Complex Numbers Problems 71-80

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Problem 71

71. For any two complex numbers z_1 and z_2, prove that
$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2}) = |z_1|^2 + |z_2|^2 + 2Re(\overline{z_1}z_2)$

Solution of Problem 71

Solution: $|z_1 + z_2|^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2x_1x_2 + 2y_1y_2$

$= |z_1|^2 + |z_2|^2 + 2(x_1x_2 + y_1y_2)$

Now, $2Re(z_1\overline{z_2}) = 2Re[(x_1 + iy_1)(x_2 - iy_2)] = 2Re[x_1x_2 + y_1y_2 - i(x_1y_2 + x_2y_1)] = 2(x_1x_2 + y_1y_2)$

Similalry, $2Re(\overline{z_1}z_2) = 2(x_1x_2 + y_1y_2)$

Thus, we have desired result.

Problem 72

72. If $|z_1| = |z_2| = 1$, then prove that $|z_1 + z_2| = \left|\frac{1}{z_1} + \frac{1}{z_2}\right|$.

Solution of Problem 72

Solution: R.H.S. = $\left|\frac{1}{z_1}+\frac{1}{z_2}\right| = \left|\frac{z_2+z_1}{z_1 z_2}\right|$

Since $|z_1| = |z_2| = 1 \therefore |z_1 z_2| = 1$ and thus $|z_1+z_2| = \left|\frac{1}{z_1}+\frac{1}{z_2}\right|$.

Problem 73

73. If $|z-2| = 2|z-1|$, then show that $|z|^2 = \frac{4}{3}Re(z)$

Solution of Problme 73

Solution: Let $z = x + iy$, then $x^2 - 4x + 4 + y^2 = 4x^2 - 8x + 4 + 4y^2 \Rightarrow 3x^2 + 3y^2 = 4x$

$\Rightarrow 3|z|^2 = 4Re(z) \Rightarrow |z|^2 = \frac{4}{3}Re(z)$

Problem 74

74. If $\sqrt[3]{a+ib} = x + iy$, then prove that $\frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2)$

Solution of Problem 74

Solution: Given $\sqrt[3]{a+ib} = x+iy \Rightarrow a+ib = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$

Comparing real and imaginary parts, we have $a = x^3 - 3xy^2, b = 3x^2y - y^3 \Rightarrow \frac{a}{x} = x^2 - 3y^2, \frac{b}{y} = 3x^2 - y^2$

$\therefore \frac{a}{x} + \frac{b}{y} = 4(x^2 - y^2)$

Problem 75

75. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

Solution of Problem 75

Solution: $x+iy=\sqrt{\frac{a+ib}{c+id}} \Rightarrow (x+iy)^2=\frac{a+ib}{c+id}$

$\Rightarrow |(x+iy)^2|=\left|\frac{a+ib}{c+id}\right|=\frac{|a+ib|}{|c+id|}$

$\Rightarrow (x^2+y^2)^2=\frac{a^2+b^2}{c+d^2}$

Problem 76

76. If $z_1, z_2, \dots, z_n$ are cube roots of unity, then prove that $|z_k| = |z_{k+1}| \forall k \in [1, n-1]$

Solution of Problem 76

Solution: Let $z = 1 = \cos 0^\circ + i \sin 0^\circ = e^{i2r\pi} \forall i \in N \Rightarrow \sqrt[n]{z} = e^{\frac{i.2r\pi}{n}}$

Clearly, $|z_k| = |z_{k+1}| = 1$

77. If n is a positive integer greater than unity and z is a complex number satisfying the equation $z^n = (z+1)^n$, then probve that $Re(z) < 0$.

Solution of Problem 77

Solution: $z^n = (z+1)^n \Rightarrow \frac{z}{z+1} = 1^{1/n}$

This means $\frac{z}{z+1}$ is nth root of unity. $\Rightarrow \left|\frac{z}{z+1}\right| = 1$

$\Rightarrow |z| = |z+1| \Rightarrow x^2 + y^2 = x^2 + 2x + 1 + y^2 \Rightarrow x = -\frac{1}{2}$

$\Rightarrow Re(z) < 0$

Problem 78

78. Prove that $x^{3m} + x^{3n-1} + x^{3r-2} \forall m, n, r \in N$, is divisible by $1 + x + x^2$.

Solution of Problem 78

Solution: Roots of $1+x+x^2=0$ are ω and ω^2. Let $f(x)=x^{3m}+x^{3n-1}+x^{3r-2}$

$f(x)=x^{3m}+\frac{x^{3n}}{x}+\frac{x^{3r}}{x^2} \Rightarrow f(\omega)=1+\frac{1}{\omega}+\frac{1}{\omega^2}=\frac{1+\omega+\omega^2}{\omega^2}=0$

Similarly $f(\omega^2)=0$

Thus, we see that $f(x)$ has same roots as $1+x+x^2=0$. Hence, $f(x)$ will be divisible by $1+x+x^2$.

Problem 79

79. If $(\sqrt{3}+i)^n = (\sqrt{3}-i)^n \forall n \in N$, then prove that minimum value of n is 6.

Solution of Problem 79

Solution: $\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2e^{i\frac{\pi}{6}}$

Similarly, $\sqrt{3}-i=2e^{-i\frac{\pi}{6}}$

Since imaginary part is what prevents equality we need to get rid of it and the least value for which it will happen is when argument is π. Thus, we need to raise to the power by 6 making $n=6$.

Problem 80

80. If $(\sqrt{3}-i)^n = 2^n, n \in I$, the set of integers, then prove that n is multiple of 12.

Solution of Problem 80

Solution: $\sqrt{3} - i = 2.\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

Thus, $(\sqrt{3} - i)^n = 2^n \Rightarrow 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{\pi}{6}\right) = 2^n$

$\Rightarrow \cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6} = 1 \Rightarrow \frac{n\pi}{6} = 2k\pi \forall k \in I \Rightarrow n = 12k$

Thus, n is a multiple of 12.