Complex Numbers Problems 121-130

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Problem 121

121. If z_1 and z_2 are complex numbers such that $a|z_1| = b|z_2|, a, b \in R$, then prove that $\frac{az_1}{bz_2} + \frac{bz_2}{az_1}$ lies on the segment $[-2, 2]$ of the real axis.

Solution of Problem 121

Solution: Let $\arg(z_1) = \theta, \arg(z_2) = \theta + \alpha$

$\Rightarrow \frac{az_1}{bz_2} = \frac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} = e^{-i\alpha}$

$\Rightarrow \frac{bz_2}{az_1} = e^{i\alpha}$

$\Rightarrow \frac{az_1}{bz_2} + \frac{bz_2}{az_1} = e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$ Thus, it will lie on the line segment $[-2, 2]$ of the real axis.

Problem 122

122. If z_1, z_2, z_3 are roots of the equation $z^3 + 3\alpha z^2 + 3\beta z + \gamma = 0$, such that they form an equilateral triangle then prove that $\alpha^2 = \beta$.

Solution of Problem 122

Solution: Since z_1, z_2, z_3 are roots of the equation $z^3 + 3\alpha z^2 + 3\beta z + \gamma = 0$

$\Rightarrow z_1 + z_2 + z_3 = -3\alpha, z_1 z_2 + z_2 z_3 + z_3 z_1 = 3\beta, z_1 z_2 z_3 = \gamma$

We know that for a triangle to be equilateral $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

$\Rightarrow (z_1 + z_2 + z_3)^2 = 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$

$\Rightarrow 9\alpha^2 = 3.3\beta \Rightarrow \alpha^2 = \beta$

Problem 123

123. If $z_1^2 + z_2^2 + 2z_1z_2\cos\theta = 0$, then prove that z_1, z_1 and the origin form an isosceles triangle.

Solution of Problem 123

Solution: Given, $z_1^2 + z_2^2 + 2z_1z_2\cos\theta = 0$

Dividing both sides with z_2^2, we get $\left(\frac{z_1}{z_2}\right)^2 + 1 + 2\frac{z_1}{z_2}\cos\theta = 0$

The above equation is a quadratic equation in $\frac{z_1}{z_2}$, $\therefore \frac{z_1}{z_2} = \frac{-2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2}$

$\Rightarrow \frac{z_1}{z_2} = -\cos\theta \pm i\sin\theta \Rightarrow \left|\frac{z_1}{z_2}\right| = 1$

$\Rightarrow |z_1| = |z_2| \Rightarrow |z_1 - 0| = |z_2 - 0|$

Thus, z_1, z_1 and the origin form an isosceles triangle.

Problem 124

124. A, B and C represent z_1, z_2 and z_3 on argnad plane. The circumcenter of this triangle lies on the origin. If the altitude AD meets circumcircle again at P, then find the complex number representing P.

Solution of Problem 124

Solution: Since origin is circumcenter $\Rightarrow |z_1| = |z_2| = |z_3| = |z|$

$\Rightarrow z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} = z\overline{z}$

$\because\ AP \perp BC \therefore\ \frac{z-z_1}{\overline{z}-\overline{z_1}} + \frac{z_2-z_3}{\overline{z_2}-\overline{z_3}} = 0$

$\Rightarrow \frac{z-z_1}{\frac{z\overline{z_1}}{z}-\overline{z_1}} + \frac{z_2-z_3}{\frac{z_3\overline{z_3}}{z}-\overline{z_3}} = 0$

$\Rightarrow \frac{z(z-z_1)}{z_1\overline{z_1}-z\overline{z_1}} + \frac{z_2(z_2-z_3)}{z_3\overline{z_3}-z_2\overline{z_3}} = 0$

$\Rightarrow \frac{-z(z_1-z)}{\overline{z_1}(z_1-z)} - \frac{z_2(z_3-z_2)}{\overline{z_3}(z_3-z_2)} = 0$

$\Rightarrow \frac{-z}{z_1} - \frac{z_2}{z_3} = 0 \Rightarrow z = -\frac{z_1 z_2}{z_3}$

Problem 125

125. If z_1 and z_2 are the roots of the equation $z^2 + pz + q = 0$, where p, q can be complex numbers. Let A, B represent z_1, z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin then find p^2.

Solution of Problem 125

Solution: Given $OA = OB, \Rightarrow |z_1| = |z_2| = l$ (let).

