + a_n| = a_1 + \cdots + a_n$. Using $a \leq |a|$ gives us $|a_1 + \cdots + a_n| = a_1 + \cdots + a_n \leq |a_1| + \cdots + |a_n|$.

13. We have $(a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = \underbrace{\left(\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right) + \cdots + \left(\frac{a_{n-1}}{a_n} + \frac{a_n}{a_{n-1}}\right)\right)}_{n(n-1)/2} + n$. We also know that $\frac{x}{y} + \frac{y}{x} \geq 2$.

    $\Rightarrow (a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \geq n + 2.\frac{n(n-1)}{2} = n^2$.

14. The given inequality is equivalent to $(a+b)\sqrt{\frac{a+b}{2}} \geq 2\sqrt{ab}\frac{\sqrt{a}+\sqrt{b}}{2}$, which can be obtained by multiplying the inequalities $a + b \geq 2\sqrt{ab}$ and $\sqrt{\frac{a+b}{2}} \geq \frac{\sqrt{a}+\sqrt{b}}{2}$.

15. We have $\frac{1}{2}(a+b)+\frac{1}{4}-\sqrt{\frac{a+b}{2}}=\left(\sqrt{\frac{a+b}{2}}-\frac{1}{2}\right)^2\geq 0$. Therefore, $\frac{1}{2}(a+b)+\frac{1}{4}\geq\sqrt{\frac{a+b}{2}}$.

16. Since $a(x+y-a)-xy=ax-xy+a(y-a)=(y-a)(a-x)$ and $y\geq a\geq x$, it follows that $(y-a)(a-x)\geq 0$. Therefore, $a(x+y-a)\geq xy$.

17. We have proven that $\frac{a+b}{2}\geq\frac{2}{\frac{1}{a}+\frac{1}{b}}$. Using this we can write that $\frac{\frac{1}{x-1}+\frac{1}{x+1}}{2}\geq\frac{2}{x-1+x+1}$ or $\frac{1}{x-1}+\frac{1}{x+1}\geq\frac{2}{x}$.

18. Following from prevvious problem we have $\frac{1}{3k+1}+\frac{1}{3k+3}=\frac{1}{(3k+2)-1}+\frac{1}{(3k+2)+1}>\frac{2}{3k+2}$. Therefore, $\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}>\frac{3}{3k+2}$.

Now we will prove that $\frac{3}{3k+2}>\frac{1}{2k+1}+\frac{1}{2k+2}$.

We find that $\frac{1}{2k+1}+\frac{1}{2k+2}-\frac{3}{3k+2}=\frac{-k}{(2k+1)(2k+2)(3k+3)}<0$.

19. The given inequality is equivalent to $\left(\frac{2}{a+b}-1\right)^2\leq\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)$. We have $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)-\left(\frac{2}{a+b}-1\right)^2=\frac{1}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{4}{(a+b)^2}+\frac{4}{a+b}=\frac{1}{ab}-\frac{4}{(a+b)^2}+\frac{4}{a+b}-\frac{a+b}{ab}=\frac{(a-b)^2[1-(a+b)]}{ab(a+b)^2}$ and $0\leq a\leq\frac{1}{2}, 0\leq b\leq\frac{1}{2}$. Then $\frac{(a-b)^2[1-(a+b)]}{ab(a+b)^2}\geq 0$, and therefore $\left(\frac{2}{a+b}-1\right)^2\leq\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)$.

20. The given inequality is equilavent to $(2k+1)\sqrt{3k+4}<(2k+2)\sqrt{3k+1}\Rightarrow(2k+1)^2(3k+4)<(2k+2)^2(3k+1)$, and it holds because $(2k+2)^2(3k+1)-(2k+1)^2(3k+4)=k>0$.

21. We know that $1<2<2^2<\cdots<2^{n-1}$ and the number of positive integers $1,2,\ldots,2^{n-1}$ is equal to n. Thus, $2^{n-1}\geq n$.

22. Consider a one meter long rope. Suppose we painted $\frac{1}{3}$m of this rope on first day, $\frac{1}{5}$ of the remaining $\frac{2}{3}$ m on second day and so on. We will find that sum of painted parts is less than 1 m.

Hence, we deduce that $\frac{1}{3}+\frac{2}{3}\cdot\frac{1}{5}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7}+\cdots+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{100}{101}\cdot\frac{1}{103}<1$.

23. Observe that $1-a\geq a$ and $1-b\geq b$. And thus,

$\frac{1-a}{1-b}+\frac{1-b}{1-a}=\frac{(1-a)^2+(1-b)^2}{(1-a)(1-b)}=\frac{[(1-a)-(1-b)]^2+2(1-a)(1-b)}{(1-a)(1-b)}=\frac{(a-b)^2}{(1-a)(1-b)}+2\leq\frac{(a-b)^2}{ab}+2=\frac{a}{b}+\frac{b}{a}$.

24. $\sum_{i=1}^{n}\frac{1}{1-a_i}\sum_{i=1}^{n}(1-a_i)=\underbrace{\left(\frac{1-a_1}{1-a_2}+\frac{1-a_2}{1-a_1}\right)+\cdots+\left(\frac{1-a_{n-1}}{1-a_n}+\frac{1-a_n}{1-a_{n-1}}\right)}_{n(n-1)/2}+n$.

Using the inequality of previous problem we have $\sum_{i=1}^{n}\frac{1}{1-a_i}\sum_{i=1}^{n}(1-a_i) \leq \underbrace{\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)+\cdots+\left(\frac{a_{n-1}}{a_n}+\frac{a_n}{a_{n-1}}\right)}_{n(n-1)/2}+n = \sum_{i=1}^{n}\frac{1}{a_i}\sum_{i=1}^{n}a_i.$

25. Observe that if $n \geq 4$, then $1+\frac{1}{2^3}+\cdots+\frac{1}{n^3} = 1+\frac{2-1}{2^3}+\cdots+\frac{n-(n-1)}{n^3}$

$= \frac{5}{4}-\left(\frac{1}{2^3}-\frac{1}{3^2}\right)-\left(\frac{1}{3^3}-\frac{1}{4^2}\right)-\cdots-\frac{n-1}{n^3} < \frac{5}{4}$

because $\frac{k}{(k+1)^3} > \frac{1}{(k+2)^2}\ \forall\, k \in \mathbb{N}$.

26. For the given conditions $(1-a)(1-b) \geq 0 \Rightarrow a+b-1 \leq ab$. Thus,

$\frac{1}{1+a+b}-\left(1-\frac{a+b}{2}\right) = \frac{a+b}{2(1+a+b)}(a+b-1) \leq \frac{1}{3}ab.$

27. Squaring $(x-y)^2 < (1-xy)^2 \Rightarrow x^2+y^2-2xy < 1-2xy+x^2y^2 \Rightarrow (1-x^2)(1-y^2) > 0$, which is true for given values of x and y.

28. Multiplying both sides by abc, $a^2+b^2+c^2 \geq 2bc+2ac-2ab \Rightarrow (a+b-c)^2 \geq 0$, which is true.

29. Multiplpying both sides by abc, $bc+ac-ab < 1 \Rightarrow ab > bc+ca-1$. We know that $ab \leq \frac{a^2+b^2}{2}$ from A.M.-G.M. inequality and $a^2+b^2 = \frac{5}{3}-c^2 \Rightarrow ab \leq \frac{5}{6}-\frac{c^2}{2}$.

Rearranging the inequality, $c(a+b) \leq \frac{11}{6}-\frac{c^2}{2}$. We know that $(a+b)^2 \leq 2(a^2+b^2)$

$\Rightarrow c(a+b) \leq c\sqrt{2\left(\frac{5}{4}-c^2\right)}$. Since $c(a+b) \leq \frac{5}{6}+ab$ and $ab < 1$, then $c(a+b) < \frac{11}{6}$. Thus, $ab > c(a+b)-1$.

30. Given $3(1+a^2+a^4) \geq (1+a+a^2)^2 \Rightarrow 2(1-a)^2(1+a+a^2) \geq 0$, which is true because $1+a+a^2 = \left(a+\frac{1}{2}\right)^2+\frac{3}{4}$, which is always positive.

31. $a^2+b^2 \geq \frac{(a+b)^2}{2} = 8$ and $c^2+d^2 \geq \frac{(c+d)^2}{2} = 18$.

Now $(ac+bd)^2+(ad-bc)^2 = (a^2+b^2)(c^2+d^2) \geq 8.18 = 144$.

32. $x_1^2+x_2^2+\cdots+x_{2n}^2+na^2-a\sqrt{2}(x_1+x_2+\cdots+x_{2n}) \geq 0$

$\Rightarrow \left(x_1-\frac{a}{\sqrt{2}}\right)^2+\left(x_2-\frac{a}{\sqrt{2}}\right)^2+\cdots+\left(x_{2n}-\frac{a}{\sqrt{2}}\right)^2 \geq 0$, which is true.

33. From A.M.-H.M. inequaltiy $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$. Thus,

$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$

Now $bc+ca+ab \leq (a+b+c)^2 \leq (\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2$

$\Rightarrow \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le \frac{1}{2}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)$

Hence proved.

34. We have to prove that $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)<0$.

$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2) = a^2b^2(a-b)-a^2c^2(a-c)+b^2c^2(b-c)$

$= a^2b^2(a-b)-a^2c^2(a-b+b-c)+b^2c^2(b-c) = (a-b)(b-c)[a^2b+a^2c-c^2b-c^2a]$

$= (a-b)(b-c)[b(a^2-c^2)+ac(a-c)] = (a-b)(b-c)(a-c)[ab+bc+ac]$, which is obviously less than zero.

35. Clearly, $2(a^3b+b^3c+c^3a) \ge a^3(b+c)+b^3(a+c)+c^3(a+b)$ using rearrangement inequality.

Using A.M.-G.M. inequality $a^3b+b^3a \ge 2a^2b^2$, $b^3c+c^3b \ge 2b^2c^2$, $c^3a+ca^3 \ge 2c^2a^2$.

From these two inequalities we have the desired inequality.

36. The given inequality is equivalent to $\frac{y}{x}+\frac{y}{z}+\frac{x}{y}+\frac{z}{y} \le \frac{x}{z}+2+\frac{z}{x}$.

We know that $\frac{y}{x}+\frac{x}{y} \ge 2$, $\frac{y}{z}+\frac{z}{y} \ge 2$ and $\frac{x}{z}+\frac{z}{x} \ge 2$.

Rewriting $\left(\frac{y}{x}-\frac{z}{x}-1\right)+\left(\frac{y}{z}-\frac{x}{z}-1\right)+\left(\frac{x}{y}+\frac{z}{y}\right) \le 0$

$\because x \le y \le z$, we have $\frac{y}{x} \le \frac{z}{x}$ and $\frac{y}{z} \le 1$. Also, $\frac{x}{y} \le 1$ and $\frac{z}{y} \ge 1$. Hence proved.

37. $\sqrt{1+\sqrt{a}} < \sqrt{\sqrt{a}+\sqrt{a}}$. Now $\sqrt{\sqrt{a}+\sqrt{a}} = \sqrt{2\sqrt{a}}$ and $\sqrt{2\sqrt{a}} < \sqrt{a\sqrt{a}} = a^{\frac{3}{4}} < a$ since $a \ge 2$.

Similarly, $\sqrt{1+\sqrt{a+\sqrt{a}^2}} < \sqrt{\sqrt{a+\sqrt{a}^2}+\sqrt{a+\sqrt{a}^2}}$ and $\sqrt{\sqrt{a+\sqrt{a}^2}+\sqrt{a+\sqrt{a}^2}} = \sqrt{2\sqrt{a+a}} = \sqrt{2\sqrt{2a}}$

$\sqrt{2\sqrt{2a}} < \sqrt{a\sqrt{2a}} = 2^{\frac{1}{4}}a^{\frac{3}{4}} <\le a$ since $a \ge 2$

Similarly for kth term, $\sqrt{1+\sqrt{a+\cdots+\sqrt{a}^k}} < \sqrt{\sqrt{a+\cdots+\sqrt{a}^k}+\sqrt{a+\cdots+\sqrt{a}^k}}$

Now $\sqrt{\sqrt{a+\cdots+\sqrt{a}^k}+\sqrt{a+\cdots+\sqrt{a}^k}} = \sqrt{2\sqrt{a+\cdots+\sqrt{a}^k}}$

$= \sqrt{2\sqrt{a+\cdots+a^{\frac{k}{2^k}}}} < \sqrt{2a} < a$

Adding all such terms we have the required inequality.

