Arithmetic Progression
Problems 61 to 70

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Problem 61

61. 25 trees are planted in a straight line at intervals of 5 meters. To water them the gardener must bring water for each tree separately from a well 10 meters from the first tree. How far he will have to travel to water all the trees beginning with the first if he starts from the well.

Solution of problem 61

Solution Since the gardener starts from the well.

$$\therefore \text{Distance covered to water first tree} = 10m$$

$$\text{Distance covered to water second tree} = 10 + 15 = 25m$$

$$\text{Distance covered to water third tree} = 15 + 20 = 35m$$

$$\text{Distance covered to water third tree} = 20 + 25 = 45m$$

$\therefore$ Total distance covered to water all trees

$$= 10 + 25 + 35 + 45 + \ldots \text{to } 25 \text{ terms}$$

$$= 10 + (25 + 35 + 45 + \ldots \text{to } 24 \text{ terms})$$

$$= 10 + \frac{24}{2}[2.25 + (24 - 1)10]$$

$$= 10 + 12.280 = 10 + 3360 = 3370m$$

Problem 62

62. If a be the first term of an A.P. and the sum of its first p terms is equal to zero, show that the sum of the next q terms is $-\frac{a(p+q)}{p-1}.q$

Solution of problem 62

Solution: Let d be the common difference and S_p be the sum of first p terms. Given,

$$S_p = \frac{p}{2}[2a + (p-1)d] = 0$$

$$\because p \neq 0, \therefore 2a + (p-1)d = 0$$

$$\therefore d = -\frac{2a}{p-1}$$

Now,

$$\begin{aligned}
\text{Sum of next } q \text{ terms} &= \text{Sum of first } (p+q) \text{ terms} - \text{ Sum of first } p \text{ terms} \\
&= S_{p+q} - S_p = S_{p+q} - 0 \\
&= \frac{p+q}{2}[2a + (p+q-1)d] \\
&= \frac{p+q}{2}\left[2a + (p+q-1).\left(-\frac{2a}{p-1}\right)\right] \\
&= \frac{p+q}{2}2a\left(1 - \frac{p+1-1}{p-1}\right) \\
&= -\frac{a(p+q)}{p-1}.q
\end{aligned}$$

Problem 63

63. The sum of the first p terms of an A.P. is equal to the sum of its first q terms, prove that the sum of its first $(p+q)$ terms is zero.

Solution of problem 63

Solution: Let the first term be a and common difference be d. Now given,

$$S_p = S_q$$

$$\frac{p}{2}[2a + (p-1)d] = \frac{q}{2}[2a + (q-1)d]$$

$$2ap + p(p-1)d = 2aq + q(q-1)d$$

$$2a(p-q) + (p^2 - p - q^2 + q)d = 0$$

$$2a(p-q) + [(p^2 - q^2) - (p-q)]d = 0$$

$$2a + (p+q-1)d = 0$$

$$S_{p+q} = \frac{p+q}{2}[2a + (p+q-1)d] = 0$$

Problem 64

64. Prove that the sum of latter half of $2n$ terms of a series in A.P. is equal to the one third of the sum of first $3n$ terms.

Solution of problem 64

Solution: Sum of latter half of $2n$ terms $= S_{2n} - S_n$. Let a be the first term and d be the common difference. Then,

$$\begin{aligned} S_{2n} - S_n &= \frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d] \\ &= \frac{n}{2}[4a + 2(2n-1)d - 2a - (n-1)d] \\ &= \frac{n}{2}[2a + (3n-1)d] \\ &= \frac{1}{3}\frac{3n}{2}[2a + (3n-1)d] \\ &= \frac{1}{3}S_{3n} \end{aligned}$$

Problem 65

65. If $S_1, S_2, S_3, \ldots, S_p$ be the sum of n terms of arithmetic progressions whose first terms are respectively $1, 2, 3, \ldots$ and common differences are $1, 2, 3, \ldots$ prove that

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{np}{4}(n+1)(p+1)$$

Solution of problem 65

Solution:

$$S_1 = \frac{n}{2}[2.1 + (n-1).1] = \frac{n(n+1)}{2}.1$$

$$S_2 = \frac{n}{2}[2.2 + (n-1).2] = \frac{n(n+1)}{2}.2$$

$$S_3 = \frac{n}{2}[2.3 + (n-1).3] = \frac{n(n+1)}{2}.3$$

$$\cdots$$

$$S_p = \frac{n}{2}[2.p + (n-1).p] = \frac{n(n+1)}{2}.p$$

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{n(n+1)}{2}[1 + 2 + 3 + \ldots + p]$$

$$S_1 + S_2 + S_3 + \ldots + S_p = \frac{n(n+1)}{2}\frac{p(p+1)}{2} = \frac{np}{4}(n+1)(p+1)$$

Problem 66

66. If a, b and c be the sum of p, q and r terms rspectively of an A.P., prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

