Complex Numbers Problems 111-120

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Problem 111

111. Find the region represented by $|z-4| < |z-2|$.

Solution of Problem 111

Solution: Let $z = x + iy$

$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2 \Rightarrow x^2 - 8x + 16 < x^2 - 4x + 4$

$\Rightarrow 4x > 12 \Rightarrow x > 3$

Problem 112

112. If $2z_1 - 3z_2 + z_3 = 0$, then find the geometrical relationship between them.

Solution of Problem 112

Solution: Given, $2z_1 - 3z_2 + z_3 = 0$

$\Rightarrow z_2 = \frac{2z_1+z_3}{3} = \frac{2z_1+z_3}{2+1}$

Thus, z_1 divides the line segement $z_1 z_3$ in the ratio of $2:1$ i.e. all three points are collinear.

Problem 113

113. If $z = x + iy$, such that $|z+1| = |z-1|$ and $\arg \frac{z-1}{z+1} = \frac{\pi}{4}$, find x and y.

Solution of Problem 113

Solution: Given, $|z+1| = |z-1| \Rightarrow (x+1)^2 + y^2 = (x-1)^2 + y^2 \Rightarrow x = 0$

Also, given that $\arg \frac{z-1}{z+1} = \frac{\pi}{4}$

$\Rightarrow z - 1 = (z+1)e^{i\pi/4} \Rightarrow -1 + iy = (1+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\Rightarrow -1 + iy = (1+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$\Rightarrow y = \sqrt{2} + 1$

Problem 114

114. If $|z|^8 = |z-1|^8$, then prove that roots of this equation are collinear.

Solution of Problem 114

Solution: Given, $|z|^8 = |z-1|^8 \Rightarrow |z| = |z-1|$

$\Rightarrow x^2 + y^2 = (x-1)^2 + y^2 \Rightarrow x = \frac{1}{2}, y \in (\infty, infty)$

which is equation of straight line parallel to y-axis at $x = 1/2$.

Problem 115

115. Prove that $z\overline{z} + a\overline{z} + \overline{a}z + b = 0$, represents a circle if $|a|^2 > b$.

Solution of Problem 115

Solution: Given, $z\overline{z} + a\overline{z} + \overline{a}z + b = 0$

$z\overline{z} + a\overline{z} + \overline{a}z + a\overline{a} = a\overline{a} - b$

$(z + a)(\overline{z} + \overline{a}) = |a|^2 - b$

which is equation of a circle if $|a|^2 - b > 0 \Rightarrow |a|^2 > b$.

Problem 116

116. If $z = (\lambda + 3) + i\sqrt{3 - lambda^2}$, where $|\lambda| < \sqrt{3}$, then prove that it represents a circle.

Solution of Problem 116

Solution: Let $z = x + iy$, comparing real and imaginary part gives us

$x = \lambda + 3, y = \sqrt{3 - \lambda^2} \Rightarrow y^2 = 3 - \lambda^2$

$\Rightarrow (x-3)^2 + y^2 = 3$

which is equation of a circle with center $(3, 0)$ and radius $\sqrt{3}$.

Problem 117

117. If z is a complex number such that $|Re(z)| + |Im(z)| = k,\ \forall k \in R$, then find the locus of z.

Solution of Problem 117

Solution: Let $z = x + iy$, then $|Re(z)| + |Im(z)| = k$ will give us four equations.

$x + y = k, x - y = k, -x + y = k$ and $-x - y = k$

These lines will intersect at $(k, 0), (0, k), (-k, 0), (0 - k)$ giving us a square as locus of z.

Problem 118

118. Consider a sequence of complex numbers such that $z_{n+1} = z_n^2 + i, \ \forall n \geq 1$, where $z_1 = 0$. Find z_{111}.

Solution of Problem 118

Solution: $z_2 = z_1^2 + i = i, z_3 = z_2^2 + i = i - 1, z_4 = z_3^2 + i = (i-1)^2 + i = -i$

$z_5 = z_4^2 + i = i - 1, z_6 = z_5^2 + i = -i$

Thus, we see that it is a cycle between $-i$ and $i - 1$ starting at z_3.

$\Rightarrow z_{111} = z_3 = i - 1 \Rightarrow |z_{111}| = \sqrt{2}$

Problem 119

119. The complex numbers whose real and imaginary parts are integers and satisfy the relation $z\overline{z}^3 + z^3\overline{z} = 350$, forms a rectangle in the argand plane. Find length of its diagonals.

Solution of Problem 119

Solution: Given, $z\overline{z}^3 + z^3\overline{z} = 350 \Rightarrow z\overline{z}(\overline{z}^2 + z^2) = 350$

Let $z = x + iy$, then given equation becomes $2(x^2 + y^2)(x^2 - y^2) = 350 \Rightarrow (x^2 + y^2)(x^2 - y^2) = 175$

Prime factors of 175 are $5, 5, 7$ so the only solution which yields integers for x and y are $x^2 + y^2 = 25, x^2 - y^2 = 7$

$\Rightarrow x = \pm 4, y = \pm 3$ which gives a rectangle with four points and digonal with a length of 10 units.

Problem 120

120. If z_1, z_2 are two complex numbers and $\arg \frac{z_1+z_2}{z_1-z_2}$ but $|z_1 + z_2| \neq |z_1 - z_2|$ then find the figure formed by $0, z_1, z_2$ and $z_1 + z_2$.

Solution of Problem 120

Solution: We know that $z_1 + z_2$ and $z_1 - z_2$ are the diagonals of a quadrilateral. Now diagonals of a parallelogram does not intersect at angle $\pi/2$ and diagonals of a square and rectangle are equal. Only rhombus satisfies the given criteria of diagonals meeting at right angle and having different lengths.

Thus, the given conditions represent a rhombus but not a square.