Miscellaneous Problems on A.P., G.P. and H.P.
Problems 71-80

Shiv Shankar Dayal

December 16, 2021

Problem 71

71. A G.P. consists of $2n$ terms. If the sum of the terms occupying the odd places is S_1, and that of the terms in even places is S_2, show that the common ratio of the progression is S_2/S_1.

Solution of Problem 71

Solution: Let a be the first term and r be the common ratio of the G.P. Then,

$$S_1 = a + ar^2 + ar^4 + ... + ar^{2n-2} = \frac{a(r^{2n} - 1)}{r^2 - 1}$$

$$S_2 = ar + ar^3 + ar^5 + ... + ar^{2n-1} = \frac{ar(r^{2n} - 1)}{r^2 - 1}$$

Thus, $S_2/S1 = r$ which is common ratio of the G.P.

Problem 72

72. If $x \neq 1, y \neq 1, x \neq y$, find the sum to n terms of the series $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$

Solution of Problem 72

Solution: Multiplying and dividing the given series by $x-y$, we get

$$S = \frac{1}{x-y}\left[(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+...\right]$$

$$= \frac{1}{x-y}\left[x^2+x^3+x^4+ \text{ to } n \text{ terms } -y^2+y^3+y^4+ \text{ to } n \text{ terms }\right]$$

$$= \frac{1}{x-y}\left[\frac{x^2(x^n-1)}{x-1}-\frac{y^2(y^n-1)}{y-1}\right]$$

Problem 73

73. Find a geometric progression of real numbers such that the sum of its first four terms is equal to 30 and sum of the squares of its first four terms is 340.

Solution of Problem 73

Solution: Let a be the first term and r be the common ratio. Then,

$$a + ar + ar^2 + a^3 = 30, a^2 + a^2r^2 + a^2r^4 + a^2r^6 = 340$$

$$\Rightarrow \frac{a(r^4-1)}{r-1} = 30, \frac{a^2(r^8-1)}{r^2-1} = 340$$

Solving these two equation will yield two possible values of $r = 2, \frac{1}{2}$ and thus $a = 2, 16$ and hence series can be found.

Problem 74

74. If S_n denotes the sum of n terms of a G.P. whose first term and common ratio are a and r respectively, show that

$$rS_n + (1-r)\sum_{n=1}^{n} S_n = na$$

Solution of Problem 74

Solution:

$$S_1 = a = \frac{a(1-r)}{1-r}$$

$$S_2 = a + ar = \frac{a(1-r^2)}{1-r}$$

$$\dots$$

$$S_n = a + ar + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$(1-r)\sum_{n=1}^{n} S_n = \frac{1-r}{1-r}\left[a(1+1+\dots \text{ to } n \text{ terms }) - a(r+r^2+\dots \text{ to } n \text{ terms })\right]$$

$$= na - \frac{ar(1-r^n)}{1-r}$$

$$rS_n = \frac{ar(1-r^n)}{1-r}$$

$$\therefore rS_n + (1-r)\sum_{n=1}^{n} S_n = na$$

Problem 75

75. Find the sum of $2n$ terms of the series where every even term if x times the term just before it and every odd term is y times the term just before it, the first term being 1.

Solution of Problem 75

Solution: The sum of series would be

$$S = 1 + x + xy + x^2y + x^2y^2 + x^3y + x^3y^3 + \ldots \text{ to } 2n \text{ terms}$$

$$\begin{aligned} S &= 1 + xy + x^2y^2 + \ldots \text{ to } n \text{ terms} + x + x^2y + x^3y + \ldots \text{ to } n \text{ terms} \\ &= \frac{(x^ny^n - 1)}{xy - 1} + \frac{x(x^ny^n - 1)}{xy - 1} \\ &= \frac{(x^ny^n - 1)(1 + x)}{xy - 1} \end{aligned}$$

Problem 76

76. Prove that in the sequence of numbers $49, 4489, 444889, \ldots$ in which every number is made by inserting 48 in the middle of previous number as indicated, each number is the square of an integer.

