Geometric Progression
Problems 91-100

Shiv Shankar Dayal

September 14, 2021

Problem 91

91. If the sum of the series $\sum_{n=0}^{\infty} r^n, |r| < 1$ is s, then find the sum of the series $\sum_{n=0}^{\infty} r^{2n}$

Solution of Problem 91

Solution:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + ... = \frac{1}{1-r} = s \Rightarrow r = \frac{1}{s} - 1$$

$$\sum_{n=0}^{\infty} r^{2n} = 1 + r^2 + r^4 + ... = \frac{1}{1-r^2} = \frac{1}{1 - \left(\frac{1}{s} - 1\right)^2}$$

Problem 92

92. If for a G.P. $t_m = \frac{1}{n^2}$ and $t_n = \frac{1}{m^2}$ then find the term $t_{\frac{m+n}{2}}$

Solution of Problem 92

Solution:

$$t_m = ar^{m-1} = \frac{1}{n^2}$$

$$t_n = ar^{n-1} = \frac{1}{m^2}$$

Dividing, we get

$$r^{m-n} = \frac{n^2}{m^2} \Rightarrow r = \left(\frac{n^2}{m^2}\right)^{\frac{1}{m-n}}$$

Now, a can be found and $t_{\frac{m+n}{2}}$ can be found.

Problem 93

93. If a, b, c be three successive terms of a G.P. with common ration r and $a < 0$ satisfying the condition $c > 4b - 3a$, then prove that $r > 3$ or $r < 1$.

Solution of Problem 93

Solution: $c = ar^2, b = ar$. We have $c > 4b - 3a \Rightarrow ar^2 > 4ar - 3a$

$$r^2 > 4r - 3 \Rightarrow (r-1)(r-3) > 0$$

$$\Rightarrow r > 3 \text{ or } r < 1$$

Problem 94

94. If $(1-k)(1+2x+4x^3+8x^3+18x^4+32x^5)=1-k^6$, where $k \neq 1$, then find $\frac{k}{x}$

Solution of Problem 94

Solution:

$$1 + 2x + 4x^3 + 8x^3 + 18x^4 + 32x^5 = \frac{1 - k^6}{1 - k}$$

$$\frac{1 - (2x)^6}{1 - 2x} = \frac{1 - k^6}{1 - k}$$

Thus, $k = 2x \Rightarrow \frac{k}{x} = 2$

Problem 95

95. If $(a^2+b^2+c^2)(b^2+c^2+d^2) \leq (ab+bc+cd)^2$, where a, b, c, d are non-zero real numbers, then show that they are in G.P.

Solution of Problem 95

Solution: Rewriting the given inequality we have

$$(b^4 - 2b^2ac + a^2c^2) + (c^4 - 2c^2bd + b^2d^2) + (a^2d^2 - abcd + b^2c^2) \leq 0$$

$$\Rightarrow (b^2 - ac)^2 + (c^2 - ad)^2 + (ad - bc)^2 \leq 0$$

This is possible if and only if $b^2 = ac, c^2 = bd, ac = bd \Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$ i.e. these numbers are in G.P.

Problem 96

96. If $a_1, a_2, \dots, a_n$ are n non-zero numbers such that $(a_1^2 + a_2^2 + \dots + a_{n-1}^2)(a_2^2 + a_3^2 + \dots + a_n^2) \leq (a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n)^2$, then show that $a_1, a_2, \dots, a_n$ are in G.P.

Solution of Problem 96

Solution: Considering terms involving a_1, a_2, a_3, we get

$$(a_1^2a_3^2 + a_2^4 - 2a_2^2a_1a_3) \leq 0 \Rightarrow (a_1a_3 - a_2^2) \leq 0$$

This cannot be -ve but only zero. Thus, a_1, a_2, a_3 are in G.P. Similarly, we can prove for rest of the terms.

Problem 97

97. α, β be the roots of $x^2 - 3x + a = 0$ and γ, δ be the roots of $x^2 - 12x + b = 0$ and the numbers $\alpha, \beta, \gamma, \delta$ form an increasing G.P., then find the values of a and b.

Solution of Problem 97

Solution: $\alpha, \beta, \gamma, \delta$ being in an increasing G.P., they may be taken as k, kr, kr^2, kr^3, where $r > 1$
Sum of the roots of the given equation, $S_1 = k(1+r) = 3, S_2 = kr^2(1+r) = 12$
Putting S_1 in S_2. we have $3r^2 = 12 \Rightarrow r = 2 \Rightarrow k = 1$
Product of roots, $P_1 = k^2 r = a, P_2 = k^2 r^5 = b$
Thus, $a = 2, b = 32$

Problem 98

98. There are $4n+1$ terms in a certain sequence of which the first $2n+1$ terms are in A.P. of common difference 2 and the last $2n+1$ terms are in G.P. of common ratio $\frac{1}{2}$. If the middle terms of both the A.P. and G.P. are same then find the mid term of the sequence.

Solution of Problem 98

Solution: Given $d = 2, r = \frac{1}{2}$
Middle term of sequence will be $\frac{1}{2}(4n + 1 + 1)$ because no. of terms is odd. Thus, T_{2n+1} is the middle term of sequence, last term of A.P. and first term of G.P.
Thus, $a + 2nd = a + 4n$
Let T_{n+1} and t_{n+1} are mid terms of A.P. and G.P>

$$T_{n+1} = a + nd = a + 2n$$

$$t_{n+1} = T_{2n+1}r^n = (a + 4n)\left(\frac{1}{2}\right)^n$$

Given, $T_{n+1} = t_{n+1}$

$$\therefore a + 2n = (a + 4n)\frac{1}{2^n} \Rightarrow a = \frac{4n - n.2^{n+1}}{2^n - 1}$$

Now mid term can be computed.

Problem 99

99. If $f(x) = 2x + 1$ and three unequal numbers $f(x), f(2x), f(4x)$ are in G.P, then find the number of values for x.

Solution of Problem 99

Solution: Given,

$$(4x+1)^2 = (2x+1)(8x+1)$$

$$2x = 0 \Rightarrow x = 0$$

Problem 100

Solution: Three distinct real numbers, a, b, c are in G.P. such that $a + b + c = xb$, then show that $x < -1$ or $x > 3$

Solution of Problem 100

Solution: Let r be the common ratio, then, we have

$$a + ar + ar^2 = x.ar \Rightarrow r^2 + r(1-x) + 1 = 0$$

Since r is real, discriminant of the quadratic equation will be greater than 0.

$$D > 0 \Rightarrow (1-x)^2 - 4 > 0 \Rightarrow x < -1, x > 3$$