Miscellaneous Problems on A.P., G.P. and H.P.
Problems 81-90

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Problem 81

81. If S_n denotes the sum to n terms of a G.P. whose first term and common ratio are a and r respectively, then prove that $S_1 + S_2 + ... + S_n = \frac{na}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

Solution of Problem 81

Solution:

$$S_1 = a = \frac{a(1-r)}{1-r}$$

$$S_2 = a + ar = \frac{a(1-r^2)}{1-r}$$

$$S_3 = \frac{a(1-r^3)}{1-r}$$

$$\dots$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_1 + S_2 + \dots + S_n = \frac{a(1-r)}{1-r} + \frac{a(1-r^2)}{1-r} + \dots + \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r}[1 + 1 + \dots + \text{ to } n \text{ terms }] - \frac{ar}{1-r}[1 + r + r^2 + \dots + r^{n-1}]$$

$$= \frac{na}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$$

Problem 82

82. If S_n denotes the sum to n terms of a G.P. whose first term and common ratio are a and r respectively, then prove that $S_1 + S_3 + S_5 + ... + S_{2n-1} = \frac{na}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}$

Solution of Problem 82

Solution:

$$S_1 = a = \frac{a(1-r)}{1-r}$$

$$S_3 = \frac{a(1-r^3)}{1-r}$$

$$S_5 = \frac{a(1-r^5)}{1-r}$$

$$\dots$$

$$S_{2n-1} = \frac{a(1-r^{2n-1})}{1-r}$$

$$S_1 + S_3 + S_5 + \dots + S_{2n-1} = \frac{a}{1-r}[1 + 1 + \dots + \text{ to } n \text{ terms }] - \frac{ar}{1-r^2}[1 + r^2 + r^4 + \dots + r^{2(n-1)}]$$

$$= \frac{na}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}$$

Problem 83

83. Let s denote the sum of terms of an infinite geometric progression and σ^2 the sum of squares of the terms. Show that the sum of first n terms of this geometric progression is given by $s\left[1-\left(\frac{s^2-\sigma^2}{s^2+\sigma^2}\right)^n\right]$, where $|r|<1$

Solution of Problem 83

Solution:

$$s=\frac{a}{1-r},\sigma^2=\frac{a^2}{1-r^2},S_n=\frac{a(1-r^n)}{1-r}$$

$$\begin{aligned}
s\left[1-\left(\frac{s^2-\sigma^2}{s^2+\sigma^2}\right)^n\right]&=\frac{a}{1-r}\left[1-\left(\frac{\frac{a^2}{(1-r)^2}-\frac{a^2}{1-r^2}}{\frac{a^2}{(1-r)^2}+\frac{a^2}{1-r^2}}\right)^n\right]\\
&=\frac{a}{1-r}\left[1-\left(\frac{\frac{1}{1-r}-\frac{1}{1+r}}{\frac{1}{1-r}+\frac{1}{1+r}}\right)^n\right]\\
&=\frac{a}{1-r}(1-r^n)=S_n
\end{aligned}$$

Problem 84

84. Let $a_1, a_2, a_3, \dots, a_n$ be a geometric progression with first term a and common ratio r, then the sum of the products $a_1, a_2, \dots, a_n$ taken two at a time i.e. $\sum_{i<j} a_i a_j = \frac{a^2 r(1-r^{n-1})(1-r^n)}{(1-r)^2(1+r)}$

Solution of Problem 84

Solution:

$$\sum_{i<j} a_i a_j = \frac{1}{2}[(a_1 + a_2 + ... + a_n)^2 - (a_1^2 + a_2^2 + ... + a_n^2)]$$

$$= \frac{1}{2}\left[(a + ar + ... + ar^{n-1})^2 - (a^2 + a^2r^2 + ... + a^2r^{2(n-1)})\right]$$

$$= \frac{1}{2}\left[\frac{a^2(1-r^n)^2}{(1-r)^2 - \frac{a^2(1-r^{2n})}{1-r^2}}\right]$$

$$= \frac{1}{2}\left[\frac{a^2(1-2r^n+r^{2n})}{(1-r)^2} - \frac{a^2(1-r^{2n})}{1-r^2}\right]$$

$$= \frac{a^2r(1-r^{n-1})(1-r^n)}{(1-r)^2(1+r)}$$

Problem 85

85. If $a_1, a_2, a_3, ...$ is a G.P. with first term a and common ratio r, show that $\frac{1}{a_1^2-a_2^2} + \frac{1}{a_2^2-a_3^2} + ... + \frac{1}{a_{n-1}^2-a_n^2}$
$= \frac{r^2(1-r^{2n-2})}{a^2r^{2n-2}(1-r^2)^2}$

Solution of Problem 85

Solution:

$$L.H.S. = \frac{1}{a^2 - a^2r^2} + \frac{1}{a^2r^2 - a^2r^4} + \frac{1}{a^2r^4 - a^2r^6} + ... + \frac{1}{a^2r^{2(n-2)} - a^2r^{2(n-1)}}$$

$$= \frac{1}{a^2(1-r^2)}\left[1 + \frac{1}{r^2} + \frac{1}{r^4} + ... + \frac{1}{r^{2(n-2)}}\right]$$

$$= \frac{1}{a^2(1-r^2)} \cdot \frac{1 - \frac{1}{r^{2(n-1)}}}{1 - \frac{1}{r^2}}$$

$$= \frac{1}{a^2(1-r^2)} \cdot \frac{1 - r^{2n-2}}{1 - r^2} \cdot \frac{r^2}{r^{2n-2}}$$

