Complex Numbers Problems 221-230

Shiv Shankar Dayal

February 8, 2024

Problem 221

221. Find the common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$.

Solution of Problem 221

Solution: $z^3 + 2z^2 + 2z + 1 = 0 \Rightarrow (z+1)(z^2 + z + 1) = 0 \Rightarrow z = -1, \omega, \omega^2.$

If $z = -1, z^{1985} + z^{100} + 1 = -1 + 1 + 1 = 1 \neq 0$, if $z = \omega, z^{1985} + z^{100} + 1 = \omega^2 + \omega + 1 = 0$ and if $z = \omega^2, z^{1985} + z^{100} + 1 = \omega + \omega^2 + 1 = 0.$

Hence ω and ω^2 are the common roots.

Problem 222

222. If $z_1 + z_2 + z_3 = \alpha, z_1 + z_2\omega + z_3\omega^2 = \beta$ and $z_1 + z_2\omega^2 + z_3\omega = \gamma$, express z_1, z_2, z_3 in terms of α, β, γ. Hence prove that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$.

Solution of Problem 222

Solution: Adding all equations $\alpha+\beta+\gamma=3z_1 \Rightarrow z_1=\frac{\alpha+\beta+\gamma}{3}$. Similarly, multiplying second equatin with ω and third equation with ω^2, and then adding we have $z_3=\frac{\alpha+\beta\omega+\gamma\omega^2}{3}$. Similarly, $z_2=\frac{\alpha+\beta\omega^2+\gamma\omega}{3}$.

$|\alpha|^2=\alpha\overline{\alpha}=(z_1+z_2+z_3)(\overline{z_1}+\overline{z_2}+\overline{z_3}), |\beta|^2=\beta\overline{\beta}=(z_1+z_2\omega+z_3\omega^2)(\overline{z_1}+\overline{z_2}\omega^2+\overline{z_3}\omega)$ and
$|\gamma|^2=\gamma\overline{\gamma}=(z_1+z_2\omega^2+z_3\omega)(\overline{z_1}+\overline{z_2}\omega+\overline{z_3}\omega^2)$ $[\because \overline{\omega}=\omega^2 \;\&\; \overline{\omega^2}=\omega]$

$\Rightarrow |\alpha|^2+|\beta|^2+|\gamma|^2=3(|z_1|^2+|z_2|^2+|z_3|^2)+z_1[\overline{z_2}(1+\omega+\omega^2)+\overline{z_3}(1+\omega+\omega^2)]+z_2[\overline{z_1}(1+\omega+\omega^2)+\overline{z_2}(1+\omega+\omega^2)]+z_3[\overline{z_1}(1+\omega+\omega^2)+\overline{z_2}(1+\omega+\omega^2)]=3(|z_1|^2+|z_2|^2+|z_3|^2)=$ R.H.S.

Problem 223

223. If n is an odd integer greater than 3, but not a multiple of 3, prove that $x^3 + x^2 + x$ is a factor of $(x+1)^n - x^n - 1$.

Solution of Problem 223

Solution: Let $f(x) = (x+1)^n - x^n - 1$. $x^3 + x^2 + x = 0 \Rightarrow x(x^2 + x + 1) = 0 \Rightarrow x = 0, \omega, \omega^2$. So for $x^3 + x^2 + x$ to be a factor of $f(x)$, $f(0) = 0$, $f(\omega) = 0$, $f(\omega^2) = 0$.

$f(0) = 1^n - 1 = 0$, $f(\omega) = (\omega + 1)^n - \omega^n - 1 = -\omega^{2n} - \omega^n - 1$ [$\because n$ is odd.] $= -(1 + \omega^n + \omega^{2n}) = 0$. Similarly, $f(\omega^2) = 0$. Hence proved.

Problem 224

224. If n is an odd integer greater than 3, but not a multiple of 3, prove that $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y)(x^2+xy+y^2)$.

Solution of Problem 224

Solution: Let $f(x, y) = (x+y)^n - x^n - y^n$. $xy(x+y)(x^2+xy+y^2) = 0 \Rightarrow x = 0, y = 0, x = -y, y = x\omega, y = x\omega^2$. When $x = 0, f(x, y) = 0; y = 0, f(x, y) = 0; y = -x \Rightarrow f(x, y) = -x^n - (-x)^n = 0[\because n = 2m+1 \ \forall \ m \in \mathbb{I}], y = xw \Rightarrow f(x, y) = [x^n(1+\omega)^n - x^n - x^n\omega^n] = -x^n\omega^{2n} - x^n - x^n\omega^n = 0$, and similarly when $y = x\omega^2, f(x, y) = 0$. Hence proved.

Problem 225

225. If $|z_1| = |z_1| = \cdots = |z_n| = 1$, prove that $|z_1 + z_2 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$.

Solution of Problem 225

Solution: R.H.S. $= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right| = \left|\frac{\overline{z_1}}{|z_1|^2} + \frac{\overline{z_2}}{|z_2|^2} + \cdots + \frac{\overline{z_n}}{|z_n|^2}\right|$
$= |\overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}| = |\overline{z_1 + z_2 + \cdots + z_n}| = |z_1 + z_2 + \cdots + z_n| =$ L.H.S.

