Miscellaneous Problems on A.P., G.P. and H.P.
Problems 101-110

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Problem 101

101. If S_n be the sum of infinite G.P.'s whose first term is n and the common ratio is $\frac{1}{n+1}$, find $\lim_{n\to\infty} \frac{S_1 S_n + S_2 S_{n-1} + ... + S_n S_1}{S_1^2 + S_2^2 + ... + S_n^2}$

Solution of Problem 101

Solution:

$$S_1 = \frac{1}{1-\frac{1}{2}} = 2, S_2 = \frac{2}{1-\frac{1}{3}} = 3, \ldots S_n = \frac{n}{1-\frac{1}{n+1}} = n+1$$

General term of numerator $t_i = S_i S_{n-i+1} = (i+1)(n-i+2) = (n+1)i - i^2 + (n+1)$

$$\therefore \text{Sum for numerator } = \sum_{i=1}^{n} t_i = \sum [(n+1)i - i^2 + (n+1)] = \frac{n(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{6} + n(n+1)$$

$$\text{Sum for denominator } = 1^2 + 2^2 + \ldots + (n+1)^2 - 1 = \frac{(n+1)(n+2)(2n+3)}{6} - 1$$

Upon simplification

$$\lim_{n\to\infty} \frac{S_1 S_n + S_2 S_{n-1} + \ldots + S_n S_1}{S_1^2 + S_2^2 + \ldots + S_n^2} = \frac{1}{2}$$

Problem 102

102. The sum of the terms of an infinitely decreasing G.P. is equal to the greatest value of the function $f(x) = x^3 + 3x - 9$ on the interval $[-5, 3]$, and the difference between the first and second terms is $f'(0)$. Prove that the common ratio of the progression is $\frac{2}{3}$.

Solution of Problem 102

Solution: $f'(x) = 3x^2 + 3$ which yields imaginary roots implying that there is no local maxima. However, $3x^2 + 3$ is positive for all values of x which means that $f(x)$ is monotonically increasing in $[-5, 3]$ implying that maximum value will be at $x = 3$

$f(3) = 27$, also let a to be the first term and r to be the common ratio then given, $a - ar = f'(0) = 3$. The sum is given as $\frac{a}{1-r} = 27$ solving these yields $r = \frac{2}{3}, -\frac{4}{3}$ but the series is decreasing so $r = \frac{2}{3}$

Problem 103

103. Find the sum of the series $\frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \ldots \infty$

Solution of Problem 103

Solution:

$$\text{Let } S = \frac{5}{13} + \frac{55}{13^2} + \frac{555}{13^3} + \ldots \infty$$

$$= \frac{5}{9}\left[\frac{10-1}{13} + \frac{100-1}{13^2} + \frac{1000-1}{13^3} + \ldots \infty\right]$$

$$= \frac{5}{9}\left[\frac{10}{13} + \frac{10^2}{13^2} + \frac{10^3}{13^3} + \ldots \infty - \frac{1}{13} - \frac{1}{13^2} - \frac{1}{13^3} - \ldots \infty\right]$$

$$= \frac{5}{9}\left[\frac{\frac{10}{13}}{1-\frac{10}{13}} - \frac{\frac{1}{13}}{1-\frac{1}{13}}\right]$$

$$= \frac{5}{9}\left[\frac{10}{13}.\frac{13}{3} - \frac{1}{13}.\frac{13}{12}\right]$$

$$= \frac{65}{36}$$

Problem 104

104. If $e^{\sin^2 x+\sin^4 x+\sin^6 x+\ldots\infty \log_e 2}$ satisfies the equation $x^2 - 9x + 8 = 0$, find the value of $\frac{\cos x}{\cos x+\sin x}, 0 < x < \frac{\pi}{2}$

Solution of Problem 104

Solution:

$$e^{\sin^2 x+\sin^4 x+\sin^6 x+\ldots\infty \log_e 2} = 2^{\frac{\sin^2 x}{1-\sin^2 x}\log_e e} = 2^{\tan^2 x}$$

$$x^2 - 9x + 8 = 0 \Rightarrow (x-1)(x-8) = 0 \Rightarrow x = 1 = 2^0, x = 8 = 2^3$$

$$\therefore 2^{\tan^2 x} = 2^0, 2^3 \because 0 < x < \frac{\pi}{2} \Rightarrow \tan x = \sqrt{3} \Rightarrow x = \frac{\pi}{3}$$

$$\frac{\cos x}{\cos x + \sin x} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{\sqrt{3}}{2}} = \frac{1}{1+\sqrt{3}}$$

Problem 105

105. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the sum to infinite number of terms of series $\cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + ...$ is finite, then show that x lies in the set $(-\frac{\pi}{2}, \frac{\pi}{2})$

Solution of Problem 105

Solution:

$$S = \cos x + \frac{2}{3}\cos x \sin^2 x + \frac{4}{9}\cos x \sin^4 x + \ldots$$
$$= \frac{\cos x}{1 - \frac{2}{3}\sin^2 x} = \frac{3\cos x}{3 - 2\sin^2 x} = \frac{3\cos x}{2 + \cos 2x}$$

The term $\frac{3\cos x}{2+\cos 2x}$ is finite for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Problem 106

106. Suppose $0 < x < \pi$ and the expression $e^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots\infty \log_e 4}$ satisfies the quadratic equation $y^2 - 20y + 64 = 0$, then find the value of x.