Problem 86

86. If $a_1, a_2, a_3, \ldots$ is a G.P. with first term a and common ratio r, show that $\frac{1}{a_1^m + a_2^m} + \frac{1}{a_2^m + a_3^m} + \ldots + \frac{1}{a_{n-1}^m + a_n^m}$ $= \frac{r^{mn-m} - 1}{a^m (1+r^m)(r^{mn-m} - r^{mn-2m})}$

Solution of Problem 86

Solution:

$$L.H.S. = \frac{1}{a^m + a^m r^m} + \frac{1}{a^m r^m + a^m r^{2m}} + ... + \frac{1}{a^m r^{m(n-2)} + a^m r^{m(n-1)}}$$

$$= \frac{1}{a^m(1+r^m)}\left[1 + \frac{1}{r^m} + \frac{1}{r^{2m}} + ... + r^{m(n-2)}\right]$$

$$= \frac{1}{a^m(1+r^m)} \cdot \frac{1 - \frac{1}{r^{m(n-1)}}}{1 - \frac{1}{r^m}}$$

$$= \frac{r^{mn-m} - 1}{a^m(1+r^m)(r^{mn-m} - r^{mn-2m})}$$

Problem 87

87. If $a_1, a_2, \ldots, a_{2n}$ are $2n$ positive real numbers which are in G.P. show that $\sqrt{a_1 a_2} + \sqrt{a_3 a_4} + \sqrt{a_5 a_6} + \ldots + \sqrt{a_{2n-1} a_{2n}} = \sqrt{a_1 + a_3 + \ldots + a_{2n-1}} \sqrt{a_2 + a_4 + \ldots + a_{2n}}$

Solution of Problem 87

Solution:

$$L.H.S. = \sqrt{a^2r} + \sqrt{a^2r^5} + \sqrt{a^2r^9} + ... + \sqrt{a^2r^{4n-3}}$$

$$= a\sqrt{r}(1 + r^2 + r^4 + ... + r^{2(n-1)}) = a\sqrt{r}.\frac{(r^{2n-1})}{r^2 - 1}$$

$$\sqrt{a_1 + a_3 + ... + a_{2n-1}} = \sqrt{a(1 + r^2 + ... + r^{2n-2})} = \sqrt{a.\frac{r^{2n-1}}{r^2 - 1}}$$

$$\sqrt{a_2 + a_4 + ... + a_{2n}} = \sqrt{ar(1 + r^2 + ... + r^{2n-2})} = \sqrt{a\sqrt{r}.\frac{r^{2n-1}}{r^2 - 1}}$$

$$\therefore \sqrt{a_1a_2} + \sqrt{a_3a_4} + \sqrt{a_5a_6} + ... + \sqrt{a_{2n-1}a_{2n}} = \sqrt{a_1 + a_3 + ... + a_{2n-1}}\sqrt{a_2 + a_4 + ... + a_{2n}}$$

Problem 88

88. Find the solution of the system of equations $1 + x + x^2 + ... + x^{23} = 0$ and $1 + x + x^2 + ... + x^{19} = 0$

Solution of Problem 88

Solution: Given

$$1 + x + x^2 + ... + x^{23} = 0, 1 + x + x^2 + ... + x^{19} = 0$$

$$\frac{x^{24} - 1}{x - 1} = 0, \frac{x^{20} - 1}{x - 1} = 0$$

$$x^{24} - 1 = 0, x^{20} - 1 = 0$$

$$\therefore x^{20}.x^4 - 1 = 0 \Rightarrow x^4 - 1 = 0$$

Thus, roots are $-1, \pm i$

Problem 89

89. A man invests $\$a$ at the end of the first year, $\$2a$ at the end of the second year, $\$3a$ at the end of the third year, and so on up to the end of nth year. If the rate of interest is $\$r$ per rupee and the interest is compounded annually, find the amount the man will receive at the end of $(n+1)$th year.

Solution of Problem 89

Solution: $\$a$ will become $a + r.(a) = a(1+r)$ at the end of second year,
$a + ar + r(a + ar) = a + 2ar + ar^2 = a(1+r)^2$ at the end of third year,
$a + 2ar + ar^2 + r(a + 2ar + ar^2) = a + 3ar + 3ar^2 + ar^3 = a(1+r)^3$ and so on. So amount received for $\$a$ will be $a(1+r)^{n+1}$

Similarly, amount receoved for $\$2a$ will be $2a(1+r)^n$ and so on.

Thus, total amount received will be $S = a(1+r)^{n+1} + 2a(1+r)^n + 3a(1+r)^{n-1} + ... + na(1+r)$

$\frac{S}{1+r} = a(1+r)^n + 2a(1+r)^{n-1} + ... + (n-1)(1+r) + na$

Writing first term of second sum against second term of first sum, second term of second sum agains third term of first sum and so on and subtracting, we get $\frac{rS}{1+r} = a(1+r)^{n+1} + a(1+r)^n + a(1+r)^{n-1} + ... + a(1+r) - na$

$\frac{rS}{1+r} = a(1+r)[(1+r)^n + (1+r)^{n-1} + ... + 1]) - na$

$S = \frac{a(1+r)^2[(1+r)^n - 1]}{r^2} - \frac{na(1+r)}{r}$

Problem 90

90. Find the value of $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right)}$

Solution of Problem 90

Solution:

$$\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right)=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}$$

$$(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\infty\right)}=\left(\frac{4}{25}\right)^{\log_{\frac{5}{2}}\frac{1}{2}}$$

$$=\left(\frac{1}{2}\right)^{\log_{\frac{5}{2}}\frac{4}{25}}=\left(\frac{1}{2}\right)^{-2}=4$$