Complex Numbers Problems 181-190

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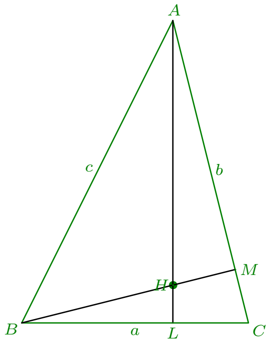
Problem 181

181. If the vertices of a $\triangle ABC$ are represented by z_1, z_2, z_3 respectively, then show that the orthocenter of $\triangle ABC$ is $\frac{z_1 a \sec A + z_2 b \sec B + z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$ or $\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$.

Solution of Problem 181

Solution: The diagram is given below:



Let AL be perpendicular on BC and H be orthocenter of the $\triangle ABC$.

$\frac{BL}{LC} = \frac{c\cos B}{b\cos C} = \frac{c\sec C}{b\sec B}$, thus L divides BC internally in the ratio of $c\sec C : b\sec B$

$L = \frac{z_3 c\sec C + z_2 b\sec B}{c\sec C + b\sec B}$

$\frac{AH}{HL} = \frac{\Delta ABH}{\Delta HBL} = \frac{\frac{1}{2}AB.BH\sin\angle ABM}{\frac{1}{2}BL.BH.\sin\angle MBC} = \frac{c\cos A}{c\cos B\cos C}[\because \angle ABM = 90^\circ - A, \angle MBC = 90^\circ - C]$

$= \frac{a\cos A}{a\cos B\cos C} = \frac{(b\cos C + c\cos B)\cos A}{a\cos B\cos C} = \frac{b\sec B + c\sec C}{a\sec A}$

$H = \frac{z_1 a\sec A + z_2 b\sec B + z_3 c\sec C}{a\sec A + b\sec B + c\sec C}$

Since the above expression is similar w.r.t. A, B and C, therefore it will also lie on the perpendiculars from B and C to opposing sides as well.

Thus, orthocenter $H = \frac{z_1 a\sec A + z_2 b\sec B + z_3 c\sec C}{a\sec A + b\sec B + c\sec C}$

$H = \frac{z_1 k\sin A\sec A + z_2 k\sin B\sec B + z_3 k\sin C\sec C}{k\sin A\sec A + k\sin B\sec B + k\sin C\sec C}$

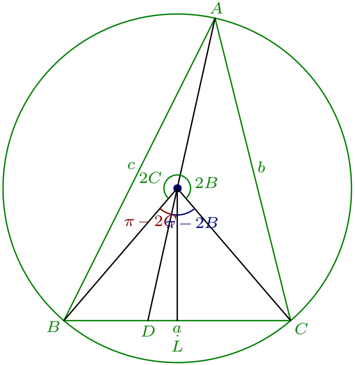
$H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$

Problem 182

182. If the vertices of a $\triangle ABC$ are represented by z_1, z_2 and z_3 respectively, show that its circumcenter is $\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$.

Solution of Problem 182

Solution: The diagram is given below:



Solution of Problem 182

Solution: Let O be the circumcenter of $\triangle ABC$ where $A = z_1, B = z_2$ and $C = z_3$.

$\frac{BD}{DC} = \frac{\frac{1}{2}BD.OL}{\frac{1}{2}DC.OL} = \frac{\Delta BOD}{\Delta COD}$

$= \frac{\frac{1}{2}OB.OD.\sin(\pi-2C)}{\frac{1}{2}OC.OD\sin(\pi-2C)} = \frac{\sin 2C}{\sin 2B}$

Thus, D divides BC internally in the ratio $\sin 2C : \sin 2B \Rightarrow D = \frac{z_3 \sin 2C + z_2 \sin 2B}{\sin 2C + \sin 2B}$

The complex number dividing AD internally in the ratio $\sin 2B + \sin 2C : \sin 2A$ is

$\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

Since the above expression is similar w.r.t. A, B and C, therefore it will also lie on the perpendicular bisectors on AC and AB as well.

