Complex Numbers Problems 171-180

Shiv Shankar Dayal

October 7, 2022

Problem 171

171. If $n > 1$, show that the roots of the equation $z^n = (1+z)^n$ are collinear.

Solution of Problem 171

Solution: Given $z^n = (z+1)^n \Rightarrow |z|^n = |z+1|^n$

$\Rightarrow |z| = |z+1| \Rightarrow x^2 = (x^2 + 2x + 1) \Rightarrow 2x + 1 = 0$

which is the equation of a straight line on which roots of the given equation will lie.

Problem 172

172. If A, B, C and D are four complex number then show that $AD.BC \leq BD.CA + CD.AB$.

Solution of Problem 172

Solution: Let z_1, z_2, z_3, z_4 be represented by the points A, B, C, D respectively.

$\therefore AD = |z_1 - z_4|$ and $BC = |z_2 - z_3|$

Let $a = (z_1 - z_4)(z_2 - z_3), b = (z_2 - z_4)(z_3 - z_1)$ and $c = (z_3 - z_4)(z_1 - z_2)$

$b + c = (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = -(z_1 - z_4)(z_2 - z_3) = -a$

$|a| = |b + c| \leq |b| + |c| \Rightarrow |-(z_1 - z_4)(z_2 - z_3)| = |(z_2 - z_4)(z_3 - z_1)| + |(z_3 - z_4)(z_1 - z_2)|$

$\Rightarrow AD.BC \leq BD.CA + CD.AB.$

Problem 173

173. If $a, b \in R$ and $a, b \neq 0$, then show that the equation of line joining a and ib is $\left(\frac{1}{2a} - \frac{i}{2b}\right) z + \left(\frac{1}{2a} + \frac{i}{2b}\right) \overline{z} = 1.$

Solution of Problem 173

Solution: Euqation of a line joining points a and ib is

$$\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ ib & = i\overline{b} & 1 \end{vmatrix} = 0 \text{ or } (\overline{a} + i\overline{b}) - (a - ib)\overline{z} - i(a\overline{b} + \overline{a}b) = 0$$

$$\Rightarrow (a + ib)z - (a - ib)\overline{z} - 2abi = 0[\because a, b \in R \therefore a = \overline{a}, b = \overline{b}]$$

$$\Rightarrow (a + ib)z - (a - ib)\overline{z} = 2abi$$

$$\Rightarrow \left(\frac{1}{2a} - \frac{i}{2b}\right) z + \left(\frac{1}{2a} + \frac{i}{2b}\right) \overline{z} = 1$$

Problem 174

174. If z_1 and z_2 are two compelx numbers such that $|z_1| - |z_2| = |z_1 - z_2|$, then show that $\arg(z_1) - \arg(z_2) = 2n\pi$ where $n \in I$.

Solution of Problem 174

Solution: Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then $r_1 - r_2 = \sqrt{(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2}$

$\Rightarrow 2r_1 r_2 = 2r_1 r_2 \cos(\theta_1 - \theta_2) \Rightarrow \cos(\theta_1 - \theta_2) = \cos 2n\pi$

$\Rightarrow \arg(z_1) - \arg(z_2) = 2n\pi$

Problem 175

175. Let A, B, C, D, E be points in the complex plane representing complex numbers z_1, z_2, z_3, z_4, z_5 respectvely. If $(z_3 - z_2)z_4 = (z_1 - z_2)z_5$, prove that $\triangle ABC$ and $\triangle DOE$ are similar.

Solution of Problem 175

Solution: $\triangle ABC$ and $\triangle DOE$ will be similar if

$\frac{AC}{AB} = \frac{DE}{DO}$ and $\angle BAC = \angle ODE$

$\Rightarrow \left|\frac{z_3-z_1}{z_2-z_1}\right| = \left|\frac{z_5-z_4}{0-z_4}\right|$ and $\arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = \arg\left(\frac{z_5-z_4}{0-z_4}\right)$

$\Rightarrow \frac{z_3-z_1}{z_2-z_1} = \frac{z_5-z_4}{0-z_4}$

Solving this yields $(z_3 - z_2)z_4 = (z_1 - z_2)z_5$ and hence triangles are similar.

