Complex Numbers Problems 201-210

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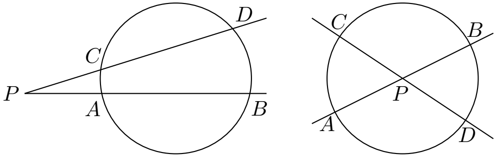
Problem 201

201. Two different non-parallel lines cut the circle $|z| = r$ at points a, b, c, d respectively. Prove that these two lines meet at point given by $\frac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{a^{-1}b^{-1}c^{-1}d^{-1}}$.

Solution of Problem 201

Solution:



Let $P(z)$ be the point of intersection and A, B, C, D represent points a, b, c, d respectively. Clearly, P, A, B are collinear. Thus,

$$\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = 0 \Rightarrow z(\overline{a} - \overline{b}) - \overline{z}(a - b) + (a\overline{b} - \overline{a}b) = 0$$

Similarly, P, C, D are collinear and thus

$$\Rightarrow z(\overline{c} - \overline{d}) - \overline{z}(c - d) + (c\overline{d} - \overline{c}d) = 0$$

Eliminating $\overline{z}$ because we have to find z, we have

$$z(\overline{a} - \overline{b})(c - d) - z(\overline{c} - \overline{d})(a - b) = (c\overline{d} - \overline{c}d)(a - b) - (a\overline{b} - \overline{a}b)(c - d)$$

$\because a, b, c, d$ lie on the circle. $|a| = |b| = |c| = |d| = r \Rightarrow a^2 = b^2 = c^2 = d^2 = r^2$

$\Rightarrow a\overline{a} = b\overline{b} = c\overline{c} = d\overline{d} = r^2$

$\Rightarrow \overline{a} = \frac{r^2}{a}, \overline{b} = \frac{r^2}{b}, \overline{c} = \frac{r^2}{c}, \overline{d} = \frac{r^2}{d}$

Putting these values in the equation we had obtained,

$z\left(\frac{r^2}{a} - \frac{r^2}{b}\right)(c-d) - z\left(\frac{r^2}{c} - \frac{r^2}{d}\right)(a-b) = \left(\frac{cr^2}{d} - \frac{dr^2}{c}\right)(a-b) - \left(\frac{ar^2}{b} - \frac{br^2}{a}\right)(c-d)$

Solving this for z, we arrive at desired answer.

Problem 202

202. If $z = 2 + t + i\sqrt{3 - t^2}$, where t is real and $t^2 < 3$, show that $\left|\frac{z+1}{z-1}\right|$ is independent of t. Also, show that the locus of point z for different values of t is a circle and find its center and radius.

Solution of Problem 202

Solution: $\frac{z+1}{z-1} = \frac{3+t+i\sqrt{3-t^2}}{1+t+i\sqrt{3-t^2}} \Rightarrow \left|\frac{z+1}{z-1}\right|^2 = \frac{(3+t)^2+(3-t^2)}{(1+t)^2+(3-t^2)} = \frac{6(t+2)}{2(t+2)} = 3$

Thus, $\left|\frac{z+1}{z-1}\right|$ is independent of t.

Let $z = x + iy = 2 + t + i\sqrt{3-t^2} \Rightarrow x = t + 2, y = \sqrt{3-t^2} = \sqrt{3-(x-2)^2}$

$\Rightarrow (x-2)^2 + y^2 = 3$, which is equation of a circle with center at $(2, 0)$ having radius $\sqrt{3}$ units.

Problem 203

203.