Geometric Progression
Problems 61-70

Shiv Shankar Dayal

September 11, 2021

Important Result

- $a^n - b^n$ is divisible by $a - b$ for any $n \in N$
  Proof:
$$\frac{a^n - b^n}{a - b} = \frac{a^n\left(1 - \frac{b^n}{a^n}\right)}{a\left(1 - \frac{b}{a}\right)}$$
$$= a^{n-1}\left(1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + \ldots + \frac{b^{n-1}}{a^{n-1}}\right)$$
$$= a^{n-1} + ba^{n-2} + b^2a^{n-3} + \ldots + b^{n-1}$$
- $a^n + b^n$ is divisible by $a + b$ where n is any odd positive natural number.
  Proof:
$$\frac{a^n + b^n}{a + b} = \frac{a^n\left(1 - \left(-\frac{b}{a}\right)^n\right)}{1 - \left(-\frac{b}{a}\right)}$$
$$= a^{n-1}\left(1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3} + \ldots + (-1)^n\frac{b^{n-1}}{a^{n-1}}\right)$$
$$= a^{n-1} - ba^{n-2} + b^2a^{n-3} - b^3a^{n-4} + \ldots + (-1)^n b^{n-1}$$

Problem 61

61. Express $0.4\dot{2}\dot{3}$ as a rational number.

Solution of Problem 61

Solution:

$$
\begin{aligned}
0.4\dot{2}\dot{3} &= 0.423232323\ldots \text{ to } \infty \\
&= .4 + .023 + .00023 + \ldots \text{ to } \infty \\
&= \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} \ldots \text{ to } \infty \\
&= \frac{4}{10} + \frac{23}{1000}\left[1 + \frac{1}{100} + \frac{1}{10000} + \ldots \text{ to } \infty\right] \\
&= \frac{4}{10} + \frac{23}{1000}\frac{1}{1 - \frac{1}{100}} \\
&= \frac{419}{990}
\end{aligned}
$$

Problem 62

62. Find $\frac{1}{5} + \frac{1}{7} + \frac{1}{5^2} + \frac{1}{7^2}$ to ∞

Solution of Problem 62

Solution: Required sum $= \left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}\right)$ to $\infty + \left(\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} +\right)$ to ∞

$$= \frac{\frac{1}{5}}{1 - \frac{1}{5}} + \frac{\frac{1}{7}}{1 - \frac{1}{7}}$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Problem 63

63. Prove that the sum of n terms of the series $11 + 103 + 1005 + \ldots$ is $\frac{10}{9}(10^n - 1) + n^2$

Solution of Problem 63

Solution: The series can be rewritten as

$$(10+1)+(100+3)+(1000+5)+\ldots$$

$$=(10+100+1000+\ldots)+(1+3+5+\ldots)$$

$$=\frac{10(10^n-1)}{10-9}+\frac{n}{2}[2.1+(n-1)2]$$

$$=\frac{10}{9}(10^n-1)+n^2$$

Problem 64

64. Find the sum to n terms of the series $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\ldots$

Solution of Problem 64

Solution: Given series on expansion gives

$$\left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right) + \left(x^6 + \frac{1}{x^6} + 2\right) + \dots$$

Rewriting the above series

$$(x^2 + x^4 + x^6 + \dots) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \dots\right) + (2 + 2 + 2 + \dots)$$

$$= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{1}{x^2} \frac{1 - \frac{1}{x^{2n}}}{1 - \frac{1}{x^2}} + 2n$$

Problem 65

65. If S be the sum, P be the product and R the sum of reciprocals of n terms in G.P., prove that $P^2 = \left(\frac{S}{R}\right)^n$

Solution of Problem 65

Solution: Let a be the first term and r be the common ratio of G.P.
Given, $S = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$