38. Let $x = n+\epsilon$, where n is the integral part and ϵ is the fractional part. Then

$5n+[5\epsilon] \ge n+\frac{2n+[2\epsilon]}{2}+\frac{3n+[3\epsilon]}{3}+\frac{4n+[4\epsilon]}{4}+\frac{5n+[5\epsilon]}{5}$

$\Rightarrow [5\epsilon] \geq \frac{[2\epsilon]}{2} + \frac{[3\epsilon]}{3} + \frac{[4\epsilon]}{4} + \frac{[5\epsilon]}{5}$

Now we consider different ranges of ϵ. If $0 \leq \epsilon < \frac{1}{5}$, the inequality becomes $0 \geq 0$.

If $\frac{1}{5} \leq \epsilon \leq \frac{1}{4}$, the inequality becomes $1 \geq 0$.

If $\frac{1}{4} \leq \epsilon \leq \frac{1}{3}$, the inequality becomes $1 \geq \frac{9}{20}$.

If $\frac{1}{3} \leq \epsilon \leq \frac{1}{2}$, the inequality becomes $1 \geq \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

if $\frac{1}{2} \leq \epsilon \leq 1$, the inequality becomes $2 \geq \frac{1}{2} + \frac{1}{3} + \frac{2}{4} + \frac{2}{5}$.

Thus, in all cases the inequality holds.

39. For $n = 1$, we have $(1!)^2 = 1^2 = 1$ and $1^1 = 1$. So $1 \geq 1$, the inequality holds for $n = 1$.

For $n = 2$, we have $(2!)^2 = 4$ and $2^2 = 4$, so the inequality holds true for $n = 2$.

Let the inequality holds for $n = k$ i.e. $(k!)^2 \geq k^k$.

For $n = k + 1$, we have to prove that $[(k+1)!]^2 \geq (k+1)^{k+1}$.

Now $[(k+1)!]^2 = [(k+1)k!]^2 = (k+1)^2(k!)^2$ and $(k+1)^{k+1} = (k+1)^k.(k+1)$.

So we have to show that $(k+1)^2(k!)^2 \geq (k+1)^k.(k+1) \Rightarrow (k+1).k^k \geq (k+1)^k$

$\Rightarrow k + 1 \geq \left(\frac{k+1}{k}\right)^k$

Now $\left(1 + \frac{1}{k}\right)^k$ has an upper bound of e, while $k + 1$ is an increasing sequence. We can prove again by induction that $k + 1 \geq \left(1 + \frac{1}{k}\right)^k$.

Hence the inequality is proved.

40. We can rewrite given expression as $\left(x^3 + \frac{x^2}{2}\right)^2 + \left(x^2 - \frac{x}{2}\right)^2 + (1 + x + x^2)$.

Now we know that $1 + x + x^2 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$, and thus, the inequality is proved.

41. Given $\alpha^2 \geq \beta\gamma$. Taking log of both sides $2\log\alpha \geq \log\beta + \log\gamma$.

From A.M.-G.M. inequality we have $\log\beta + \log\gamma \geq 2\sqrt{\log\beta\log\gamma}$

$\therefore \log\alpha \geq \sqrt{\log\beta\log\gamma}$. Squaring we prove the inequality.

42. $\log_4 5 > 1.1$ because $4^{1.1} \approx 4.86$. $\log_5 6 > 1.1$ because $5^{1.1} \approx 5.59$. $\log_6 7 > 1.1$ because $6^{1.1} \approx 7.32$, and $\log_7 8 > 1.1$ because $7^{1.1} \approx 8.48$.

Thus, $\log_4 5 + \log_5 6 + \log_6 7 + \log_7 8 > 4.4$.

43. $\frac{n}{3.5...(2n+1)} = \frac{1}{2} \cdot \frac{2n}{3.5...(2n+1)} = \frac{1}{2}\left(\frac{1}{3.5...(2n-1) - \frac{1}{3.5...(2n+1)}}\right)$

Let $S = \frac{1}{3} + \frac{2}{3.5} + \cdots + \frac{n}{3.5...(2n+1)} = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{3.5}\right) + \cdots + \left(\frac{1}{3.5....(2n-1)} - \frac{1}{3.5...(2n+1)}\right)\right]$

$= \frac{1}{2}\left(1 - \frac{1}{3.5...(2n+1)}\right) < \frac{1}{2}$. Hence proved.

44. $\prod_{i=2}^{n} \frac{i^3+1}{i^3-1} = \prod_{i=2}^{n} \frac{i+1}{i-1} \cdot \frac{i^2-i+1}{i^2+i+1}$

Now $\prod_{i=2}^{n} \frac{i+1}{i-1} = \frac{n(n+1)}{2}$ and $\prod_{i=2}^{n} \frac{i^2-i+1}{i^2+i+1} = \frac{3}{n^2+n+1}$

If P is the given product then $P = \frac{3n(n+1)}{2(n^2+n+1)}$

Now $\lim_{n\to\infty} \frac{3n(n+1)}{2(n^2+n+1)} = \frac{3}{2}$. Thus, $P < \frac{3}{2}$.

45. For $n = 1$, $1.1! < (1+1)!$, which is true. Let the inequality be true for $n = k$ i.e. $1.1! + 2.2! + \cdots + k.k! < (k+1)!$

Now, for $n = k+1$ the inequality becomes $1.1! + 2.2! + \cdots + k.k! + (k+1)(k+1)! < (k+1)! + (k+1)(k+1)! = (k+1)![k+1+1] = (k+2)!$

Thus, the inequality is proven by mathematical induction.

46. $1 + \frac{1}{k^2} = \frac{k^2+1}{k^2} = \frac{k^2-1}{k^2} + \frac{2}{k^2}$.

$\prod_{k=2}^{n} \frac{k^2-1}{k^2} = \frac{1.3}{2^2} \cdot \frac{2.4}{3^2} \cdots . \frac{(n-1)(n+1)}{n^2}$

$= \frac{1.2.3...(n-1)}{2.3.4...n} \cdot \frac{3.4.5...(n+1)}{2.3.4...n} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$

Now $\frac{1}{2} + \frac{1}{2n} \leq \frac{1}{2} + \frac{1}{4}$ because for $n \geq 2$, $\frac{1}{2n} \leq \frac{1}{4}$.

Now $\sum_{k=1}^{\infty} \frac{2}{k^2} = \frac{\pi^2}{3}$, which is upper bound of the sum. $\Rightarrow \sum_{k=2}^{\infty} \frac{2}{k^2} = \frac{\pi^2}{3} - \frac{2}{1^2}$. We see that the sum of two components is less than 2.

Thus, the inequality is proven.

47. We have proven in previous example that $\prod_{k=2}^{n} \frac{k^2-1}{k^2} = \frac{1}{2} + \frac{1}{2n}$. If we want to maximize L.H.S. then we pick continuous natural numbers starting from 2. For $n \geq 2$, $\frac{1}{2n} \leq \frac{1}{4}$, however, sum is greater than $\frac{1}{2}$.

48. We can rewrite the inequality as $\frac{1}{2}-\left(\frac{1}{3}-\frac{1}{4}\right)-\cdots-\left(\frac{1}{999}-\frac{1}{1000}\right)$. Considering first three terms we have $\frac{1}{2}-\frac{1}{12}-\frac{1}{30}=\frac{23}{60}$, which is less than $\frac{2}{5}$ and further terms will make it lesser. Hence proved.

49. Given $\frac{a+b}{1+a+b}\leq\frac{a}{1+a}+\frac{b}{1+b}\Rightarrow\frac{a}{1+a+b}-\frac{a}{1+a}+\frac{b}{1+a+b}-\frac{b}{1+b}\leq 0$

$\Rightarrow\frac{a+a^2-a-a^2-ab}{(1+a+b)(1+a)}+\frac{b+b^2-b-b^2-ab}{(1+a+b)(1+b)}\leq 0$, which is clearly true.

50. Multiplying both sides by $2(1+a)(1+b)(2+a+b)$, we get

$2(a+b)(1+a)(1+b)\geq[a(1+b)+b(1+a)](2+a+b)\Rightarrow(a-b)^2\geq 0$, which proves the inequality.

51. Let $S_i=\sum\limits_{j=1}^{i}a_j$ and $b_i=\frac{S_i}{i}$, which makes L.H.S. $\sum\limits_{i=1}^{n}\frac{b_i}{i}$.

Clearly, $a_i=b_i-\frac{i-1}{i}b_{i-1}$.

$\sum\limits_{i=1}^{n}a_i=b_1+\left(b_2-\frac{b_1}{2}\right)+\cdots+\left(b_n-\frac{n-1}{n}b_{n-1}\right)$

$=b_n+\frac{b_{n-1}}{n}+\frac{b_{n-2}}{n-1}+\cdots+\frac{b_1}{2}=b_n+\sum\limits_{i=1}^{n}\frac{b_i}{i+1}$

Now $2\sum\limits_{i=1}^{n}a_i-\sum\limits_{i=1}^{n}\frac{b_i}{i}=\left(2-\frac{1}{n}\right)+\sum\limits_{i=1}^{n}\left(\frac{2}{i+1}-\frac{1}{i}\right)b_i$

$\frac{2}{i+1}-\frac{1}{i}=\frac{i-1}{i(i+1)}$ so the sum $2\sum\limits_{i=1}^{n}a_i-\sum\limits_{i=1}^{n}\frac{b_i}{i}\geq 0$

Hence proved.

52. We observe that if $a=2,b=3$ and $c=7$ then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{41}{42}$. If $a=2,b=3,c>7$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{41}{42}$, if $a=2,b>3,c>4$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{41}{42}$, and if $a>2,b>3,c>4$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{41}{42}$.

Thus, we see that maximum value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is $\frac{41}{42}$.

53. *Nesbitt's inequality:* From A.M.-H.M. inequality $\frac{(x+y)+(y+z)+(z+x)}{3}\geq\frac{3}{\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}}$

$\Rightarrow[(x+y)+(y+z)+(z+x)]\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)\geq 9$

$\Rightarrow 2\frac{x+y+z}{x+y}+2\frac{x+y+z}{y+z}+2\frac{x+y+z}{z+x}\geq 9$

$\Rightarrow\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}$.

Now $\frac{4x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} > 2$ is to be proven.

$\Rightarrow \frac{3x}{y+z} \geq \frac{1}{2}$

If $x = y = z$ then it is greater than $\frac{1}{2}$.

If we make $y >> x$ and $z >> x$ to so that it is less than $\frac{1}{2}$ then the other terms of the inequality becomes greater than 2.

If $x >> y = z$ then it is greater than $\frac{1}{2}$.

Thus, inequality holds in all cases.

54. Since $a, b, c, d > 0$ we can write $a + b + c, a + b + d, a + c + d, b + c + d > a + b + c + d$.

So $\frac{a}{a+b+d} > \frac{a}{a+b+c+d}, \frac{b}{a+b+c} > \frac{b}{a+b+c+d}, \frac{c}{b+c+d} > \frac{c}{a+b+c+d}$, and $\frac{d}{a+c+d} > \frac{d}{a+b+c+d}$

Adding we get $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} > 1$.

Similarly, $\frac{a}{a+b+d} < \frac{a}{a+b}$, and $\frac{b}{a+b+c} < \frac{b}{a+b}$. Adding $\frac{a}{a+b+d} + \frac{b}{a+b+c} < 1$.

Also, $\frac{c}{b+c+d} < \frac{c}{c+d}$, and $\frac{d}{a+c+d} < \frac{d}{c+d}$. Adding $\frac{c}{b+c+d} + \frac{d}{c+d} < 1$.

Adding all these we prove the required inequality.

55. If $a + b \leq c + d$ then $a \leq c$ or $b \leq d$. If $a \leq c$ then from first condition $b - d > (c - d)(c + a) > c - a \Rightarrow a + b > c + d$.

Similarly for other condition same can be proved. And thus, the inequality is proven.

56. $(b-a)(9-a^2) + (c-a)(9-b^2) + (c-b)(9-c^2) = 9b + c(9-b^2) + (c-b)(9-c^2) = 18c - c^3 + bc(c-b) \leq 18c - c^3 + \frac{1}{4}c^3 = 18c - \frac{3}{4}c^2$

We can use calculus to apply maxima and minima to prove the inequality.

57. We assume that all of them are greater than $\frac{1}{4}$ so $abc(1-a)(1-b)(1-c) > \frac{1}{16}$.

From A.M.-G.M. inequality $\frac{a+1-a}{1} \geq \sqrt{a(1-a)} \Rightarrow a(1-a) \leq \frac{1}{4}$. Similarly, $b(1-b) \leq \frac{1}{4}$ and $c(1-c) \leq \frac{1}{4}$.

Multiplying $abc(1-a)(1-b)(1-c) \leq \frac{1}{16}$, which leads to a contradiction. Thus, at least one of the assumed numbers is less that $\frac{1}{4}$.

58. $\sqrt{a + \frac{1}{4}(b-c)^2} \leq a + \frac{b+c}{2}$ if $a, b, c > 0$ and $a + b + c = 1$.

Adding for all the terms we get $2(a + b + c) = 2$.

