# Complex Numbers Problems 131-140

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# Problem 131

**131.** Find the intercept made by the circle $z\overline{z} + \overline{\alpha}z + \alpha\overline{z} + r = 0$ on real axis on the complex plane.

# Solution of Problem 131

**Solution:** Let $z_1$ and $z_2$ be points on real axis which circle cuts with. Since these are on real axis and if $z$ represents this points then $z = \overline{z}[\because z = x + i.0]$

Substituting $z = \overline{z}$ in the equation of the circle, we get $z^2 + (\overline{\alpha} + \alpha)z + r = 0$

Since $z_1, z_2$ are the roots $\therefore z_1 + z_2 = -\alpha, z_1 z_2 = r$

Length of intercept $= |z_1 - z_2| = \sqrt{(z_1 - z_2)^2} = \sqrt{(z_1 + z_2)^2 - 4z_1 z_2} = \sqrt{(\overline{\alpha} + \alpha)^2 - 4r}$

# Problem 132

**132.** If $a = \cos\alpha + i\sin\alpha, b = \cos\beta + i\sin\beta, c = \cos\gamma + i\sin\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then find the value of $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)$.

# Solution of Problem 132

**Solution:** Clearly, $a = e^{i\alpha}, b = e^{i\beta}, c = e^{i\gamma}$

Also given, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1 \Rightarrow e^{i(\alpha-\beta)} + e^{i(\beta-\gamma)} + e^{i(\gamma-\alpha)} = 1$

Comparing real parts, we get $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = 1$

# Problem 133

**133.** Find the locus of the center of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally.

# Solution of the Problem 133

**Solution:** Let $A(z_1), B(z_2)$ be the centers of given circles and $P$ be the center of the variable circle which touches given circles externally, then

$|AP| = a + r$ and $|BP| = b + r$ where $r$ is the radius of the variable circle. Clearly,

$|AP| - |BP| = a - b \Rightarrow ||AP| - |BP|| = |a - b| =$a constant.

Hence, locus of $P$ is a right bisector if $a = b$, a hyperbola if $|a - b| < |AB|$ an empty set of $|a - b| > |AB|$, set of all points on line $AB$ except those which lie between $A$ and $B$ if $|a - b| = |AB| \neq 0$.

# Problem 134

**134.** Prove that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\frac{2ab}{a^2-b^2}$.

# Solution of Problem 134

**Solution:** Let $a+ib = re^{i\theta}, r^2 = a^2+b^2 \Rightarrow a-ib = e^{-i\theta}, \tan\theta = \frac{b}{a}$

$\frac{a-ib}{a+ib} = e^{-2i\theta} \Rightarrow i\log\left(\frac{a-ib}{a+ib}\right) = i\log e^{-2i\theta} = 2\theta$

$\Rightarrow \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2b/a}{1-b^2/a^2} = \frac{2ab}{a^2-b^2}.$

# Pronlem 135

**135.** $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $Re(z_1\overline{z_2}) = 0$. Also, $w_1 = a + ic, w_2 = b + id$ then prove that $|w_1| = |w_2| = 1$ and $Re(w_1\overline{w_2}) = 0$.

# Solution of Problem 135

**Solution:** Given, $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1$

$Re(z_1\overline{z_2}) = 0 \Rightarrow Re[(a + ib)(c - id)] = 0 \Rightarrow ac + bd = 0$

$a^2 + b^2 = c^2 + d^2 \Rightarrow (a + ic)^2 = (d - ib)^2[\because ac == bd] \Rightarrow a + ic = d - ibor - d + ib$

$\Rightarrow a = d$ and $c = -b$ or $a = -d, c = b$

$\Rightarrow a^2 + c^2 = b^2 + d^2 = 1 \Rightarrow |w_1| = |w_2| = 1$

$Re(w_1\overline{w2}) = Re[(a + ic)(b - id)] = ab + cd = 0$

# Problem 136

**136.** If $\left|\frac{z_1}{z_2}\right| = 1$ and $\arg(z_1 z_2) = 0$, then prove that $|z_2|^2 = z_1 z_2$.

# Solution of Problem 136

**Solution:** Let $z_1 = r(\cos\theta + i\sin\theta)$. Given, $\left|\frac{z_1}{z_2}\right| = 1$

$\Rightarrow |z_1| = |z_2| = r$. Also given, $\arg(z_1 z_2) = 0 \Rightarrow \arg(z_1) + \arg(z_2) = 0$

$\Rightarrow \arg(z_2) = -\theta \Rightarrow z_2 = r[\cos(-\theta) + i\sin(-\theta)] = r[\cos\theta - i\sin\theta] = \overline{z_1}$

$\Rightarrow \overline{z_2} = z_1 \Rightarrow |z_2|^2 = z_1 z_2$.

# Problem 137

**137.** Find the value of the expression
$2\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right)+3\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right)+4\left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right)+...+(n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right).$

# Solution of Problem 137

**Solution:** $t_n = (n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$

$= n^3 + n^2\left(1+\frac{1}{\omega}+\frac{1}{\omega^2}\right) + n\left(1+\frac{1}{\omega}+\frac{1}{\omega^2}\right) + 1$

$= n^3 + n^2(1+\omega+\omega^2) + n(1+\omega+\omega^2) + 1 = n^3 + 1$

$\therefore S_n = \sum_{i=1}^{n} t_i = \sum_{i=1}(i^3+1) = \frac{n^2(n+1)^2}{4} + 1.$

# Problem 138

**138.** If $z_1$ and $z_2$ are two complex numbers satisfying the equation $\left|\frac{z_1+iz_2}{z_1-iz_2}\right| = 1$, then prove that $\frac{z_1}{z_2}$ is purely real.

# Solution of Problem 138

**Solution:** Given $|z_1 + iz_2| = |z_1 - iz_2|$

$\Rightarrow (z_1 + iz_2)(\overline{z_1} - i\overline{z_2}) = (z_1 - iz_2)(\overline{z_1 + i\overline{z_2}})$

$\Rightarrow \overline{z_1}z_2 = z_1\overline{z_2} \Rightarrow \frac{z_1}{z_2} = \frac{\overline{z_1}}{\overline{z_2}}$

Thus, $\frac{z_1}{z_2}$ is purely real.

# Problem 139

**139.** If $z = -2 + 2\sqrt{3}i$, then find values of $z^{2n} + 2^{2n} z^n + 2^{4n}$.

# Solution of Problem 139

**Solution:** $z = -2 + 2\sqrt{3}i = 4\omega$

$z^{2n} + 2^{2n}z^n + 2^{4n} = 4^{2n}[\omega^{2n} + \omega^n + 1]$

The above expression has value of $0$ if $n$ is not a multiple of $3$ and $3.4^{2n}$ if $n$ is multiple of $3$.

# Problem 140

**140.** If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then find the values of $\frac{x}{y} + \frac{y}{x}, xy + \frac{1}{xy}$.

# Solution of Problem 140

**Solution:** $x + \frac{1}{x} = 2\cos\theta, \Rightarrow x^2 - 2\cos\theta x + 1 = 0$

$\Rightarrow x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

Similarly, $y = e^{\pm i\phi}$

$\frac{x}{y} + \frac{y}{z} = 2\cos(\theta - \phi)$

and $xy + \frac{1}{xy} = 2\cos(\theta + \phi)$