Also given, $\arg(z_1) = \alpha + \arg(z_2) \Rightarrow z_1 = le^{i(\alpha+\arg(z_2))} = le^{i\arg(z_2)}.e^{i\alpha} = z_2 e^{i\alpha}$

Now, $z_1 z_2 = q \Rightarrow z_2^2 e^{i\alpha} = q$ and $z_1 + z_2 = -p \Rightarrow z_2(1 + e^{i\alpha}) = -p$

$\Rightarrow 2z_2 \cos\frac{\alpha}{2}.e^{i\alpha/2} = -p \Rightarrow p^2 = 4z_2^2\cos^2\frac{\alpha}{2}.e^{i\alpha}$

$\Rightarrow p^2 = 4q\cos^2\frac{\alpha}{2}$

Problem 126

126. If $Re\left(\frac{z+4}{2x-1}\right) = \frac{1}{2}$ then prove that locus of z is a straight line.

Solution of Problem 126

Solution: Let $z + iy$, then $Re\left(\frac{z+4}{2x-i}\right) = Re\left(\frac{x+4+iy}{2x+i(2y-1)}\right)$

$\Rightarrow Re\left(\frac{[(x+4)+iy][(2x-i(2y-1))]}{4x^2+(2y-1)^2}\right) = \frac{1}{2}$

$\Rightarrow \frac{2x(x+4)+y(2y-1)}{4x^2+(2y-1)^2} = \frac{1}{2} \Rightarrow 16x + 2y - 1 = 0$

which is equation of a straight line.

Problem 127

127. If z_1, z_2 and z_3 are vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If z_1, z_2, z_3 are in clockwise sense then find z_2 and z_3.

Solution of Problem 127

Solution: Since the circle is inscribed in $|z| = 2$ so center is origin. Also, since z_1, z_2 and z_3 are in clockwise direction $z_2 = z_1 e^{-i120^\circ}, z_3 = z_2 e^{-i120^\circ}$

$\Rightarrow z_2 = (1 + \sqrt{3}i)[(\cos. -120^\circ + i. \sin -120^\circ)] = 1 - \sqrt{3}i$

$\Rightarrow z_3 = -2$

Problem 128

128. If $z_1 = \frac{a}{1-i}, z_2 = \frac{b}{2+i}, z_3 = a - bi$ for $a, b \in R$ and $z_1 - z_2 = 1$. Then find the centroid of the triangle formed by z_1, z_2 and z_3.

Solution of Problem 128

Solution: Given $z_1 = \frac{a}{1-i} \Rightarrow z_1 = \frac{a+ia}{2}, z_2 = \frac{b}{2+i} = \frac{2b-ib}{5}$

Also given, $z_1 - z_2 = 1 \Rightarrow 5a + i5a - 4b + i2b = 10$

Comparing real and imaginary parts, we get $5a - 4b = 10, 5a + 2b = 0 \Rightarrow a = \frac{2}{3}, b = -\frac{5}{3}$

Cnetroid is $\frac{z_1+z_2+z_3}{3} = \frac{1}{3}(1 + 7i)$

Pronlem 129

129. If $\lambda \in R$. If the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$ form three vertices of an equilateral triangle in the argand plane, then find λ.

Solution of Problem 129

Solution: From the quadratic equation we have $z_1 + z_2 = -1$ and $z_1 z_2 = \frac{\lambda}{2}$

Since $0, z_1, z_2$ form an equilateral triangle, $\Rightarrow z_1 z_2 + z_2.0 + z_1.0 = z_1^2 + z_2^2 + 0^2$

$\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2 \Rightarrow (-1)^2 = 3.\frac{\lambda}{2} \Rightarrow \lambda = \frac{2}{3}$

Problem 130

130. If a, b, c and u, v, w are complex numbers such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number then prove that the triangles are similar.

Solution of Problem 130

Solution: Let A, B, C represent a, b, c and U, V, W represent u, v, w.

$\Rightarrow AB = b - c, BC = c - b = (a - b)(1 - r), CA = a - c = r(a - b)$

$\Rightarrow UV = v - u, VW = w - v = (u - v)(1 - r), WU = u - w = r(u - v)$

$\Rightarrow \frac{AB}{UV} = \frac{BC}{VW} = \frac{CA}{WU}$

Thus, the triangles are similar.