Miscellaneous Problems on A.P., G.P. and H.P.
Problems 91-100

Shiv Shankar Dayal

January 10, 2022

Problem 91

91. One side of an equilateral triangle is 24 cm. The mid-point of its sides are joined to form another triangle, whose mid-points are in turn joined for form still another triangle. The process continues indefinitely. Find the sum of perimeter of all the triangles.

Solution of Problem 91

Solution: Perimeter of first triangle will be $24 * 3$ i.e. 72 cm. The triangle which will be formed by joining mid-points of this triangle will have sides of 12 cm each.

Thus, perimeter of second trianble will be $12 * 3$ i.e. 36 cm. The triangle which will be formed by joining mid-points of this triangle will have sides of 6 cm each and so on.

Thus sum of perimeter of all such triangles will be

$$72 + 36 + 18 + 9 + \ldots \infty$$

$$= \frac{72}{1 - \frac{1}{2}} = 144 \text{ cm}$$

Problem 92

92. A ball is dropped from a height of 900 cm. Each time is rebounces, it rises to $2/3$ of the height it has fallen through. Find the total distance travelled by the ball before it comes to rest.

Solution of Problem 92

Solution: First the ball will fall a distance for 900 cm. Then it will rise and fall for $2/3$rd of that distance travelling a total of $2 * 900 * 2/3$ cm. Then it will rise and fall for $2/3$rd of that distance travelling a total of $2 * 900 * (2/3)^2$ cm. This process will go on till infinity. Thus we can write following equation for total distance travelled, S(say)

$$S = 900 + 2 * 900 * \frac{2}{3} + 2 * 900 * \frac{2^2}{3^2} + 2 * 900 * \frac{2^3}{3^3} + \ldots \infty$$

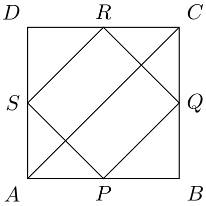
$$= 900 + \frac{1200}{1 - \frac{2}{3}} = 4500 \text{ cm}$$

Problem 93

93. A square is drawn by joining the mid-points of the sides of a given square. A third square is drawn inside the second square in the same way and this process continutes indeinitely. If the sides of the first square is 4 cm, determine the sum of the areas of all the squares.

Solution of Problem 93

Solution:



Let $ABCD$ be the first sqaure. Let $AB = a \Rightarrow AC = \sqrt{2}a \therefore PQ = \frac{AC}{2} = \frac{a}{\sqrt{2}}$

$\therefore$ Area of first square $= a^2$

Area of second sqaure $= \frac{a^2}{2}$

Area of third square $= \frac{a^2}{4}$

Sum of areas of all squares $= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + ... = \frac{a^2}{1-\frac{1}{2}} = 2a^2 = 32$ sq.cm.

Problem 94

94. In an increasing G.P., the sum of the first and the last term is 66, the product of the second and the last term but one term is 128, and the sum of all the terms is 126. How many terms are there in the progression?

Solution of Problem 94

Solution: Let a be the first term and r be the common ratio. Also, let that there are n terms in the G.P. Then according to the question

$$a + ar^{n-1} = 66, ar.ar^{n-2} = 128, \frac{a(1-r^n)}{1-r} = 126$$

$$\Rightarrow a^2r^{n-1} = 128 \Rightarrow a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0 \Rightarrow a = 2, 64$$

$$\text{For } a = 2, a - ar^n = 126(1-r) \Rightarrow 2 - \frac{128}{2}.r = 126(1-r) \Rightarrow r = 2$$

$$\text{For } a = 64, a - ar^n = 126(1-r) \Rightarrow 64 - \frac{128}{64}r = 126(1-r) \Rightarrow r = \frac{1}{2}$$

However, the G.P. is increasing so $a = 2, r = 2$

$$\Rightarrow ar.ar^{n-2} = 128 \Rightarrow 2^2.2^{n-1} = 128 \Rightarrow 2^{n-1} = 32 \Rightarrow n = 6$$

Problem 95

95. The sum of an infinite G.P. is 2 and the sum of the G.P. made from the cubes of the terms of this infinite series is 24. Then find the series.

