Complex Numbers

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Theory

A complex number comprises of two numbers: a real number and an imaginary number. An imaginary number is square root of a negative number, for example, $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}$. These are called imaginary numbers because they do not exist in real life in the sense that like ordinary numbers they cannot be used for counting.

A real number like 1 can also be represented as a complex number having a 0 imaginary part. The value $\sqrt{-1}$ is denoted by the Greek letter ι, which stands for *iota*. Typically, we use either i or j to denote this.

Clearly we have following:

$$i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1, ...$$

If you examine carefully you will find that following holds true

$$i^{4m} = 1, i^{4m+1} = i, i^{4m+2} = -1 \text{ and } i^{4m+3} = -i \; \forall \; m \in P$$

P is the set of positive integers including zero.
Note: $1 = \sqrt{1} = \sqrt{-1 * -1} = i * i = -1$
However, the above result is wrong because for any two real numbers a and b the result $\sqrt{a} * \sqrt{b} = \sqrt{ab}$ holds good if and only if the two numbers are zero or positive. Thus $1 = \sqrt{-1 * -1}$ is wrong because power of $-$ is -1 which makes the set of equalities go wrong.

Definitions

A complex number is commonly written as $a + ib$ or $x + iy$. Here a, b, x and y are all real numbers. The complex number itself is denoted by z, like $z = x + iy$. Here x is called the *real* part and is also denote by $Re(z)$ and y is called the imaginary part and is also denoted by $Im(z)$.

A complex number is purely real if its imaginary part or y or $Im(z)$ is zero. Similarly, a complex number is purely imaginary if its real part or x or $Re(z)$ is zero. Clearly, as you can fathom that there can exist only one number which has both the parts as zero and certainly that is 0. That is, $0 = 0 + i0$.

The set of all complex number is typically denoted by C. Two complex numbers z_1 and z_2 are said to be true if there real parts are equal and imaginary parts are equal. That is if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then for $z_1 1$ to be equal to z_2, x_1 must be equal to x_2 and y_1 must be equal to y_2.

Simple Operations

1. **Addition:** $(a+ib)+(c+id)=(a+c)+i(b+d)$
2. **Subtraction:** $(a+ib)-(c+id)=(a-c)+i(b-d)$
3. **Multiplication:** $(a+ib)*(c+id)=ac+ibc+iad+bdi^2=(ac-bd)+i(bc+ad)$
4. **Division:** $\frac{a+ib}{c+id}=\frac{a+ib}{c+id}.\frac{c-id}{c-id}=\frac{ac+bd+i(bc+ad)}{c^2+d^2}$

Conjugate of a Complex Number

Let $z=x+iy$ be a complex number then its complex conjugate is a number with imaginary part made negative and it is written as $\overline{z}=x-iy$. $\overline{z}$ is the typical representation for a conjugate of a complex number z.

Properties of Conjugates

1. $z_1=z_2 \Leftrightarrow \overline{z_1}=\overline{z_2}$
   Clearly as we know for two complex numbers to be equal both parts must be equal so this is very easy to understand that if $x_1=x_2$ and $y_1=y_2$ then this bidirectional condition is always satisfied.
2. $\overline{(\overline{z})}=z$.
   $z=x+iy$, hence, $\overline{z}=x-iy$, hence $\overline{(\overline{z})}=x-(-iy)=x+iy=z$
3. $z+\overline{z}=2Re(x)$
   Clearly, $z+\overline{z}=x+iy+x-iy=2x=2Re(x)$
4. $z-\overline{z}=2iIm(x)$
   Clearly, $z-\overline{z}=x+iy-(x-iy)=2iy=2iIm(x)$

Conjugate contd.

5. $z + \overline{z} = 0 \Leftrightarrow z$ is purely imaginary.
   $z + \overline{z} = x + iy + x - iy = 2x = 0$ which means rela part is zero and hence z is purely imaginary.
6. $z = \overline{z} \Leftrightarrow z$ is purely real.
   $x + iy = x - iy \Rightarrow 2iy = 0$ and thus z is purely real.
7. $z\overline{z} = [x^2 + y^2]$
   Clearly, $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$
8. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_1}\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$
   $= (x_1 + x_2) - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{z_1} + \overline{z_2}$
9. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
   It can be proven like item 8.
10. $\overline{z_1 z_2} = \overline{z_1} - \overline{z_2}$
    It can be proven like item 8.
11. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$ You can rationalize the base by multiplying it from its conjugate and apply division formula given above to prove it.
12. If $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$. where $s_0, a_1, \dots, a_n$ and z are complex numbers, then
    $\overline{P(z)} = \overline{a_0} + \overline{a_1 z} + \overline{a_2}(\overline{z})^2 + ... + \overline{a_n}\overline{(z)^n} = \overline{P}(\overline{z})$ where
    $\overline{P}(z) = \overline{a_0} + \overline{a_1} z + \overline{a_2} z^2 + ... + \overline{a_n} z^n$

Conjugate contd.

13. If $R(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomilas in z, and $Q(z) \neq 0$, then
$\overline{R(z)} = \frac{\overline{P}(\overline{z})}{\overline{Q}(\overline{z})}$
14. If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\overline{z} = \begin{vmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{vmatrix}$ where $a_i, b_i, c_i (i = 1, 2, 3)$ are complex numbers.

