Complex Numbers Problems
181-190

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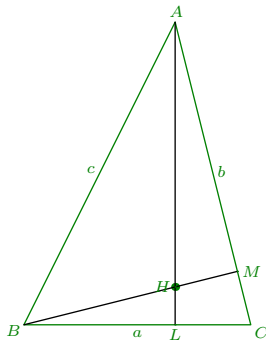
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181. If the vertices of a $\triangle ABC$ are represented by z_1, z_2, z_3 respectively, then show that the orthocenter of $\triangle ABC$ is $\frac{z_1 a \sec A + z_2 b \sec B + z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$ or $\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$.

Solution of Problem 181

Solution: The diagram is given below:

Let AL be perpendicular on BC and H be orthocenter of the $\triangle ABC$.

$\frac{BL}{LC} = \frac{c\cos B}{b\cos C} = \frac{c\sec C}{b\sec B}$, thus L divides BC internally in the ratio of $c\sec C : b\sec B$

$L = \frac{z_3 c\sec C + z_2 b\sec B}{c\sec C + b\sec B}$

$\frac{AH}{HL} = \frac{\Delta ABH}{\Delta HBD} = \frac{\frac{1}{2}AB.BH\sin\angle ABM}{\frac{1}{2}BL.LH.\sin\angle MBC} = \frac{c\cos A}{c\cos B\cos C}[\because \angle ABM = 90° - A, \angle MBC = 90° - B]$

$= \frac{a\cos A}{a\cos B\cos C} = \frac{(b\cos C + c\cos B)\cos A}{a\cos B\cos C} = \frac{b\sec B + c\sec C}{a\sec A}$

$H = \frac{z_1 a\sec A + z_2 b\sec B + z_3 c\sec C}{a\sec A + b\sec B + c\sec C}$

Since the above expression is similar w.r.t. A, B and C, therefore it will also lie on the perpendiculars from B and C to opposing sides as well.

Thus, orthocenter $H = \frac{z_1 a\sec A + z_2 b\sec B + z_3 c\sec C}{a\sec A + b\sec B + c\sec C}$

$H = \frac{z_1 k\sin A\sec A + z_2 k\sin B\sec B + z_3 k\sin C\sec C}{k\sin A\sec A + k\sin B\sec B + k\sin C\sec C}$
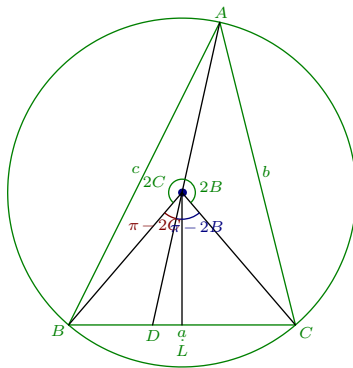
$H = \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$

182. If the vertices of a $\triangle ABC$ are represented by z_1, z_2 and z_3 respectively, show that its circumcenter is $\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$.

Solution of Problem 182

Solution: The diagram is given below:

Solution of Problem 182

Solution: Let O be the circumcenter of $\triangle ABC$ where $A = z_1, B = z_2$ and $C = z_3$.

$$\frac{BD}{DC} = \frac{\frac{1}{2} BD.OL}{\frac{1}{2} DC.OL} = \frac{\Delta BOD}{\Delta COD}$$

$$= \frac{\frac{1}{2} OB.OD.\sin(\pi - 2C)}{\frac{1}{2} OC.OD \sin(\pi - 2C)} = \frac{\sin 2C}{\sin 2B}$$

Thus, D divides BC internally in the ratio $\sin 2C : \sin 2B \Rightarrow D = \frac{z_3 \sin 2C + z_2 \sin 2B}{\sin 2C + \sin 2B}$

The complex number dividing AD internally in the ratio $\sin 2B + \sin 2C : \sin 2A$ is

$\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

Since the above expression is similar w.r.t. A, B and C, therefore it will also lie on the perpendicular bisectors on AC and AB as well.