Let BO produced meet AC at E and CO produced meet AB at F. We can show that, the complex numner representing the point dividing the line segment BE internally in the ratio $(\sin 2C + \sin 2A) : \sin 2B$ and the complex number representing the point dividing the line segment CF internally in the ratio $(\sin 2A + \sin 2B) : \sin 2C$ will be each $= \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

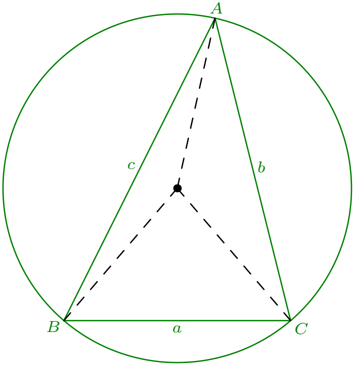
Thus, circumcenter is $\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

Problem 183

183. Show that the circumcenter of the triangle whose vertices are given by the complex numbers z_1, z_2, z_3 is given by $z = \frac{\sum z_1\overline{z_1}(z_2-z_3)}{\sum \overline{z_1}(z_2-z_3)}$.

Solution of Problem 183

Solution: Consider the diagram given below:



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Let z be the circumcenter of the triangle represented by $A(z_1), B(z_2)$ and $C(z_3)$ respectively, then

$|z-z_1| = |z-z_2| = |z-z_3|$ so we have $|z-z_1| = |z-z_2|$

$\Rightarrow |z-z_1|^2 = |z-z_2|^2 \Rightarrow (z-z_1)(\overline{z}-\overline{z_1}) = (z-z_2)(\overline{z}-\overline{z_2})$

$\Rightarrow z\overline{z} + z_1\overline{z_1} - \overline{z}z_1 - z\overline{z_1} = z\overline{z} + z_2\overline{z_1} - \overline{z}z_2 - z\overline{z_2}$

$$\Rightarrow z(\overline{z_1}-\overline{z_2}) + \overline{z}(z_1-z_2) = z_1\overline{z_1} - z_2\overline{z_2} \tag{1}$$

Similarly considering $|z-z_1| = |z-z_3|$, we will have

$$\Rightarrow z(\overline{z_1}-\overline{z_3}) + \overline{z}(z_1-z_3) = z_1\overline{z_1} - z_3\overline{z_3} \tag{2}$$

We have to eliminate $\overline{z}$ from equation (1) and (2) i.e. multiplying equation (1) with (z_1-z_3) and (2) with (z_1-z_2), we get following

$z[\overline{z_1}(z_2-z_3) + \overline{z_2}(z_3-z_1) + \overline{z_3}(z_1-z_2)] = z_1\overline{z_1}(z_2-z_3) + z_2\overline{z_2}(z_3-z_1) + z_3\overline{z_3}(z_1-z_2)$

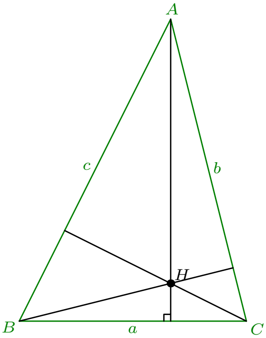
$\Rightarrow z = \frac{\sum z_1\overline{z_1}(z_2-z_3)}{\sum \overline{z_1}(z_2-z_3)}$

Problem 184

184. Find the orthocenter of the triangle with vertices z_1, z_2, z_3.

Solution of Problem 184

Solution:



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Let z be the orthocenter of $\triangle A(z_1)B(z_2)C(z_3)$ i.e. the intersection point of perpendiculars on sides from opposite vertices.

Since $AH \perp BC \therefore \arg\left(\frac{z_1-z}{z_3-z_2}\right) = \pm\frac{\pi}{2}$

$\Rightarrow \frac{z_1-z}{z_3-z_2}$ is purely imaginary.

