Miscellaneous Problems on A.P., G.P. and H.P.
Problems 141-150

Shiv Shankar Dayal

January 14, 2022

Some useful results

$$\sum_{n=1}^{n} 1 = n = {}^{n}C_1$$

$$\sum_{n=1}^{n} n = \frac{n(n+1)}{2} = {}^{n+1}C_2$$

$$\sum_{n=1}^{n} {}^{n+1}C_2 = {}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + ... + {}^{n+1}C_2$$

$$= ({}^{3}C_3 + {}^{3}C_2) + {}^{4}C_2 + ... + {}^{n+1}C_2$$

$$= {}^{4}C_3 + {}^{4}C_2 + {}^{5}C_2 + ... + {}^{n+1}C_2$$

$$...$$

$$= {}^{n+1}C_3$$

Similarly

$$\sum_{n=1}^{n} {}^{n+2}C_3 = {}^{n+3}C_4$$

To find $t_1 + t_2 + ... + t_n$

Let $S_n = t_1 + t_2 + ... + t_n$

Terms of the given series are: $t_1, t_2, t_3, \dots, t_{n-1}, t_n$

First order differences are: $\Delta t_1, \Delta t_2, \dots, \Delta t_{n-1}$

Second order differences are: $\Delta^2 t_1, \Delta^2 t_2, \dots, \Delta^2 t_{n-1}$

Then, $t_n = t_1 + {}^{n-1}C_1 \Delta t_1 + {}^{n-1}C_2 \Delta^2 t_1 + ... + {}^{n-1}C_{n-1} \Delta^{n-1} t_1$

When $n = 1$, L.H.S $= t_1$ and R.H.S. $= t_1$ so the theorem is true for $n = 1$. Let the theorem be true for $n = m$

$t_m = t_1 + {}^{m-1}C_1 \Delta t_1 + {}^{m-1}C_2 \Delta^2 t_1 + ... + {}^{m-1}C_{m-1} \Delta^{m-1} t_1$

$\therefore \Delta t_m = \Delta t_1 + {}^{m-1}C_1 \Delta^2 t_1 + {}^{m-1}C_2 \Delta^3 t_1 + ... + {}^{m-1}C_{m-1} \Delta^m t_1$

$t_m + \Delta t_m = t_1 + ({}^{m-1}C_1 + {}^{m-1}C_0) \Delta t_1 + ({}^{m-1}C_2 + {}^{m-1}C1) \Delta^2 t_1 + ... + ({}^{m-1}C_{m-1} + {}^{m-1}C_{m-2}) \Delta^{m-1} t_1 + {}^{m-1}C_{m-1} \Delta^m t_1$

$\therefore t_{m+1} = t_1 + {}^{m}C_1 \Delta t_1 + {}^{m}C_2 \Delta^2 t_1 + {}^{m}C_{m-1} \Delta^{m-1} t_1 + {}^{m}C_m \Delta^m t_1$

Thus, theorem is true for $n = m + 1$ whenever it is true for $n = m$

Thus, $t_n = t_1 + {}^{n-1}C_1 \Delta t_1 + {}^{n-1}C_2 \Delta^2 t_1 + ... + {}^{n-1}C_{n-1} \Delta^{n-1} t_1$

Problem 141

141. Two trains A and B start from the same station P at the same time. A covers half the distance between first station P and second station Q with speed x and other half distance with speed y. Train B covers the whole distance with speed $\frac{x+y}{2}$. Which train will reach Q earlier.

Solution of Problem 141

Solution: Let s be the distance between P and Q.

Time taken by train $A = \frac{s}{2x} + \frac{s}{2y} = \frac{s(x+y)}{2xy} = \frac{s}{\text{H.M of } x \text{ and } y}$

Time taken by train $B = \frac{2s}{x+y} = \frac{s}{\text{A.M of } x \text{ and } y}$

So, second train wil reach earlier as A.M. $\geq$ H.M.

Problem 142

142. If n is a root of equation $x^2(1-ac)-x(a^2+c^2)-(1+ac)=0$ and if n H.M.'s are inserted between a and c, show that the difference between the first and last mena is equal to $ac(a-c)$.

