

Complex Numbers Problems

31-40

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Important Identities Related to Cube Root of Unity

- ▶ $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
- ▶ $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
- ▶ $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- ▶ $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- ▶ $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
- ▶ $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
- ▶ $x^2 + y^2 + z^2 - xy - yz - zx =$
 $(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega) \text{ or } (x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z) \text{ or } (x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$
- ▶ $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

Problem 31

31. Find the square root of $7 + 8i$.

Solution of Problem 31

Solution: Let $z = x + iy = \sqrt{7+8i}$, $\Rightarrow (x^2 - y^2) + 2ixy = 7 + 8i$

Comparing real and imaginary parts, we get

$$x^2 - y^2 = 7, 2xy = 8 \Rightarrow x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{49 + 64} = \sqrt{113} [\because x, y \in R, x^2 + y^2 > x^2 - y^2 \Rightarrow x^2 + y^2 \neq -\sqrt{113}]$$

$$\Rightarrow 2x^2 = 7 + \sqrt{113} \Rightarrow x = \pm \sqrt{\frac{7+\sqrt{113}}{2}}$$

$$2y^2 = \sqrt{113} - 7 \Rightarrow y = \pm \sqrt{\frac{\sqrt{113}-7}{2}}$$

Problem 32

32. Find the square root of $a^2 - b^2 + 2iab \forall a, b \in \mathbb{R}$.

Solution of Problem 32

Solution: Let $z = x + iy = \sqrt{a^2 - b^2 + 2iab} \Rightarrow x^2 - y^2 + 2ixy = a^2 - b^2 + 2iab$

Comparing real and imaginary parts, we get

$$\Rightarrow x^2 - y^2 = a^2 - b^2, 2xy = 2ab \Rightarrow x^2 + y^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = a^2 + b^2 [\because x, y \in R, x^2 + y^2 > x^2 - y^2 \Rightarrow x^2 + y^2 \neq -(a^2 + b^2)]$$

$$\Rightarrow 2x^2 = 2a^2 \Rightarrow x = \pm a \text{ and similarly, } y = \pm b$$

Problem 33

33. Find the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i} \left(\frac{x}{y} + \frac{y}{x} \right) + \frac{31}{16}$.

Solution of Problem 34

Solution: Given equation can be rewritten as $\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2 \cdot \frac{i}{4} \left(\frac{x}{y} + \frac{y}{z} \right) + \frac{i^2}{4^2} - \frac{i^2}{4^2} + \frac{31}{16}$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2} - 2 \cdot \frac{i}{4} \left(\frac{x}{y} + \frac{y}{z} \right) + \frac{i^2}{4^2} + \frac{1}{16} + \frac{31}{16}$$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2} - 2 \cdot \frac{i}{4} \left(\frac{x}{y} + \frac{y}{z} \right) + \frac{i^2}{4^2} + 2 \cdot \frac{x}{y} \cdot \frac{y}{x}$$

$$= \left(\frac{x}{y} + \frac{y}{x} - \frac{i}{4} \right)^2$$

Therefore, square root is $\pm \left(\frac{x}{y} + \frac{y}{x} - \frac{i}{4} \right)$

Problem 35

35. Find the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} - \frac{1}{i} \left(\frac{x}{y} - \frac{y}{x} \right) - \frac{9}{4}$

Solution of Problem 35

Solution: Like previous problem the given equation can be rewritten as $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 2 \cdot \frac{i}{2} \left(\frac{x}{y} - \frac{y}{x} \right) + \frac{i^2}{2^2} - \frac{i^2}{2^2} - \frac{9}{4}$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2} + 2 \cdot \frac{i}{2} \left(\frac{x}{y} - \frac{y}{x} \right) + \frac{i^2}{2^2} - 2$$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2} + 2 \cdot \frac{i}{2} \left(\frac{x}{y} - \frac{y}{x} \right) + \frac{i^2}{2^2} - 2 \cdot \frac{x}{y} \cdot \frac{y}{x}$$

$$= \left(\frac{x}{y} - \frac{y}{x} + \frac{i}{2} \right)^2$$

Therefore, square root is $\pm \left(\frac{x}{y} - \frac{y}{x} + \frac{i}{2} \right)$

Problem 36

36. Find the square root of $x^2 + \frac{1}{x^2} + 4i\left(x - \frac{1}{x}\right) - 6$

Solution of Problem 36

Solution: Given equation can be written as $x^2 + \frac{1}{x^2} + 2.2i \left(x - \frac{1}{x}\right) + (2i)^2 - (2i)^2 - 6$

$$= x^2 + \frac{1}{x^2} + 2.2i \left(x - \frac{1}{x}\right) + 4i^2 - 2 = x^2 + \frac{1}{x^2} + 2.2i \left(x - \frac{1}{x}\right) + 4i^2 - 2.x.\frac{1}{x}$$
$$= \left(x - \frac{1}{x} + 2i\right)^2$$

Thus square root is $\pm \left(x - \frac{1}{x} + 2i\right)$

Problem 37

37. Find the minimum value of $|z| + |z - 2|$

Solution of Problem 37

Solution: We know that for two complex numbers z_1 and z_2 , $|z_1| + |z_2| \geq |z_1 - z_2|$

$$|z| + |z - 2| \geq |z - (z - 2)| = |2| = 2$$

Therefore, minimum value is 2.