Miscellaneous Problems on A.P., G.P. and H.P.
Problems 131-140

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Problem 131

131. The second, third and sixth terms of an A.P. are consecutive terms of a geometric progression. Find the common ratio of the G.P.

Solution of Problem 131

Solution: Let a be the first term and d be the common difference of A.P., then

$$(a+2d)^2 = (a+d)(a+5d) \Rightarrow a^2+4ad+4d^2 = a^2+6ad+5d^2$$

$$\Rightarrow d^2+2ad = 0 \Rightarrow d = -2a$$

Hence, common ratio of the G.P. $= \frac{a+2d}{a+d} = \frac{-3a}{-a} = 3$

Problem 132

132. A sequence of numbers is formed by adding together corresponding terms of an A.P. and a G.P. with common ratio 2. The first term of the sequence is 57, the second term is 94 and the third term is 171. Find the fourth term. Find also an expression for the nth term of the sequence.

Solution of Problem 132

Solution: Let a be the first term and d be the common difference of A.P. Also, let a' to be the first term of G.P. Then, given

$$a + a' = 57, a + d + 2a' = 94, a + 2d + 4a' = 171$$

$$\Rightarrow 2a' + d = 94 - a \Rightarrow a + 2(94 - a) = 171 \Rightarrow a = 17$$

$$a' = 40 \Rightarrow d = -3$$

$$\Rightarrow t_4 = a + 3d + 8a' = 328$$

$$\Rightarrow t_n = 17 + (n-1).-3 + 40.2^{n-1} = 20 - 3n + 10.2^{n+1}$$

Problem 133

133. The first, eighth and twenty second terms of an arithmetic progression are three consecutive terms of a geometric progression. Find the common ratio of the geometric progression. If sum of the first twenty two terms of arithmetic progression is 275, find its first term.

Solution of Problem 133

Solution: Let a be the first term and d be the common ratio of the A.P. then given

$$a(a+21d) = (a+7d)^2 \Rightarrow 49d^2 - 7ad = 0 \Rightarrow d = a/7[\because d \neq 0]$$

$$\therefore r = \frac{a+7d}{a} = 2$$

Also, given

$$S_{22} = \frac{22}{2}[2a + 21.d] = 275 \Rightarrow 5a = 25 \Rightarrow a = 5$$

Problem 134

134. An arithmetic progression has common difference 2 and a geometric progression has common ratio 2. A new sequence is formed by adding together the corresponding terms of these progressions. Given that the first term of this new sequence is 8 and the fifth term is 91, find the first terms.

Solution of Problem 134

Solution: Let a and a' be the first term of the A.P. and G.P. respectively. Then, given

$$a + a' = 8, a + 8 + 16a' = 91$$

$$\Rightarrow 15a' = 91 - 16 = 75 \Rightarrow a' = 5 \Rightarrow a = 3$$

Problem 135

135. If a, b, c are in A.P. and b, c, d are in H.P., prove that $ad = bc$

Solution of Problem 135

Solution:

$$2b = a + c, c = \frac{2bd}{b+d}$$
$$\Rightarrow bc + cd = d(a+c) \Rightarrow bc = ad$$

Problem 136

136. If a, b, c are in H.P., b, c, d are in G.P. and c, d, e are in A.P., show that $e = \frac{ab^2}{(2a-b)^2}$

Solution of Problem 136

Solution:

$$b = \frac{2ac}{a+c}, c^2 = bd, 2d = c + e$$

$$\Rightarrow ab + bc = 2ac \Rightarrow c = \frac{ab}{2a-b}$$

$$e = 2d - c = \frac{2c^2}{b} - c = \frac{2a^2b^2}{b(2a-b)^2} - \frac{ab}{2a-b}$$

$$= \frac{2a^2b - 2a^2b + ab^2}{(2a-b)^2} = \frac{ab^2}{(2a-b)^2}$$

Problem 137

137. If an A.P. and a G.P. have the same 1st and 2nd terms then show that every other term of the A.P. will be less than the corresponding term of G.P. all the terms being positive.

Solution of Problem 137

Solution:

$$ar^n - a - nd = a\left(1 + \frac{d}{a}\right)^n - a - nd \left[\because r = \frac{a+d}{a}\right]$$

$$= a\left[1 + {}^nC_1\left(\frac{d}{a}\right) + {}^nC_2\left(\frac{d}{a}\right)^2 + \ldots + {}^nC_n\left(\frac{d}{a}\right)^n\right] - a - nd$$

$$= a\left[{}^nC_2\frac{d^2}{a^2} + {}^nC_3\frac{d^3}{a^3} + \ldots + {}^nC_n\frac{d^n}{a^n}\right] > 0 \left(\because \frac{d}{a} > 0\right)$$

Problem 138

138. If three unequal numbers are in H.P. and their squares are in A.P. show that they are in the ratio $1+\sqrt{3} : -2 : 1-\sqrt{3}$ or $1-\sqrt{3} : -2 : 1+\sqrt{3}$

Solution of Problem 138

Solution: Let a, b, c be three numbers in H.P. Then, given that

$$b = \frac{2ac}{a+c}, 2b^2 = a^2 + c^2$$

Let $a = ck$

$$\Rightarrow \frac{8a^2c^2}{(a+c)^2} = a^2 + c^2$$

$$8a^2c^2 = (a^2+c^2)(a+c)^2 \Rightarrow (1+k^2)(1+k)^2 - 8k^2 = 0$$

$$(k-1)^2(k^2+4k+1) = 0$$

$$\because k \neq 1 k = -2 \pm 3$$

So $a : c = 1 : -2 \pm 3$ and now the ratio for b can be found.

Problem 139

139. If A, G, H are the arithmetic, geometric and harmonic means of two positive real numbers a and b, and if $A = kh$, prove that $A^2 = kG^2$. Find the ratio of a to b. For what value of k does the ratio exist.

Solution of Problem 139

Solution:

$$A = \frac{a+b}{2}, H = \frac{2ab}{a+b}, G = \sqrt{ab}$$

$$A = kH \Rightarrow (a+b)^2 = 4kab \Rightarrow A = kG^2$$

Let $b = ma$

$$\Rightarrow a^2(1+m^2) = 4kma^2 \Rightarrow 1+m^2 = 4km \Rightarrow m = \frac{4k \pm \sqrt{16k^2-4}}{2} = 2k \pm \sqrt{4k^2-1}$$

Also, $(a+b)^2 = 4kab \Rightarrow (a-b)^2 = 4kab - 4ab \because (a-b)^2 \geq 0 \therefore k \geq 1$

Problem 140

140. If p be the rth term when n A.M.'s are inserted between a and b and q be the rth term when n H.M.'s are inserted between a and b, then show that $\frac{p}{a} + \frac{b}{q}$ is independent of n and r.

Solution of Problem 140

Solution: Since n means are inserted therefore total no. of terms will be $n+2$. Let d be the c.d. of A.P. and d' be the c.d of H.P.

$$\Rightarrow d = \frac{b-a}{n+1}, d' = \frac{a-b}{(n+1)ab}$$

$$\Rightarrow p = a + rd = \frac{(n+1)a + r(b-a)}{n+1}, \frac{1}{q} = \frac{1}{a} + r\frac{a-b}{(n+1)ab} \Rightarrow q = \frac{(n+1)ab}{r(a-b)+(n+1)b}$$

$$\frac{p}{a} + \frac{b}{q} = \frac{(n+1)a + r(b-a)}{a(n+1)} + \frac{r(a-b)+(n+1)b}{(n+1)a}$$

$$= \frac{a+b}{a}$$

which is independent of n and r