Miscellaneous Problems on A.P., G.P. and H.P. Problems 1-10

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Problem 1

1. If $a_1, a_2, a_3, \dots, a_{2n}$ are in A.P., show that $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \dots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$

Solution of Problem 1

Solution:

$$L.H.S. = (a_1^2 - a_2^2) + (a_3^2 - a_4^2) + ... + (a_{2n-1}^2 - a_{2n}^2)$$

$$= (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + ... + (a_{2n-1} - a_{2n})(a_{2n-1} + a_{2n})$$

$$= -d(a_1 + a_2 + a_3 + ... + a_{2n})$$

$$= -\left(\frac{a_{2n} - a_1}{2n-1}\right)\frac{2n}{2}(a_1 + a_{2n}) = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$$

Problem 2

2. If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are in A.P., whose common difference is d show that $\sin d[\sec\alpha_1 \sec\alpha_2 + \sec\alpha_2 \sec\alpha_3 + \dots + \sec\alpha_{n-1} \sec\alpha_n] = \tan\alpha_n - \tan\alpha_1$

Solution of Problem 2

Solution: $t_1 = \sin d \sec \alpha_1 \sec \alpha_2 = \frac{\sin(\alpha_2 - \alpha_1)}{\cos \alpha_1 \cos \alpha_2}$

$$= \frac{\sin \alpha_2 \cos \alpha_1}{\cos \alpha_1 \cos \alpha_2} - \frac{\cos \alpha_2 \sin \alpha_1}{\cos \alpha_1 \cos \alpha_2} = \tan \alpha_2 - \tan \alpha_1$$

Similarly,

$$t_2 = \tan \alpha_3 - \tan \alpha_2$$

$$\dots$$

$$t_{n-1} = \tan \alpha_n - \tan \alpha_{n-1}$$

Adding, we get $t_1 + t_2 + \dots + t_{n-1} = \tan \alpha_n - \tan \alpha_1$

Problem 3

3. If $a_1, a_2, a_3, \dots, a_n$ be in A.P., prove that $\frac{1}{a_1 a_n} + \frac{1}{a_2 a_{n-1}} + \dots + \frac{1}{a_n a_1} = \frac{2}{a_1 + a_n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)$

Solution of Problem 3

Solution:

$$
\begin{aligned}
L.H.S. &= \frac{1}{a_1 + a_n}\left(\frac{a_1 + a_n}{a_1 a_n} + \frac{a_1 + a_n}{a_2 a_{n-1}} + \ldots + \frac{a_1 + a_n}{a_n a_1}\right)\\
&= \frac{1}{a_1 + a_n}\left(\frac{a_1 + a_n}{a_1 a_n} + \frac{a_2 + a_{n-1}}{a_2 a_{n-1}} + \ldots + \frac{a_n + a_1}{a_n a_1}\right)\\
&= \frac{1}{a_1 + a_n}\left[\left(\frac{1}{a_n} + \frac{1}{a_1}\right) + \left(\frac{1}{a_2} + \frac{1}{a_{n-1}}\right) + \ldots + \left(\frac{1}{a_n} + \frac{1}{a_1}\right)\right]\\
&= \frac{2}{a_1 + a_n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)
\end{aligned}
$$

Problem 4

4. If $a_1, a_2, a_3, ...$ be in A.P. such that $a_i \neq 0$, show that $S = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + ... + \frac{1}{a_n a_{n+1}} = \frac{n}{a_1 a_{n+1}}$

Solution of Problem 4

Solution:

$$t_1 = \frac{1}{a_1 a_2} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)$$

$$t_2 = \frac{1}{d}\left(\frac{1}{a_2} - \frac{1}{a_3}\right)$$

$$\cdots$$

$$t_n = \frac{1}{d}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)$$

Adding, we get

$$S = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{n+1}}\right) = \frac{n}{a_1 a_{n+1}}$$

Problem 5

5. If $a_1, a_2, a_3, \dots, a_n$ be in A.P. and $a_1 = 0$, show that $\frac{a_3}{a_2} + \frac{a_4}{a_3} + \dots + \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$

Solution of Problem 5

Solution: $\because a_1 = 0 \therefore a_2 = d, a_3 = 2d, \dots, a_n = (n-1)d$

$$L.H.S. = \frac{2}{1} + \frac{3}{2} + \dots + \frac{n-1}{n-2} - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-3}\right)$$

$$= (1+1) + \left(1 + \frac{1}{2}\right) + \dots + \left(1 + \frac{1}{n-2}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-3}\right)$$

$$= n - 2 + \frac{1}{n-2} = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$$

