Arithmetic, Geometric and Harmonic Means
Theory and Problems 1-10

Shiv Shankar Dayal

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Arithmetic Mean

Let a and b be the two given quantities and A be the A.M. between them. Then, a, A, b will be in A.P.

$$\therefore A - a = b - A \Rightarrow A = \frac{a+b}{2}$$

Let $A_1, A_2, \ldots, A_n$ be the n A.M. between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ will be in A.P.
Now $b = a + (n+2-1)d \Rightarrow d = \frac{b-1}{n+1}$

$$A_1 = a + d = \frac{an+b}{n+1}$$

$$A_2 = a + 2d = \frac{a(n-1)+2b}{n+1}$$

$$\ldots$$

$$A_n = a + nd = \frac{a+nb}{n+1}$$

Geometric Mean

Let a and b be two positive numbers and G be the G.M. between them. Then, a, G, b will be in G.P.

$$\therefore \frac{G}{a} = \frac{b}{G} \Rightarrow G = \sqrt{ab}$$

Let $G_1, G_2, \ldots, G_n$ be the n G.M. between two given quantities a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ will be in G.P. Clearly, b is $(n+2)$th term of the G.P.

$$b = ar^{n+1}$$

where r is the common ratio of the G.P.

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$\ldots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Harmonic Mean

Let a and b be two given numbers and H be the H.M. between them. Then, a, H, b will be in H.P. This implies that $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ will be in H.P.

$$\frac{1}{B} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \therefore H = \frac{2ab}{a+b}$$

Let $H_1, H_2, \dots, H_n$ be the H.M. between two given quantities a and b. Also, let d be the common difference of the corresponding A.P. Then, $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \dots, \frac{1}{H_n}, \frac{1}{b}$ will be in A.P.

$$t_{n+1} = \frac{1}{b} = \frac{1}{a} + (n+1)d \Rightarrow d = \frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{a+bn}{ab(n+1)}$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d = \frac{2a+b(n-1)}{ab(n+1)}$$

$$\dots$$

$$\frac{1}{H_n} = \frac{na+b}{ab(n+1)}$$

Relation Between A.M., G.M. and H.M.

Let a and b be two real, positive and unequal quantities and A, G and H be the single A.M., G.M. and H.M. respectively.
Then $A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$
Now, $AH = ab = G^2 \therefore \frac{G}{A} = \frac{H}{G}$
Hence, A, G and H are in G.P.
Also, $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} > 0[\because a \neq b, a, b, > 0]$
Thus, $A - G > 0 \Rightarrow A > G$
Since $\frac{H}{G} = \frac{G}{A} \Rightarrow \frac{H}{G} < 1$
Thus, $A > G > H$
If $a = b,$ it can be proven that $A = G = H$

Problem 1

1. If n arithmetic means are inserted between 20 and 80 such that first mean : last mean $= 1 : 3$, find n.

Solution of Problem 1

Solution: Let the n means be $x_1, x_2, \dots, x_n$
Then $20, x_1, x_2, \dots, x_n, 80$ are in A.P.

$$80 = 20 + (n+1)d$$

$$d = \frac{60}{n+1}$$

$$x_1 = 20 + d = \frac{20n + 80}{n+1}$$

$$x_n = 20 + nd = \frac{20 + 80n}{n+1}$$

Given, $x_1 : x_n = 1 : 3 \Rightarrow \dfrac{20n + 80}{80n + 20} = \dfrac{1}{n} \Rightarrow n = 11$

Problem 2

2. Prove that the sum of n arthmetic means between two given numbers is n times the single arithmetic between them.

Solution of Problem 2

Solution: Single A.M. $= \frac{a+b}{2}$
Let the n arithmetic means are $x_1, x_2, \ldots, x_n$, then $a, x_1, x_2, \ldots, x_n, b$ will be in A.P.
$\therefore x_1 = a + d$ and $x_n = a + d$, where d is the common difference.

$$x_1 + x_2 + \ldots + x_n = \frac{n}{2}(a+b)[\because \text{sum of } n \text{ terms } = \frac{n}{2}(\text{first term + last term})]$$

Thus, we have proven the desired condition.