Solution of Problem 106

Solution:

$$e^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots\infty \log_e 4} = 4^{\frac{1}{1-|\cos x|}} \log_e e = 4^{\frac{1}{1-|\cos x|}}$$

$$y^2 - 20y + 64 = 0 \Rightarrow (y-16)(y-4) = 0 \Rightarrow y = 4^1, 4^2 \Rightarrow \frac{1}{1-|\cos x|} = 1, 2$$

$$\text{If } \frac{1}{1-|\cos x|} = 1 \text{ then } |\cos x| = 0 \Rightarrow x \notin (0, \pi)$$

$$\therefore 1 - |\cos x| = \frac{1}{2} \Rightarrow x = n\pi + \frac{\pi}{3}, n \in I$$

Problem 107

107. An A.P. and a G.P. with positive terms have the same number of terms and their first terms as well as the last terms are equal. Show that the sum of A.P. is greater than or equal to the sum of the G.P.

Solution of Problem 107

Solution: Let a be the first term, b be the last term and n be the number of terms of A.P. and G.P.

Then c.d. of A.P. $= \frac{b-a}{n-1}$
and c.r. of the G.P. $= \left(\frac{b}{a}\right)^{n-1}$. Let S be the sum of n terms of A.P. and S' the sum of n terms of G.P. then $S = \frac{n}{2}(a+b)$

$$S' = a(1 + r + r^2 + ... + r^{n-1})$$
$$S' = a(r^{n-1} + r^{n-2} + ... + 1)$$
$$\therefore S' = \frac{a}{2}[(1 + r^{n-1}) + (r + r^{n-2}) + (r^k + r^{n-k-1}) + ... + (r^{n-1} + 1)]$$

Now,

$$(r^k + r^{n-k-1}) - (r^{n-1} + 1) = (r^k - 1) + r^{n-1}(r^{-k} - 1)$$
$$= (r^k - 1)\left(1 - \frac{r^{n-1}}{r^k}\right) = (r^k - 1)(1 - r^{n-k-1}) \leq 0$$
$$\therefore S' \leq \frac{an}{2}(1 + r^{n-1}) = \frac{an}{2}\left(1 + \frac{b}{a}\right) = \left(\frac{a+b}{2}\right)n = S$$
$$\therefore S \geq S'$$

Problem 108

108. Given a G.P. and A.P. of positive terms $a, a_1, a_2, \dots, a_n, \dots$ and $b, b_1, b_2, \dots, b_n, \dots$ respectively, with the common ratio of the G.P. being different from 1, prove that there exists $x \in R, x > 0$ such that
$\log_x a_n - b_n = \log_x a - b, \ \forall n \in N.$

Solution of Problem 108

Solution: Given $a, a_1, a_2, a_3, ...$ are in G.P. so $\log a, \log a_1, \log a_2, ...$ are in A.P. Let the common difference of this A.P. be d_1. Now $\log a_n = \log a + nd_1$. Further if d be the common difference of the A.P. $b, b_1, b_2, ...$ then $b_n = b + nd$

$$\therefore \frac{\log a_n - \log a}{b_n - b} = \frac{nd_1}{nd} = \frac{d_1}{d}$$

Let $\log x = \frac{d_1}{d}$ for a fixed positive real number x.

$$\Rightarrow \frac{\log a_n - \log a}{b_n - b} = \log x \Rightarrow b_n - b = \log_x \left(\frac{a_n}{a}\right)$$

$$\Rightarrow \log_x a_n - \log_x a = b_n - b \Rightarrow \log_x a_n - b_n = \log_x a - b$$

Problem 109

109. If the $(m+1)$th, $(n+1)$th and $(r+1)$th terms of an A.P. are in G.P., and m, n, r are in H.P., show that the ratio of the first term to the common difference of the A.P. is $-n/2$.

Solution of Problem 109

Solution: Given $a+md, a+nd, a+rd$ are in G.P., where a is the first term and d is the c.d. of A.P.

$$\Rightarrow (a+nd)^2 = (a+md)(a+rd)$$

$$\Rightarrow d(n^2d+2an) = d(am+ar+mrd) \Rightarrow (n^2-mr)d = a(m+r-rn)$$

$$\frac{d}{a} = \frac{m+r-2n}{n^2-mr}$$

Given, m, n, r are in H.P. $\therefore n = \frac{2mr}{m+r} \Rightarrow m+r = \frac{2mr}{n}$

$$\therefore \frac{d}{a} = \frac{\frac{2mr}{n}-2n}{n^2-mr} = -\frac{2}{n} \therefore \frac{a}{d} = -\frac{n}{2}$$

Problem 110

110. If a, b, c are in G.P. and $a-b, c-a, b-c$ are in H.P., then show that $a+4b+c=0$

Solution to Problem 110

Solution: Let r be the common ratio of the G.P., then $b = ar, c = ar^2$. Given, $a - b, c - a, b - c$ are in H.P.

$$\therefore c - a = \frac{2(a-b)(b-c)}{a-b+b-c}$$

$$(c-a)^2 = 2(a-b)(b-c) \Rightarrow (ar^2 - a)^2 = 2(a-ar)(ar-ar^2)$$

$$a^2(r^2-1)^2 = -2a^2(1-r)r(1-r) \Rightarrow (r+1)^2 = -2r \Rightarrow 1 + 4r + r^2 = 0$$

$$\Rightarrow a + 4ar + ar^2 = 0 \Rightarrow a + 4b + c = 0$$