Complex Numbers Problems 151-160

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Problem 151

151. Consider an equilateral triangle $A\left(\frac{2}{\sqrt{3}}e^{i\pi/2}\right)$, $B\left(\frac{2}{\sqrt{3}}e^{-i\pi/6}\right)$ and $C\left(\frac{2}{\sqrt{3}}e^{-i5\pi/6}\right)$. If $P(z)$ is any point on the incircle then find the value of $AP^2+BP^2+CP^2$.

Solution of Problem 151

Solution: $A(z_1) = \frac{2i}{\sqrt{3}}, B(z_2) = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 1 - \frac{i}{\sqrt{3}}, C(z_3) = \frac{2}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -1 - \frac{i}{\sqrt{3}}$

Clearly, the points lie on the circle $z = 2/\sqrt{3}$ and $\triangle ABC$ is equilateral and its centroid coincides with circumcentre. Hence,

$z_1 + z_2 + z_3 = 0$ and $\overline{z_1} + \overline{z_2} + \overline{z_3} = 0$ Clearly, radius of incircle $= \frac{1}{\sqrt{3}}$ hence any point on circle is $\frac{1}{\sqrt{3}}(\cos\alpha + i\sin\alpha)$.

$AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\overline{z_1} + \overline{z}z_1)$
$\Rightarrow AP^2 + BP^2 + CP^2 = 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\overline{z_1} + \overline{z_2} + \overline{z_3}) - \overline{z}(z_1 + z_2 + z_3)$

$= 3 \times \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{4}{3} - 0 - 0 = 5$

Problem 152

152. If $A_1, A_2, \ldots, A_n$ be the vertices of a regular polygon of n sides in a circle of unit radius and $a = |A_1A_2|^2 + |A_1A_3|^2 + \ldots + |A_1A_n|^2, b = |A_1A_2||A_1A_3| \ldots |A_1A_n|$, then find $\frac{a}{b}$.

Solution of Problem 152

Solution: Let O be the center of the polygon and $z_0, z_1, \ldots, z_{n-1}$ represent the vertices $A_1, A_2, \ldots, A_n$.

$\therefore z_0 = 1, z_1 = \alpha, z_2 = \alpha^2, \ldots, z_{n-1} = \alpha^{n-1}$ where $\alpha = e^{i2\pi/n}$

$|A_1A_2|^2 = |\alpha^r - 1|^2 = |1 - \alpha^r|^2 = \left|1 - \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right|^2$

$= \left(1 - \cos\frac{2r\pi}{n}\right)^2 + \sin^2\frac{2r\pi}{n} = 2 - 2\cos\frac{2r\pi}{n}$

$\sum_{r=1}^{n} |A_1A_2|^2 = 2(n-1) - 2\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{3} + \ldots + \cos\frac{2(n-1)\pi}{n}\right]$

$= 2(n-1) - 2.$ real part of $(\alpha + \alpha^2 + \ldots + \alpha^{n-1}) = 2n[\because 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0]$

$|A_1A_2||A_1A_3| \ldots |A_1A_n| = |1-\alpha||1-\alpha^2| \ldots |1-\alpha^{n-1}|$

$= |(1-\alpha)(1-\alpha^2)\ldots(1-\alpha^{n-1})|$
Since $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are roots of $z^n - 1 = 0$

$(z-1)(z-\alpha)(z-\alpha^2)\ldots(z-\alpha^{n-1}) = z^n - 1$

$\Rightarrow (z-\alpha)(z-\alpha^2)\ldots(z-\alpha^{n-1}) = \frac{z^n-1}{z-1} = 1 + z + z^2 + \ldots + z^{n-1}$

Putting $z = 1$, we get

$|(1-\alpha)(1-\alpha^2)\ldots(1-\alpha^{n-1})| = n \Rightarrow \frac{a}{b} = 2$

Problem 153

153. If $\left(1+i\frac{x}{a}\right)\left(1+i\frac{x}{b}\right)\left(1+i\frac{x}{c}\right)... = A + iB$, then prove that $\left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)... = A^2 + B^2$.

Solution of Problem 153

Solution: Let $L.H.S. = z_1$ and $R.H.S. = z_2$ then $\overline{z_1} = \overline{z_2}$

$\Rightarrow z_1\overline{z_1} = z_2\overline{z_2} \Rightarrow z_1^2 = z_2^2$

$\Rightarrow \left(1 + \frac{x^2}{a^2}\right)\left(1 + \frac{x^2}{b^2}\right)\left(1 + \frac{x^2}{c^2}\right) ... = A^2 + B^2.$

Problem 154

154. Find the range of real number α for which the equations $z + \alpha|z-1| + 2i = 0; z = x + iy$ has a solution. Also, find the solution.

