Complex Numbers

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June 24, 2022

Theory

A complex number comprises of two numbers: a real number and an imaginary number. An imaginary number is square root of a negative number, for example, $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$. These are called imaginary numbers because they do not exist in real life in the sense that like ordinary numbers they cannot be used for counting.

A real number like 1 can also be represented as a complex number having a 0 imaginary part. The value $\sqrt{-1}$ is denoted by the Greek letter ι, which stands for *iota*. Typically, we use either i or j to denote this.

Clearly we have following:

$$i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1, ...$$

If you examine carefully you will find that following holds true

$$i^{4m} = 1, i^{4m+1} = i, i^{4m+2} = -1 \text{ and } i^{4m+3} = -i \ \forall \ m \in P$$

P is the set of positive integers including zero.
Note: $1 = \sqrt{1} = \sqrt{-1 * -1} = i * i = -1$
However, the above result is wrong because for any two real numbers a and b the result $\sqrt{a} * \sqrt{b} = \sqrt{ab}$ holds good if and only if the two numbers are zero or positive. Thus $1 = \sqrt{-1 * -1}$ is wrong because power of $-$ is -1 which makes the set of equalities go wrong.

Definitions

A complex number is commonly written as $a + ib$ or $x + iy$. Here a, b, x and y are all real numbers. The complex number itself is denoted by z, like $z = x + iy$. Here x is called the *real* part and is also denote by $Re(z)$ and y is called the imaginary part and is also denoted by $Im(z)$.

A complex number is purely real if its imaginary part or y or $Im(z)$ is zero. Similarly, a complex number is purely imaginary if its real part or x or $Re(z)$ is zero. Clearly, as you can fathom that there can exist only one number which has both the parts as zero and certainly that is 0. That is, $0 = 0 + i0$.

The set of all complex number is typically denoted by C. Two complex numbers z_1 and z_2 are said to be true if there real parts are equal and imaginary parts are equal. That is if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then for $z_1 1$ to be equal to z_2, x_1 must be equal to x_2 and y_1 must be equal to y_2.

Simple Operations

1. **Addition:** $(a+ib)+(c+id)=(a+c)+i(b+d)$
2. **Subtraction:** $(a+ib)-(c+id)=(a-c)+i(b-d)$
3. **Multiplication:** $(a+ib)*(c+id)=ac+ibc+iad+bdi^2=(ac-bd)+i(bc+ad)$
4. **Division:** $\frac{a+ib}{c+id}=\frac{a+ib}{c+id}.\frac{c-id}{c-id}=\frac{ac+bd+i(bc+ad)}{c^2+d^2}$

Conjugate of a Complex Number

Let $z=x+iy$ be a complex number then its complex conjugate is a number with imaginary part made negative and it is written as $\overline{z}=x-iy$. $\overline{z}$ is the typical representation for a conjugate of a complex number z.

Properties of Conjugates

1. $z_1=z_2 \Leftrightarrow \overline{z_1}=\overline{z_2}$
   Clearly as we know for two complex numbers to be equal both parts must be equal so this is very easy to understand that if $x_1=x_2$ and $y_1=y_2$ then this bidirectional condition is always satisfied.
2. $\overline{(\overline{z})}=z$.
   $z=x+iy$, hence, $\overline{z}=x-iy$, hence $\overline{(\overline{z})}=x-(-iy)=x+iy=z$
3. $z+\overline{z}=2Re(x)$
   Clearly, $z+\overline{z}=x+iy+x-iy=2x=2Re(x)$
4. $z-\overline{z}=2iIm(x)$
   Clearly, $z-\overline{z}=x+iy-(x-iy)=2iy=2iIm(x)$

Conjugate contd.

5. $z + \overline{z} = 0 \Leftrightarrow z$ is purely imaginary.
   $z + \overline{z} = x + iy + x - iy = 2x = 0$ which means rela part is zero and hence z is purely imaginary.
6. $z = \overline{z} \Leftrightarrow z$ is purely real.
   $x + iy = x - iy \Rightarrow 2iy = 0$ and thus z is purely real.
7. $z\overline{z} = [x^2 + y^2]$
   Clearly, $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$
8. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_1}\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$
   $= (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{z_1} + \overline{z_2}$
9. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
   It can be proven like item 8.
10. $\overline{z_1 z_2} = \overline{z_1} - \overline{z_2}$
    It can be proven like item 8.
11. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$ You can rationalize the base by multiplying it from its conjugate and apply division formula given above to prove it.
12. If $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$. where $s_0, a_1, ..., a_n$ and z are complex numbers, then
    $\overline{P(z)} = \overline{a_0} + \overline{a_1}\,\overline{z} + \overline{a_2}(\overline{z})^2 + ... + \overline{a_n}\overline{(z)^n} = \overline{P}(\overline{z})$ where
    $\overline{P}(z) = \overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + ... + \overline{a_n} z^n$

Conjugate contd.

13. If $R(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomilas in z, and $Q(z) \neq 0$, then
$\overline{R(z)} = \frac{\overline{P}(\overline{z})}{\overline{Q}(\overline{z})}$
14. If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\overline{z} = \begin{vmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{vmatrix}$ where $a_i, b_i, c_i (i = 1, 2, 3)$ are complex numbers.

Modulus of a Complex Number

Modulus of a complex numbe z is denoted by $|z|$ and is equalt to the real number $\sqrt{x^2 + y^2}$. Note that $|z| \geq 0 \; \forall \; z \in C$

Properties of Modulus

1. $|z| = 0 \Leftrightarrow z = 0.$
$x^2 + y^2 = 0 \Leftrightarrow x = 0, y = 0 \Rightarrow z = 0$
2. $|z| = |\overline{z}| = |-z| = |-\overline{z}| = x^2 + y^2$
3. $-|z| \leq Re(x) \leq |z|$ Clearly, $-(x^2 + y^2) \leq x^2 \leq (x^2 + y^2)$
4. $-|z| \leq Im(x) \leq |z|$ Clearly, $-(x^2 + y^2) \leq y^2 \leq (x^2 + y^2)$
5. $z\overline{z} = |z|^2$ Clearly, $(x + iy)(x - iy) = (x^2 + y^2) = |z|^2$
6. $|z_1 z_2| = |z_1||z_2|$ Clearly, $|z_1 z_2| = |x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1))|$
$= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = |z_1||z_2|$

Modulus contd.

13. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{z_2}$, if $z_2 \neq 0$
14. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \overline{z_1}z_2 + z_1\overline{z_2} = |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2})$
15. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \overline{z_1}z_2 - z_1\overline{z_2} = |z_1|^2 + |z_2|^2 - 2Re(z_1\overline{z_2})$
16. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
17. If a amd b are real numbers and z_1 and z_2 are complex numbers, then $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$
18. If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.
19. If z_1 and z_2 are complex numbers then $|z_1 + z_2| \leq |z_2| + |z_2|$. This expression can be generalized to n terms as well.
20. Similalrly, these can be proven that $|z_1 - z_2| \leq |z_1| + |z_2|, |z_1| - |z_2| \leq |z_1| + |z_2|$ and $|z_1 - z_2| \geq ||z_1| - |z_2||$