

Complex Numbers Problems

191-200

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Problem 191

191. If $z - 6 - 8i \leq 4$, then find the least and greatest value of z .

Solution of Problem 191

Solution: $|z - 6 - 8i| \leq |4| \Rightarrow -4 \leq ||z| - |6 + 8i|| \leq 4$

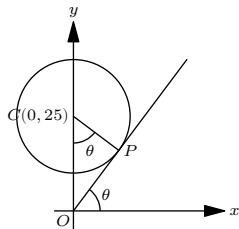
$$\Rightarrow -4 \leq |z| - 10 \leq 10 \Rightarrow 6 \leq |z| \leq 14$$

Problem 192

192. If $z - 25i \leq 15$ then find the least positive value of $\arg(z)$.

Solution of Problem 192

Solution: The diagram is given below:



Given $|z - 25i| \leq 15$, which represents a circle having center $(0, 25)$ and a radius 15.

Let OP be tangent to the circle at point P , then $\angle XOP$ will represent least value of $\arg(z)$.

Let $\angle XOP = \theta$ then $\angle OCP = \theta$. Now $OC = 25$, $CP = 15 \therefore OP = 20$

$$\therefore \tan \theta = \frac{OP}{CP} = \frac{4}{3}. \therefore \text{Least value of } \arg(z) = \theta = \tan^{-1} \frac{4}{3}$$

Problem 193

193. Show that the equation $|z - z_1|^2 + |z - z_2|^2 = k$ where $k \in \mathbb{R}$ will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$.

Solution of Problem 193

193. Given, $|z - z_1|^2 + |z - z_2|^2 = k$

$$\Rightarrow |z|^2 + |z_1|^2 - 2z\overline{z_1} + |z|^2 + |z_2|^2 - 2z\overline{z_2} = k$$

$$\Rightarrow 2|z|^2 - 2z(\overline{z_1} + \overline{z_2}) = k - (|z_1|^2 + |z_2|^2)$$

$$\Rightarrow |z|^2 - 2z\left(\frac{\overline{z_1} + \overline{z_2}}{2}\right) + \frac{1}{4}|z_1 + z_2|^2 = \frac{k}{2} + \frac{1}{4}[|z_1 + z_2|^2 - 2|z_1|^2 - 2|z_2|^2]$$

$$\Rightarrow \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}\left[k - \frac{1}{2}|z_1 - z_2|^2\right]$$

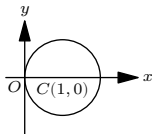
The above equation represents a circle with center at $\frac{z_1 + z_2}{2}$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{|z_1 - z_2|^2}{2}$.

Problem 194

194. If $|z - 1| = 1$, prove that $\frac{z-2}{z} = i \tan[\arg(z)]$.

Solution of Problem 194

Solution: Since $|z - 1| = 1$, z represents a circle with center $(1, 0)$ and a radius of 1. It is shown below:



Now $|z - 1| = 1$. Let $z = x + iy$ then $x^2 + y^2 = 2x$. Also,

$$\frac{z - 2}{z} = \frac{x - 2 + iy}{x + iy} = \frac{x^2 - 2x + y^2 + 2iy}{x^2 + y^2} = i \frac{y}{x}$$

Case I. When z lies in the first quadrant. This implies $\arg(z) = \theta$, where $\tan \theta = \frac{y}{x} \therefore i \tan[\arg(z)] = i \tan \theta = i \frac{y}{x}$.

Case II. When z lies in the fourth quadrant. Thus, $\arg(z) = 2\pi - \theta$, where $\tan \theta = \frac{-y}{x}$

$\therefore i \tan[\arg(z)] = i \tan(2\pi - \theta) = i \frac{y}{x}$.

Problem 195

195. Find the locus of z if $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$.

Solution of Problem 195

Solution: Let $z = x + iy$. Now we have $\frac{z-1}{z+1} = \frac{(x^2-1)+y^2}{(x+1)^2+y^2} + i \frac{2y}{(x+1)^2+y^2}$

$$\therefore \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \tan\left(\arg\left(\frac{z-1}{z+1}\right)\right) = \frac{2y}{x^2-1+y^2}$$

$\Rightarrow x^2 + y^2 - 1 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 2$, which is equation of a circle having center at $(0, 1)$ and radius $\sqrt{2}$.

Problem 196

196. If α is real and z is a complex number and u and v be the real and imaginary parts of $(z - 1)(\cos \alpha - i \sin \alpha) + (z - 1)^{-1}(\cos \alpha + i \sin \alpha)$. Prove that the locus of the points representing the complex numbers such that $v = 0$ is a circle of unit radius with center at a point $(1, 0)$ and a straight line passing through the center of the circle.

