Miscellaneous Problems on A.P., G.P. and H.P.
Problems 51-60

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Problem 51

51. If $\int_0^{\frac{\pi}{2}} \cos^n x \cos(nx) dx$, then prove that $I_1, I_2, I_3, ...$ are in G.P.

Solution of Problem 51

Solution:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos(nx) dx$$

$$I_{n+1} = \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x [\cos x \cos(n+1)x] dx$$

Now, $\cos(nx) = \cos[(n+1)-1]x = \cos(n+1)x \cos x + \sin(n+1)x \sin x$
$\Rightarrow \cos(n+1)x \cos x = \cos nx - \sin(n+1)x \sin x$

$$\therefore I_{n+1} = \int_0^{\frac{\pi}{2}} \cos^n x [\cos(nx) - \sin(n+1)x \sin x] dx$$

$$= I_n - \int_0^{\frac{\pi}{2}} \cos^n x \sin x \sin(n+1)x dx$$

Taking $u = \sin(n+1)x$ and $v = \cos^n x \sin x$, we get

$$= I_n + \left[\frac{\cos^{n+1} x \sin(n+1)x}{n+1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos^{n+1} x \cos(n+1)x dx$$

$$= I_n + 0 - 0 - I_{n+1} \Rightarrow 2I_{n+1} = I_n \Rightarrow \frac{I_{n+1}}{I_n} = \frac{1}{2} \forall n \in N$$

Thus, $I_1, I_2, I_3, ...$ are in G.P.

Problem 52

52. Let $I_n = \int_0^{\pi} \frac{\sin(2n-1)x}{\sin x}\,dx$. Show that $I_1, I_2, I_3, \ldots$ are in A.P. as well as in G.P.

Solution of Problem 52

Solution:

$$I_{n+1} - I_n = \int_0^{\pi} \frac{\sin(2n+1)x - \sin(2n-1)x}{dx} = \int_0^{\pi} \frac{2\cos 2nx \sin x}{\sin x} dx$$

$$= 2\int_0^{\pi} \cos 2nx dx = \frac{2}{2n}[\sin 2nx]_0^{\pi} = 0 \Rightarrow I_{n+1} = I_n$$

Also, $I_1 = \int_o^{\pi} \frac{\sin x}{\sin x} dx = \pi$ Thus, $I_1 = I_2 = I_3 = ... = \pi$ Thus, $I_1, I_2, I_3, ...$ are in A.P. as well as in G.P.

Problem 53

53. Prove that the three successive terms of a G.P. will form sides of a triangle if the common ratio r satisfied the inequality $\frac{1}{2}(\sqrt{5}-1) < r < \frac{1}{2}(\sqrt{5}+1)$

Solution of Problem 53

Solution: Let the sides of the triangle be a, ar, ar^2.

When $r = 1$ the triangle formed will be equilateral triangle.

If $r > 1$ the triangle will be formed if $a + ar > ar^2$

$$r^2 - r - 1 < 0 \Rightarrow \frac{1-\sqrt{5}}{2} < r < \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow r < \frac{1+\sqrt{5}}{2} [\because r > 1]$$

Similarly when $r < 1$ the triangle will be formed if $r > \frac{1-\sqrt{5}}{2}$

Thus, the range is $\frac{1}{2}(\sqrt{5}-1) < r < \frac{1}{2}(\sqrt{5}+1)$

Problem 54

54. Find out whether $111\ldots1(91 \text{ digits })$ is a prime number.

Solution of Problem 54

Solution: $111\ldots 1 = 10^{90} + 10^{89} + 10^{88} + 10^{2} + 10 + 1 = \frac{10^{91}}{10-1}$ Since $91 = 13 \times 7$ we can multiply or divide by $10^{13} - 1$ or $10^{7} - 1$

$$\frac{10^{91}}{10-1} = \frac{10^{91}}{10^{7}-1}\,\frac{10^{7}}{10-1}$$

$$= (10^{84} + 10^{77} + 10^{70} + \ldots + 1)(10^{6} + 10^{5} + \ldots + 1)$$

Therefore, given no. is not a prime nnumber.

Problem 55

55. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfied the relation $f(x+y) = f(x)f(y)$ for all natural nuumbers x, y and further $f(1) = 2$

Solution of Problem 55

Solution: Given $f(x+y) = f(x)f(y)$ for all natural number x and y

$$\therefore f(a+k) = f(a)f(k)$$

$$\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)f(k) = f(a)[f(1)+f(2)+...+f(k)]$$

$$f(2) = f(1)+f(1) \Rightarrow f(2) = [f(1)]^2 = 2^2$$

$$f(3) = f(1)f(2) = 2^3$$

Given

$$\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$$

$$2^a[2+2^2+...+2^n] = 16(2^n - 1)$$

$$2^a.2.(2^n - 1) = 16.(2^n - 1) \Rightarrow a = 3$$

Problem 56

56. In a certain test, there are n questions. In this test 2^{n-i} students give wrong answers to at least i questions $(1 \leq i \leq n.)$ If total no. of wrong answers given is 2047, find the value of n.

