Miscellaneous Problems on A.P., G.P. and H.P.
Problems 41-50

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Problem 41

41. Find the sum of the product of the first n natural numbers takes two at a time.

Solution of Problem 41

Solution: Required sum is $S = 1.2 + 2.3 + 3.4 + ... + (n-1).n$

$$= \frac{(1+2+...+n)^2 - (1^2+2^2+...+n^2)}{2}$$

$$= \frac{\frac{n^2(n+1)^2}{2^2} - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{1}{24}n(n^2-1)(3n+2)$$

Problem 42

42. A postman delivered daily for 42 days 4 more letters each day than on the previous day. The total delivery made for the first 24 days of the period was the same as that for the last 18 days. How many letters did he deliver during the whole period?

Solution of Problem 42

Solution: Let the postman deliver a letters on the first day. Given, $d = 4$. Also, according to the question

$$\frac{24}{2}[2a + (24 - 1)4] = \frac{18}{2}[2(a + 24.4) + (18 - 1)4]$$

$$24a + 48.23 = 18a + 24.72 + 36.17$$

$$a = 206$$

Thus, total no. of letters delivered $= \frac{42}{2}[2.206 + (42 - 1).4] = 12096$

Problem 43

43. If S_n denotes the sum to n terms of an A.P. and $S_n = n^2p, S_m = m^2p, m \neq n$, prove that $S_n = p^3$

Solution of Problem 43

Solution: Let a be the first term and d be the common difference of A.P. Then

$$S_n = \frac{n}{2}[2a + (n-1)d] = n^2 p \Rightarrow 2a + (n-1)d = 2np$$

$$S_m = \frac{m}{2}[2a + (m-1)d] = m^2 p \Rightarrow 2a + (m-1)d = 2mp$$

Subtracting, we get

$$d = 2p$$

Substituting this in any of the equations we get $a = p$

$$S_p = \frac{p}{2}[2a + (p-1)d] = p^3$$

Problem 44

44. There are n A.P.'s whose common difference are $1, 2, 3, \dots, n$ respectively the first term of each being unity. Prove that the sum of their nth terms is $\frac{n}{2}(n^2+1)$

Solution of Problem 44

Solution: nth term of first A.P. $= 1 + (n-1).1 = n$

nth term of second A.P. $= 1 + (n-1).2 = 2n - 1$

nth term of third A.P. $= 1 + (n-1).3 = 3n - 2$

...

nth term of nth A.P. $= 1 + (n-1).n = n^2 - n + 1$

Since all the nth terms are in A.P., Sum of all these $= \frac{n}{2}(n + n^2 - n + 1) = \frac{n}{2}(n^2 + 1)$

Problem 45

45. If $S_1, S_2, \dots, S_m$ are the sum of n terms of m A.P.s whose first terms are $1, 2, \dots, m$ and whose common differences are $1, 3, 5, \dots, 2m-1$ respectively, show that $S_1 + S_2 + \dots + S_m = \frac{1}{2}mn(mn+1)$

Solution of Problem 45

Solution: Sum of first A.P. $S_1 = \frac{n}{2}[2.1 + (n-1).1]$

Sum of second A.P. $S_2 = \frac{n}{2}[2.2 + (n-1).3]$

Sum of third A.P. $S_3 = \frac{n}{2}[2.3 + (n-1).5]$

...

Sum of mth A.P. $S_m = \frac{n}{2}[2.m + (n-1)(2m-1)]$

Adding all these, we get

$$S_1 + S_2 + ... + S_m = \frac{n}{2}[2(1+2+...+m) + (n-1)(1+3+...+2m-1)]$$

$$= \frac{n}{2}\left(2.\frac{m(m+1)}{2} + (n-1).\frac{m}{2}.2m\right)$$

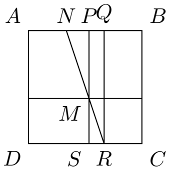
$$= \frac{1}{2}mn(mn+1)$$

Problem 46

46. A straight line is drawn through the center of a square $ABCD$ intersecting side AB at point N so that $AN : NB = 1 : 2$. On this line take an arbitrary point M lying inside the square. Prove that the distances from M to the sides AB, AD, BC, CD of the square taken in that order, form an A.P.

