Miscellaneous Problems on A.P., G.P. and H.P.
Problems 121-130

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Problem 121

121. The sum of first ten terms of an A.P. is equal to 155, and the sum of first two terms of a G.P. is 9. Find these progressions if the first term of the A.P. euqals the common ratio of the G.P. and the first term of G.P. equals the common difference of A.P.

Solution of Problem 121

Solution: Let a be the first term and d be the common difference of A.P. and thus d will be the first term and a be the common ratio of the G.P. Given,

$$155 = \frac{10}{2}[2a + (10-1)d] \Rightarrow 2a + 9d = 31$$

$$d + ad = 9$$

$$\Rightarrow a = \frac{25}{2}, 2 \Rightarrow d = \frac{2}{3}, 3$$

Thus, A.P. is $2, 5, 8, ...$ or $\frac{25}{2}, \frac{79}{6}, \frac{83}{6}, ...$ and the G.P. is $3, 6, 12, ...$ or $\frac{2}{3}, \frac{25}{3}, \frac{625}{6}, ...$

Problem 122

122. If a, b, c be in H.P., prove that $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$

Solution of Problem 122

Solution: Since a, b, c are in H.P. therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c} \Rightarrow \frac{3}{b} - \frac{2}{c} = \frac{1}{a} + \frac{1}{b} - \frac{1}{c}$ and $\frac{3}{b} - \frac{2}{a} = \frac{1}{b} + \frac{1}{c} - \frac{1}{a}$

$$\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right) = \left(\frac{3}{b} - \frac{2}{c}\right)\left(\frac{3}{b} - \frac{2}{a}\right)$$

$$= \frac{9ac - 6ab - 6bc + 4b^2}{acb^2} = \frac{4}{ac} + \frac{9}{b^2} - \frac{6b(a+c)}{acb^2}$$

$$= \frac{4}{ac} + \frac{9}{b^2} - \frac{6b}{acb^2} \cdot \frac{2}{b}$$

$$= \frac{4}{ac} - \frac{3}{b^2}$$

Problem 123

123. If a, b, c are positive real numbers which are in H.P. show that $\frac{a+b}{2a-b} + \frac{b+c}{2c-b} \geq 4$

Solution of Problem 123

Solution: Because a, b, c are in H.P. therefore $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

$$\frac{a+b}{2a-b} + \frac{b+c}{2c-b} = \frac{\frac{1}{b}+\frac{1}{a}}{\frac{2}{b}-\frac{1}{a}} + \frac{\frac{1}{b}+\frac{1}{c}}{\frac{2}{b}-\frac{1}{c}}$$

$$= \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} = \frac{c^2+a^2}{ac} + \frac{a+c}{b}$$

$$= \frac{c^2+a^2}{ac} + \frac{(a+c)^2}{2ac} = \frac{c^2+a^2}{ac} - 2 + \frac{(a+c)^2}{2ac} - 2 + 4$$

$$= \frac{(c-a)^2}{ac} + \frac{(a-c)^2}{2ac} + 4 \geq 4$$

Problem 124

124. If $(a+b)/(1-ab), b, (b+c)/(1-bc)$ are in A.P., then prove that a, b^{-1}, c are in H.P.

Solution of Problem 124

Solution:

$$b - \frac{a+b}{1-ab} = \frac{b+c}{1-bc} - b$$

$$\Rightarrow \frac{b - ab^2 - a - b}{1-ab} = \frac{b + c - b + b^2 c}{1-bc}$$

$$\Rightarrow \frac{-a(1+b^2)}{1-ab} = \frac{c(1+b^2)}{1-bc} \Rightarrow -a(1-bc) = c(1-ab)$$

$$\Rightarrow a + c = 2abc \Rightarrow 2b = \frac{a+c}{ac}$$

$\therefore a, b^{-1}, c$ are in H.P.

Problem 125

125. Suppose a, b, c are in A.P. and $|a|, |b|, |c| < 1$ if $x = 1 + a + a^2 + \dots \infty, y = 1 + b + b^2 + \dots \infty,$ $z = 1 + c + c^2 + \dots \infty$ then prove that x, y, z are in H.P.

