# Miscellaneous Problems on A.P., G.P. and H.P.
# Problems 61-70

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# Problem 61

**61.** The pollution in a normal atmosphere is less that $0.01\%$. Due to leakage of gas from a factory the pollution increased to $20\%$. If everyday $80\%$ of the pollution us neutralised, in how many days the atmosphere will be normal?

# Solution of Problem 61

**Solution:** Each day pollution decreases by $80\%$ i.e. $20\%$ pollution remains. Thus common ratio for pollution remaining is $\frac{20}{100} = \frac{1}{5}$

Let it takes $n$ days for pollution to become normal. Then,

$$a = 20, 20.\frac{1}{5^n - 1} < .01$$

$$\Rightarrow n > 4$$

Thus, atmosphere becomes normal on $5$th day.

# Problem 62

**62.** The sides of a triangle are in G.P. and its largest angle is twice the smallest one. Prove that the common ratio of the G.P. lies in the interval $(1, \sqrt{2})$

# Solution of Problem 62

**Solution:** Let $a, ar, ar^2$ be the sides of the G.P. with $r > 1$. Let smallest angle be $\alpha$. The according to question largest angle will be $2\alpha$. Applying sine rule to smallest and largest angle

$$\frac{a}{\sin\alpha} = \frac{ar^2}{\sin 2\alpha} \Rightarrow \frac{\sin 2\alpha}{\sin\alpha} = r^2$$

$$\Rightarrow 2\cos\alpha = r^2$$

$$\because \alpha \neq 0 \therefore \cos\alpha < 1 \Rightarrow r^2 < 2 \Rightarrow r < \sqrt{2}$$

Thus, common ratio lies in the range $(1, \sqrt{2})$

# Problem 63

**63.** If $a, b, c, d$ are in G.P., then prove that $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$.

# Solution of Problem 63

**Solution:** Let $r$ be the common ratio of G.P. Then, $b = ar, c = ar^2, d = ar^3$

$$ax^3 + bx^2 + cx + d = ax^3 + arx^2 + ar^2x + ar^3 = a(x^2 + r^2)(x + r)$$

$$ax^2 + c = a(x^2 + r^2)$$

Clearly, $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$.

# Problem 64

**64.** If $a, b, c$ are three distinct real numbers and they are in G.P. If $a + b + c = xb$, then prove that $x < -1$ or $x > 3$.

# Solution of Problem 64

**Solution:** Let $r$ be the common ratio of the G.P. Then $b = ar, c = ar^2$ Given that

$$a + ar + ar^2 = xar \Rightarrow r^2 + (1 - x)r + 1 = 0$$

$$\because r \in R \Rightarrow D \geq 0 \Rightarrow (1 - x)^2 - 4 \geq 0 \Rightarrow x^2 - 2x - 3 \geq 0$$

$$(x + 1)(x - 3) \geq 0 \Rightarrow x \leq -1, x \geq 3$$

# Problem 65

**65.** If $a, b, c, d, p$ are real and $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \leq 0$. Show that $a, b, c, d$ are in G.P. whose common ratio is $p$

# Solution of Problem 65

**Solution:** Given, $(a^2+b^2+c^2)p^2-2(ab+bc+cd)p+(b^2+c^2+d^2)\leq 0$ and $p$ is real.

$$D=0, (ab+bc+cd)^2-(a^2+b^2+c^2)(b^2+c^2+d^2)=0 \Rightarrow (b^2-ac)^2+(c^2-bd)^2+(bc-ad)^2=0$$

$$b^2=ac, c^2=ad, bc=ad$$

Thus, $a,b,c,d$ are in G.P. Let $r$ be the common ratio then

$$p=\frac{ab+bc+cd}{a^2+b^2+c^2}=\frac{a^2r+a^2r^3+a^2r^5}{a^2+a^2r^2+a^2r^4}=r$$

# Problem 66

**66.** If $2x^4 = y^4 + z^4, xyz = 8$ and $\log_y x, \log_z y, \log_x z$ are in G.P., show that $x = y = z = 2$.

# Solution of Problem 66

**Solution:** $\log_y x, \log_z y, \log_x z$ are in G.P.

$$\Rightarrow 2\log_z y = \log_y x \log_x z \Rightarrow \left(\frac{\log y}{\log z}\right)^2 = \frac{\log x}{\log y} . \frac{\log z}{\log x}$$

$$\Rightarrow (\log y)^3 = (\log z)^3 \Rightarrow y = z$$

Also, $2x^4 = y^4 + z^4 \Rightarrow 2x^4 = 2y^4 \Rightarrow x = y \therefore x = y = z$

Also, $xyz = 8, \Rightarrow x = y = z = 2$

## Problem 67

**67.** If $a, b, c, d$ are in both A.P. and G.P. and $b = 2$, then find the number of such sequences.

# Solution of Problem 67

**Solution:** Let $r$ be the common ratio. Since $a, b, c, d$ are in A.P.

$$\therefore b - a = c - b \Rightarrow a(r-1) = ar(r-1) \Rightarrow a(r-1)(r-1) = 0$$

$$\Rightarrow r = 1$$

Thus one such series is possible.

# Problem 68

**68.** If $\log_x a, a^{\frac{x}{2}}, \log_b x$ are in G.P., then find $x$.

# Solution of Problem 68

**Solution:** Given, $\log_x a, a^{\frac{x}{2}}, \log_b x$ are in G.P.

$$\therefore \left(a^{\frac{x}{2}}\right)^2 = \log_x a . \log_b x$$

$$\Rightarrow a^x = \log_b a \Rightarrow x = \log_a(\log_b a)$$

# Problem 69

**69.** The $(m+n)$th and $(m-n)$th terms of a G.P. are $p$ and $q$ respectively. Show that $m$th and $n$th terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$ respectively.

# Solution of Problem 69

**Solution:** Let $x$ be the first term and $y$ be the common difference. Then,

$$t_{m+n} = xy^{m+n-1} = p, t_{m-n} = xy^{m-n-1} = q$$

$$\Rightarrow pq = x^2y^2(m-1) = (xy^{m-1})^2 = t_m^2 \Rightarrow t_p = \sqrt{pq}$$

$$\frac{q}{p} = y^{-2n} \Rightarrow y = \left(\frac{p}{q}\right)^{\frac{1}{2n}}$$

Now $x$ and $t_n$ can be found easily.

# Problem 70

**70.** If the $p$th, $q$th and $r$th terms of an A.P. are in G.P., then find the common ratio of the G.P.

# Solution of Problem 70

**Solution:** Let $a$ be the first term and $d$ be the common difference of A.P. Also, let $t_p = x$ and common ratio be $r$ of the G.P. Then

$$a + (p-1)d = t_p = x, a + (q-1) = xr, a + (r-1) = xr^2$$

$$(q-p)d = x(r-1), (r-q)d = xr(r-1)$$

$$\Rightarrow r = \frac{r-q}{q-p}$$