Summation of Series
Problems 21-30

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Problem 21

21. Find the sum of the series $1^3 + 3^3 + 5^3 + ...$ to n terms.

Solution of Problem 21

Solution:

$$
\begin{aligned}
t_n &= (2n-1)^3 = 8n^3 - 12n^2 + 6n - 1 \\
S_n &= \sum t_n = 8\sum n^3 - 12\sum n^2 + 6\sum n - \sum 1 \\
&= 8\left(\frac{n(n+1)}{2}\right)^2 - 12\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} - n \\
&= 4n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n \\
&= n^2(2n^2-1)
\end{aligned}
$$

Problem 22

22. Find the sum of the series $1^2 + 4^2 + 7^2 + 10^2 + ...$ to n terms.

Solution of Problem 22

Solution:

$$\begin{aligned} t_n &= (3n-2)^2 = 9n^2 - 12n + 4 \\ S_n &= \sum t_n = 9\sum n^2 - 12\sum n + 4\sum 1 \\ &= 9.\frac{n(n+1)(2n+1)}{6} - 12.\frac{n(n+1)}{2} + 4n \\ &= \frac{1}{2}n(6n^2 - 3n - 1) \end{aligned}$$

Problem 23

23. Find the sum of the series $1^2 + 2 + 3^2 + 4 + 5^2 + 6 + ...$ to $2n$ terms.

Solution of Problem 23

Solution:

$$S_{2n} = 1^2 + 3^2 + 5^2 + \ldots \text{ to } n \text{ terms} + 2 + 4 + 6 + \ldots \text{ to } n \text{ terms}$$

$$t_{2n} = (2n-1)^2 + 2n = 4n^2 - 4n + 1 + 2n = 4n^2 - 2n + 1$$

$$S_{2n} = 4\sum n^2 - 2\sum n + \sum 1$$

$$= 4.\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n$$

$$= \frac{1}{3}n(4n^2 + 3n + 2)$$

Problem 24

24. Find the sum of the series $1^2 - 2^2 + 3^2 - 4^2 + ...$ to n terms.

Solution of Problem 24

Solution: When n is odd i.e. $2m + 1 \ \forall m \in N$

$$S_n = 1^2 + 3^2 + 5^2 + ... \text{ to } m+1 \text{ terms } - (2^2 + 4^2 + 6^2 + ... \text{ to } m \text{ terms})$$

$$= \sum_{i=1}^{m+1} (2i-1)^2 - \sum_{i=1}^{m} (2i)^2$$

$$= \sum_{i=1}^{m} 4i^2 + 4(m+1)^2 - 4\sum_{i=1}^{m+1} i + \sum_{i=1}^{m+1} 1 - \sum_{i=1}^{m} 4i^2$$

$$= 4(m+1)^2 - 4.\frac{(m+1)(m+2)}{2} + m + 1$$

$$= \frac{1}{2}n(n+1)$$

When n is even i.e. $2m \ \forall m \in N$

$$S_n = 1^2 + 3^2 + 5^2 + ... \text{ to } m \text{ terms } - (2^2 + 4^2 + 6^2 + ... \text{ to } m \text{ terms})$$

$$= \sum_{i=1}^{m} (2i-1)^2 - \sum_{i=1}^{m} (2i)^2$$

$$= -4\sum_{i=1}^{m} i + \sum_{i=1}^{m} 1$$

$$= -4.\frac{m(m+1)}{2} + m$$

$$= -\frac{1}{2}n(n+1)$$

Problem 25

25. Find the sum of the series $1.3 + 3.5 + 5.7 + ...$ to n terms.

Solution of Problem 25

Solution:

$$t_n = (2n-1)(2n+1) = 4n^2 - 1$$

$$S_n = 4\sum n^2 - \sum 1 = 4\frac{n(n+1)(2n+1)}{6} - n$$

$$= \frac{1}{3}n(4n^2 + 6n - 1)$$

Problem 26

26. Find the sum of the series $1.2 + 2.3 + 3.4 + ...$ to n terms.

Solution of Problem 26

Solution:

$$t_n = n(n+1)$$

$$S_n = \sum n^2 + \sum n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{1}{3}n(n+1)(n+2)$$

Problem 27

27. Find the sum of the series $1.2^2 + 2.3^2 + 3.4^2 + ...$ to n terms.

Solution of Problem 27

Solution:

$$t_n = n(n+1)^2 = n^3 + 2n^2 + n$$

$$S_n = \sum n^3 + 2\sum n^2 + \sum n$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+5)$$

Problem 28

28. Find the sum of the series $2.1^2 + 3.2^2 + 4.3^2 + ...$ to n terms.

Solution of Problem 28

Solution:

$$t_n = (n+1).n^2 = n^3 + n^2$$

$$S_n = \sum n^3 + \sum n^2 = \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+1)$$

Problem 29

29. Find the sum of the series $1 + (1 + 3) + (1 + 3 + 5) + ...$ to n terms.

Solution of Problem 29

Solution:

$$t_n = 1 + 3 + 5 + \ldots \text{ to } n \text{ terms}$$

$$= \frac{n}{2}[2.1 + (n-1)2] = \frac{n(2n)}{2} = n^2$$

$$S_n = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 30

30. Find the sum of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$ to n terms.

Solution of Problem 30

Solution:

$$t_n = 1^2 + 2^2 + 3^2 + \ldots \text{ to } n \text{ terms}$$
$$= \frac{n(n+1)(2n+1)}{6}$$
$$S_n = \frac{1}{6}[\sum 2n^3 + \sum 3n^2 + \sum n]$$
$$= \frac{1}{12}n(n+1)^2(n+2)$$