59. Using Cauchy-Schwarz inequality $\left(\sqrt{a+\frac{1}{4}(b-c)^2}+\sqrt{b}+\sqrt{c}\right)^2 \le (1^2+1^2+1^2)\Big(a+\frac{1}{4}(b-c)^2+b+c\Big) = 3\Big(1+\frac{1}{4}(b^2+c^2-2bc)\Big) = 4\Big(1+\frac{1}{4}[(1-a)^2-4bc]\Big)$

From A.M.-G.M. inequality $bc \le \frac{(b+c)^2}{4} \Rightarrow bc \le \frac{(1-a)^2}{4}$.

Thus above inequality becomes $\left(\sqrt{a+\frac{1}{4}(b-c)^2}+\sqrt{b}+\sqrt{c}\right)^2 \le 3\Big(1+\frac{1}{4}[(1-a)^2-(1-a)^2]\Big) = 3.$

Hence proved.

60. We know that $\frac{x}{y}+\frac{y}{x} \ge 2$. Given expression is $\frac{a^4}{b^4}+\frac{b^4}{a^4}-\frac{a^2}{b^2}-\frac{b^2}{a^2}+\frac{a}{b}+\frac{b}{a}$

$=\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-2-\left(\frac{a}{b}+\frac{b}{a}\right)^2+2+\frac{a}{b}+\frac{b}{a}$

Substityting $\frac{a}{b}+\frac{b}{a}=y$ we have the expression as y^4-5y^2+y+4. Applying maxima-minima we find the minimum value as 2.

61. Since $x_1+x_2+\cdots+x_n=1$, it follows by problem 9 that there are two numbers such that one of them is not greater than $\frac{1}{n}$, and the other one is not less than $\frac{1}{n}$. WLOG we can assume that $x_1 \le x_2$. Substituting x_1 by $\frac{1}{n}$, x_2 by $x_1+x_2-\frac{1}{n}$. Then we obtain numbers $\frac{1}{n}, x_1+x_2-\frac{1}{n}, x_3, \ldots, x_n$ such that

$\frac{(1-x_1)\cdots(1-x_n)}{x_1\cdots x_n} \ge \frac{\left(1-\frac{1}{n}\right)\left(1-x_1-x_2+\frac{1}{n}\right)\cdots(1-x_n)}{\frac{1}{n}\left(x_1+x_2-\frac{1}{n}\right)\cdots x_n}$ from problem 16. $\frac{(1-x_1)(1-x_2)}{x_1x_2} = 1+\frac{1-(x_1+x_2)}{x_1x_2}$. Repeating this step, we obtain n numbers less than $\frac{1}{n}$. For these numbers L.H.S. of the inequality is equal to $(n-1)^n$ and is not greater than $\frac{(1-x)1)\cdots(1-x_n)}{x_1\cdots x_n}$.

62. Let $\sqrt[n]{x_1\ldots x_n}=y$. According to problem 11, WLOG one can assume that $x_1 \le y \le x_2$, and therefore, for numbers $y, \frac{x_1x_2}{y}, x_3, \ldots, x_n$

$\frac{1}{x_1}+\cdots+\frac{1}{1+x_n} \ge \frac{1}{1+y}+\frac{1}{1+\frac{x_1x_2}{y}}+\cdots+\frac{1}{1+x_n}$. Since $\frac{1}{1+x_1}+\frac{1}{1+x_2} \ge \frac{1}{1+y}+\frac{1}{1+\frac{x_1x_2}{y}}$ and $\frac{1}{1+x_1}+\frac{1}{1+x_2} = 1+\frac{1-x_1x_2}{1+x_1+x_2+x_1x_2}$.

After a finite numbere of steps, we deduce that $\frac{1}{1+x_1}+\frac{1}{1+x_2}+\cdots+\frac{1}{1+x_n} \ge \frac{1}{1+y}+\cdots\frac{1}{1+y} = \frac{n}{1+\sqrt[n]{x_1\ldots x_n}}$.

63. If $a=b=c=d=\frac{1}{4}$, then we have equality. Let $a<\frac{1}{4}<b$, then

If $c+d-\frac{176}{27}cd<0$, then $A=ab\Big(c+d-\frac{176}{27}cd\Big)+cd(a+b) \le cd(a+b) \le \left(\frac{a+b+c+d}{3}\right)^3 = \frac{1}{27}$

If $c+d-\frac{176}{27}cd \geq 0$, then $A \leq \frac{1}{4}\left(a+b-\frac{1}{4}\right)\left(c+d-\frac{176}{27}cd\right)+cd\left(\frac{1}{4}+a+b-\frac{1}{4}\right)$

Hence, we must prove inequality for numbers $a_1=\frac{1}{4}, b_1=a+b-\frac{1}{4}, c_1=d, d_1=d$.

Similarly, either one can prove the inequality for numbers a_1, b_1, c_1, d_2 or it will be sufficient to prove the inequality for the case among a_1, b_1, c_1, d_1 two numbers are equaal to $\frac{1}{4}$. Continuing in this way we obtain that it is sufficient to prove the inequality for numbers $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$. In this case the inequality holds.

64. Let $x \geq y \geq z$, then $y \leq \frac{1}{2}$. Therefore, $0 \leq y(x+z)+xz(1-2y) \leq y\left(\frac{1}{3}+x+z-\frac{1}{3}\right)+\frac{1}{3}\left(x+z-\frac{1}{3}\right)(1-2y)$

Therefore, if we substitute the numbers x, y, z by the numbers $\frac{1}{3}, y, x+z-\frac{1}{3}$ in the expression $xy+yz+xz-2xyz$, then its value does not decrease. Continuing similarly, we can substitute the numbers $\frac{1}{3}, y, x+z-\frac{1}{3}$ by $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. Hence, we deduce that $xy+yz+zx-2xyz \leq \frac{1}{9}+\frac{1}{9}+\frac{1}{9}-\frac{2}{27}=\frac{7}{27}$.

65. Let us first prove the following lemma.

Lemma If $a \leq b$ and $x>0$, then $(a-x)^{12}+(b+x)^{12}>a^{12}+b^{12}$

$(a-x)^{12}+(b+x)^{12}-a^{12}-b^{12}=C_1^{12}x(b^{11}-a^{11})+C_2^{12}x^2(b^{10}-a^{10})+\cdots+2x^{12}>0$

Let $y_i=\sqrt{3}x_i, i=1,2,\ldots,1997$. We have $-1 \leq y_i \leq 3$ and $y_1+\cdots+y_{1997}=-954$ and $x_1^{12}+\cdots+x_{1997}^{12}=\frac{y_1^{12}+\cdots+y_{1997}^{12}}{3^6}$

If any two numbers among the numbers $y_1,\ldots,y_{1997}$ belong to $(-1,3)$ then aaccording to lemma, we can substitute these two numbers such that one of them is equal to -1 or 3, then first two conditions hold, and $y_1^{12}+\cdots+y_{1997}^{12}$ increases.

Therefore, the sum $y_1^{12}+\cdots+y_{1997}^{12}$ is maximum if we substitute these by either $-1,\ldots,-1,3,\ldots,3$ or $-1,\ldots,-1,3,\ldots,3,a$ where $a \in (-1,3)$.

Taking into consideration the second condition, we obtain that only the second case is possible so that $k=\frac{a+2}{4}+1735$, where k is the number of -1's. Since $\frac{a+2}{4} \in \mathbb{Z}$ and $a \in (-1,3)$ we must have $a=2$.

Therefore, the greatest value of $x_1^{12}+\cdots+x_{1997}^{12}$ is $\frac{1736+260.3^{12}+2^{12}}{3^6}=189548$.

66. Let $\frac{\alpha_1+\cdots+\alpha_n}{n}=\phi$. If $\alpha_1=\cdots=\alpha_n=\phi$ then

$\cos\alpha_1\cos\alpha_2\cdots\cos\alpha_n(\tan\alpha_1+\tan\alpha_2+\cdots+\tan\alpha_n)=\cos^n\phi.n.\tan\phi=n\sin\phi\cos^{n-1}\phi$

$= \ n\sqrt{(n-1)^{n-1}\sin^2\phi\Big(\frac{\cos^2\phi}{n-1}\Big)^{n-1}} \ \le \ n\sqrt{(n-1)^{n-1}\left(\frac{\sin^2\phi+\frac{\cos^2\phi}{n-1}+\cdots+\frac{\cos^2\phi}{n-1}}{n}\right)^n} \ =$
$n\sqrt{(n-1)^{n-1}\frac{1}{n^n n}} = \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}.$

Let there be two numbers α_1 and α_2 such that $\alpha_1 < \phi < \alpha_2$.

$\cos\alpha_1\cos\alpha_2 = \frac{1}{2}[\cos(\alpha_1+\alpha_2)+\cos(\alpha_1-\alpha_2)] < \frac{1}{2}[\cos(\alpha_1+\alpha_2)+\cos(2\phi-(\alpha_1+\alpha_2))]$

$= \cos\phi\cos(\alpha_1+\alpha_2-\phi)$

We have $\cos\alpha_1\cos\alpha_2\cdots\cos\alpha_n(\tan\alpha_1+\tan\alpha_2+\cdots+\tan\alpha_n)$

$= \sin(\alpha_1+\alpha_2)\cos\alpha_3\ldots\cos\alpha_n + \cos\alpha_1\cos\alpha_2\cos\alpha_3\ldots\cos\alpha_n(\tan\alpha_3+\cdots+\tan\alpha_n) <$ $\sin(\alpha_1+\alpha_2)\cos\alpha_3\ldots\cos\alpha_n + \cos\phi\cos(\alpha_1+\alpha_2-\phi)\cos\alpha_3\ldots\cos\alpha_n(\tan\alpha_3+\cdots+\tan\alpha_n)$

$= \cos\phi\cos(\alpha_1+\alpha_2-\phi)\cos\alpha_3\ldots\cos\alpha_n(\tan\phi+\tan(\alpha_1+\alpha_2-\phi)+\tan\alpha_3+\ldots+\tan\alpha_n)$

Continuing similarly for the numbers ϕ, $\alpha_1+\alpha_2-\phi$, $\alpha_3, \ldots, \alpha_n$ we obtain a new sequence, two of whose terms are equal to ϕ. Repeating these steps $n-1$ times, we obtain a sequence, $n-1$ of whose terms are equal to ϕ and the nth term is equalt to $n\phi-(n-1)\phi = \phi$. Hence, we have

$\cos\alpha_1\cos\alpha_2\cdots\cos\alpha_n(\tan\alpha_1+\tan\alpha_2+\cdots+\tan\alpha_n) < \cos^n\phi . n . \tan\phi \le \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n-2}{2}}}.$

Equallity holds if and only if $\alpha_1 = \cdots = \alpha_n = \phi$, where $\phi = \tan^{-1}\frac{1}{\sqrt{n-1}}$.

67. Consider $x \ge 0, y \ge 0, x+y \le \frac{2}{3}$ and $k \ge 2, k \in \mathbb{N}$, then

$x^k(1-x) + y^k(1-y) \le (x+y)^k(1-x-y)$

If $x+y=0$, then above inequality holds, while if $x+y \ne 0$, then

$\frac{x^k}{(1+x)^k}(1-x) + \frac{y^k}{(x+y)^k}(1-y) \le \left(\frac{x}{x+y}\right)^2(1-x) + \left(\frac{y}{x+y}\right)^2(1-y) = \frac{(x+y)^2(1-x-y)+xy[3(x+y)-2]}{(x+y)^2} \le 1-x-y.$

Let $x_{i+1} \ge x_i \ge 0$, $i = 1, \ldots, n-1$, $x_1+\cdots+x_n = 1$ and $n \ge 3$. Then $(n-2)x_1 + (n-2)x_2 \le (x_3+\cdots+x_n)+(x_3+\cdots+x_n) = 2-2x_1-2x_2$, and therefore,

$x_1 + x_2 \le \frac{2}{n} \le \frac{2}{3}.$

Therefore, if we substitute the numbers $x_1, \ldots, x_n$ by $0, x_1+x_2, x_3, \ldots, x_n$ then their sum will be equal to 1. Note that

$$\sum_{i=1}^{k} x_i^k(1-x_i) \le (x_1+x_2)^k(1-x_1-x_2) + x_3^k(1-x_3)+\cdots+x_n^k(1-x_n).$$

Repeating this step a finite number of times, we end ip with case $n=2$, i.e., $\sum_{i=1}^{n} x_i^k(1-x_i) \le x^k(1-x)+(1-x)^k x$, and therefore, $\sum_{i=1}^{n} x_i^k(1-x_i) \le a_k$.

Note that $a_1 = \max_{[0;1]}[2x(1-x)] = \frac{1}{2}$, $a_2 = \max_{[0;1]}[x(1-x)] = \frac{1}{2}$, $a_3 = \max_{[0;1]}[x(1-x)(1-2x(1-x))] = \frac{1}{8}$ and so on.

