Miscellaneous Problems on A.P., G.P. and H.P.
Problems 111-120

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Problem 111

111. If S_1, S_2 and S_3 denote the sum to $n(> 1)$ terms of three sequences in A.P., whose first terms are unity and common differences are in H.P., prove that $n = \frac{2S_3S_1 - S_1S_2 - S_2S_3}{S_1 - 2S_2 + S_3}$

Solution to Problem 111

Solution: Let d_1, d_2, d_3 be the common differences of the A.P.'s.

$$\Rightarrow S_1 = \frac{n}{2}[2 + (n-1)d_1] \Rightarrow d = \frac{2(S_1 - n)}{n(n-1)}$$

$$\text{Similalrly } d_2 = \frac{2(S_2 - n)}{n(n-1)}, d_3 = \frac{2(S_3 - n)}{n(n-1)}$$

$\because d_1, d_2, d_3$ are in H.P. $\therefore \frac{1}{d_2} - \frac{1}{d_1} = \frac{1}{d_3} - \frac{1}{d_2}$

$$\Rightarrow \frac{n(n-1)}{2(S_2 - n)} - \frac{n(n-1)}{2(S_1 - n)} = \frac{n(n-1)}{2(S_3 - n)} - \frac{n(n-1)}{2(S_2 - n)}$$

$$\Rightarrow \frac{1}{S_2 - n} - \frac{1}{S_1 - n} = \frac{1}{S_3 - n} - \frac{1}{S_2 - n}$$

$$\Rightarrow \frac{S_1 - S_2}{(S_1 - n)(S_2 - n)} = \frac{S_2 - S_3}{(S_3 - n)(S_2 - n)}$$

$$\Rightarrow n = \frac{2S_3S_1 - S_1S_2 - S_2S_3}{S_1 - 2S_2 + S_3}$$

Problem 112

112. Find a three-digit number such that its digits are in G.P. and the digits of the number obtained from it by subtracting 400 form an A.P.

Solution of Problem 112

Solution: Let the digits at hundreds, tens and units places be a, ar and ar^2 and the required number be x, then $x = 100a + 10a + ar^2$

Let $y = x - 400 \Rightarrow y = 100(a-4) + 1 - ar + ar^2$ In the number y, the digit at hundreds place is $a - 4$. Clearly

$1 \le a - 4 \le 5$ $[\because 1 \le a \le 9$ and $a - 4 \ge 1] \Rightarrow 5 \le a \le 9$

According to question $a - 4, ar, ar^2$ are in A.P. $\therefore 2ar = a - 4 + ar^2 \Rightarrow a(r-1)^2 = 4 \Rightarrow r - 1 = \pm\frac{2}{\sqrt{a}}$

$\because a$ and ar are integers. $\therefore r$ is a rational number. Thus, a must be a perfect square. $\therefore a = 9$

Thus, $r = \frac{5}{3}, \frac{1}{3}$ but $r \neq \frac{5}{3}$ othereise $ar = 15 \therefore r = \frac{1}{3} \therefore ar = 3, ar^2 = 1$

Hence required number is 931.

Problem 113

113. If a, b, c be distinct positive numbers in G.P. and $\log_c a, \log_b c, \log_a b$ be in A.P., prove that the common difference of the progression is $3/2$.

Solution of Problem 113

Solution: Given a, b, c are in G.P. Let r be the common ratio of this G.P. then $b = ar$ and $c = ar^2$.

Given, $\log_c a, \log_b c, \log_a b$ are in A.P.

$$\Rightarrow \frac{\log a}{\log c}, \frac{\log c}{\log b}, \frac{\log b}{\log a} \text{ are in A.P.}$$

$$\Rightarrow \frac{\log a}{\log a + 2\log r}, \frac{\log a + 2\log r}{\log a + \log r}, \frac{\log a + \log r}{\log a} \text{ are in A.P.}$$

$$\frac{1}{1+2x}, \frac{1+2x}{1+x}, 1+x \text{ are in A.P. where } \frac{\log r}{\log a} = x$$

$$2\left(\frac{1+2x}{1+x} = \frac{1}{1+2x} + 1 + x\right) \Rightarrow x(2x^2 - 3x - 3) = 0$$

$$2x^2 - 3x - 3 = 0[\because x \neq 0, \text{ else } \log r = 0 \Rightarrow r = 1 \text{ which is not possible as } a, b, c \text{ are distinct}]$$

$$2d = 1 + x - \frac{1}{1+2x} = \frac{2x^2+3x}{1+2x} = \frac{3x+3+3x}{1+2x} = 3 \Rightarrow d = \frac{3}{2}$$

Problem 114

114. If p be the first of the n arithmetic means between two numbers a and b and q the first of the n harmonic means between the same two numbers, prove that the value of q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$

Solution to Problem 114

Solution: Let the two numbers be a and b. Since n A.M.'s have been inserted between a and b $\therefore$ common difference of A.P., $d = \frac{b-a}{n+1}$

Now $p =$ first A.M. $=$ 2nd term of A.P. $= a + d = \frac{an+b}{n+1}$

Similarly for harmonic series $q = \frac{ab(n+1)}{bn+a}$

We know that x will not lie between α and β if $(x-\alpha)(x-\beta) > 0$

$$q - p = -\frac{n(a-b)^2}{(bn+a)(n+1)}$$

$$q - \left(\frac{n+1}{n-1}\right)^2 p = -\frac{(n+1)(a+b)^2 n}{(n-1)^2(bn+a)}$$

$$\Rightarrow (q-p)\left[q - \left(\frac{n+1}{n-1}\right)^2 p\right] = \frac{n^2(a-b)^2(a+b)^2}{(n-1)^2(bn+a)^2} > 0$$

Problem 115

115. Find a three digit number whose consecutive numbers form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now if we increase the second digit of the required number by 2, the resulting number will form an A.P.

