Complex Numbers Problems 21-30

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Problem 21

21. Find the number of complex numbers satisfying $z^3 + \overline{z} = 0$

Solution of Problem 21

Solution: Given, $z^3 = -\overline{z} \Rightarrow |z|^3 = |z|$

$\Rightarrow |z|(|z|-1)(|z|+1) = 0 \Rightarrow |z| = 0, |z| = 1[\because |z|+1 > 0]$

If $|z| = 0$, then $z = 0$. If $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\overline{z} = 1$

$z^3 + \frac{1}{z} = 0 \Rightarrow z^4 + 1 = 0$, which has four distinct roots. Thus, given equation has five roots.

Problem 22

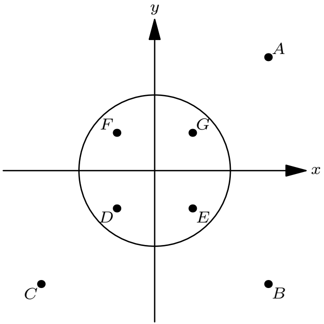
22. Find the number of real roots of the equation $z^3 + iz - 1 = 0$.

Solution of Problem 22

Solution: Since we have to find real roots, let $z = x$, a real value. The given equation becomes $x^3 + ix - 1 = 0 \Rightarrow x^3 = 1, x = 0$ which is not possible. So there are no real solutions.

Problem 23

23. In the following diagram, if given circle is unit circle then find the reciprocal of point A.



Solution of Problem 23

Solution: Let $z = x + iy$, then $\sqrt{x^2 + y^2} > 1$, because point A is outside circle.

$\frac{1}{z} = \frac{x - iy}{\sqrt{x^2+y^2}}$ so $\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{x^2+y^2} < 1$

This leads to the fact that point E is reciprocal of point A.

Problem 24

24. If $z = (3 + 7i)(p + iq)$, where $p, q \in I$, is purely imaginary, then find the minimum value of $|z|^2$.

Solution of Problem 24

Solution: $z = (3p - 7q) + i(3q + 7p)$, which is purely imaginary, $\Rightarrow 3p - 7q = 0$

$\Rightarrow \frac{p}{q} = \frac{7}{3} \Rightarrow \frac{p}{q} + i = \frac{7}{3} + i \Rightarrow \frac{p+iq}{q} = \frac{7+3i}{3}$

$\Rightarrow p + iq = 7 + 3i \Rightarrow z = 21 + 9i + 49i - 21 = 58i \Rightarrow |z|^2 = 3364.$

Problem 25

25. If $\alpha = \left(\frac{a-ib}{a+ib}\right)^2 + \left(\frac{a+ib}{a-ib}\right)^2$, $\forall\ a, b \in R$ then prove that α is purely real.

Solution of Problem 25

Solution: Given, $\alpha = \left(\frac{a-ib}{a+ib}\right)^2 + \left(\frac{a+ib}{a-ib}\right)^2 = \frac{(a-ib)^4+(a+ib)^4}{(a-ib)^2(a+(ib)^2)}$

$= \frac{a^4-4a^3.ib+6a^2i^2b^2-4ai^3b^3+b^4+a^4+4a^3ib+6a^2i^2b^2+4ai^3b^3+b^4}{(a^2+b^2)^2}$

$= \frac{2a^4-12a^2b^2+2b^4}{(a^2+b^2)^2}$ which is purely real.

Problem 26

26. If $\beta = \frac{z-1}{z+1}$ such that $|z| = 1$, then prove that β is purely imaginary.

Solution of Problme 26

Solution: Let $z = x + iy$ then given $|z| = 1 \Rightarrow x^2 + y^2 = 1$

$\beta = \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{x^2-1+y^2+iy(x+1-x+1)}{(x+1)^2+y^2} = \frac{2iy}{(x+1)^2+y^2}$ which is purely imaginary.

Problem 27

27. If $|z-3i|=3$ such that its argument $arg(z) \in (0, \frac{\pi}{2})$, then find the value of $\cot(arg(z)) - \frac{6}{z}$.

Solution of Problem 27

Solution: Let $z = x + iy \Rightarrow x^2 + (y-3)^2 = 9 \Rightarrow x = 3\cos\theta, y = 3\sin\theta + 3$

$z = 3[\cos\theta + i(\sin\theta + 1)] = 3\left[\sin\left(\frac{\pi}{2} - \theta\right) + i\left(1 + \cos\left(\frac{\pi}{2} - \theta\right)\right)\right]$

$= 3\left[2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$

$= 6\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right] = 6\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)e^{i\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}$

$\cot\left(arg(z)\right) = \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

$\frac{6}{z} = \sec\left(\frac{\pi}{4} - \frac{\theta}{2}\right)e^{-i\left(\frac{\pi}{4} + \frac{\theta}{2}\right)} = \sec\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\left[\sin\left(\frac{\pi}{4} - \frac{\theta}{2} - i\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)\right]$

$= \tan\left(\frac{\pi}{4} - \frac{A}{2}\right) - i \Rightarrow \cot(arg(z)) - \frac{6}{z} = i$

Problem 28

28. Find the polar form of the complex number $\frac{-16}{1+i\sqrt{3}}$

Solution of Problem 28

Solution: Let $z = r(\cos\theta + i\sin\theta) = \frac{-16}{1+\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{-16(1-i\sqrt{3})}{1+3}$

$= -4 + i4\sqrt{3}$ then $r\cos\theta = 4, r\sin\theta = 4\sqrt{3} \Rightarrow r^2 = 64 \Rightarrow r = 4, \cos\theta = \frac{-1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$

$\Rightarrow z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Problem 29

29. Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $arg(z) + arg(w) = \pi$ then prove that $z = -\overline{w}$.

Solution of Problem 29

Solution: Let $z = r(\cos\theta + i\sin\theta)$ then because $arg(z) + arg(w) = \pi \Rightarrow arg(w) = \pi - \theta$

$\Rightarrow w = r(-\cos\theta + i\sin\theta) = -r(\cos\theta - i\sin\theta) \therefore r = -\overline{w}$

Problem 30

30. If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ then prove that $(x^2 + y^2) = \frac{a^2+b^2}{c^2+d^2}$

Solution of Problem 30

Solution: $x - iy = \sqrt{\frac{a-ib}{c-id}} \Rightarrow x^2 - y^2 - 2ixy = \frac{a-ib}{c-id} = \frac{(a-ib)(c+id)}{c^2+d^2}$

$x^2 - y^2 - 2ixy = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$

Comparing real and imaginary parts, we get $x^2 - y^2 = \frac{ac+bd}{c^2+d^2}, 2xy = \frac{bc-ad}{c^2+d^2}$

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = \frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)} = \frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2}$

$= \frac{a^2+b^2}{c^2+d^2}$