Geometric Progression
Problems 101-113

Shiv Shankar Dayal

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Problem 101

101. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in A.P., such that $|a| < 1, |b| < 1, |c| < 1$, then show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P. as well.

Solution of Problem 101

Solution.

$$x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$$

$$\therefore \frac{1}{x} = 1-a, \frac{1}{y} = 1-b, \frac{1}{z} = 1-c$$

which are in A.P. because a, b, c are in A.P.

Problem 102

102. Given that $0 < x < \frac{\pi}{4}, \frac{\pi}{4} < y < \frac{\pi}{2}$ and $\sum_{k=0}^{\infty}(-1)^k \tan^{2k} x = p, \sum_{k=0}^{\infty}(-1)^k \cot^{2k} y = q$ then prove that $\sum_{k=0}^{\infty} \tan^{2k} x \cot^{2k} y$ is $\frac{1}{\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}}$

Solution of Problem 102

Solution: For $p, a = 1, r = -\tan^2 x$

$$\therefore p = \frac{1}{1+\tan^2 x} = \cos^2 x$$

For $q, a = 1, r = -\cot^2 y$

$$\therefore q = \frac{1}{1+\cot^2 y} = \sin^2 y$$

$$\therefore S = \frac{1}{1-\tan^2 x \cot^2 y} = \frac{1}{1-\frac{1-\cos^2 x}{\cos^2 x}\frac{1-\sin^y}{\cos^2 y}}$$

$$= \frac{pq}{p+q-1} = \frac{1}{\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}}$$

Problem 103

103. An equilateral triangle is drawn by joining the mid-points of a given equilateral triangle. A third equilateral triangle is drawn inside the second in the same manner and the process is continued indefinitely. If the side of first equilateral triangle is $3^{1/4}$ inch, then find the sum of areas of all these triangles.

Solution of Problem 103

Solution: Let side of outermost equilateral triangle is a, then its area is $\frac{\sqrt{3}}{4}a^2$. The sides of subsequent internal triangles will be $\frac{a}{2}, \frac{a}{4}, \frac{a}{8}, ...$

Therefore, total area is $\frac{\sqrt{3}}{4}a^2\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+...\right)$

$$= \frac{\sqrt{3}}{4}a^2.\frac{1}{4}\frac{1}{1-\frac{1}{4}} = 1$$

Problem 104

104. If $S = exp(1 + |\cos x| + \cos^2 x + |\cos^3 x| + \cos^4 x \ldots$ to $\infty) \log_c 4$ satisfies the roots of the equation $t^2 - 20t + 64 = 0$ for $0 < x < \pi$ then find the values of x.

Solution of Probolem 104

Solution: $\cos^2 x = |\cos^2 x|$
Sum of infinite series is $S = \frac{1}{1-|\cos x|}$ where $|\cos x| < 1$
$E = e^{S log_e 4} = 4^S$
E satisfie the equation $t^2 - 20t + 64 = 0 \therefore t = 16, 6$
$\Rightarrow S = 1, 2 \Rightarrow |\cos x| = 0, \pm\frac{1}{2}$
$x = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$

Problem 105

105. If $S \subset (-\pi, \pi)$, denote the set of values of x satisfying the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots \text{ to } \infty} = 4^3$ then find the value of S.

Solution of Problem 105

Solution: The given equation may be written as

$$8^{1+|\cos x|+|\cos^2 x|+|\cos^3 x|+\dots \text{ to } \infty} = 4^3 = 8^2$$

$$1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \dots \text{ to } \infty = 2$$

To sum the G.P., we must observer that for $-\pi < x < \pi$, we have $|\cos x| < 1$

$$\therefore \frac{1}{1 - |\cos x|} = 2 \Rightarrow \cos x = \pm 1/2$$

Problem 106

106. If $0 < x < \frac{\pi}{2}$ and $2^{\sin^2 x + \sin^4 x + \ldots \text{ to } \infty}$ satisfies the roots of the equation $x^2 - 9x + 8 = 0$, then find the value of $\cos x / (\cos x + \sin x)$

Solution of Problem 106

Solution:

$$S_\infty = \frac{\sin^x}{1-\sin^x} = \tan^2 x$$

$$L.H.S. = 2^{\tan^2 x}$$

The roots of the equattion $x^2 - 9x + 8 = 0$ are 1 and 8

$$2^{\tan^2 x} = 1 = 2^0, 2^{\tan^2 x} = 8 = 2^3$$

$$\therefore \tan^2 x = 0, \tan^2 x = 3$$

$\therefore x = \dfrac{\pi}{3}$ is the only value of x satisfying the condition $0 < x < \dfrac{\pi}{2}$

$$\frac{\cos x}{\cos x + \sin x} = \frac{1}{1 + \tan x} = \frac{1}{1 + \sqrt{3}}$$

