

Complex Numbers Problems

181-190

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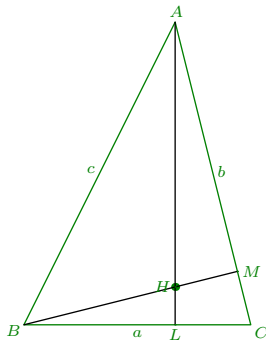
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Problem 181

181. If the vertices of a $\triangle ABC$ are represented by z_1, z_2, z_3 respectively, then show that the orthocenter of $\triangle ABC$ is $\frac{z_1 a \sec A + z_2 b \sec B + z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$ or $\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$.

Solution of Problem 181

Solution: The diagram is given below:



Let AL be perpendicular on BC and H be orthocenter of the $\triangle ABC$.

$$\frac{BL}{LC} = \frac{c \cos B}{b \cos C} = \frac{c \sec C}{b \sec B}, \text{ thus } L \text{ divides } BC \text{ internally in the ratio of } c \sec C : b \sec B$$

$$L = \frac{z_3 c \sec C + z_2 \sec B}{c \sec C + b \sec B}$$

$$\frac{AH}{HL} = \frac{\frac{1}{2} AB \cdot BN \sin \angle ABM}{\frac{1}{2} BL \cdot LH \cdot \sin \angle MBC} = \frac{c \cos A}{c \cos B \cos C} [\because \angle ABM = 90^\circ - A, \angle MBC = 90^\circ - B]$$

$$= \frac{a \cos A}{a \cos B \cos C} = \frac{b \sec B + c \sec C}{a \sec A}$$

$$H = \frac{z_1 a \sec A + z_2 b \sec B + z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$$

Since the above expression is similar w.r.t. A, B and C , therefore it will also lie on the perpendiculars from B and C to opposing sides as well.

$$\text{Thus, orthocenter } H = \frac{z_1 a \sec A + z_2 b \sec B + z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$$

$$H = \frac{z_1 k \sin A \sec A + z_2 k \sin B \sec B + z_3 k \sin C \sec C}{k \sin A \sec A + k \sin B \sec B + k \sin C \sec C}$$

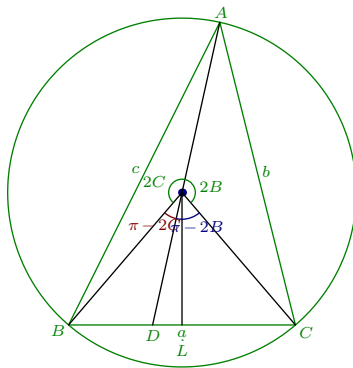
$$H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

Problem 182

182. If the vertices of a $\triangle ABC$ are represented by z_1, z_2 and z_3 respectively, show that its circumcenter is
$$\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}.$$

Solution of Problem 182

Solution: The diagram is given below:



Solution of Problem 182

Solution: Let O be the circumcenter of $\triangle ABC$ where $A = z_1, B = z_2$ and $C = z_3$.

$$\begin{aligned}\frac{BD}{DC} &= \frac{\frac{1}{2}BD \cdot OL}{\frac{1}{2}DC \cdot OL} = \frac{\triangle BOD}{\triangle COD} \\ &= \frac{\frac{1}{2}OB \cdot OD \cdot \sin(\pi - 2C)}{\frac{1}{2}OC \cdot OD \sin(\pi - 2C)} = \frac{\sin 2C}{\sin 2B}\end{aligned}$$

Thus, D divides BC internally in the ratio $\sin 2C : \sin 2B \Rightarrow D = \frac{z_3 \sin 2C + z_2 \sin 2B}{\sin 2C + \sin 2B}$

The complex number dividing AD internally in the ratio $\sin 2B + \sin 2C : \sin 2A$ is

$$\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

Since the above expression is similar w.r.t. A, B and C , therefore it will also lie on the perpendicular bisectors on AC and AB as well.

