Summation of Series
Problems 31-40

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Problem 31

31. Find the sum of the series $1.2.3 + 2.3.5 + 3.4.7 + ...$ to n terms.

Solution of Problem 31

Solution:

$$t_n = n(n+1)(2n+1) = 2n^3 + 3n^2 + n$$

$$S_n = \sum t_n = 2\sum n^3 + 3\sum n^2 + \sum n$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{1}{2}n(n+1)^2(n+2)$$

Problem 32

32. Find the sum of the series $1.2.3 + 2.3.4 + 3.4.5 + ...$ to n terms

Solution of Problem 32

Solution:

$$t_n = n(n+1)(n+2) = n^3 + 3n^3 + 2n$$

$$S_n = \sum t_n = \sum n^3 + 3\sum n^2 + 2\sum n$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 3\frac{n(n+1)(n+2)}{6} + 2\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

Problem 33

33. Find the sum of the series $1.3^2 + 2.5^2 + 3.7^2 + ...$ to 20 terms.

Solution of Problem 33

Solution:

$$t_n = n(2n+1)^2 = 4n^3 + 4n^2 + n$$

$$S_{20} = 4\sum_{n=1}^{20} n^3 + 4\sum_{n=1}^{20} n^2 + \sum_{n=1}^{20} n$$

$$= 4\left[\frac{20.21}{2}\right]^2 + 4.\frac{20.21.41}{6} + \frac{20.21}{2}$$

$$= 188090$$

Problem 34

34. Find the sum of the series $(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + ...$ to n terms.

Solution of Problem 34

Solution:

$$t_i = i(n^2 - i^2) = n^2 i - i^3$$

$$S_n = n^2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^3$$

$$S_n = n^2 \frac{n(n+1)}{2} - \left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{1}{4} n^2 (n^2 - 1)$$

Problem 35

35. Find the sum of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$ to 10 terms.

Solution of Problem 35

Solution:

$$t_n = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$S_n = \frac{1}{3}\sum n^3 + \frac{1}{2}\sum n^2 + \frac{1}{6}\sum n$$

$$= \frac{1}{3}\left[\frac{n(n+1)}{2}\right]^2 + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{12}n(n+1)$$

Substituting $n = 10$, we get

$$S_{10} = 1210$$

Problem 36

36. Find the sum of the series $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + ...$ to 10 terms.

Solution of Problem 36

Solution:

$$t_n = [(2n+1)^3 - (2n)^3] = 12n^2 + 6n + 1$$

$$S_n = 12\sum n^2 + 6\sum n + \sum 1$$

$$S_n = 2n(n+1)(2n+1) + 3n(n+1) + n$$

Substituting $n = 10$, we get

$$S_{10} = 4960$$

Problem 37

37. Find the sum of the series $1 + \frac{1}{1.2} + \frac{1}{1+2+3} + ...$ to n terms.

Solution of Problem 37

37.

$$t_n = \frac{1}{1+2+3+...+n} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$t_{n-1} = 2\left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$\dots$$

$$t_3 = 2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$t_2 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$t_1 = 2\left(1 - \frac{1}{2}\right)$$

Adding, we get

$$S_n = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$$

Problem 38

38. Find the sum to infinity of the series $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} + ...$

Solution of Problem 38

Solution:

$$t_n = \frac{1}{2n.2(n+1)} = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$t_1 = \frac{1}{4}\left(1 - \frac{1}{2}\right)$$

$$t_2 = \frac{1}{4}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$\dots$$

$$t_{n-1} = \frac{1}{4}\left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$t_n = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

As n approaches ∞, t_n aproaches zero. Thus, $S_\infty = \frac{1}{4}$

Problem 39

39. Find the sum of the series $2 + 6 + 12 + 20 + ...$ to n terms.

Solution of Problem 39

Solution:

$$S_n = 2 + 6 + 12 + 20 + ... + t_n$$
$$S_n = 2 + 6 + 12 + 20 + ... + t_n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = 2 + 4 + 6 + 8 + ... \text{ to } n \text{ terms } - t_n)$$

$$t_n = \frac{n}{2}[2.2 + (n-1)2] = n(n+1)$$

$$S_n = \sum t_n = \sum n^2 + \sum n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{1}{3}n(n+1)(n+2)$$

Problem 40

40. Find the sum of the series $3 + 6 + 11 + 18 + ...$ to n terms.

Solution of Problem 40

Solution:

$$S_n = 3 + 6 + 11 + 18 + ... + t_n$$

$$S_n = 3 + 6 + 11 + 18 + ... + t_n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = 3 + [3 + 5 + 7 + ... \text{ to } n \text{ terms } - t_n]$$

$$t_n = 3 + \frac{n-1}{2}[2.3 + (n-1).2] = 3 + (n-1)(n+2)$$

$$= n^2 + n + 1$$

$$S_n = \sum t_n = \sum n^2 + \sum n + \sum 1$$

$$= \frac{1}{6} n(2n^2 + 3n + 13)$$