Problem 3

3. Between two numbers whose sum is $\frac{13}{6}$, an even number of arithmetic means are inserted. If the sum of means exceeds their number by unity, find the number of means.

Solution of Problem 3

Solution: Let $2n$ be the number of means between two number a and b

$$\text{Sum of the } 2n \text{ means} = \frac{a+b}{2}.2n = (a+b)n$$

Given,

$$(a+b)n = 2n+1 \Rightarrow \frac{13}{6} = 2n+1 \Rightarrow 2n = 12$$

Problem 4

4. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$, $a \neq b$ is the A.M. of a and b.

Solution of Problem 4

Solution: A.M. between a and $b = \frac{a+b}{2}$
Given,

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{a+b}{2}$$

$$\Rightarrow (a-b)(a^n - b^2) = 0$$

$$\because a \neq b \therefore a^n = b^n \;\; \Rightarrow n = 0$$

Problem 5

5. Insert 4 G.M. between 5 and 160.

Solution of Problem 5

Solution: Let x_1, x_2, x_3, x_4 be the four G.M. between 5 and 160.

Thus, $5, x_1, x_2, x_3, x_4, 160$ will be in G.P.

$$160 = 5r^5 \Rightarrow r = 2$$

Thus, means are $10, 20, 40, 80$.

Problem 6

6. Show that the product of n geometric means inserted between two positive quantities is equal to the nth power of the single geometric mean between them.

Solution of Problem 6

Solution: Let $x_1, x_2, \ldots, x_n$ be n G.M. between two numbers a and b.

$$x_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \; x_2 = a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, x_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Thus,

$$x_1 x_2 \ldots x_n = a^n\left(\frac{b}{a}\right)^{\frac{1+2+\ldots+n}{n+1}} = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}} = (\sqrt{ab})^n$$

Thus, we have proven the desired result as single G.M. is $\sqrt{ab}$

Problem 7

7. Insert 6 harmonic means between 3 and $\frac{6}{23}$.

Solution of Problem 7

Solution: Let $x_1, x_2, \dots, x_6$ be the six H.M. between 3 and $\frac{6}{23}$
Thus, $\frac{1}{3}, \frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_6}, \frac{23}{6}$ are in A.P.
Let d be the common denominator.

$$\Rightarrow \frac{23}{6} = \frac{1}{3} + 7d \Rightarrow d = \frac{1}{2}$$

Now the means can be easily computed.

Problem 8

8. If the A.M. and G.M. between two numbers be 5 and 3 respectively, find the numbers.

Solution of Problem 8

Solution: Let a and b be the two numbers. Thus, $\frac{a+b}{2} = 5 \Rightarrow a + b = 10$
Since 3 is the G.M. so $a, 3, b$ are in G.P. Let r be the common ratio then $ar = 3$ and $b = ar^2$
Given,

$$a + ar^2 = 10, ar = 3 \Rightarrow \frac{1 + r^2}{r} = \frac{10}{3}$$

Solving we get $r = 3, \frac{1}{3}$ which gives us two numbers as $9, 1$ and $1, 9$

Problem 9

9. If the A.M. between two numbers be twice their G.M., show that the ratio of numbers is $2 + \sqrt{3} : 2 - \sqrt{3}$.

Solution of Problem 9

Solution: Let A be the A.M and G be the G.M. between two numbers a and b, then $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

$$\text{Given, } \frac{a+b}{2} = 2\sqrt{ab} \Rightarrow \frac{a+b}{\sqrt{ab}} = \frac{2}{1}$$

By component and dividendo, we get

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3}{2} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$$

Again by componendo and dividendo, we get

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

Problem 10

10. If a be one A.M. and g_1 and g_1 be two G.M. between b and c, prove that $g_1^3 + g_2^3 = 2abc$

Solution of Problem 10

Solution: Clearly, $a = \frac{b+c}{2}$. Let r be the common ratio then $g_1 = br$ and $g_2 = br^2$

$$g_1^3 + g_2^3 = (br)^3 + (br^2)^3 = b^3r^3(1+r^3) = b^3\frac{c}{b}\left(1+\frac{c}{b}\right) = bc(b+c) = 2abc$$