Logarithm Problem 31-40

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Problem 31

31. Prove that $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \ldots + \frac{1}{\log_{1988} N} = \frac{1}{\log_{1988!} N}$

Solution of Problem 31

Solution:

$$L.H.S. = \frac{1}{\log_2 N} + \frac{1}{\log_3 N} + ... + \frac{1}{\log_{1988} N}$$

$$= \log_N 2 + \log_N 3 + ... + \log_N 1988$$

$$= \log_N (2.3. \dots 1988) = \log_N 1988! = \frac{1}{\log_{1988!} N} = R.H.S.$$

Problem 32

32. If $0 < x < 1$, prove that $\log(1+x) + \log(1+x^2) + \log(1+x^4) \ldots \infty = -\log(1-x)$

Solution of Problem 32

Solution: Given equation can be rewritten as

$$\log(1-x)+\log(1+x)+\log(1+x^2)+\log(1+x^4)+\ldots\infty=0$$

$$=\log(1-x^2)+\log(1+x^2)+\log(1+x^4)+\ldots\infty$$
$$=\log(1-x^4)+\log(1+x^4)+\ldots\infty$$

The powers of x will grow till infinity and since $0<x<1$ it will aproach 0 leaving $\log 1=0$

Problem 33

33. Find the sum of the series $\frac{1}{\log_2 a} + \frac{1}{\log_4} a + ...$ up to n terms.

Solution of Problem 33

Solution:

$$
\begin{aligned}
L.H.S. &= \log_a 2 + \log_a 4 + \log_a 8 + \dots \text{ upto } n \text{ terms} \\
&= (1 + 2 + 3 + \dots + n) \log_a 2 \\
&= \frac{n(n+1)}{2} \log_a 2
\end{aligned}
$$

Problem 34

34. If $\log_4 10 = x, \log_2 20 = y$ and $\log_5 8 = z$, prove that $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$

Solution of Problem 34

Solution:

$$L.H.S. = \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$$

$$= \frac{1}{\log_4 10 + \log_4 4} + \frac{1}{\log_2 20 + \log_2 2} + \frac{1}{\log_5 8 + \log_5 5}$$

$$= \frac{1}{\log_4 40} + \frac{1}{\log_2 40} + \frac{1}{\log_5 40}$$

$$= \log_{40} 4 + \log_{40} 2 + \log_{40} 2$$

$$= \log_{40} 40 = 1 = R.H.S.$$

Problem 35

35. If $x = \log_a bc, y = \log_b ca, z = \log_c ab$, prove that $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$

Solution of Problem 35

Solution:

$$L.H.S. = \frac{1}{\log_a bc + 1} + \frac{1}{\log_b ca + 1} + \frac{1}{\log_c ab + 1}$$

$$= \frac{1}{\log_a bc + \log_a a} + \frac{1}{\log_b ca + \log_b b} + \frac{1}{\log_c ab + \log_c c}$$

$$= \frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$$

$$= \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1 = R.H.S.$$

Problem 36

36. Prove that $\frac{1}{1+\log_b a+\log_b c}+\frac{1}{1+\log_c a+\log_c b}+\frac{1}{1+\log_a b+\log_a c}=1$

Solution of Problem 36

Solution:

$$L.H.S. = \frac{1}{1+\log_b a + \log_b c} + \frac{1}{1+\log_c a + \log_c b} + \frac{1}{1+\log_a b + \log_a c}$$

$$= \frac{1}{\log_b b + \log_b a + \log_b c} + \frac{1}{\log_c c + \log_c a + \log_c b} + \frac{1}{\log_a a + \log_a b + \log_a c}$$

$$= \frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc}$$

$$= \log_{abc} b + \log_{abc} c + \log_{abc} a = \log_{abc} abc = 1 = R.H.S.$$

Problem 37

37. Prove that $x^{\log y-\log z}y^{\log z-\log x}z^{\log x-\log y}=1$

Solution of Problem 37

Solution: We have to prove that

$$x^{\log y - \log z} y^{\log z - \log x} z^{\log x - \log y} = 1$$

Taking $\log$ of both sides, we get

$$(\log y - \log z)\log x + (\log z - \log x)\log y + (\log x - \log y)\log z = 0$$

$$0 = 0$$

Problem 38

38. If $\frac{\log a}{y-z} = \frac{\log b}{z-x} = \frac{\log c}{x-y}$, prove that $a^x b^y c^z = 1$

Solution of Problem 38

Solution: We have to prove that $a^x b^y c^z = 1$

Taking $\log$ of both sides, we get $x \log a + y \log b + z \log c = 0$

$$\frac{\log a}{y - z} = \frac{\log b}{z - x} = \frac{\log c}{x - y} = k(\text{say})$$

$$x \log a = k(xy - zx), y \log b = k(yz - xy), z \log c = k(zx - yz)$$

Adding all these, we get

$$x \log a + y \log b + z \log c = k(xy - zx + yz - xy + zx - yz) = 0$$

Problem 39

39. If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, prove that $y^z z^y = z^x x^z = x^y y^x$

Solution of Problem 39

Solution:

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z} = \frac{1}{k}\text{(let)}$$

$$\log x = kx(y+z-x), \log y = ky(z+x-y) = \log z = kz(x+y-z)$$

$$\text{Let } y^z z^y = z^x x^z = x^y y^x = c$$

Taking $\log$ of both sides, we get

$$z\log y + y\log z = x\log z + z\log x = y\log x + x\log y = \log c$$

$$\Rightarrow zky(z+x-y) + ykz(x+y-z) = xkz(x+y-z) + zkx(y+z-x) = ykx(y+z-x) + xky(x+z-y)$$

$$\Rightarrow yz^2 + xyz - y^2z + xyz + y^2z - z^2y = x^2z + xyz - xz^2 + xyz + xz^2 - x^2z = xy^2 + xyz - x^2y + xyz - xy^2$$

$$2xyz = 2xyz = 2xyz$$

Problem 40

40. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, prove that $a^{b+c} b^{c+a} c^{a+b} = 1$

Solution of Problem 40

Solution:

$$\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k(\text{let})$$

$$\log a = k(b-c), \log b = k(c-a), \log c = k(a-b)$$

We have to prove that

$$a^{b+c} b^{c+a} c^{a+b} = 1$$

Taking $\log$ of both sides, we get

$$(b+c)\log a + (c+a)\log b + (a+b)\log c = 0$$

$$\Rightarrow k(b^2-c^2) + k(c^2-a^2) + k(a^2-b^2) = 0$$

$$0 = 0$$