Problem 226

226. If $\alpha, \beta \in \mathbb{C}$, show that $|\alpha + \sqrt{\alpha^2 - \beta^2}| + |\alpha - \sqrt{\alpha^2 - \beta^2}| = |\alpha + \beta| + |\alpha - \beta|$.

Solution of Problem 226

Solution: For any two complex numbers z_1 and z_2, we know that $|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2$. Let $z_1=\alpha+\sqrt{\alpha^2-\beta^2}$ and $z_2=\alpha-\sqrt{\alpha^2-\beta^2}$.

Now $(|z_1|+|z_2|)^2=|z_1|^2+|z_2|^2+2|z_1||z_2|=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\beta|^2=|\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha+\beta||\alpha-\beta|$

$=(|\alpha+\beta|+|\alpha-\beta|)^2 \Rightarrow |z_1|+|z_2|=|\alpha+\beta|+|\alpha-\beta|=$ R.H.S.

Problem 227

227. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\Re(z_1\overline{z_2}) = 0$, then show that the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfy i. $|\omega_1| = 1$ ii. $|\omega_2| = 1$ iii. $\Re(\omega_1\overline{\omega_2}) = 0$.

Solution of Problem 227

Solution:

$|z_1| = |z_1| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1, z_1\overline{z_2} = ac + bd + i(bc - ad) \because \Re(z_1\overline{z_2}) = 0 \Rightarrow ac + bd = 0 \Rightarrow \frac{a}{d} = -\frac{b}{c} = k$ (say). $\therefore a = kd, b = -kc.$

$\therefore k^2d^2 + k^2c^2 = 1 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1.$ Now
$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1, |\omega_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1, \omega_1\overline{\omega_2} = (a + ic)(b - id) \therefore \Re(\omega\overline{\omega_2}) = ab + cd = 0.$

Problem 228

228. Prove that $\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right| < 1$ if $|z_1| < 1, |z_2| < 1$.

Solution of Problem 228

Solution: Given, $\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right| < 1 \Leftrightarrow \left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|^2 < 1 \Leftrightarrow |z_1 - z_2|^2 < |1 - \overline{z_1}z_2|^2$

$\Leftrightarrow (z_1 - z_2)\overline{(z_1 - z_2)} < (1 - \overline{z_1}z_2)\overline{(1 - \overline{z_1}z_2)} \Leftrightarrow (z_1 - z_2)(\overline{z_1} - \overline{z_2}) < (1 - \overline{z_1}z_2)((1 - z_1\overline{z_2}))$

$\Leftrightarrow |z_1|^2 + |z_2|^2 > 1 + |z_1|^2|z_2|^2 \Leftrightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 > 0 \Leftrightarrow (1 - |z_1|^2)(1 - |z_2|^2) > 0 \Rightarrow$
$(1 + |z_1|)(1 - |z_1|)(1 + |z_2|)(1 - |z_2|) > 0$

$\Leftrightarrow (1 - |z_1|)(1 - |z_2|) > 0$ which is true as $|z_1| < 1$ and $|z_2| < 1$.

229. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\frac{z-z_1}{z-z_2}$ is $\frac{\pi}{2}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

Solution of Problem 229

Solution: Let $z = x + iy$ then $\frac{z-z_1}{z-z_2} = \frac{(x-10)+i(y-6)}{(x-4)+i(y-6)}$. Rationalizing $\frac{x^2-14x+40+(y-6)^2}{(x-4)^2+(y-6)^2} + \frac{i6(y-6)}{(x-4)^2+(y-6)^2} = a + ib$ (say) $\because \arg(a+ib) = \frac{\pi}{4} \Rightarrow x^2 - 14x + 40 + (y-6)^2 = 6(y-6) \Rightarrow x^2 + y^2 - 14x - 18y + 112 = 0 \Rightarrow |z - 7 - 9i|^2 = 18$. Hence proved.

230. Find all complex numbers z for which $\arg\left(\frac{3z-6-3i}{2z-8-6i}\right) = \frac{\pi}{4}$ and $|z-3+i| = 3$.

Solution of Problem 230

Solution: Let $z = x + iy$ then $\frac{3z-6-3i}{2z-8-6i} = \frac{x-6+i(3y-3)}{2x-8+i(2y-6)}$.

Rationalizing $\frac{6x^2+6y^2-36x-24y+66+i(12x-12y-12)}{(2x-8)^2+(2y-6)^2} = a + ib$ (let)

$\because \arg(a+ib) = \frac{\pi}{4} \Rightarrow 6x^2 + 6y^2 - 36x - 24y + 66 = 12x - 12y - 12 \Rightarrow x^2 + y^2 - 8x - 2y + 13 = 0.$ Also given, $|z - 3 + i| = 3 \Rightarrow x = -2y + 6.$

Substituting this in previously obtained equation, we have

$5y^2 - 10y + 1 = 0 \Rightarrow y = 1 \pm \frac{2}{\sqrt{5}} \Rightarrow x = 4 \mp \frac{4}{\sqrt{5}}$. Hence we have our z.