Miscellaneous Problems on A.P., G.P. and H.P.
Problems 61-70

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Problem 61

61. The pollution in a normal atmosphere is less that 0.01%. Due to leakage of gas from a factory the pollution increased to 20%. If everyday 80% of the pollution us neutralised, in how many days the atmosphere will be normal?

Solution of Problem 61

Solution: Each day pollution decreases by 80% i.e. 20% pollution remains. Thus common ratio for pollution remaining is $\frac{20}{100} = \frac{1}{5}$

Let it takes n days for pollution to become normal. Then,

$$a = 20, 20.\frac{1}{5^{n-1}} < .01$$

$$\Rightarrow n > 4$$

Thus, atmosphere becomes normal on 5th day.

Problem 62

62. The sides of a triangle are in G.P. and its largest angle is twice the smallest one. Prove that the common ratio of the G.P. lies in the interval $(1, \sqrt{2})$

Solution of Problem 62

Solution: Let a, ar, ar^2 be the sides of the G.P. with $r > 1$. Let smallest angle be α. The according to question largest angle will be 2α. Applying sine rule to smallest and largest angle

$$\frac{a}{\sin\alpha} = \frac{ar^2}{\sin 2\alpha} \Rightarrow \frac{\sin 2\alpha}{\sin\alpha} = r^2$$

$$\Rightarrow 2\cos\alpha = r^2$$

$$\because \alpha \neq 0 \therefore \cos\alpha < 1 \Rightarrow r^2 < 2 \Rightarrow r < \sqrt{2}$$

Thus, common ratio lies in the range $(1, \sqrt{2})$

Problem 63

63. If a, b, c, d are in G.P., then prove that $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$.

Solution of Problem 63

Solution: Let r be the common ratio of G.P. Then, $b = ar, c = ar^2, d = ar^3$

$$ax^3 + bx^2 + cx + d = ax^3 + arx^2 + ar^2x + ar^3 = a(x^2 + r^2)(x + r)$$

$$ax^2 + c = a(x^2 + r^2)$$

Clearly, $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$.

Problem 64

64. If a, b, c are three distinct real numbers and they are in G.P. If $a + b + c = xb$, then prove that $x < -1$ or $x > 3$.

Solution of Problem 64

Solution: Let r be the common ratio of the G.P. Then $b = ar, c = ar^2$ Given that

$$a + ar + ar^2 = xar \Rightarrow r^2 + (1 - x)r + 1 = 0$$

$$\because r \in R \Rightarrow D \geq 0 \Rightarrow (1 - x)^2 - 4 \geq 0 \Rightarrow x^2 - 2x - 3 \geq 0$$

$$(x + 1)(x - 3) \geq 0 \Rightarrow x \leq -1, x \geq 3$$

Problem 65

65. If a, b, c, d, p are real and $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \leq 0$. Show that a, b, c, d are in G.P. whose common ratio is p

Solution of Problem 65

Solution: Given, $(a^2+b^2+c^2)p^2-2(ab+bc+cd)p+(b^2+c^2+d^2)\leq 0$ and p is real.

$$D=0,(ab+bc+cd)^2-(a^2+b^2+c^2)(b^2+c^2+d^2)=0\Rightarrow (b^2-ac)^2+(c^2-bd)^2+(bc-ad)^2=0$$

$$b^2=ac, c^2=ad, bc=ad$$

Thus, a,b,c,d are in G.P. Let r be the common ratio then

$$p=\frac{ab+bc+cd}{a^2+b^2+c^2}=\frac{a^2r+a^2r^3+a^2r^5}{a^2+a^2r^2+a^2r^4}=r$$

Problem 66

66. If $2x^4 = y^4 + z^4, xyz = 8$ and $\log_y x, \log_z y, \log_x z$ are in G.P., show that $x = y = z = 2$.

Solution of Problem 66

Solution: $\log_y x, \log_z y, \log_x z$ are in G.P.

$$\Rightarrow 2\log_z y = \log_y x \log_x z \Rightarrow \left(\frac{\log y}{\log z}\right)^2 = \frac{\log x}{\log y}.\frac{\log z}{\log x}$$

$$\Rightarrow (\log y)^3 = (\log z)^3 \Rightarrow y = z$$

Also, $2x^4 = y^4 + z^4 \Rightarrow 2x^4 = 2y^4 \Rightarrow x = y \therefore x = y = z$

Also, $xyz = 8, \Rightarrow x = y = z = 2$

Problem 67

67. If a, b, c, d are in both A.P. and G.P. and $b = 2$, then find the number of such sequences.

Solution of Problem 67

Solution: Let r be the common ratio. Since a, b, c, d are in A.P.

$$\therefore b - a = c - b \Rightarrow a(r-1) = ar(r-1) \Rightarrow a(r-1)(r-1) = 0$$

$$\Rightarrow r = 1$$

Thus one such series is possible.

Problem 68

68. If $\log_x a, a^{\frac{x}{2}}, \log_b x$ are in G.P., then find x.

Solution of Problem 68

Solution: Given, $\log_x a, a^{\frac{x}{2}}, \log_b x$ are in G.P.

$$\therefore \left(a^{\frac{x}{2}}\right)^2 = \log_x a . \log_b x$$

$$\Rightarrow a^x = \log_b a \Rightarrow x = \log_a(\log_b a)$$

Problem 69

69. The $(m+n)$th and $(m-n)$th terms of a G.P. are p and q respectively. Show that mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$ respectively.

Solution of Problem 69

Solution: Let x be the first term and y be the common difference. Then,

$$t_{m+n} = xy^{m+n-1} = p, t_{m-n} = xy^{m-n-1} = q$$

$$\Rightarrow pq = x^2y^2(m-1) = (xy^{m-1})^2 = t_m^2 \Rightarrow t_p = \sqrt{pq}$$

$$\frac{q}{p} = y^{-2n} \Rightarrow y = \left(\frac{p}{q}\right)^{\frac{1}{2n}}$$

Now x and t_n can be found easily.

Problem 70

70. If the pth, qth and rth terms of an A.P. are in G.P., then find the common ratio of the G.P.

Solution of Problem 70

Solution: Let a be the first term and d be the common difference of A.P. Also, let $t_p = x$ and common ratio be r of the G.P. Then

$$a + (p-1)d = t_p = x, a + (q-1) = xr, a + (r-1) = xr^2$$

$$(q-p)d = x(r-1), (r-q)d = xr(r-1)$$

$$\Rightarrow r = \frac{r-q}{q-p}$$