Geometric Progression
Problems 21-30

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Choosing numbers in G.P. when product is given

1. If three numbers in G.P. are given with a product then you should take the numbers as $\frac{a}{r}, a, ar$. If the product is not given then you should take them as a, ar, ar^2
2. If four nummbers in G.P. are given with a product then you should take the numbers as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$. If the product of numbers is not given then take them as a, ar, ar^2, ar^3.

Problem 21

21. If the product of three numbers in G.P. be 216 and their sum is 19, find the numbers.

Solution of problem 21

Solution: Let the three numbers be $\frac{a}{r}, a, ar$
Given that product it 216.

$$\therefore \frac{a}{r}.a.ar = 216 \Rightarrow a^3 = 216 \Rightarrow a = 6$$

Also, given that sum of these numbers is 19

$$\therefore \frac{a}{r} + a + ar = 19 \Rightarrow \frac{6}{r} + 6 + 6r = 19$$

$$6r^2 - 13r + 6 = 0 \Rightarrow 6r^2 - 9r - 4r + 6 = 0$$

$$3r(2r-3) - 2(2r-3) = 0 \Rightarrow (2r-3)(3r-2) = 0$$

$$r = \frac{2}{3}, \frac{3}{2}$$

When $r = \frac{2}{3}$, numbers are $9, 6, 4$ and when $r = \frac{3}{2}$, numbers are $4, 6, 9$

Problem 22

22. A number consists of three digits in G.P. The sum of the right hand and left hand digits exceed twice the middle digit by 1 and the sum of left hand and middle digit is two-third of the sum of the middle and right hand digits. Find the number.

Solution of problem 22

Solution: Let the three digits be a, ar, ar^2
Given,

$$a + ar^2 = 2ar + 1 \Rightarrow a(r-1)^2 = 1$$

Also given that,

$$a + ar = \frac{2}{3}(ar + ar^2) \Rightarrow 3a(1+r) = 2ar(1+r)$$

$$(1+r)(3-2r) = 0 \therefore r = \frac{3}{2}, -1$$

$$r = \frac{3}{2} \Rightarrow a = \frac{1}{(r-1)^2} = \frac{1}{(\frac{3}{2}-1)^2} = 4$$

r cannot be -1 as that will make $a = \frac{1}{4}$ which is not possible as digits of a number are integers.
Hence, $a = 4, ar = 4\frac{3}{2} = 6, ar^2 = 4\left(\frac{3}{2}\right)^2 = 9$
Thus, our number is 469

Problem 23

23. In a set of four numbers, the first three are in G.P. and the last three are in A.P. with a common difference of 6. If the first number is same as fourth, find the four numbers.

Solution of problem 23

Solution: Let the last three numbers in A.P. be $b, b+6, b+12$ and the first number be a.
Thus,

$$a = b + 12, b^2 = a(b+6) \Rightarrow b^2 = (b+12)(b+6)$$

$$18b + 72 = 0 \Rightarrow b = -4 \Rightarrow a = -4 + 12 = 8$$

Thus, numbers are $8, -4, 2, 8$

Problem 24

24. The sum of three numbers in G.P. is 21 and the sum of their squares is 189. Find the numbers.

Solution of problem 24

Solution: Let the three numbers be a, ar, ar^2 where a is the first term and r is the common ratio. Thus, we have

$$a + ar + ar^2 = 21 \Rightarrow a(1 + r + r^2) = 21$$

$$a^2 + a^2r^2 + a^2r^4 = 189 \Rightarrow a^2(1 + r^2 + r^4) = 189$$

Squaring first equation and dividing it by second, we get

$$\frac{a^2(1 + r + r^2)^2}{a^2(1 + r^2 + r^4)} = \frac{441}{189} = \frac{7}{3}$$

$$\frac{(1 + r + r^2)^2}{1 + 2r^2 + r^4 - r^2} = \frac{7}{3} \Rightarrow \frac{(1 + r + r^2)^2}{(1 + r^2)^2 - r^2} = \frac{7}{3}$$

$$\frac{1 + r + r^2}{1 - r + r^2} = \frac{7}{3} \Rightarrow 2r^2 - 5r + 2 = 0$$

$$\therefore r = 2, \frac{1}{2}$$

When $r = 2, a = 3$, when $r = \frac{1}{2}, a = 12$
Thus, numbers are either $3, 6, 12$ or $12, 6, 3$

Problem 25

25. The prodduct of three consecutive terms of a G.P. is -64 and the first term is four times the third. Find the terms.

Solution of problem 25

Solution: Let the numbers are $\frac{a}{r}, a, ar$, where a be the first term and r be the common ratio. Given,

$$\frac{a}{r}.a.ar = -64 \Rightarrow a^3 = -64 \Rightarrow a = -4$$

$$\frac{a}{r} = 4ar \Rightarrow \frac{1}{r} = 4r, \Rightarrow r^2 = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2}$$

When $r = \frac{1}{2}$ numbers are $-8, -4, -2$ and when $r = -\frac{1}{2}$ numbers are $8, -4, 2$

Problem 26

26. Three numbers whose sum is 15 are in A.P. If $1, 4, 19$ be added to them respectively the resulting numbers are in G.P. Find the numbers.

