Logarithm Problem 101-110

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Problem 101

101. Solve $x^{\log_{10} x} > 10$

Solution of Problem 101

Solution:

$$\text{Given, } x^{\log_{10} x} > 10$$

$$\Rightarrow \log_{10} x \log_{10} x > 1$$

$$\Rightarrow \log_{10x} < -1, \log_{10} x > 1$$

Thus range for values of x would be $(0, 0.1) \cup (10, \infty]$

Problem 102

102. Solve $\log_x 2 \log_{2x} 2 \log_2 4x > 1$

Solution of Problem 102

Solution:

$$\text{Given, } \log_x 2 \log_{2x} 2 \log_2 4x > 1$$

$$\Rightarrow \frac{1}{\log_2 x} \frac{1}{\log_2 2x} \log_2 2^2 x > 1$$

$$\Rightarrow \frac{1}{\log_x 2} \frac{1}{1 + \log_2 x} (2 + \log_2 x) > 1$$

$$\text{Let } z = \log_2 x, \text{ then we have}$$

$$\Rightarrow \frac{1}{z} \frac{1}{1+z} (2+z) > 1$$

$$\Rightarrow z^2 - 2 < 0 \Rightarrow -\sqrt{2} < z < \sqrt{2}$$

However, for logarithm to be defined $x > 0, 2x \neq 1 \Rightarrow x \neq \frac{1}{2}$, and thus the ranges is $(2^{-\sqrt{2}}, \frac{1}{2}) \cup (\frac{1}{2}, 2^{\sqrt{2}})$

Problem 103

103. Solve $\log_2 x \log_3 2x + \log_3 x \log_2 4x > 0$

Solution of Problem 103

Solution:

$$\text{Given, } \log_2 x \log_3 2x + \log_3 x \log_2 4x > 0$$
$$\Rightarrow \log_3 x \log_2 2x + \log_3 x \log_2 4x > 0$$
$$\Rightarrow \log_3 x(\log_2 2 + \log_2 x + \log_2 4 + \log_2 x) > 0$$
$$\Rightarrow \log_3 x(3 + 2\log_2 x) > 0$$

This can be true if $\log_3 x > 0 \Rightarrow x > 1$ and $3 + 2\log_2 x > 0, x > 2^{-\frac{3}{2}}$ i.e $x > 1$

This is also true if $\log_3 x < 0 \Rightarrow x < 1$ and $3 + 2\log_2 x < 0, x < 2^{\frac{-3}{2}}$ i.e. $x < 2^{\frac{-3}{2}}$

However, for logarithm to be defined $x > 0$.

So the range is $(0, 2^{\frac{-3}{2}}) \cup (1, \infty)$

Problem 104

104. Find the value of $\log_{12} 60$ if $\log_6 30 = a$ and $\log_{15} 24 = b$

Solution of Problem 104

Solution:

$$\log_{12} 60 = \frac{\log_2 60}{\log_2 12} = \frac{\log_2(2^2.3.5)}{\log_2(2^2.3)} = \frac{2+\log_2 3+\log_2 5}{2+2\log_2 3}$$

Let, $\log_2 3 = x$ and $\log_2 5 = y$

$$\Rightarrow \log_{12} 60 = \frac{2+x+y}{2+x}$$

Given, $a = \log_6 30 = \frac{\log_2 30}{\log_2 6} = \frac{\log_2(2.3.5)}{\log_2(2.3)} = \frac{1+\log_2 3+\log_2 5}{1+\log_2 3} = \frac{1+x+y}{1+x}$

Also, $b = \log_{15} 24 = \frac{\log_2 24}{\log_2 15} = \frac{\log_2(2^3.3)}{\log_2(3.5)} = \frac{3+\log_2 3}{\log_2 3+\log_2 5} = \frac{3+x}{x+y}$

Solving these equations, we get $x = \frac{b+3-ab}{ab-3}, y = \frac{2a-b-2+ab}{ab-1}$

Substituting these values of a and b for $\log_{12} 60$, we get $\log_{12} 60 = \frac{2ab+2a-1}{ab+b+1}$

Problem 105

105. If $\log_a x, \log_b x$ and $\log_c x$ be in A.P. and $x \neq 1$, prove that $c^2 = (ac)^{\log_a b}$

Solution of Problem 105

Solution: Since $\log_a x, \log_b x$ and $\log_c x$ are in A.P.

$$\Rightarrow 2\log_b x = \log_a x + \log_c x$$

$$\Rightarrow \frac{2}{\log_x b} = \frac{1}{\log_x a} + \frac{1}{\log_x c}$$

$$\Rightarrow \frac{2}{\log_x b} = \frac{\log_x a + \log_x c}{\log_x a \log_x c}$$

$$\Rightarrow 2\log_x c = \log_x ac \frac{\log_x b}{\log_x a}$$

$$\Rightarrow \log_x c^2 = \log_x ac \log_a b$$

$$\Rightarrow c^2 = ac^{\log_a b}$$

Problem 106

106. If $a = \log_{\frac{1}{2}}(\sqrt{0.125})$ and $b = \log_3\left(\frac{1}{\sqrt{24}-\sqrt{17}}\right)$ then find $a > 0, b > 0$ or not.