Let BO produced meet AC at E and CO produced meet AB at F. We can show that, the complex numner representing the point dividing the line segment BE internally in the ratio $(\sin 2C + \sin 2A) : \sin 2B$ and the complex number representing the point dividing the line segment CF internally in the ratio $(\sin 2A + \sin 2B) : \sin 2C$ will be each $= \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$
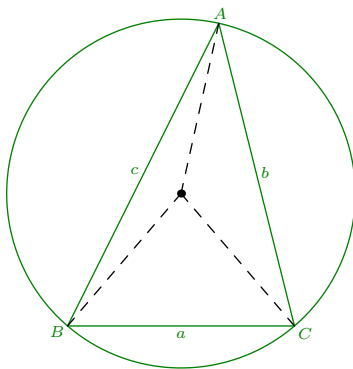
Thus, circumcenter is $\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

183. Show that the circumcenter of the triangle whose vertices are given by the complex numbers z_1, z_2, z_3 is given by $z = \frac{\sum z_1 \overline{z_1}(z_2 - z_3)}{\sum \overline{z_1}(z_2 - z_3)}$.

Solution of Problem 183

Solution: Consider the diagram given below:

Let z be the circumcenter of the triangle represented by $A(z_1)$, $B(z_2)$ and $C(z_3)$ respectively, then

$|z - z_1| = |z - z_2| = |z - z_3|$ so we have $|z - z_1| = |z - z_2|$

$\Rightarrow |z - z_1|^2 = |z - z_2|^2 \Rightarrow (z - z_1)(\overline{z} - \overline{z_1}) = (z - z_2)(\overline{z} - \overline{z_2})$

$\Rightarrow z\overline{z} + z_1\overline{z_1} - \overline{z}z_1 - z\overline{z_1} = z\overline{z} + z_2\overline{z_1} - \overline{z}z_2 - z\overline{z_2}$

$\Rightarrow z(\overline{z_1} - \overline{z_2}) + \overline{z}(z_1 - z_2) = z_1\overline{z_1} - z_2\overline{z_2}$ \hfill (1)

Similarly considering $|z - z_1| = |z - z_3|$, we will have

$\Rightarrow z(\overline{z_1} - \overline{z_3}) + \overline{z}(z_1 - z_3) = z_1\overline{z_1} - z_3\overline{z_3}$ \hfill (2)

We have to eliminate $\overline{z}$ from equation (1) and (2) i.e. multiplying equation (1) with $(z_1 - z_3)$ and (2) with $(z_1 - z_2)$, we get following

$z[\overline{z_1}(z_2 - z_3) + \overline{z_2}(z_3 - z_1) + \overline{z_3}(z_1 - z_2)] = z_1\overline{z_1}(z_2 - z_3) + z_2\overline{z_2}(z_3 - z_1) + z_3\overline{z_3}(z_1 - z_2)$
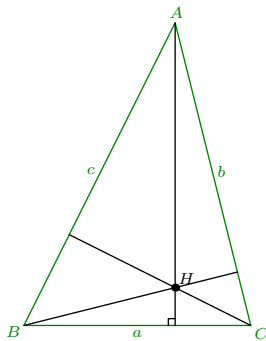
$\Rightarrow z = \frac{\sum z_1\overline{z_1}(z_2 - z_3)}{\sum \overline{z_1}(z_2 - z_3)}$

184. Find the orthocenter of the triangle with vertices z_1, z_2, z_3.

Solution of Problem 184

Solution:

Contd

Let z be the orthocenter of $\triangle A(z_1)B(z_2)C(z_3)$ i.e. the intersection point of perpendiculars on sides from opposite vertices.

Since $AH \perp BC \therefore \arg\left(\frac{z_1-z}{z_3-z_2}\right) = \pm\frac{\pi}{2}$

$\Rightarrow \frac{z_1-z}{z_3-z_2}$ is purely imaginary.

