

# Geometric Progression Problems 61-70

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## Important Result

- $a^n - b^n$  is divisible by  $a - b$  for any  $n \in \mathbb{N}$

**Proof:**

$$\begin{aligned}\frac{a^n - b^n}{a - b} &= \frac{a^n \left(1 - \frac{b^n}{a^n}\right)}{a \left(1 - \frac{b}{a}\right)} \\&= a^{n-1} \left(1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + \dots + \frac{b^{n-1}}{a^{n-1}}\right) \\&= a^{n-1} + ba^{n-2} + b^2a^{n-3} + \dots + b^{n-1}\end{aligned}$$

- $a^n + b^n$  is divisible by  $a + b$  where  $n$  is any odd positive natural number.

**Proof:**

$$\begin{aligned}\frac{a^n + b^n}{a + b} &= \frac{a^n \left(1 - \left(-\frac{b}{a}\right)^n\right)}{1 - \left(-\frac{b}{a}\right)} \\&= a^{n-1} \left(1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3} + \dots + (-1)^n \frac{b^{n-1}}{a^{n-1}}\right) \\&= a^{n-1} - ba^{n-2} + b^2a^{n-3} - b^3a^{n-4} + \dots + (-1)^n b^{n-1}\end{aligned}$$

## Problem 61

**61.** Express  $0.\dot{4}\dot{2}\dot{3}$  as a rational number.

## Solution of Problem 61

**Solution:**

$$\begin{aligned}0.4\dot{2}\dot{3} &= 0.423232323 \dots \text{ to } \infty \\&= .4 + .023 + .00023 + \dots \text{ to } \infty \\&= \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} \dots \text{ to } \infty \\&= \frac{4}{10} + \frac{23}{1000} \left[ 1 + \frac{1}{100} + \frac{1}{10000} + \dots \text{ to } \infty \right] \\&= \frac{4}{10} + \frac{23}{1000} \frac{1}{1 - \frac{1}{100}} \\&= \frac{419}{990}\end{aligned}$$

## Problem 62

**62.** Find  $\frac{1}{5} + \frac{1}{7} + \frac{1}{5^2} + \frac{1}{7^2}$  to  $\infty$

## Solution of Problem 62

**Solution:** Required sum =  $\left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}\right)$  to  $\infty + \left(\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \dots\right)$  to  $\infty$

$$= \frac{\frac{1}{5}}{1 - \frac{1}{5}} + \frac{\frac{1}{7}}{1 - \frac{1}{7}}$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

## Problem 63

**63.** Prove that the sum of  $n$  terms of the series  $11 + 103 + 1005 + \dots$  is  $\frac{10}{9}(10^n - 1) + n^2$

## Solution of Problem 63

**Solution:** The series can be rewritten as

$$\begin{aligned} & (10 + 1) + (100 + 3) + (1000 + 5) + \dots \\ &= (10 + 100 + 1000 + \dots) + (1 + 3 + 5 + \dots) \\ &= \frac{10(10^n - 1)}{10 - 9} + \frac{n}{2}[2 \cdot 1 + (n - 1)2] \\ &= \frac{10}{9}(10^n - 1) + n^2 \end{aligned}$$



## Problem 64

**64.** Find the sum to  $n$  terms of the series  $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \dots$

## Solution of Problem 64

**Solution:** Given series on expansion gives

$$\left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right) + \left(x^6 + \frac{1}{x^6} + 2\right) + \dots$$

Rewriting the above series

$$\begin{aligned} & (x^2 + x^4 + x^6 + \dots) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \dots\right) + (2 + 2 + 2 + \dots) \\ &= \frac{x^2(x^{2n} - 1)}{x^2 - 1} + \frac{1}{x^2} \frac{1 - \frac{1}{x^{2n}}}{1 - \frac{1}{x^2}} + 2n \end{aligned}$$

## Problem 65

**65.** If  $S$  be the sum,  $P$  be the product and  $R$  the sum of reciprocals of  $n$  terms in G.P., prove that  $P^2 = \left(\frac{S}{R}\right)^n$

## Solution of Problem 65

**Solution:** Let  $a$  be the first term and  $r$  be the common ratio of G.P.

$$\text{Given, } S = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$\text{Also, } P = a \cdot ar \cdot ar^2 \cdot \dots \cdot ar^{n-1} = a^n r^{1+2+\dots+n-1} = a^n r^{\frac{n(n-1)}{2}}$$

$$\text{Also, } R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots + \frac{1}{ar^{n-1}}$$

$$= \frac{1}{a} \frac{1-\frac{1}{r^n}}{1-\frac{1}{r}} = \frac{1}{a} \frac{r^n-1}{r^n} \frac{r}{r-1}$$

$$= \frac{1-r^n}{1-r} \frac{1}{ar^{n-1}}$$

$$\frac{S}{R} = a^2 r^{n-1}$$

$$\left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = (a^n r^{\frac{n(n-1)}{2}})^2 = P^2$$

## Problem 66

**66.** Find  $1 + \frac{x}{1+x} + \frac{x^2}{(1+x)^2} + \dots$  to  $\infty$  if  $x > 0$

## Solution of Problem 66

**Solution:** Here terms of a given series are in G.P. where  $a = 1, r = \frac{x}{1+x}$  Also,  $|r| < 1$

$$S_{\infty} = \frac{a}{1-r} = \frac{1}{1-\frac{x}{1+x}} = 1+x$$

## Problem 67

**67.** Prove that in an infinite G.P. whose common ratio is  $r$  is numerically less than one, the ratio of any term to the sum of all the succeeding terms is  $\frac{1-r}{r}$ .

## Solution of Problem 67

**Solution:** The sum of all terms  $= S_{\infty}$  If we consider  $t_n$  in the ratio then sum of rest of terms will be  $S_{\infty} - S_n$ , thus, required ratio is

$$\begin{aligned}\frac{t_n}{S_{\infty} - S_n} &= \frac{ar^{n-1}}{\frac{a}{1-r} - \frac{a(1-r^n)}{1-r}} \\ &= \frac{ar^{n-1}}{\frac{a}{1-r}(1 - 1 + r^n)} = \frac{1-r}{r}\end{aligned}$$



## Problem 68

**68.** If  $S_1, S_2, S_3, \dots, S_p$  are the sum of infinite geometric series whose first terms are  $1, 2, 3, \dots, p$  and whose common ratios are  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{p+1}$  respectively, prove that  $S_1 + S_2 + S_3 + \dots + S_p = p(p+3)/2$