Geometric Progression
Problems 71-80

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Problem 71

71. After striking the floor a certain ball rebound to $\frac{4}{5}$th of the height from which it has fallen. Find the total distance it travels before coming to rest if it is gently dropped from a height of 120 meters.

Solution of Problem 71

Solution: Distance covered before first bounce $= 120$ meters.
After striking the floor the ball will go up for $120\frac{4}{5}$ meters and then fall the same distance so distance covered $= 2.120.\frac{4}{5}$ meteres.
For the next bounce distance covered would be $2.120.\frac{4^2}{5^2}$ meters.
This will keep happening till ball comes to rest.
Thus, total distance covered would be $= 120 + 240.\frac{4}{5} + 240.\frac{4^2}{5^2} + \dots$ to ∞

$$= 120 + 240.\frac{4}{5}\left[1 + \frac{4}{5} + \frac{4^2}{5^2} + \dots \text{ to } \infty\right]$$

$$= 120 + 240.\frac{4}{5}.5 = 1080$$

meters.

Problem 72

72. If a be the first term and b be the nth term and p be the product of n terms of a G.P., show that $p^2 = (ab)^n$

Solution of Problem 72

Solution: Let r be the common ratio. $b = ar^{n-1} \Rightarrow ab = a^2 r^{n-1} \Rightarrow (ab)^n = a^{2n} r^{n(n-1)}$

$$p = a.ar.ar^2.\dots.ar^{n-1} = a^n r^{\frac{n(n-1)}{2}} \Rightarrow p^2 = a^{2n} r^{n(n-1)}$$

Thus, $p^2 = (ab)^n$

Problem 73

73. Show that the ratio of sum of n terms of two G.P.'s having the same common ratio is equal to the ratio of their nth terms.

Solution of Problem 73

Solution: Let a and b be first terms and r be the common ratio of two G.P.

Ratio of sums $= \frac{\frac{a(r^n-1)}{r-1}}{\frac{b(r^n-1)}{r-1}} = \frac{a/b}{}$

Ratio of nth terms $= \frac{ar^{n-1}}{br^{n-1}} = \frac{a}{b}$

Hence, proved.

Problem 74

74. If S_1, S_2, S_3 be the sum of $m, 2n, 3n$ terms respectively of a G.P. show that $(S_2 - S_1)^2 = S_1(S_3 - S_2)$

Solution of Problem 74

Solution: Let a be the first term and r be the common ratio of the G.P. Then,

$$S_1 = \frac{a(r^n - 1)}{r - 1}, S_2 = \frac{a(r^{2n} - 1)}{r - 1}, S_3 = \frac{a(r^{3n} - 1)}{r - 1}$$

$$S_2 - S_1 = \frac{ar^n(r^n - 1)}{r - 1}$$

$$S_3 - S_2 = \frac{ar^{2n}(r^n - 1)}{r - 1}$$

$$(S_2 - S_1)^2 = \frac{a^2 r^{2n}(r^n - 1)^2}{(r - 1)^2}$$

$$S_1(S_3 - S_2) = \frac{a(r^n - 1)}{r - 1}\left(\frac{a(r^{3n} - 1)}{r - 1} - \frac{a(r^{2n} - 1)}{r - 1}\right)$$

$$= \frac{a(r^n - 1)}{(r - 1)^2}[a(r^{3n} - r^{2n})] = \frac{a^2 r^{2n}(r^n - 1)^2}{(r - 1)^2}$$

Hence, proved.

Problem 75

75. If S_n denotes the sum of n terms of a G.P.,whose first term is a and common ratio is r, find
$S_1 + S_2 + \ldots + S_{2n-1}$

Solution of Problem 75

Solution:

$$S_1 = a = \frac{a(r-1)}{r-1}$$

$$S_2 = a + ar = \frac{a(r^2-1)}{r-1}$$

$$\dots$$

$$S_{2n-1} = a + ar + \ldots + ar^{2n-2} = \frac{a(r^{2n-1}-1)}{r-1}$$

$$S_1 + S_2 + \ldots + S_{2n-1} = \frac{a}{1-r}\left[(r-1) + (r^2-1) + \ldots + (r^{2n-1}-1)\right]$$

$$= \frac{a}{r-1}\left[\frac{r(r^{2n-1-1})}{r-1} - 2n + 1\right]$$

Problem 76

77. The sum of n terms of a series is $a.2^n - b$, find its nth term. Are the terms of this series in G.P.

Solution of Problem 76

Solution: Given $S_n = a.2^n - b \Rightarrow S_{n-1} = a.2^{n-1} - b \Rightarrow t_n = a2^{n-1}$ Since the ratio of terms will be 2 as evident from t_n the series is in G. P.

Problem 77

77. Find the nth term and the sum of n terms of the series $1 + (1 + 2) + (1 + 2 + 2^2) + \ldots$

Solution of Problem 77

Solution: $t_n = 1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$
Thus, we can rewrite series as $S_n = (2-1) + (2^2-1) + (2^3-1) + \ldots + (2^n-1)$

$$= 2(1 + 2 + 2^2 + \ldots + 2^{n-1}) - 1 - 1 \ldots \text{ to } n \text{ terms}$$
$$= 2.\frac{2^n-1}{2-1} - n = 2^{n+1} - 2 - n$$

Problem 78

78. Find $\frac{1}{1+x^2}\left[1+\frac{2x}{1+x^2}+\left(\frac{2x}{1+x^2}\right)^2+\ldots \text{ to } \infty\right]$ where $x \geq 0$

Solution to Problem 78

Solution: Let $S = \frac{1}{1+x^2}\left[1 + \frac{2x}{1+x^2} + \left(\frac{2x}{1+x^2}\right)^2 + \ldots \text{ to } \infty\right]$

$$S = \frac{1}{1+x^2} \cdot \frac{1}{1 - \frac{2x}{1+x^2}} = \frac{1}{(1-x)^2}$$

Problem 79

79. The sum of an infinite G.P. whose common ratio is numerically less than 1 is 32 and the sum of their first two terms is 24. Find the terms of the G.P.

Solution of Problem 79

Solution: Let a be the first term and r be the common ratio of the G.P.

$$S_\infty = \frac{a}{1-r} = 32 \& a + ar = 24$$

$$32 - 32r + 32r - 32r^2 = 24 \Rightarrow 4 - 4r^2 = 3 \Rightarrow 4r^2 = 1 \Rightarrow r = \pm\frac{1}{2} \Rightarrow a = 16, 48$$

Thus, terms can be found now.

Problem 80

80. The sum of infinite number of terms of a decreasing G.P. is 4 and the sum of the squares of its terms to infinity is $\frac{16}{3}$, find the G.P.

Solution to Problem 80

Solution: Let a be the first term and r be the common ratio of the G.P.
Let $P = a + ar + ar^2 + \ldots = \frac{a}{1-r} = 4$
Let $Q = a^2 + a^2r^2 + a^2r^4 + \ldots = \frac{a^2}{1-r^2} = \frac{16}{3}$
Solving these, we get $a = 2, r = \frac{2}{1}$ so the G.P. is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$