Miscellaneous Problems on A.P., G.P. and H.P. Problems 191-200

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January 17, 2022

Problem 191

191. If $a, b, c > 0$, show that $(a+b)(b+c)(a+c) \geq 8abc$

Solution of Problem 191

Solution: We know that A.M. $\geq$ G.M.

$$\frac{a+b}{2} \geq \sqrt{ab}, \frac{b+c}{2} \geq \sqrt{bc}, \frac{a+c}{2} \geq \sqrt{ac}$$

Multiplying, we get

$$(a+b)(b+c)(c+a) \geq 8abc$$

Problem 192

192. If $x + y + z = a$, show that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{a}$

Solution of Problem 192

Solution: We know that A.M $\geq$ H.M.

$$\frac{x+y+z}{3} \geq \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{a}$$

Problem 193

193. If n is a positive integer, show that $n^n \geq 1.3.5 \ldots (2n-1)$

Solution of Problem 193

Solution: We know that A.M $\geq$ G.M.

$$\frac{1+3+5+\ldots+(2n-1)}{n} \geq (1.3.5.\ldots(2n-1))^{\frac{1}{n}}$$

$$\Rightarrow \frac{n^2}{n} \geq (1.3.5.\ldots(2n-1))^{\frac{1}{n}}$$

$$\Rightarrow n^n \geq 1.3.5\ldots(2n-1)$$

Problem 194

194. Find the greatest value of $(7-x)^4(2+x)^5$ if $-2 < x < 7$

Solution of Problem 194

Solution: We consider seven numbers five of which are $2 + x$ and remaining four are $7 - x$. Now, we know that A.M $\geq$ G.M.

$$\frac{4.\frac{7-x}{4} + 5.\frac{2+x}{5}}{9} \geq \left[\left(\frac{7-x}{4}\right)^4 \left(\frac{2+x}{5}\right)^5\right]^{\frac{1}{9}}$$

$$\frac{9}{9} \geq \left[\left(\frac{7-x}{4}\right)^4 \left(\frac{2+x}{5}\right)^5\right]^{\frac{1}{9}}$$

$$(7-x)^4(2+x)^5 \leq 4^4.5^5$$

So the greatest value would be $4^4.5^5$

Problem 195

195. If $x, y > 0$, find the least value of $3x + 4y$, when $x^2y^3 = 6$.

Solution of Problem 195

Solution: We consider five numbers two of which are $\frac{3x}{2}$ and remaining three are $\frac{4y}{3}$. Now we know that, A.M. $\geq$ G.M.

$$\frac{2.\frac{3x}{2}+3.\frac{4y}{3}}{5} \geq \left[\left(\frac{3x}{2}\right)^2\left(\frac{4y}{3}\right)^3\right]^{\frac{1}{5}}$$

$$\Rightarrow \frac{3x+4y}{5} \geq \left[\frac{9x^2}{4}.\frac{64y^3}{27}\right]^{\frac{1}{5}}$$

$$\Rightarrow 3x+4y \leq 5.32^{\frac{1}{5}} = 10$$

So the least value of $3x+4y$ is 10.

Problem 196

196. If $a, b, c > 0$, show that $\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leq \frac{a+b+c}{2}$

Solution of Problem 196

Solution: We know that A.M $\geq$ H.M.

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}, \frac{b+c}{2} \geq \frac{2bc}{b+c}, \frac{c+a}{2} \geq \frac{2ca}{c+a}$$

$$\frac{a+b+c}{2} \geq \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}$$

Problem 197

197. If $a, b, c > 0$, show that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$

Solution of Problem 197

Solution:

$$(a-b)^2 \geq 0, (b-c)^2 \geq 0, (c-a)^2 \geq 0$$

$$\Rightarrow \frac{(a-b)^2}{ab} \geq 0, \frac{(b-c)^2}{bc} \geq 0, \frac{(c-a)^2}{ac} \geq 0$$

$$\Rightarrow \frac{a^2+b^2}{ab} \geq 2, \frac{b^2+c^2}{bc} \geq 2, \frac{c^2+a^2}{ca} \geq 2$$

$$\Rightarrow \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} \geq 6$$

$$\Rightarrow \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

Problem 198

198. If $x_i > 0, i = 1, 2, 3, \ldots, n$ show that $(x_1 + x_2 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right) \geq n^2$

Solution of Problem 198

Solution: We know that A.M. $\geq$ H.M.

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \frac{n}{\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right)}$$

$$\Rightarrow (x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \geq n^2$$

Problem 199

199. If x, y are positive real numbers and m, n are positive integers, then show that $\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})} \leq \frac{1}{4}$

Solution of Problem 199

Solution: We know that A.M $\geq$ G.M.Considering 1 and x^{2n}

$$\Rightarrow \frac{1+x^{2n}}{2} \geq \sqrt{1.x^{2n}} = x^n$$

Considering 1 and y^{2m}

$$\Rightarrow \frac{1+y^{2m}}{2} \geq \sqrt{1.y^{2m}} = y^m$$

Myltiplying. we get

$$(1+x^{2n})(1+y^{2m}) \geq 4x^n y^m$$

$$\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})} \leq \frac{1}{4}$$

Problem 200

200. If the arithmetic mean of $(b-c)^2, (c-a)^2$ and $(a-b)^2$ is the same as that of $(b+c-2a)^2, (c+a-2b)^2$ and $(a+b-2c)^2$, show that $a=b=c$

Solution of Problem 200

Solution: Let $b-c=x, c-a=y$ and $a-b=z, \Rightarrow x+y+z=0$. This also implies that $a+b-2c=x-y, b+c-2a=y-z, c+a-2b=z-x$

Clearly, $x+y+z=0$

Given,

$$\frac{(x-y)^2+(y-z)^2+(z-x)^2}{3}=\frac{x^2+y^2+z^2}{3}$$

$$\Rightarrow x^2+y^2+z^2-2xy-2yz-2zx=0$$

$$\Rightarrow (x+y+z)^2=4(xy+yz+zx)$$

$$\Rightarrow xy+yz+zx=0$$

$$\Rightarrow (c-a)(a-b)+(a-b)(b-c)+(c-a)(b-c)=0$$

$$\Rightarrow ca-bc-a^2+ab+ab-ca-b^2+bc+bc-c^2-ab+ca=0$$

$$\Rightarrow ab+bc+ca-a^2-b^2-c^2=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow a=b=c$$