Miscellaneous Problems on A.P., G.P. and H.P.
Problems 21-30

Shiv Shankar Dayal

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Problem 21

21. Prove that $\tan 70^\circ, \tan 50^\circ + \tan 20^\circ, \tan 20^\circ$ are in A.P.

Solution of Problem 21

Solution:

$$\tan 70^\circ = \tan(50^\circ + 20^\circ) = \frac{\tan 50^\circ + \tan 20^\circ}{1 - \tan 50^\circ \tan 20^\circ}$$

$$\tan 70^\circ - \tan 70^\circ \tan 50^\circ \tan 20^\circ = \tan 50^\circ + \tan 20^\circ$$

$$\because \tan 70^\circ . \tan 20^\circ = \tan 70^\circ \cot 70^\circ = 1$$

$$\Rightarrow \tan 70^\circ = \tan 20^\circ + 2 \tan 50^\circ$$

Thus, $\tan 70^\circ$, $\tan 50^\circ + \tan 20^\circ$, $\tan 20^\circ$ are in A.P. with common difference of $\tan 50^\circ$.

Problem 22

22. If $\log_l x, \log_m x, \log_n x$ are in A.P. and $x \neq 1$, prove that $n^2 = (nl)^{\log_l m}$

Solution of Problem 22

Solution: $\because \log_l x, \log_m x, \log_n x$ are in A.P.

$$2\log_m x = \log_l x + \log_n x \Rightarrow 2\frac{\log x}{\log m} = \frac{\log x}{\log l} + \frac{\log x}{\log n}$$

$$\Rightarrow \frac{2}{\log m} = \frac{\log nl}{\log l \log n} \Rightarrow 2\log n = \log nl \log_l m$$

$$\Rightarrow \log n^2 = \log(nl)^{\log_l m}$$

Taking anti-log $n^2 = (nl)^{\log_l m}$

Problem 23

23. The length of sides of a right angled triangle are in A.P., show that their ratio is $3 : 4 : 5$

Solution of Problem 23

Solution: Let the sides are $a, a+d, a+2d$ of the right angled triangle where $d > 0$. Clearly, $a+2d$ is largest and hence hypotenuse. Thus, we can write

$$(a+2d)^2 = a^2 + (a+d^2) \Rightarrow a^2 - 2ad - 3d^2 = 0 \Rightarrow a = -d, a = 3d$$

However, a side cannot be -ve so $a = 3d$ and hence ratio of sides are $3 : 4 : 5$

Problem 24

24. If $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in A.P., determine the value of x.

Solution of Problem 24

Solution: $\because \log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in A.P.

$$\log_3(2^x - 5) - \log_3 2 = \log_3\left(2^x - \frac{7}{2}\right) - \log_3(2^x - 5)$$

$$(2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right) \Rightarrow 2^{2x} - 12.2^x + 32 = 0$$

$$2^x = 8, 4 \Rightarrow x = 3, 2$$

However, if $d = 2, 2^x - 5 < 0$ and logarithm of that wont be real. Thus, $x = 3$

Problem 25

25. Find the values of a for which $5^{1+x} + 5^{1-x}, \frac{a}{2}, 25^x + 25^{-x}$ are in A.P.

Solution of Problem 25

Solution:

$$a = 5^{1+x} + 5^{1-x} + 25^{x} + 25^{-x} = 5(5^{x} + 5^{-x}) + 25^{x} + 25^{-x} \geq 5.2 + 2 = 12$$

This solution used the fact that for real no. $x + \frac{1}{x} \geq 2$ which can be proved trivially.

Problem 26

26. If $\log 2, \log(2^x - 1)$ and $\log(2^x + 3)$ are in A.P., then find x.

Solution of Problem 26

Solution:

$$\log(2^x - 1) - \log 2 = \log(2^x + 3) - \log(2^x - 1) \Rightarrow (2^x - 1)^2 = 2(2^x + 3)$$

$$2^{2x} - 4.2^x - 5 = 0 \Rightarrow 2^x = -1, 5$$

However, for real $x, 2^x$ cannot be -1. Thus, $x = \log_2 5$ is the solution.

Problem 27

27. If $1, \log_y x, \log_z y, -15 \log_x z$ are in A.P., then prove that $x = z^3$ and $y = z^{-3}$

Solution of Problem 27

Solution: Let d be the common difference of the A.P. Then,

$$\log_y x = 1 + d \Rightarrow x = y^{1+d}$$

$$\log_z y = 1 + 2d \Rightarrow y = z^{1+2d} \Rightarrow x = y^{(1+d)(1+2d)}$$

$$-15 \log_x z = 1 + 3d \Rightarrow z = x^{-\frac{1+3d}{15}}$$

$$\therefore x = x^{-\frac{(1+d)(1+2d)(1+3d)}{15}}$$

$$(1+d)(1+2d)(1+3d) = -15$$

$$(d+2)(6d^2 - d + 8 = 0)$$

$\therefore d = -2[\because 6d^2 - d + 8 = 0$ has complex roots.$]$

$\therefore x = z^3, y = z^{-3}$

Problem 28

28. Show that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannot be terms of a single A.P.

Solution of Problem 28

Solution: Let $\sqrt{2}, \sqrt{3}, \sqrt{5}$ be pth, qth and rth term of an A.P. Then,

$$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{2}} = \frac{r-q}{q-p} = k$$

Since p, q, r are integers therefore k will be a rational number.

$$\sqrt{5}-\sqrt{3} = k(\sqrt{3}-\sqrt{2})$$

Squaring, we get

$$5+3-2\sqrt{15} = k(3+2-2\sqrt{6})$$

$$k = \frac{8-2\sqrt{15}}{5-2\sqrt{6}}$$

Multiplying both numberator and denominator with $5+2\sqrt{6}$, we obtain that k is not a rational number. Thus, we have proved the required condition.

Problem 29

29. A circle of one centimeter radius is drawn on a piece of paper and with the same center $3n - 1$ other circles are drawn of radii 2 cm, 3 cm, 4 cm and so on. The inner circle is painted blue, the ring between that and next circle is painted red, the next ring yellow then other rings blue, red, yellow and so on in this order. Show that the successive aread of each color are in A.P.

Solution of Problem 29

Solution: The blue color circles will have radius of $1, 4, 7, 10, \ldots, 3n-2$ cm each.

So the areas of blue rings are $\pi.1^2, \pi(4^2-3^2), \pi(7^2-6^2), \ldots, \pi[(3n-2)^2-(3n-3)^2]$

$\pi[(3n-2)^2-(3n-3)^2] = \pi(6n-5)$ which is t_n and forms an A.P. with common ratio 6

Similarly, it can be shown that red and yellow rings form an A.P.

Problem 30

30. If $x, y, z (x, y, z \neq 0)$ are in A.P. and $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are also in A.P., then prove that $x = y = z$

Solution of Problem 30

Solution: Since x, y, z are in A.P. $2y = x + z$ Similarly,

$$2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$$

$$2\tan^{-1} y = \tan^{-1} \frac{x+z}{1-xz}$$

$$\tan(2\tan^{-1} y) = \frac{x+z}{1-xz}$$

$$\frac{2y}{1-y^2} = \frac{x+z}{1-xz}$$

$$(x+z)\left[\frac{1}{1-y^2} - \frac{1}{1-xz} = 0\right]$$

$$\Rightarrow y^2 = xz$$

So x, y, z are in G.P. as well. Thus,

$$4y^2 = x^2 + z^2 + 2xz = x^2 + z^2 + 2y^2 \Rightarrow x^2 + z^2 - 2xz = 0 \Rightarrow x = z \Rightarrow x = y = z$$