Summation of Series
Theory and Problems 1-10

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Sum of $\sum_{i=1}^{n} i^2$

We observe that

$$i^3 - (i-1)^3 = 3i^2 - 3i + 1$$

$$\sum_{i=1}^{n} i^3 - (i-1)^3 = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$n^3 = 3\sum_{i=1}^{n} i^2 - 3\frac{n(n+1)}{2} + n$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of $\sum_{i=1}^{n} i^3$

Following like previously, we observe that

$$i^4 - (i-1)^4 = 4i^3 - 6i^2 + 4i - 1$$

$$\sum_{i=1}^{n} i^4 - (i-1)^4 = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$n^4 = 4\sum_{i=1}^{n} i^3 - 6.\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - n$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Arithmetico Geometric Series

If the terms of an A. P. are multiplied by the corresponding terms of a G. P., then the new series obtained is called an Arithmetico Geometric series.

Example: If the terms of the arithmetic series $2 + 5 + 8 + 11 + ...$ are multiplied with the corresponding terms of the geometric series $x + x^2 + x^3 + ...$ then the following arithmetico-geometric series is formed

$$2x + 5x^2 + 8x^3 + 11x^4 + ...$$

Sum of an Arithmetic Geometric Series

Let $a_1, a_2, \dots, a_n$ be an A.P. and $b_1, b_2, \dots, b_2$ be a G.P. Let d be the common difference of the A.P. and r be the common ratio of the G.P. Let

$$S_n = ab + (a+d)br + (a+2d)br^2 + \dots + [a+(n-1)d]br^{n-1}$$

We multiply each term by r and write first term below second, second term below thirs and so on

$$rS_n = abr + (a+d)br^2 + (a+2d)br^3 + \dots [a+(n-1)d]br^n$$

Subtracting, we get

$$(1-r)S_n = ab + dbr + dbr^2 + \dots + dbr^{n-1} - [a+(n-1)d]br^2$$

$$= ab + \frac{dbr(1-r^{n-1})}{1-r} - [a+(n-1)d]br^n$$

$$S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r} (r \neq 1)$$

If $-1 < r < 1$, then $\lim_{n \to \infty} r^n = 0$

$$S_\infty = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$$

Problem 1

1. Find the sum of n terms of the series whose nth term is $12n^2 - 6n + 5$

Solution of Problem 1

Solution: We have $t_n = 12n^2 - 6n + 5$

$$S_n = \sum_{i=1}^{n} t_i = 12\sum_{i=1}^{n} i^2 - 6\sum_{i=1}^{n} i + \sum_{i=1}^{n} 5$$

$$= 12\frac{n(n+1)(2n+1)}{6} - 6\frac{n(n+1)}{2} + 5n$$

$$= 2n(n+1)(2n+1) - 3n(n+1) + 5n$$

$$= 4n^3 + 3n^2 + 4n$$

Problem 2

2. Find the sum to n terms of the series $1^2 + 3^2 + 5^2 + 7^2 + ...$

Solution of Problem 2

Solution: Clearly, nth term $= 2n - 1$ for A.P. $1, 3, 5, 7, ...$

Thus, nth term of the given series $= t_n = (2n-1)^2 = 4n^2 - 4n + 1$

Thus,

$$
\begin{aligned}
S_n &= 4\sum_{i=1}^{n} t_i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\
&= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\
&= \frac{n}{3}(4n^2 - 1)
\end{aligned}
$$

Problem 3

3. Find the sum to n terms of the series $1.2.3 + 2.3.4 + 3.4.5 + ...$

Solution of Problem 3

Solution: nth term of the series $= [1+(n-1).1].[2+(n-1).1].[3+(n-1).1] = n^3+3n^2+2n$

$$S_n = \sum_{i=1}^{n} t_i = \sum n^3 + 3\sum n^2 + 2\sum n$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 3.\frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n+1+2\right]$$

$$= \frac{n(n+1)}{2}(\frac{n^2+5n+6}{2})$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