Solution of Problem 115

Solution: Let a be the first digit and r be the common ratio and x be the required number.

$$\therefore 100a + 10ar + ar^2 = x \Rightarrow x - 792 = 100(a-7) + 10(ar-9) + 2ar^2 = 100ar^2 + 10ar + a$$

Also,

$$2(ar+2) = a + ar^2 \Rightarrow ar^2 - 2ar + a = 4 \Rightarrow a(1-r)^2 = 4 \Rightarrow r - 1 = \pm\frac{2}{\sqrt{a}}$$

Clearly, $a - 7 \geq 0 \Rightarrow a \geq 7$

$\because a$ and ar are integers. $\therefore r$ is a rational number. Thus, a must be a perfect square. $\therefore a = 9 \Rightarrow r = \frac{1}{3}, \frac{5}{3}$ but $r \neq \frac{5}{3}$ othereise $ar = 15 \therefore r = \frac{1}{3} \therefore ar = 3, ar^2 = 1$

Hence required number is 931.

Problem 116

116. An A.P. and a G.P. each has p as first term and q as second term where $0 < q < p$. Find the sum to infinity, s of the G.P., and prove that the sum of first n terms of the A.P. may be written as $np - \frac{n(n-1)}{2} \cdot \frac{p^2}{s}$

Solution of Problem 116

Solution: Common difference of A.P. $= q - p$ and common ratio of G.P. $= \frac{q}{p} < 1$

$$s = \frac{p}{1 - \frac{q}{p}} = \frac{p^2}{p - q}$$

Let S_n be the sum of n terms of A.P., then

$$S_n = \frac{n}{2}[2p + (n-1)d] = np + \frac{n(n-1)d}{2} = np + \frac{n(n-1)(q-p)p^2}{2p^2} = np - \frac{n(n-1)}{2}.\frac{p^2}{s}$$

Problem 117

117. If $\log_x y, \log_z x, \log_y z$ are in G.P., $xyz = 64$ and x^3, y^3, z^3 are in A.P., then find x, y and z.

Solution of Problem 117

Solution: $\because \log_x y, \log_z x, \log_y z$ are in G.P.

$$\Rightarrow (\log_z x)^2 = \log_x y . \log_y z \Rightarrow \left(\frac{\log x}{\log z}\right)^2 = \frac{\log y}{\log x} . \frac{logz}{\log y}$$

$$\Rightarrow (\log x)^3 = (\log z)^3 \Rightarrow x = z$$

$$\Rightarrow x = y = z = 4 \because xyz = 64 \text{ and } 2y^3 = x^3 + z^3$$

Problem 118

118. Find all complex numbers x and y such that $x, x+2y, 2x+y$ are in A.P. and $(y+1)^2, xy+5, (x+1)^2$ are in G.P.

Solution of Problem 118

118.

$$2(x+2y) = x + 2x + y \Rightarrow 3y = x$$

$$(xy+5)^2 = (y+1)^2(x+1)^2 \Rightarrow (3y^2+5) = \pm(y+1)(3y+1)$$

$$\Rightarrow y = 1, \frac{-1 \pm 2\sqrt{2}i}{3}$$

$$x = 3, -1 \pm 2\sqrt{2}i$$

Problem 119

119. Find A.P. of distinct terms whose first term is 3 and second, tenth and thirty fourth terms form a G.P.

Solution of Problem 119

Solution: Let $a = 3$ be the first term and d be the common difference of the G.P. then, given

$$(a + 9d)^2 = (a + d)(a + 33d) \Rightarrow a^2 + 18ad + 81d^2 = a^2 + 34ad + +33d^2 \Rightarrow d = \frac{a}{3} = 1$$

So the A.P. is $3, 4, 5, \ldots$

Problem 120

120. Let a, b, c, d be four positive real numbers such that the geometric mean of a and b is equal to the gerometric mean of c and d and the arithmetic mean of a^2 and b^2 is equal to the arithmetic mean of c^2 and d^2. Show that the arithmetic mean of a^n and b^n is equal to the arithmetic mean of c^n and d^n for every integral value of n.

Solution of Problem 120

Solution: Given,

$$\sqrt{ab} = \sqrt{cd}, \frac{a^2+b^2}{2} = \frac{c^2+d^2}{2}$$

$$\Rightarrow ab = cd, a^2+b^2 = c^2+d^2$$

$$\Rightarrow (a-b)^2 = (c-d)^2, (a+b)^2 = (c+d)^2$$

$$\Rightarrow a = c, b = d$$

Thus, arithmetic mean of a^n and b^n is equal to the arithmetic mean of c^n and d^n for every integral value of n.