Miscellaneous Problems on A.P., G.P. and H.P.
Problems 151-160

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Problem 151

151. Find $\sum_{k=n}^{n} \tan^{-1} \frac{2k}{2+k^2+k^4}$

Solution of Problem 151

Solution:

$$t_n = \tan^{-1}\frac{2n}{2+n^2+n^4} = \tan^{-1}\frac{2n}{1+1+n^2+n^4}$$

$$= \tan^{-1}\frac{2n}{1+1+(n^2+1)^2-n^2} = \tan^{-1}\frac{2n}{1+(n^2+n+1)(n^2-n+1)}$$

$$= \tan^{-1}\frac{(n^2+n+1)-(n^2-n+1)}{1+(n^2+n+1)(n^2-n+1)}$$

$$= \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$$

$$\therefore t_1 = \tan^{-1} 3 - \tan^{-1} 1$$

$$t_2 = \tan^{-1} 7 - \tan^{-1} 3$$

$$\cdots$$

$$t_{n-1} = \tan^{-1}(n^2-n+1) - \tan^{-1}[(n-1)^2-(n-1)+1]$$

$$t_n = \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$$

Adding, we get

$$S_n = \tan^{-1}(n^2+n+1) - \tan^{-1} 1 = \tan^{-1}\frac{n^2+n}{n^2+n+2}$$

Problem 152

152. Show that $\frac{1^4}{1.3}+\frac{2^4}{3.5}+\frac{3^4}{5.7}+\ldots+\frac{n^4}{(2n-1)(2n+1)}=\frac{n(4n^2+6n+5)}{48}+\frac{n}{16(2n+1)}$

Solution of Problem 152

Solution:

$$t_n = \frac{n^4}{4n^2-1} = \frac{1}{16}\left[\frac{16n^4}{4n^2-1}\right]$$

$$= \frac{1}{16}\left[\frac{16n^4-1+1}{4n^2-1}\right] = \frac{1}{16}\left[4n^2+1+\frac{1}{(2n-1)(2n+1)}\right]$$

$$= \frac{1}{16}\left[4n^2+1+\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$S_n = \sum t_n = \frac{1}{4}\sum n^2 + \frac{1}{16}\sum 1 + \frac{1}{32}\sum\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{n}{16} + \frac{1}{32}\left(1-\frac{1}{2n+1}\right)$$

$$= \frac{n}{48}(4n^2+6n+5) + \frac{1}{16}\frac{n}{2n+1}$$

$$= \frac{n(4n^2+6n+5)}{48} + \frac{n}{16(2n+1)}$$

Problem 153

153. If $a_1, a_2, \dots, a_n, \dots$ are in A.P. with first term a and common difference d, find the sum for $r > 1$ of $a_1 a_2 \dots a_r + a_2 a_3 \dots a_{r+1} + \dots$ to n terms

Solution of Problem 153

Solution:

$$t_k = a_k a_{k+1} \dots a_{k+r-1}, t_{k+1} = a_{k+1} a_{k+2} \dots a_{k+r}$$

$$\therefore a_{k+r} t_k = a_k t_{k+1}$$

$$[a_1 + (k+r-1)d]t_k = [a_1 + (k-1)d]t_{k+1}$$

$$\Rightarrow [a_1 + (k-2)d]t_k - [a_1 + (k-1)d]t_{k+1} = -(1+r)dt_k$$

Thus,

$$(a-d)t_1 - (a_1 + 0d)t_2 = -(1+r)dt_1$$

$$(a+0d)t_2 - (a_1 + d)t_3 = -(1+r)dt_2$$

$$\dots$$

$$[a_1 + (n-2)d]t_n - [a_1 + (n-1)d]t_{n+1} = -(1+r)dt_n$$

$$(a-d)t_1 - [a_1 + (n-1)d]t_{n+1} = -(1+r)d[t_1 + t_2 + \dots + t_n]$$

$$\therefore t_1 + t_2 + \dots + t_n = \frac{a_n a_{n+1} \dots a_{n+r} - a_0 a_1 \dots a_r}{(r+1)d}$$