Problem 107

107. If $S_\lambda = \sum_{r=0}^{\infty} \frac{1}{\lambda^r}$, then find $\sum_{\lambda=1}^{n} (\lambda - 1) S_\lambda$

Solution of Problem 107

Solution:

$$S_\lambda = \frac{\lambda}{\lambda - 1}$$

$$\sum_{\lambda=1}^{n} (\lambda - 1) S_\lambda = \sum_{\lambda=1}^{n} \lambda = \frac{n(n+1)}{2}$$

Problem 108

108. If a, b, c are in A.P. then prove that $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ are in G.P. $\forall x \neq 0$

Solution of Problem 108

Solution:

$$\frac{2^{bx+1}}{2^{ax+1}} = \frac{2^{cx+1}}{2^{bx+1}}$$

$$(b-a)x = (c-b)x \Rightarrow b-a = c-b \;\forall x \neq 0$$

Above is true as a, b, c are in A.P.

Problem 109

109. If $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} = \frac{c+de^x}{c-de^x}$ then prove that a, b, c, d are in G.P.

Solution of Problem 109

Solution: Writing $a + be^x = 2a - (a - be^x)$, we have

$$\frac{2a}{a - be^x} - 1 = \frac{2b}{b - ce^x} - 1 = \frac{2c}{c - de^x} - 1$$

$$\Rightarrow \frac{a - be^x}{a} = \frac{b - ce^x}{b} = \frac{c - de^x}{c}$$

$$1 - \frac{b}{a}e^x = 1 - \frac{c}{b}e^x = 1 - \frac{d}{c}e^x$$

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Thus, a, b, c, d are in G.P.

Problem 110

110. If x, y, z arein G.P. and $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in A.P. then prove that $x = y = z$ but their common values are not necessarily zero.

Solution of Problem 110

Solution: Since x, y, z are in G.P. $y^2 = xz$ and $2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

$$\frac{2y}{1-y^2} = \frac{x+z}{1-xz} \Rightarrow 2y = x+z$$

$$4y^2 = (x+z)^2 \Rightarrow (x-z)^2 = 0 \Rightarrow x = z$$

$$\therefore x = y = z$$

Problem 111

111. If a, b, c are three unequal numbers such that a, b, c are in A.P. and $b-a, c-b, a$ are in G.P. then prove that $a:b:c=1:2:3$

Solution of Problem 111

Solution:

$$b - a = c - b, (c - b)^2 = a(b - a)$$

$$\Rightarrow (b - a)^2 = a(b - a) \Rightarrow b = 2a$$

$$c = 2b - a = 3a$$

$$\therefore a : b : c = 1 : 2 : 3$$

Problem 112

112. The sides a, b, c of a triangle are in G.P. sych that $\log a - \log 2b, \log 2b - \log 3c, \log 3c - a$ are in A.P., then prove that $\triangle ABC$ is an obtuse angled triangle.

Solution of Problem 112

Solution: $\log \frac{a}{2b}, \log \frac{2b}{3c}, \log \frac{3c}{a}$ are in A.P.

$$\therefore 2\log \frac{2b}{3c} = \log\left(\frac{a}{2b}.\frac{3c}{a}\right)$$

$$\Rightarrow \left(\frac{2b}{3c}\right)^2 = \frac{3c}{2b} \Rightarrow 8b^3 = 27c^3 \therefore 2b = 3c$$

Also, a, b, c are in G.P. i.e. $b^2 = ac$

$$\frac{9c^2}{4} = ac \therefore a = \frac{9}{4}c$$

Thus, sides are $\frac{9}{4}c, \frac{6}{4}c, c$. Clearly, a is greatest side so corresponding angle will be largest.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{29}{48} < 0$$

Therefore $\angle A$ is obtuse so the triangle is obtuse angled triangle.

Problem 113

113. If the roots of the equation $ax^3 + bx^2 + cx + d = 0$ be in G.P. then prove that $c^3 a = b^3 d$

Solution of Problem 113

Solution: Given, three roots are in G.P. so we take them as $\frac{p}{r}, p, pr$
Product of roots is $p^3 = -\frac{d}{a} \Rightarrow ap^3 + d = 0$
Also, p is a root of the equation, therefore, $ap^3 + bp^2 + cp + d = 0$
$\Rightarrow bp^2 + cp = 0 \Rightarrow bp = -c \Rightarrow b^3p^3 + c^3 = 0$
$\Rightarrow b^3\left(-\frac{d}{a}\right) + c^2 \Rightarrow b^3d = c^3a$