68. Let $x_1 \le x_2 \le \cdots \le x_n$, $n \ge 3$. Now if $x_2 \cdots x_{n-1} = \frac{2(n-1)}{n^{n-1}}$, then we have

$2(n-1)(x_2x_3+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n)-n^{n-1}x_1x_2\ldots x_n = 2(n-1)[(x_1+x_n)(x_2+\cdots+x_{n-1})+x_2x_3+\cdots+x_2x_{n-1}]+x_1x_n[2(n-1)-n^{n-1}x_2\ldots x_{n-1}] \le 2(n-1)[x(1-x)+x_2x_3+\cdots+x_2x_{n-1}+\cdots+x_{n-2}x_{n-1}]$, where $x = x_2+\cdots+x_{n-1}$.

From R.M.S. inequality, it follows that $\frac{x_2^2+\cdots+x_{n-1}^2}{n-2} \ge \left(\frac{x}{n-2}\right)^2$, and therefore, $x_2x_3+\cdots+x_2x_{n-1}+\cdots+x_{n-2}x_{n-1} \le \frac{n-3}{2(n-2)}x^2$, whence

$$A \le 2(n-1)\left[x(1-x)+\frac{n-3}{2(n-2)x^2}\right] = 4(n-2)\cdot\frac{n-1}{2(n-2)}x\left(1-\frac{n-1}{2(n-2)}x\right) \le n-2.$$

If $x_2\ldots x_{n-1} \le \frac{2(n-1)}{n^{n-1}}$, then for $x_1=x_2=\cdots=x_n=\frac{1}{n}$, we have $A=n-2$.

Otherwise if $x_i \ne \frac{1}{n}$ for some value of i, then $x_1 < \frac{1}{n} < x_n$.

Substituting x_1 by $\frac{1}{n}$ and x_n by $x_1+x_n-\frac{1}{n}$, we see that the value of the given expression increases.

Continuing similalry, either we can end the proof of inequality or it will be sufficient to prove the inequality for $x_1=\cdots=x_n=\frac{1}{n}$.

69. If $y_1, y_2, \ldots, y_n \ge 0$ and $y_1+y_2+\cdots+y_n > 0$ then according to problem 67, for $x_i = \frac{y_i}{y_1+y_2+\cdots+y_n}$, $i=1,2,\ldots,n$ we have

$2(n-1)q_np_n^{n-2} - n^{n-1}y_1y_2\ldots y_n \le (n-2)p_n^n$, where $p_n = y_1+y_2+\cdots+y_n$, $q_n = y_1y_2+y_1y_3+\cdots+y_1y_n+\cdots+y_{n-1}y_n$. Therefore, $y_1y_2\ldots y_n = 1$ we have

$$q_n \le \frac{(n-1)p_n^n+n^{n-1}}{2(n-1)p_n^{n-2}}$$

If $x_1 = 0$, then we have following inequality

$$\frac{x_1+\cdots+x_n}{n} \le \frac{(\sqrt{x_2}-\sqrt{x_3})^2+\cdots+(\sqrt{x_2}-\sqrt{x_n})^2+\cdots+(\sqrt{x_{n-1}}-\sqrt{x_n})^2+x_2+\cdots+x_n}{n}$$

or $2\sqrt{x_2x_3}+2\sqrt{x_2x_4}+\cdots+2\sqrt{x_2x_n}+\cdots+2\sqrt{x_{n-2}x_n} \le (n-2)(x_2+\cdots+x_n)$.

The last ineqeuality can be proved using $2\sqrt{ab} \le a + b(a, b) \ge 0$.

If $x_i > 0$, let $y_i = \frac{\sqrt{x_i}}{\sqrt[2n]{x_1 x_2 \dots x_n}}$, $i = 1, 2, \dots, n$, then $y_1 y_2 \dots y_n = 1$, and we need to prove that $\frac{p_n^2 - 2q_n}{n} - 1 \le \frac{(n-1)(p_n^2 - 2q_n) - 2q_n}{n}$, or $q_n \le \frac{(n-2)p_n^2 + n}{2(n-1)}$.

It follows that $q_n \le \frac{(n-2)P_n^n + n^{n-1}}{2(n-1)p_n^{n-1}} \le \frac{(n-2)p_n^2 + n}{2(n-1)}$.

70. For numbers $y_1 = x_1^2, \dots, y_n = x_n^2$, using the inequality of problem 68, we deduce that

$\frac{x_1^2 + \dots + x_n^2}{n} - \sqrt[n]{x_1^2 \cdots x_n^2} \le \frac{(|x_1| - |x_2|)^2 + \dots + (|x_1| - |x_n|)^2 + \dots + (|x_{n-1}| - |x_n|)^2}{n}$

or $(n-1)(x_1^2 + \dots + x_n^2) + n\sqrt[n]{x_1^2 \dots x_n^2} \ge (|x_1| + \dots + |x_n|)^2 \ge (x_1 + \dots + x_n)^2$.

71. WLOG we can assume that $c \ge a, c \ge b$. From A.M.-G.M. inequality $\frac{a+b}{2} \ge \sqrt{ab}$ and we know that $\frac{a}{b} + \frac{b}{a} \ge 2$.

Note that $\frac{b}{c} + \frac{c}{a} - \frac{b}{a} - 1 = \frac{(c-a)(c-b)}{ac} \ge 0$, whence $\frac{b}{c} + \frac{c}{a} - \frac{b}{a} - 1 \ge 0$.

Adding the inequalities we obtain $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3$.

72. WLOG we can assume that $a \ge b \ge c$. We have

$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} + \sqrt{\frac{b+c}{a}} - \sqrt{\frac{b+c}{a}} \ge 2 + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} - \sqrt{\frac{b+c}{a}} \ge 2 + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{2a}} - \sqrt{\frac{b+c}{a}} = 2 + \frac{1}{\sqrt{a}}\left(\sqrt{\frac{b}{1+\frac{c}{a}}} + \sqrt{\frac{c}{2}} - \sqrt{b+c}\right) \ge 2 + \frac{1}{\sqrt{a}}\left(\sqrt{\frac{b}{1+\frac{c}{a}}} + \sqrt{\frac{c}{2}} - \sqrt{b+c}\right) = 2 + \frac{\sqrt{c}}{\sqrt{2a(b+c)}}\left(\sqrt{b+c} - \sqrt{2c}\right) \ge 2$.

73. For positive numbers a, b, c we have from A.M.-G.M. inequality $\frac{a+b}{2} \ge \sqrt{ab}$ and so on. Multiplying these we get the required inequality.

74. Note that three factors on the L.H.S. of the given inequality are positive. If only one factor on the L.H.S. is not positive, then the proof is obvious. Consider the case when all four factors are positive. In this case, from A.M.-G.M. inequality for the numbers $a+b+c-d$ and $b+c+d-a$, $a+b+c-d$ and $d+a+b-c$, $b+c+d-a$ and $c+d+a-b$, $c+d+a-b$ and $d+a+b-c$, we have the following inequalities:

$\sqrt{(a+b+c-d)(b+c+d-a)} \le \frac{(a+b+c-d)+(b+c+d-a)}{2} = b+c$,

$\sqrt{(a+b+c-d)(d+a+b-c)} \le a+b$, $\sqrt{(b+c+d-a)(c+d+a-b)} \le c+d$,

$\sqrt{(c+d+a-b)(d+a+b-c)} \le a+d$.

Multiplying these inequalities we obtain the given inequality.

75. The given inequality is equivalent to $abc \ge (a+b-c)(c+a-b)(b+c-a)$. See problem 76.

76. The given inequality is equivalent to $7abc+a^3+b^3+c^3 > a^2b+ab^2+c^2a+ca^2+b^2c+bc^2$ and the proof follows from previous problem.

77. The given inequality is equivalent to the following inequality:

$\log(a-1)\log(a+1) < \log^2 a$. If $\log(a-1) \leq 0$, then $\log(a-1)\log(a+1) \leq 0 \leq \log^2 a$.

Otherwise, if $\log(a+1) > 0$, then using A.M.-G.M. inequality, we have $2\sqrt{\log(a-1)\log(a+1)} \leq \log(a-1)\log(a+1) = \log(a^2-1) < \log a^2$, which is equilavent to $\log(a-1)\log(a+1) < \log a^2$.

78. Since $a>0, b>0, c>0$, at least two factors on the R.H.S. of the inequality are positive. If only one factor is negative then the proof is obvious. Consider the case in which all three factors on R.H.S. are positive. Using A.M.-G.M. inequality for nonnegative numbers, we have

$\sqrt{(a+b-c)(a+c-b)} \leq \frac{a+b-c+a+c-b}{2} = a$, $\sqrt{(a+b-c)(b+c-a)} \leq b$, $\sqrt{(a+c-b)(b+c-a)} \leq c$.

Multiplying these inequalities we get required inequality.

79. We have $\frac{x^8+y^8}{2} \geq \left(\frac{x^4+y^4}{2}\right)^2 \geq \left(\frac{x^2+y^2}{2}\right)^4 \geq \left(\frac{x+y}{2}\right)^8 = \frac{1}{128}$.

80. Since $a+b=1$, from A.M.-G.M. inequality, we have $\frac{1}{ab} \geq 4$. Using inequality $\frac{a_1+\cdots+a_n}{n} \leq \sqrt{\frac{a_1^2+\cdots+a_n^2}{n}}$, we have

$\frac{\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2}{2} \geq \left(\frac{a+\frac{1}{a}+b+\frac{1}{b}}{2}\right)^2 = \left(\frac{1+\frac{1}{ab}}{2}\right)^2 \geq \left(\frac{1+4}{2}\right)^2 = \frac{25}{4}$.

81. We know that $\frac{a_1+\cdots+a_n}{n} \geq \sqrt[n]{a_1\ldots a_n} \geq \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}$, whence

$$(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right) \geq n^2$$

Using above inequality(since $x_1+\cdots+x_n=1$), we deduce that $\frac{1}{x_1}+\cdots+\frac{1}{x_n} \geq n^2$. Using $\frac{a_1+\cdots+a_n}{n} \leq \sqrt{\frac{a_1^2+\cdots+a_n^2}{n}}$, we have

$\left(x_1+\frac{1}{x_1}\right)^2+\cdots+\left(x_n+\frac{1}{x_n}\right)^2 \geq n\left(\frac{x_1+\frac{1}{x_1}+\cdots+x_n+\frac{1}{x_n}}{n}\right)^2 = n\left(\frac{1+x_1+\cdots+x_n}{n}\right)^2 \geq n\left(\frac{1+n^2}{n}\right)^2 = \frac{(1+n^2)^2}{n}$.

82. From A.M.-G.M. inequality, we have $a^4+b^4 \geq 2a^2b^2$, $b^4+c^4 \geq 2b^2c^2$, $c^4+a^4 \geq 2c^2a^2$.

Adding these, we have $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2$.

Also, from A.M.-G.M. inequality, we have $a^2b^2+b^2c^2 \geq 2ab^2c$, $b^2c^2+c^2a^2 \geq 2abc^2$, $a^2b^2+c^2a^2 \geq 2a^2bc$.

Adding these, we have $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$.

From these two inequalities $a^4 + b^4 + c^4 \geq abc(a+b+c)$.

83. Using that $xy = 1$, we have $x^2 + y^2 = (x-y)^2 + 2xy = (x-y)^2 + 2 \geq 2\sqrt{2}(x-y)$.

84. From A.M.-G.M. inequality, for every value of λ, we have following inequalities

$(6a_i + 1)\lambda^2 \geq 2\lambda\sqrt{6a_i + 1}, i = 1, 2, \dots, 5$

Adding these inequalities and using the fact that $a_1 + \dots + a_5 = 1$, for $\lambda > 0$, we have

$\frac{11+5\lambda^2}{2\lambda} \geq \sum_{i=1}^{5} \sqrt{6a_i + 1}$.

Taking $\lambda = \sqrt{\frac{11}{5}}$, we have the desired inequality.

85. From A.M.-G.M. inequality, we have $5\sqrt{ab} + 3\sqrt{bc} + 7\sqrt{ca} \leq \frac{5(a+b)}{2} + \frac{3(b+c)}{2} + \frac{7(c+a)}{2} = 6a + 4b + 5c$.

86. Since $a^4 + b^4 \geq 2a^2b^2$, we have $2(a^4+b^4) + 17 \geq 4a^2b^2 + 17 > 4(a^2+b^2+4) \geq 16ab$. Therefore, $2(a^4 + b^4) + 17 > 16ab$.

87. We have $1 + \underbrace{b + \dots + b}_{n} \geq (n+1)\sqrt[n+1]{b^n}$, and thus, $\left(\frac{1+nb}{n+1}\right)^{n+1} \geq b^n$.

88. Using the A.M.-G.M. inequality for the numbers $\underbrace{(1+1/n), \dots, (1+1/n)}_{n}, 1$, we have $\frac{\left(1+\frac{1}{n}\right)+\dots+\left(1+\frac{1}{n}\right)+1}{n+1} > \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$ or $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$.

