Logarithm Problem 21-30

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Problem 21

21. Simplify $\frac{\log_9 11}{\log_5 13} \div \frac{\log_3 11}{\log_{\sqrt{5}} 13}$

Solution of Problem 21

Solution: Given

$$
\begin{aligned}
&\frac{\log_9 11}{\log_5 13} \div \frac{\log_3 11}{\log_{\sqrt{5}} 13} \\
&= \frac{\log_{3^2} 11}{\log_5 13} \cdot \frac{\log_{5^{\frac{1}{2}}} 13}{\log_3 11} \\
&= \frac{\frac{1}{2}\log_3 11}{\log_5 13} \cdot \frac{2\log_5 13}{\log_3 11} = 1
\end{aligned}
$$

Problem 22

22. Simplify $3^{\sqrt{\log_3 2}} - 2^{\sqrt{\log_2 3}}$

Solution of Problem 22

Solution: Taking $\log$ with base 10, we get

$$
\begin{aligned}
&= \sqrt{\log_3 2}\log 3 - \sqrt{\log_2 3}\log 2 \\
&= \sqrt{\frac{\log 2}{\log 3}(\log 3)^2} - \sqrt{\frac{\log 3}{\log 2}(\log 2)^2} \\
&= \sqrt{\log 2 \log 3} - \sqrt{\log 3 \log 2} = 0
\end{aligned}
$$

Problem 23

23. Find the least integer n such that $7^n > 10^5$, given that $\log_{10} 343 = 2.5353$

Solution of Problem 23

Solution:

$$\log_{10} 343 = 2.5353 \Rightarrow \log_{10} 7^3 = 2.5353 \Rightarrow \log_{10} 7 = 0.8451$$

$$7^n > 10^5 \Rightarrow n \log_{10} 7 > 5 \Rightarrow n > \frac{5}{0.8451}$$

Thus, least value of such integer is 6.

Problem 24

24. If a, b, c are in G.P. then prove that $\log_a x, \log_b x, \log_c x$ are in H.P.

Solution of Problem 24

Solution: Since a, b, c are in G.P. therefore we can write $b^{=}ac$

Taking log, on both sides, we get $2\log b = \log a + \log c$. Thus, $\log a, \log b, \log c$ are in A.P.

$\therefore \frac{1}{\log a}, \frac{1}{\log b}, \frac{1}{\log c}$ are in H.P.

$\therefore \frac{\log x}{\log a}, \frac{\log x}{\log b}, \frac{\log x}{\log c}$ are in H.P.

$\therefore \log_x a, \log_x b, \log_x c$ are in H.P.

Problem 25

25. Prove that $\log \sin 8x = 3 \log 2 + \log \sin x + \log \cos x + \log \cos 2x + \log \cos 4x$

Solution of Problem 25

Solution:

$$
\begin{aligned}
L.H.S. - \log \sin 8x &= \log 2 \sin 4x \cos 4x \\
&= \log 2 + \log \sin 4x + \log \cos 4x \\
&= \log 2 + \log 2 \sin 2x \cos 2x + \log \cos 4x \\
&= 2 \log 2 + \log \sin 2x + \log \cos 2x + \log \cos 4x \\
&= 2 \log 2 + \log 2 \sin x \cos x + \log \cos 2x + \log \cos 4x \\
&= 4 \log 2 + \log \sin x + \log \cos x + \log \cos 2x + \log \cos 4x = R.H.S.
\end{aligned}
$$

Problem 26

26. If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, then prove that $xyz + 1 = 2yz$

Solution of Problem 26

Solution: We have to prove that $xyz + 1 = 2yx$

Dividing both sides with yz, we get $x + \frac{1}{yz} = 2$

$$L.H.S. = \log_{2a} a + \frac{1}{\log_{3a} 2a \log_{4a} 3a}$$

$$= \frac{\log a}{\log 2a} + \frac{\log 3a \log 4a}{\log 2a \log 3a}$$

$$= \frac{\log a}{\log 2a} + \frac{\log 4a}{\log 2a}$$

$$= \frac{\log a + \log 4a}{\log 2a} = \frac{\log (2a)^2}{\log 2a} = 2 = R.H.S.$$

Problem 27

27. If a and b are lengths of the sides and c be the length of hypotenuse of a right angle triangle and $c - b \neq 1$ and $c + b \neq 1$, prove that $\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a$

Solution of Problem 27

Solution: We have to prove that $\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a$

Dividing both side by $\log_{c+b} a \log_{c-b} a$, we get

$$L.H.S. = \frac{1}{\log_{c-b} a} + \frac{1}{\log_{c+b} a} = 2$$

$$\log_a (c-b) + \log_a (c+b) = 2$$

$$\log_a (c^2 - b^2) = a \Rightarrow c^2 - b^2 = a^2$$

$$\Rightarrow c^2 = a^2 + b^2$$

which is true for the given right angle triangle.

Problem 28

28. If $\frac{\log z}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, then prove that $x^x y^y z^x = 1$

Solution of Problem 28

Solution: Let $\frac{\log z}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$

$$\Rightarrow \log x = k(y-z), \log y = k(z-x), \log z = k(x-y)$$

We have to prove that $x^x y^y z^z = 1$ Taking $\log$ of both sides, we get

$$x \log x + y \log y + z \log z = 0$$

$$kx(y-z) + ky(z-x) + kz(x-y) = 0$$

$$0 = 0$$

Problem 29

29. If $\frac{yz\log(yz)}{y+z} = \frac{zx\log(zx)}{z+x} = \frac{xy\log(xy)}{x+y}$, then prove that $x^x = y^y = z^z$

Solution of Problem 29

Solution: Given, $\frac{yz\log(yz)}{y+z} = \frac{zx\log(zx)}{z+x} = \frac{xy\log(xy)}{x+y}$

Dividing each term by xyz, we get

$$\frac{\log y + \log z}{xy + zx} = \frac{\log z + \log x}{yz + xy} = \frac{\log x + \log y}{zx + yz} = k(\text{let})$$

$$\log y + \log z = k(xy + zx), \log z + \log x = k(yz + zx), \log x + \log y = k(zx + yz)$$

$$2(\log x + \log y + \log z) = 2k(xy + yz + zx)$$

$$\therefore\ \log x = kyz \Rightarrow x\log x = kxyz$$

$$\text{Similarly, } \log y = kzx \Rightarrow y\log y = kxyz$$

$$\log z = kxy \Rightarrow z\log z = kxyz$$

$$\therefore x\log x = y\log y = z\log z$$

$$\therefore x^x = y^y = z^z$$

Problem 30

30. Prove that $(yz)^{\log y - \log z} zx^{\log z - \log x} (xy)^{\log x - \log y} = 1$

Solution of Problem 30

Solution: Taking log of both sides, we get

$$(\log y - \log z)(\log y + \log z) + (\log z - \log x)(\log z + \log x) + (\log x - \log y)(\log x + \log y) = 0$$

$$(\log y)^2 - (\log z)^2 + (\log z)^2 - (\log x)^2 + (\log x)^2 - (\log y)^2 = 0$$

$$0 = 0$$