Complex Numbers Problems 161-170

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Problem 161

161. If z is any complex number, then show that $\left|\frac{z}{|z|} - 1\right| \leq |arg(z)|$

Solution of Problem 161

Solution: Let $z = re^{i\theta}$, then $\frac{z}{|z|} = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \left|\frac{z}{|z|} - 1\right| = |(\cos\theta - 1) + i\sin\theta| = \sqrt{\cos\theta^2 - 2\cos\theta + 1 + \sin^2\theta}$

$= \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}} = 2\sin\frac{\theta}{2} \leq \theta$

$\Rightarrow \left|\frac{z}{|z|} - 1\right| \leq |arg(z)|$

Problem 162

162. If z is any complex number, then show that $|z-1| \leq ||z|-1| + |z||arg z|$

Solution of Problem 162

Solution: Clearly, $|z-1| = |z-|z|+|z|-1| \leq |z-|z|| + ||z|-1|$

$= |z|\left|\frac{z}{|z|}-1\right| + ||z|-1|$

Using the result of previous problem, we get

$|z-1| \leq ||z|-1| + |z||argz|$

Problem 163

163. If $\left|z + \frac{1}{z}\right| = a$, where z is a complex number and $a > 0$, find the greatest and least values of $|z|$.

Solution of Problem 163

Solution: Let $z = r(\cos\theta + i\sin\theta)$, then $\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$\left|z + \frac{1}{z}\right| = \left|\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right|$

$\Rightarrow \left(r + \frac{1}{r}\right)^2 \cos^2\theta + i\left(r - \frac{1}{r}\right)^2 \sin^2\theta = a^2$

$\Rightarrow \left(r - \frac{1}{r}\right)^2 = a^2 - 4\cos^2\theta$

r will be greatest when $r - \frac{1}{r}$ will be greatets i.e. $\cos\theta = 0 \Rightarrow r - \frac{1}{r} = a$

$\Rightarrow r_{max} = \frac{a+\sqrt{a^2+4}}{2}$

Similarly, for lowest value of $r, \cos\theta = 1 \Rightarrow r - \frac{1}{r} = a^2 - 4 \Rightarrow r^2 - (a^2-4)r - 1 = 0$

$r_{min} = \frac{a^2-4-\sqrt{a^4-8a^2+20}}{2}$

Problem 164

164. If z_1, z_2 be complex numebrs and c is a positive number, prove that $|z_1 + z_2|^2 < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$.

Solution of Problem 164

Solution: We have to prove that $|z_1 + z_2|^2 < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$

$\Rightarrow (z_1 + z_2)(\overline{z_1} + \overline{z_2}) < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$

$\Rightarrow |z_1|^2 + z_1\overline{z_2} + z_2\overline{z_1} + |z_1|^2 < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$

$\Rightarrow z_1\overline{z_2} + z_2\overline{z_1} < (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$

$\Rightarrow (x_1 + iy_1)(x_2 - iy_2) + (x_2 + iy_2)(x_1 - iy_1) < \frac{1}{c}[c^2(x_1^2 + y_1^2) + (x_2^2 + y_2^2)]$

$\Rightarrow 2cx_1x_2 + 2cy_1y_2 < c^2x_1^2 + c^2y_1^2 + x_2^2 + y_2^2$

$\Rightarrow (cx_1 - x_2)^2 + (cy_1 - y_2)^2 > 0$ which is true.

Problem 165

165. If z_1 and z_2 are two complex numbers such that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$, prove that $\frac{iz_1}{z_2} = x$ where x is a real number. Find the angle between the lines from origin to the points $z_1 + z_2$ and $z_1 - z_2$ in terms of x.