Problem 176

176. Let z and z_0 are two complex numbers and $z, z_0, z\overline{z_0}, 1$ are represented by points P, P_0, Q, A respectively. If $|z| = 1$, show that the triangle POP_0 and AOQ are congruent and hence $|z - z_0| = |z\overline{z_0} - 1|$, where O represents the origin.

Solution of Problem 176

Solution: Given $OA = 1$ and $|z| = 1 = OP \Rightarrow OA = OP$

$OP_0 = |z_0|$ and $OQ = |z\overline{z_0}| = |z||\overline{z_0}| = |z_0|$

$\Rightarrow OP_0 = OQ$. Also give that $\angle P_0OP = \arg \frac{z_0}{z}$

$\angle AOQ = \arg\left(\frac{1}{z\overline{z_0}}\right) = \arg\left(\frac{\overline{z}}{\overline{z_0}}\right) [\because z\overline{z} = 1]$

$= -\arg\left(\frac{\overline{z_0}}{\overline{z}}\right) = -\arg\overline{\left(\frac{z_0}{z}\right)} = \arg\left(\frac{z_0}{z}\right) = \angle P_0OP$ and thus the triangles are congruent.

Problem 177

177. If the line segment joining z_1 and z_2 is divided by P and Q in the ratio $a : n$ internally and externally, then find $OP^2 + OQ^2$ where O is origin.

Solution of Problme 177

Solution: $P = \frac{az_2+bz_1}{a+b}, Q = \frac{az_2-bz_1}{a-b}$

$OP^2 = \left|\frac{az_2+bz_1}{a+b}\right|^2 = \left(\frac{az_2+bz_1}{a+b}\right)\left(\frac{a\overline{z_2}+b\overline{z_1}}{a+b}\right)$

$= \frac{1}{a^2+b^2}[a^2|z_2|^2 + b^2|z_1|^2 + ab(z_1\overline{z_2} + \overline{z_1}z_2)]$

Similalry OQ^2 can be computed and sum found.

Problem 178

178. Let z_1, z_2, z_3 be three complex numbers and a, b, c be real numbers not all zero such that $a+b+c=0$ and $az_1+bz_2+cz_3=0$, then show that z_1, z_2, z_3 are collinear.

Solution of Problem 178

Solution: Let $c \neq 0$, then $c = -(a+b)$ so we can write

$az_1 + bz_2 - (a+b)z_3 = 0 \Rightarrow z_3 = \frac{az_1+bz_2}{a+b}$

Thus, we see that z_3 divides line segment $z_1 z_2$ in the ratio of $a : b$ making all three of them collinear.

Problem 179

179. If $z_1 + z_2 + ... + z_n = 0$, prove that if a line passes through origin then all these do not lie of the same side of the line provided they do not lie on the line.

Solution of Problem 179

Solution: Equation of a line passing through origin is $a\overline{z} + \overline{a}z = 0$. Let us assume that all the points lie on the same side of the above line, so we have

$a\overline{z_i} + \overline{a}z_i > 0$ or < 0 for $i = 1, 2, 3, \ldots, n$

Thus, $a\sum_{i=1}^{n} \overline{z_i} + \overline{a}\sum_{i=1}^{n} z_i > 0$ or < 0

But it is given that $\sum_{i=1}^{n} z_i = 0 \Rightarrow \sum_{i=1}^{n} \overline{z_i} = 0$

$\therefore a\sum_{i=1}^{n} \overline{z_i} + \overline{a}\sum_{i=1}^{n} z_i = 0$

which in contradiction with equation above. So all points cannot lie on the same side of line.

Problem 180

180. The points $z_1 = 9 + 12i$ and $z_2 = 6 - 8i$ are given on a complex plane. Find the equation of the angle formed by the vector representing z_1 and z_2.

Solution of Problem 180

Solution: Let OA and OB be the unit vectors representing z_1 and z_2, then we have

$\vec{OA} = \frac{z_1}{|z_1|}, \vec{OB} = \frac{z_2}{|z_2|}$

Therefore equation of bisector will be $z = t\left(\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right) = \frac{6}{5}t$, where is an arbitrary positive integer.