Solution of problem 26

Solution: Let the three numbers in A.P. are $a-d, a, a+d$ where a is the first term and d is the common difference. Given,

$$a-d+a+a+d=15 \Rightarrow 3a=15 \Rightarrow a=5$$

After adding $1, 4, 19$ the numbers are in G.P.,

$$\therefore (a+4)^2=(a-d+1)(a+d+19)$$

$$\Rightarrow a^2+8a+16=a^2-d^2+19+ad-ad+19a+a-19d+d$$

$$\Rightarrow 8a+16=20a-d^2-18d+19 \Rightarrow d^2+18d+12a+3=0$$

$$d^2+18d+63=0 \Rightarrow (d+21)(d-3)=0$$

$$\therefore d=-21, 3$$

When $d=3$, numbers are $2, 5, 8$ and when $d=-21$, numbers are $26, 5, -16$

Problem 27

27. From three numbers in G.P. other three numbers in G.P. are subtracted. Resulting numbers are found to be in G.P. again. Prove that the three sequences have the same common ratio.

Solution of problem 27

Solution: Let a, x be the first terms and r, y be the common ratios of the first two G.P. in question. Then we are given that,

$$(ar - xy)^2 = (a - x)(ar^2 - xy^2) \Rightarrow a^2r^2 + x^2y^2 - 2arxy = a^2r^2 + x^2y^2 - axr^2 - axy^2$$

$$\Rightarrow axr^2 + axy^2 - 2arxy = 0 \Rightarrow ax(r^2 + y^2 - 2ry) = 0$$

$$\because a, x \neq= 0, (r - y)^2 = 0 \Rightarrow r = y$$

Common ratio of the third G. P. $\frac{ar-xy}{a-x} = \frac{r(a-x)}{a-x} = r$

Problem 28

28. If a, b, c, d are in G.P., show that $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$

Solution of problem 28

Solution: Let r be the common ratio of the G.P., then $b = ar, c = ar^2, d = ar^3$

$$L.H.S. = (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2 = a^2[(r - r^2)^2 + (r^2 - 1)^2 + (r^3 - r)^2]$$

$$= a^2[r^2 + r^4 - 2r^3 + r^4 + 1 - 2r^2 + r^4 + r^2 - 2r^4]$$

$$= a^2[r^6 - 2r^3 + 1] = (ar^3 - a)^2 = (d - a)^2 = R.H.S$$

Problem 29

29. If a, b, c, d are in G. P., then show that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ad + bc + cd)^2$$

Solution of problem 29

Solution: Let r be the common ratio of the given G. P. Since a, b, c, d are in G. P.

$$\therefore b = ar, c = ar^2, d = ar^3$$

$$\begin{aligned} L.H.S. &= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) \\ &= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6) \\ &= a^2(1 + r^2 + r^4)a^2r^2(1 + r^2 + r^4) \\ &= [a^2r(1 + r^2 + r^4)]^2 = (a.ar + ar.ar^2 + ar^2ar^3)^2 \\ &= (ab + bc + cd)^2 = R.H.S. \end{aligned}$$

Problem 30.

30. If $a^x = b^y = c^2$ where x, y, z are in G.P., show that $\log_b a = \log_c b$.

Solution of problem 30

Solution: Given, $a^x = b^y = c^2 = k(\text{say})$
Taking logarithm, we get $x \log a = y \log b = z \log c = \log k$

$$e \therefore x = \frac{\log k}{\log a}, y = \frac{\log k}{\log b}, z = \frac{\log k}{\log c}$$

Also given that x, y, z are in G.P.

$$\therefore \frac{y}{x} = \frac{z}{y} \Rightarrow \frac{\log k \log a}{\log b \log k} = \frac{\log k \log b}{\log c \log k}$$

$$\Rightarrow \frac{\log b}{\log a} = \frac{\log c}{\log b} \Rightarrow \log_b a = \log_c b$$