$\Rightarrow \overline{\left(\frac{z_1-z}{z_3-z_2}\right)} = -\left(\frac{z_1-z}{z_3-z_2}\right) \Rightarrow \frac{\overline{z_1}-\overline{z}}{\overline{z_3}-\overline{z_2}} = \frac{z-z_1}{z_3-z_2}$

$\Rightarrow \overline{z_1} - \overline{z} = \frac{(z-z_1)(\overline{z_3}-\overline{z_2})}{z_3-z_2}$ Similarly for $BH \perp AC, \overline{z_2} - \overline{z} = \frac{(z-z_2)(\overline{z_1}-\overline{z_2})}{z_1-z_3}$

Eliminating $\overline{z}$ like last problem we arrive at the desired result.

Problem 185

185. $ABCD$ is a rhombus described in clockwise direction. Suppose that the vertices A, B, C, D are given by z_1, z_2, z_3, z_4 respectively and $\angle CBA = 2\pi/3$. Show that $2\sqrt{3}z_2 = (\sqrt{3} - i)z_1 + (\sqrt{3} + i)z_3$ and $2\sqrt{3}z_4 = (\sqrt{3} + i)z_1 + (\sqrt{3} - i)z_3$.

Solution of Problem 185

Solution: We have $\angle CBA = \frac{2\pi}{3}$, therefore

$$\frac{z_3-z_2}{z_1-z_2} = \frac{|z_3-z_2|}{|z_1-z_2|}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$$

$$\frac{z_3-z_2}{z_1-z_2} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}[\because BC = AB]$$

$$z_3 + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z_1 = \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) z_2$$

Solving this yields $2\sqrt{3}z_2 = (\sqrt{3}-i)z_1 + (\sqrt{3}+i)z_3$

Also, since diagonals bisect each other $\Rightarrow \frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$

$$z_4 = z_1 + z_3 - z_2$$

Substituting the value of z_2, we get

$$2\sqrt{3}z_4 = (\sqrt{3}+i)z_1 + (\sqrt{3}-i)z_3$$

Problem 186

186. The points P, Q and R represent the numbers z_1, z_2 and z_3 respectively and the angles of the $\triangle PQR$ at Q and R are both $\frac{1}{2}(\pi - \alpha)$. Prove that $(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$.

Solution of Problem 186

Solution: Since $\angle PQR = \angle PRQ = \frac{1}{2}(\pi - \alpha) \therefore PQ = PR$ Also, $\angle QPR = \pi - 2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \alpha$

$\therefore arg\frac{z_3 - z_1}{z_2 - z_1} = \alpha \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \frac{PR}{RQ}(\cos\alpha + i\sin\alpha)$

$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} - 1 = (\cos\alpha - 1) + i\sin\alpha \Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = -2\sin^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$\Rightarrow \left(\frac{z_3 - z_2}{z_2 - z_1}\right)^2 = -4\sin^2\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]^2 = -4\sin^2\frac{\alpha}{2}[\cos\alpha + i\sin\alpha] = -4\sin^2\frac{\alpha}{2}.\frac{z_3 - z_1}{z_2 - z_1}$

$\Rightarrow (z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$

Problem 187

187. Points z_1 and z_2 are adjacent vertices of a regular polygon of n sides. Find the vertex z_3 adjacent to $z_2 (z_1 \neq z_3)$.

Solution of Problem 187

Solution: Let C be the center of a regular polygon of n sides. Let $A_1(z_1), A_2(z_2)$ and $A_3(z_3)$ be its three consecutive vertices.

