Miscellaneous Problems on A.P., G.P. and H.P.
Problems 161-170

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Problem 161

161. Find the sum of the series $1+\frac{x}{b_1}+\frac{x(x+b_1)}{b_1 b_2}+\frac{x(x+b_1)(x+b_2)}{b_1 b_2 b_3}+...+\frac{x(x+b_1)...(x+b_{n-1})}{b_1 b_2 ... b_n}$

Solution of Problem 161

Solution: Let t_n denote the nth term of the series.

$$t_1 = 1$$

$$t_2 = \frac{x}{b_1} = \frac{(x+b_1)-b_1}{b_1} = \frac{x+b_1}{b_1} - 1$$

$$t_3 = \frac{x(x+b_1)}{b_1 b_2} = \frac{[(x+b_2)-b_2](x+b_1)}{b_1 b_2} = \frac{(x+b_1)(x+b_2)}{b_1 b_2} - \frac{x+b_1}{b_1}$$

$$\dots$$

$$t_{n+1} = \frac{(x+b_1)\dots(x+b_n)}{b_1 b_2 \dots b_n} - \frac{(x+b_1)\dots(x+b_{n-1})}{b_1 b_2 \dots b_{n-1}}$$

$$\therefore S_n = \frac{(x+b_1)\dots(x+b_n)}{b_1 b_2 \dots b_n}$$

Problem 162

162. Let $S_k(n) = 1^k + 2^k + ... + n^k$, show that $nS_k(n) = S_{k+1}(n) + S_k(n-1) + S_k(n-2) + ... + S_k(2) + S_k(1)$

Solution of Problem 162

Solution:

$$nS_k(n) = n[1^k + 2^k + ... + n^k]$$
$$= 1^k + (1^k + 2.2^k) + (1^k + 2^k + 3.3^k) + ... + (1^k + 2^k + ... + n.n^k)$$
$$= 1^{k+1} + [S_k(1) + 2^{k+1}] + [S_k(2) + 3^{k+1}] + ... + [S_k(n-1) + n^{k+1}]$$

$$= S_k(1) + S_k(2) + ... + S_k(n-1) + S_{k+1}(n)$$

Problem 163

163. Find the sum of all the numbers of the form n^3 which lie between 100 and 10000.

Solution of Problem 163

Solution: $n^3 > 100 \Rightarrow n > 4, n^3 < 100000 \Rightarrow n < 22$
So

$$S = 5^3 + 6^3 + ... + 21^3$$

$$S' = 1^3 + 2^3 + 3^3 + 4^3$$

$$S' + S - S' = 1^3 + 2^2 + ... + 21^3 - (1^3 + 2^3 + ... + 4^3)$$

$$= 53261$$

Problem 164

164. If S be the sum of the n consecutive integers beginning with a and t the sum of their squares, show that $nt - S^2$ is independent of a

Solution of Problem 164

Solution:

$$S = a + (a+1) + ... + (a+n-1)$$

$$= na + \frac{n(n-1)}{2}$$

$$S^2 = n^2a^2 + n^2(n-1)a + \frac{n^2(n-1)^2}{4}$$

$$t = a^2 + (a+1)^2 + ... + (a+n-1)^2$$

$$nt = na^2 + n^2(n-1)a + n\sum_{i=1}^{n-1} i^2$$

Clearly, $nt - S^2$ is independent of a.

Problem 165

165. If $\sum_{x=5}^{n+5} 4(x-3) = Pn^2 + Qn + R$, find the value of $P + Q$.

Solution of Problem 165

Solution:

$$\sum_{x=5}^{n+5} 4(x-3) = \sum_{x=1}^{n+5} 4(x-3) - \sum_{x=1}^{4} 4(x-3)$$

$$= \frac{4(n+5)(n+6)}{2} - 12(n+5) - \frac{4.4.5}{2} + 12.4$$

$$= 2n^2 + 10n + 8$$

$$\therefore P + Q = 12$$

Problem 166

166. Find the sum to $2n$ terms of the series $5^3 + 4.6^3 + 7^3 + 4.8^3 + 9^3 + 4.10^3 + ...$

Solution of Problem 166

Solution: Let S be the sum of series, then

$$S = 5^3 + 7^3 + 9^3 + \ldots \text{ to } n \text{ terms } + 2^5(3^3 + 4^3 + 5^3 + \ldots \text{ to } n \text{ terms })$$

$$= 1^3 + 3^3 + 5^3 + \ldots \text{ to } n+2 \text{ terms } - 1^3 - 3^3 + 2^5(1^3 + 3^3 + 5^3 + \ldots \text{ to } n+1 \text{ terms }) - 2^5$$

