Geometric Progression

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Definition: A succession of numbers is said to be in G.P. if the ratio of any term and the preceding term is constant throughout. The constant term is known as *common ratio* of the G.P.
nth term of a G.P.: Let a be the first term and r be the common ratio of the G.P.

Now, first term of G.P., $t_1 = a$
second term of G.P., $t_2 = ar$
third term of G.P., $t_3 = ar^2$
...nth term of G.P., $t_n = ar^{n-1}$

Properties of G.P.

1. If each term of a G.P. is multiplied with a non-zero number then the sequence thus obtained is also in G.P.

   Let $a, ar, ar^2, ar^3, \ldots$ be a sequence in G.P. where a is the first term and r is the common ratio.

   Upon multiplying the terms of this sequence with a non-zero number, say k, it becomes $ak, ark, ar^2k, ar^3k, \ldots$

   Thus, we see that the resulting sequence is still G.P. with first term as ak and common ratio r
2. If each term of a G.P. is divided with a non-zero number then the sequence thus obtained is also in G.P.

   Following as above we will have our sequence as $\frac{a}{k}, \frac{ar}{k}, \frac{ar^2}{k}, \frac{ar^3}{k}, \ldots$

   We see that this sequence is also in G.P.
3. The reciprocals of the terms of of a G.P. are also in G.P.

   The reciprocals of the terms of a G.P. $a, ar, ar^2, ar^3, \ldots$ is $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \frac{1}{ar^3}, \ldots$ which we see is a G.P. with first term as $\frac{1}{a}$ and common ratio $\frac{1}{r}$

Sum of first n terms of a G.P.

Let a be the first term and r be the common ratio of a G.P. and S_n be the sum of first n terms.

Case I: When $r \neq 1$

$$S_n = a + ar + ar^2 + ar^2 + \ldots + ar^{n-1}$$
$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

Upon subtratcion,

$$(1-r)S_n = a - ar^n$$
$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Case II: When $r = 1$

$$S_n = a + a + a + \ldots \text{ up to } n \text{ terms} = na$$

Sum of a G.P. when $|r| < 1$

Let a be the first term, r be the common ratio and S_n be the sum of n terms of the G.P. in question.

Now, we have already found that $S_n = \frac{a(1-r^n)}{1-r}$. However, when $n = \infty$, $r^n = 0$ if $|r| < 1$

$$\therefore S_\infty = \frac{a}{1-r}$$

Recurring Decimal

Recurring decimal is a very good example of an infinite G.P. and its value can be obtained from the formula for sum to infinity of a G.P. For example, let us find the value of $.\dot{3}$

Now,

$$
\begin{aligned}
.\dot{3} &= .33333\ldots \text{ to infinity} \\
&= .3 + .03 + .003 + .0003 + \ldots \text{ to infinity} \\
&= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots \text{ to infinity} \\
&= \frac{3}{10}\left[1 + \frac{1}{10} + \frac{1}{100} + \ldots \text{ to infinity}\right] \\
&= \frac{3}{10}.\frac{1}{1-\frac{1}{10}} \\
&= \frac{3}{10}.\frac{10}{9} \\
&= \frac{1}{3}
\end{aligned}
$$

If the terms of an A.P. is multiplied by the corresponding terms of a G.P., then the new series obtained is called an Arithmetico Geometric series.
Example: If the terms of the arithmetic series $2 + 5 + 8 + 11 + \dots$ is myltiplied by the corresponsing terms of the geometric series $x + x^2 + x^3 + x^4 + \dots$, then the following arithmetic geometric series is obtained.

$$2x + 5x^2 + 8x^3 + 11x^4 + \dots$$

Sum of an Arithmetico Geometric Series
Let S be the sum of the arithmetic geometric series. Then each terms of the series is multiplied by r(the common ratio of G.P.) and are written shifting each term one step rightward and then we can subtract rS from S to get $(1 - r)S$. Then the sum can be obtained.
Example:

$$\begin{aligned} S &= 2x + 5x^2 + 8x^3 + 11x^4 + \dots \\ xS &= \quad\quad 2x^2 + 5x^3 + 8x^4 + \dots \\ (1 - x)S &= 2x + 3x^2 + 3x^3 + 3x^4 + \dots \end{aligned}$$