Complex Numbers Problems 91-100

Shiv Shankar Dayal

September 6, 2022

Problem 91

91. Find the general equation of the straight line joining the points $z_1 = 1 + i$ and $z_1 = 1 - i$.

Solution of Problem 91

Solution: We know that equation of the straight line is given by

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow z(\overline{z_1} - \overline{z_2}) - \overline{z}(z_1 - z_2) + z_1\overline{z_2} - \overline{z_1}z_2 = 0$$

$$\Rightarrow z(1 + i - 1 - i) - \overline{z}(1 + i - 1 + i) + (1 + i)^2 - (1 - i)^2 = 0$$

$$\Rightarrow z + \overline{z} - 2 = 0$$

Problem 92

92. If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2 + 8z_3 = 0$, then prove that

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$$

Solution of Problem 92

Solution: Given, $5z_1 - 13z_2 + 8z_3 = 0 \Rightarrow z_2 = \frac{5z_1+8z_3}{5+8}$

This means z_1 divides the line segment joining z_1 and z_2 in the ratio of $5:8$ which also implies that these three points are collinear. Thus,

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$$

Problem 93

93. Find the length of perpedicular from $P(2-3i)$ to the line $(3+4i)z+(3-4i)\overline{z}+9=0$.

Solution of Problem 93

Solution: We know that length of perpendicular from z_1 to $\overline{a}z + a\overline{z} + b = 0$ is given by $\frac{|\overline{a}z_1 + a\overline{z_1} + b|}{2|a|}$.

Thus desired length $= \frac{|(2-3i)(3+4i)+(2+3i)(3-4i)+9|}{2|3-4i|}$

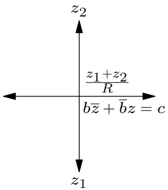
$= \frac{45}{10} = \frac{9}{2}$

Problem 94

94. If a point z_1 is a reflection of a point z_2 through the line $b\overline{z} + \overline{b}z = c, b \neq 0$ in the argand plane, then prove that $\overline{b}z_2 + b\overline{z_1} = c$.

Solution of Problem 94

Solution:



Since mid-point lies on the given line, therefore $b\left(\frac{\overline{z_1}+\overline{z_2}}{2}\right)+\overline{b}\left(\frac{z_1+z_2}{2}\right)=c$

Since line segment joining z_1 and z_2 is perpedicular to the given line therefore, Slope of z_1z_2 + Slope of line = 0

$\Rightarrow \frac{z_2-z_1}{\overline{z_2}-\overline{z_1}}-\frac{b}{\overline{b}}=0$

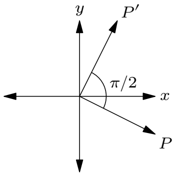
Solving these two equations, we get $\overline{b}z_2+b\overline{z_1}=c$.

Problem 95

95. The point represented by the complex number $2 - i$ is rotated about origin by an angle $\pi/2$ in the anti-clockwise direction. Find the new coordinates.

Solution of Problem 95

Solution:



Let $z = 2 - i$ then after rotation new point would be $z.e^{i\pi/2} = (2-i)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= (2-i)i = 1 + 2i$

Problem 96

96. A particle P starts from the point $z_0 = 1 + 2i$. It first moves horizantally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1, the particle moves $\sqrt{2}$ units in the direction of vector $\hat{i} + \hat{j}$ and it then rotaes about origin in anti-clockwise direction for an angle $\pi/2$ to reach z_2. Find the coordinates of z_2.

Solution of Problem 96

Solution: Coordinate of z_0 after moving 5 points horizontally and 3 points vertically away from origin would be $6 + 5i$.

It then moves in the direction of vecor $\hat{i} + \hat{j}$ for $\sqrt{2}$ units. This vector makes angle $\pi/4$ with x-axis. So new coordinate would be $6 + \sqrt{2}\cos\pi/4 + 5 + \sqrt{2}\sin\pi/4 = 7 + 6i$.

It then rotates by angle $\pi/2$ so new coordinate would be $(7 + 6i)e^{i\pi/2} = (7 + 6i)i = -6 + 7i$

Problem 97

97. A man walks a distance of 3 units from the origin in North-East direction. Then he walks 4 units in North-West direction. Find the final coordinates.

Solution of Problem 97

Solution: North-East direction makes angle of $\pi/4$ with x-axis. So coordinates of point 3 units from origin in North-East direction $= 3.e^{i\pi/4} = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$

North-West direction makes angle of $3\pi/4$ with x-axis. A disaplacement of 4 units in this direction will mean a shift in coordinates by $4.e^{i3\pi/4} = 4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{4}{\sqrt{2}} + i\sin\frac{4}{\sqrt{2}}$

Thus, final coordiate would be sum of the above two i.e. $-\frac{1}{\sqrt{2}} + i\frac{7}{\sqrt{2}}$

Problem 98

98. If three complex numbers satisfty the relationship $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$, then prove that z_1, z_2 and z_3 form an equilateral triangle.

Solution of Problem 98

Solution: Given, $\frac{z_1-z_3}{z_2-z_3} = \frac{1-i\sqrt{3}}{2} = \frac{1-i\sqrt{3}}{2} . \frac{1+i\sqrt{3}}{2}$

$= \frac{1+3}{2(1+i\sqrt{3})} = \frac{2}{1+i\sqrt{3}}$

$\Rightarrow \frac{z_2-z_3}{z_1-z_3} = \frac{1+i\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\Rightarrow \left|\frac{z_2-z_3}{z_1-z_3}\right| = 1$ and $arg\left(\frac{z_2-z_3}{z_1-z_3}\right) = \frac{\pi}{3}$

Hence, the triangle is equilateral.

Problem 99

99. If z_1, z_2 and z_3 form an equilateral triangle then prove that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$. and hence $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$

Solution of Problem 99

99. Since sides of an equilateral triangle make an angle of 60° with each other, therefore

$\frac{z_3-z_1}{z_2-z_1} = \cos 60^\circ \pm \sin 60^\circ = \frac{1 \pm i\sqrt{3}}{2}$

$\Rightarrow 2z_3 - 2z_1 + z1 - z_2 = \pm i(z_2 - z_1)\sqrt{3}$

$\Rightarrow (2z_3 - z_1 - z_2)^2 = 3(z_2 - z_1)^2$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

$\Rightarrow z_1 z_2 + z_2 z_3 + z_3 z_1 - z_z^2 - z_2^2 - z_3^2 + z_1 z_2 - z_1 z_2 + z_2 z_3 - z_2 z_3 + z_1 z_3 - z_1 z_3 = 0$

$\Rightarrow (z_1 - z_2)(z_2 - z_3) + (z_2 - z_3)(z_3 - z_1) + (z_3 - z_1)(z_1 - z_2) = 0$

$\Rightarrow \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$

Problem 100

100. If z_1, z_2 and z_3 are vertices of an equilateral triangle and z_0 is the circumcenter then prove that $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

Solution of Problem 100

Solution: Since it is an equilateral triangle centroid and circumcenters would be identical. $\therefore z_0 = \frac{z_1+z_2+z_3}{3}$

Since it is an equilateral triangle, we have just proven that $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

From first equation, we have $\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$

$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2) \Rightarrow 3z_0^2 = z_1^2 + z_2^2 + z_3^2$