Modulus of a Complex Number

Modulus of a complex numbe z is denoted by $|z|$ and is equalt to the real number $\sqrt{x^2 + y^2}$. Note that $|z| \geq 0 \; \forall \; z \in C$

Properties of Modulus

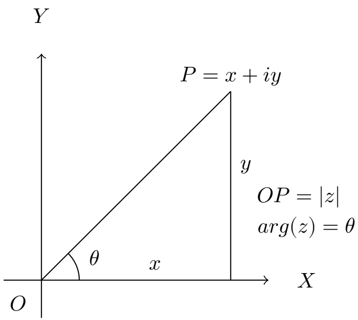
1. $|z| = 0 \Leftrightarrow z = 0.$
$x^2 + y^2 = 0 \Leftrightarrow x = 0, y = 0 \Rightarrow z = 0$
2. $|z| = |\overline{z}| = |-z| = |-\overline{z}| = x^2 + y^2$
3. $-|z| \leq Re(x) \leq |z|$ Clearly, $-(x^2 + y^2) \leq x^2 \leq (x^2 + y^2)$
4. $-|z| \leq Im(x) \leq |z|$ Clearly, $-(x^2 + y^2) \leq y^2 \leq (x^2 + y^2)$
5. $z\overline{z} = |z|^2$ Clearly, $(x + iy)(x - iy) = (x^2 + y^2) = |z|^2$
6. $|z_1 z_2| = |z_1||z_2|$ Clearly, $|z_1 z_2| = |x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1))|$
$= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = |z_1||z_2|$

Modulus contd.

13. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{z_2}$, if $z_2 \neq 0$
14. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \overline{z_1}z_2 + z_1\overline{z_2} = |z_1|^2 + |z_2|^2 + 2Re(z_1\overline{z_2})$
15. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \overline{z_1}z_2 - z_1\overline{z_2} = |z_1|^2 + |z_2|^2 - 2Re(z_1\overline{z_2})$
16. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
17. If a amd b are real numbers and z_1 and z_2 are complex numbers, then $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$
18. If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.
19. If z_1 and z_2 are complex numbers then $|z_1 + z_2| \leq |z_2| + |z_2|$. This expression can be generalized to n terms as well.
20. Simialrly, these can be proven that $|z_1 - z_2| \leq |z_1| + |z_2|, |z_1| - |z_2| \leq |z_1| + |z_2|$ and $|z_1 - z_2| \geq ||z_1| - |z_2||$

Theory contd

A complex number z which we have considered to be equal to $x + iy$ can be represented by a point P whose caretesian coordinates are (x, y) referred to rectangular axes Ox and Oy where O is origin i.e. $(0, 0)$ and are called *real* and *imaginary* axis respectively. The xy two dimensional plane is also called *Argand plane, complex plane or Gaussian plane*. The point P is also called the *image* of the complex number and z is also called the *affix* or *complex coordinate* of point P.
The modulus is given by the length of segment OP which is equal to $OP = \sqrt{x^2 + y^2} = |z|$. Thus, $|z|$ is the length of OP.



In the diagram θ is known as the argument of z. it is the angle made with positive direction(i.e. counter-clockwise) of real axis. This arhument is not unique. If θ is an argument of a complex number z then $2n\pi + \theta$ where $n \in I$ where I is the set of integers will be arguments as well. The value of argument for which $-\pi < \theta \leq \pi$ is called the *principal argument*.

Different Arguments of a Complex Number

In the digram given in previous slide the argument is given as

$$arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

this value is for when z is in first quadrant. When z will lie in second, third and fourth quadrants then arguments will be

$$arg(z) = \pi - \tan^{-1}\left(\frac{y}{|x|}\right), arg(z) = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right), arg(z) = -\tan^{-1}\left(\frac{|y|}{x}\right)$$

Polar Form of a Complex Number

If z is a non-zero complex number, then we can write $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = arg(z)$

In this case z is also given by $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ where $n \in I$.

Euler's Formula

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$.

Properties of Arguments

If z, z_1 and z_2 are complex numbers then

1. $arg(\overline{z}) = -arg(z)$. This can be easily proven as $z = x + iy$ and $\overline{z} = x - iy$ so sign of argument will get a -ve sign as y gets one.
2. $arg(z_1 z_2) = arg(z_1) + arg(z_2) + 2k\pi$ where

$$k = \begin{cases} 0 & -\pi < arg(z_1) + arg(z_2) \leq \pi \\ 1 & -2\pi < arg(z_1) + arg(z_2) \leq -\pi \\ -1 & -\pi < arg(z_1) + arg(z_2) \leq 2\pi \end{cases}$$

3. $arg(z_1 \overline{z_2}) = arg(z_1) - arg(z_2)$
4. $\arg\left(\frac{z_1}{z_2}\right) = arg(z_1) + arg(z_2) + 2k\pi$ where k is same as item 2 with $+$ sign between z_1 and z_2 are replaced with $-$ sign.
5. $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow arg(z_1) - arg(z_2) = \pi/2$
6. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow arg(z_1) = arg(z_2)$
7. $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$
8. $|z_1 - z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 + \theta_2)$

Vector Representation

Complex numbers can also be represented as vectors. Length of the vector is nothing but modulus of complex number and argument is the angle which the vector makes with the real axis. It is denoted as $\overrightarrow{OP}$ where OP represents the vector of the complex number z.