$\Rightarrow \overline{\left(\frac{z_1-z}{z_3-z_2}\right)} = -\left(\frac{z_1-z}{z_3-z_2}\right) \Rightarrow \frac{\overline{z_1}-\overline{z}}{\overline{z_3}-\overline{z_2}} = \frac{z-z_1}{z_3-z_2}$

$\Rightarrow \overline{z_1} - \overline{z} = \frac{(z-z_1)(\overline{z_3}-\overline{z_2})}{z_3-z_2}$ Similarly for $BH \perp AC$, $\overline{z_2} - \overline{z} = \frac{(z-z_2)(\overline{z_1}-\overline{z_2})}{z_1-z_3}$

Eliminating $\overline{z}$ like last problem we arrive at the desired result.

185. $ABCD$ is a rhombus described in clockwise direction. Suppose that the vertices A, B, C, D are given by z_1, z_2, z_3, z_4 respectively and $\angle CBA = 2\pi/3$. Show that $2\sqrt{3}z_2 = (\sqrt{3} - i)z_1 + (\sqrt{3} + i)z_3$ and $2\sqrt{3}z_4 = (\sqrt{3} + i)z_1 + (\sqrt{3} - i)z_3$.

Solution of Problem 185

Solution: We have $\angle CBA = \frac{2\pi}{3}$, therefore

$\frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} \left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right]$

$\frac{z_3 - z_2}{z_1 - z_2} = -\frac{1}{2} + \frac{i\sqrt{3}}{2} [\because BC = AB]$

$z_3 + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) z_1 = \left(\frac{3}{2} - \frac{i\sqrt{3}}{2} \right) z_2$

Solving this yields $2\sqrt{3} z_2 = (\sqrt{3} - i) z_1 + (\sqrt{3} + i) z_3$

Also, since diagonals bisect each other $\Rightarrow \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$

$z_4 = z_1 + z_3 - z_2$

Substituting the value of z_2, we get

$2\sqrt{3} z_4 = (\sqrt{3} + i) z_1 + (\sqrt{3} - i) z_3$

186. The points P, Q and R represent the numbers z_1, z_2 and z_3 respectively and the angles of the $\triangle PQR$ at Q and R are both $\frac{1}{2}(\pi - \alpha)$. Prove that $(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{\alpha}{2}$.

Solution of Problem 186

Solution: Since $\angle PQR = \angle PRQ = \frac{1}{2}(\pi - \alpha) \therefore PQ = PR$ Also, $\angle QPR = \pi - 2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \alpha$

$\therefore arg\frac{z_3-z_1}{z_2-z_1} = \alpha \Rightarrow \frac{z_3-z_1}{z_2-z_1} = \frac{PR}{RQ}(\cos\alpha + i\sin\alpha)$

$\Rightarrow \frac{z_3-z_1}{z_2-z_1} - 1 = (\cos\alpha - 1) + i\sin\alpha \Rightarrow \frac{z_3-z_2}{z_2-z_1} = -2\sin^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$\Rightarrow \left(\frac{z_3-z_2}{z_2-z_1}\right)^2 = -4\sin^2\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]^2 = -4\sin^2\frac{\alpha}{2}[\cos\alpha + i\sin\alpha] = -4\sin^2\frac{\alpha}{2}.\frac{z_3-z_1}{z_2-z_1}$

$\Rightarrow (z_3-z_2)^2 = 4(z_3-z_1)(z_1-z_2)\sin^2\frac{\alpha}{2}$

187. Points z_1 and z_2 are adjacent vertices of a regular polygon of n sides. Find the vertex z_3 adjacent to $z_2(z_1 \neq z_3)$.

Solution of Problem 187

Solution: Let C be the center of a regular polygon of n sides. Let $A_1(z_1)$, $A_2(z_2)$ and $A_3(z_3)$ be its three consecutive vertices.