Problem 4

4. Find the sum of the series $1.n + 2.(n-1) + 3.(n-2) + ... + n.1$

Solution of Problem 4

Solution:

$$t_i = i.[n - (i-1)] = ni - i^2 + i$$

$$S_n = n\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)}{2}\left[n - \frac{2n+1}{3} + 1\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$

Problem 5

5. Find the sum to n terms of the series $1 + (1 + 2) + (1 + 2 + 3) + ...$

Solution of Problem 5

Solution:

$$t_i = 1 + 2 + 3 + ... + i = \frac{i(i+1)}{2}$$

$$\Rightarrow S_n = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

Problem 6

6. Find the sum to n terms of the series $1 + (2+3) + (4+5+6) + ...$

Solution of Problem 6

Solution: First term has 1 number, second term has 2 numbers, third term has 3 numbers and so on. Thus, nth term will have n numbers. Total no. of numbers for n terms $= 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. Thus,

$$S_n = 1 + 2 + 3 + ... + \frac{n(n+1)}{2}$$

$$= \frac{\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} + 1\right)}{2}$$

$$= \frac{n(n+1)(n^2+n+2)}{8}$$

Problem 7

7. Find the sum of series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$ to 16 terms.

Solution of Problem 7

7.

$$t_i = \frac{1^3 + 2^3 + 3^3 + ... + i^3}{1 + 3 + 5 + ... + (2i - 1)}$$

$$= \frac{\left\{\frac{i(i+1)}{2}\right\}^2}{\frac{i}{2}[2.1 + (i - 1)2]}$$

$$= \frac{(i + 1)^2}{4}$$

$$S_{16} = \sum_{i=1}^{16} \frac{i^2 + 2i + 1}{4}$$

$$= \frac{1}{4}\left[\frac{16(16 + 1)(2.16 + 1)}{6} + \frac{2.16(16 + 1)}{2} + 16\right]$$

$$= 446$$

Problem 8

8. Find $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$ to 10 terms.

Solution of Problem 8

Solution:

$$t_i = [(2i+1)^3 - (2n)^3]$$
$$= 12i^2 + 6i + 1$$
$$S_{10} = \sum_{i=1}^{10} (12i^2 + 6i + 1)$$
$$= 12.\frac{10(10+1)(20+1)}{6} + 6.\frac{10.11}{2} + 10$$
$$= 4960$$

Problem 9

9. Find $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ...$ to n terms.

Solution to Problem 9

Solution: Rewriting terms

$$t_1 = \frac{1}{1} - \frac{1}{2}$$
$$t_2 = \frac{1}{2} - \frac{1}{3}$$
$$t_3 = \frac{1}{3} - \frac{1}{4}$$
$$\cdots$$
$$t_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding all the terms

$$S_n = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

Problem 10

10. Find the sum of $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ...$ to infinity.

Solution of Problem 10

Solution:$t_n = \frac{1}{n(n+1)(n+2)}$
Let $t_n = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$
$A =$ values of $\frac{1}{(n+1)(n+2)}$ when $n = 0 \therefore A = \frac{1}{2}$
$B =$ values of $\frac{1}{n(n+2)}$ when $n = -1 \therefore B = -1$
$C =$ values of $\frac{1}{n(n+1)}$ when $n = -2 \therefore C = \frac{1}{2}$

$$t_n = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$t_1 = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.3}$$
$$t_2 = \frac{1}{2.2} - \frac{1}{3} + \frac{1}{2.4}$$
$$t_3 = \frac{1}{2.3} - \frac{1}{4} + \frac{1}{2.5}$$
$$\cdots$$
$$t_{n-2} = \frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n}$$
$$t_{n-1} = \frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}$$
$$t_n = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

Solution of Problem 10

Adding, we get

$$S_n = \frac{1}{2.1} - \frac{1}{2} + \frac{1}{2.2} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$S_\infty = \frac{1}{4}$$