

Miscellaneous Problems on A.P., G.P. and H.P. Problems 11-20

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Problem 11

11. In an A.P. of $2n$ terms the middle pair of terms are $p + q$ and $p - q$. Show that the sum of cubes of the terms in A.P. are $2np[p^2 + (4n^2 - 1)q^2]$

Solution of Problem 11

Solution: Let t_r denote the r th term of the A.P.

Given, $t_n = p + q$ and $t_{n+1} = p - q \therefore d = -2q$

Also, $t_1 + t_{2n+1} = t_2 + t_{2n} = \dots = t_n + t_{n+1} = 2p$

Let S be the sum of cubes of the terms of A.P., then $S = (t_1^3 + t_{2n}^3) + (t_2^3 + t_{2n-1}^3) + \dots + (t_n^3 + t_{n+1}^3)$

$$\begin{aligned} t_1^3 + t_{2n}^3 &= (t_1 + t_{2n})^3 - 3t_1 t_{2n} (t_1 + t_{2n}) = 8p^3 - 6pt_1 t_{2n} = 8p^3 - \frac{6p}{4} [(t_1 + t_{2n})^2 - (t_1 - t_{2n})^2] \\ &= 8p^3 - \frac{3p}{2} [4p^2 - (2n-1)^2 \cdot 4q^2] [\because t_{2n} = t_1 + (2n-1)d \text{ and } d = -2q] \\ &= 2p^3 + 6pq^2(2n-1)^2 \end{aligned}$$

Similarly

$$t_2^3 + t_{2n-1}^3 = 2p^3 + 6pq^2(2n-3)^2, t_3^3 + t_{2n-2}^3 = 2p^3 + 6pq^2(2n-5)^2, \dots, t_n^3 + t_{n+1}^3 = 2p^3 + 6pq^2 \cdot 1^2$$

Adding all these, we get

$$\begin{aligned} S &= 2np^3 + 6pq^2[1^2 + 3^2 + 5^2 + \dots \text{ to } n \text{ terms}] \\ &= 2np[p^2 + (4n^2 - 1)q^2] \end{aligned}$$

Problem 12

12. Find the sum S_n of the cubes of the first n terms of an A.P. and show that the sum of the first n terms of the A.P. is a factor of S_n .

Solution of Problem 12

Solution: Let S be the sum of first n terms of the A.P. $a, a + d, a + 2d, \dots$ then $S = \frac{n}{2}[2a + (n - 1)d]$

$$\begin{aligned}S_n &= a^3 + (a + d)^3 + (a + 2d)^3 + \dots + [a + (n - 1)d]^3 \\&= na^3 + 3a^2d[1 + 2 + 3 + \dots + (n - 1)] + 3ad^2[1^2 + 2^2 + \dots + (n - 1)^2] + d^3[1^3 + 2^3 + \dots + (n - 1)^3] \\&= na^3 + 3a^2d \cdot \frac{n(n - 1)}{2} + 3ad^2 \cdot \frac{(n - 1) \cdot n \cdot (2n - 1)}{6} + d^3 \frac{n^2(n - 1)^2}{4} \\&= \frac{n}{2} \left(2a^3 + 3(n - 1)a^2d + (n - 1)(2n - 1)ad^2 + \frac{1}{2}n(n - 1)^2d^3 \right) \\&= \frac{n}{2} \left[a^2(2a + (n - 1)d) + (n - 1)ad(2a + (n - 1)d) + \frac{n(n - 1)}{2}d^2(2a + (n - 1)d) \right] \\&= \frac{n}{2}[2a + (n - 1)d] \left[a^2 + (n - 1)ad + \frac{n(n - 1)}{2}d^2 \right] \\&= S \left[a^2 + (n - 1)ad + \frac{n(n - 1)}{2}d^2 \right]\end{aligned}$$

Hence, S is a factor of S_n

Problem 13

13. Show that any positive integral power (greater than 1) of a positive integer m , is the sum of m consecutive odd positive integers. Find the first odd integer for m^r ($r > 1$)

Solution of Problem 13

Solution: Let r be a positive integer and $r > 1$.

Let $m^r = (2k + 1) + (2k + 3) + \dots + (2k + 2m - 1)$

$$m^r = \frac{m}{2}[4k + 2 + (m - 1)2] \Rightarrow m^{r-1} = 2k + m \Rightarrow k = \frac{m^{r-1} - m}{2}$$

Clearly for $r > 1$, m^{r-1} and m are both odd or both even. $\therefore m^{r-1} - m$ is an even number. Thus, such integer k exists.

First off interger = $2k + 1 = m^{r-1} - m + 1$

Problem 14

14. If a be the sum of n terms and b^2 the sum of the square of n terms of an A.P., find the first term and common difference of the A.P.

