Complex Numbers Problems 201-210

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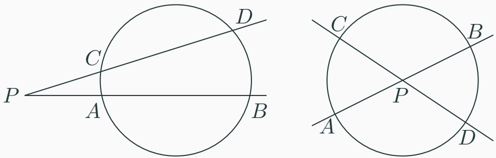
201. Two different non-parallel lines cut the circle $|z| = r$ at points a, b, c, d respectively. Prove that these two lines meet at point given by $\frac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{a^{-1}b^{-1}c^{-1}d^{-1}}$.

Solution:



Let $P(z)$ be the point of intersection and A, B, C, D represent points a, b, c, d respectively. Clearly, P, A, B are collinear. Thus,

$$\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = 0 \Rightarrow z(\overline{a} - \overline{b}) - \overline{z}(a - b) + (a\overline{b} - \overline{a}b) = 0$$

Similarly, P, C, D are collinear and thus

$$\Rightarrow z(\overline{c} - \overline{d}) - \overline{z}(c - d) + (c\overline{d} - \overline{c}d) = 0$$

Eliminating $\overline{z}$ because we have to find z, we have

$$z(\overline{a} - \overline{b})(c - d) - z(\overline{c} - \overline{d})(a - b) = (c\overline{d} - \overline{c}d)(a - b) - (a\overline{b} - \overline{a}b)(c - d)$$

$\because a, b, c, d$ lie on the circle. $|a| = |b| = |c| = |d| = r \Rightarrow a^2 = b^2 = c^2 = d^2 = r^2$

$\Rightarrow a\overline{a} = b\overline{b} = c\overline{c} = d\overline{d} = r^2$

$\Rightarrow \overline{a} = \frac{r^2}{a}, \overline{b} = \frac{r^2}{b}, \overline{c} = \frac{r^2}{c}, \overline{d} = \frac{r^2}{d}$

Putting these values in the equation we had obtained,
$z\left(\frac{r^2}{a} - \frac{r^2}{b}\right)(c-d) - z\left(\frac{r^2}{c} - \frac{r^2}{d}\right)(a-b) = \left(\frac{cr^2}{d} - \frac{dr^2}{c}\right)(a-b) - \left(\frac{ar^2}{b} - \frac{br^2}{a}\right)(c-d)$

Solving this for z, we arrive at desired answer.

Problem 202

202. If $z = 2 + t + i\sqrt{3 - t^2}$, where t is real and $t^2 < 3$, show that $\left|\frac{z+1}{z-1}\right|$ is independent of t. Also, show that the locus of point z for different values of t is a circle and find its center and radius.

Solution of Problem 202

Solution: $\frac{z+1}{z-1} = \frac{3+t+i\sqrt{3-t^2}}{1+t+i\sqrt{3-t^2}} \Rightarrow \left|\frac{z+1}{z-1}\right|^2 = \frac{(3+t)^2+(3-t^2)}{(1+t)^2+(3-t^2)} = \frac{6(t+2)}{2(t+2)} = 3$

Thus, $\left|\frac{z+1}{z-1}\right|$ is independent of t.

Let $z = x + iy = 2 + t + i\sqrt{3-t^2} \Rightarrow x = t+2, y = \sqrt{3-t^2} = \sqrt{3-(x-2)^2}$

$\Rightarrow (x-2)^2 + y^2 = 3$, which is equation of a circle with center at $(2,0)$ having radius $\sqrt{3}$ units.

203. Let z_1, z_2, z_3 be three non-zero complex numbers such that $z_2 \neq 1, |z_1| = a, |z_2| = b$ and $|z_3| = c$. Let

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0,$$

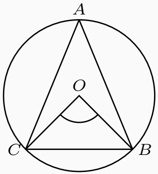
then show that $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$.

Solution of Problem 203

Solution: Given $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \Rightarrow a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$

$\because z_1, z_2, z_3$ are three non-zero complex numbers, hence $a^2+b^2+c^2-ab-bc-ca = 0$

$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0 \Rightarrow a = b = c$. This can be represented by following diagram



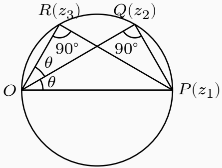
Now $OA = OB = OC$, where O is the origin and A, B and C are the points representing z_1, z_2 and z_3 respectively.
$\therefore O$ is the circumcenter of $\triangle ABC$.

Now $\arg\left(\frac{z_3}{z_2}\right) = \angle BOC = 2\angle BAC = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$.

Problem 204

204. P is such a point that on a circle with OP as diameter, two points Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q and R are represented by complex numbers z_1, z_2 and z_3 respectively, show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2 \theta$.

Solution:



$z_2 = \frac{OQ}{OP} z_1 e^{i\theta} = \cos\theta z_1 e^{i\theta}$ and $z_3 = \frac{OR}{OP} z_1 e^{i2\theta} = \cos 2\theta z_1 e^{i2\theta}$

$\Rightarrow z_2^2 = \cos^2\theta z_1^2 e^{i2\theta} \Rightarrow z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$

Problem 205

205. Find the equation in complex variables of all the circles which are orthogonal to $|z| = 1$ and $|z - 1| = 4$.

Solution of Problem 205

Solution: Given circles are $|z| = 1 \Rightarrow x^2 + y^2 - 1 = 0$ and $|z - 1| = 4 \Rightarrow x^2 - 2x + y^2 - 15 = 0.$

Let the circles cut by these two orthogonally is $x^2 + y^2 + 2gx + 2fy + c = 0$

Since first circle cuts this family of circles orthoginally, therefore

$2g.0 + 2f.0 = c - 1 \Rightarrow c = 1$ and $2g(-1) + 2f.0 = c - 15 \Rightarrow g = 7$

Thus, required circles are $x^2 + y^2 + 14x + 2fy + 1 = 0 \Rightarrow |z + 7 + if| = \sqrt{48 + f^2}$

Problem 206

206. Find the real values of the parameter t for which there is at least one complex number $z = x + iy$ satisfying the condition $|z+3| = t^2 - 2it + 6$ and the inequality $z - 3\sqrt{3}i < t^2$.

