

Summation of Series Problems 31-40

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Problem 31

31. Find the sum of the series $1.2.3 + 2.3.5 + 3.4.7 + \dots$ to n terms.

Solution of Problem 31

Solution:

$$\begin{aligned}t_n &= n(n+1)(2n+1) = 2n^3 + 3n^2 + n \\S_n &= \sum t_n = 2 \sum n^3 + 3 \sum n^2 + \sum n \\&= 2 \left[\frac{n(n+1)}{2} \right]^2 + 3 \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\&= \frac{1}{2} n(n+1)^2 (n+2)\end{aligned}$$

Problem 32

32. Find the sum of the series $1.2.3 + 2.3.4 + 3.4.5 + \dots$ to n terms

Solution of Problem 32

Solution:

$$\begin{aligned}t_n &= n(n+1)(n+2) = n^3 + 3n^2 + 2n \\S_n &= \sum t_n = \sum n^3 + 3 \sum n^2 + 2 \sum n \\&= \left[\frac{n(n+1)}{2} \right]^2 + 3 \frac{n(n+1)(n+2)}{6} + 2 \frac{n(n+1)}{2} \\&= \frac{1}{4} n(n+1)(n+2)(n+3)\end{aligned}$$

Problem 33

33. Find the sum of the series $1.3^2 + 2.5^2 + 3.7^2 + \dots$ to 20 terms.

Solution of Problem 33

Solution:

$$t_n = n(2n + 1)^2 = 4n^3 + 4n^2 + n$$

$$S_{20} = 4 \sum_{n=1}^{20} n^3 + 4 \sum_{n=1}^{20} n^2 + \sum_{n=1}^{20} n$$

$$\begin{aligned} &= 4 \left[\frac{20 \cdot 21}{2} \right]^2 + 4 \cdot \frac{20 \cdot 21 \cdot 41}{6} + \frac{20 \cdot 21}{2} \\ &= 188090 \end{aligned}$$

Problem 34

34. Find the sum of the series $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \dots$ to n terms.

Solution of Problem 34

Solution:

$$t_i = i(n^2 - i^2) = n^2i - i^3$$

$$S_n = n^2 \sum_{i=1}^n i - \sum_{i=1}^n i^3$$

$$\begin{aligned} S_n &= n^2 \frac{n(n+1)}{2} - \left[\frac{n(n+1)}{2} \right]^2 \\ &= \frac{1}{4} n^2 (n^2 - 1) \end{aligned}$$

Problem 35

35. Find the sum of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$ to 10 terms.

Solution of Problem 35

Solution:

$$t_n = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$\begin{aligned} S_n &= \frac{1}{3} \sum n^3 + \frac{1}{2} \sum n^2 + \frac{1}{6} \sum n \\ &= \frac{1}{3} \left[\frac{n(n+1)}{2} \right]^2 + \frac{1}{2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{12} n(n+1) \end{aligned}$$

Substituting $n = 10$, we get

$$S_{10} = 1210$$

Problem 36

36. Find the sum of the series $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \dots$ to 10 terms.

Solution of Problem 36

Solution:

$$t_n = [(2n+1)^3 - (2n)^3] = 12n^2 + 6n + 1$$

$$S_n = 12 \sum n^2 + 6 \sum n + \sum 1$$

$$S_n = 2n(n+1)(2n+1) + 3n(n+1) + n$$

Substituting $n = 10$, we get

$$S_{10} = 4960$$

Problem 37

37. Find the sum of the series $1 + \frac{1}{1 \cdot 2} + \frac{1}{1+2+3} + \dots$ to n terms.

Solution of Problem 37

37.

$$t_n = \frac{1}{1+2+3+\dots+n} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$t_{n-1} = 2\left(\frac{1}{n-1} - \frac{1}{n}\right)$$

...

$$t_3 = 2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$t_2 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$t_1 = 2\left(1 - \frac{1}{2}\right)$$

Adding, we get

$$S_n = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$$

Problem 38

38. Find the sum to infinity of the series $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} + \dots$

Solution of Problem 38

Solution:

$$t_n = \frac{1}{2n \cdot 2(n+1)} = \frac{1}{4} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$t_1 = \frac{1}{4} \left(1 - \frac{1}{2} \right)$$

$$t_2 = \frac{1}{4} \left(\frac{1}{2} - \frac{1}{3} \right)$$

...

$$t_{n-1} = \frac{1}{4} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$t_n = \frac{1}{4} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

As n approaches ∞ , t_n approaches zero. Thus, $S_\infty = \frac{1}{4}$

Problem 39

39. Find the sum of the series $2 + 6 + 12 + 20 + \dots$ to n terms.

Solution of Problem 39

Solution:

$$S_n = 2 + 6 + 12 + 20 + \dots + t_n$$

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Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = 2 + 4 + 6 + 8 + \dots \text{ to } n \text{ terms} - t_n)$$

$$t_n = \frac{n}{2}[2.2 + (n-1)2] = n(n+1)$$

$$\begin{aligned} S_n = \sum t_n &= \sum n^2 + \sum n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{1}{3}n(n+1)(n+2) \end{aligned}$$

Problem 40

40. Find the sum of the series $3 + 6 + 11 + 18 + \dots$ to n terms.

Solution of Problem 40

Solution:

$$S_n = 3 + 6 + 11 + 18 + \dots + t_n$$

$$S_n = 3 + 6 + 11 + 18 + \dots + t_n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = 3 + [3 + 5 + 7 + \dots \text{ to } n \text{ terms} - t_n]$$

$$t_n = 3 + \frac{n-1}{2}[2 \cdot 3 + (n-1) \cdot 2] = 3 + (n-1)(n+2)$$

$$= n^2 + n + 1$$

$$S_n = \sum t_n = \sum n^2 + \sum n + \sum 1$$

$$= \frac{1}{6}n(2n^2 + 3n + 13)$$