Complex Numbers Problems 81-90

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September 5, 2022

Problem 81

81. If $z^4 + z^3 + 2z^2 + z + 1 = 0$, then prove that $|z| = 1$.

Solution of Problem 81

Solution: Given, $z^4 + z^3 + 2z^2 + z + 1 = 0 \Rightarrow z^2(z^2 + z + 1) + z^2 + z + 1 = 0$

$\Rightarrow (z^2 + 1)(z^2 + z + 1) = 0$

If $z^2 + 1 = 0 \Rightarrow z = i \Rightarrow |z| = 1$

If $z^2 + z + 1 = 0 \Rightarrow z = \omega, \omega^2 \Rightarrow |z| = 1$

Problem 82

82. If $z = \sqrt[7]{-1}$, then find the value of $z^{86} + z^{175} + z^{289}$.

Solution of Problem 82

Solution: $\because z = \sqrt[7]{-1} \Rightarrow z^7 = -1$

$$z^{86} + z^{175} + z^{289} = (z^7)^{14}.z^2 + (z^7)^{25} + (z^7)^{41}z^2 = z^2 - 1 - z^2 = -1$$

Problem 83

83. If $z^3 + 2z^2 + 3z + 2 = 0$, then find all the non-real, complex roots of this equation.

Solution of Problem 83

Solution: Given, $z^3 + 2z^2 + 3z + 2 = 0 \Rightarrow z^3 + z^2 + 2z + z^2 + z + 2 = 0$

$\Rightarrow (z+1)(z^2 + z + 2) = 0$

If $z + 1 = 0 \Rightarrow z = -1$, which is real and is of no interest for us.

If $z^2 + z + 2 = 0 \Rightarrow z = \frac{-1 \pm i\sqrt{7}}{2}$ which are complex roots of the given equation.

Problem 84

84. If z is a non-real root of $z = \sqrt[5]{1}$ then find the value of $2^{|1+z+z^2+z^{-2}-z^{-1}|}$

Solution of Problem 84

Solution: $z = \sqrt[5]{1} \Rightarrow z^5 = 1$

$$2^{|1+z+z^2+z^{-2}-z^{-1}|} = 2^{|1+z+z^2+z^3-z^4|}[\because z^4 = 1 \Rightarrow z^{-1} = \frac{z^5}{z} = z^4]$$

$$= 2^{|1+z+z^2+z^3+z^4-2z^4|} = 2^{\left|\frac{1-z^5}{1-z}-2z^4\right|} = 2^{|2z^4|} = 2^2 = 4[\because |z| = 1]$$

Problem 85

85. If z is a non-real root of unity then find the value of $1 + 3z + 5z^2 + ... + (2n-1)z^{n-1}$.

Solution of Problem 85

Solution: Let $S = 1 + 3z + 5z^2 + ... + (2n-1)z^{n-1}$

$\Rightarrow zS = z + 3z^2 + 5z^3 + ... + (2n-3)z^{n-1} + (2n-1)z^n$

$\Rightarrow (1-z)S = 1 + 2z + 2z^2 + 2z^3 + ... + 2z^{n-1} + (2n-1)z^n$

$\Rightarrow (1-z)S = 1 + 2n - 1 + 2[z + z^2 + ... z^{n-1}][\because z^n = 1]$

$= 2n + 2. -1[\because 1 + z + z^2 + ... + z^{n-1} = 0] \Rightarrow S = \frac{2(n-1)}{1-z}$

Problem 86

86. Find the value of $\sqrt{-1-\sqrt{-1-\sqrt{-1-\infty}}}$.

Solution of Problem 86

Solution: Let $z = \sqrt{-1 - \sqrt{-1 - \sqrt{-1 - \infty}}} \Rightarrow z = \sqrt{-1 - z}$

$\Rightarrow z^2 = -1 - z \Rightarrow z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm i\sqrt{3}}{2} \Rightarrow z = \omega, \omega^2$

Problem 87

87. If $z = e^{\frac{i2\pi}{n}}$, then find the vaule of $(11 - z)(11 - z^2) \dots (11 - z^{n-1})$.

Solution of Problem 87

Solution: Given, $z = e^{\frac{i2\pi}{n}}$, which is nth root of unity.

$\therefore x^n - 1 = (x-1)(x-z)(x-z^2(x-z^3)...(x-z^{n-1})$

Putting $x = 11, (11-z)(11-z^2)\ldots(11-z^{n-1}) = \frac{11^n - 1}{10}$

Problem 88

88. If $\frac{3}{2+\cos\theta+i\sin\theta} = a+ib$, then prove that $a^2+b^2 = 4a-3$.

Solution of Problem 88

Solution: Given, $\frac{3}{2+\cos\theta+i\sin\theta} = a+ib \Rightarrow a+ib\frac{3(2+\cos\theta-i\sin\theta)}{5+4\cos\theta}$

Comparing real and imaginary parts, we get $a = \frac{6+3\cos\theta}{5+4\cos\theta}, b = \frac{-3\sin\theta}{5+4\cos\theta}$

$\Rightarrow a^2+b^2 = \frac{36+36\cos\theta+9\cos^2\theta+9\sin^2\theta}{(5+4\cos\theta)^2}$

$= \frac{45+36\cos\theta}{(5+\cos\theta)^2} = \frac{9(5+4\cos\theta)}{(5+4\cos\theta)^2} = \frac{9}{5+4\cos\theta}$

$4a-3 = \frac{24+12\cos\theta-15-12\cos\theta}{5+4\cos\theta} = \frac{9}{5+4\cos\theta}$

$\Rightarrow a^2+b^2 = 4a-3$

Problem 89

89. If $|2z - 1| = |z - 2|$, then prove that $|z| = 1$.

Solution of Problem 89

Solution: Let $z = x + iy, \Rightarrow |(2x - 1) + 2iy| = |(x - 2) + iy|$

$\Rightarrow 4x^2 - 4x + 1 + 4y^2 = x^2 - 4x + 4 + y^2 \Rightarrow 3x^2 + 3y^2 = 3$

$\Rightarrow x^2 + y^2 = 1 \Rightarrow |z| = 1$

Problem 90

90. If x is real and $\frac{1-ix}{1+ix} = m + in$, then prove that $m^2 + n^2 = 1$.

Solution of Problem 90

Solution: Given, $\frac{1-ix}{1+ix} = m + in \Rightarrow m + in = \frac{1-ix}{1+ix} \cdot \frac{1-ix}{1-ix}$

$m + in = \frac{1-x^2-2ix}{1+x^2}$

Comparing real and imaginary parts, we get $m = \frac{1-x^2}{1+x^2}, n = \frac{-2x}{1+x^2}$

$m^2 + n^2 = \frac{(1-x^2)^2+4x^2}{(1+x^2)^2} = 1$