89. This problem can be solved like previous problem.

90. Using the A.M.-G.M. inequality for numbers $\underbrace{1+\frac{m}{n-1}, \cdots, 1+\frac{m}{n-1}}_{n-1}, 1$, we have $\left(1 + \frac{m}{n-1}\right)^{\frac{n-1}{m}} < \left(1+\frac{m}{n}\right)^{\frac{n}{m}}$.

Using the A.M.-G.M. inequality for numbers $\underbrace{1+\frac{m}{n}\cdots, 1+\frac{m}{n}}_{m-1}, 1$, we have $\left(1+\frac{m}{n}\right)^{\frac{n}{m}} < \left(1+\frac{m-1}{n}\right)^{\frac{n}{m-1}}$.

91. Using the A.M.-G.M. inequality, we have $\sqrt[n]{n!} = \sqrt[n]{1\cdots n} < \frac{1+2+\dots+n}{n} = \frac{n+1}{2}$.

92. $S_n + n = n + 1 + \frac{1}{2} + \dots + \frac{1}{n} = 2 + \frac{3}{2} + \dots + \frac{n+1}{n} > n\sqrt[n]{2.\frac{3}{2}.\cdots\frac{n+1}{n}} = n\sqrt[n]{n+1}$.

93. $n - S_n = \frac{1}{2} + \dots + \frac{n-1}{n} > (n-1)\sqrt[n-1]{\frac{1}{2}.\frac{2}{3}.\dots.\frac{n-1}{n}} = (n-1)n^{\frac{1}{1-n}}$.

94. We know that $q^n - 1 = (q-1)(q^{n-1} + q^{n-2} + \cdots + 1)$ and $q > 1$. So the given inequality is reduced to $(q^{n-1} + q^{n-2} + \cdots + 1)(q^{n+1} + 1) \geq 2nq^n$.

Using the A.M.-G.M. inequality for $n > 1$, we have $q^{n-1} + q^{n-2} + \cdots + 1 \geq n.\sqrt[n]{q^{n-1}q^{n-1} \ldots 1}$ and $q^{n+1} + 1 \geq 2q^{\frac{n+1}{2}}$. Multiplying these we get desired inequality.

95. Using the A.M.-G.M. inequality $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10.\sqrt[10]{(abcd)^5} = 10$.

96. Since $a, b, c > 0$, at least two factors on the L.H.S. are positive(see Problem 7). If only one factor on L.H.S. is non-positive, then the proof is obvious. Consider the case in which all three factors on L.H.S. are positive. In this case, we note that

$3 + 3abc = b\left(a - 1 + \frac{1}{b}\right) + c\left(b - 1 + \frac{1}{c}\right) + a\left(c - 1 + \frac{1}{a}\right) + bc\left(a - 1 + \frac{1}{b}\right) + ac\left(b - 1 + \frac{1}{c}\right) + ab\left(c - 1 + \frac{1}{a}\right)$

From the A.M.-G.M. inequality, we have

$3 + 3abc \geq 6\sqrt[3]{a^3b^3c^3\left(a - 1 + \frac{1}{b}\right)^2\left(b - 1 + \frac{1}{c}\right)^2\left(c - 1 + \frac{1}{a}\right)^2}$, which is the given inequality.

97. Let $a + b + c = A$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = B$, then we need to prove that

$$t^3 + (AB - A - B)t^2 + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + A + B - 2AB\right)t + AB + 2 - A - B \geq 0$$

For $t = 1$, inequality holds, since for $t = 1$ it is equivalent to earlier problem for $abc = 1$.

Therefore, for $t = 1$, from above it follows that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq A + B - 3$.

Since $t > 0$, $A = a + b + c \geq 3.\sqrt[3]{abc} = 3$, $B = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3.\sqrt[3]{\frac{1}{abc}} = 3$, and $AB + 2 - A - B = (A-1)(B-1) + 1 > 0$, it follows that

$t^3 + (AB - A - B)t^2 + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + A + B - 2AB\right)t + AB + 2 - A - B \geq t^3 + (AB - A - B)t^2 + (2A + 2B - 2AB - 3)t + AB + 2 - A - B = (t-1)^2(t + AB + 2 - A - B) \geq 0.$

98. $a_n + \underbrace{\sqrt[n-1]{a_1 \ldots a_{n-1}} + \cdots + \sqrt[n-1]{a_1 \ldots a_{n-1}}}_{n-1} \geq n.\sqrt[n]{a_n \underbrace{\sqrt[n-1]{a_1 \ldots a_{n-1}} \cdots \sqrt[n-1]{a_1 \ldots a_{n-1}}}_{n-1}}$,

and therefore, $n\sqrt[n]{a_1 \ldots a_n} - (n-1)\sqrt[n-1]{a_1 \ldots a_{n-1}} \leq a_n$.

99. $\sqrt[n]{\frac{a_1}{a_1+b_1+\cdots+k_1} \cdot \frac{a_2}{a_2+b_2+\cdots+k_2} \cdots \frac{a_n}{a_n+b_n+\cdots+k_n}} +$

$\sqrt[n]{\frac{b_1}{a_1+b_1+\cdots+k_1} \cdot \frac{b_2}{a_2+b_2+\cdots+k_2} \cdots \frac{b_n}{a_n+b_n+\cdots+k_n}} + \cdots + \sqrt[n]{\frac{k_1}{a_1+b_1+\cdots+k_1} \cdot \frac{k_2}{a_2+b_2+\cdots+k_2} \cdots \frac{k_n}{a_n+b_n+\cdots+k_n}} \leq$

$\frac{1}{n}\left(\frac{a_1}{a_1+b_1+\cdots+k_1} \cdot \frac{a_2}{a_2+b_2+\cdots+k_2} \cdots \frac{a_n}{a_n+b_n+\cdots+k_n}\right) +$

$\frac{1}{n}\left(\frac{b_1}{a_1+b_1+\cdots+k_1}\cdot\frac{b_2}{a_2+b_2+\cdots+k_2}\cdots\frac{b_n}{a_n+b_n+\cdots+k_n}\right)+\cdots+\frac{1}{n}\left(\frac{k_1}{a_1+b_1+\cdots+k_1}\cdot\frac{k_2}{a_2+b_2+\cdots+k_2}\cdots\frac{k_n}{a_n+b_n+\cdots+k_n}\right)=$ 1.

100. We observe that $n-k+ka^n=\underbrace{1+\cdots+1}_{n-k}+\underbrace{a^n+\cdots+a^n}_{k}\geq n.\sqrt[n]{1\cdots 1.\underbrace{a^n\cdots a^n}_{k}}=na^k$. Thus, the inequality is proved.

101. We observe that $\frac{x^k}{y^{k-1}}+(k-1)y\geq k.\sqrt[k]{\frac{x^k}{y^{k-1}}y^{k-1}}=kx$, where $k\in\mathbb{N}$, $k\geq 2$, whence

$\frac{x^k}{y^{k-1}}\geq kx-(k-1)y.$

Using above inequality, we have $\frac{x_1^2}{x_2}+\frac{x_2^3}{x_3^2}+\cdots+\frac{x_n^{n+1}}{x_1^n}\geq(2x_1-x_2)+(3x_2-2x_3)+\cdots+[nx_{n-1}-(n-1)x_n]+[(n+1)x_n-nx_1]=2(x_1+\cdots+x_n)-nx_1\geq x_1+\cdots+x_n$, whence

$\frac{x_1^2}{x_2}+\frac{x_2^3}{x_3^2}+\cdots+\frac{x_n^{n+1}}{x_1^n}\geq x_1+x_2+\cdots+x_n.$

102. We observe that $\frac{a^{x_1-x_2}}{x_1+x_2}+\frac{a^{x_2-x_3}}{x_2+x_3}+\cdots+\frac{a^{x_n-x_1}}{x_n+x_1}\geq n.\sqrt[n]{\frac{a^{x_1-x_2}}{x_1+x_2}\cdot\frac{a^{x_2-x_3}}{x_2+x_3}\cdot\cdots\cdot\frac{a^{x_n-x_1}}{x_n+x_1}}=\frac{n}{\sqrt[n]{(x_1+x_2)(x_2+x_3)\cdots(x_n+x_1)}}\geq\frac{n}{\underbrace{(x_1+x_2)+\cdots+(x_n+x_1)}_{n}}=\frac{n^2}{2\sum\limits_{i=1}^{n}x_i}$.

103. Using the A.M.-G.M. inequality, for the numbers x_i+1 and $(p-1)$ 1's, where $i=1,2,\ldots,n$, we deduce that, $\sqrt[p]{x_i+1}=\sqrt[p]{(x_i+1)\underbrace{1\cdots 1}_{p-1}}\leq\frac{x_i+1+(p-1).1}{p}=1+\frac{x_i}{p}$, whence, $\sqrt[p]{x_1+1}+\cdots+\sqrt[p]{x_n+1}\leq\left(1+\frac{x_1}{p}\right)+\cdots+\left(1+\frac{x_n}{p}\right)=n+\frac{x_1+\cdots+x_n}{p}=n+1.$

104. $x^k(1-x^m)=[x^{km}(1-x^m)^m]^{\frac{1}{m}}=\left(mx^k\cdots mx^k(k-kx^m)\cdots(k-kx^m)\right).\frac{1}{m^{\frac{m}{k}}.k}\leq\frac{1}{m^{\frac{m}{k}}.k}\left(\frac{mx^k+\cdots+mx^k+(k-kx^m)+\cdots+(k-kx^m)}{m+k}\right)=\frac{k^{\frac{k}{m}}.m}{(m+k)^{\frac{k}{m}+1}}$.

105. According to previous problem we see that $x(1-x^2)\leq\frac{2\sqrt{3}}{9}$, whence

$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{x^2}{x(1-x^2)}+\frac{y^2}{y(1-y^2)}+\frac{z^2}{z(1-z^2)}\geq\frac{9}{2\sqrt{3}}(x^2+y^2+z^2)=\frac{3\sqrt{3}}{2}.$

106. We have $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=\frac{1+x}{1-x^2}+\frac{1+y}{1-y^2}+\frac{1+z}{1-z^2}=\frac{1}{1-x^2}+\frac{1}{1-y^2}+\frac{1}{1-z^2}+\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}\geq\frac{9}{3-x^2-y^2-z^2}+\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}\geq\frac{9}{2}+\frac{3\sqrt{3}}{2}$(from previous problem).

107. Using A.M.-G.M. inequality, we have $f(x)=(1+x)^{-\frac{1}{n}}+(1-x)^{-\frac{1}{n}}\geq 2\sqrt{(1+x)^{-\frac{1}{n}}(1-x)^{-\frac{1}{n}}}=\frac{2}{\sqrt[2n]{1-x^2}}\geq 2[\because x\in[0;1)]$.

On the other hand $f(0)=2$. Thus, the minimum value of the function $f(x)$ is 2.

108. Let us represent the function $f(x)$ as $f(x) = (m+n).\frac{n.\frac{ax^m}{n}+m.\frac{b}{mx^n}}{m+n}$.

Using the A.M.-G.M. inequality, we have $f(x) \geq (m+n)\sqrt[m+n]{\left(\frac{ax^m}{n}\right)^n\left(\frac{b}{mx^n}\right)^m} = (m+n)\sqrt[m+n]{\frac{a^nb^m}{n^nm^m}}$, where equality holds if $\frac{ax^m}{n} = \frac{b}{mx^n}$. Therefore, $x = \sqrt[m+n]{\frac{bn}{am}}$.

109. Let us represent the function $f(x)$ as $f(x) = \frac{(x-a)(x-a)\alpha(b-x)\beta(b+x)}{\alpha\beta}$, where $\alpha > 0, \beta > 0$.

Using the A.M.-G.M. inequality, we have

$4\sqrt[4]{(x-a)(x-a)\alpha(b-x)\beta(b+x)} \leq (x-a)+(x-a)+\alpha(b-x)+\beta(b+x) = (2-\alpha+\beta)x+(\alpha+\beta)b-2a$.

Note that the R.H.S. does not depend on x if $\alpha - \beta = 2$, and equality holds if $x-a = \alpha(b-x) = \beta(b+x)$. Thus, it follows that $\alpha = \frac{x-a}{b-x}, \beta = \frac{x-a}{b+x}$.

Hence, from the equation $\alpha - \beta = 2$, we deduce that $2x^2 - ax - b^2 = 0$. This equation has only one positive root, $x_0 = \frac{a+\sqrt{a^2+8b^2}}{4}$. We can easily prove that $x_0 \in [a, b]$. Therefore, $f(x)$ attains it maaximum value in $[a, b]$ at x_0.