Problem 154

154. If $a_1, a_2, \dots, a_n, \dots$ are in A.P. and none of them is zero. Then prove that

$$\frac{1}{a_1 a_2 \dots a_r} + \frac{1}{a_2 a_3 \dots a_{r+1}} + \dots + \frac{1}{a_n a_{n+1} \dots a_{n+r-1}} = \frac{1}{(r-1)(a_2-a_1)}\left[\frac{1}{a_1 a_2 \dots a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-1}}\right]$$

Solution of Problem 154

Solution: Let a be the first term and d be the common difference of A.P. Let t_k be the kth term of the given sequence. Then,

$$t_k = \frac{1}{a_k a_{k+1} \dots a_{k+r-1}}, t_{k+1} = \frac{1}{a_{k+1} a_{k+2} \dots a_{k+r}}$$

$$\Rightarrow a_k t_k = a_{k+r} t_{k+1}$$

$$[a + (k-1)d]t_k - (a + kd)t_{k+1} = d(r-1)t_{k+1}$$

$$\therefore (a + 0d)t_1 - (a + d)t_2 = d(r-1)t_2$$

$$(a + d)t_2 - (a + 2d)t_3 = d(r-1)t_3$$

$$\dots$$

$$[a + (n-2)d]t_{n-1} - [a + (n-1)d]t_n = d(r-1)t_n$$

Adding, we get

$$at_1 - [a + (n-1)d]t_n = d(r-1)[t_2 + t_3 + \dots + t_n]$$

$$[a + (r-d)d]t_1 - [a + (n-1)d]t_n = d(r-1)S[t_1 + t_2 + \dots + t_n]$$

$$t_1 + t_2 + \dots + t_n = \frac{1}{(r-1)d}\left(\frac{a_r}{a_1 a_2 \dots a_r} - \frac{a_n}{a_n a_{n+1} \dots a_{n+r-1}}\right)$$

$$S_n = \frac{1}{(r-1)(a_2 - a_1)}\left(\frac{1}{a_1 a_2 \dots a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-1}}\right)$$

Problem 155

155. Find the sum to n terms of the series $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \ldots$

Solution of Problem 155

Solution: Let t_i be the ith term of the series, then

$$t_i = \frac{1}{i(i+1)(i+2)(i+3)}, t_{i+1} = \frac{1}{(i+1)(i+2)(i+3)(i+4)}$$

$$\Rightarrow it_i = (i+4)t_{i+1} \Rightarrow it_i - (i+1)t_{i+1} = 3t_{i+1}$$

$$\therefore 1.t_1 - 2t_2 = 3t_2$$

$$2.t_2 - 3.t_3 = 3t_3$$

$$\dots$$

$$(n-1).t_i - nt_n = 3t_n$$

Adding, we get

$$t_1 - nt_n = 3(t_1 + t_2 + \dots + t_n) \Rightarrow 4t_1 - nt_n = 3[t_1 + t_2 + \dots + t_n]$$

$$t_1 + t_2 + \dots + t_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$$

Problem 156

156. Find the sum to n terms of the series $\frac{3}{2.4.6} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + ...$

Solution of Problem 156

Solution:

$$t_n = \frac{n+2}{n(n+1)(n+3)} = \frac{(n+2)^2}{n(n+1)(n+2)(n+3)}$$

$$= \frac{n^2+4n+4}{n(n+1)(n+2)(n+3)} = \frac{n(n+4)}{n(n+1)(n+2)(n+3)} + \frac{4}{n(n+1)(n+2)(n+3)}$$

$$= \frac{n(n+1)+3n}{n(n+1)(n+2)(n+3)} + \frac{4}{n(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} + \frac{3}{(n+1)(n+2)(n+3)} + \frac{4}{n(n+1)(n+2)(n+3)}$$

Now that we have found t_n we can find S_n like previous problem.

$$S_n = \frac{29}{36} - \frac{1}{n+3} - \frac{3}{2(n+2)(n+3)} - \frac{4}{3(n+1)(n+2)(n+3)}$$

