Complex Numbers Problems 61-70

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Problem 61

61. If $z_n = \cos \frac{\pi}{(2n+1)(2n+3)} + i \sin \frac{\pi}{(2n+1)(2n+3)}$ then find $z_1 z_2 z_3 \ldots \infty$.

Solution of Problem 61

Solution: $z_n = \cos\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) \cdot \frac{\pi}{2} + i \sin\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) \cdot \frac{\pi}{2}$

$\therefore z_1 z_2 z_3 \ldots \infty = \cos\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} \ldots \infty\right) \cdot \frac{\pi}{2} + i \sin\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} \ldots \infty\right) \cdot \frac{\pi}{2}$

$= \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$

Problem 62

62. If z_1, z_2 be two complex numbers and a, b are two real numbers, then prove that $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$.

Solution of Problem 62

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (ax_1 - bx_2)^2 + (ay_1 - by_2)^2 + (bx_1 + ax_2)^2 + (by_1 + ay_2)^2$

$= a^2x_1^2 + b^2x_2^2 - 2abx_1x_2 + a^2y_1^2 + b^2y_2^2 - 2aby_1y_2 + b^2x_1^2 + a^2x_2^2 + 2abx_1x_2 + b^2y_1^2 + a^2y_2^2 + 2aby_1y_2$

$= (a^2 + b^2)(x_1^2 + y_1^2 + x_2^2 + y_2^2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

Problem 63

63. Show that the equation $\frac{A^2}{x-a} + \frac{B^2}{x-b} + ... + \frac{H^2}{x-h} = x + l$ where $A, B, ..., H; a, b, ..., h$ and l are real; cannot have imaginary roots.

Solution of Problem 63

Solution: Let $x = y + iz$, then given expression becomes

$$\frac{A^2}{y+iz-a} + \frac{B^2}{y+iz-b} + ... + \frac{H^2}{y+iz-h} = y + iz + l$$

$$\frac{A^2(y-a-iz)}{(y-a)^2+z^2} + \frac{B(y-b-iz)}{(y-b)^2+z^2} + ... + \frac{H^2(y-iz-h)}{(y-h)^2+z^2} = y + iz + l$$

Comparing imaginary parts, we have

$$-iz\left[\frac{A^2}{(y-a)^2+z^2} + \frac{B^2}{(y-a)^2+z^2} + ... + \frac{H^2}{(y-a)^2+z^2}\right] = iz$$

$$iz\left[1 + \frac{A^2}{(y-a)^2+z^2} + \frac{B^2}{(y-a)^2+z^2} + ... + \frac{H^2}{(y-a)^2+z^2}\right] = 0$$

Clearly the term inside brackets is non-zero. So $z = 0$.

Problem 64

64. Find all real numbers x, such that $|1 + 4i - 2^{-x}| \leq 5$.

Solution of Problem 64

Solution: Let $2^{-x} = p$, then

$|1 + 4i - p| \leq 5 \Rightarrow (1 - p)^2 + 16 \leq 25$

$1 - p \leq \pm 3 \Rightarrow p \geq 4, -2 \Rightarrow x \geq -2 \;\because p \nless 0 \Rightarrow p \in [-2, \infty]$

Problem 65

65. Show that a unimodular complex number, not purely real can always be expressed as $\frac{c+i}{c-i}$ for some real c.

Solution of Problem 65

65. Since the number is unimodular $\Rightarrow |z| = 1$, so let $z = \cos\theta + i\sin\theta$.

Given, $\cos\theta + i\sin\theta = \frac{c+i}{c-i} = \frac{c+i}{c-i}.\frac{c+i}{c-i} = \frac{c^2-1+2ic}{c^2+1}$

Comparing real and imaginary parts, we get

$\cos\theta = \frac{c^2-1}{c^2+1} \Rightarrow c = \pm\cot\frac{\theta}{2}$

and $\sin\theta = \frac{2c}{c^2+1} \Rightarrow c = \cot\frac{\theta}{2}, \tan\frac{\theta}{2}$

So the common value is $c = \cot\frac{\theta}{2}$.

Problem 66

66. If $(z^3+3)^2=-16$, then find $|z|$.

Solution of Problem 66

Solution: $(z^3+3)^2 = -16 = 16i^2 \Rightarrow z^3 = -3 \pm 4i$

$|z^3| = 5 \Rightarrow |z| = 5^{1/3}$

Problem 67

67. If $\frac{\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x}{1+2i\sin\frac{x}{2}}$ is real, the find the set of all possible values of x.

Solution of Problem 67

Solution: Let $z = \frac{\sin\frac{x}{2} + \cos\frac{x}{2} - i\tan x}{1+2i\sin\frac{x}{2}}$

$= \frac{\sin\frac{x}{2} + \cos\frac{x}{2} - i\tan x}{1+2i\sin\frac{x}{2}} \cdot \frac{1-2i\sin\frac{x}{2}}{1-2i\sin\frac{x}{2}}$

Since it is real so imaginary part of this will be 0.

$\Rightarrow -\tan x - 2\sin\frac{x}{2}\cos\frac{c}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} = 0$

$2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos x} = 0$

$\Rightarrow \sin\frac{x}{2} = 0 \Rightarrow x = 2n\pi$ where $n = 0, 1, 2, 3 \ldots$

or $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)\cos x + \cos\frac{x}{2} = 0$

$\Rightarrow \tan^3\frac{x}{2} - \tan\frac{x}{2} - 2 = 0$

If α is a solution of above then the set of possible values are $x = 2n\pi + 2\alpha$

Problem 68

68. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Solution of Problem 68

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= 2(x_1^2 + y_1^2 + x_2^2 + y_2^2)$

$= 2(|z_1|^2 + |z_2|^2)$

Problem 69

69. If $x^2 - x + 1 = 0$ then find the value of $\sum_{n=1}^{5} \left(x^n + \frac{1}{x^n}\right)^2$.

Solution of Problem 69

Solution: Given, $x^2 - x + 1 = 0 \Rightarrow x = -\omega, -\omega^2$

$\sum_{n=1}^{5} \left(x^n + \frac{1}{x^n}\right)^2 = \sum_{n=1}^{5} \left(x^{2n} + \frac{1}{x^{2n}} + 2\right)$

$= \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right) + \left(x^6 + \frac{1}{x^6} + 2\right) + \left(x^8 + \frac{1}{x^8} + 2\right) + \left(x^{10} + \frac{1}{x^{10}} + 2\right)$

$= (x^2 + x^4 + x^6 + x^8 + x^{10}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \frac{1}{x^8} + \frac{1}{x^{10}}\right) + 10$

$= (\omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^{10}) + \left(\frac{1}{\omega^2} + \frac{1}{\omega^4} + \frac{1}{\omega^6} + \frac{1}{\omega^8} + \frac{1}{\omega^{10}}\right) + 10$

$= -1 - 1 + 10 = 8$

Problem 70

70. If $3^{49}(x+iy) = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)^{100}$ then find x and y.

Solution of Problem 70

Solution: $3^{49}(x+iy) = \left[i\sqrt{3}\left(\frac{1-i\sqrt{3}}{2}\right)\right]^{100}$

$= i^{100}3^{50}(-\omega)^{100} \Rightarrow 3^{49}(x+iy) = 3^{50}.\omega$

$x+iy = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

$\Rightarrow x = -\frac{3}{2}, y = \frac{3\sqrt{3}}{2}$