Complex Numbers

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Geometrical Results contd

Length of a Perpendicular from a Point to a Line

Length of a perpendicular of point $A(\omega)$ from the line $\overline{a}z + a\overline{z} + b = 0, (a \in C, b \in R)$ is given by

$$p = \frac{|\overline{a}\omega + a\overline{\omega} + b|}{2|a|}$$

Equation of a Circle

The equation of a circle with center z_0 and radius r is $|z - z_0| = r$ or $z = z_0 + re^{i\theta}, 0 \leq \theta \leq 2\pi$ or $z\overline{z} - z_0\overline{z} - \overline{z_0}z + z_0\overline{z_0} - r^2 = 0$

General equation of a circle is $z\overline{z} - a\overline{z} + \overline{a}z + b = 0, (a \in C, b \in R)$ such that $\sqrt{a\overline{a} - b} \geq 0$. Center of this circle is $-a$ and radius is $a\overline{a} - b$.

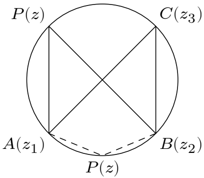
An equation of the circle, one of whose diameter is the line segment joining z_1 and z_2 is $(z - z_1)(\overline{z} - \overline{z_2}) + (\overline{z} - \overline{z_1})(z - z_2) = 0$ An equation of the the circle passing through two points z_1 and z_2 is

$$(z - z_1)(\overline{z} - \overline{z_2}) + (\overline{z} - \overline{z_1})(z - z_2) + k\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$$

where k is a parameter.

Geometrical Results contd

Equation of a Circle Passing through Three Points



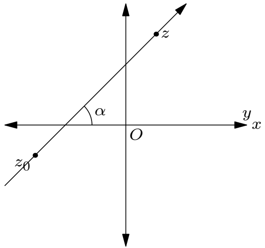
We choose any point $P(z)$ on the circle. Two such points are shown in the figure above one is in same segment with C and the other one in different segement. So we have

$$\angle ACB = \angle APB \text{ or } \angle ACB + \angle APB = \pi$$

$$\arg \frac{z_3 - z_2}{z_3 - z_1} - arg \frac{z - z_2}{z - z_1} = 0 \text{ or } \arg \frac{z_3 - z_2}{z_3 - z_1} + arg \frac{z - z_2}{z - z_1} = \pi$$

Clearly, in both cases the fraction must be purely real. Thus we can apply the property of conjugates i.e. $z = \overline{z}$ which also gives us the condition for four concyclic points.

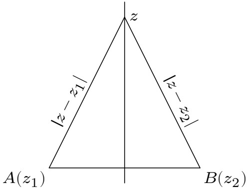
Finding Loci by Examination



The argument above line can be represented by equation $arg(z - z_0) = \alpha$ where α is a real number and z_0 is a fixed point. If z_0 is origin then the equation becomes $arg(z) = \alpha$ which is a vector starting at origin and making angle α with x-axis.

Finding Loci by Examination

If z_1 and z_2 are two fixed points such that $|z - z_1| = |z - z_2|$ then z represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$. And z, z_1, z_2 will form an isoscles triangle.



If z_1 and z_2 are two fixed points and $k > 0, k \neq 1$ is a real number then $\frac{|z-z_1|}{|z-z_2|} = k$ represents a circle.

Finding Loci by Examination

Consider $|z - z_1| + |z - z_2| = k$. Let z_1 and z_2 be two fixed points and k be a positive real number.

1. If $k > |z - z_2|$, then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at z_1 and z_2 and k is length of major axis.
2. If $k = |z - z_2|$ then it represents the line segment joining z_1 and z_2.
3. If $k < |z - z_2|$ then it does not represent any curve.

Consider $|z - z_1| - |z - z_2| = k$ like previous case.

1. If $k \neq |z - z_2|$ then it represents parabolas with foci at z_1 and z_2.
2. If $k = |z - z_2|$, then it represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the the segment AB.

$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$ represents a circle with z_1 and z_2 representing the diameter.

Finding Loci by Examination

Consider $arg \frac{z-z_1}{z-z_2} = \alpha$ where z_1 and z_2 are two fixed points and α be a real number such that $0 \leq \alpha \leq \pi$.

1. If $0 < \alpha < \pi$ and $\alpha \neq \pi/2$, then it represents a segment of a circle passing through z_1 and z_2.
2. If $\alpha = \pi/2$, then it represents a circle with diameter as the line segment joining z_1 and z_2.
3. If $\alpha = 0$, then it represents the line segment joining z_1 and z_2.
4. If $\alpha = \pi$, then it represents the line segment joining z_1 and z_2 but excluding $z_1 z_2$.