Solution of Problem 196

Solution: Let $z = x + iy$. Now, $u + iv = (z - 1)(\cos \alpha - i \sin \alpha) + \frac{1}{z-1}(\cos \alpha + i \sin \alpha)$

$$= (x - 1) \cos \alpha + y \sin \alpha + i[y \cos \alpha - (x - 1) \sin \alpha] + \frac{x-1-iy}{(x-1)^2+y^2}(\cos \alpha + i \sin \alpha) = 0$$

Equating imaginary parts, we get

$$v = y \cos \alpha - (x - 1) \sin \alpha + \frac{(x-1) \sin \alpha - y \cos \alpha}{(x-1)^2+y^2} = 0 \Rightarrow [y \cos \alpha - (x - 1) \sin \alpha][(x - 1)^2 + y^2] = 0$$

\therefore Either $y \cos \alpha - (x - 1) \sin \alpha = 0 \Rightarrow y = \tan \alpha (x - 1)$, which is a straight line passing through $(1, 0)$ or $(x - 1)^2 + y^2 - 1 = 0$ which is a circle with center $(1, 0)$ and unit radius.

Problem 197

197. If $|a_n| < 2$ for $n = 1, 2, 3, \dots$ and $1 + a_1z + a_2z^2 + \dots + a_nz^n = 0$, show that z does not lie in the interior of the circle $|z| = \frac{1}{3}$.

Solution of Problem 197

197. Given, $1 + a_1z + a_2z^2 + \cdots + a_nz^n = 0 \Rightarrow |a_1z| + |a_2z^2| + \cdots + |a_nz^n| \geq 1$ and

L.H.S. $< 2|z| + 2|z|^2 + \cdots$ to $\infty [\because |a_n|] < 2$.

Let $|z| < 1$ then $\frac{2|z|}{1-|z|} < 1 \Rightarrow |z| > \frac{1}{3}$

When $|z| > 1$, clearly $|z| > \frac{1}{3}$; hence, z does not lie in the interior of the circle with radius $\frac{1}{3}$.

Problem 198

198. Show that all the roots of the equation $z^n \cos \theta_0 + z^{n-1} \cos \theta_1 + \cdots + \cos \theta_n = 2$, where $\theta_0, \theta_1, \dots, \theta_n \in \mathbb{R}$ lie outside the circle $|z| = \frac{1}{2}$.

Solution of Problem 198

Solution: Given, $z^n \cos \theta_0 + z^{n-1} \cos \theta_1 + \cdots + \cos \theta_n = 2$

$$\Rightarrow 2 = |z^n \cos \theta_0 + z^{n-1} \cos \theta_1 + \cdots + \cos \theta_n|$$

$$< |z^n \cos \theta_0| + |z^{n-1} \cos \theta_1| + \cdots + |\cos \theta_n|$$

$$= |z^n| |\cos \theta_0| + |z^{n-1}| |\cos \theta_1| + \cdots + |\cos \theta_n|$$

$$\leq |z|^n + |z|^{n-1} + \cdots + 1 < 1 + |z| + |z|^2 + \cdots \text{ to } \infty$$

$$\Rightarrow 2 = \frac{1}{1-|z|} \Rightarrow |z| > \frac{1}{2} \text{ [when } |z| < 1 \text{]}$$

Hence z lies outside the circle $|z| = \frac{1}{2}$.

Thus all roots of the given equation lie outside the circle $|z| = \frac{1}{2}$.

Problem 199

199. z_1, z_2, z_3 are non-zero, non-collinear complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$. Show that z_1, z_2, z_3 lie on a circle passing through origin.

Solution of Problem 199

Solution: Recall that points z_1, z_2, z_3 are concyclic if $\left(\frac{z_2-z_4}{z_1-z_4}\right)\left(\frac{z_1-z_3}{z_2-z_3}\right)$ is real. We assume that z_4 is origin.

$$\text{Given, } \frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} = \frac{z_2+z_3}{z_2z_3} \therefore z_1 = \frac{2z_2z_3}{z_2+z_3}.$$

Putting the value of z_1 and z_4 in the concyclic condition expression we obtain

$$\left(\frac{z_2-z_4}{z_1-z_4}\right)\left(\frac{z_1-z_3}{z_2-z_3}\right) = \frac{1}{2}.$$

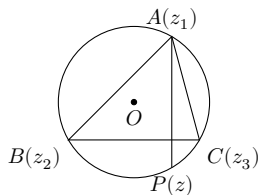
Thus, z_1, z_2, z_3 lie on a circle passing through origin.

Problem 200

200. A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane and the circumcenter of the $\triangle ABC$ lies on the origin. If the altitude of the triangle through vertex A meets the circle again at P , prove that P represents the complex number $\frac{z_2 z_3}{z_1}$.

Solution of Problem 200

Solution: The origin O is the circumcenter of $\triangle ABC$ and AP is perpendicular to BC . Let $P = z$.



We have $OP = OA = OB = OC \therefore |z| = |z_1| = |z_2| = |z_3| \Rightarrow |z|^2 = |z_1|^2 = |z_2|^2 = |z_3|^2 \Rightarrow z\bar{z} = z_1\bar{z}_1 = z\bar{z}_2 = z\bar{z}_3$.

Since AP is perpendicular to BC , therefore

$$\arg\left(\frac{z_1 - z}{z_2 - z_3}\right) = \frac{\pi}{2} \text{ or } \frac{-\pi}{2} \Rightarrow \frac{z_1 - z}{z_2 - z_3} \text{ is purely imaginary.}$$

$$\Rightarrow \overline{\left(\frac{z_1 - z}{z_2 - z_3}\right)} = -\frac{z_1 - z}{z_2 - z_3}$$

Solving the above equation gives $z = \frac{z_2 z_3}{z_1}$.