$$= \sum_{i=1}^{n+2}(2i-1)^3 - 28 + 2^5\sum_{i=1}^{n+1}(2i-1)^3 - 32$$

$$= n(10n^3 + 96n^2 + 243n + 540)$$

Problem 167

167. Find the sum to n terms of the series $\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\ldots$

Solution of Problem 167

Solution: Let S be the sum of the series and $x = \frac{2n+1}{2n-1}$, then

$$S = x + 3x^2 + 5x^3 + ...$$

$$xS = x^2 + 3x^3 + ... + (2n-1)x^{n+1}$$

$$(1-x)S = x + 2x^2 + 2x^3 + ... = x + 2x^2(1 + x + x^2 + ... \text{ to } n-1 \text{ terms}) - (2n-1)x^{n+1}$$

$$= x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$$

$$S = \frac{x}{1-x} + \frac{2x^2(1-x^{n-1})}{(1-x)^2} - \frac{(2n-1)x^{n+1}}{1-x}$$

$$= \frac{x^2 - x + 2x^{n+1} - 2x^2 + (x-1).(2n-1)x^{n+1}}{(x-1)^2}$$

$$= n(2n+1)$$

Problem 168

168. Find the sum to n terms of the series $1+5\left(\frac{4n+1}{4n-3}\right)+9\left(\frac{4n+1}{4n-3}\right)^2+13\left(\frac{4n+1}{4n-3}\right)^3+\ldots$

Solution of Problem 168

Solution: Let S be the sum to n terms and $x = \frac{4n+1}{4n-3}$, then

$$S = 1 + 5x + 9x^2 + 13x^3 + ...$$

$$xS = x + 5x^2 + 9x^3 + ... + (4n+1)x^n$$

$$(1-x)S = 1 + 4x + 4x^2 + 4x^3 + ... + 4x^{n-1} - (4n+1)x^n$$

$$S = \frac{1}{x-1} + \frac{4x(x^{n-1}-1)}{(x-1)^2} - \frac{(4n+1)x^n}{(x-1)}$$

$$= 4n^2 - 3n$$

Problem 169

169. Prove that the numbers of the sequence $121, 12321, 1234321, \ldots$ are each a perfect square of an odd integer.

Solution of Problem 169

Solution:

$$t_n = 1.10^{2n} + 2.10^{2n-1} + 3.10^{n-2} + ... + n.10^{n+1} + (n+1)10^n + n.10^n + (n-1)10^{n-2} + ... + 3.10^2 + 2.10 + 1$$

$$= 10^{2n}\left[1 + 2.\frac{1}{10} + 3.\frac{1}{10^2} + ... + n.\frac{1}{10^{n-1}}\right] + (1 + 2.10 + 3.10^2 + ... + n.10^{n-1} + (n+1)10^n)$$

$$= 10^{2n}S_1 + S_2$$

$$S_1 = 1 + 2.\frac{1}{10} + 3.\frac{1}{10^2} + ... + n.\frac{1}{10^{n-1}}$$

$$\frac{S_1}{10} = \frac{1}{10} + 2\frac{1}{10^2} + ... + (n-1)\frac{1}{10^{n-1}} + n.\frac{1}{10^n}$$

$$S_1 = \frac{100}{81}\left(1 - \frac{1}{10^n}\right) - \frac{90n}{81.10^n}$$

$$S_2 = 1 + 2.10 + 3.10^2 + ... + (n+1)10^n$$

$$10S_2 = 10 + 2.10^2 + ... + n.10^n + (n+1)10^{n+1}$$

$$S_2 = \frac{1 - 10^{n+1}}{81} + \frac{(n+1)10^{n+1}}{9}$$

Substituting S_1 and S_2 we obtain t_n as

$$t_n = \left(\frac{10^{n+1} - 1}{9}\right)^2$$

Thus, the numbers in the sequence will be square of odd positive integer.

Problem 170

170. Prove that the sum to n terms of the series $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + ...$ is $6n/(n+1)$

Solution of Problem 170

Solution:

$$t_n = \frac{2n+1}{1^2+2^2+...+n^2}$$

$$= \frac{2n+1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$$

$$\therefore t_1 = \frac{6}{1.2} = 6\left(1-\frac{1}{2}\right)$$

$$t_2 = \frac{6}{2.3} = 6\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$\dots$$

$$t_n = \frac{6}{n(n+1)} = 6.\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Adding, we get

$$S = \frac{6n}{n+1}$$