Solution of Problem 46

Solution:



Consider the above diagram in which line NR passes through center of square $ABCD$ and divides AB such that $AN : NB = 1 : 2$. Also, let each side has length equal to $3a$.

$\Rightarrow AN = NQ = QB = a$ So in $\triangle NQR, NQ = \frac{1}{3}QR \Rightarrow NP = \frac{1}{3}PM = \frac{1}{3}x$, where x is distance of M from AB, since the triangles NPM and NQR are similar.

Distance of M from $AD = AN + NP = a + \frac{1}{3}x$

Distance of M from $BC = BN - NP = 2a - \frac{1}{3}x$

Distance of M from $CD = QR - PM = 3a - x$ and thus these are in A.P.

Problem 47

47. If the sides of a right-angled triangle are in G.P., find the cosine of the greater acute angle.

Solution of Problem 47

Solution: Let the sides of right-angled triangle be a, ar, ar^2 where ar^2 is the hypotenuse.

$$\therefore a^2r^4 = a^2 + a^2r^2 \therefore r2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\because r > 1 \Rightarrow r^2 > 1 \Rightarrow r^2 = \frac{1 + \sqrt{5}}{2}$$

Thus, cosine of greater acute angle

$$= \frac{a}{ar^2} = \frac{1}{r^2} = \frac{2}{1 + \sqrt{5}}$$

Problem 48

48. If a, b, c, d are non-zero real numbers and $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$, prove that a, b, c, d are in G.P.

Solution of Problem 48

Solution:

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$$

$$\Rightarrow a^2b^2+b^4+b^2c^2+a^2c^2+b^2c^2+c^4+a^2d^2+b^2d^2+c^2d^2=a^2b^2+b^2c^2+c^2d^2+2acb^2+2bdc^2+2abcd$$

$$\Rightarrow (b^4+a^2c^2-2acb^2)+(c^4+b^2d^2-2bdc^2)+(a^2d^2+b^2c^2-2abcd)=0$$

$$\Rightarrow (b^2-ac)^2+(c^2-bd)^2+(ad-bc)^2=0$$

$$\Rightarrow b^2=ac, c^2=bd, ad=bc$$

$\Rightarrow a,b,c,d$ are in G.P.

Problem 49

49. Does there exist a geometric progression containing $27, 8$ and 12 as three of its terms? If it exists, how many such progressions are possible?

Solution of Problem 49

Solution: If possible let $27, 8$ and 12 be the pth, qth and kth terms respectively of a G.P. whose first term is a and common ratio is r. Then

$$27 = ar^{p-1}, 8 = ar^{q-1}, 12 = ar^{k-1}$$

$$\Rightarrow r^{p-q} = \frac{27}{8} = \frac{3^3}{2^3}$$

$$\Rightarrow r^{k-q} = \frac{12}{8} = \frac{3}{2}$$

$$\Rightarrow 3k - 3q = p - q \Rightarrow p + 2q - 3k = 0$$

The posible set of solutions is $p = 4n, q = n, k = 2n$ where $n \in N$. Thus, infinite such G.P.s are possible.

Problem 50

50. Show that $10, 11, 12$ cannot be terms of a G.P.

Solution of Problem 50

Solution: If possible, let $10, 11$ and 12 be the pth, qth and kth terms respectively of a G.P. whose first term is a and common ratio is r. Then

$$10 = ar^{p-1}, 11 = ar^{q-1}, 12 = ar^{k-1}$$
$$\Rightarrow \frac{11}{10} = r^{q-p}, \frac{12}{11} = r^{k-q}$$
$$\Rightarrow \left(\frac{11}{10}\right)^{k-q} = \left(\frac{12}{11}\right)^{q-p}$$
$$11^{k-p} = 5^{k-q}2^{k+q-2p}.3^{q-p}$$

This is possible only when $k-p=0, k-q=0, k+q-2p=0$ and $q-p=0$ or $k=p=q$ which is not possible since p, q, k are distinct.