Miscellaneous Problems on A.P., G.P. and H.P.
Problems 31-40

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Problem 31

31. If θ and α are two real numbers such that $\frac{\cos^4\theta}{\cos^2\alpha}, \frac{1}{2}, \frac{\sin^4\theta}{\sin^2\alpha}$ are in A.P., prove that $\frac{\cos^{2n+2}\theta}{\cos^{2n}\alpha}, \frac{1}{2}, \frac{\sin^{2n+2}\theta}{\sin^{2n}\alpha}$

Solution of Problem 31

Solution:

$$\frac{\cos^4\theta}{\cos^2\alpha}+\frac{\sin^4\theta}{\sin^2\alpha}=1=\cos^2\theta+\sin^2\theta$$

$$\Rightarrow \frac{\cos^2\theta}{\cos^2\alpha}(\cos^2\theta-\cos^2\alpha)=\frac{\sin^2\theta}{\sin^2\alpha}(\sin^2\alpha-\sin^2\theta)=\frac{\sin^2\theta}{\sin^2\alpha}(\cos^2\theta-\cos^2\alpha)$$

$$\Rightarrow \cos^2\theta=\cos^2\alpha,\ \frac{\cos^2\theta}{\cos^2\alpha}=\frac{\sin^2\theta}{\sin^2\alpha}$$

In both the cases required consition is satisfied.

Problem 32

32. If $a_n = \int_0^\pi (\sin 2nx / \sin x) dx$, show that $a_1, a_2, a_3, ...$ are in A.P.

Solution of Problem 32

Solution:

$$a_n + a_{n+2} - 2a_{n+1} = \int_0^{\pi} \frac{\sin 2nx + \sin 2(n+2)x - 2\sin 2(n+1)x}{\sin x} dx$$

$$= \int_0^{\pi} \frac{2\sin 2(n+1)x \cos 2x - 2\sin 2(n+1)x}{\sin x} dx$$

$$= \int_0^{\pi} \frac{-2\sin^2 x.2\sin 2(n+1)x}{\sin x} dx$$

$$= -2\int_0^{\pi} 2\sin 2(n+1)x \sin x dx$$

$$= -2\int_0^{\pi} [\cos(2n+1)x - \cos(2n+3)x] dx = 0$$

Therefore, $a_1, a_2, a_3, ...$ are in A.P.

Problem 33

33. If $l_n = \int_0^{\frac{\pi}{4}} tan^n x dx$, show that $\frac{1}{l_2+l_4}, \frac{1}{l_3+l_5}, \frac{1}{l_4+l_6}, ...$ are in A.P. Find the common difference of A.P.

Solution of Problem 33

Solution:

$$l_n + l_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x(1+\tan^2 x)dx = \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n+1}$$

Thus, $\frac{1}{l_2+l_4} = 3, \frac{1}{l_3+l_5} = 4, \ldots$ and common differene is 1.

Problem 34

34. If α, β, γ are in A.P.and $\alpha = \sin(\beta+\gamma), \beta = \sin(\gamma+\alpha)$ and $\gamma = \sin(\alpha+\beta)$. Prove that $\tan\alpha = \tan\beta = \tan\gamma$

Solution of Problem 34

Solution:

$$\because \beta-\alpha=\gamma-\beta \Rightarrow \cos(\beta-\alpha)=\cos(\gamma-\beta)$$

$$\sin(\gamma+\alpha)-\sin(\beta+\gamma)=\sin(\alpha+\beta)-\sin(\gamma+\alpha)$$

$$2\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}+\gamma=2\cos\frac{\beta-\gamma}{2}\sin\frac{\beta+\gamma}{2}+\alpha$$

$$\Rightarrow \frac{\alpha+\beta}{2}+\gamma=\frac{\beta+\gamma}{2}+\alpha$$

$$\Rightarrow \gamma=\alpha$$

$$\Rightarrow \alpha=\beta=\gamma \Rightarrow \tan\alpha=\tan\beta=\tan\gamma$$

Problem 35

35. Suppose a, b, c are three positive real numbers in A.P., such that $abc = 4$. Prove that the minimum value of b is $4^{\frac{1}{3}}$

Solution of Problem 35

Solution: Let d be the common difference. Then $(b-d)b(b+d) = 4 \Rightarrow b(b^2 - d^2) = 4$

$$b^2 - d^2 < b^2 \Rightarrow b^3 > 4$$

Thus, minimum value of b is $4^{\frac{4}{3}}$

Problem 36

36. The sixth term of an A.P. is 2, and its common difference is greater than 1. Show that the value of the common difference of the progression so that the product of first, fourth and fifth terms is greatest is $\frac{8}{5}$.