Solution of Problem 165

Solution: Given $\left|\frac{z_1-z_2}{z_1+z_2}\right| = 1 \Rightarrow |z_1 - z_2|^2 = |z_1 + z_2|^2$

$\Rightarrow (z_1 - z_2)(\overline{z_1} - \overline{z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$\Rightarrow 2z_1\overline{z_2} = -2z_2\overline{z_1} \Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = -\frac{z_1}{z_2}$

$\Rightarrow \frac{z_1}{z_2} =$ purely imaginary $\Rightarrow i\frac{z_1}{z_2} =$ real $= x$

Now $\frac{z_1+z_2}{z_1-z_2} = \frac{z_1/z_2+1}{z_1/z_2-1} = \frac{-ix+1}{-ix-1} = \frac{-1+x^2+2ix}{1+x^2}$

If θ is the angle between given lines then

$\tan\theta = \arg\frac{z_1+z_2}{z_1-z_2} = \frac{2x}{x^2-1}$

Problem 166

166. Let z_1, z_2 be any two complex numbers and a, b be two real numbers such that $a^2 + b^2 \neq 0$. Prove that $|z_1|^2 + |z_2|^2 - |z_1^2 + z_2^2| \leq 2\frac{|az_1+bz_2|^2}{a^2+b^2} \leq |z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2|$

Solution of Problem 166

Solution: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Also let $a = r\cos\alpha, b = r\sin\alpha$

$|az_1 + bz_2|^2 = |rr_1(\cos\theta_1 + i\sin\theta_1)\cos\alpha + rr_2(\cos\theta_2 + i\sin\theta_2)\sin\alpha|^2$

$= r^2(r1\cos\theta_1\cos\alpha + r_2\cos\theta_2\sin\alpha)^2 + r^2(r_1\sin\theta_1\cos\alpha + r_2\sin\theta_2\sin\alpha)^2$

$= r^2[r_1^2\cos^2\alpha + r_2^2\sin^2\alpha + 2r_1r_2\cos\alpha\sin\alpha\cos(\theta_1 - \theta_2)]$

$= \frac{r^2}{2}[r_1^2(1+\cos 2\alpha) + r_2^2(1-\cos 2\alpha) + 2r_1r_2\sin 2\alpha\cos(\theta_1 - \theta_2)]$

$\frac{2|az_1+bz_2|^2}{a^2-b^2} = r_1^2 + r_2^2 + (r_1^2 - r_2^2)\cos 2\alpha + 2r_2r_2\cos(\theta_1 - \theta_2)\sin 2\alpha$

$= A + B\cos 2\alpha + C\sin 2\alpha$ where $A = r_1^2 + r_2^2, B = r_1^2 - r_2^2, C = 2r_1r_2\cos(\theta_1 - \theta_2)$

Clearly, $-\sqrt{B^2 + C^2} \le B\cos 2\alpha + C\sin 2\alpha \le \sqrt{B^2 + C^2}$

$\therefore A - \sqrt{B^2 + C^2} \le A + B\cos 2\alpha + C\sin 2\alpha \le A + \sqrt{B^2 + C^2}$

$\therefore A - \sqrt{B^2 + C^2} \le \frac{2|az_1+bz_2|^2}{a^2+b^2} \le A + \sqrt{B^2 + C^2}$

Now $B^2 + C^2 = r_1^4 + r_2^4 - 2r_1^2r_2^2 + 4r_1^2r_2^2\cos^2(\theta_1 - \theta_2)$

Solution contd.

Again $|z_1^2+z_2^2| = |r_1^2(\cos 2\theta_1 + i\sin 2\theta_1) + r_2^2(\cos 2\theta_2 + i\sin 2\theta_2)|$

$= \sqrt{(r_1^2\cos 2\theta_1 + r_2^2\cos 2\theta_2)^2 + (r_1^2\sin 2\theta_1 + r_2^2\sin 2\theta_2)^2}$

$= \sqrt{r_1^4 + r_2^4 + 2r_1^2r_2^2\cos 2(\theta_1-\theta_2)}$

$= \sqrt{r_1^4 + r_2^4 + 2r_1^2r_2^2[2\cos^2(\theta_1-\theta_2)-1]} = \sqrt{B^2+C^2}$

$A = r_1^2 + r_2^2 = |z_1|^2 + |z_2|^2$

Hence, $|z_1|^2 + |z_2|^2 - |z_1^2+z_2^2| \le 2\frac{|az_1+bz_2|^2}{a^2+b^2} \le |z_1|^2 + |z_2|^2 + |z_1^2+z_2^2|$.