110. Let us represent the product xyz as $\frac{1}{2\sqrt{3}\pi}.2x.\sqrt{3}y.\pi z$, and from the A.M.-G.M. inequality, $xyz = \frac{1}{2\sqrt{3}\pi}.2x.\sqrt{3}y.\pi z \leq \frac{1}{2\sqrt{3}\pi}.\left(\frac{2x+\sqrt{3}y+\pi z}{3}\right)^3 = \frac{1}{54\sqrt{3}\pi}$, where equality holds if $2x = \sqrt{3}y = \pi z$.

Thus, we have the maximum value as $\frac{1}{54\sqrt{3}\pi}$.

111. Note that $f(0) = 0$, and if $x < 0$, then $f(x) < 0$. On the other hand, if $x > 0$, then $f(x) > 0$. Therefore, the function $f(x)$ attains it maximum value in $(0, \infty)$ and its minimum value in $(-\infty, 0)$.

From the A.M.-G.M. inequality for the numbers ax^2 and b, we have $\frac{ax^2+b}{2} \geq |x|\sqrt{ab}$, where equality holds if $ax^2 = b$.

From the above inequality, it follows that $\frac{x}{ax^2+b} \leq \frac{1}{2\sqrt{ab}}$. Therefore, the maximum value of the function $f(x)$ is equal to $\frac{1}{2\sqrt{ab}}$, and it attains this value at the point $x = \sqrt{\frac{b}{a}}$. Since $f(x)$ is an odd function, its minimum value is equal to $-\frac{1}{2\sqrt{ab}}$, which the function attains at the point $x = -\sqrt{\frac{b}{a}}$.

112. We have $y = \frac{5\sqrt{(x+2)(x+4)}+12}{x+3}$, and thus if $x \geq -2$, then $y = \frac{\sqrt{(25x+50)(x+4)}+12}{x+3} \leq \frac{\frac{25x+50+x+4}{2}+12}{x+3} = 13$, and $y = 13$ if $25x+50 = x+4$, i.e. $x = -\frac{23}{12}$. If $x \leq -4$ then $y < 0$.

Therefore, the maximum value of y is 13.

113. We have $y = \frac{\sqrt[3]{(x^2+1)(x^2+1)\left(\frac{2}{5}x^2+\frac{6}{5}\right)\cdot\frac{5}{2}}}{3x^2+4} = \sqrt[3]{\frac{5}{2}}\cdot\frac{\sqrt[3]{(x^2+1)(x^2+1)\left(\frac{2}{5}x^2+\frac{6}{5}\right)}}{3x^2+4} \leq \sqrt[3]{\frac{5}{2}}\cdot\frac{x^2+1+x^2+1+\frac{2}{5}x^2+\frac{6}{5}}{3(3x^2+4)} = \sqrt[3]{\frac{5}{2}}\cdot\frac{12x^2+16}{15(3x^2+4)} = \frac{4}{15}\sqrt[3]{\frac{5}{2}}$

Note that $y = \frac{4}{15}\sqrt[3]{\frac{5}{2}}$, if $x^2+1 = \frac{2}{5}x^2+\frac{6}{5}$ i.e. if $x^2 = \frac{1}{3}$.

114. From A.M.-G.M. inequality it follows that $1 \geq z^2+1$, and therefore, $z = 0, x = y = 1$.

115. We observe that the inequality $\frac{x^2+y^2+z^2}{3} \geq \left(\frac{x+y+z}{3}\right)^2$ becomes an inequality because $x+y+z = x^2+y^2+z^2 = 3$. Hence, the solution is $x = y = z = 1$.

116. We have $\frac{a^2+b^2+c^2+d^2}{4} \geq \left(\frac{a+b+c+d}{4}\right)^2 \Rightarrow \frac{16-e^2}{4} \geq \left(\frac{8-e}{4}\right)^2 \Rightarrow 5e^2 - 16e \leq 0 \Rightarrow 0 \leq e \leq 3.2$.

Thus, the maximum value of e is 3.2.

117. We have $\frac{x_1}{x_2}+\frac{x_3}{x_4}+\frac{x_5}{x_6} \geq \frac{1}{x_2}+\frac{x_2}{x_4}+\frac{x_4}{x_6} \geq 3.\sqrt[3]{\frac{1}{x_2}\cdot\frac{x_2}{x_4}\cdot\frac{x_4}{x_6}} = 3\sqrt[3]{\frac{1}{x_6}} \geq 0.3$.

Thus, minimum value of the given expression is 0.3.

118. Using the A.M.-G.M. inequality for the numbers $x^4, y^4, 1, 1$, we have $x^4+y^4+1+1 \geq 4\sqrt[4]{x^4y4 = 4|x||y| \Rightarrow} x^4+y^4+2 \geq 4xy$ and therefore, $x^4 = 1, y^4 = 1 \Rightarrow x, y = \pm 1$.

119. We have $(xy)^2+(yz)^2+(zx)^2 = 3xyz$, whence $xyz > 0$. On the other hand, $xyz \in \mathbb{Z}$, and hence, $xyz \geq 1$. From the A.M.-G.M. inequality, we have $3xyz = (xy)^2+(yz)^2+(zx)^2 \geq 3xyz.\sqrt[3]{xyz \geq 3xyz, \text{ and}}$ hence, $xyz = 1$, and the inequality becomes equality. It follows that $(xy)^2 = (yz)^2 = (zx)^2$. Therefore $x^2 = y^2 = z^2 = 1$.

$(x, y, z) = (1, 1, 1) = (1, -1, -1) = (-1, 1, -1) = (-1, -1, 1)$.

120. We observe that for $x > 0, \alpha > \beta \geq 0$, we have $x^\alpha - x^\beta \geq x^{\alpha-\beta} - 1$, since $(x^{\alpha-\beta} - 1)(x^\beta - 1) \geq 0$. Therefore,

$x_1^\alpha + \cdots + x_n^\alpha - (x_1^\beta + \cdots + x_n^\beta) \geq (x_1^{\alpha-\beta} - 1) + \cdots + (x_n^{\alpha-\beta} - 1) = x_1^{\alpha-\beta} + \cdots + x_n^{\alpha-beta} - n \geq n.\sqrt[n]{x_1^{\alpha-\beta} \ldots x_n^{\alpha-\beta}} - n = 0$.

Thus, $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha \geq x_1^\beta + x_2^\beta + \cdots + x_n^\beta$.

121. If $\beta \geq 0$, then from previous probelm, we have $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha \geq x_1^\beta + x_2^\beta + \cdots + x_n^\beta$.

If $\beta < 0$, then $x_1^\beta + \cdots + x_n^\beta = \frac{1}{x_1^{-\beta}} + \cdots + \frac{1}{x_n^{-\beta}} = x_2^{-\beta} \ldots x_n^{-\beta} + \cdots + x_1^{-\beta} \ldots x_{n-1}^{-\beta} \leq \frac{x_2^{-\beta(n-1)} + \cdots + x_n^{-\beta(n-1)}}{n-1} + \cdots + \frac{x_1^{-\beta(n-1)} + \cdots + x_{n-1}^{-\beta(n-1)}}{n-1} = x_1^{-\beta(n-1)} + \cdots + x_n^{-\beta(n-1)} \leq x_1^\alpha + \cdots + x_n^\alpha$.

Therefore, $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha \geq x_1^\beta + x_2^\beta + \cdots + x_n^\beta$.

122. WLOG we can assume that $\max(x, y, z) = x$, in which case, $x^2y + y^2z + z^2x \leq x^2y + xyz + 0.5z^2x + 0.5zx^2 = 0.5x(x+z)(2y+z)$.

Using the A.M.-G.M. inequality, we have $0.5x(x+z)(2y+z) \leq 0.5\left(\frac{x+(x+z)+(2y+z)}{3}\right)^3 = \frac{4}{27}$, and thus, the given inequality is proven.

123. We have $\frac{1+a}{1+ab} + \frac{1+b}{1+bc} + \frac{1+c}{1+cd} + \frac{1+d}{1+da} = \frac{cd+acd}{cd+abcd} + \frac{ad+adb}{ad+abcd} + \frac{1+c}{1+cd} + \frac{1+d}{1+da} = 1 + \frac{c(1+ad)}{1+cd} + 1 + \frac{d(1+ab)}{1+da} \geq 2 + 2\sqrt{\frac{cd(1+ab)}{1+cd}} = 4.$

124. $\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} = \frac{cd+abcd}{cd+acd} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{bc+abcd}{bc+bcd} \geq (1+cd)\frac{4}{cd+acd+1+c} + (1+bc)\frac{4}{1+b+bc+bcd} = (1+cd)\frac{4b}{bcd+abcd+b+bc} + (1+bc)\frac{4}{1+b+bc+bcd} = \frac{4[b(1+cd)+1+bc]}{1+b+bc+bcd} = 4.$

125. We observe that $(c-b)(c-d) + (e-f)(e-d) + (e-f)(c-b) < 0$, and therefore, $(bd+df+fb) - (ac+ce+ea) < (c+e)(b+d+f-a-c-e)$, or $\alpha - \beta < \gamma(T-S)$, where $\alpha = bd+df+fb$, $\beta = ac+ce+ea$, $\gamma = c+e$.

We have $S\alpha + T\beta = S(\alpha-\beta) + (S+T)\beta < S\gamma(T-S) + (S+T)(ce+a\gamma) \leq S\gamma(T-S) + (S+T)\left(\frac{\gamma^2}{4}+a\gamma\right) = \gamma\left(2ST - \frac{3}{4}(S+T)\gamma\right)$.

Thus, $\sqrt{\frac{3}{4}(S+T)(S\alpha+T\beta)} < \sqrt{\frac{3}{4}(S+T)\gamma\left(2ST-\frac{3}{4}(S+T)\gamma\right)} \leq \frac{1}{2}\left(\frac{3}{4}(S+T)\gamma + \left(2ST - \frac{3}{4}(S+T)\gamma\right)\right) = ST$.

Therefore, $\sqrt{3(S+T)[S(bd+df+fb)+T(ac+ce+ea)]} < 2ST$.

126. We observe that $\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{\sqrt[4]{a.\frac{a+b}{2}\cdot\frac{a+b+c}{3}\cdot\frac{a+b+c+d}{4}}} = \sqrt[4]{1.\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}\cdot\frac{4a}{a+b+c+d}} +$

$\sqrt[4]{1.\frac{2a}{a+b}\cdot\frac{3b}{a+b+c}\cdot\frac{4b}{a+b+c+d}} +$

$\sqrt[12]{1.1.1.\frac{2b}{a+b}\cdot\frac{2b}{a+b}\cdot\frac{2b}{a+b}\cdot\frac{3a}{a+b+c}\cdot\frac{3b}{a+b+c}\cdot\frac{3c}{a+b+c}\cdot\frac{4c}{a+b+c+d}\cdot\frac{4c}{a+b+c+d}\cdot\frac{4c}{a+b+c+d}} +$

$\sqrt[4]{1.\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}\cdot\frac{4d}{a+b+c+d}} \leq \frac{1}{4}\left(1+\frac{2a}{a+b}+\frac{3a}{a+b+c}+\frac{4a}{a+b+c+d}\right) +$

$\frac{1}{4}\left(1+\frac{2a}{a+b}+\frac{3b}{a+b+c}+\frac{4b}{a+b+c+d}\right) + \frac{1}{12}\left(3+\frac{6b}{a+b}+\frac{3a}{a+b+c}+\frac{3b}{a+b+c}+\frac{3c}{a+b+c}+\frac{12c}{a+b+c+d}\right) +$

$\frac{1}{4}\left(1+\frac{2b}{a+b}+\frac{3c}{a+b+c}+\frac{4d}{a+b+c+d}\right) = 4.$

127. WLOG we assume that $a, b, c, d \geq 0$. Let x, y, z be arbitrary positive numbers. Then using A.M.-G.M. inequality, we observe that

$a^{12} + (ab)^6 + (abc)^4 + (abcd)^3 = a^{12} + \frac{1}{x^6}(xab)^6 + \frac{1}{x^4y^8}(xya.yb.c)^4 + \frac{1}{x^3y^6z^9}(xyza.yzb.zc.d)^3 \leq a^{12} + \frac{1}{2x^6}(x^{12}a^{12} + b^{12}) + \frac{1}{3x^4y^8}(x^{12}y^{12}a^{12} + y^{12}b^{12} + c^{12}) + \frac{1}{4x^3y^6z^9}(x^{12}y^{12}z^{12}a^{12} + y^{12}z^{12}b^{12} + z^{12}c^{12} + d^{12}) = A(a^{12} + b^{12} + c^{12} + d^{12})$.

We take numbers x, y, z such that $1 + \frac{x^6}{2} + \frac{x^8y^4}{3} + \frac{x^9y^6z^3}{4} = \frac{1}{2x^6} + \frac{y^4}{3x^4} + \frac{y^6z^3}{4x^3} = \frac{1}{4x^3y^4z^9} = A$.

