Harmonic Progression
Theory and Problems 1-10

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Harmonic Progresion

Unequal numbers $a_1, a_2, a_3, ...$ are said to be in H.P., if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ...$ are in A.P. Thus, you can observe that no term in an H.P. can be 0 because that will make reciprocal infinite.

nth term of an H.P. $= \frac{1}{\text{corresponding term in corresponsing A.P}}$

If a is the first term and b is the nth term then c.d. $d = \frac{\frac{1}{b} - \frac{1}{a}}{n-1}$

There are no special properties of an H.P. but when we solve problems related to H.P. we treat its reciprocals as an A.P.

Problem 1

1. Find the 100th term of the sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

Solution of Problem 1

Solution: Clearly, we have corresponding A.P. as $1, 3, 5, 7, ...$
Thus, first term $a = 1$ and common difference $d = 2$

$$t_{100} = a + (100 - 1)d = 199$$

Problem 2

2. If pth term of an H.P. is qr, and qth term is rp, prove that rth term is pq.

Solution of Problem 2

Solution: Let a be the first term and d be the common difference of corresponding A.P. The pth and qth term of the A.P. will be $\frac{1}{qr}$ and $\frac{1}{qr}$ respectively.
For A.P. nth term $= a + (n-1)d$

$$\frac{1}{qr} = a + (p-1)d$$

$$\frac{1}{pr} = a + (q-1)d$$

Subtracting $\frac{q-p}{pqr} = (q-p)d \therefore d = \frac{1}{pqr}$

$$\Rightarrow \frac{1}{qr} = a + (p-1)d \Rightarrow a = \frac{1}{pqr}$$

Now it is trivial to find rth term.

Problem 3

3. If the pth, qth and rth terms of an H.P. be respectively a, b and c, then prove that $(q-r)bc + (r-p)ca + (p-q)ab = 0$

Solution of Problem 3

Solution: $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. and are pth, qth and rth term respectively. Let x be the first term and d be the common difference of this A.P.

$$\frac{1}{a} = x + (p-1)d$$

Multiplying with abc, we get

$$bc = abc[x + (p-1)]d$$

$$(q-r)bc = (q-r)abc[x + (p-1)d]$$

Similarly for qth term, we have

$$(r-p)ca = (r-p)abc[x + (q-1)d]$$

and for rth term

$$(p-q)ab = (p-q)abc[x + (r-1)d]$$

Now we can add all the terms and prove the result.

Problem 4

4. If a, b, c are in H.P., prove that $\frac{a-b}{b-c} = \frac{a}{c}$

Solution of Problem 4

Solution: Since a, b, c are in H.P

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

$$b = \frac{2ca}{c+a}$$

Substituting in $\frac{a-b}{b-c}$

$$\frac{a - \frac{2ca}{c+a}}{\frac{2ca}{c+a} - c} \Rightarrow \frac{a^2 - ac}{ac - c^2} = \frac{a}{c}$$

Problem 5

5. If a, b, c, d are in H.P., then, prove that $ab + bc + cd = 3ad$

Solution of Problem 5

Solution: Since a, b, c, d are in H.P., therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{c}$ are in A.P.
Let x the common difference.

$$\frac{1}{b} - \frac{1}{a} = x \Rightarrow ab = \frac{1}{x}(a-b)$$

Similarly,

$$bc = \frac{1}{x}(b-c)$$

$$cd = \frac{1}{x}(c-d)$$

Adding, we get

$$ab + bc + cd = \frac{1}{x}(a-d) = \frac{1}{\frac{\frac{1}{d}-\frac{1}{a}}{4-1}}(a-d) = 3ad$$

Problem 6

6. If $x_1, x_2, x_3, \dots, x_n$ are in H.P., prove that $x_1x_2 + x_2x_3 + x_3x_4 + \dots + x_{n-1}x_n = (n-1)x_1x_n$

Solution of Problem 6

Solution: Let d be the common difference of corresponding A.P. Following like previous probelm

$$\frac{1}{x_2} - \frac{1}{x_1} = d \Rightarrow x_1x_2 = \frac{1}{d}(x_1 - x_2)$$

$$\frac{1}{x_3} - \frac{1}{x_2} = d \Rightarrow x_2x_3 = \frac{1}{d}(x_2 - x_3)$$

$$\dots$$

$$\frac{1}{x_n} - \frac{1}{x_{n+1}} = d \Rightarrow x_{n-1}x_n = \frac{1}{d}(x_{n-1} - x_n)$$

Adding all these, we get

$$x_1x_2 + x_2x_3 + \dots x_{n-1}x_n = \frac{1}{d}(x_1 - x_n)$$

Also,

$$\frac{1}{x_n} - \frac{1}{x_1} = (n-1)d$$

Sunstituting the value of d we can obtain desired result.

Problem 7

7. If a, b, c are in H.P., show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

Solution of Problem 7

Solution: Given a, b, c are in H.P. which implies $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \text{are in A.P.}$$

$$1+\frac{b+c}{a}, 1+\frac{c+a}{b}, 1+\frac{a+b}{c} \text{are in A.P.}$$

$$\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c} \text{are in A.P.}$$

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{are in H.P.}$$

Problem 8

8. If a^2, b^2, c^2 are in A.P. show that $b+c, c+a, a+b$ are in A.P.

Solution of Problem 8

Solution:

$$a^2, b^2, c^2 \text{ are in A.P.}$$

$$\Rightarrow b^2 - a^2 = c^2 - b^2$$

$$\Rightarrow \frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in H.P.}$$

$$\Rightarrow b+c, c+a, a+b \text{ are in H.P.}$$

Problem 9

9. Find the sequence whose nth term is $\frac{1}{3n-2}$. Is tihs sequence an H.P.?

Solution of Problem 9

Solution: $t_1 = 1, t_2 = \frac{1}{4}, t_3 = \frac{1}{7}$
nth term of correspoding A.P. $t_n = 3n - 2$ and $n - 1$th term of corresponding A.P. $t_{n-1} = 3n - 5$ Thus, common difference $d = t_{n-1} - t_n = 3$ which is a constant and thus, we can say that corresponding reciprocals will form an H.P.

Problem 10

10. If mth term of an H.P. be n and nth term be m, prove that $(m+n)$th term $= \frac{mn}{m+n}$ and (mn)th term $= 1$

Solution of Problem 10

Solution: Reciprocals in A.P. would be $t_m = \frac{1}{n}$ and $t_n = \frac{1}{m}$. Let a be the first term and d be the common difference.

$$t_m = a + (m-1)d = \frac{1}{n}$$

and

$$t_n = a + (n-1)d = \frac{1}{m}$$

Subtracting

$$(m-n)d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$$

Substituting d in t_m, we have

$$a = \frac{1}{n} - \frac{m-1}{mn} = \frac{1}{mn}$$

Now,

$$t_{m+n} = \frac{1}{mn} + (m+n-1)\frac{1}{mn} = \frac{m+n}{mn}$$

Reciprocal is desired $\frac{mn}{m+n}$
Similarly,

$$t_{mn} = a + (mn-1)d = \frac{1}{mn} + (mn-1)\frac{1}{mn} = 1$$