Solution of Problem 142

142. Let d be the common difference of corresponding A.P. Also, let H_1 and H_n be first and last H.M.

$$\Rightarrow d = \frac{\frac{1}{c} - \frac{1}{a}}{n+1} = \frac{ac}{ac(n+1)}$$

$$\frac{1}{H_1} = \frac{1}{a} + \frac{a-c}{ac(n+1)} \Rightarrow H_1 = \frac{ac(n+1)}{nc+a}$$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-n)}{ac(n+1) \Rightarrow H_n = \frac{ac(n+1)}{na+c}}$$

$$H_1 - H_n = \frac{ac(n+1)}{nc+a} - \frac{ac(n+1)}{na+c} = \frac{ac(n^2-1)(a-c)}{(n^1+1)ac + n(a^2+c^2)}$$

Also, given that n is a root of equation $x^2(1-ac) - x(a^2+c^2) - (1+ac) = 0$

$$\therefore n^2(1-ac) - n(a^2+c^2) - 1 - ac = 0 \Rightarrow n^2 - 1 = (n^2+1)ac + n(a^2+c^2)$$

$$\therefore H_1 - H_n = ac(a-c)$$

Problem 143

143. If $A_1, A_2, \dots, A_n$ are the n A.M.'s and $H_1, H_2, \dots, H_n$ the n H.M.'s between a and b, show that $A_r H_{n-r+1} = ab$ for $1 \leq r \leq n$

Solution of Problem 143

Solution: Let d be the common difference for A.P. and d' be the common difference for H.P., then

$$d = \frac{b-a}{n+1}, d' = \frac{\frac{1}{b} - \frac{1}{a}}{n+1} = \frac{a-b}{(n+1)ab}$$

$$A_r = a + rd = a + \frac{r(b-a)}{n+1} = \frac{(n-r+1)a + rb}{n+1}$$

$$\frac{1}{H_{n-r+1}} = \frac{1}{a} + \frac{(n-r+1)(a-b)}{(n+1)ab} = \frac{(n-r+1)a + rb}{(n+1)ab}$$

$$\Rightarrow H_{n-r+1} = \frac{(n+1)ab}{(n-r+1)a + rb}$$

$$\Rightarrow A_r H_{n-r+1} = ab$$

Problem 144

144. Find the coefficient of x^{99} and x^{98} in the polynomial $(x-1)(x-2)(x-3)\dots(x-100)$.

Solution of Problem 144

Solution: Consider the equation $(x-1)(x-2)(x-3)\dots(x-100)=0$. Its roots are $1,2,3,\dots,100$

So the equation is a polynomial of x of degree 100. Coefficient of $x^{100}=1$

Now sum of roots of equation taken one at a time

$1+2+3+\dots+100=(-1)^1\,\frac{\text{coeff. of } x^{99}}{\text{coeff. of } x^{100}}=-\text{coeff. of } x^{99}$

$\therefore$ coeff. of $x^{99}=-(1+2+3+\dots+100)=-5050$

Sum of products of roots taken two at a time = coeff. of $x^{98}=\frac{1}{2}[(1+2+3+\dots+100)^2-(1^2+2^2+\dots+100^2)]$

$=\frac{1}{2}\left[5050^2-\frac{100\times101\times102}{6}\right]=12582075$

Problem 145

145. Find the nth term and sum to n terms of the series $12, 40, 90, 168, 280, 432, \ldots$

Solution of Problem 145

Solution:

$$t_1 = 12, 40, 90, 168, 280, 432, \ldots$$

$$\Delta t_1 = 28, 50, 78, 112, 152, \ldots$$

$$\Delta^2 t_1 = 22, 28, 34, 40, \ldots$$

$$\Delta^3 t_1 = 6, 6, 6, \ldots$$

$$t_n = 12 + 28^{n-1}C_1 + 22.^{n-1}C_2 + 6.^{n-1}C_3$$

$$S_n = \sum_{n=1}^{n}(12 + 28^{n-1}C_1 + 22.^{n-1}C_2 + 6.^{n-1}C_3)$$

$$S_n = 12n + 28.^{n}C_2 + 22.^{n}C_3 + 6.^{n}C_4$$

$$= 12n + 28.\frac{n(n-1)}{2!} + 22.\frac{n(n-1)(n-2)}{3!} + 6.\frac{n(n-1)(n-2)(n-3)}{4!}$$

$$= \frac{n}{12}(n+1)(3n^2 + 23n + 46)$$

Problem 146

146. Find the nth term and the sum to n terms of the series $10, 23, 60, 169, 494, \ldots$

Solution of problem 146

Solution: The series and the successive order differences are:

$$10, 23, 60, 169, 494, ...$$

$$13, 37, 109, 325, ...$$

$$24, 72, 216, ...$$

Here second order differences are in G.P. whose common ratio is 3. Let $t_n = a + bn + c.3^{n-1}$

$$\therefore a + b + c = t_1 = 10, a + 2b + 3c = t_2 = 23, a + 3b + 9c = t_3 = 60$$

