Complex Numbers Problems 101-110

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Problem 101

101. If z_1, z_2 and z_3 form a right-angled, isosceles triangle with right angle at z_3, then prove that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Solution of Problem 101

Solution: Since right angle is at z_3, therefore $\frac{z_2-z_3}{z_1-z_3} = e^{i\pi/2} = i$

$\Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2 \Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2 + 2z_1z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = -2z_3^2 + 2z_2z_3 + 2z_1z_3 - 2z_1z_2$

$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Problem 102

102. Find the equation of the circle whose center is z_0 and radius is r.

Solution of Problem 102

Solution: Clearly, $|z - z_0|^2 = r^2 \Rightarrow (z - z_0)(\overline{z - z_0}) = r^2$

$\Rightarrow (z - z_0)(\overline{z} - \overline{z_0}) = r^2$

$\Rightarrow z\overline{z} - \overline{z}z_0 - z\overline{z_0} + z_0\overline{z_0} = r^2$

Problem 103

103. If $z = 1 - t + i\sqrt{t^2 + t + 2}$, where t is a real parameter. Prove that locus of z in argand plane is a hyperbola.

Solution of Problem 103

Solution: Given, $z = 1 - t + i\sqrt{t^2 + t + 2}$; comparing real and imaginary parts, we get

$x = 1 - t, y = \sqrt{t^2 + t + 1} \Rightarrow y^2 = t^2 + t + 2$

$\Rightarrow y^2 = (1 - x)^2 + (1 - x) + 2 = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$

which is equation of a parabola.

Problem 104

104. Find the locus of z if $\overline{z} = \overline{a} + \frac{r^2}{z-a}$.

Solution of Problem 104

Solution: Given, $\overline{z} = \overline{a} + \frac{r^2}{z-a}$

$\Rightarrow (\overline{z} - \overline{a})(z - a) = r^2$

which is equation of a circle with center at a and radius r.

Problem 105

105. If the equation $|z - z_1|^2 + |z - z_2|^2 = k$ represents the equation of a cirlce, where $z_1 = 2 + 3i$, $z_2 = 4 + 3i$ are the ends of a diameter, then find the value of k.

Solution of Problem 105

Solution: Since z_1 and z_2 are ends of diameter

$\Rightarrow |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow k = |z_1 - z_2|^2 = |2 + 3i - 4 - 3i|^2 = 4$

Problem 106

106. If $|z+1| = \sqrt{2}|z-1|$, then show that locus of z is a circle.

Solution of Problem 106

Solution: $z = x + iy$, then $|(x+1) + iy| = \sqrt{2}|(x-1) + iy|$

Squaring both sides, we get

$(x+1)^2 + y^2 = 2[(x-1)^2 + y^2] \Rightarrow x^2 + y^2 - 6x + 1 = 0$

which is equation of a circle.

Problem 107

107. Prove that the locus of z given by $\left|\frac{z-1}{z-i}\right| = 1$ is a straight line.

Solution of Problem 107

Solution: Given, $\left|\frac{z-1}{z-i}\right| = 1 \Rightarrow |z-1| = |z-i|$

Let $z = x + iy$, then we have

$|(x-1) + iy| = |x + i(y-1)|$

Squaring both sides, we get

$\Rightarrow (x-1)^2 + y^2 = x^2 + (y-1)^2 \Rightarrow 2x = 2y \Rightarrow x = y$

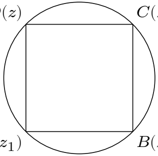
which is equation of a straight line.

Problem 108

108. Find the condition for four complex numbers z_1, z_2, z_3 and z_4 to lie on a cyclic quadrilateral.

Solution of Problem 108

Solution:



$\angle z_1 = \arg\left(\frac{z_1-z_2}{z_1-z_4}\right)$, $\angle z_2 = \arg\left(\frac{z_3-z_2}{z_1-z_2}\right)$, $\angle z_3 = \arg\left(\frac{z_3-z_4}{z_3-z_2}\right)$, and $\angle z_4 = \arg\left(\frac{z_1-z_4}{z_3-z_4}\right)$

$\angle z_1 + \angle z_3 = \pi \Rightarrow \arg\frac{z_1-z_2}{z_1-z_4} + \arg\left(\frac{z_3-z_4}{z_3-z_2}\right) = \pi$

$\Rightarrow \arg\left(\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}\right) = \pi$

$\Rightarrow \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is real number.

Problem 109

109. If z_1, z_2 and z_3 are complex numbers, such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, then show that these points lie on a circle passing through origin.

Solution of Problem 109

Solution: Given, $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$

$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$

$\Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi - \arg\frac{z_3}{z_2}$

$\Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) + arg\left(\frac{z_3 - 0}{z_2 - 0}\right) = \pi$

Thus, the given points and origin are concyclic.

Problem 110

110. If $|z-\omega|^2+|z-\omega^2|^2=r^2$, where r is radius and ω, ω^2 are cube roots of unity and ends of diameter of the circle then find radius.

Solution of Problem 110

Solution: From the equation of circle, $r^2 = |\omega - \omega^2|^2$

$\Rightarrow r^2 = |i\sqrt{3}|^2 = 3 \Rightarrow r = \sqrt{3}$