Therefore, $x^{12} = 1 - \frac{1}{A}$, $y^{12} = 1 - \frac{1}{2\sqrt{A(A-1)}}$, $z^{12} = 1 - \frac{1}{3\sqrt[3]{A[\sqrt{A(A-1)} - 0.5]^2}}$, and $\frac{256}{27}A\left(3\sqrt[3]{A[A(A-1) - 0.5]^2} - 1\right)^3 = 1$.

Let $f(A) = \frac{256}{27}A\left(3\sqrt[3]{A[A(A-1) - 0.5]^2} - 1\right)^3 - 1$

By applying limits on this function between the limits 1.42 and 1.43 we find a value such that $f(A) = 0$ and thus the equality is proven.

128. Observing the first two terms $C_0^n.1^n + C_1^n.1^{n-1}.\frac{1}{n} = 2$. Thus, first two terms only are enough to set the bound to 2. Rest of the terms make it definitely greater than 2.

129. We see that L.H.S. will have terms $E_1 = 1, E_2 = \sum a_i, E_3 = \sum a_ia_j$ and so on. All these terms are square-free i.e. a_i does not repeat and power is always one.

Using multinomial theorem $S_k^2 = (a_1 + a_2 + \cdots + a_k)^2 = \sum \frac{k!}{i_1!i_2!\ldots i_n!}a_1^{i_1}\cdots a_k^{i_n}$

Thus, $\frac{S_k^2}{k!}$ will have all terms on L.H.S. and also more terms which won't be square-free making R.H.S. greater than L.H.S.

130. From A.M.-G.M. inequality we have $1 + a \geq 2\sqrt{a}, 1 + b \geq 2\sqrt{b}, 1 + c \geq 2\sqrt{c} \Rightarrow (1+a)(1+b)(1+c) \geq 8\sqrt{abc}$.

Also $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = \frac{1+a}{a}.\frac{1+b}{b}.\frac{1+c}{c} \geq \frac{8\sqrt{abc}}{abc} = \frac{8}{\sqrt{abc}}$.

Also $\frac{a+b+c}{3} \geq \sqrt{abc} \Rightarrow \frac{1}{3} \geq \sqrt[3]{abc} \Rightarrow \frac{1}{27} \geq abc$.

Now $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 1 + \frac{ab+bc+ca+2}{abc}$, and $\frac{ab+bc+ca}{3} \geq \sqrt[3]{(abc)^2}$

$\Rightarrow \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \geq \Rightarrow 1 + \frac{3.\frac{1}{9}+2}{\frac{1}{27}} = 64.$

131. $\frac{a^n-1}{a-1} = 1 + a + a^2 + \cdots + a^{n-1}$.

Thus, $\frac{a^n-1}{a^n(a-1)} = \sum_{k=1}^{n} a^{-k}$.

Applying A.M.-G.M. inequality for $\sum_{k=1}^{n} a^{-k}$ and $a^{\frac{n(n+1)}{2}}$, we have

$$\frac{\sum_{k=1}^{n} a^{-k} + a^{\frac{n(n+1)}{2}}}{n+1} \geq (a^{-1}.a^{-2}.\ldots.a^{-n}.a^{\frac{n(n+1)}{2}}) = 1$$

$\Rightarrow \sum_{k=1}^{n} a^{-k} + a^{\frac{n(n+1)}{2}} \geq n+1$, and hence, the inequality is proven.

132. Using A.M.-G.M. inequality, we have

$$\frac{a^{n+1}+a^{n+1}+\cdots+a^{n+1}+1}{n+1} \geq \sqrt[n+1]{a^{(n+1)n}.1} = a^n$$

$\Rightarrow na^{n+1} + 1 \geq (n+1)a^n$.

133. We observe that $\sqrt{i} + \sqrt{i+1} = \frac{1}{\sqrt{i+1}-\sqrt{i}}$

$$\text{L.H.S.} = \prod_{i=1}^{n} \frac{1}{\sqrt{i+1}-\sqrt{i}}$$

Using A.M.-G.M. inequality, we have

$$\frac{(\sqrt{k+1}-\sqrt{k})+(\sqrt{k+2}-\sqrt{k+1})+\cdots+(\sqrt{n+1}-\sqrt{n})}{n-k+1} \leq \sqrt[n-k+1]{\sqrt{\prod_{i=k}^{n}(\sqrt{i+1}-\sqrt{i})}}$$

$$\Rightarrow \left(\frac{\sqrt{n+1}-\sqrt{k}}{n-k+1}\right)^{n-k+1} \leq \prod_{i=k}^{n}(\sqrt{i+1}+\sqrt{i})$$

We also have, $\frac{n-k+1}{\sqrt{n+1}-\sqrt{k}} = \sqrt{n+1} + \sqrt{k}$

$\Rightarrow (\sqrt{n+1}+\sqrt{k})^{n-k+1} \leq \prod_{i=k}^{n}(\sqrt{i+1}+\sqrt{i})$

Since $n-k+1 \geq 2$,w e have

$$(\sqrt{n+1}+\sqrt{k})^2 \leq \prod_{i=k}^{n}(\sqrt{i+1}+\sqrt{i})$$

Now, $(\sqrt{n+1}+\sqrt{k})^2 = n+1+k+2\sqrt{k(n+1)}$ and R.H.S. $n-k-\sqrt{n}+\sqrt{k}+2$

We need to prove that $n+1+k+2\sqrt{k(n+1)} \geq n-k-\sqrt{n}+\sqrt{k}+2$

$\Rightarrow 2k-1+\sqrt{n}+2\sqrt{k(n+1)} \geq \sqrt{k}$, which is true for given conditions. Hence, the inequality is proven.

134. Using A.M.-G.M. inequality we have

$$\frac{\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}}{n} \geq \sqrt[n]{\frac{a_1}{a_2}.\frac{a_2}{a_3}.\cdots.\frac{a_{n-1}}{a_n}.\frac{a_n}{a_1}} = 1$$

$\Rightarrow \frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} \geq n$.

135. We see that $a_1 + (a_2 - a_1) + \cdots + (a_{n+1} - a_n) = a_{n+1}$. Using A.M.-G.M. for these terms and $\frac{1}{a_1(a_2-a_1)\cdots(a_{n+1}-a_n)}$

$$\frac{a_1+(a_2-a_1)+\cdots+(a_{n+1}-a_n)+\frac{1}{a_1(a_2-a_1)\cdots(a_{n+1}-a_n)}}{n+2} \geq$$

$$\sqrt[n+2]{a_1(a_2-a_1)\cdots(a_{n+1}-a_n).\frac{1}{a_1(a_2-a_1)\cdots(a_{n+1}-a_n)}} = 1$$

$$\Rightarrow a_{n+1} + \frac{1}{a_1(a_2-a_1)\cdots(a_{n+1}-a_n)} \geq n+2.$$

136. We have to prove that $1 + \frac{x}{2} \leq \frac{1}{\sqrt{1-x}}$. Both sides are positive for given conditions.

$$\Rightarrow \left(1 + \frac{x}{2}\right)\sqrt{1-x} \leq 1$$

Squaring both sides $\left(1 + \frac{x}{2}\right)^2(1-x) \leq 1$.

Applying A.M.-G.M. inequality we have

$$\frac{1+\frac{x}{2}+1+\frac{x}{2}+1-x}{3} \geq \sqrt[3]{\left(1-\frac{x}{2}\right)^2(1-x)}$$

$$\Rightarrow 1 \geq \sqrt[3]{\left(1-\frac{x}{2}\right)^2(1-x)}$$

Cubing boht sides we have $\left(1+\frac{x}{2}\right)^2(1-x) \leq 1$, and hence, the inequlity is proven.

137. We assume that all of a, b, c, d, e have the same sign. Let $x_1 = \frac{a}{b}, x_2 = \frac{b}{c}, x_3 = \frac{c}{d}, x_4 = \frac{d}{e}, x_5 = \frac{e}{a}$.

So the inequality becomes $\sum_{i=1}^{5} x_i^4 \geq \sum_{i=1}^{5} x_i$.

We use the A.M.-G.M. inequality for $\frac{x^4+1+1+1}{4} \geq \sqrt[4]{x^4} = x \Rightarrow x^4 + 3 \geq 4x \Rightarrow x^4 - x \geq 3(x-1)$. Thus,

$$\sum_{i=1}^{5}(x_i^4 - x_i) \geq 3\sum_{i=1}^{5}(x_i) - 15$$

Again from A.M.-G.M. inequality $\sum_{i=1}^{5} x_i \geq 5\sqrt[5]{\prod_{i=1}^{5} x_i} = 5$. Thus, $\sum_{i=1}^{5} x_i - 5 = 0$. And therefore,

$\sum_{i=1}^{5}(x_i^4 - x_i) \geq 0$. And hence, the inequality is proven.

138. We will prove that for any $x > 0$ and a positive integer $n \geq 1$, the inequality $x^n \geq nx - (n-1)$ holds.

Consider n positive integers $x^n, 1, 1, \ldots, 1(n-1)$ times. By A.M.-G.M. inequality

$\frac{x^n + \overbrace{1+1+\cdots+1}^{n-1 \text{ times}}}{n} \geq \sqrt[n]{x^n}$

$\Rightarrow x^n + (n-1) \geq nx \Rightarrow x^n \geq nx - (n-1)$

Setting $n = 1999$ and for $a, b, c, d > 0$

$\left(\frac{a}{b}\right)^{1999} + \left(\frac{b}{c}\right)^{1999} + \left(\frac{c}{d}\right)^{1999} + \left(\frac{d}{a}\right)^{1999} \geq 1999\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) - 4.1988$

Let $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = S$ so we want to prove that L.H.S $\geq S$.

If we can show that $1999S - 7992 \geq S \Rightarrow S \geq 4$ then the desired inequality is proven.

Using A.M.-G.M. ineqeuality again $\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}{4} \geq \sqrt[4]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \cdot \frac{d}{a}}$

$\Rightarrow S \geq 4$. And hence, the inequality is proven.

139. Let L.H.S. be S, then using A.M.-G.M. inequality we have

$\frac{S}{n} \geq \left(\sqrt{\frac{a_1+a_2}{a_3}} \cdot \sqrt{\frac{a_2+a_3}{a_4}} \cdots \sqrt{\frac{a_n+a_1}{a_2}}\right)^{\frac{1}{n}}$

$S \geq \left[\left(\frac{a_1+a_2}{a_3} \cdot \frac{a_2+a_3}{a_4} \cdots \frac{a_n+a_1}{a_2}\right)\right]^{\frac{1}{2n}}$

Using A.M.-G.M again on terms of numerator we have

$(a_1 + a_2)(a_2 + a_3) \cdots (a_n + a_1) \geq 2^n (a_1 a_2 \ldots a_n)$

Putting this back in previour inequality we have the proof.

140. We will first prove the inequality $\frac{t}{1+t^2} \leq \frac{3\sqrt{3}}{16}(1 + t^2)$.

When $t \leq 0$ the inequality is obviously satisfied. For $t > 0$ we rearrange the inequality as $(1 + t^2)^2 \geq \frac{16t}{3\sqrt{3}}$.

Consider four numbers $t^2, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. We use A.M.-G.M. inequality on these to get

$\frac{t^2+1}{4} \geq \sqrt[4]{t^2 \cdot \frac{1}{27}}$. Squaring both sides we have, $(1 + t^2)^2 \geq \frac{16t}{3\sqrt{3}}$.

Thus, $\frac{t}{1+t^2} \leq \frac{3\sqrt{3}}{16}(1 + t^2)$ is true for all t.

Putting $t = x, y, z$ and adding proves the desired inequality.

141. $\because \sum_{i=1}^{n} a_i = 1$, we have $1 - a_j = \sum_{i \neq j} a_i$

Using A.M.-G.M. inequality on the $n-1$ terms of $\sum_{i \neq j} a_i$, we get

$$\sum_{i \neq j} a_i \geq (n-1)\left(\prod_{i \neq j} a_i\right)^{\frac{1}{n-1}}$$

$$\therefore \frac{1}{a_j} - 1 \geq \frac{(n-1)\left(\prod_{i \neq j} a_i\right)^{\frac{1}{n-1}}}{a_j}$$

$$\Rightarrow \prod_{j=1}^{n}\left(\frac{1}{a_j} - 1\right) \geq \prod_{j=1}^{n} \frac{(n-1)\left(\prod_{i \neq j} a_i\right)^{\frac{1}{n-1}}}{a_j}$$

$$= (n-1)^n \frac{\prod_{j=1}^{n}\left(\prod_{i] \neq j} a_i\right)^{\frac{1}{n-1}}}{\prod_{j=1}^{n} a_j}$$

Let $P = \prod_{i=1}^{n} a_i \Rightarrow \prod_{i \neq j} a_i = \frac{P}{a_j}$ so $\left(\prod_{i \neq j} a_i\right)^{\frac{1}{n-1}} = \left(\frac{P}{a_j}\right)^{\frac{1}{n-1}}$

So the numerator becomes $\prod_{j=1}^{n}\left(\frac{P}{a_j}\right)^{\frac{1}{n-1}} = \left(\frac{P^n}{P}\right)^{\frac{1}{n-1}} = P$

Thus, $\prod_{j=1}^{n}\left(\frac{1}{a_j} - 1\right) \geq (n-1)^n \frac{P}{P} = (n-1)^n$

For the other half of the product we want to show that $\prod_{i=1}^{n}\left(\frac{1}{a_i} + 1\right) \geq (n+1)^n$

Let $f(x) = \log(\frac{1}{x} + 1)$ so $f'(x) = -\frac{1}{x(1+x)}$ and $f''(x) = \frac{1+2x}{x^2(1+x)^2}$.

