Complex Numbers Problems 141-150

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Problem 141

141. The complex numbers z_1 and z_2 such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, prove that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

Solution of Problem 141

Solution: Given, $|z_1| = |z_2|$, $Re(z_1) > 0$ and $Im(z_1) < 0$

$$Re\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{1}{2}\left(\frac{z_1+z_2}{z_1-z_2} + \frac{\overline{z_1}+\overline{z_2}}{\overline{z_1}-\overline{z_2}}\right)$$

$$= \frac{1}{2}\left(\frac{2(|z_1|^2-|z_2|^2)}{|z_1-z_2|^2}\right) = 0$$

Thus, $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

Problem 142

142. If $A(z_1), B(z_1)$ and $C(z_3)$ are the vertices of a $\triangle ABC$ in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then prove that the value of $z_2 = z_3 + i(z_1 - z_3)$.

Solution of Problem 142

Solution: Given, $\frac{AB}{BC} = \sqrt{2} \Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = \frac{|z_1 - z_2|}{|z_3 - z_2|} . e^{i\pi/4}$

$= \frac{AB}{BC} . e^{i\pi/4} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 1 + i$

$\Rightarrow z_1 - z_2 = (1+i)(z_3 - z_2) \Rightarrow z_2 = z_3 + i(z_1 - z_3)$

Problem 143

143. If $z_1 z_2 \in C, z_1^2 + z_2^2 \in R, z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then find the value of $z_1^2 + z_2^2$.

Solution of Problem 143

Solution: Given, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$

$\Rightarrow z_1^3 - 3z_1z_2^2 + iz_2(3z_1^2 - z_2^2) = 2 + 11i \Rightarrow (z_1 + iz_2)^3 = 2 + 11i$ and

$\Rightarrow z_1^3 - 3z_1z_2^2 - iz_2(3z_1^2 - z_2^2) = 2 - 11i \Rightarrow (z_1 - iz_2)^3 = 2 - 11i$

Multiplying above equations, we get

$(z_1^2 + z_2^2)^3 = 4 + 121 = 125 \Rightarrow z_1^2 + z_2^2 = 5$

Problem 144

144. If $\sqrt{1-c^2} = nc - 1$ and $z = e^{i\theta}$, then find the value of $\frac{c}{2n}(1+nz)\left(1+\frac{n}{z}\right)$.

Solution of Problem 144

Solution: Given $\sqrt{1-c^2} = nc - 1 \Rightarrow 1 - c^2 = n^2c^2 - 2nc + 1 \Rightarrow \frac{c}{2n} = \frac{1}{1+n^2}$

$\frac{c}{2n}(1+nz)\left(1+\frac{n}{z}\right) = \frac{1}{1+n^2}\left[1+n^2+n\left(z+\frac{1}{z}\right)\right]$

$= \frac{1}{1+n^2}\left[1+n^2+2\cos\theta+n\right] = 1 + \frac{2n}{1+n^2}\cos\theta = 1 + c\cos\theta$

Problem 145

145. Consider an eclipse having its foci at $A(z_1)$ and $B(z_2)$ in the argand plane. If the eccentricity of the ellipse is e and it is known that origin is an interior point of the ellipse, then prove that $e \in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|}\right)$

Solution of Problem 145

Solution: If $P(z)$ is any point of the ellipse, then equation of ellipse is given by

$|z - z_1| + |z - z_2| = \frac{|z_1 - z_2|}{e}$

If we put z_1 or z_2 in the above equation then L.H.S. becomes $|z_1 - z_2|$.

Thus, for any interior point of the ellipse, we have $|z - z_1| + |z - z_2| < \frac{|z_1 - z_2|}{e}$

If $P(z)$ lies on the ellipse, we have $|z - z_1| + |z - z_2| = \frac{|z_1 - z_2|}{e}$

It is given that origin is an internal point, so $|0 - z_1| + |0 - z_2| < \frac{|z_1 - z_2|}{e}$

$e \in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|}\right)$

Problem 146

146. If $|z-2-i| = |z| \left|\sin\left(\frac{\pi}{4} - \arg(z)\right)\right|$, then find the locus of z.

Solution of Problem 146

Solution: Let $z = x + iy$, then we have

$$|(x-2)+i(y-1)| = |z|\left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right|$$

whhere, $\theta = \arg(z)$

$$\Rightarrow \sqrt{(x-2)^2+(y-1)^2} = \frac{1}{\sqrt{2}}|x-y|$$

which is equation of a parabola.

