Miscellaneous Problems on A.P., G.P. and H.P.
Problems 171-180

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Problem 171

171. Find the sum to n terms of the series $\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + \dots$

Solution of Problem 171

Solution:

$$t_n = \frac{1}{(1+nx)[1+(n+1)x]} = \frac{1}{x}\left(\frac{1}{1+nx} - \frac{1}{1+(n+1)x}\right)$$

$$t_1 = \frac{1}{x}\left(\frac{1}{1+x} - \frac{1}{1+2x}\right)$$

$$t_2 = \frac{1}{x}\left(\frac{1}{1+2x} - \frac{1}{1+3x}\right)$$

$$\cdots$$

$$t_n = \frac{1}{x}\left(\frac{1}{1+nx} - \frac{1}{1+(n+1)x}\right)$$

Adding, we get

$$S_n = \frac{1}{x}\left(\frac{1}{1+x} - \frac{1}{1+(n+1)x}\right) = \frac{n}{(1+x)[1+(n+1)x]}$$

Problem 172

172. Find the sum to n terms of the series $\frac{1}{(1+x)(1+ax)} + \frac{a}{(1+ax)(1+a^2x)} + \frac{1}{(1+a^2x)(1+a^3x)} + \ldots$

Solution of Problem 172

Solution:

$$t_n = \frac{a^{n-1}}{(1+a^{n-1}x)(1+a^n x)} = \frac{1}{(a-1)x}\left(\frac{1}{1+a^{n-1}x} - \frac{1}{1+a^n x}\right)$$

$$t_1 = \frac{1}{(a-1)x}\left(\frac{1}{1+x} - \frac{1}{1+ax}\right)$$

$$t_2 = \frac{1}{(a-1)x}\left(\frac{1}{1+ax} - \frac{1}{1+a^2 x}\right)$$

$$\cdots$$

$$t_n = \frac{1}{(a-1)x}\left(\frac{1}{1+a^{n-1}x} - \frac{1}{1+a^n x}\right)$$

Adding, we get

$$S = \frac{1}{(a-1)x}\left(\frac{1}{1+x} - \frac{1}{1+a^n x}\right)$$

Problem 173

173. Find the nth term of the series $-1, -3, 3, 23, 63, 129, \ldots$

Solution of Problem 173

Solution First order differences are: $-2, 6, 20, 40, 66, ...$

Secodn order differences are: $8, 14, 20, 26, ...$

Third order differences are; $6, 6, 6, ...$

$$\begin{aligned}\Rightarrow t_n &= -1 - 2.^{n-1}C_1 + 8.^{n-1}C_2 + 6.^{n-1}C_3 \\ &= n^3 - 2n^2 - 3n + 3\end{aligned}$$

Problem 174

174. Find the sum to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \ldots$

Solution of Problem 174

Solution:

$$t_n = \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}} = \frac{\sqrt{2n+1}-\sqrt{2n-1}}{2}$$

$$\therefore t_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$t_2 = \frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2}$$

$$\cdots$$

$$t_n = \frac{\sqrt{2n+1}}{2} - \frac{\sqrt{2n-1}}{2}$$

Adding, we get

$$S = \frac{\sqrt{2n+1}-1}{2}$$

Problem 175

175. If $a_1, a_2, \dots, a_n, \dots$ are in A.P. with first term a and common difference d, then prove that $a_1a_2 + a_2a_3 + \dots + a_na_{n+1} = \frac{[a+(n-1)d](a+nd)-(a-d)a(a+d)}{3d} = \frac{n}{3}[3a^2 + 2and + (n^2-1)d^2]$

Solution of Problem 175

Solution:

$$t_k = a_k a_{k+1}, t_{k+1} = a_{k+1} a_{k+2}$$

$$a_{k+2} t_k = a_k t_{k+1}$$

$$[a_1 + (k+1)d]t_k - [a_1 + (k-1)d]t_{k+1} = 0$$

$$[a_1 + (k-2)d]t_k - [a_1 + (k-1)d]t_{k+1} = -3dt_k$$

$$\therefore (a_1 - d)t_1 - (a_1 + 0d)t_2 = -3dt_1$$

$$(a_1 + 0d)t_2 - (a_1 + d)t_3 = -3dt_2$$

$$\dots$$

$$[a_1 + (n-2)d]t_n - [a_1 + (n-1)]t_{n+1} = -3dt_n$$

Adding, we get

$$-3d(t_1 + t_2 + \dots + t_n) = (a_1 - d)t_1 - [a_1 + (n-1)]t_{n+1}$$

$$S = \frac{[a + (n-1)d](a + nd)[a + (n+1)d] - (a-d)a(a+d)}{3d}$$

$$= \frac{n}{3}[3a^2 + 3nad + (n^2 - 1)d^2]$$

Problem 176

176. If $a_1, a_2, \ldots, a_n, \ldots$ are in A.P. with first term a and common difference d, then prove that $a_1a_2a_3 + a_2a_3a_4 + \ldots + a_na_{n+1}a_{n+2} = \frac{[a+(n-1)d](a+nd)[a+(n+1)d][a+(n+2)d]-(a-d)a(a+d)(a+2d)}{4d} = \frac{n}{4}[4a^3 + 6(n+1)a^2d + 2(2n^2+3n-1)ad^2 + (n^3-2n^2-n-2)d^3]$