Solution of Problem 56

Solution: Number of students giving wrong answers to at least i questions $= 2^{n-i}$

Number of students gicing wrong answers to at least $i+1$ questions $= 2^{n-i-1}$

$\therefore$ Number of students giving wrong answers to exactly i questions $= 2^{n-i} - 2^{n-i-1}$. Also, number of students giving wrong answers to exactly n questions $= 2^{n-n} = 1$

$\therefore$ Total no. of wrong answers $1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + ... + (n-1)(2^1 - 2^0) + n(2^0)$

$= 2^{n-1} + (-2^{n-2} + 2.2^{n-2}) + (-2.2^{n-3} + 3.2^{n-3}) + ... + [-(n-1)2^0] + n.2^0$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + ... + 2^0 = 2^n - 1$

Given, $2^n - 1 = 2047 \Rightarrow n = 11$

Problem 57

57. If $S_1, S_2, S_3, \dots, S_2n$ are the sums of infinite geometric series whose first terms are respectively $1, 2, 3, \dots, 2n$ and common ratio are respectively $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2n+1}$, find the value of $S_1^2 + S_2^2 + \dots + S_{2n-1}^2$

Solution of Problem 57

Solution: $S_1 = \frac{1}{1-\frac{1}{2}} = 2, S_2 = \frac{2}{1-\frac{1}{3}} = 3, S_3 = \frac{3}{1-\frac{1}{4}} = 4$ and so on.

$$S_1^2 + S_2^2 + ... + S_{2n-1}^2 = 2^2 + 3^2 + ... + 2n^2 = 1^2 + 2^2 + 3^2 + ... + 2n^2 - 1^2$$

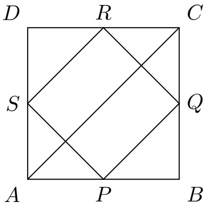
$$= \frac{2n(2n+1)(4n+1)}{6} - 1 = \frac{n(2n+1)(4n+1)}{3} - 1$$

Problem 58

58. A sqaure is given, a second square is made by joining the middle points of the first square and then a third square is made by joining the middle points of the sides of second square and so on till infinity. Show that the area of first square is equal to sum of the areas of all the succeeding squares.

Solution of Problem 58

Solution:



Let $ABCD$ be the first sqaure. Let $AB = a \Rightarrow AC = \sqrt{2}a \therefore PQ = \frac{AC}{2} = \frac{a}{\sqrt{2}}$

$\therefore$ Area of first square $= a^2$

Area of second sqaure $= \frac{a^2}{2}$

Area of third sqaure $= \frac{a^2}{4}$

Sum of areas of all squares except first $= \frac{a^2}{2} + \frac{a^2}{4} + ... = \frac{\frac{a^2}{2}}{1-\frac{1}{2}} = a^2$

Problem 59

59. If a is the value of x for which the function $7 + 2x \log 25 - 5^{x-1} - 5^{2-x}$ has the greatest value and $r = lim_{x\to 0} \int_0^x \frac{t^2}{x^2 \tan(\pi + x)} dt$, find $\lim_{n\to\infty} \sum_{n=1}^{n} ar^{n-1}$

Solution of Problem 59

Solution: Let $f(x) = 7 + 2x\log 25 - 5^{x-1} - 5^{2-x}$

$$f'(x) = 4\log 5 - 5^{x-1}\log 5 + 5^{2-x}\log 5 = \frac{\log 5}{5^{x+1}}(5^x - 25)(5^x + 5)$$

$\because \frac{\log 5}{5^{x+1}}(5^x + 5) > 0$ for all real $x > 0$

$f'(x) < 0$ or > 0 as $x > 2$ or $x < 2$

So $f(x)$ has local max. at $x = 2$ and has no local min. Hence, $f(x)$ has greatest value at $x = 2$ i.e. $a = 2$. Now

$$r = lim_{x\to 0}\int_0^x \frac{t^2}{x^2\tan(\pi + x)}dt = lim_{x\to 0}\frac{\int_0^x t^2}{x^2\tan(\pi + x)}dt$$

$$= \lim_{x\to 0}\frac{x^3}{3x^2\tan x} = \frac{1}{3}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{n=1}^{n} ar^{n-1} = a + ar + ar^2 + \ldots \text{ to } \infty = \frac{a}{1-r}$$

$$= 3$$

Problem 60

60. If pth, qth, rth terms of a G.P. are positive numbers a, b, c respectively, show that the vectors $(\log a).\vec{i} + (\log b)\vec{j} + (\log c)\vec{k}$ and $(q - r)\vec{i} + (r - p)\vec{j} + (p - q)\vec{k}$ are perpendicular.

Solution of Problem 60

Solution: Let x be the first term and y be the common ratio of the G.P. Then

$$a = xy^{p-1}, b = xy^{q-1}, c = xy^{r-1}$$

$$\Rightarrow \log a = \log x + (p-1)\log y, \log b = \log x + (q-1)\log y, \log c = \log x + (r-1)\log y$$

If the vectors are perpendicular the dot product will be zero.

$$\therefore (q-r)\log a + (r-p)\log b + (p-q)\log c = 0$$

$$(q-r+r-p+p-q)\log x + [(q-r)(p-1) + (r-p)(q-1) + (p-q)(r-1)]\log y = 0$$