Solution of Problem 106

Solution: Given, $a = \log_{\frac{1}{2}}(\sqrt{0.125})$ in this case both base and the number are less than 1 so the logarithm i.e. $a > 0$.

Also, $b = \log_3\left(\frac{1}{\sqrt{24}-\sqrt{17}}\right) = \log_3\left(\frac{\sqrt{24}+\sqrt{17}}{7}\right)$ where both base and the number are greater than 1 so the logarithm i.e $b > 0$.

Problem 107

107. Which one is greater among $\cos(\log_e \theta)$ or $\log_e(\cos\theta)$ if $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$

Solution of Problem 107

Solution:

$$\text{Given, } e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$$

$$\Rightarrow \log_e e^{-\frac{\pi}{2}} < \log_e \theta < \log_e \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} < \log_e \theta < 1 < \frac{\pi}{2} [\because \log_e \frac{\pi}{2} < \log_e e]$$

$$\Rightarrow -\frac{\pi}{2} < \log_e \theta < \frac{\pi}{2}$$

$$\Rightarrow \cos(\log_e \theta) > 0$$

$$\text{Again, } e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$$

$$\Rightarrow 0 < \theta < \frac{\pi}{2}$$

$$\Rightarrow 0 < \cos\theta < 1$$

$$\Rightarrow \log_e \cos\theta < 0$$

$$\therefore \cos(\log_e \theta) > \log_e(\cos\theta)$$

Problem 108

108. If $\log_2 x + \log_2 y \geq 6$, prove that $x + y \geq 16$

Solution of Problem 108

Solution: Given, $\log_2 x + \log_2 y \geq 6 \Rightarrow \log_2 xy \geq 6 \Rightarrow xy \geq 64$

In the given inequality both x and y are positive values as negative values will make the logarithm invalid.

We know that $A.M. \geq G.M. \Rightarrow \frac{x+y}{2} \geq \sqrt{xy} \Rightarrow x + y \geq 16$

Problem 109

109. If a, b, c be three distinct positive numbers, each different from 1 such that $\log_b a \log_c a - \log_a a + \log_a b \log_c b - \log_b b + \log_a c \log_b c - \log_c c = 0$ then prove that $abc = 1$

Solution of Problem 109

Solution:

$$\text{Given, } \log_b a \log_c a - \log_a a + \log_a b \log_c b - \log_b b + \log_a c \log_b c - \log_c c = 0$$

$$\Rightarrow \frac{(\log a)^2}{\log b \log c} - 1 + \frac{(\log b)^2}{\log a \log c} - 1 + \frac{(\log c)^2}{\log a \log b} - 1 = 0$$

$$\text{Let } \log a = x, \log b = y, \log c = z, \text{ then we have}$$

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} - 3 = 0$$

$$\Rightarrow \frac{x^3 + y^3 + z^3 - 3xyz}{xyz} = 0$$

$$\Rightarrow (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 0$$

$$\Rightarrow (x + y + z)\frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0$$

$\because x, y, z$ are different the term inside brackets will be always positive. Thus,

$$x + y + z = 0 \Rightarrow \log a + \log b + \log c = 0$$

$$\log abc = 0 \Rightarrow abc = 1$$

Problem 110

110. If $y = 10^{\frac{1}{1-\log x}}$ and $z = 10^{\frac{1}{1-\log y}}$, prove that $x = 10^{\frac{1}{1-\log x}}$

Solution of Problem 110

Solution:

$$\text{Given, } y = 10^{\frac{1}{1-\log x}}$$

$$\Rightarrow \log y = \frac{1}{1-\log x}$$

$$z = 10^{\frac{1}{1-\log y}}$$

$$\Rightarrow \log z = \frac{1}{1-\log y}$$

$$\Rightarrow \log y = 1 - \frac{1}{\log z}$$

$$\Rightarrow \frac{1}{1-\log x} = 1 - \frac{1}{\log z}$$

$$\Rightarrow \log x = \frac{1}{1-\log z}$$

$$x = 10^{\frac{1}{1-\log z}}$$