Problem 167

167. If $b + ic = (1 + a)z$ and $a^2 + b^2 + c^2 = 1$, prove that $\frac{a+ib}{1+c} = \frac{1+iz}{1-iz}$, where a, b, c are real numbers and z is a complex number.

Solution of Problem 167

Solution: Given $z = \frac{b+ic}{1+a} \therefore iz = \frac{-c+ib}{1+a} \Rightarrow \frac{1}{iz} = \frac{1+a}{-c+ib}$

Using componendo and dividendo, we get

$\Rightarrow \frac{1+iz}{1-iz} = \frac{1+a-c+ib}{1+a+c-ib}$

Also, given $a^2 + b^2 + c^2 = 1 \Rightarrow a^2 + b^2 = 1 - c^2$

$\Rightarrow (a+ib)(a-ib) = (1+c)(1-c) \Rightarrow \frac{a+ib}{1-c} = \frac{1+c}{a-ib} = \frac{1}{u}$(say)

$\therefore \frac{1+iz}{1-iz} = \frac{a+ib+1-c}{1+c+a-ib} = \frac{a+ib+u(a+ib)}{1+c+u(1+c)}$

$= \frac{a+ib}{1+c}$

Problem 168

168. If $a, b, c, \dots, k$ are all n real roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = 0w$, where $p_1, p_2, \dots, p_n$ are real, show that $(1 + a^2)(1 + b^2) \dots (1 + k^2) = (1 - p_2 + p_4 + \dots)^2 + (p_1 - p_3 + \dots)^2$.

Solution of Problem 168

Solution: We can write that $(x-a)(x-b)\dots(x-k) = x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n$

Substituting $x = i$, we get

$(i-a)(i-b)\dots(i-k) = i^n + p_1i^{n-1} + p_2i^{n-2} + \dots + p_{n-1}i + p_n$

Dividing both sides by i^n, we get $(1+ia)(1+ib)\dots(1+ik) = 1 + \frac{p_1}{i} + \frac{p_2}{i^2} + \dots$

Taking modulus and squarin, we get

$(1+a^2)(1+b^2)\dots(1+k^2) = (1-p_2+p_4+\dots)^2 + (p_1-p_3+\dots)^2$

Problem 169

169. If $f(x) = x^4 - 8x^3 + 4x^2 + 4x + 39$ and $f(3+2i) = a + ib$, find $a : b$.

Solution of Problem 169

Solution: $3+2i$ is one value of x for which $f(3+2i)=a+ib$

$\Rightarrow x=3+2i \Rightarrow x^2-6x+13=0$

$f(x)=x^4-8x^3+4x^2+4x+39=(x^2-6x+13)(x^2-2x-21)-96x+312$

$\Rightarrow f(3+2i)=-96(3+2i)+312=24-192i=a+ib$

$\Rightarrow a:b=1:-8$

Problem 170

170. Let A and B be two complex numbers such that $\frac{A}{B} + \frac{B}{A} = 1$, prove that the triangle formed by origin and these two points is equilateral.

Solution of Problem 170

Solution: Given $\frac{A}{B}+\frac{B}{A}=1 \Rightarrow A^2-AB+B^2=0$

$A=\frac{B\pm\sqrt{3}iB}{2}=-\omega B,-\omega^2 B \Rightarrow |A|=|B|$

$|A-B|=|-\omega B-B|$ or $|-\omega^2 B-B|=|\omega^2 B|$ or $|\omega B|$

$\Rightarrow |A-B|=|B|$

Thus, $|A|=|B|=|A-B|$ making the triangle equilateral.