Let BO produced meet AC at E and CO produced meet AB at F . We can show that, the complex number representing the point dividing the line segment BE internally in the ratio $(\sin 2C + \sin 2A) : \sin 2B$ and the complex number representing the point dividing the line segment CF internally in the ratio

$$(\sin 2A + \sin 2B) : \sin 2C \text{ will be each } = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

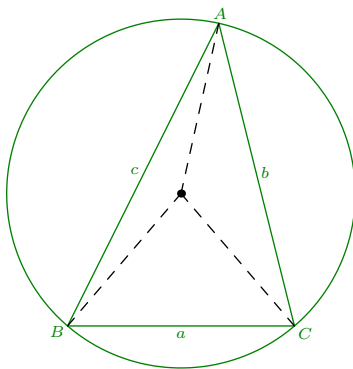
Thus, circumcenter is $\frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

Problem 183

183. Show that the circumcenter of the triangle whose vertices are given by the complex numbers z_1, z_2, z_3 is given by $z = \frac{\sum z_1 \bar{z}_1 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)}$.

Solution of Problem 183

Solution: Consider the diagram given below:



Contd

Let z be the circumcenter of the triangle represented by $A(z_1)$, $B(z_2)$ and $C(z_3)$ respectively, then

$$|z - z_1| = |z - z_2| = |z - z_3| \text{ so we have } |z - z_1| = |z - z_2|$$

$$\Rightarrow |z - z_1|^2 = |z - z_2|^2 \Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$

$$\Rightarrow z\bar{z} + z_1\bar{z}_1 - \bar{z}z_1 - z\bar{z}_1 = z\bar{z} + z_2\bar{z}_1 - \bar{z}z_2 - z\bar{z}_2$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = z_1\bar{z}_1 - z_2\bar{z}_2 \quad (1)$$

Similarly considering $|z - z_1| = |z - z_3|$, we will have

$$\Rightarrow z(\bar{z}_1 - \bar{z}_3) + \bar{z}(z_1 - z_3) = z_1\bar{z}_1 - z_3\bar{z}_3 \quad (2)$$

We have to eliminate \bar{z} from equation (1) and (2) i.e. multiplying equation (1) with $(z_1 - z_3)$ and (2) with $(z_1 - z_2)$, we get following

$$z[\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)] = z_1\bar{z}_1(z_2 - z_3) + z_2\bar{z}_2(z_3 - z_1) + z_3\bar{z}_3(z_1 - z_2)$$

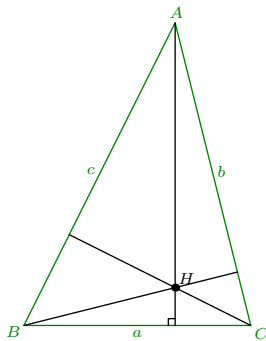
$$\Rightarrow z = \frac{\sum z_1\bar{z}_1(z_2 - z_3)}{\sum \bar{z}_1(z_2 - z_3)}$$

Problem 184

184. Find the orthocenter of the triangle with vertices z_1, z_2, z_3 .

Solution of Problem 184

Solution:



Let z be the orthocenter of $\triangle A(z_1)B(z_2)C(z_3)$ i.e. the intersection point of perpendiculars on sides from opposite vertices.

$$\text{Since } AH \perp BC \therefore \arg\left(\frac{z_1 - z}{z_3 - z_2}\right) = \pm \frac{\pi}{2}$$

$$\Rightarrow \frac{z_1 - z}{z_3 - z_2} \text{ is purely imaginary.}$$

$$\Rightarrow \overline{\left(\frac{z_1 - z}{z_3 - z_2}\right)} = -\left(\frac{z_1 - z}{z_3 - z_2}\right) \Rightarrow \frac{\overline{z_1 - z}}{\overline{z_3 - z_2}} = \frac{z - z_1}{z_3 - z_2}$$

$$\Rightarrow \overline{z_1} - \overline{z} = \frac{(z - z_1)(\overline{z_3} - \overline{z_2})}{z_3 - z_2} \quad \text{Similarly for } BH \perp AC, \overline{z_2} - \overline{z} = \frac{(z - z_2)(\overline{z_1} - \overline{z_3})}{z_1 - z_3}$$

Eliminating \overline{z} like last problem we arrive at the desired result.

