Harmonic Progression
Problems 11-20

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September 18, 2021

Problem 11

11. The sum of three rational numbers in H.P. is 37 and the sum of their reciprocals is $\frac{1}{4}$, find the numbers.

Solution of Problem 11

Solution: Let the three numbers be $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

$$\text{Thus sum of reciprocals } = 3a = \frac{1}{4} \Rightarrow a = \frac{1}{12}$$

$$\text{Sum of three terms } = \frac{a(a+d)+(a-d)(a+d)+a(a-d)}{a(a^2-d^2)} = 37$$

$$\Rightarrow \frac{3a^2-d^2}{a(a^2-d^2)} = 37$$

Now we can substitute for a and find d, and thus we will have the required numbers.

Problem 12

12. If a, b, c are in H.P., prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$

Solution of Problem 12

Solution: Since a, b, c are in H.P.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$$

Substituting this in given equation

$$\frac{1}{b-c} + \frac{1}{b-a} = \frac{1}{\frac{2ac}{a+c} - a} + \frac{1}{\frac{2ac}{c+a} - c}$$

$$= \frac{ac}{a(c-a)} + \frac{a+c}{c(a-c)} = \frac{a+c}{a(c-a)} - \frac{a+c}{c(c-a)}$$

$$= \frac{ac + c^2 - a^2 - ac}{ac(c-a)} = \frac{a+c}{ac} = \frac{1}{c} + \frac{1}{a}$$

Problem 13

13. If a, b, c are in H.P., prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

Solution of Problem 13

Solution: We know that $b = \frac{2ac}{a+c}$, substituting for b

$$\frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{3ac+a^2}{a(c-a)} + \frac{3ac+c^2}{c(a+c)} = \frac{3c+a}{c-a} - \frac{3a+c}{c-a} = \frac{2c-2a}{c-a} = 2$$

Problem 14

14. If x_1, x_2, x_3, x_4, x_5 are in H.P., prove that $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = 4x_1x_5$

Solution of Problem 14

Solution: Since x_1, x_2, x_3, x_4, x_5 are in H.P. $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, \frac{1}{x_5}$ are in A.P. Let d be the common difference.

$$\frac{1}{x_2} - \frac{1}{x_1} = d \Rightarrow x_1x_2 = \frac{x_1 - x_2}{d}$$

$$\frac{1}{x_3} - \frac{1}{x_2} = d \Rightarrow x_2x_3 = \frac{x_2 - x_3}{d}$$

$$\frac{1}{x_4} - \frac{1}{x_3} = d \Rightarrow x_3x_4 = \frac{x_3 - x_4}{d}$$

$$\frac{1}{x_5} - \frac{1}{x_4} = d \Rightarrow x_4x_5 = \frac{x_4 - x_5}{d}$$

Adding all these

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 = \frac{x_1 - x_5}{d} = \frac{x_1 - x_5}{\frac{\frac{1}{x_5} - \frac{1}{x_1}}{4}} = 4x_1x_5$$

Problem 15

15. If x_1, x_2, x_3, x_4 are in H.P., prove that $(x_1 - x_3)(x_2 - x_4) = 4(x_1 - x_2)(x_3 - x_4)$

Solution of Problem 15

Solution: Let d be the common difference.

$$\frac{1}{x_3} - \frac{1}{x_1} = 2d \Rightarrow x_1 - x_3 = 2dx_1x_3$$

$$\frac{1}{x_4} - \frac{1}{x_2} = 2d \Rightarrow x_2 - x_4 = 2dx_2x_4$$

$$\frac{1}{x_2} - \frac{1}{x_2} = d \Rightarrow x_1 - x_2 = dx_1x_2$$

$$\frac{1}{x_4} - \frac{1}{x_3} = d \Rightarrow x_3 - x_4 = dx_3x_4$$

Clearly,

$$(x_1 - x_3)(x_2 - x_4) = 4(x_1 - x_2)(x_3 - x_4)$$

Problem 16

16. If $b+c, c+a, a+b$ are in H.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Solution of Problem 16

Solution: Since $b+c, c+a, a+b$ are in H.P.

$$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P.}$$

$$\Rightarrow \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b} \text{ are in A.P.}$$

$$\Rightarrow 1+\frac{a}{b+c}, 1+\frac{b}{c+a}, 1+\frac{c}{a+b} \text{ are in A.P.}$$

$$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in A.P.}$$

Problem 17

17. If $b + c, c + a, a + b$ are in H.P., prove that a^2, b^2, c^2 are in A.P.

Solution of Problem 17

Solution: Given, $b+c, c+a, a+b$ are in H.P.

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b}$$

$$\Rightarrow (b+a)(b-a) = (c+b)(c-b)$$

$$\Rightarrow b^2 - a^2 = c^2 - b^2$$

$$\Rightarrow a^2, b^2, c^2 \text{ are in A.P.}$$

Problem 18

18. If a, b, c are in A.P., prove that $\frac{bc}{ab+ac}, \frac{ca}{bc+ab}, \frac{ab}{ca+cb}$ are in H.P.

Solution of Problem 18

Solution: Given a, b, c are in A.P. Multiplying with $ab + bc + ca$

$$a(ab + bc + ca), b(ab + bc + ca), c(ab + bc + ca) \text{ are in A.P.}$$

Dividing by abc

$$\frac{ab + bc + ca}{bc}, \frac{ab + bc + ca}{ac}, \frac{ab + bc + ca}{ab} \text{ are in A.P.}$$

$$\frac{ab + ca}{bc}, \frac{bc + ab}{ca}, \frac{ca + cb}{ab} \text{ are in A.P.}$$

Problem 19

19. If a, b, c are in H.P., prove that $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

Solution of Problem 19

Solution: Given, a, b, c are in H.P.

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P.}$$

Multiplying with $a + b + c$

$$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \text{ are in A.P.}$$

Subtracting 2 from each term

$$\frac{b+c-a}{a}, \frac{a+c-b}{b}, \frac{a+b-c}{c} \text{ are in A.P.}$$

Problem 20

20. If a, b, c are in H.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

Solution of Problem 20

Solution: