Complex Numbers Problems 31-40

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Important Identities Related to Cube Root of Unity

- $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
- $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
- $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
- $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
- $x^2 + y^2 + z^2 - xy - yz - zx =$
  $(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$ or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$ or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

Problem 31

31. Find the square root of $7 + 8i$.

Solution of Problem 31

Solution: Let $z = x + iy = \sqrt{7 + 8i}, \Rightarrow (x^2 - y^2) + 2ixy = 7 + 8i$

Comparing real and imaginary parts, we get

$x^2 - y^2 = 7, 2xy = 8 \Rightarrow x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{49 + 64} = \sqrt{113}[\because x, y \in R, x^2 + y^2 > x^2 - y^2 \Rightarrow x^2 + y^2 \neq -\sqrt{113}]$

$\Rightarrow 2x^2 = 7 + \sqrt{113} \Rightarrow x = \pm\sqrt{\frac{7+\sqrt{113}}{2}}$

$2y^2 = \sqrt{113} - 7 \Rightarrow y = \pm\sqrt{\frac{\sqrt{113}-7}{2}}$

Problem 32

32. Find the square root of $a^2 - b^2 + 2iab \ \forall \ a, b \in R$.

Solution of Problem 32

Solution: Let $z = x + iy = \sqrt{a^2 - b^2 + 2iab} \Rightarrow x^2 - y^2 + 2ixy = a^2 - b^2 + 2iab$

Comparing real and imaginary parts, we get

$\Rightarrow x^2 - y^2 = a^2 - b^2, 2xy = 2ab \Rightarrow x^2 + y^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = a^2 + b^2 [\because x, y \in R, x^2 + y^2 > x^2 - y^2 \Rightarrow x^2 + y^2 \neq -(a^2 + b^2)]$

$\Rightarrow 2x^2 = 2a^2 \Rightarrow x = \pm a$ and similarly, $y = \pm b$

Problem 33

33. Find the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{31}{16}$.

Solution of Problem 33

Solution: Given equation can be rewritten as $\frac{x^2}{y^2}+\frac{y^2}{x^2}-2.\frac{i}{4}\left(\frac{x}{y}+\frac{y}{z}\right)+\frac{i^2}{4^2}-\frac{i^2}{4^2}+\frac{31}{16}$

$=\frac{x^2}{y^2}+\frac{y^2}{x^2}-2.\frac{i}{4}\left(\frac{x}{y}+\frac{y}{z}\right)+\frac{i^2}{4^2}+\frac{1}{16}+\frac{31}{16}$

$=\frac{x^2}{y^2}+\frac{y^2}{x^2}-2.\frac{i}{4}\left(\frac{x}{y}+\frac{y}{z}\right)+\frac{i^2}{4^2}+2.\frac{x}{y}.\frac{y}{x}$

$=\left(\frac{x}{y}+\frac{y}{x}-\frac{i}{4}\right)^2$

Therefore, square root is $\pm\left(\frac{x}{y}+\frac{y}{x}-\frac{i}{4}\right)$

Problem 34

34. Find the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} - \frac{1}{i}\left(\frac{x}{y} - \frac{y}{x}\right) - \frac{9}{4}$

Solution of Problem 34

Solution: Like prevous problem the given equation can be rewritten as $\frac{x^2}{y^2}+\frac{y^2}{x^2}+2.\frac{i}{2}\left(\frac{x}{y}-\frac{y}{x}\right)+\frac{i^2}{2^2}-\frac{i^2}{2^2}-\frac{9}{4}$

$$=\frac{x^2}{y^2}+\frac{y^2}{x^2}+2.\frac{i}{2}\left(\frac{x}{y}-\frac{y}{x}\right)+\frac{i^2}{2^2}-2$$

$$=\frac{x^2}{y^2}+\frac{y^2}{x^2}+2.\frac{i}{2}\left(\frac{x}{y}-\frac{y}{x}\right)+\frac{i^2}{2^2}-2.\frac{x}{y}.\frac{y}{x}$$

$$=\left(\frac{x}{y}-\frac{y}{x}+\frac{i}{2}\right)^2$$

Therefore, square root is $\pm\left(\frac{x}{y}-\frac{y}{x}+\frac{i}{2}\right)$

Problem 35

35. Find the square root of $x^2 + \frac{1}{x^2} + 4i\left(x - \frac{1}{x}\right) - 6$

Solution of Problem 35

Solution: Given equation can be written as $x^2 + \frac{1}{x^2} + 2.2i\left(x - \frac{1}{x}\right) + (2i)^2 - (2i)^2 - 6$

$= x^2 + \frac{1}{x^2} + 2.2i\left(x - \frac{1}{x}\right) + 4i^2 - 2 = x^2 + \frac{1}{x^2} + 2.2i\left(x - \frac{1}{x}\right) + 4i^2 - 2.x.\frac{1}{x}$

$= \left(x - \frac{1}{x} + 2i\right)^2$

Thus square root is $\pm\left(x - \frac{1}{x} + 2i\right)$

Problrm 36

36. Find the minimum value of $|z| + |z - 2|$

Solution of Problem 36

Solution: We know that for two complex numbers z_1 and z_2, $|z_1| + |z_2| \geq |z_1 - z_2|$

$|z| + |z - 2| \geq |z - (z - 2)| = |2| = 2$

Therefore, minimum value is 2.

Problem 37

37. If $|z_1 - 1| < 1, |z_2 - 2| < 2$ and $|z_3 - 3| < 3$ then prove that maximum value of $|z_1 + z_2 + z_3|$ is 12.

Solution of Problem 37

Solution: $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6 \leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$

$< 1 + 2 + 3 + 6 = 12$

Thus, maximum value of $|z_1 + z_2 + z_3|$ is 12.

Problem 38

38. If α, β are two complex numbers then prove that $|\alpha|^2 + |\beta|^2 = \frac{1}{2}(|\alpha+\beta|^2 + |\alpha-\beta|^2)$.

Solution of Problem 38

Solution: $|\alpha+\beta|^2 = (\alpha+\beta)(\overline{\alpha+\beta}) = (\alpha+\beta)(\overline{\alpha}+\overline{\beta})$

$= \alpha\overline{\alpha} + \alpha\overline{\beta} + \overline{\alpha}\beta + \beta\overline{\beta} = |\alpha|^2 + |\beta|^2 + \alpha\overline{\beta} + \overline{\alpha}\beta$

Similarly, $|\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 - \alpha\overline{\beta} - \overline{\alpha}\beta$

Thus, $|\alpha|^2 + |\beta|^2 = \frac{1}{2}(|\alpha+\beta|^2 + |\alpha-\beta|^2)$

Problem 39

39. Find $\sqrt{i}\sqrt{-i}$

Solution of Problem 39

Solution: Let $z = \sqrt{i}\sqrt{-i} = \sqrt{-i^2} = \sqrt{-1.i^2} = i^2 = -1$

Problem 40

40. Simplify $i^{n+80} + i^{n+50}$ in the form of $A + iB$.