Summation of Series
Problems 11-20

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Problem 11

11. Find the sum of n terms of the series $1 + 5 + 11 + 19 + \ldots$

Solution of Problem 11

Solution: Let

$$S_n = 1 + 5 + 11 + 19 + ... + t_n$$

$$S_n = 1 + 5 + 11 + 19 + ... + t_{n-1} + t_n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = 1 + [4 + 6 + 8 + ... \text{to } n - 1 \text{ terms}] - t_n$$

$$t_n = 1 + \frac{n-1}{2}[4 + (n-2)2] = n^2 + n - 1$$

$$S_n = \sum t_n = \sum n^2 + \sum n - \sum 1$$

$$= \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n\right]$$

$$= \frac{n(n^2 + 3n - 1)}{3}$$

Problem 12

12. A sum is distributed among certain number of persons. Second person gets one rupee more than the first, third person gets two rupees more than the second, fourth person gets three rupees more than the third and so on. If the first person gets one rupee and the last person get 67 rupees, find the number of persons.

Solution of Problem 12

Solution: Given, first person gets 1 rupees, second person gets $1+1=2$ rupees, third person gets $2+2=4$ rupees, fourth person gets $4+3=7$ rupees and so on.
Let there be n persons and t_n be the amount last person gets which is given as 67 rupees.

$$S_n = 1+2+4+7+\ldots+t_n$$

$$S_n = 1+2+4+7+\ldots+t_n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = 1 + [1+2+3+\ldots \text{to } n-1 \text{ terms}] - t_n$$

$$t_n = 1 + \frac{n(n-1)}{2} = \frac{n^2-n+2}{2} = 67$$

$$n^2 - n + 132 = 0 \Rightarrow (n-12)(n+11) = 0 \Rightarrow n = 12, -11$$

But no. of persons cannot be negative, therefore no. of persons is 12.

Problem 13

13. Natural numbers have been grouped in the following way $1, (2, 3), (4, 5, 6), (7, 8, 9, 10), ...$ Show that the sum of the numbers in the nth group is $\frac{n(n^2+1)}{2}$.

Solution of Problem 13

Solution: Since 1st group contains 1 number, 2nd group contains 2 numbers, 3rd group contains 3 numbers, and so on, therefore, nth group will contain n numbers.
Here, we observe that numbers in each group are in A.P. whose c.d. is $1.(n-1)$th group will have $(n-1)$ numbers. Total numbers of numbers till the end of $(n-1)$th group is

$$N = 1 + 2 + 3 + \ldots \text{ for } (n-1) \text{ terms}$$
$$= \frac{n(n-1)}{2}$$

Thus, first term of nth group will be $N+1$ i.e $\frac{n^2-n+2}{2}$
Thus, sum of numbers in nth group is

$$S = \frac{n}{2}\left(2.\frac{n^2-n+2}{2} + (n-1).1\right) = \frac{n(n^2+1)}{2}$$

Problem 14

14. Find $1 + 3 + 7 + 15 + ...$ to n terms.

Solution of Problem 14

Solution:

$$S = 1 + 3 + 7 + 15 + ... + t_n$$

$$S = 1 + 3 + 7 + 15 + ... + t_{n-1} + t_n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper series and so on, we get

$$0 = [1 + 2 + 4 + 8 + ... \text{ to } n \text{ terms}] - t_n$$

$$t_n = 2^n - 1$$

$$S = \sum t_n = [2^1 + 2^2 + ... + 2^n] - \sum 1 = 2(2^n - 1) - n = 2^{n+1} - 2 - n$$

Problem 15

15. Find $1 + 2x + 3x^2 + 4x^3 + ...$ to n terms.

Solution of Problem 15

Solution:

$$S_n = 1 + 2x + 3x^2 + ... + n.x^{n-1}$$

$$xS_n = x + 2x^2 + 3xs + ... + (n-1)x^{n-1} + n.x^n$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper

$$(1-x)S_n = [1 + x + x^2 + ... \text{ to } n \text{ terms}] - nx^n$$

$$S_n = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}$$

Problem 16

16. Find $1 + 2.2 + 3.2^2 + 4.3^3 + ... + 100.2^{99}$

Solution of Problem 16

Solution:

$$S = 1 + 2.2 + 3.2^2 + 4.3^3 + ... + 100.2^{99}$$

$$2S = 1.2 + 2.2^2 + 3.2^3 + ... + 99.2^{99} + 100.2^{100}$$

Subtracting first term of lower series with second term of upper series, second term of lower series from third term of upper

$$-S = 1 + 2 + 2^2 + 2^3 + ... + 2^{99} - 100.2^{100}$$

$$S = 100.2^{100} - (2^{100} - 1)$$

$$= 99.2^{100} + 1$$

Problem 17

17. find $1 + 2^2x + 3^2x^2 + 4^2x^4 + \ldots$ to $\infty, |x| < 1$

Solution of Problem 17

Solution:

$$S = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots \text{ to } \infty$$

$$xS = x + 2^2x^2 + 3^2x^3 + \dots \text{ to } \infty$$

$$(1-x)S = 1 + 3x + 5x^2 + 7x^3 + \dots \text{ to } \infty$$

Repeating the process again

$$(1-x)^2S = 1 + 2x(1 + x + x^2 + \dots \text{ to } \infty)$$

$$S = \frac{1 + \frac{2x}{1-x}}{(1-x)^2} = \frac{1+x}{(1-x)^3}$$

Problem 18

18. If the sum of n terms of a sequence be $2n^2 + 4,$ find its nth term. Is this sequence in A.P.?

Solution of Problem 18

Solution:

$$S_n = 2n^2 + 4 \Rightarrow S_{n-1} = 2(n-1)^2 + 4 = 2n^2 - 4n + 6$$

$$t_n = S_n - S_{n-1} = 4n - 6$$

$$\Rightarrow t_{n-1} = 4n - 2$$

$$d = t_n - t_{n-1} = 4$$

Since common difference is a constant the sequence is in A.P.

Problem 19

19. Find the sum of n terms of the series whose nth term is $n(n-1)(n+1)$

Solution of Problem 19

Solution:

$$t_n = n(n-1)(n+1) = n^3 - n$$

$$S_n = \sum n^3 - \sum n = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - 1\right]$$

$$= \frac{n(n+1)(n^2+n-2)}{4}$$

Problem 20

20. Find the sum of 80 terms of the series whose nth term is $n(n^2 - 1)$

Solution of Problem 20

Solution: The nth term is same as problem 19. Thus,

$$S_{80} = \frac{80.81.(80^2 + 80 - 2)}{4} = 10494360$$