Complex Numbers Problems 51-60

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Problem 51

51. If $|z+6| = |2z+3|$, then prove that $|z| = 3$

Solution of Problem 51

Solution: Given, $|z+6| = |2z+3|$, let $z = x + iy$

$\Rightarrow (x+6)^2 + y^2 = (2x+3)^2 + 4y^2$

$\Rightarrow x^2 + 12x + 36 + y^2 = 4x^2 + 12x + 9 + 4y^2$

$\Rightarrow 3x^2 + 2y^2 = 27 \Rightarrow x^2 + y^2 = 9 \Rightarrow |z| = 3$

Problem 52

52. If $\sqrt{a - ib} = x - iy$, then prove that $\sqrt{a + ib} = x + iy$

Solution of Problem 52

Solution: Given $\sqrt{a-ib} = x - iy$, squaring we get

$$a - ib = x^2 - y^2 - 2ixy$$

Comparing real and imaginary parts, we get

$$a = x^2 - y^2, b = 2xy \Rightarrow a + ib = x^2 - y^2 + 2ixy = x^2 + i^2y^2 + 2ixy$$

$$\Rightarrow \sqrt{a+ib} = x + iy$$

Problem 53

53. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, then find the value of $x_1x_2x_3\ldots$ to ∞.

Solution of Problem 53

Solution: $x_1 x_2 x_3 \dots \infty = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\dots\infty$

$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots\infty\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots\infty\right)$

$= \cos\frac{\pi}{2}.\frac{1}{1-\frac{1}{2}} + i\sin\frac{\pi}{2}.\frac{1}{1-\frac{1}{2}}$

$= \cos\pi + i\sin\pi = -1$

Problem 54

54. Find the value of $\frac{(\cos\theta+i\sin\theta)^4}{(\sin\theta+i\cos\theta)^2}$

Solutiobn of Problem 54

Solution: Given, $\frac{(\cos\theta+i\sin\theta)^4}{(\sin\theta+i\cos\theta)^5}$

$= \frac{(\cos\theta+i\sin\theta)^4}{i^5\left(\frac{1}{i}\sin\theta+\cos\theta\right)^5}$

$= \frac{(\cos\theta+i\sin\theta)^4}{i(\cos\theta-i\sin\theta)^5}$

$= \frac{(\cos\theta+i\sin\theta)^4}{i(\cos\theta+i\sin\theta)^{-5}}$

$= \frac{1}{i}(\cos\theta + i\sin\theta)^9 = \sin 9\theta - i\cos 9\theta$

Problem 55

55. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$ then find $Im(z)$.

Solution of Problem 55

Solution: $z = \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^5 + \left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]^5$

$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$

$= 2\cos\frac{5\pi}{6} \therefore Im(z) = 0$

Problem 56

56. Find the product of all values of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$

Solution of Problem 56

Solution: $z = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}$, thus general root is $\cos\frac{2n\pi+\pi}{4} + i\sin\frac{2n\pi+\pi}{4}$

Thus, substituting $n = 0, 1, 2, 3$ we find four roots and the product is

$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$= \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= \left(-\frac{1}{2} - \frac{1}{2}\right)\left(\frac{-1}{2} - \frac{1}{2}\right)$

$= -1. - 1 = 1$

Problem 57

57. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then find $arg(z_1) - arg(z_2)$.

Solution of Problem 57

Solution: Let $z_1 = r_1(\cos x + i\sin x)$ and $z_2 = r_2(\cos y + i\sin y)$

Then $(r_1\cos x + r_2\cos y)^2 + (r_1\sin x + r_2\sin y)^2 = r_1^2 + r_2^2 + 2r_2r_2$

$\Rightarrow 2r_1r_2(\cos x\cos y + \sin x\sin y) = 2r_2r_2$

$\Rightarrow \cos(x-y) = 1 \Rightarrow x - y = 0 \Rightarrow \arg(z_1) - arg(z_2) = 0$

Problem 58

58. If $z = 1 - \sin\alpha + i\cos\alpha$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$, then find the modulus and principal value of its argument.

Solution of Problem 58

Solution: Let $z = 1 - \sin\alpha + i\cos\alpha = r(\cos\theta + i\sin\theta)$, then

$r = \sqrt{(1-\sin\alpha)^2 + \cos^2\alpha} = \sqrt{2 - 2\sin\alpha}$

$\tan\theta = \frac{\cos\alpha}{1-\sin\alpha} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2} - 2\tan\frac{\alpha}{2}}$

$= \frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}} = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

$\Rightarrow \theta = \frac{\pi}{4} - \frac{\alpha}{2}$

Problem 59

59. Find the value of expression $\left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^8$.

Solution of Problem 59

Solution: Let $z = \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]$

$= \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right] \cdot \left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}\right]$

$= \frac{\left(1+\sin\frac{\pi}{8}\right)^2-\cos^2\frac{\pi}{8}+2i(1+\sin\frac{\pi}{8})\cos\frac{\pi}{8}}{\left(1+\sin\frac{\pi}{8}\right)^2+\cos^2\frac{\pi}{8}}$

$= \frac{2\sin\frac{\pi}{8}+2\sin^2\frac{\pi}{8}+2i(1+\sin\frac{\pi}{8})\cos\frac{\pi}{8}}{2+2\sin\frac{\pi}{8}}$

$= \sin\frac{\pi}{8} + i\cos\frac{\pi}{8} = i\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

$z^8 = i^8(\cos\pi - i\sin\pi) = -1$

Problem 60

60. If $z_r = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}, r = 0, 1, 2, 3, 4$ then find $z_1 z_2 z_3 z_4 z_5$.

Solution of Problem 60

Solution: $z_1 z_2 z_3 z_4 z_5 = \cos\left(\frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5} + \frac{10\pi}{5}\right) + i\sin\left(\frac{2\pi}{5} + \frac{4\pi}{5} + \frac{6\pi}{5} + \frac{8\pi}{5} + \frac{10\pi}{5}\right)$

$= \cos\frac{30\pi}{5} + i\sin\frac{30\pi}{5} = \cos 6\pi + i\sin 6\pi = 1$