Miscellaneous Problems on A.P., G.P. and H.P. Problems 181-190

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January 18, 2022

Problem 181

181. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \infty = \frac{\pi^2}{6}$, then find $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty$

Solution of Problem 181

Solution: In problem 180 we have proved that $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\infty = \frac{\pi^2}{24}$ and $\frac{1}{1^2} + \frac{1}{3^3} + \frac{1}{5^2} + \ldots\infty = \frac{\pi^2}{8}$

$$\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\infty = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}$$

Problem 182

182. If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$, then prove that $H_n = n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + ... + \frac{n-1}{n}\right)$

Solution of Problem 182

Solution:

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \\ &= n - n + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \\ &= n - (1-1) - \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{3}\right) + \ldots + \left(1 - \frac{1}{n}\right) \\ &= n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n-1}{n}\right) \end{aligned}$$

Problem 183

183. Show that $\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+...+\frac{2^n}{x^{2^n}+1}=\frac{1}{x+1}-\frac{2^{n+1}}{x^{2^{n+1}}-1}$

Solution of Problme 183

Solution: We can rewrite the question like $\frac{1}{x+1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} - ... - \frac{2^n}{x^{2^n}+1} = \frac{2^{n+1}}{x^{2^{n+1}}-1}$

$$L.H.S. = \left(\frac{1}{x+1} - \frac{1}{x+1}\right) - \frac{2}{x^2+1} - \frac{4}{x^4+1} - ... - \frac{2^n}{x^{2^n}+1}$$

$$= \left(\frac{2}{x^2-1} - \frac{2}{x^2+1}\right) - \frac{4}{x^4+1} - ... - \frac{2^n}{x^{2^n}+1}$$

$$= \left(\frac{4}{x^4-1} - \frac{4}{x^4+1}\right) - ... - \frac{2^n}{x^{2^n}+1}$$

Processing similarly we obtain R.H.S.

Problem 184

184. Show that $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)...\left(1+\frac{1}{3^{2^n}}\right)=\frac{3}{2}\left(1-\frac{1}{3^{2^{n+1}}}\right)$

Solution of Problem 184

Solution: Multiplying and dividing by $1 - \frac{1}{3}$, we get

$$L.H.S. = \frac{\left(1-\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)}\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)...\left(1+\frac{1}{3^{2^n}}\right)$$

$$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1-\frac{1}{3^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)...\left(1+\frac{1}{3^{2^n}}\right)$$

$$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1-\frac{1}{3^4}\right)\left(1+\frac{1}{3^4}\right)...\left(1+\frac{1}{3^{2^n}}\right)$$

Proceeding similarly we obtain the R.H.S.

Problem 185

185. If $x + y + z = 1$ and x, y, z are positive numbers show that $(1 - x)(1 - y)(1 - z) \geq 8xyz$

Solution of Problem 185

Solution: Since A.M $\geq$ G.M.

$$\therefore \frac{x+y}{2} \geq \sqrt{xy}, \frac{y+z}{2} \geq \sqrt{yz}, \frac{x+z}{2} \geq \sqrt{zx}$$

$$\frac{(x+y)(y+z)(z+x)}{8} \geq xyz \Rightarrow (1-x)(1-y)(1-z) \geq 8xyz$$

Problem 186

186. If $a > 0, b > 0$ and $c > 0$, prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

Solution of Problem 186

Solution: Since A.M geq H.M.

$$\therefore \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

Problem 187

187. If $a + b + c = 3$ and $a > 0, b > 0, c > 0$, find the greatest value of $a^2b^3c^2$.

Solution of Problem 187

Solution: Taking A.M. and G.M of 7 numbers $\frac{a}{2}, \frac{a}{2}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2}$, we get

$$\frac{2.\frac{a}{2} + 3.\frac{b}{3} + 2.\frac{c}{2}}{7} \geq \left[\left(\frac{a}{2}\right)^2 \left(\frac{b}{3}\right)^3 \left(\frac{c}{2}\right)^2\right]^{\frac{1}{7}}$$

$$\frac{3}{7} \geq \left(\frac{a^2b^3c^2}{2^2 3^3 2^2}\right)^{\frac{1}{7}} \Rightarrow \frac{3^7}{7^7} \geq \frac{a^2b^3c^2}{2^2 3^3 2^2}$$

$$a^2b^3c^2 \leq \frac{3^{10}2^4}{7^7}$$

Problem 188

188. Let $a_i + b_i = 1 (i = 1, 2, \ldots, n)$ and $a = \frac{1}{n}(a_1 + a_2 + \ldots + a_n), b = \frac{1}{n}(b_1 + b_2 + \ldots + b_n)$, show that $a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = nab - (a_1 - a)^2 - (a_2 - a)^2 - \ldots - (a_n - a)^2$

Solution of Problem 188

Solution:

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i(1-a_i) = \sum_{i=1}^{n} a)i - \sum_{i=1}^{n} a_i^2 = na - \sum_{i=1}^{n}(a_i - a + a)^2$$

$$= na - \sum_{i=1}^{n}[(a_i - a)^2 + a^2 + 2a(ai - a)] = na - \sum_{i=1}^{n}(ai - a)^2 - na^2 + 2a\sum_{i=1}^{n}(a_i - na)$$

$$= na(1-a) - \sum_{i=1}^{n}(a_i - a)^2 = nab - \sum_{i=1}^{n}(a_i - a)^2$$

$$\because na + nb = \sum_{i=1}^{n}(a_i + b_i) = n \therefore a + b = 1$$

Problem 190

190. A sequence $a_1, a_2, a_3, \dots, a_n$ of real numbers is such that $a_1 = 0, |a_2| = |a_1 + 1|, |a_3| = |a_2 + 1|, \dots,$ $|a_n| = |a_{n-1} + 1|$. Prove that the arithmetic mean $(a_1 + a_2 + \dots + a_n)/n$ of these numbers cannot be less than $-1/2$.

Solution of Problem 190

Solution: Let a_{n+1} be a number such that $|a_{n+1}| = |a_n + 1|$

Squaring all the numbers, we get

$$\begin{aligned} a_1^2 &= 0 \\ a_2^2 &= a_1^2 + 2a_1 + 1 \\ a_3^2 &= a_2^2 + 2a_2 + 1 \\ &\dots \\ a_n^2 &= a_{n-1}^2 + 2a_{n-1} + 1 \\ a_{n+1}^2 &= a_n^2 + 2a_n + 1 \end{aligned}$$

Adding, we get

$$a_1^2 + a_2^2 + \dots + a_n^2 + a_{n+1}^2 = a_1^2 + a_2^2 + \dots + a_n^2 + 2(a_1 + a_2 + \dots + a_n) + n$$

$$\Rightarrow 2(a_1 + a_2 + \dots + a_n) = -n + a_{n+1}^2 \geq -n \Rightarrow (a_1 + a_2 + \dots + a_n)/n \geq -1/2$$