Problem 185

185. $ABCD$ is a rhombus described in clockwise direction. Suppose that the vertices A, B, C, D are given by z_1, z_2, z_3, z_4 respectively and $\angle CBA = 2\pi/3$. Show that $2\sqrt{3}z_2 = (\sqrt{3} - i)z_1 + (\sqrt{3} + i)z_3$ and $2\sqrt{3}z_4 = (\sqrt{3} + i)z_1 + (\sqrt{3} - i)z_3$.

Solution of Problem 185

Solution: We have $\angle CBA = \frac{2\pi}{3}$, therefore

$$\frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} \left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right]$$

$$\frac{z_3 - z_2}{z_1 - z_2} = -\frac{1}{2} + \frac{i\sqrt{3}}{2} [\because BC = AB]$$

$$z_3 + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) z_1 = \left(\frac{3}{2} - \frac{i\sqrt{3}}{2}\right) z_2$$

$$\text{Solving this yields } 2\sqrt{3}z_2 = (\sqrt{3} - i)z_1 + (\sqrt{3} + i)z_3$$

$$\text{Also, since diagonals bisect each other} \Rightarrow \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$z_4 = z_1 + z_3 - z_2$$

Substituting the value of z_2 , we get

$$2\sqrt{3}z_4 = (\sqrt{3} + i)z_1 + (\sqrt{3} - i)z_3$$

Problem 186

186. The points P, Q and R represent the numbers z_1, z_2 and z_3 respectively and the angles of the $\triangle PQR$ at Q and R are both $\frac{1}{2}(\pi - \alpha)$. Prove that $(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{\alpha}{2}$.

Solution of Problem 186

Solution: Since $\angle PQR = \angle PRQ = \frac{1}{2}(\pi - \alpha) \therefore PQ = PR$ Also, $\angle QPR = \pi - 2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \alpha$

$$\therefore \arg \frac{z_3 - z_1}{z_2 - z_1} = \alpha \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \frac{PR}{RQ}(\cos \alpha + i \sin \alpha)$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} - 1 = (\cos \alpha - 1) + i \sin \alpha \Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = -2 \sin^2 \frac{\alpha}{2} + i 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$$

$$\Rightarrow \left(\frac{z_3 - z_2}{z_2 - z_1} \right)^2 = -4 \sin^2 \frac{\alpha}{2} \left[\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right]^2 = -4 \sin^2 \frac{\alpha}{2} [\cos \alpha + i \sin \alpha] = -4 \sin^2 \frac{\alpha}{2} \cdot \frac{z_3 - z_1}{z_2 - z_1}$$

$$\Rightarrow (z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{\alpha}{2}$$

Problem 187

187. Points z_1 and z_2 are adjacent vertices of a regular polygon of n sides. Find the vertex z_3 adjacent to z_2 ($z_1 \neq z_3$).

Solution of Problem 187

Solution: Let C be the center of a regular polygon of n sides. Let $A_1(z_1)$, $A_2(z_2)$ and $A_3(z_3)$ be its three consecutive vertices.

$$\angle CA_2A_1 = \frac{1}{2} \left(\pi - \frac{2\pi}{n} \right) \therefore A_1A_2A_3 = \pi - \frac{2\pi}{n}$$

Case I: When z_1, z_2, z_3 are in anticlockwise order. $\Rightarrow z_1 - z_2 = (z_3 - z_2)e^{i(\pi - 2\pi/n)} [\because A_1A_2 = A_3A_2]$

$$z_1 - z_2 = (z_2 - z_3)e^{-i2\pi/n} [\because e^{i\pi} = -1] \Rightarrow z_3 = z_2 - (z_1 - z_2)e^{i2\pi/n}$$

