Geometric Progression
Problems 81-90

Shiv Shankar Dayal

September 15, 2021

Problem 81

81. If $p(x) = (1 + x^2 + x^4 + ... + x^{2n-2})/(1 + x + x^2 + ... + x^{n-1})$ is a polynomial in x, then find the possible values of n.

Solution of Problem 81

Solution:

$$p(x) = \frac{1-x^{2n}}{1-x^2}\frac{1-x}{1-x^n} = \frac{(1+x^n)}{1+x}$$

Since $p(x)$ is a polynomial, thus, $x+1=0$ must be a root of $1+x^n$ i.e. $1+(-1)^n=0$. Hence, n is odd.

Problem 82

82. If each term in a G.P. is twice the terms following it, then find the common ratio of the G.P.

Solution of Problem 82

Solution: Let a be the first term and r be the common ratio of the G.P. Thus,

$$a_n = 2[a_{n+1} + a_{n+2} + ...]\forall n \in N$$

$$ar^{n-1} = 2[ar^n + ar^{n+1} + ...] = \frac{2ar^n}{1-r}$$

$$1 = \frac{2r}{1-r} \Rightarrow r = \frac{1}{3}$$

Problem 83

83. If $x = a + \frac{a}{r} + \frac{a}{r^2} + \ldots\infty, y = b - \frac{b}{r} + \frac{b}{r^2} - \ldots\infty$ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + \ldots\infty$, then prove that $\frac{xy}{z} = \frac{ab}{c}$

Solution of Problem 83

Solution: $x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$

$$y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$$

$$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$xy = \frac{abr^2}{r^2-1}$$

$$\frac{xy}{z} = \frac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}} = \frac{ab}{c}$$

Problem 84

84. A G.P. consists of an even number of terms. If the sum of all terms is 5 times the sum of the terms occupying odd places, then find the common ratio.

Solution of Problem 84

Solution: Let a be the first term and r be the common ratio of the G.P.
Sum of all terms $S = \frac{a(r^n-1)}{r-1}$
Sum of all odd terms $S_{odd} = \frac{a(r^{2\cdot\frac{n}{2}}-1)}{r^2-1} = \frac{a(r^n-1)}{r^2-1}$
Given $S = 5S_{odd} \Rightarrow \frac{a(r^n-1)}{r-1} = \frac{5a(r^n-1)}{r^2-1}$
$\Rightarrow \frac{1}{r-1} = \frac{5}{r^2-1} \Rightarrow r^2 - 5r + 4 = 0 \Rightarrow r = 1, 4$
But r cannot be 1 so $r = 4$

Problem 85

85. If sum of n terms of a G.P. is $3 - \frac{3^{n+1}}{4^{2n}}$, then find the common ratio.

Solution of Problem 85

Solution: Let $S_n = 3 - \frac{3^{n+1}}{4^{2n}}$ be sum of n terms. Then,

$$S_{n-1} = 3 - \frac{3^n}{4^{2(n-1)}}$$

$$t_n = S_n - S_{n-1} = \frac{3^n}{4^{2n-2}} - \frac{3^{n+1}}{4^{2n}} = \frac{16.3^n - 3^{n+1}}{4^{2n}} = \frac{13.3^n}{4^{2n}}$$

$$t_{n-1} = \frac{13.3^{n-1}}{4^{2(n-1)}}$$

$$r = \frac{t_n}{t_{n-1}} = \frac{3}{16}$$

Problem 86

86. In an infinite G.P. whose terms are all positive, the common ratio being less than unity, prove that any term $>, =, <$ the sum of all the succeeding terms according as the common ratio $<, =, \frac{1}{2}$

Solution of Problem 87

Solution: Let s be the first term and r be the common ratio of the G.P. Then, $t_n = ar^{n-1}$
Sum of succeeding terms $S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n n)}{1-r} = \frac{ar^n}{1-r}$
Equating, we get $ar^{n-1} = \frac{ar^n}{1-r} \Rightarrow 1 = \frac{r}{1-r} \Rightarrow r = \frac{1}{2}$
Similarly we can prove for conditions of greater than and less than.

Problem 87

87. Prove that $(666 \ldots n \text{ digits})^2 + 888 \ldots n \text{ digits} = 444 \ldots 2n \text{ digits}$

Solution of Problem 87

Solution:

$$\frac{36}{81}(999\ldots n \text{ digits})^2 + \frac{8}{9}999\ldots n \text{ digits} = \frac{4}{9}999\ldots 2n \text{ digits}$$

$$\frac{36}{81}(10^{2n} - 2.10^n + 1) + \frac{8}{9}(10^n - 1) = \frac{4}{9}(10^{2n} - 1)$$

$$\frac{4}{9}(10^{2n} - 2.10^n + 1) + \frac{4}{9}(2.10^n - 2) = \frac{4}{9}(10^{2n} - 1)$$

Problem 88

88. Find the sum $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+ ...$ to n terms.

Solution of Problem 88

Solution: Multiplying and dividing by $x-y$, we get

$$\frac{1}{x-y}[(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots$$

Now it is trivial to isolate two G.P. and find the difference of their sums.

Problem 89

89. Find the sum of the series $\frac{4}{3} + \frac{10}{9} + \frac{29}{27} + \dots$

Solution of Problem 89

Solution: Given series can be rewritten as $\frac{3+1}{3} + \frac{9+1}{9} + \frac{27+1}{27} + \ldots$

$$= 1 + \frac{1}{3} + 1 + \frac{1}{9} + 1 + \frac{1}{9}$$

$$= 1 + \frac{1}{3} . \frac{1}{1 - \frac{1}{3}} = 3$$

Problem 90

90. In a geometric series consisting of positive terms, each term equals the sum of next two terms. Find the common ratio.

Solution of Problem 90

Solution: Let a be the first term and r be the common ratio. Then, $a = ar + ar^2 \Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{5}}{2}$
However, r cannot be negative, thus, $r = \frac{\sqrt{5}-1}{2}$