Also, $P = a.ar.ar^2.\ldots.ar^{n-1} = a^n r^{1+2+\ldots+n-1} = a^n r^{\frac{n(n-1)}{2}}$
Also, $R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots + \frac{1}{ar^{n-1}}$
$= \frac{1}{a}\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}} = \frac{1}{a}\frac{r^n-1}{r^n}\frac{r}{r-1}$
$= \frac{1-r^n}{1-r}\frac{1}{ar^{n-1}}$

$$\frac{S}{R} = a^2 r^{n-1}$$

$$\left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = (a^n r^{\frac{n(n-1)}{2}})^2 = P^2$$

Problem 66

66. Find $1 + \frac{x}{1+x} + \frac{x^2}{(1+x)^2} + \ldots$ to ∞ if $x > 0$

Solution of Problem 66

Solution: Here terms of a given series are in G.P. where $a = 1, r = \frac{x}{1+x}$ Also, $|r| < 1$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{x}{1+x}} = 1 + x$$

Problem 67

67. Prove that in an infinite G.P. whose common ratio is r is numerically less than one, the ratio of any term to the sum of all the succeediing terms is $\frac{1-r}{r}$.

Solution of Problem 67

Solution: The sum of all terms $= S_\infty$ If we consider t_n in the ratio then sum of rest of terms will be $S_\infty - S_n$, thus, required ratio is

$$\frac{t_n}{S_\infty - S_n} = \frac{ar^{n-1}}{\frac{a}{1-r} - \frac{a(1-r^n)}{1-r}}$$

$$= \frac{ar^{n-1}}{\frac{a}{1-r}(1 - 1 + r^n)} = \frac{1-r}{r}$$

Problem 68

68. If $S_1, S_2, S_3, \ldots, S_p$ are the sum of infinite geometric series whose first terms are $1, 2, 3, \ldots, p$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{p+1}$ respectively, prove that $S_1 + S_2 + S_3 + \ldots + S_p = p(p+3)/2$

Solution of Problem 68

Solution: Let us find out sums one by one.

$$S_1 = \frac{1}{1-\frac{1}{2}} = 2$$

$$S_2 = \frac{2}{1-\frac{1}{3}} = 3$$

$$S_3 = \frac{3}{1-\frac{1}{4}} = 4$$

$$\ldots$$

$$S_p = \frac{p}{1-\frac{1}{p+1}} = p+1$$

$$\text{L.H.S.} = S_1 + S_2 + S_3 + \ldots + S_p$$

$$= 2+3+4+\ldots+p+1 = \frac{p}{2}[2.2+(p-1)] = \frac{p(p+1)}{3}$$

Problem 69

69. If $x = 1 + a + a^2 + a^3 + \dots$ to ∞ and $y = 1 + b + b^2 + b^3 + \dots$ to ∞, show that $1 + ab + a^2b^2 + a^3b^3 + \dots$ to $\infty = \frac{xy}{x+y-1}$, where $0 < a < 1$ and $0 < b < 1$

Solution of Problem 69

Solution:

$$x = \frac{1}{1-a}, y = \frac{1}{1-b}$$

$$\Rightarrow a = \frac{x-1}{x}, b = \frac{y-1}{y}$$

$$1 + ab + a^2b^2 + a^3b^3 + \dots \text{ to } \infty = \frac{1}{1-ab}$$

$$= \frac{1}{1 - \frac{x-1}{x}\frac{y-1}{y}} = \frac{xy}{x+y-1}$$

Problem 70

70. Find the sum to infinity for the series $1 + (1 + a)r + (1 + a + a^2)r^2 + \dots$, where $0 < a < 1$ and $0 < r < 1$

Solution:

$$1 + (1 + a)r + (1 + a + a^2)r^2 + \ldots \text{ to } \infty$$

$$= \frac{1 - a}{1 - a} + \frac{1 - a^2}{1 - a} r + \frac{1 - a^3}{1 - a} + \ldots \text{ to } \infty$$

$$= \frac{1}{1 - a}[1 + r + r^2 + \ldots \text{ to } \infty - a(1 + ar + a^2 r^2 + \ldots \text{ to } \infty)]$$

$$= \frac{1}{1 - a}\left[\frac{1}{1 - r} - a\left(\frac{1}{1 - ar}\right)\right]$$

$$= \frac{1}{(1 - r)(1 - ar)}$$