Problem 6

6. If $a_1, a_2, \dots, a_n$ are in A.P., whose common difference is d, show that $\sum_{k=1}^{n} \frac{a_k a_{k+1} a_{k+2}}{a_k + a_{k+2}}$
$= \frac{n}{2}\left[a_1^2 + (n+1)a_1 d + \frac{(n-1)(2n+5)}{6} d^2\right]$

Solution of Problem 6

Solution:

$$L.H.S. = \sum_{k=1}^{n} \frac{a_k a_{k+1} a_{k+2}}{(a_{k+1}-d)+(a_{k+1}+d)} = \frac{1}{2}\sum_{n=1}^{k} a_k a_{k+2}$$

$$= \frac{1}{2}\sum_{k=1}^{n}(a_{k+1}-d)(a_{k+1}+d) = \frac{1}{2}\sum_{k=1}^{n}(a_{k+1}^2-d^2)$$

$$= \frac{1}{2}\sum_{k=1}^{n}[(a_1+kd)^2-d^2] = \frac{1}{2}\sum_{k=1}^{n}[a_1^2+2a_1kd+(k^2-1)d^2]$$

$$= \frac{n}{2}\left[a_1^2+(n+1)a_1d+\frac{(n-1)(2n+5)}{6}d^2\right]$$

Problem 7

7. If x, y and z are positive real numbers different from 1, and $x^{18} = y^{21} = z^{28}$, show that $3, 3\log_y x, 3\log_z y, 7\log_x z$ are in A.P.

Solution of Problem 7

Solution:

$$x^{18} = y^{21} \Rightarrow 18 \log x = 21 \log y \Rightarrow \log_y x = \frac{21}{18} = \frac{7}{6}$$

$$y^{21} = z^{28} \Rightarrow \log_z y = \frac{4}{3}$$

$$x^{18} = z^{28} \Rightarrow \log_x z = \frac{9}{14}$$

$$3 \log_y x = \frac{7}{2}, 3 \log_z y = 4, 7 \log_x z = \frac{9}{2}$$

Clearly, $3, \frac{7}{2}, 4, \frac{9}{2}$ are in A.P.

Problem 8

8. If $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x} dx$, then $I_1, I_2, I_3, ...$ are in A.P.

Solution of Problem 8

Solution:

$$I_{n+2} + I_n - 2I_{n+1} = \int_0^{\frac{\pi}{2}} \frac{\sin^2(n+2)x + \sin^2 nx - 2\sin^2(n+1)x}{\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 - \cos(2n+4)x + 1 - \cos 2nx - 2 + 2\cos(2n+2)x}{2\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos(2n+2)x - 2\cos(2n+2)x\cos 2x}{2\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos(2n+2)x.2\sin^2 x}{2\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{2}} 2\cos(2n+2)x dx$$

$$= 0$$

Thus, $I_1, I_2, I_3, ...$ are in A.P.

Problem 9

9. Can there be an A.P. whose terms are distinct prime numbers?

Solution of Problem 9

Solution: Let $a_1, a_2, a_3, ...$ be an A.P., whose terms are ditinct prime numbers.

Clearly, a_1 is a positive integer greater than 1.

Also, c.d. of A.P. i.e. $d = a2 - a1$, then $d \geq 1$

Now $(a_1 + 1)$th term of A.P. $= a_1 + a_1 d = a_1(1 + d)$

Since a_1 is a positive no. and $1 + d$ is a positive integer greater or equal than two it is a composite number. Thus, an A.P. of distinct prime no. is not possible.

Problem 10

10. Four distinct no. are in A.P. If one of these integers is sum of the squares of remaining three, then 0 must be one of the numbers in A.P.

Solution of Problem 10

Solution: Let the four distinct integers in A.P. be $a, a+d, a+2d, a+3d$ where $d>0$

Let $a+3d = a^2+(a+d)^2+(a+2d)^2 = 3a^2+6ad+5d^2$

$\Rightarrow 5d^2+3(2a-1)d+3a^2-a=0$

$\because d$ is real $\therefore 9(2a-1)^2-20(3a^2-a)\geq 0$

$\Rightarrow -24a^2-16a+9\geq 0 \Leftrightarrow \frac{-4-\sqrt{70}}{12}\leq a\leq \frac{-4+\sqrt{70}}{12}$

$\therefore a=-1,0 \Rightarrow d=1,\frac{4}{5}$

We find that $-1,0,1,2$ to be the sequence of numbers.