Complex Numbers

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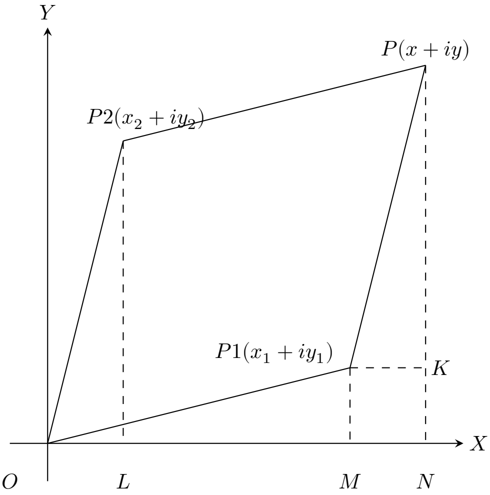
Geometrical Representation

Lte $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers which are represented by two points P_1 and P_2 in the following diagrams.

Additon



Addition of Two Complex Numbers

Clearly, $z = z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$.

Let P_1M, P_2L and PN be parallel to the y-axis; P_1K be parallel to the x-axis. This implied that triangle OP_2L and PP_1K are congruent.

We have $P_1K = OL = x_1$ and $P_2L = PK = y_1$

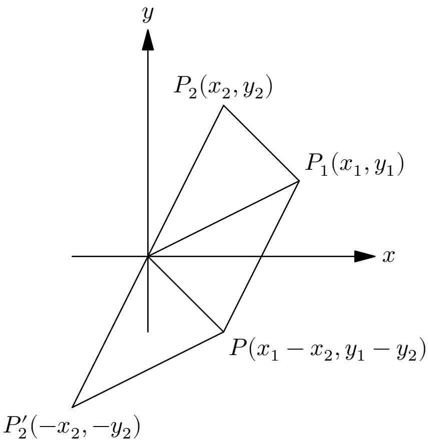
Thus, $ON = OM + MN = OL + P_1K = x_1 + x_2$

and $PN = PK + KN = P_2L + P_1M = y_2 + y_1$

So we can say that coordinates of P are $(x_1 + x_2, y_1 + y_2)$ which represents the complex number z.

We also see that this obeys vector addition i.e. $OP_1 + OP_2 = OP_1 + P_1P = OP$

Subtraction



Subtraction

We first represent $-z_2$ by P_2' so that P_2P_2' is bisected at O. Complete the parallelogram OP_1PP_2'. Then it can be easily seen that P representd the difference $z_1 - z_2$.

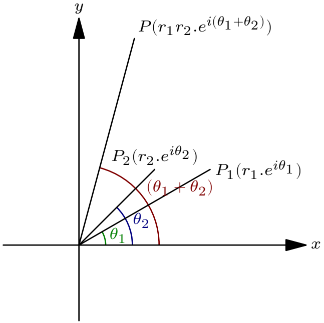
As OP_1PP_2' is a parallelogram so $P_1P = OP_2'$. Using vetor notation, we have,

$$z_1 - z_2 = OP_1 - OP_2 = OP_1 + OP_2' = OP_1 + P_1P = P_2P1$$

It follows that the complex number $z_1 - z_2$ is represented by the vector P_1P_2, where points P_1 and P_2 represent the complex numbers z_1 and z_2 respectively.

It should be noted that $arg(z_1 - z_2)$ is the angle through which OX must be rotated in the anticlockwise direction to make it parallel with P_1P_2.

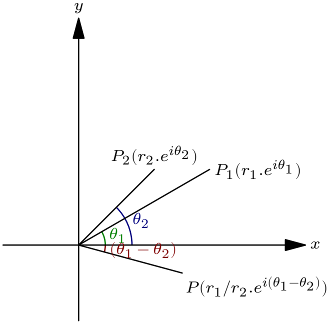
Multiplication



For multiplication it is convenient to use Euler's formula of complex numbers.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then clealry, $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

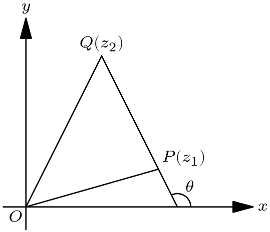
Division



For division also it is convenient to use Euler's formula of complex numbers.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then clealry, $z_1/z_2 = r_1/r_2 e^{i(\theta_1-\theta_2)}$

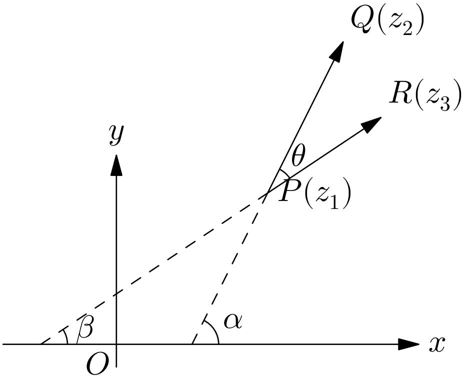
Three Important Results



$z_1 - z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP}$

$\therefore |z_1 - z_2| = |\overrightarrow{QP}| = QP$ which is nothing but distance between P and Q.

$arg(z_1 - z_2)$ is the angle made by $\overrightarrow{QP}$ with x-axis whis is nothing but θ.



$\theta = \alpha - \beta = arg(z_3 - z_1) - arg(z_2 - z_1) \Rightarrow \theta = arg\frac{z_3 - z_1}{z_2 - z_1}$

Similarly if three complex numbers are vertices of a triangle then angles of those vertices can also be computed using previous results.

Similarly, for four points to be concyclic where those points are represented by z_1, z_2, z_3 and z_4 if

$$arg\left(\frac{z_2 - z_4}{z_1 - z_4} . \frac{z_1 - z_3}{z_2 - z_4}\right) = 0$$

Any Root of an Imaginary Number is an Imaginary Number

Let iy be an imaginary number such that $y \neq= 0$

Let $\sqrt[n]{iy} = a, \therefore iy = a^n$

If a is real then a^n will also be real which is not possib;e as iy is an imaginary number so a will also be imaginary.

Square Root of a Complex Number

Consider a complex number $z = x + iy$ and let us say that $\sqrt{x+iy} = a + ib \Rightarrow x + iy = (a^2 - b^2) + 2abi$

$$\Rightarrow x = a^2 - b^2, y = 2ab$$

then we can write

$$a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2}$$

Thus, from these two equations we can write

$$a = \pm\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}}, b = \pm\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$$

Cube Roots of Unity

Let $x = 1^{1/3} \Rightarrow x^3 = 1 \Rightarrow x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0$

$x = -1, \frac{-1 \pm \sqrt{-3}}{2}$

It can be easily verified that if $\omega = \frac{-1+\sqrt{3}i}{2}$ then $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

Thus, three roots of cube root of unity are $1, \omega$ and ω^2.

It can be easily verified that $1 + \omega + \omega^2 = 0$(because ω is one of the roots) and $\omega^3 = 1$.