Solution of Problem 36

Solution: Let a be the first term and d be the common ratio. Then $a + 5d = 2$. Also, let

$$a_1 a_4 a_5 = p \Rightarrow (2-5d)(2-2d)(2-d) = p = 2[4 - 16d + 17d^2 - 5d^3]$$

Let $S = 4 - 16d + 17d^2 - 5d^3$ Differentiating w.r.t. d, we get

$$S' = -15d^2 + 34d - 16 = 0 \Rightarrow d = \frac{2}{3}, \frac{8}{5}$$

Since $d > 1, \Rightarrow d = \frac{8}{5}$

Problem 37

37. Find the sum of n terms of the series: $\log a + \log \frac{a^3}{b} + \log \frac{a^5}{b^2} + \log \frac{a^7}{b^3} + ...$

Solution of Problem 37

Solution: Let

$$
\begin{aligned}
S &= \log a + 3\log a + 5\log a + \ldots - [\log b + 2\log b + 3\log b + \ldots] \\
&= \frac{n}{2}[2\log a + (n-1).2\log a] - \frac{n-1}{2}[\log b + (n-2)\log b] \\
&= n^2 \log a - \frac{n(n-1)}{2}\log b \\
&= \log\left(\frac{a^{2n}}{b^{n-1}}\right)^{\frac{n}{2}}
\end{aligned}
$$

Problem 38

38. The first, second and the last terms of an A.P. are a, b, c respectively. Prove that the sum of al the terms is $\frac{(b+c-2a)(a+c)}{2(b-a)}$

Solution of Problem 37

Solution: Let d be the common difference. $d = b - a, c = a + (n-1)d \Rightarrow n - 1 = \frac{c-a}{b-a} \Rightarrow n = \frac{b+c-2a}{b-a}$

Let Sum of n terms be S, then

$$S = \frac{n}{2}(a+c) = \frac{(b+c-2a)(a+c)}{2(b-a)}$$

Problem 38

38. If S_n denotes the sum of n terms of an A.P., show that $S_{n+3} = 3(S_{n+2} - S_{n+1}) + S_n$

Solution of Problem 38

Solution:

$$
\begin{aligned}
3(S_{n+2} - S_{n+1}) + S_n &= 3t_{n+2} + S_n = 3a + 3(n+1)d + S_n \\
&= S_n + a + nd + 2a + (2n+3)d = S_n + t_{n+1} + 2a + (2n+3)d \\
&= S_{n+1} + (a + (n+1)d) + (a + (n+2)d) \\
&= S_{n+1} + t_{n+2} + t_{n+3} = S_{n+3}
\end{aligned}
$$

Problem 39

39. If $a_1, a_2, \ldots, a_n$ are in arithmetic progression with common difference d, prove that $\sum_{r<s} a_r a_s = \frac{1}{2}n(n-1)[a_1^2 + (n-1)a_1 d + \frac{1}{12}(3n^2 - 7n + 2)d^2]$

Solution of Problem 39

Solution: The expression can be written as $\sum_{i=1}^{n} = a_i[a_{i+1} + a_{i+2} + ... + a_n]$

$$= \frac{n-i}{2}[a_1 + (i-1)d][2a_1 + (n+i-1)d]$$

Solving this yields desired result.

Problem 40

40. Balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second of two balls and so on. If 669 more balls are added, then all balls can be arranged in the shape of a square and each of the sides contained 8 balls less than each side of the triangle did. Determine the initial no. of balls.

Solution of Problem 40

Solution: Let n be the no. of rows for triangle. Then,

$$(n-8)^2 = \frac{n(n+1)}{2} + 669 \Rightarrow n = 55$$

Thus, total no. of initital balls $= \frac{55.56}{2} = 1540$