Solution of Problem 76

Solution:

$$49 = (4 \times 10) + 9$$

$$4489 = (4 \times 10^3 + 4 \times 10^2) + (8 \times 10) + 9$$

$$\cdots$$

$$t_k = 4\frac{10^k - 1}{9} \cdot 10^k + 8\frac{10^k - 1}{9} + 1$$

$$= 4\frac{10^k - 1}{9} \cdot 10^k - 4\frac{10^k - 1}{9} + 12\frac{10^k - 1}{9} + 1$$

$$= 36\frac{10^{2k} - 2 \cdot 10^k + 1}{81} + 12\frac{10^k - 1}{9} + 1$$

$$= \left(6\frac{10^k - 1}{9} + 1\right)^2$$

Problem 77

77. If there be m quantities in a G.P., whose common ratio is r and S_m denotes the sum of the first m terms then prove that the sum of their products taken two and two together is $\frac{r}{r+1} S_m S_{m-1}$

Solution of Problem 77

Solution:

$$S_m = a + ar + ar^2 + ... + ar^{m-1} = \frac{a(r^m - 1)}{r - 1}$$

Let S be the required sum then

$$S = \frac{(\sum a_i)^2 - \sum a_i^2}{2} = \frac{\left(\frac{a(r^m-1)}{r-1}\right)^2 - [a^2 + a^2r^2 + ... + a^{2(m-1)}]}{2}$$

$$2S = \frac{a^2(r^m - 1)}{r - 1}\left[\frac{r^m - 1}{r - 1} - \frac{r^m + 1}{r + 1}\right]$$

$$2S = \frac{r}{r+1}\frac{a(r^m - 1)}{r - 1}\frac{a(r^{m-1} - 1)}{r - 1} = \frac{r}{r+1}S_m S_{m-1}$$

Problem 78

78. Solve the following equations for x and y

$$\log_{10} x + \log_{10} x^{\frac{1}{2}} + \log_{10} x^{\frac{1}{4}} + ... = y$$

$$\frac{1 + 3 + 5 + (2y - 1)}{4 + 7 + 10 + ... + 3y + 1} = \frac{20}{7 \log_{10} x}$$

Solution of Problem 78

Solution:

$$\log_{10} x + \log_{10} x^{\frac{1}{2}} + \log_{10} x^{\frac{1}{4}} + ... = y$$

$$\log_{10} x + \frac{1}{2}\log_{10} x + \frac{1}{4}\log_{10} x + ... = y$$

$$y = \frac{\log_{10} x}{1 - \frac{1}{2}} = 2\log_{10} x$$

$$\frac{1 + 3 + 5 + (2y - 1)}{4 + 7 + 10 + ... + 3y + 1} = \frac{20}{7\log_{10} x}$$

$$\frac{y^2}{\frac{y}{2}[8 + (y - 1).3]} = \frac{40}{7y}$$

$$\Rightarrow y = 10, x = 10^5$$

Problem 79

79. If $a_1, a_2, \ldots, a_n$ are in G.P. and $S = a_1 + a_2 + \ldots + a_n, T = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$ and $P = a_1.a_2.\ldots.a_n$ show that $P^2 = \left(\frac{S}{T}\right)^n$

Solution of Problem 79

Solution: Let $a = a_1$ be the first term and r to be the common ratio of the G.P., then

$$S = \frac{a(r^n - 1)}{r - 1}$$

$$P = a^n r^{1+2+\ldots+n-1} = a^n r^{\frac{(n-1)n}{2}}$$

$$T = \frac{1}{a} \frac{1 - \frac{1}{r^n}}{1 - \frac{1}{r}} = \frac{1}{a} \frac{r^n - 1}{r - 1} \cdot \frac{1}{r^{n-1}}$$

Clearly,

$$P^n = \left(\frac{S}{T} \right)^n$$

Problem 80

80. Let a, b, c be respectively the sums of the first n terms, the next n terms and the next n terms of a G.P. show that a, b, c are in G.P.

Solution of Problem 80

Solution: Let x to be the first term and r to be the common ratio of the G.P.

$$a = \frac{x(y^n - 1)}{y - 1}$$

$$b = \frac{xy^n(y^n - 1)}{y - 1}$$

$$c = \frac{xy^{2n}(y^n - 1)}{y - 1}$$

Clearly, a, b, c are in G.P.