$\angle CA_2A_1 = \frac{1}{2}\left(\pi - \frac{2\pi}{n}\right) \therefore A_1A_2A_3 = \pi - \frac{2\pi}{n}$

Case I: When z_1, z_2, z_3 are in anticlockwise order. $\Rightarrow z_1 - z_2 = (z_3 - z_2)e^{i(\pi - 2\pi/n)} [\because A_1A_2 = A_3A_2]$

$z_1 - z_2 = (z_2 - z_3)e^{-i2\pi/n} [\because e^{i\pi} = -1] \Rightarrow z_3 = z_2 - (z_1 - z_2)e^{i2\pi/n}$

Case II: When z_1, z_2, z_3 are in clockwise order. $\Rightarrow z_3 - z_2 = (z_1 - z_2)e^{i(\pi - i2\pi/n)}$

$z_3 = z_2 + (z_2 - z_1)e^{-i2\pi/n}$

188. Let $A_1, A_2, \ldots, A_n$ be the vertices of an n sided regular polygon such that $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}$, find the value of n.

Solution of Problem 188

Solution: Let O be the origin and the complex number representing A_1 be z, then A_2, A_3, A_4 will be represented by $ze^{i2\pi/n}, ze^{i4\pi/n}, ze^{i6\pi/n}$. Let $|z| = a$

$$A_1 A_2 = \left| z - ze^{i2\pi/n} \right| = |z| \left| 1 - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \right|$$

$$= a\sqrt{\left(1 - \cos \frac{2\pi}{n}\right)^2 + \sin^2 \frac{2\pi}{n}} = a\sqrt{2\left(1 - \cos \frac{2\pi}{n}\right)} = 2a \sin \frac{\pi}{n}$$

Similarly, $A_1 A_3 = 2a \sin \frac{2\pi}{n}$ and $A_1 A_4 = 2a \sin \frac{3\pi}{n}$

Given $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4} \therefore \frac{1}{2a \sin \frac{\pi}{n}} = \frac{1}{2a \sin \frac{2\pi}{n}} + \frac{1}{2a \sin \frac{3\pi}{n}}$

$\Rightarrow \sin \frac{\pi}{n} \left(\sin \frac{3\pi}{n} + \sin \frac{2\pi}{n} \right) = \sin \frac{2\pi}{n} \sin \frac{3\pi}{n}$

$\Rightarrow \sin \frac{3\pi}{n} + \sin \frac{2\pi}{n} = 2 \cos \frac{2\pi}{n} \sin \frac{3\pi}{n} = \sin \frac{4\pi}{n} + \sin \frac{2\pi}{n}$

$\Rightarrow \sin \frac{3\pi}{n} = \sin \frac{4\pi}{n} \Rightarrow \frac{3\pi}{n} = m\pi + (-1)^n \frac{4\pi}{n}, m = 0, \pm 1, \pm 2, ...$

If $m = 0 \Rightarrow \frac{3\pi}{n} = \frac{4\pi}{n} \Rightarrow 3 = 4$ (not possible)

If $m = 1 \Rightarrow \frac{3\pi}{n} = \pi - \frac{4\pi}{n} \Rightarrow n = 7$

If $m = 2, 3 ..., -1, -2, ...$ gives values of n which are not possible. Thus $n = 7$.

189. If $|z| = 2$, then show that the points representing the complex numbers $-1 + 5z$ lie on a circle.

Solution: Given, $|z| = 2$. Let $z_1 = -1 + 5z \Rightarrow z_1 + 1 = 5z$

$|z_1 + 1| = |5z| = 5|z| = 10$

$\Rightarrow z_1$ lies on a circle with center $(-1, 0)$ having radius 10.

190. If $|z - 4 + 3i| \leq 2$, find the least and tghe greatest values of $|z|$ and hence find the limits between which $|z|$ lies.

Solution of Problem 190

Solution: Given, $|z - 4 + 3i| \leq 2 \Rightarrow ||z| - |4 - 3i|| \leq 2$

$\Rightarrow ||z| - 5| \leq 2 \Rightarrow -2 \leq |z| - 5 \leq 2 \Rightarrow 3 \leq |z| \leq 7$