Theory of Logarithm

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Theory of Logarithm

Definition: A number x is called the logartihm of a number y to the base b if $b^x = y$ where $b > 0, b \neq, y > 0$

Mathematically, it is represented by the equation $\log_b y = x or b^x = y$

Notes:

1. The conditions $b > 0, b \neq 1$ and $y > 0$ are necessary in the definition of logarithm.
2. When $b = 1$ suppose logarithm is defined, and we have to find the value of $\log_1 y = x$. Let $\log_1 y = x \Rightarrow 1^x = y \Rightarrow 1 = y$
   If $\log_1 2$ is defined then $1 = 2$. So we see that $b = 1$ leads to meaningless result.
3. Similalrlh if $y < 0$, then $b^x = y$ which is meaningless as L.H.S. is positive and R.H.S. is negative.
4. Let the condition to be true when $b = 0$. Thus, $0^x = N$ i.e. if $\log_0 2$ is defined will mean that $0 = 2$ which signifies that our assumption is false.
5. No number can have two different logarithms to a given base. Assume that a number N has two different logarithms x and y with base b. Then, $\log_b N = x, \log_b N = y \Rightarrow N = b^x, N = b^y \Rightarrow b^x = b^y \Rightarrow x = y$
6. When the number or base is negative the value of logarithm comes out to be a complex number with non-zero imaginary part. Let $\log_e(-5) = x \Rightarrow \log_e 5.e^{i\pi} = x \Rightarrow x = \log_e 5 + i\pi$

Important Results

1. $\log_b 1 = 0$

   Proof: Let $\log_b 1 = x \Rightarrow b^x = 1 \Rightarrow x = 0$
2. $\log_b b = 1$

   Proof: Let $\log_b b = x \Rightarrow b^x = b \Rightarrow x = 1$
3. $b^{\log_b N} = N$

   Proof: Let $\log_b N = x \Rightarrow b^x = N$

   $b^{\log_b N} = N$

Important Formulas

1. $\log_b(x.y) = \log_b x + \log_b y (x > 0, y > 0)$

   Proof: Let $\log_b x = m \Rightarrow b^m = x$ and $\log_b y = n \Rightarrow b^n = y$

   $x.y = b^m.b^n = b^{m+n} = b^0$(say)

   $m + n = o$

   $\log_b x.y = \log_b x + \log_b y$

   Corollary: $\log_b(x.y.z) = \log_b x + \log_b y + \log_b z$

   If $x < 0, y < 0, \log_b(x.y) = \log_b |x| + \log_b |y|$

2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y (x > 0, y > 0)$

   Proof: Let $\log_b x = m \Rightarrow b^m = x$ and $\log_b y = n \Rightarrow b^n = y$

   $\log_b \left(\frac{x}{y}\right) = o \Rightarrow b^o = \frac{x}{y}$

   $\frac{x}{y} = \frac{b^m}{b^n} = b^{m-n} = b^o \Rightarrow m - n = o$

   $\log_b \left(\frac{x}{y}\right) = \log_b |x| - \log_b |y| (x < 0, y < 0)$

Important Formulas

3. $\log_b N^k = k \log_b N$

   Proof: Let $\log_b N = x \Rightarrow b^x = N$

   Lety $\log_b N^k = y \Rightarrow b^y = N^k \Rightarrow b^y = (b^x)^k = b^{kx}$

   $\Rightarrow y = kx \Rightarrow \log_b N^k = k \log_b N$
4. $\log_b a = \log_c a \log_b c$

   Proof: Let $\log_b a = x \therefore b^x = a$

   $\log_c a = y \therefore c^y = a$

   $\log_b c = z \therefore b^z = c$

   $b^x = a = c^y = b^{yz} \Rightarrow x = yz[\because b \neq 1]$

   Alternatively, we can also write it as $\log_b a = \frac{\log_c a}{\log_c b}$

Important Formulas

5. $\log_{(b^k)} N = \frac{1}{k} \log_b N [b > 0]$

   Proof: From previous point we can infer that $\log_{(b^k)} N = \frac{\log N}{\log b^k} = \frac{\log N}{k \log b} = \frac{1}{k} \log_b N$

6. $\log_b a = \frac{1}{\log_a b}$

   Proof: Let $\log_b a = x \therefore b^x = a$

   $\log_a b = y \therefore a^y = b$

   $a = b^y = a^{xy} \Rightarrow xy = 1$

   $\Rightarrow \log_b a = \frac{1}{\log_a b}$

Typically a logarithm will have an integral part and a fractional part. The integral part is called *characteristics* and the fractional part is called *mantissa*.

For exmaple, if $\log x = 4.7$, then 4 is the characteristics and $.7$ is the mantissa. If characteristics is less than zero then at times it is written with a bar above. For example, $\log x = -5.3 = \overline{5}.3$

Bases of Logarithms

There are two popular bases of logarithms. Common base is 10 and another is e, When base is 10, logarithm is knows as common logarithms and when base is e, logarithms is known as *natual* or *Napierian* logarithm.

$\log_{10} x$ is also written as lgx and $\log_e x$ as lnx

Inequality of Logarithms

If $b > 1$, and $\log_b x_1 > \log_b x_2$ then $x_1 > x_2$. If $b < 1$, and $\log_b x_1 > \log_b x_2$ then $x_1 < x_2$.

Expansion of Logarithm and its Graph

The logarithm series is given below:

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{r} + ...$ Given below is an example how logarithm function behaves: