

Complex Numbers Problems

21-30

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July 26, 2022

Problem 21

21. Find the number of complex numebers satisfying $z^3 + \bar{z} = 0$

Solution of Problem 21

Solution: Given, $z^3 = -\bar{z} \Rightarrow |z|^3 = |z|$

$$\Rightarrow |z|(|z| - 1)(|z| + 1) = 0 \Rightarrow |z| = 0, |z| = 1 [\because |z| + 1 > 0]$$

If $|z| = 0$, then $z = 0$. If $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1$

$z^3 + \frac{1}{z} = 0 \Rightarrow z^4 + 1 = 0$, which has four distinct roots. Thus, given equation has five roots.

Problem 22

22. Find the number of real roots of the equation $z^3 + iz - 1 = 0$.

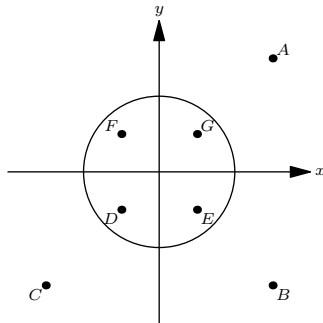
Solution of Problem 22

Solution: Since we have to find real roots, let $z = x$, a real value. The given equation becomes

$x^3 + ix - 1 = 0 \Rightarrow x^3 = 1, x = 0$ which is not possible. So there are no real solutions.

Problem 23

23. In the following diagram if given circle is unit circle then find the reciprocal of point A .



Solution of Problem 23

Solution: Let $z = x + iy$, then $\sqrt{x^2 + y^2} > 1$, because point A is outside circle.

$$\frac{1}{z} = \frac{x-iy}{\sqrt{x^2+y^2}} \text{ so } \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} < 1$$

This leads to the fact that point E is reciprocal of point A .

Problem 24

24. If $z = (3 + 7i)(p + iq)$, where $p, q \in I$, is purely imaginary, then find the minimum value of $|z|^2$.

Solution of Problem 24

Solution: $z = (3p - 7q) + i(3q + 7p)$, which is purely imaginary, $\Rightarrow 3p - 7q = 0$

$$\Rightarrow \frac{p}{q} = \frac{7}{3} \Rightarrow \frac{p}{q} + i = \frac{7}{3} + i \Rightarrow \frac{p+iq}{q} = \frac{7+3i}{3}$$

$$\Rightarrow p + iq = 7 + 3i \Rightarrow z = 21 + 9i + 49i - 21 = 58i \Rightarrow |z|^2 = 3364.$$

Problem 25

25. If $\alpha = \left(\frac{a-ib}{a+ib}\right)^2 + \left(\frac{a+ib}{a-ib}\right)^2$, $\forall a, b \in \mathbb{R}$ then prove that α is purely real.

Solution of Problem 25

Solution: Given, $\alpha = \left(\frac{a-ib}{a+ib}\right)^2 + \left(\frac{a+ib}{a-ib}\right)^2 = \frac{(a-ib)^4 + (a+ib)^4}{(a-ib)^2(a+ib)^2}$

$$= \frac{a^4 - 4a^3ib + 6a^2i^2b^2 - 4ai^3b^3 + b^4 + a^4 + 4a^3ib + 6a^2i^2b^2 + 4ai^3b^3 + b^4}{(a^2+b^2)^2}$$
$$= \frac{2a^4 - 12a^2b^2 + 2b^4}{(a^2+b^2)^2} \text{ which is purely real.}$$

Problem 26

26. If $\beta = \frac{z-1}{z+1}$ such that $|z| = 1$, then prove that β is purely imaginary.

Solution of Problem 26

Solution: Let $z = x + iy$ then given $|z| = 1 \Rightarrow x^2 + y^2 = 1$

$$\begin{aligned}\beta &= \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy} \\ &= \frac{x^2-1+y^2+iy(x+1-x+1)}{(x+1)^2+y^2} = \frac{2iy}{(x+1)^2+y^2} \text{ which is purely imaginary.}\end{aligned}$$

Problem 27

27. If $|z - 3i| = 3$ such that its argument $\arg(z) \in (0, \frac{\pi}{2})$, then find the value of $\cot(\arg(z)) - \frac{6}{z}$.

Solution of Problem 27

Solution: Let $z = x + iy \Rightarrow x^2 + (y - 3)^2 = 9 \Rightarrow x = 3 \cos \theta, y = 3 \sin \theta + 3$

$$z = 3[\cos \theta + i(\sin \theta + 1)] = 3\left[\sin\left(\frac{\pi}{2} - \theta\right) + i(1 + \cos\left(\frac{\pi}{2} - \theta\right))\right]$$

$$= 3\left[2 \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i2 \cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$= 6 \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right] = 6 \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) e^{i(\frac{\pi}{4} + \frac{\theta}{2})}$$

$$\cot(\arg(z)) = \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\frac{6}{z} = \sec\left(\frac{\pi}{4} - \frac{\theta}{2}\right) e^{-i(\frac{\pi}{4} + \frac{\theta}{2})} = \sec\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - i \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$= \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - i \Rightarrow \cot(\arg(z)) - \frac{6}{z} = i$$

Problem 28

28. Find the polar form of the complex number $\frac{-16}{1+i\sqrt{3}}$

Solution of Problem 28

Solution: Let $z = r(\cos \theta + i \sin \theta) = \frac{-16}{1+\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{-16(1-i\sqrt{3})}{1+3}$

$= -4 + i4\sqrt{3}$ then $r \cos \theta = 4, r \sin \theta = 4\sqrt{3} \Rightarrow r^2 = 64 \Rightarrow r = 4, \cos \theta = \frac{-1}{2}, \sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$

$\Rightarrow z = 8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$

Problem 29

29. Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$ then prove that $z = -\overline{w}$.

Solution of Problem 29

Solution: Let $z = r(\cos \theta + i \sin \theta)$ then because $\arg(z) + \arg(w) = \pi \Rightarrow \arg(w) = \pi - \theta$

$$\Rightarrow w = r(-\cos \theta + i \sin \theta) = -r(\cos \theta - i \sin \theta) \therefore r = -\overline{w}$$

Problem 30

30. If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ then prove that $(x^2 + y^2) = \frac{a^2+b^2}{c^2+d^2}$

Solution of Problem 30

Solution: $x - iy = \sqrt{\frac{a-ib}{c-id}} \Rightarrow x^2 - y^2 - 2ixy = \frac{a-ib}{c-id} = \frac{(a-ib)(c+id)}{c^2+d^2}$

$$x^2 - y^2 - 2ixy = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

Comparing real and imaginary parts, we get $x^2 - y^2 = \frac{ac+bd}{c^2+d^2}$, $2xy = \frac{bc-ad}{c^2+d^2}$

$$\begin{aligned}(x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 = \frac{(ac+bd)^2 + (bc-ad)^2}{(c^2+d^2)^2} = \frac{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2}{(c^2+d^2)^2} \\ &= \frac{a^2+b^2}{c^2+d^2}\end{aligned}$$