Arithmetic, Geometric and Harmonic Means Problems 11-20

Shiv Shankar Dayal

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Problem 11

11. If a, b, c are in G.P. and x, y be the A.M. between a, b and b, c respectively, show that $\frac{a}{x} + \frac{b}{y} = 2, \frac{1}{x} + \frac{1}{y} = \frac{2}{b}$

Solution of Problem 11

Solution: Given a, b, c are in G.P., if we let r to be the common ratio then $b = ar, c = ar^2$. Also, given $x = \frac{a+b}{2}, y = \frac{b+c}{2}$

$$\frac{a}{x} + \frac{b}{y} = \frac{2a}{a+b} + \frac{2c}{b+c} = \frac{2a}{a(1+r)} + \frac{2ar^2}{a(1+r)} = 2$$

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c} = \frac{2}{a(1+r)}\left(1 + \frac{1}{r}\right) = \frac{2}{b}$$

Problem 12

12. If A be the A.M. and H be the H.M. between two numbers, a and b, prove that $\frac{a-A}{a-H}\frac{b-A}{b-H} = \frac{A}{H}$

Solution of Problem 12

Solution: We know that, $A = \frac{a+b}{2}, H = \frac{2ab}{a+b}$

$$\frac{a-A}{a-H}.\frac{b-A}{b-H} = \frac{(a-b)(b-a)(a+b)^2}{4(a^2+ab-2ab)(ab+b^2-2ab)} = \frac{(a+b)^2}{4ab} = \frac{A}{H}$$

Problem 13

13. If A_1, A_2 be the A.M., G_1, G_2 be the G.M. and H_1, H_2 be the H.M. between any two numbers, show that $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}$

Solution of Problem 13

Solution: Let the two numbers be a and b

$$\therefore A_1 + A_2 = a + b, G_1G_2 = ab$$

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

Thus, $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}$

Problem 14

14. The arithmetic mean of two numbers exceed their geometric mean by $\frac{3}{2}$ and the geometric mean exceeds their harmonic mean by $\frac{6}{5}$, find the numbers.

Solution of Problem 14

Solution: Let the two numbers be a and b and A, G, H be the respective A.M., G.M., H.M. between them.

$$A = G + \frac{3}{2}, G = H + \frac{6}{5}$$

$$AH = G^2 \Rightarrow \left(G + \frac{3}{2}\right)\left(G - \frac{6}{5}\right) = G^2 \Rightarrow G = 6$$

$$\Rightarrow a + b = 15, ab = 36$$

So the numbers are 12 and 3.

Problem 15

15. If a, b, c, d be four distinct numbers in H.P., show that $a + d > b + c$ and $ad > bc$

Solution of Problem 15

Solution: Since a, b, c, d are in H.P. thus, b is H.M. of a and c and c is H.M. of b and d.
Since A.M. > H.M. $\therefore \frac{a+c}{2} > b$ or $a + c > 2b$
Similarly $b + d < 2c$. Adding these two $a + b + c + d > 2(b + c) \Rightarrow a + d > b + c$
Also, since G.M. > H.M. $\sqrt{ac} > b$ and $\sqrt{bd} > c$ multiplying these two $ad > bc$

Problem 16

16. If three positibve unequal numbers a, b, c be in H.P., prove that $a^n + c^n > 2b^n$, where n is a positive integer.

Solution of Problem 16

Solution: Since a, b, c are in H.P. b is H.M. of a and d

Since G.M. > H.M. $\therefore \sqrt{ac} > b$

Now, A.M. of a^n and $c^n = \frac{a^n+c^n}{2}$ and G.M. $= (\sqrt{ac})^n$

But A.M. > G.M. $\therefore \frac{a^n+b^n}{2} > (\sqrt{ac})^n > b^n$

Problem 17

17. $x+y+z=15,$ if a,x,y,z,b are in A.P., and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ if a,x,y,z,b are in H.P., find a and b.

Solution of Problem 17

Solution: $a + b = x + y + z = \frac{a+b}{2}.3 \Rightarrow a + b = 10$ when they are in A.P.

When they are in H.P. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{(\frac{1}{a} + \frac{1}{b})}{2} \Rightarrow ab = 9$

Thus, numbers are $a = 9, b = 1$

Problem 18

18. If $x > 0$, prove that $x + \frac{1}{x} \geq 2$

Solution of Problem 18

Solution: We know that A.M. $\geq$ G.M

$$\therefore \frac{x + \frac{1}{x}}{2} \geq \sqrt{x.\frac{1}{x}} \Rightarrow x + \frac{1}{x} \geq 2$$

Problem 19

19. Insert 8 A.M. between 5 and 32.

Solution of Problem 19

Solution: Let $x_1, x_2, \dots, x_8$ are 8 arithmetic means between 5 and 32. Let d be the common difference. Then,

$$32 = 5 + 9d \Rightarrow d = 3$$

Thus, the means are $8, 11, 14, 17, 20, 23, 26, 29$

Problem 20

20. Insert 7 A.M. between 2 and 34.

Solution of Problem 20

Solution: Let $x_1, x_2, \dots, x_7$ are 7 arithmetic means between 2 and 34. Let d be the common difference. Then,

$$34 = 2 + 8d \Rightarrow d = 4$$

Thus, means are $6, 10, 14, 18, 22, 26, 30$