$$\Rightarrow a = 3, b = 1, c = 6$$

$$t_n = 3 + n + 6.3^{n-1}$$

$$S_n = \sum_{n=1}^{n} t_n = \frac{1}{2}(n^2 + 7n - 6) + 3^{n+1}$$

Problem 147

147. Find the sum of the series $3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + \ldots \infty$

Solution of Problem 147

Solution: Here one factor of the terms is in G.P. i.e. x.
Now the series of the coeff. of terms together with successive order differences are

$$3, 5, 9, 15, 23, 33, \ldots$$

$$2, 4, 6, 8, 10, \ldots$$

$$2, 2, 2, , 2, \ldots$$

$$0, 0, 0, \ldots$$

Hence third order differences are constant. Now,

$$S = 3 + 5x + 9x^2 + 15x^3 + 23x^4 + 33x^5 + \ldots \infty$$

$$-3xS = -9x - 15x^2 - 27x^3 - 45x^4 - 69x^5 - \ldots$$

$$3x^2S = 9x^2 + 15x^3 + 27x^4 + 45x^5 + \ldots$$

$$-x^3S = -3x^3 - 5x^4 - 9x^5 - \ldots$$

Adding, we get $(1-x)^3S = 3 - 4x + 3x^2$

$$\therefore S = \frac{3 - 4x + 3x^2}{(1-x)^3}$$

Problem 148

148. If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ and $H'_n = \frac{n+1}{2} - \{\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + ... + \frac{n-2}{2.3}\}$, show that $H_n = H'_n$

Solution of Problem 148

Solution: Let t_r denote the rth term of the series $\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + ... + \frac{n-2}{2.3}$, then

$$t_1 = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

$$t_2 = \frac{2}{n-2} - \frac{2}{n-1} = \frac{2}{n-2} - \frac{1}{n-1} - \frac{1}{n-1}$$

$$t_3 = \frac{3}{n-3} - \frac{3}{n-2} = \frac{3}{n-3} - \frac{2}{n-2} - \frac{1}{n-2}$$

$$...$$

$$t_{n-2} = \frac{n-2}{2} - \frac{n-2}{3} = \frac{n-2}{2} - \frac{n-3}{3} - \frac{1}{3}$$

$$t_1 + t_2 + ... t_n = \frac{n-2}{2}\left(-\frac{1}{n} - \frac{1}{n-1} - \frac{1}{n-2} - ... - \frac{1}{3}\right)$$

$$= \frac{n+1}{2} - \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)$$

$$\therefore H_n' = \frac{n+1}{2} - (t_1 + t_2 + ... + t_n) = 1 + \frac{1}{2} + ... + \frac{1}{n} = H_n$$

Problem 149

149. Show that $\tan^{-1}\left(\frac{x}{1+1.2x^2}\right)+\tan^{-1}\left(\frac{x}{1+2.3x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)=\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$

Solution of Problem 149

Solution:

$$\tan^{-1}\left(\frac{x}{1+1.2x^2}\right) = \tan^{-1}\left(\frac{2x-x}{1+x.2x}\right) = \tan^{-1}2x - \tan^{-1}x$$

$$\tan^{-1}\left(\frac{x}{1+2.3x^2}\right) = \tan^{-1}\left(\frac{3x-2x}{1+2x.3x}\right) = \tan^{-1}3x - \tan^{-1}2x$$

$$\cdots$$

$$\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right) = \tan^{-1}\left(\frac{(n+1)x-nx}{1+nx.(n+1)x}\right) = \tan^{-1}(n+1)x - \tan^{-1}nx$$

Adding, we get

$$L.H.S. = \tan^{-1}(n+1)x - \tan^{-1}x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right) = R.H.S.$$

Problem 150

150. Find the sum to n terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$

Solution of Problem 150

Solution: The nth term of the given series is

$$t_n = \frac{n}{1+n^2+n^4} = \frac{n}{(1+n^2)^2-n^2} = \frac{1}{2}\left(\frac{1}{1+n^2-n} - \frac{1}{1+n^2+n}\right)$$

$$\therefore t_1 = \frac{1}{2}\left(1-\frac{1}{3}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right)$$

$$t_3 = \frac{1}{2}\left(\frac{1}{7}-\frac{1}{13}\right)$$

$$\cdots$$

$$t_n = \frac{1}{2}\left(\frac{1}{1+n^2-n} - \frac{1}{1+n^2+n}\right)$$

Adding, we get

$$S = \frac{1}{2}\left(1-\frac{1}{1+n^2+n}\right) = \frac{n(n+1)}{2(1+n+n^2)}$$