Complex Numbers Problems 131-140

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131. Find the intercept made by the circle $z\overline{z} + \overline{\alpha}z + \alpha\overline{z} + r = 0$ on real axis on the complex plane.

Solution: Let z_1 and z_2 be points on real axis which circle cuts with. Since these are on real axis and if z represents this points then $z = \overline{z}[\because z = x + i.0]$

Substituting $z = \overline{z}$ in the equation of the circle, we get $z^2 + (\overline{\alpha} + \alpha)z + r = 0$

Since z_1, z_2 are the roots $\therefore z_1 + z_2 = -\alpha, z_1 z_2 = r$

Length of intercept $= |z_1 - z_2| = \sqrt{(z_1 - z_2)^2} = \sqrt{(z_1 + z_2)^2 - 4z_1 z_2} = \sqrt{(\overline{\alpha} + \alpha)^2 - 4r}$

132. If $a = \cos\alpha + i\sin\alpha, b = \cos\beta + i\sin\beta, c = \cos\gamma + i\sin\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then find the value of $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)$.

Solution: Clearly, $a = e^{i\alpha}, b = e^{i\beta}, c = e^{i\gamma}$

Also given, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1 \Rightarrow e^{i(\alpha-\beta)} + e^{i(\beta-\gamma)} + e^{i(\gamma-\alpha)} = 1$

Comparing real parts, we get $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = 1$

133. Find the locus of the center of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally.

Solution: Let $A(z_1), B(z_2)$ be the centers of given circles and P be the center of the variable circle which touches given circles externally, then

$|AP| = a + r$ and $|BP| = b + r$ where r is the radius of the variable circle. Clearly,

$|AP| - |BP| = a - b \Rightarrow ||AP| - |BP|| = |a - b|$ =a constant.

Hence, locus of P is a right bisector if $a = b$, a hyperbola if $|a - b| < |AB|$ an empty set of $|a - b| > |AB|$, set of all points on line AB except those which lie between A and B if $|a - b| = |AB| \neq 0$.

134. Prove that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$.

Solution of Problem 134

Solution: Let $a + ib = re^{i\theta}, r^2 = a^2 + b^2 \Rightarrow a - ib = e^{-i\theta}, \tan\theta = \frac{b}{a}$

$\frac{a-ib}{a+ib} = e^{-2i\theta} \Rightarrow i\log\left(\frac{a-ib}{a+ib}\right) = i\log e^{-2i\theta} = 2\theta$

$\Rightarrow \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2b/a}{1-b^2/a^2} = \frac{2ab}{a^2-b^2}$.

135. $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $Re(z_1\overline{z_2}) = 0$. Also, $w_1 = a + ic, w_2 = b + id$ then prove that $|w_1| = |w_2| = 1$ and $Re(w_1\overline{w_2}) = 0$.

Solution: Given, $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1$

$Re(z_1 \overline{z_2}) = 0 \Rightarrow Re[(a + ib)(c - id)] = 0 \Rightarrow ac + bd = 0$

$a^2 + b^2 = c^2 + d^2 \Rightarrow (a + ic)^2 = (d - ib)^2 [\because ac == bd] \Rightarrow a + ic = d - ib \, or - d + ib$

$\Rightarrow a = d$ and $c = -b$ or $a = -d, c = b$

$\Rightarrow a^2 + c^2 = b^2 + d^2 = 1 \Rightarrow |w_1| = |w_2| = 1$

$Re(w_1 \overline{w2}) = Re[(a + ic)(b - id)] = ab + cd = 0$

136. If $\left|\frac{z_1}{z_2}\right| = 1$ and $\arg(z_1 z_2) = 0$, then prove that $|z_2|^2 = z_1 z_2$.

Solution: Let $z_1 = r(\cos\theta + i\sin\theta)$. Given, $\left|\frac{z_1}{z_2}\right| = 1$

$\Rightarrow |z_1| = |z_2| = r$. Also given, $\arg(z_1 z_2) = 0 \Rightarrow \arg(z_1) + \arg(z_2) = 0$

$\Rightarrow \arg(z_2) = -\theta \Rightarrow z_2 = r[\cos(-\theta) + i\sin(-\theta)] = r[\cos\theta - i\sin\theta] = \overline{z_1}$

$\Rightarrow \overline{z_2} = z_1 \Rightarrow |z_2|^2 = z_1 z_2$.

137. Find the value of the expression

$2\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right) + 3\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right) + 4\left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right) + ... + (n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right).$

Solution: $t_n = (n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$

$= n^3 + n^2\left(1 + \frac{1}{\omega} + \frac{1}{\omega^2}\right) + n\left(1 + \frac{1}{\omega} + \frac{1}{\omega^2}\right) + 1$

$= n^3 + n^2(1 + \omega + \omega^2) + n(1 + \omega + \omega^2) + 1 = n^3 + 1$

$\therefore S_n = \sum_{i=1}^{n} t_i = \sum_{i=1}(i^3 + 1) = \frac{n^2(n+1)^2}{4} + 1.$

138. If z_1 and z_2 are two complex numbers satisfying the equation $\left| \frac{z_1 + iz_2}{z_1 - iz_2} \right| = 1$, then prove that $\frac{z_1}{z_2}$ is purely real.

Solution: Given $|z_1 + iz_2| = |z_1 - iz_2|$

$\Rightarrow (z_1 + iz_2)(\overline{z_1} - i\overline{z_2}) = (z_1 - iz_2)(\overline{z_1 + i\overline{z_2}})$

$\Rightarrow \overline{z_1}z_2 = z_1\overline{z_2} \Rightarrow \frac{z_1}{z_2} = \frac{\overline{z_1}}{\overline{z_2}}$

Thus, $\frac{z_1}{z_2}$ is purely real.

139. If $z = -2 + 2\sqrt{3}i$, then find values of $z^{2n} + 2^{2n}z^n + 2^{4n}$.

Solution: $z = -2 + 2\sqrt{3}i = 4\omega$

$z^{2n} + 2^{2n}z^n + 2^{4n} = 4^{2n}[\omega^{2n} + \omega^n + 1]$

The above expression has value of 0 if n is not a multiple of 3 and 3.4^{2n} if n is multiple of 3.

Problem 140

140. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then find the values of $\frac{x}{y} + \frac{y}{x}, xy + \frac{1}{xy}$.

Solution: $x + \frac{1}{x} = 2\cos\theta, \Rightarrow x^2 - 2\cos\theta x + 1 = 0$

$\Rightarrow x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

Similarly, $y = e^{\pm i\phi}$

$\frac{x}{y} + \frac{y}{z} = 2\cos(\theta - \phi)$

and $xy + \frac{1}{xy} = 2\cos(\theta + \phi)$