Solution of Problem 206

Solution: Given, $|z+3| = t^2 - 2t + 6$ which is equation of a circle having center $(-3, 0)$ and radius $t^2 - 2t + 6$. Let $A = (-3, 0)$ and $r_1 = t^2 - 2t + 6$. In this case z lies on the circle.

Also, $|z - 3\sqrt{3}i| < t^2$ implies z lies on the interior of the circle having center $(0, 3\sqrt{3})$ and radius t^2. Let $B = (0, 3\sqrt{3})$ and $r_2 = t^2$. $AB = \sqrt{3^2 + 27} = 6$. $r_2 - r_1 = 2(t - 3)$

Clearly, when the two circles are disjoint or touching each other no solution is possible. This leads to following cases:

Case I: When $t > 3$ i.e. $r2 > r_1$

In this case at least one z is possible if $AB < r_1 + r_2 \Rightarrow 6 < 2(t^2 - t + 3) \Rightarrow t < 0$ or $t > 1 \Rightarrow 3 < t < \infty$

Case II: When $t \leq 3$ i.e. $r_1 > r_2$

In this case at least one z will be possible if $|r_1 - r_2| \leq AB < r_1 + r_2$

$2(3 - t) \leq 6 < 2(t^2 - t + 3)$ i.e. $t \leq 0$ and $t < 0$ or $t > 1$

Combining all solutions we gace $1 < t < \infty$

Problem 207

207. If a, b, c and d are real values and $ad > bc$, show that the imginary parts of the complex number z and $\frac{az+b}{cz+d}$ have the same sign.

Solution of Problem 207

Solution: Let $z = x + iy$. $\frac{az+b}{cz+d} = \frac{ax+b+iay}{cx+d+icy} = \frac{(ax+b+iay)(cx+d-icy)}{(cx+d)^2+c^2y^2}$

$\Im\left(\frac{az+b}{cz+d}\right) = \frac{ay(cx+d)-cy(ax+b)}{(cx+d)^2+c^2y^2} = \frac{ady-bcy}{(cx+d)^2+c^2y^2}$

$\because ad > bc$, therfore the signs of imaginary parts of z and $\frac{az+b}{cz+d}$ are the same.

208. If $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ and $z_1 = \frac{i(z_2+1)}{z_2-1}$, prove that

$$x_1^2 + y_1^2 - x_1 = \frac{x_2^2 + y_2^2 + 2x_2 - 2y_2 + 1}{(x_2 - 1)^2 + y_2^2}$$

Solution of Problem 208

Solution: Given, $z_1 = \frac{i(z_2+1)}{z_2-1} \Rightarrow x_1 + iy_1 = \frac{-y_2+i(x_2+1)}{(x_2-1)+iy_2} = \frac{[-y_2+i(x_2+1)][(x_2-1)+iy_2]}{(x_2-1)^2+y_2^2}$

Comparing real and imaginary parts, we have

$x_1 = \frac{-y_2(x_2-1)-(x_2+1)y_2}{(x_2-1)^2+y_2^2} = \frac{-2x_2y_2}{(x_2-1)^2+y_2^2}$ and $y_1 = \frac{x_2^2-1-y_2^2}{(x_2-1)^2+y_2^2}$

Substituting for x_1 and y_1 in $x_1^2 + y_1^2 - x_1$ we will arrive at the desired result.

209. Simplify the following:

$$\frac{(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3}{(\cos 2\theta + i\sin 2\theta)^5}.$$

Solution of Problem 209

Solution: $(\cos 3\theta - i\sin 3\theta)^6 = (e^{-i3\theta})^6 = e^{-i18\theta}$

$(\cos 2\theta + i\sin 2\theta)^5 = (e^i 2\theta)^5 = e^{i10\theta}$

$(\sin\theta - i\cos\theta)^3 = [(-i)^3(\cos\theta + i\sin\theta)^3] = i.e^{i\theta}$

$\frac{(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3}{(\cos 2\theta + i\sin 2\theta)^5} = i.e^{-i25\theta}$

$= \sin 25\theta + i\cos 25\theta$

Problem 210

210. Find all complex numbers such that $z^2 + |z| = 0$.

Solution of Problem 210

Solution: Let $z = x + iy$, then we have $x^2 - y^2 + 2ixy + \sqrt{x^2 + y^2} = 0$

Equating imaginary parts, we have $2xy = 0$ i.e. either $x = 0$ or $y = 0$.

If $x = 0$, then $-y^2 + \sqrt{y^2} = 0 \Rightarrow y^4 + y^2 = 0 \Rightarrow y = 0, y = \pm 1$.

If $y = 0$, then $x^2 + \sqrt{x^2} = 0$ Since x is real only one solution is possible i.e. $x = 0$.

Hence, $z = 0, \pm i$.