Solution of Problem 125

Solution:

$$x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$$

$$a, b, c \text{ are in A.P.}$$

$$\Rightarrow 1-a, 1-b, 1-c \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c} \text{ are in H.P.}$$

$$\Rightarrow x, y, z \text{ are in H.P.}$$

Problem 126

126. If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and a, b, c are in G.P. prove that x, y, z are in A.P.

Solution of Problem 126

Solution: Let

$$a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$$

$$\Rightarrow a = k^x, b = k^y, c = k^z$$

$$\because a, b, c \text{ are in G.P.} \Rightarrow b^2 = ac \Rightarrow k^{2y} = k^{x+z} \Rightarrow 2y = x + z$$

$\therefore x, y, z$ are in A.P.

Problem 127

127. If a, b, c be in A.P., l, m, n be in H.P. and al, bm, cn be in G.P. with common ratio not equal to 1 and a, b, c, l, m, n are positive show that $a : b : c = \frac{1}{n} : \frac{1}{m} : \frac{1}{l}$

Solution of Problem 127

Solution:

$$2b = a + c, m = \frac{2ln}{l+n}, b^2m^2 = acln$$

$$\Rightarrow \left(\frac{a+c}{2} . \frac{2ln}{l+n}\right)^2 = acln$$

$$\Rightarrow \frac{ln}{(l+n)^2} = \frac{ac}{(a+c)^2}$$

$$\Rightarrow \frac{(a+c)^2}{ac} = \frac{(l+n)^2}{ln}$$

$$\Rightarrow \frac{a}{c} + \frac{c}{a} = \frac{l}{n} + \frac{n}{l}$$

$$\Rightarrow a : c = \frac{1}{n} : \frac{1}{l}$$

Now it can be proven that $a : b : c = \frac{1}{n} : \frac{1}{m} : \frac{1}{l}$

Problem 128

128: Find three numbers a, b, c between 2 and 18 such that their sum is 25, the numbers $2, a, b$ are consecutive terms of an A.P. and the numbers $b, c, 18$ are consecutive terms of a G.P.

Solution of Problem 128

Solution:

$$a + b + c = 25, 2a = 2 + b, c^2 = 18b$$

$$\Rightarrow b = 2(a-1), c = \sqrt{18b} = \sqrt{36(a-1)} = 6\sqrt{a-1}$$

$$\Rightarrow a + 2(a-1) + 6\sqrt{a-1} = 25 \Rightarrow a + 2\sqrt{a-1} = 9$$

$$\Rightarrow a = 17, 5 \text{ however if } a = 17 \text{ then } b = 32 > 18$$

$$\therefore a = 5, b = 8, c = 12$$

Problem 129

129. If a, b, c are in A.P. and a, mb, c are in G.P.; prove that a, m^2b, c are in H.P.

Solution of Problem 129

Solution:

$$2b = a + c, m^2b^2 = ac$$

$$m^2b = \frac{ac}{b} = \frac{2ac}{a+c}$$

$\Rightarrow a, m^2b, c$ are in H.P.

Problem 130

130. An A.P., a G.P. and an H.P. have the same first term a abd same second term b, show that $n+2$th terms will be in G.P. is $\frac{b^{2n+2}-a^{2n+2}}{ab(b^{2n}-a^{2n})} = \frac{n+1}{n}$

Solution of Problem 130

Solution: The common difference of A.P. $= b - a$, common ratio of G.P. is b/a and common difference for corresponding A.P. of H.P. is $(a-b)/ab$

$n+2$th term of A.P. $= a + (n+1)(b-a) = (n+1)b - na$

$n+2$th term of G.P. $= ar^{n+1} = \frac{b^{n+1}}{a^n}$

$n+2$th term of H.P. $= \frac{1}{\frac{1}{a}+\frac{(n+1)(a-b)}{ab}} = \frac{ab}{(n+1)a-nb}$

These will be in G.P. if

$$\frac{[(n+1)b-na]ab}{(n+1)a-nb} = \frac{b^{2n+2}}{a^{2n}}$$

$$(n+1)a^{2n+1}b^2 - na^{2n+2}b = (n+1)ab^{2n+2} - nb.b^{2n+2}$$

$$(n+1)ab^2[a^{2n}-b^{2n}] = nb[a^{2n+2}-b^{2n+2}]$$

$$\frac{b^{2n+2}-a^{2n+2}}{ab(b^{2n}-a^{2n})} = \frac{n+1}{n}$$