Problem 147

147. Find the maximum area of the triangle formed by the complex coordinates zz_1 and z_2, which satisfy the relation $|z - z_1| = |z - z_2|$ and $\left|z - \frac{z_1+z_2}{2}\right| \leq r$, where $r > |z_1 - z_2|$.

Solution of Problem 147

Solution: Since $|z - z_1| = |z - z_2|$, therefore z will be one of the vertices of the isosceles triangle where base will be formed by z_1 and z_2.

Also, since $\left|z - \frac{z_1+z_2}{2}\right| \leq r$ so z will lie on the circle whose center is $\frac{z_1+z_2}{2}$ and radius is r. Thus, the distance between segment $z_1 z_2$ will be r.

Thus, the maximum area of the triangle will be $\frac{1}{2}|z_1 - z2|.r$

Problem 148

148. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are complex numbers such that $|z_1| = 1, |z_1| = 2$ and $Re(z_1 z_2) = 0$, and $\omega_1 = a_1 + \frac{ia_2}{2}$ and $\omega_2 = 2b_1 + ib2$, then prove that $|\omega_1| = 1, |\omega_2| = 2$ and $Re(\omega_1 \omega_2) = 0$.

Solution of Problem 148

Solution: Given $|z_1| = 1 \Rightarrow a_1^2 + b_1^2 = 1, |z_2| = 2 \Rightarrow a_2^2 + b_2^2 = 4.$

Also given $Re(z_1 z_2) = 0 \Rightarrow a_1 a_2 - b_1 b_2 = 0 \Rightarrow a_1 a_2 = b_1 b_2$

$\Rightarrow a_2^2 + b_2^2 = 4a_1^2 + 4b_1^2 \Rightarrow a_2^2 - 4a_1^2 = 4b_1^2 - b_2^2 \Rightarrow a_2^2 - 4a_1^2 + 4ia_1 a_2 = 4b_1^2 - b_2^2 + 4ib_1 b_2$

$\Rightarrow (a_2 + 2ia_1)^2 = (2b_1 + ib_2)^2 \Rightarrow a_2 = \pm 2b_1$

$\omega_1 = a_1 + \frac{ia_2}{2} = a_1 \pm b_1 \Rightarrow |\omega_1| = \sqrt{a_1^2 + b_1^2} = 1$

$\omega_2 = 2b_1 + ib_2 = \pm a_2 + ib_2 \Rightarrow |\omega_2| = \sqrt{a_2^2 + b_2^2} = 2$

$Re(\omega_1 \omega_2) = 2a_1 b_1 - 2a_2 b_2 = 0$

Problem 149

149. Let z be a complex number and a be a be a real number such that $z^2 + az + a^2 = 0$, then prove that i) locus of z is a pair of straight lines ii) $\arg(z) = \pm\frac{2\pi}{3}$ iii) $|z| = |a|$

Solution of Problem 149

Solution: Given $z^2 + az + a^2 = 0 \Rightarrow z = a\omega, a\omega^2$ where ω is cube-root of unity.

Thus, it represents a pair of straight lines and $|z| = |a|$

$\arg(z) = \arg(a) + \arg(\omega)$ or $\arg(a) + \arg(\omega^2) = \pm\frac{2\pi}{3}$

Problem 150

150. If $x + \frac{1}{x} = 1$ and $p = x^{4000} + \frac{1}{x^{4000}}$ and q is the the digit at units place in $2^{2^n} + 1, n \in N$ and $n > 1$, then find $p + q$.

Solution of Problem 150

Solution: Given $x+\frac{1}{x}=1 \Rightarrow x^2-x+1=0 \therefore x=-\omega,-\omega^2$

Now, for $x=-\omega, p=\omega^{4000}+\frac{1}{\omega^{4000}}=\omega+\frac{1}{\omega}=-1$

Similarly, for $x=-\omega^2, p=-1$

$2^{2^n}=2^{4k}=16^k=$ a number with last digit as $6 \Rightarrow q=6+1=7$

$\Rightarrow p+q=-1+7=6.$