Case II: When z_1, z_2, z_3 are in clockwise order. $\Rightarrow z_3 - z_2 = (z_1 - z_2)e^{i(\pi - i2\pi/n)}$

$$z_3 = z_2 + (z_2 - z_1)e^{-i2\pi/n}$$

Problem 188

188. Let A_1, A_2, \dots, A_n be the vertices of an n sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, find the value of n .

Solution of Problem 188

Solution: Let O be the origin and the complex number representing A_1 be z , then A_2, A_3, A_4 will be represented by $ze^{i2\pi/n}, ze^{i4\pi/n}, ze^{i6\pi/n}$. Let $|z| = a$

$$\begin{aligned} A_1 A_2 &= |z - ze^{i2\pi/n}| = |z| \left| 1 - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \right| \\ &= a \sqrt{\left(1 - \cos \frac{2\pi}{n}\right)^2 + \sin^2 \frac{2\pi}{n}} = a \sqrt{2 \left(1 - \cos \frac{2\pi}{n}\right)} = 2a \sin \frac{\pi}{n} \end{aligned}$$

Similarly, $A_1 A_3 = 2a \sin \frac{2\pi}{n}$ and $A_1 A_4 = 2a \sin \frac{3\pi}{n}$

$$\text{Given } \frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4} \therefore \frac{1}{2a \sin \frac{\pi}{n}} = \frac{1}{2a \sin \frac{2\pi}{n}} + \frac{1}{2a \sin \frac{3\pi}{n}}$$

$$\Rightarrow \sin \frac{\pi}{n} \left(\sin \frac{3\pi}{n} + \sin \frac{2\pi}{n} \right) = \sin \frac{2\pi}{n} \sin \frac{3\pi}{n}$$

$$\Rightarrow \sin \frac{3\pi}{n} + \sin \frac{2\pi}{n} = 2 \cos \frac{2\pi}{n} \sin \frac{3\pi}{n} = \sin \frac{4\pi}{n} + \sin \frac{2\pi}{n}$$

$$\Rightarrow \sin \frac{3\pi}{n} = \sin \frac{4\pi}{n} \Rightarrow \frac{3\pi}{n} = m\pi + (-1)^n \frac{4\pi}{n}, m = 0, \pm 1, \pm 2, \dots$$

$$\text{If } m = 0 \Rightarrow \frac{3\pi}{n} = \frac{4\pi}{n} \Rightarrow 3 = 4 \text{ (not possible)}$$

$$\text{If } m = 1 \Rightarrow \frac{3\pi}{n} = \pi - \frac{4\pi}{n} \Rightarrow n = 7$$

If $m = 2, 3, \dots, -1, -2, \dots$ gives values of n which are not possible. Thus $n = 7$.

Problem 189

189. If $|z| = 2$, then show that the points representing the complex numbers $-1 + 5z$ lie on a circle.

Solution of Problem 189

Solution: Given, $|z| = 2$. Let $z_1 = -1 + 5z \Rightarrow z_1 + 1 = 5z$

$$|z_1 + 1| = |5z| = 5|z| = 10$$

$\Rightarrow z_1$ lies on a circle with center $(-1, 0)$ having radius 10.

Problem 190

190. If $|z - 4 + 3i| \leq 2$, find the least and the greatest values of $|z|$ and hence find the limits between which $|z|$ lies.

Solution of Problem 190

Solution: Given, $|z - 4 + 3i| \leq 2 \Rightarrow ||z| - |4 - 3i|| \leq 2$

$$\Rightarrow ||z| - 5| \leq 2 \Rightarrow -2 \leq |z| - 5 \leq 2 \Rightarrow 3 \leq |z| \leq 7$$