Solution of problem 66

Solution: Let x be the first term and d be the common difference of the A.P. Given,

$$a = \frac{p}{2}[2x + (p-1)d] \Rightarrow \frac{a}{p} = x + \frac{p-1}{2}d \tag{1}$$

$$b = \frac{q}{2}[2x + (q-1)d] \Rightarrow \frac{b}{q} = x + \frac{q-1}{2}d \tag{2}$$

$$c = \frac{r}{2}[2x + (r-1)d] \Rightarrow \frac{c}{r} = x + \frac{r-1}{2}d \tag{3}$$

Multiplying (1) by $(q-r)$, (2) by $(r-p)$ and (3) by $(p-q)$ and adding,

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = \left(x + \frac{q-1}{d}\right)(q-r) + \left(x + \frac{q-1}{2}\right)(r-p) + \left(x + \frac{r-1}{2}\right)(p-q)$$

$$= x[q-r+r-p+p-q] + \frac{d}{2}[(q-r)(p-1) + (r-p)(q-q) + (p-q)(r-1)]$$

$$= 0$$

Problem 67

67. If the sum of m terms of an A.P. is equal to half the sum of $(m+n)$ terms and is also equal to half the sum of $(m+p)$ terms, prove that $(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$

Solution of problem 67

Solution: Let a be the first term and d be the common difference of the A.P.
According to question, $S_m = \frac{1}{2} S_{m+n}$

$$\frac{m}{2}[2a + (m-1)d] = \frac{m+n}{2}[2a + (m+n-1)d] \tag{1}$$

Let $2a + (m-1)d = x$, then from (1) we have

$$2mx = (m+n)(x+nd) \Rightarrow 2mx - mx - nx = (m+n)nd$$

$$x(m-n) = (m+n)nd \tag{2}$$

Also,

$$S_m = \frac{1}{2} S_{m+p}$$

Replacing n by p in (2)

$$x(m-p) = (m+p)pd \tag{3}$$

Dividing (2) by (3), we get

$$\frac{m-n}{m-p} = \frac{(m+n)n}{(m+p)p}$$

$$(m+n)(m-p)n = (m+p)(m-n)p$$

Dividing both sides by mnp

$$(m+n)\left(\frac{1}{p} - \frac{1}{m} = (m+p)\left(\frac{1}{n} - \frac{1}{m}\right)\right)$$

$$(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

Problem 68

68. If there are $(2n+1)$ terms in an A.P., then prove that the ratio of sum of odd terms and the sum of even terms is $n+1 : n$.

Solution of problem 68

Solution: Let the A.P. be $a, a+d, a+2d, \ldots, a+2nd$
Sum of its odd terms $= a + (a+2d) + (a+4d) + \ldots + (a+2nd)$

$$= \frac{n+1}{2}[2a + (n+1-1).2d] = (n+1)(a+nd)$$

Sum of even terms $= (a+d) + (a+3d) + \ldots + (a+(2n-1)d)$

$$= \frac{n}{2}[2(a+d) + (n-1)d] = n(a+nd)$$

$$\therefore \frac{\text{Sum of odd terms}}{\text{Sum of even terms}} = \frac{n+1}{n}$$

Problem 69

69. The sum of n terms of two series in A.P. are in the ration $(3n - 13) : (5n + 21)$. Find the ratio of their 24th terms.

Solution of problem 69

Solution: Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of the two series in A.P.
Now, ratio of 24th terms

$$= \frac{a_1 + 23d_1}{a_2 + 23d_2} \tag{1}$$

Ratio of sum of n terms

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{3n-13}{5n+21}$$

$$\frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{3n-13}{5n+21}$$

Putting $n = 47$, we get

$$\frac{2a_1 + 46d_1}{2a_2 + 46d_2} = \frac{3.47-13}{5.47+21}$$

$$\frac{a_1 + 23d_1}{a_2 + 23d_2} = \frac{1}{2}$$

Therefore, from (1) we get the ratio of 24th terms as $\frac{1}{2}$

Problem 70

70. If the mth term of an A.P. is $\frac{1}{n}$ and nth term of an A.P. is $\frac{1}{m}$ then prove that the sum to mn terms is $\frac{mn+1}{2}$

Solution of problem 70

Solution: Let a be the first term and d be the common difference of the A.P.

$$t_m = a + (m-1)d = \frac{1}{n} \tag{1}$$

$$t_n = a + (n-1)d = \frac{1}{m} \tag{2}$$

Subtracting (1) from (2), we get

$$(n-m)d = \frac{1}{m} - \frac{1}{n} = \frac{n-m}{mn}$$

$$d = \frac{1}{mn}$$

Substituting the value of d in (1), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n}$$

$$a = \frac{1}{n}(1 - \frac{m-1}{m}) = \frac{1}{mn}$$

$$S_{mn} = \frac{mn}{2}[2a + (mn-1)d]$$

Substituting the value of d, we get

$$= \frac{mn}{2}\left(\frac{2}{mn} + (mn-1)\frac{1}{mn}\right)$$

$$= \frac{mn+1}{2}$$