Solution of Problem 95

Solution: Let a be the first term and r be the common ratio of the G.P. Then according to question

$$\frac{a}{1-r} = 2, \frac{a^3}{1-r^3} = 24$$

$$\Rightarrow 8(1-r)^3 = 24(1-r^3) \Rightarrow 1-2r+r^2 = 3+3r+3r^2$$

$$\Rightarrow 2r^2+5r+2 = 0 \Rightarrow (r+2)(2r+1) = 0 \Rightarrow r = -2, -1/2$$

Since it is an infinite G.P. $|r| < 1 \Rightarrow r = -1/2 \Rightarrow a = 3$

Thus, G.P. is $3, -3/2, 3/4, -3/8, \ldots$

Problem 96

96. The sum of an infinite G.P. is 3 and the sum of squares of the terms of this series is also 3. Find the sequence.

Solution of Problem 96

Solution: Let a be the first term and r be the common ratio of the G.P. Then according to question

$$\frac{a}{1-r} = 3, \frac{a^2}{1-r^2} = 3$$

$$\Rightarrow 3^2(1-r)^2 = 3(1-r^2) \Rightarrow 3 - 3r = 1 + r \Rightarrow r = 1/2$$

$$\Rightarrow a = 3/2$$

Thus, G.P. is $3/2, 3/4, 3/8, \ldots$

Problem 97

97. If the sum of an infinitely decreasing G.P. is 3.5 and the sum of the squares of its terms is $147/16$. Show that the sum of the cubes of the terms is $1029/38$.

Solution of Problem 97

Solution: Let a be the first term and r be the common ratio of the G.P. Then according to question

$$\frac{a}{1-r} = \frac{7}{2}, \frac{a^2}{1-r^2} = \frac{147}{16}$$

$$\Rightarrow \frac{49}{4}(1-r)^2 = \frac{147}{16}(1-r^2) \Rightarrow 1-r = \frac{3}{4}(1+r) \Rightarrow r = \frac{1}{7}$$

$$\Rightarrow a = 3 \Rightarrow \frac{a^3}{1-r^3} = \frac{27}{\frac{342}{343}} = \frac{1029}{38}$$

Problem 98

98. Find the value of x in $]-\pi, \pi[$ which satisfy the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots \text{ to } \infty} = 64$

Solution to Problem 98

Solution:

$$8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\ldots \text{ to } \infty} = 8^{\frac{1}{1-|\cos x|}} = 8^2$$

$$\Rightarrow 1-|\cos x| = \frac{1}{2} \Rightarrow \cos x = \pm\frac{1}{2} \Rightarrow x = n\pi + \frac{\pi}{3} \ \forall\, n \in I$$

Problem 99

99. If $A = 1 + r^a + r^{2a} + ...$ to ∞ and $B = 1 + r^b + r^{2b} + ...$ to ∞, prove that $r = \left(\frac{A-1}{A}\right)^{\frac{1}{a}} = \left(\frac{B-1}{B}\right)^{\frac{1}{b}}$

Solution of Problem 99

Solution:

$$A = 1 + r^a + r^{2a} + \ldots \text{ to } \infty$$

$$A = \frac{1}{1 - r^a} \Rightarrow r = \left(\frac{A-1}{A}\right)^{\frac{1}{a}}$$

$$B = 1 + r^b + r^{2b} + \ldots \text{ to } \infty$$

$$B = \frac{1}{1 - r^b} \Rightarrow r = \left(\frac{B-1}{B}\right)^{\frac{1}{b}}$$

Problem 100

100. If $s_1, s_2, \dots, s_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \dots, n$ and common ratios are $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+1}$ respectively, then prove that $s_1 + s_2 + \dots + s_n = \frac{1}{2}n(n+3)$

Solution of Problem 100

Solution:

$$s_1 = \frac{1}{1-\frac{1}{2}} = 2$$

$$s_2 = \frac{2}{1-\frac{1}{3}} = 3$$

$$\dots$$

$$s_n = \frac{n}{1-\frac{1}{n+1}} = n+1$$

$$s_1 + s_2 + \dots + s_n = 2 + 3 + \dots + (n+1) = \frac{1}{2}n(n+3)$$