Problem 157

157. Find $\frac{1}{1.3} + \frac{2}{1.3.5} + \frac{3}{1.3.5.7} + ...$ to n terms

Solution of Problem 157

Solution:

$$t_n = \frac{n}{1.3.5.7\ldots(2n-1)(2n+1)}$$

$$= \frac{1}{2}\left[\frac{1}{1.3.5.7\ldots(2n-1)} - \frac{1}{1.3.5.7\ldots(2n+1)}\right]$$

$$\therefore t_1 = \frac{1}{2}\left(1 - \frac{1}{1.3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{1.3} - \frac{1}{1.3.5}\right)$$

$$\ldots$$

$$t_n = \frac{1}{2}\left(\frac{1}{1.3.5.7\ldots(2n-1)} - \frac{1}{1.3.5.7\ldots(2n+1)}\right)$$

$$S_n = \frac{1}{2}\left[1 - \frac{1}{1.3.5.7\ldots(2n+1)}\right]$$

Problem 158

158. Find $\frac{2}{1.3}\cdot\frac{1}{3}+\frac{3}{3.5}\cdot\frac{1}{3^2}+\frac{4}{5.7}\cdot\frac{1}{3^3}+...$ to n terms

Solution of Problem 158

Solution:

$$t_n = \frac{n+1}{(2n-1)(2n+1)} \cdot \frac{1}{3^n}$$

$$= \frac{1}{4}\left[\frac{3}{2n-1} - \frac{1}{2n+1}\right] \cdot \frac{1}{3^n} = \frac{1}{4}\left[\frac{1}{2n-1} \cdot \frac{1}{3^{n-1}} - \frac{1}{2n+1} \cdot \frac{1}{3^n}\right]$$

$$\therefore t_1 = \frac{1}{4}\left(\frac{1}{1.1} - \frac{1}{3} \cdot \frac{1}{3}\right)$$

$$t_2 = \frac{1}{4}\left(\frac{1}{3.3} - \frac{1}{5} \cdot \frac{1}{3^2}\right)$$

$$t_3 = \frac{1}{4}\left(\frac{1}{5} \cdot \frac{1}{3^2} - \frac{1}{7} \cdot \frac{1}{3^3}\right)$$

$$\cdots$$

$$t_n = \frac{1}{4}\left(\frac{1}{2n-1} \cdot \frac{1}{3^{n-1}} - \frac{1}{2n+1} \cdot \frac{1}{3^n}\right)$$

$$S_n = \frac{1}{4}\left[1 - \frac{1}{2n+1} \cdot \frac{1}{3^n}\right]$$

Problem 159

159. Find the sum of n terms of the series $\frac{1}{3} + \frac{3}{3.7} + \frac{5}{3.7.11} + \frac{7}{3.7.11.15} + \ldots$

Solution of Problem 159

Solution:

$$t_n = \frac{2n-1}{3.5.7.11\ldots(4n-1)}$$

$$= \frac{1}{2}\left[\frac{1}{3.5.7\ldots(4n-5) - \frac{1}{3.5.7\ldots(4n+1)}}\right]$$

$$t_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{3.7}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{3.7} - \frac{1}{3.7.11}\right)$$

$$\ldots$$

$$t_n = \frac{1}{2}\left(\frac{1}{3.7.11\ldots(4n-5) - \frac{1}{3.7.11\ldots(4n-1)}}\right)$$

$$t_1 + t_2 + \ldots t_n = \frac{1}{3} + \frac{1}{2}\left[\frac{1}{3} - \frac{1}{3.7.11\ldots(4n-1)}\right]$$

$$S_n = \frac{1}{2} - \frac{1}{2}.\frac{1}{3.7.11\ldots(4n-1)}$$

Problem 160

160. Find the sum of the series: $1 + 2(1-a) + 3(1-a)(1-2a) + 4(1-a)(1-2a)(1-3a) + \ldots$ to m terms

Solution of Problem 160

Solution:

$$t_n = n(1-a)(1-2a)\ldots[a-(n-1)a]$$

$$t_n = -\frac{1}{a}(1-na-1)(1-a)(1-2a)\ldots[a-(n-1)a]$$

$$= -\frac{1}{a}[(1-a)(1-2a)\ldots(1-na)-(1-a)(1-2a)\ldots\{a+(n-1)a\}]$$

$$\therefore t_1 = -\frac{1}{a}[(1-a)-1]$$

$$t_2 = -\frac{1}{qa}[(1-a)(1-2a)-(1-a)]$$

$$\ldots$$

$$= -\frac{1}{a}[(1-a)(1-2a)\ldots(1-na)-(1-a)(1-2a)\ldots\{a+(n-1)a\}]$$

Adding, we get

$$S_n = \frac{1}{a}[1-(1-a)(1-2a)\ldots(1-na)]$$