Since $a_i > 0, x > 0$, so $f''(x) > 0$. Thus, $f(x)$ is a convex function.

Using Jensen's inequality(which is a consequence of A.M.-G.M. inequality)

$$\frac{1}{n}\sum_{i=1}^{n} f(a_i) \geq f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)$$

Substituting $f(a_i) = \log\left(\frac{1}{a_i} + 1\right)$ and $\sum_{i=1}^{n} a_i = 1$, we have

$$\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{1}{a_i} + 1\right) \geq \log\left(\frac{1}{\sum_{i=1}^{n} a_i} + 1\right) = \log\left(\frac{1}{1/n} + 1\right) = \log(n+1)$$

$$\Rightarrow \log\left(\left(\prod_{i=1}^{n}\left(\frac{1}{a_i}+1\right)\right)\right) \geq \log(n+1)$$

Since log is an increasing function we take antilog of both sides to get

$$\prod_{i=1}^{n}\left(\frac{1}{a_i}+1\right) \geq (n+1)^n$$

Combining the two inequalities obtained we have proven the desireed inequality.

142. $u = 1-(v+w) \geq 1-\left(\frac{7}{16}+\frac{7}{16}\right) = \frac{1}{8}$. So the constraint becomes $\frac{1}{8} \leq u, v, z \leq \frac{7}{16}$

By A.M.-G.M. inequality $\frac{(1+u)+(1+v)+(1+w)}{3} \geq \sqrt[3]{(1+u)(1+v)(1+w)}$

$\Rightarrow (1+u)(1+v)(1+w) \leq \frac{64}{27}$, which is the maximum value of the product.

Let $1+u = x, 1+v = y$ and $1+w = z$. We need to minimize the product $P = xyz$ such that $x+y+z = 4$ and $\frac{9}{8} \leq x, y, z \leq \frac{23}{16}$.

A.M.-G.M. inequaltiy states that the product is maximized when $x = y = z$. Conversely the product is minimized when they are as far apart as possible.

Let k variables be $\frac{9}{8}$ and m variables be $\frac{23}{16}$. The sum is $k.\frac{9}{8}+m.\frac{23}{16} = 4 \Rightarrow 18k+23m = 64$ and $k+m = 3$.

So for (k, m) we have four possibilities and we see that with the exception of case $k = 1, m = 2$ the sum condition is not satisfied. And the product in this case is $\frac{4761}{2048}$.

143. Consider A.M.-G.M. inequality of $p, \frac{x}{p}$s and $q, \frac{y}{q}$s.

$$\frac{\underbrace{\frac{x}{p}+\cdots+\frac{x}{p}}_{p\text{ times}}+\underbrace{\frac{y}{q}+\cdots+\frac{y}{q}}_{q\text{ times}}}{p+q} \geq \sqrt[p+q]{\underbrace{\frac{x}{p}\cdots\frac{x}{p}}_{p\text{ times}}\underbrace{\frac{y}{q}\cdots\frac{y}{q}}_{q\text{ times}}} \Rightarrow \frac{x+y}{p+q} \geq \sqrt[p+q]{\frac{x^p y^q}{p^p q^q}} \Rightarrow x^p y^q \leq \frac{p^p q^q a^{p+q}}{(p+q)^{p+q}}.$$

144. Putting $x = \frac{1}{\sqrt{2}}$ and $x = -\frac{1}{\sqrt{2}}$ we have

$$\frac{a}{2}+\frac{b}{\sqrt{2}}+c \leq \frac{1}{\sqrt{1-\frac{1}{2}}} = \sqrt{2} \text{ and } \frac{a}{2}-\frac{b}{\sqrt{2}}+c \leq \frac{1}{\sqrt{1-\frac{1}{2}}} = \sqrt{2}$$

Adding we get $a+2c \leq 2\sqrt{2}$.

145. We have $\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) = 2+\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{a}{c}+\frac{c}{a}$.

Using A.M.-G.M. inequality on $\frac{a}{b}, \frac{a}{c}, \frac{b}{c}$ we have $\frac{\frac{a}{b}+\frac{a}{c}+\frac{b}{c}}{3} \geq \sqrt[3]{\frac{a^2}{bc}} = \frac{a}{\sqrt[3]{abc}}$

Taking cyclic sum of this A.M.-G.M. inequality we prove the result.

146. Let $x_i = \frac{1+a_i}{1-a_i}$ so the inequality becomes $\prod_{i=1}^{n+1} x_i \geq n^{n+1}$.

Now, $a_i = \frac{x_i-1}{x_i+1}$. So our assumption becomes $\sum_{i=1}^{n+1} \frac{x_i-1}{x_i+1} \geq n-1$

$$\Rightarrow \sum_{i=1}^{n+1} \frac{x_i-1}{x_i+1} \geq n-1 \Rightarrow \sum_{i=1}^{n+1}\left(1-\frac{2}{x_i+1}\right) \geq n-1 \Rightarrow (n+1)-2\sum_{i=1}^{n+1}\frac{1}{x_i+1} \geq n-1$$

$$\Rightarrow \sum_{i=1}^{n+1} \frac{1}{x_i+1} \leq 1.$$

Considering symmetry. Suppose all $x_i = x$, then: $(n+1).\frac{1}{x+1} \leq 1 \Rightarrow x \geq n$

Thus, $\prod_{i=1}^{n+1} x_i \geq n^{n+1}$, and hence, the inequality is proved.

147. Let $A = (a+b)(b+c)(c+d)(d+a)$. Using A.M.-G.M. inequality we have

$a+b \geq 2\sqrt{ab}$ and so on. Hence, $(a+b)(b+c)(c+d)(d+a) \geq 16abcd$. So inequality becomes $A^3 \geq 4096a^3b^3c^3d^3$.

Using A.M.-G.M. again we have $(a+b+c+d)^4 \geq 256abcd \Rightarrow abcd \leq \frac{(a+b+c+d)^3}{256}$

$$A^3 \geq 4096a^3b^3c^3d^3 \geq 4096.\frac{(a+b+c+d)^3}{256}.a^2b^2c^2d^2 = 16a^2b^2c^2d^2(a+b+c+d)^4.$$

148. We have $\left(1+\frac{a}{b}\right)^2+\left(1+\frac{b}{c}\right)^2+\left(1+\frac{c}{a}\right)^2 \geq \frac{1}{3}\left(1+\frac{a}{b}+1+\frac{b}{c}+1+\frac{c}{a}\right)^2$ and similar relation for the other term of the L.H.S.

Using A.M.-G.M. on $1, 1, 1, \frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ we have

$$\frac{3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{6} \geq \sqrt[6]{1} \Rightarrow \frac{1}{3}\left(1+\frac{a}{b}+1+\frac{b}{c}+1+\frac{c}{a}\right)^2 \geq \frac{6^2}{3}$$

For both the terms of L.H.S. the R.H.S. of above inequality becomes $\frac{6^4}{3^2} = 4.6^2$.

Using A.M.-G.M. on terms of R.H.S.

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}{6} \geq \sqrt[6]{1}$$

$$\Rightarrow 4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^2 \geq 4.6^2$$

And hence, the inequality is proven.

149. Using A.M.-G.M. inequality $a^2+bc \geq 2a\sqrt{bc}$ and so on. So L.H.S. $\geq 512a^6b^6c^6$

Similarly, R.H.S. $\geq 512(a^3b^3c^3)^{3/2}$.

Thus, given inequality becomes $a^3b^3c^3 \geq 1$. Since the ienquality is homogeneous we can assume that $abc = 1$ WLOG. And then the inequality holds for equality condition and otherwise.

150. Let $x = \frac{a}{a+b+c}$, $y = \frac{b}{a+b+c}$, and $z = \frac{c}{a+b+c}$ so $x + y + z = 1$

$\Rightarrow x + \sqrt{xy} + \sqrt[3]{xyz} \leq x + \frac{x+y}{2} + \frac{1}{3} = \frac{9x+3y+2}{6}$. Comparing with $\frac{4}{3}$, we get

$3x + y \leq 2$, which is always true under the constraint $x + y + z = 1$, and is maximized when $x = y = z = \frac{1}{3}$. And thus, $x + \sqrt{xy} + \sqrt[3]{xyz} = 1 < \frac{4}{3}$. And thus, the inequality is proven.

151. Using A.M.-G.M. inequality we have $a = a$, $\sqrt{ab} \leq \frac{a+b}{2}$, $\sqrt[3]{abc} \leq \frac{a+b+c}{3}$

Thus, we have $a + \sqrt{ab} + \sqrt[3]{abc} \leq a + \frac{a+b}{2} + \frac{a+b+c}{3}$

Applying A.M.-G.M. on R.H.S. we have

$a + \frac{a+b}{2} + \frac{a+b+c}{3} \leq \sqrt[3]{a.\frac{a+b}{2}.\frac{a+b+c}{3}}$.

Combining we have desired inequality.

152. By A.M.-G.M. inequality we have L.H.S. $\leq \left(\frac{a+b}{2}\right)^{5/2} + \left(\frac{b+c}{2}\right)^{5/2} + \left(\frac{c+a}{2}\right)^{5/2}$

Let $x = a + b, y = b + c, z = c + a \Rightarrow x + y + z = 2$

Consider the function $f(x) = \left(\frac{x}{2}\right)^{5/2}$. This function is convex on $x > 0$, since the second derivative $f''(x) = \frac{15}{8\sqrt{2}x^{3/2}} > 0$.

So by Jensen's inequality we have

$f(x) + f(y) + f(z) \leq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{2}{3}\right) = \frac{\sqrt{3}}{9}$. And hence, the inequality is proven.

153. Using Cauchy-Schwarz inequality, we have $(a^2+b^2+c^2)(1^2+2^2+3^2) \geq (a+2b+3c)^2 \geq 14^2$, and thus, $a^2 + b^2 + c^2 \geq 14^2$.

154. We have $ab + \sqrt{(1-a^2)(1-b^2)} \leq \sqrt{a^2 + (\sqrt{1-a^2})^2}.\sqrt{b^2 + (\sqrt{1-b^2})^2} = 1$.

155. We have $\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{(\sqrt{c})^2 + (\sqrt{b-c})^2}.\sqrt{(\sqrt{a-c})^2 + (\sqrt{c})^2} = \sqrt{ab}$.

156. We have $a\sqrt{a^2+c^2} + b\sqrt{b^2+c^2} \leq \sqrt{a^2 + \left(\sqrt{b^2+c^2}\right)^2}.\sqrt{b^2 + \left(\sqrt{a^2+c^2}\right)^2} = a^2 + b^2 + c^2$.

157. $\frac{1}{\sqrt{a}}.\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{b}}.\frac{1}{\sqrt{c}} + \frac{1}{\sqrt{c}}.\frac{1}{\sqrt{a}} \leq \sqrt{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right)} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

158. We have $\sqrt{a}(a+c-b)+\sqrt{b}(a+b-c)+\sqrt{c}(b+c-a)=$

$\sqrt{a(a+c-b)}\sqrt{a+c-b}+\sqrt{b(a+b-c)}\sqrt{a+b-c}+\sqrt{c(b+c-a)}\sqrt{b+c-a}$

$\leq\sqrt{a(a+c-b)+b(a+b-c)+c(b+c-a)}.\sqrt{(a+c-b)+a+b-c+b+c-a}$

$=\sqrt{a^2+b^2+c^2}\sqrt{a+b+c}.$

159. We have $\left[\left(\sqrt{a_1}\right)^2+\cdots+\left(\sqrt{a_n}\right)^2\right]\left[\left(\frac{1}{\sqrt{a_1}}\right)^2+\cdots+\left(\frac{1}{\sqrt{a_n}}\right)^2\right]\geq\left(\sqrt{a_1}.\frac{1}{\sqrt{a_1}}\cdots\sqrt{a_n}.\frac{1}{\sqrt{a_n}}\right)^2=$ n^2.

160. We have $(a_1^2+\cdots+a_n^2)\left(\underbrace{1^2+\cdots+1^2}_{n}\right)\geq(a_1+\cdots+a_n)^2$ whence

$\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}\geq\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^2.$

III
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