$\angle CA_2A_1 = \frac{1}{2}\left(\pi - \frac{2\pi}{n}\right) \therefore A_1A_2A_3 = \pi - \frac{2\pi}{n}$

Case I: When z_1, z_2, z_3 are in anticlockwise order. $\Rightarrow z_1 - z_2 = (z_3 - z_2)e^{i(\pi - 2\pi/n)}[\because A_1A_2 = A_3A_2]$

$z_1 - z_2 = (z_2 - z_3)e^{-i2\pi/n}[\because e^{i\pi} = -1] \Rightarrow z_3 = z_2 - (z_1 - z_2)e^{i2\pi/n}$

Case II: When z_1, z_2, z_3 are in clockwise order. $\Rightarrow z_3 - z_2 = (z_1 - z_2)e^{i(\pi - i2\pi/n)}$

$z_3 = z_2 + (z_2 - z_1)e^{-i2\pi/n}$

Problem 188

188. Let $A_1, A_2, \ldots, A_n$ be the vertices of an n sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, find the value of n.

Solution of Problem 188

Solution: Let O be the origin and the complex number representing A_1 be z, then A_2, A_3, A_4 will be represented by $ze^{i2\pi/n}, ze^{i4\pi/n}, ze^{i6\pi/n}$. Let $|z| = a$

$$A_1A_2 = \left|z - ze^{i2\pi/n}\right| = |z|\left|1 - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right|$$

$$= a\sqrt{\left(1 - \cos\frac{2\pi}{n}\right)^2 + \sin^2\frac{2\pi}{n}} = a\sqrt{2\left(1 - \cos\frac{2\pi}{n}\right)} = 2a\sin\frac{\pi}{n}$$

Similarly, $A_1A_3 = 2a\sin\frac{2\pi}{n}$ and $A_1A_4 = 2a\sin\frac{3\pi}{n}$

Given $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4} \therefore \frac{1}{2a\sin\frac{\pi}{n}} = \frac{1}{2a\sin\frac{2\pi}{n}} + \frac{1}{2a\sin\frac{3\pi}{n}}$

$$\Rightarrow \sin\frac{\pi}{n}\left(\sin\frac{3\pi}{n} + \sin\frac{2\pi}{n}\right) = \sin\frac{2\pi}{n}\sin\frac{3\pi}{n}$$

$$\Rightarrow \sin\frac{3\pi}{n} + \sin\frac{2\pi}{n} = 2\cos\frac{2\pi}{n}\sin\frac{3\pi}{n} = \sin\frac{4\pi}{n} + \sin\frac{2\pi}{n}$$

$$\Rightarrow \sin\frac{3\pi}{n} = \sin\frac{4\pi}{n} \Rightarrow \frac{3\pi}{n} = m\pi + (-1)^n\frac{4\pi}{n}, m = 0, \pm 1, \pm 2, \ldots$$

If $m = 0 \Rightarrow \frac{3\pi}{n} = \frac{4\pi}{n} \Rightarrow 3 = 4$ (not possible)

If $m = 1 \Rightarrow \frac{3\pi}{n} = \pi - \frac{4\pi}{n} \Rightarrow n = 7$

If $m = 2, 3 \ldots, -1, -2, \ldots$ gives values of n which are not possible. Thus $n = 7$.

Problem 189

189. If $|z| = 2$, then show that the points representing the complex numbers $-1 + 5z$ lie on a circle.

Solution of Problem 189

Solution: Given, $|z| = 2$. Let $z_1 = -1 + 5z \Rightarrow z_1 + 1 = 5z$

$|z_1 + 1| = |5z| = 5|z| = 10$

$\Rightarrow z_1$ lies on a circle with center $(-1, 0)$ having radius 10.

Problem 190

190. If $|z-4+3i| \leq 2,$ find the least and tghe greatest values of $|z|$ and hence find the limits between which $|z|$ lies.

Solution of Problem 190

Solution: Given, $|z-4+3i| \leq 2 \Rightarrow ||z|-|4-3i|| \leq 2$

$\Rightarrow ||z|-5| \leq 2 \Rightarrow -2 \leq |z|-5 \leq 2 \Rightarrow 3 \leq |z| \leq 7$