Solution of Problem 14

14. Let $x = x_1$ be the first term and d be the common difference. Then,

$$x + (x + d) + \dots + [x + (n - 1)d] = a \Rightarrow nx + \frac{d \cdot (n - 1)n}{2} = a$$

Squaring both sides of the above equation

$$nx^2 + \frac{d^2(n - 1)^2n}{4} + n(n - 1)xd = \frac{a^2}{n}$$

Also,

$$\begin{aligned} x^2 + (x + d)^2 + \dots + [x + (n - 1)d]^2 &= b^2 \\ \Rightarrow nx^2 + d^2[1^2 + 2^2 + \dots + (n - 1)^2] + 2xd[1 + 2 + 3 + \dots + (n - 1)] &= b^2 \\ \Rightarrow nx^2 + d^2 \frac{(n - 1)n(2n - 1)}{6} + 2xd \frac{n(n - 1)}{2} &= b^2 \end{aligned}$$

Subtracting the two obtained equations we get

$$\begin{aligned} d^2 \frac{n(n - 1)(n + 1)}{12} &= \frac{nb^2 - a^2}{n} \Rightarrow d = \pm \frac{2\sqrt{3(nb^2 - a^2)}}{n\sqrt{n^2 - 1}} \\ \Rightarrow x &= \frac{1}{n} \left[a \mp \frac{-(n - 1)\sqrt{3(nb^2 - a^2)}}{\sqrt{n^2 - 1}} \right] \end{aligned}$$

Problem 15

15. If a_1, a_2, \dots, a_n are in A.P, whose common difference is d , then find the sum of the series

$$\sin d [\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \dots + \csc a_{n-1} \csc a_n]$$

Solution of Problem 15

Solution:

$$t_1 = \sin d(\csc a_1 \csc a_2) = \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} = \cot a_1 - \cot a_2$$

$$t_2 = \cot a_2 - \cot a_3$$

...

$$t_{n-1} = \cot a_{n-1} - \cot a_n$$

Adding, we get $\sin d[\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \dots + \csc a_{n-1} \csc a_n] = \cot a_1 - \cot a_n$

Problem 16

16. If a_1, a_2, \dots, a_n are in A.P. where $a_i > 0 \forall i$, show that

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

Solution of Problem 16

Solution:

$$t_1 = \frac{1}{\sqrt{a_1} + \sqrt{a_2}} = \frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} = \frac{1}{d}(\sqrt{a_2} - \sqrt{a_1})$$

$$t_2 = \frac{1}{d}(\sqrt{a_3} - \sqrt{a_2})$$

...

$$t_{n-1} = \frac{1}{d}(\sqrt{a_n} - \sqrt{a_{n-1}})$$

Adding, we get

$$S = \frac{1}{d}(\sqrt{a_n} - \sqrt{a_1}) = \frac{1}{d} \frac{a_n - a_1}{\sqrt{a_1} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

Problem 17

17. If a_1, a_2, \dots, a_n are in A.P., whose common difference is d show that $\sum_2^n \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \tan^{-1} \frac{a_n - a_1}{1+a_n a_1}$

Solution of Problem 17

Solution: We have to prove that $\tan^{-1} \frac{d}{1+a_1a_2} + \tan^{-1} \frac{d}{1+a_2a_3} + \dots + \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \tan^{-1} \frac{a_n - a_1}{1+a_na_1}$

$$t_1 = \tan^{-1} \frac{d}{1+a_1a_2} = \tan^{-1} \frac{a_2 - a_1}{1+a_1a_2} = \tan^{-1} a_2 - \tan^{-1} a_1$$

$$t_2 = \tan^{-1} \frac{d}{1+a_2a_3} = \tan^{-1} a_3 - \tan^{-1} a_2$$

...

$$t_{n-1} = \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \tan^{-1} a_n - \tan^{-1} a_{n-1}$$

Adding, we get

$$\tan^{-1} \frac{d}{1+a_1a_2} + \tan^{-1} \frac{d}{1+a_2a_3} + \dots + \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \tan^{-1} a_n - \tan^{-1} a_1 = \tan^{-1} \frac{a_n - a_1}{1+a_na_1}$$

Problem 18

18. If a_1, a_2, \dots, a_n are the first n items of an A.P. with first term a and common difference d such that $ad > 0$. Let $S_n = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_{n-1} a_n}$. Prove that the product $a_1 a_n S_n$ does not depend on a or d .

Solution of Problem 18

Solution:

$$\frac{1}{a_1 a_2} = \frac{a_2 - a_1}{d a_1 a_2} = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_2} \right)$$

$$\frac{1}{a_2 a_3} = \frac{1}{d} \left(\frac{1}{a_2} - \frac{1}{a_3} \right)$$

...

$$\frac{1}{a_{n-1} a_n} = \frac{1}{d} \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right)$$

Adding, we get

$$S_n = \frac{1}{d} \left(\frac{1}{a_1} - \frac{1}{a_n} \right) = \frac{n-1}{a_1 a_n}$$

$\therefore a_1 a_n S_n = n - 1$ which is independent of a and d .

Problem 19

19. If $a_1, a_2, \dots, a_n, a_{n+1}, \dots$ be in A.P., whose common difference is d and $S_1 = a_1 + a_2 + \dots + a_n$,
 $S_2 = a_{n+1} + \dots + a_{2n}$, $S_3 = a_{2n+1} + \dots + a_{3n}$ Show that S_1, S_2, S_3, \dots are in A.P. whose common difference is $n^2 d$

Solution of Problem 19

Solution:

$$S_2 - S_1 = \frac{n}{2}[2a_{n+1} + (n-1)d - 2a_1 - (n-1)d] = \frac{n}{2}[2(a_1 + nd) - 2a_1] = n^2d$$

Similarly,

$$S_3 - S_2 = S_4 - S_3 = \dots = n^2d$$

Problem 20

20. If a, b, c are three terms of an A.P. such that $a \neq b$, show that $(b - c)/(a - b)$ is a rational number.

Solution of Problem 20

Solution: $b = \frac{a+c}{2} \therefore (b-c)/(a-b) = 1$