Complex Numbers

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nth Root of Unity

$$1 = \cos 0 + i \sin 0 \Rightarrow 1^{\frac{1}{n}} = (\cos 0 + i \sin 0)^{\frac{1}{n}}$$

$$= \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \ \forall\ k = 0, 1, 2, 3, \ldots (n-1)$$

$$= e^{i2k\pi/n}$$

$= 1, e^{i2\pi/n}, e^{i4\pi/n}, \ldots, e^{i2(k-1)\pi/n} = 1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ where $\alpha = e^{i2k\pi/n}$

Since α is one of the roots $\Rightarrow 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$ and $1\alpha.\alpha^2. \ldots \alpha^{n-1} = \alpha^{n(n-1)/2} = 1$

De Moivre's Theorem

Statement: If n is any integer then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof: By Euler's formula $\cos\theta + i\sin\theta = e^{i\theta} \Rightarrow (\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

Proof by Induction:

For $n = 1, (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

Let it be true for $n = m$ i.e. $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

For $n = m + 1, (\cos\theta + i\sin\theta)^{m+1} = (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$

$= \cos m\theta\cos\theta - \sin m\theta\sin\theta + i(\sin m\theta\cos\theta + \cos m\theta\sin\theta) = \cos(m+1)\theta + i\sin(m+1)\theta$

Thus, it is true for $n = m + 1$. Hence, we have proven the theorem by induction.

It is now trivial to prove it for fractional and negative powers.

Important Geometrical Results

Section Formula

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ then if $z = x + iy$ which divides the previous two points in the ratio $m : n$ can be given by using the results from coordinate geometry as below:

$$x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n} \therefore z = \frac{mz_2 + nz_1}{m+n}$$

Extending this section formula we can say that if there is a point which divides this line in two equal parts i.e. the point is mid-point then $m = 1$ and $n = 1$ and z is given my $\frac{1}{2}(z_1 + z_2)$

Distance Formula

Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_1 - z_2|$

Equation of Line Passing Through Two points

The equation between two point z_1 and z_2 is given by the determinant

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_1} & 1 \end{vmatrix} = 0$$

or,

$$\frac{z - z_1}{\overline{z} - \overline{z_1}} = \frac{z_1 - z_2}{\overline{z_1} - \overline{z_2}}$$

Geometrical Results contd

Collinear Points

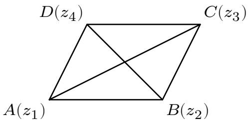
Three points z_1, z_2 and z_3 are collinear if and only if

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0$$

The above formula comes from the equation of line passing through two points.

Parallelogram

Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a parallelogram if $z_1 + z_3 = z_2 + z_4$

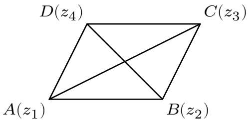


The diagonals of a paralleogram bisect each other i.e. mid-point of AC and BD are same i.e. $\frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4) \Rightarrow z_1 + z_3 = z_2 + z_4$

Geometrical Results contd

Rhombus

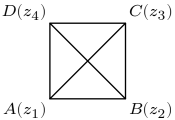
Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a rhombus if $z_1 + z_3 = z_2 + z_4$ and $|z_4 - z_1| = |z_2 - z_1|$



The diagonals must bisect each other. Thus, $z_1 + z_3 = z_2 + z_4$. Also, four sides of a rhombus are equal i.e. $AD = AB \Rightarrow |z_4 - z_1| = |z_2 - z_1|$

Square

Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a square if $z_1 + z_3 = z_2 + z_4, |z_4 - z_1| = |z_2 - z_1|$ and $|z_3 - z_1| = |z_4 - z_2|$

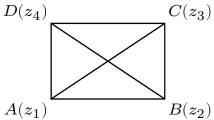


The diagonals must bisect each other. Thus, $z_1 + z_3 = z_2 + z_4$. Also, four sides of a square are equal i.e. $AD = AB \Rightarrow |z_4 - z_1| = |z_2 - z_1|$. Also the digonals are equal in length so $|z_3 - z_1| = |z_4 - z_2|$

Geometricla Results contd

Rectangle

Four complex numbers $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ represent the vertices of a square if $z_1 + z_3 = z_2 + z_4$ and $|z_3 - z_1| = |z_4 - z_2|$

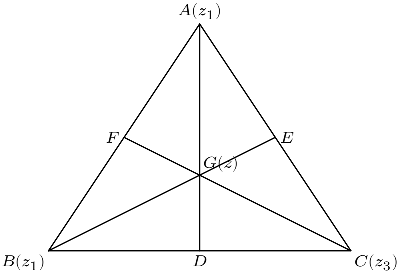


The diagonals must bisect each other. Thus, $z_1 + z_3 = z_2 + z_4$. Also the digonals are equal in length so $|z_3 - z_1| = |z_4 - z_2|$

Centroid of a Triangle

Let $A(z_1), B(z_2)$ and $C(z_3)$ be the vertices of a $\triangle ABC$. Centroid $G(z)$ of the $\triangle ABC$ is the point of concurrence of the medians of all three sides and is given by

$$z = \frac{z_1 + z_2 + z_3}{3}$$



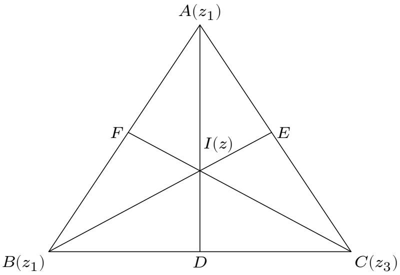
Geometricla Results contd

Incenter of a Triangle

Let $A(z_1), B(z_2)$ and $C(z_3)$ be the vertices of a $\triangle ABC$. inceneter $I(z)$ of the $\triangle ABC$ is the point of concurrence of the internal bisectors of and is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a+b+c}$$

where a, b, c are the lengths of the sides.



Geometricla Results contd

Circumcenter of a Triangle

Circumcenter $S(z)$ of a $\triangle ABC$ is the point of concurrence of perpendicular bisectors of sides of the triangle. It is given by

$$z = \frac{(z_2 - z_3)|z_1|^2 + (z_3 - z_1)|z_2|^2 + (z_1 - z_2)|z_3|^2}{\overline{z_1}(z_2 - z_3) + \overline{z_2}(z_3 - z_1) + \overline{z_3}(z_1 - z_2)}$$

$$= \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z_1} & z_1 & 1 \\ \overline{z_2} & z_2 & 1 \\ \overline{z_3} & z_3 & 1 \end{vmatrix}}$$

Also,

$$z = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

Geometricla Results contd

Circumcenter of a Triangle

The orthocenter $H(z)$ of the $\triangle ABC$ is the point of concurrence of altitudes of the side. It is given by

$$z = \frac{\begin{vmatrix} z_1^2 & \overline{z_1} & 1 \\ z_2^2 & \overline{z_2} & 1 \\ z_3^2 & \overline{z_3} & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z_1} & z_1 & 1 \\ \overline{z_2} & z_2 & 1 \\ \overline{z_3} & z_3 & 1 \end{vmatrix}}$$

$$= \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

$$= \frac{z_1 a \sec A + b z_2 \sec B + c z_3 \sec C}{a \sec A + b \sec B + c \sec C}$$

Euler's Line

The centroid G of a triangle lies on the segment joining the orthocenter H and the circumcenter S of the triangle. G divides the line H and S in the ratio $2:1$.