Complex Numbers Problems 41-50

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July 31, 2022

Problem 41

41. Simplify $\left(i^{17} + \frac{1}{i^{15}}\right)^3$ in the form of $A + iB$.

Solution of Problem 41

Solution: Given, $\left(i^{17}+\frac{1}{i^{15}}\right)^3$

$= i^{51} + 3.i^{34}\frac{1}{i^{15}} + 3i^{17}.\frac{1}{i^{30}} + \frac{1}{i^{45}}$

$= i^{51} + 3.i^{19} + 3.\frac{1}{i^{13}} + \frac{1}{i^{45}}$

$= i^{3} + 3.i^{3} + 3.\frac{1}{i} + \frac{1}{i}$

$= -i - 3i - 3i - i = -8i$

Problem 42

42. Simplify $\frac{(1+i)^2}{2+3i}$ in the form of $A + iB$.

Solution of Problem 42

Solution: Given, $\frac{(1+i)^2}{2+3i} = \frac{1+i^2+2i}{2+3i} = \frac{2i}{2+3i}$

$= \frac{2i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{6-4i}{2^2+3^2}$

$= \frac{1}{13}(6-4i)$

Problem 43

43. Simplify $\left(\frac{1}{1+i}+\frac{1}{1-i}\right)\frac{7+8i}{7-8i}$ the form of $A+iB$.

Solution of Problem 43

Solution: $\frac{1}{1+i} + \frac{1}{1-i} = \frac{1-i+1+i}{1-i^2} = \frac{2}{2} = 1$

$\frac{7+8i}{7-8i} = \frac{7+8i}{7-8i} \cdot \frac{7+8i}{7+8i}$

$= \frac{49-64+112i}{113} = \frac{-15+112i}{113}$

Problem 44

44. Simplify $\frac{(1+i)^{4n+7}}{(1-i)^{4n-1}}$ in the form of $A + iB$.

Solution of Problem 44

Solution: $\frac{1+i}{1-i} = \frac{1+i}{1-i}.\frac{1+i}{1+i} = \frac{1+i^2+2i}{2} = i$

$\frac{(1+i)^{4n+7}}{(1-i)^{4n-1}} = \frac{(1+i)^{4n-1}}{(1-i)^{4n-1}}.(1+i)^8 = i^{4n-1}.(1+i^2+2i)^4 = \frac{1}{i}.16i^4 = -16i$

Problem 45

45. Simplify $\frac{1}{1-\cos\theta+i\sin\theta}$ in the form of $A+iB$.

Solution of Problem 45

Solution: Given $\frac{1}{1-\cos\theta+i\sin\theta} = \frac{1}{1-\cos\theta+i\sin\theta}\cdot\frac{1-\cos\theta-i\sin\theta}{1-\cos\theta-i\sin\theta}$

$= \frac{1-\cos\theta-i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \frac{1-\cos\theta-i\sin\theta}{2-2\cos\theta}$

$= \frac{2\sin^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2.2\sin^2\frac{\theta}{2}}$

$= \frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$

Problem 46

46. Simplify $\frac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cot u+i)(i+\tan u)}$ in the form of $A+iB$.

Solution of Problem 46

Solution: Given fraction can be rewritten as $\frac{(\cos x\cos y - i\sin x\sin y) + i(\sin x\cos y + \cos x\sin y)}{\frac{(\cos u + i\sin u)(\sin v + i\cos v)}{\sin u\cos v}}$

$$= \sin u\cos v\,\frac{\cos(x+y) + i\sin(x+y)}{\sin(v-u) + i\cos(v-u)}$$

$$= \sin u\cos v\,\frac{\cos(x+y) + i\sin(x+y)}{\sin(v-u) + i\cos(v-u)} \cdot \frac{\sin(v-u) - i\cos(v-u)}{\sin(v-u) - i\cos(v-u)}$$

$$= \sin u\cos v.[\sin(v-u+x+y) - i\cos(v-u+x+y)]$$

Problem 47

47. Show that for $z \in C, |z| = 0$ if and only if $z = 0$.

Solution of Problem 47

Solution: If $|z| = 0$ then $\sqrt{x^2 + y^2} = 0 \Rightarrow x^2 + y^2 = 0$

Above is possible if and only if $x = 0$ and $y = 0 \Rightarrow z = 0$

Problem 48

48. If z_1 and z_2 are $1 - i$ and $2 + 7i$ then find $Im\left(\frac{z_1 z_2}{\bar{z}_1}\right)$.

Solution of Problem 48

Solution: $\frac{z_1 z_2}{\overline{z_1}} = \frac{(1-i)(2+7i)}{1+i} = \frac{2+7-2i+7i}{1+i} = \frac{9+5i}{1+i}$

$= \frac{9+5i}{1+i} . \frac{1-i}{1-i} = \frac{9+5+5i-9i}{2} = 7 - 2i$

$\therefore Im\left(\frac{z_1 z_2}{\overline{z_1}}\right) = -2$

Problem 49

49. Find x and y if $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)+i}{3-i} = i$.

Solution of Problem 49

Solution: Given, $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

$\frac{x+i(x-2)}{3+i}+\frac{2y+i(1-3y)}{3-i}=i$

$3x+(x-2)+i[3(x-2)-x]+6y+(3y-1)+i[3-9y+2y]=10i$

Comparing real and imaginary parts, we get $4x-2+9y-1=0, 2x-6+3-7y=10$

$4x+9y=3, 2x-7y=13 \Rightarrow y=-1, x=3$

Problem 50

50. If $|z - i| < 1$, then prove that $|z + 12 - 6i| < 14$.

Solution of Problem 50

Solution: $|z + 12 - 6i| \leq |z - i| + |12 - 5i| < 1 + 13 = 14$