Solution of Problem 154

Solution: Given, $x + iy + \alpha\sqrt{(x-1)^2 + y^2} + 2i = 0$

Equating real and imaginary parts, we get

$y + 2 = 0 \Rightarrow y = -2$ and $x + \alpha\sqrt{(x-1)^2 + y^2} = 0$

Substituting the value of y, we get $\alpha\sqrt{x^2 - 2x + 5} = -x \Rightarrow (\alpha^2 - 1)x^2 - 2\alpha^2 x + 5\alpha^2 = 0$

Because x is real, the discriminant has to be greater than zero. $\Rightarrow 4\alpha^4 - 20\alpha^2(\alpha^2 - 1) \geq 0$

$\Rightarrow \alpha^2 - 5\alpha^2 + 5 \geq 0 \Rightarrow -\frac{\sqrt{5}}{2} \leq \alpha \leq \frac{\sqrt{5}}{2}$

Problem 155

155. For every real number $a \geq 0$, find all the complex numbers satisfying the equation $2|z| - 4az + 1 + ia = 0$.

Solution of Problem 155

Solution: Let $z = x + iy \Rightarrow 2\sqrt{x^2 + y^2} - 4a(x + iy) + 1 + ia = 0$

Equating real and imaginary parts, we get

$2\sqrt{x^2 + y^2} - 4ax + 1 = 0$ and $-4ay + a = 0 \Rightarrow y = \frac{1}{4}$

$2\sqrt{x^2 + \frac{1}{16}} - 4ax + 1 = 0 \Rightarrow 4\left(x^2 + \frac{1}{16}\right) = 16a^2x^2 - 8ax + 1$

$x^2(4 - 16a^2) + 8ax - \frac{3}{4} = 0 \Rightarrow x = \frac{-a}{1-4a^2} \pm \frac{1}{4}\frac{\sqrt{4a^2+3}}{1-4a^2}$

Problem 156

156. Show that $(x^2+y^2)^5 = (x^5 - 10x^3y^2 + 5xy^4) + (5x^4y - 10x^2y^3 + y^5)^2$

Solution of Problem 156

Solution: $(x+iy)^5 = (x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)$

Taking modulus and squaring, we get

$(x^2+y^2)^5 = (x^5 - 10x^3y^2 + 5xy^4) + (5x^4y - 10x^2y^3 + y^5)^2$

Problem 157

157. Express $(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)$ as sum of two squares.

Solution of Problem 157

Solution:
$(x+ia)(x+ib)(x+ic) = [(x^2-ab)+i(a+b)x](x+ic) = (x^3-abx-acx-bcx)+i(cx^2-abc+ax^2+bx^2)$

Taking modulus and squaring, we get

$(x^2+a^2)(x^2+b^2)(x^2+c^2) = [x^3-(ab+bc+ca)x]+[(a+b+c)x^2-abc]^2$

Problem 158

158. If $(1+x)^n = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, then prove that $2^n = (a_0 - a_2 + a_4 - ...)^2 + (a_1 - a_3 + a_5 - ...)^2$.

Solution of Problem 158

Solution: Given, $(1+x)^n = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

Substituting $x = i$,we get

$(1+i)^n = a_o + ia_1 - a_2 - ia_3 + a_4 + ... = (a_0 - a_2 + a_4 - ...) + i(a_1 - a_3 + a_5 - ...)$

Taking modulus and squaring, we get

$2^n = (a_0 - a_2 + a_4 - ...)^2 + (a_1 - a_3 + a_5 - ...)^2$

Problem 159

159. Dividing $f(z)$ by $z - i$, we get i as remainder and if we divide by $z + i$, we get $1 + i$ as remainder. Find the remainder upon division of $f(z)$ by $z^2 + 1$.

Solution of Problem 159

Solution: Let $f(z) = m(z-i) + i$ and $f(z) = n(z+i) + 1 + i$ where m and n are quotients upon division.

Substituting $z = i$ in the first equation and $z = -i$ in the second we obtain $f(i) = i$ and $f(-i) = 1 + i$.

Let $g(z)$ be the quotient and $az + b$ be the remainder upong division of $f(z)$ by $z^2 + 1$. Hence we have

$f(z) = g(z)(z^2 + 1) + az + b$. Substituting $z = i$ and $z = -i$, we get

$f(i) = i = ai + b$ and $f(-i) = 1 + i = -ai + b$

Adding, we get $2b = 1 + 2i \Rightarrow b = \frac{1+2i}{2} \Rightarrow ai = i - \frac{1+2i}{2}$

Problem 160

160. If $|z| \leq 1, |w| \leq 1$, show that $|z-w|^2 \leq (|z|-|w|)^2 + [\arg(z) - \arg(w)]^2$.

Solution of Problem 160

Solution: Let $z = r_1 e^{i\theta_1}, w = r_2 e^{i\theta_2}$. $\because |z| \leq 1$ and $|w| \leq 1 \Rightarrow r_1 \leq 1$ and $r_2 \leq 1$

$|z - w|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$

$= r_1^2 + r_2^2 - 2r_2r_2 \cos(\theta_1 - \theta_2) = (r_1 - r_2)^2 + 2r_2r_2 - 2r_2r_2 \cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 4r_1r_2 \sin\left(\frac{\theta_1 - \theta_2}{2}\right)^2 \leq (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2 [\because r_1, r_2 \leq 1$ and $\sin\theta \leq \theta]$

$= (|z| - |w|)^2 + [\arg(z) - \arg(w)]^2.$