Complex Numbers

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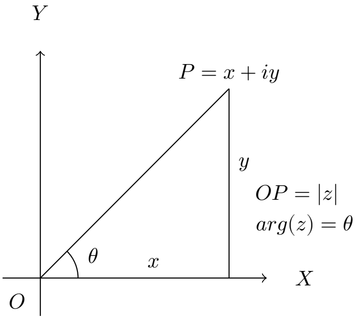
Theory contd

A complex number z which we have considered to be equal to $x + iy$ can be represented by a point P whose caretesian coordinates are (x, y) referred to rectangular axes Ox and Oy where O is origin i.e. $(0, 0)$ and are called *real* and *imaginary* axis respectively. The xy two dimensional plane is also called *Argand plane, complex plane or Gaussian plane*. The point P is also called the *image* of the complex number and z is also called the *affix* or *complex coordinate* of point P.
The modulus is given by the length of segment OP which is equal to $OP = \sqrt{x^2 + y^2} = |z|$. Thus, $|z|$ is the length of OP.



In the diagram θ is known as the argument of z. it is the angle made with positive direction(i.e. counter-clockwise) of real axis. This arhument is not unique. If θ is an argument of a complex number z then $2n\pi + \theta$ where $n \in I$ where I is the set of integers will be arguments as well. The value of argument for which $-\pi < \theta \leq \pi$ is called the *principal argument*.

Different Arguments of a Complex Number

In the digram given in previous slide the argument is given as

$$arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

this value is for when z is in first quadrant. When z will lie in second, third and fourth quadrants then arguments will be

$$arg(z) = \pi - \tan^{-1}\left(\frac{y}{|z|}\right), arg(z) = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right), arg(z) = -\tan^{-1}\left(\frac{|y|}{x}\right)$$

Polar Form of a Complex Number

If z is a non-zero complex number, then we can write $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = arg(z)$

In this case z is also given by $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ where $n \in I$.

Euler's Formula

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$.

Properties of Arguments

If z, z_1 and z_2 are complex numbers then

1. $arg(\overline{z}) = -arg(z)$. This can be easily proven as $z = x + iy$ and $\overline{z} = x - iy$ so sign of argument will get a -ve sign as y gets one.
2. $arg(z_1 z_2) = arg(z_1) + arg(z_2) + 2k\pi$ where
$$k = \begin{cases} 0 & -\pi < arg(z_1) + arg(z_2) \leq \pi \\ 1 & -2\pi < arg(z_1) + arg(z_2) \leq -\pi \\ -1 & -\pi < arg(z_1) + arg(z_2) \leq 2\pi \end{cases}$$
3. $arg(z_1 \overline{z_2}) = arg(z_1) - arg(z_2)$
4. $\arg\left(\frac{z_1}{z_2}\right) = arg(z_1) + arg(z_2) + 2k\pi$ where k is same as item 2 with $+$ sign between z_1 and z_2 are replaced with $-$ sign.
5. $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow arg(z_1) - arg(z_2) = \pi/2$
6. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow arg(z_1) = arg(z_2)$
7. $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$
8. $|z_1 - z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 + \theta_2)$

Vector Representation

Complex numbers can also be represented as vectors. Length of the vector is nothing but modulus of complex number and argument is the angle which the vector makes with the real axis. It is denoted as $\overrightarrow{OP}$ where OP represents the vector of the complex number z.