Solution of Problem 176

Solution:

$$t_k = a_k a_{k+1} a_{k+2}, t_{k+1} = a_{k+1} a_{k+2} a_{k+3}$$

$$a_{k+3} t_k = a_k t_{k+1}$$

$$[a_1 + (k+2)d] t_k = [a_1 + (k-1)d] t_{k+1}$$

$$[a_1 + (k-2)d] t_k - [a_1 + (k-1)d] t_{k+1} = -4d t_k$$

$$(a_1 - d) t_1 - (a_1 + 0d) t_2 = -4d t_1$$

$$(a_1 + 0d) t_2 - (a_1 + d) t_3 = -4d t_2$$

$$\ldots$$

$$[a_1 + (n-2)d] t_n - [a_1 + (n-1)] t_{n+1} = -4d t_n$$

Adding, we get

$$-4d(t_1 + t_2 + \ldots + t_n) = (a_1 - d) t_1 - [a_1 + (n-1)] t_{n+1}$$

$$S = \frac{[a + (n-1)d](a + nd)[a + (n+1)d][a + (n+2)d] - (a-d)a(a+d)(a+2d)}{4d}$$

$$= \frac{n}{4}[4a^3 + 6(n+1)a^2 d + 2(2n^2 + 3n - 1)ad^2 + (n^3 - 2n^2 - n - 2)d^3]$$

Problem 177

177. Find the sum to n terms of the series $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + ...$

Solution of Problem 177

Solution:

$$t_n = \frac{2n+1}{n^2.(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$t_1 = \frac{1}{1} - \frac{1}{2^2}$$

$$t_2 = \frac{1}{2^2} - \frac{1}{3^2}$$

$$\cdots$$

$$t_n = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

Adding, we get

$$S = 1 - \frac{1}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

Problem 178

178. Let S_n denote the sum to n terms of the series $1.2 + 2.3 + 3.4 + ...$ and σ_{n-1} that to $n-1$ terms of the series $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + ...$ Then prove that $18S_n\sigma_{n-1} - S_n = -2$

Solution of Problem 178

Solution:

$$t_n = n(n+1), S_n = \sum(n^2+n) = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\Rightarrow S_n = \frac{n(n+1)(n+2)}{3}$$

We have proved in problem 155 that $\sigma_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$

$$\therefore \sigma_{n-1} = \frac{1}{18} - \frac{1}{3n(n+1)(n+2)}$$

Now it is trivial to prove that $18S_n\sigma_{n-1} - S_n = -2$

Problem 179

179. Find $\frac{5}{1.2}\cdot\frac{1}{3}+\frac{7}{2.3}\cdot\frac{1}{3^2}+\frac{9}{3.4}\cdot\frac{1}{3^3}+\ldots$ to n terms

Solution of Problem 179

Solution:

$$t_n = \frac{2n+3}{n(n+1)} \cdot \frac{1}{3^n} = \left(\frac{3}{n} - \frac{1}{n+1}\right) \cdot \frac{1}{3^n}$$

$$\therefore t_1 = \left(3 - \frac{1}{2}\right) \cdot \frac{1}{3}$$

$$t_2 = \left(\frac{3}{2} - \frac{1}{3}\right) \cdot \frac{1}{3^2}$$

$$t_3 = \left(\frac{3}{3} - \frac{1}{4}\right) \cdot \frac{1}{3^3}$$

$$\cdots$$

$$t_n = \left(\frac{3}{n} - \frac{1}{n+1}\right) \cdot \frac{1}{3^n}$$

Adding, we get

$$S_n = 1 - \frac{1}{n+1} \cdot \frac{1}{3^n}$$

Problem 180

180. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\infty = \frac{\pi^2}{6}$ then find $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty$

Solution of Problem 180

Solution:

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \infty$$

$$S' = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots \infty$$

$$4S' = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \infty$$

$$4S' = S \Rightarrow S' = \frac{S}{4}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = S - S' = \frac{3}{4}S = \frac{\pi^2}{8}$$