Arithmetic, Geometric and Harmonic Means Problems 31-40

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Problem 31

31. If A be the A.M. and G be the G.M. between two numbers, show that the numbers are $A + \sqrt{A^2 - G^2}$ and $A - \sqrt{A^2 - G^2}$

Solution of Problem 31

Solution:

$$A = \frac{a+b}{2}, G = \sqrt{ab}$$

$$A + \sqrt{A^2 - G^2} = \frac{a+b}{2} + \sqrt{\frac{(a+b)^2}{4} - ab}$$

$$= \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2 - 2ab}{4}}$$

$$= \frac{a+b}{2} + \frac{a-b}{2} = a$$

Similarly

$$A - \sqrt{A^2 - G^2} = \frac{a+b}{2} - \frac{a-b}{2} = b$$

Problem 32

32. if the ratio of A.M and G.M. of two numbers a and b is $m : n$, prove that $a : b = m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$

Solution of Problem 32

Solution: Given, A.M. : G.M. $= m : n$

$$\Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{m}{n} \Rightarrow \frac{(a+b)^2}{4ab} = \frac{m^2}{n^2}$$

$$m + \sqrt{m^2 - n^2} = \frac{(a+b)n}{2\sqrt{ab}} + \sqrt{\frac{(a+b)^2 n^2}{4ab} - n^2}$$

$$= \frac{(a+b)n}{2\sqrt{ab}} + \frac{(a-b)n}{2\sqrt{ab}} = \frac{an}{\sqrt{ab}}$$

Similarly,

$$m - \sqrt{m^2 - n^2} = \frac{bn}{\sqrt{ab}}$$

Thus, $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2} = a : b$

Problem 33

33. If one G.M. G and two A.M p and q are inserted between two numbers, show that $G^2 = (2p - q)(2q - p)$

Solution of Problem 33

Solution: Let a and b be two numbers. $G = \sqrt{ab}$. Let d be the common difference, then, $d = \frac{b-a}{3}$ as two A.M. are inserted making no. of terms four.

$$p = a + d = \frac{b+2a}{3}, q = \frac{2b+a}{3}$$

$$(2p-q)(2q-p) = \frac{2b+4a-2b-a}{3} \cdot \frac{4b+2a-b-2a}{3}$$

$$= ab = G^2$$

Problem 34

34. If one A.M. A and two G.M. p and q be inserted between two numbers, show that $\frac{p^2}{q} + \frac{q^2}{p} = 2A$

Solution of Problem 34

Solution: We have $A = \frac{a+b}{2}$. Let r be the common ratio then $r = \sqrt[3]{\frac{b}{a}}$ because there are four terms in G.P.

$$p = ar = a\sqrt[3]{\frac{b}{a}}$$

$$q = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{3}}$$

Clearly,

$$\frac{p^2}{q} + \frac{q^2}{p} = \frac{a^2.\left(\frac{b}{q}\right)^{\frac{2}{3}}}{a\left(\frac{b}{a}\right)^{\frac{2}{3}}} + \frac{a^2\left(\frac{b}{a}\right)^{\frac{4}{3}}}{a\left(\frac{b}{a}\right)^{\frac{1}{3}}}$$

$$= a + b = 2A$$

Problem 35

35. if A.M. between a and b is equal to m times the H.M., prove that $a : b = \sqrt{m} + \sqrt{m-1} : \sqrt{m} - \sqrt{m-1}$

Solution of Problem 35

Solution: Given, $A = mH$

$$\Rightarrow \frac{a+b}{2} = \frac{2abm}{a+b} \Rightarrow m = \frac{(a+b)^2}{4ab}$$

$$\sqrt{m} = \frac{a+b}{2\sqrt{ab}}, \sqrt{m-1} = \frac{a-b}{2\sqrt{ab}}$$

Clearly, $a : b = \sqrt{m} + \sqrt{m-1} : \sqrt{m} - \sqrt{m-1}$

Problem 36

36. If 9 arithmetic means and 9 harmonic means be inserted between 2 and 3, prove that $A + \frac{6}{H} = 5$, where A is any arithmetic mean and H, the corresponding mean.

Solution of Problem 36

Solution: Let d and h be common difference for the A.P. and H.P. respectively.

$$d = \frac{3-1}{10} = \frac{1}{10}, h = \frac{\frac{1}{3}-\frac{1}{2}}{10} = -\frac{1}{60}$$

$$A = a + rd = \frac{20+r}{10}, \frac{1}{H} = \frac{1}{2} - \frac{r}{60} \Rightarrow \frac{1}{H} = \frac{30-r}{60}$$

$$A + \frac{6}{H} = 5 \Rightarrow \frac{20+r}{10} + \frac{30-r}{10} = 5$$

Problem 37

37. If a is the A.M. between b and c, b the G.M. between a and c, then show that c is the H.M. between a and b.

Solution of Problem 37

Solution:

$$a = \frac{b+c}{2}, b = \sqrt{ac} \Rightarrow c = \frac{b^2}{a}$$

Substituting for c in the A.M.,

$$a = \frac{b(a+b)}{2a}$$

Substituting for a for H.M. between a and b

$$\frac{2ab}{a+b} = \frac{2b(a+b)b}{2a(a+b)} = \frac{b^2}{a} = c$$

Problem 38

38. If a_1, a_2 be the two A.M., g_1, g_2 be the two G.M. and h_1, h_2 be the two H.M.between any two numbers x and y, show that $a_1 h_2 = a_2 h_1 = g_1 g_2 = xy$

Solution of Problem 38

Solution: Clearly

$$a_1 = \frac{2x-y}{3}, a_2 = \frac{x-2y}{3}$$

$$g_1 = x\left(\frac{y}{x}\right)^{\frac{1}{3}}, g_2 = a\left(\frac{y}{x}\right)^{\frac{2}{3}}$$

$$h_1 = \frac{3xy}{x-2y}, h_2 = \frac{3xy}{2x-y}$$

Substituting these values, we get $a_1h_2 = a_2h_1 = g_1g_2 = xy$

Problem 39

39. If between any two numbers, there be inserted $2n-1$ arithmetic, geometric and harmonic means show that nth means inserted are in G.P.

Solution of Problem 39

Solution: Let the two numbers be a and b. Common diff. would be $d = \frac{b-a}{2n}$, common ratio would be $r = \left(\frac{b}{a}\right)^{\frac{1}{2n}}$ and c.d. for harmonic progression would be $= \frac{a-b}{2nab}$

nth arithmetic mean $= a + \frac{b-a}{2n}.n = \frac{a+b}{2}$

nth geometric mean $= \sqrt{ab}$

nth harmonic mean $\frac{2ab}{a+b}$

Clearly, these are in G.P.

Problem 40

40. If the A.M. between two numbers exceed their G.M. by 2, and the G.M. exceeds the H.M. by $\frac{8}{5}$, find the numbers.

Solution of Problem 40

Solution: Let the two numbers be a and b.

$$\frac{a+b}{2} = 2 + \sqrt{ab}, \sqrt{ab} = \frac{2ab}{a+b} + \frac{8}{5}$$

$$AH = G^2 \Rightarrow (2+G)\left(G - \frac{8}{5}\right) = G^2$$

$$\Rightarrow 2G - \frac{16}{5} + G^2 - \frac{8G}{5} = G^2$$

Now G can be computed and thus A can be computed which will give a and b