Miscellaneous Problems on A.P., G.P. and H.P.
Problems 11-20

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Problem 11

11. In an A.P. of $2n$ terms the middle pair of terms are $p+q$ and $p-q$. Show that the sum of cubes of the terms in A.P. are $2np[p^2+(4n^2-1)q^2]$

Solution of Problem 11

Solution: Let t_r denote the rth term of the A.P.

Given, $t_n = p + q$ and $t_{n+1} = p - q \therefore d = -2q$

Also, $t_1 + t_{2n} = t_2 + t_{2n-1} = ... = t_n + t_{n+1} = 2p$

Let S be the sum of cubes of the terms of A.P., then $S = (t_1^3 + t_{2n}^3) + (t_2^3 + t_{2n-1}^3) + ... + (t_n^3 + t_{n-1}^3)$

$$t_1^3 + t_{2n}^3 = (t_1 + t_{2n})^3 - 3t_1 t_{2n}(t_1 + t_{2n}) = 8p^3 - 6pt_1 t_{2n} = 8p^3 - \frac{6p}{4}[(t_1 + t_{2n})^2 - (t_1 - t_{2n})^2]$$

$$= 8p^3 - \frac{3p}{2}[4p^2 - (2n-1)^2.4q^2][\because t_{2n} = t_1 + (2n-1)d \text{ and } d = -2q]$$

$$= 2p^3 + 6pq^2(2n-1)^2$$

Similarly

$$t_2^3 + t_{2n-1}^3 = 2p^3 + 6pq^2(2n-3)^2, t_3^3 + t_{2n-2}^3 = 2p^3 + 6pq^2(2n-5)^2, \dots, t_n^3 + t_{n+1}^3 = 2p^3 + 6pq^2.1^2$$

Adding all these, we get

$$S = 2np^3 + 6pq^2[1^2 + 3^2 + 5^2 + ... \text{ to } n \text{ terms}]$$

$$= 2np[p^2 + (4n^2 - 1)q^2]$$

Problem 12

12. Find the sum S_n of the cubes of the first n terms of an A.P. and show that the sum of the first n terms of the A.P. is a factor of S_n.

Solution of Problem 12

Solution: Let S be the sum of first n terms of the A.P. $a, a+d, a+2d, ...$ then $S = \frac{n}{2}[2a+(n-1)d]$

$$S_n = a^3 + (a+d)^3 + (a+2d)^3 + ... + [a+(n-1)d]^3$$

$$= na^3 + 3a^2d[1+2+3+...+(n-1)] + 3ad^2[1^2+2^2+...+(n-1)^2] + d^3[1^3+2^3+...+(n-1)^3]$$

$$= na^3 + 3a^2d.\frac{n(n-1)}{2} + 3ad^2.\frac{(n-1).n.(2n-1)}{6} + d^3\frac{n^2(n-1)^2}{4}$$

$$= \frac{n}{2}\left(2a^3 + 3(n-1)a^2d + (n-1)(2n-1)ad^2 + \frac{1}{2}n(n-1)^2d^3\right)$$

$$= \frac{n}{2}\left[a^2(2a+(n-1)d) + (n-1)ad(2a+(n-1)d) + \frac{n(n-1)}{2}d^2(2a+(n-1)d)\right]$$

$$= \frac{n}{2}[2a+(n-1)d]\left[a^2 + (n-1)ad + \frac{n()n-1}{2}d^2\right]$$

$$= S\left[a^2 + (n-1)ad + \frac{n()n-1}{2}d^2\right]$$

Hence, S is a factor of S_n

Problem 13

13. Show that any positive integral power (greater than 1) of a positive integer m, is the sum of m consecutive odd positive integers. Find the first odd integer for $m^r (r > 1)$

Solution of Problem 13

Solution: Let r be a positive integer and $r > 1$.

Let $m^r = (2k+1) + (2k+3) + ... + (2k+2m-1)$

$$m^r = \frac{m}{2}[4k + 2 + (m-1)2 \Rightarrow m^{r-1} = 2k + m] \Rightarrow k = \frac{m^{r-1} - m}{2}$$

Clearly for $r > 1, m^{r-1}$ and m are both odd or both even. $\therefore m^{r-1} - m$ is an even number. Thus, such integer k exists.

First off interger $= 2k + 1 = m^{r-1} - m + 1$

Problem 14

14. If a be the sum of n terms and b^2 the sum of the square of n terms of an A.P., find the first term and common difference of the A.P.

Solution of Problem 14

14. Let $x = x_1$ be the first term and d be the common difference. Then,

$$x + (x+d) + ... + [x + (n-1)d] = a \Rightarrow nx + \frac{d.(n-1)n}{2} = a$$

Squaring both sides of the above equation

$$nx^2 + \frac{d^2(n-1)^2 n}{4} + n(n-1)xd = \frac{a^2}{n}$$

Also,

$$x^2 + (x+d)^2 + ... + [x + (n-1)d]^2 = b^2$$

$$\Rightarrow nx^2 + d^2[1^2 + 2^2 + ... + (n-1)^2] + 2xd[1 + 2 + 3 + ... + (n-1)] = b^2$$

$$\Rightarrow nx^2 + d^2\frac{(n-1)n(2n-1)}{6} + 2xd\frac{n(n-1)}{2} = b^2$$

Subtracting the two obtained equations we get

$$d^2\frac{n(n-1)(n+1)}{12} = \frac{nb^2 - a^2}{n} \Rightarrow d = \pm\frac{2\sqrt{3(nb^2 - a^2)}}{n\sqrt{n^2 - 1}}$$

$$\Rightarrow x = \frac{1}{n}\left[a \mp \frac{-(n-1)\sqrt{3(nb^2 - a^2)}}{\sqrt{n^2 - 1}}\right]$$

Problem 15

15. If $a_1, a_2, \dots, a_n$ are in A.P., whose common diference is d, then find the sum of the series

$$\sin d[\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \dots + \csc a_{n-1} \csc a_n]$$

Solution of Problem 15

Solution:

$$t_1 = \sin d(\csc a_1 \csc a_2) = \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} = \cot a_1 - \cot a_2$$

$$t_2 = \cot a_2 - \cot a_3$$

$$\ldots$$

$$t_{n-1} = \cot a_{n-1} - \cot a_n$$

Adding, we get $\sin d[\csc a_1 \csc a_2 + \csc a_2 \csc a_3 + \ldots + \csc a_{n-1} \csc a_n] = \cot a_1 - \cot a_n$

Problem 16

16. If $a_1, a_2, \dots, a_n$ are in A.P. where $a_i > 0\ \forall i$, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution of Problem 16

Solution:

$$t_1 = \frac{1}{\sqrt{a_1} + \sqrt{a_2}} = \frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} = \frac{1}{d}(\sqrt{a_2} - \sqrt{a_1})$$

$$t_2 = \frac{1}{d}(\sqrt{a_3} - \sqrt{a_2})$$

...

$$t_{n-1} = \frac{1}{d}(\sqrt{a_n} - a_{n-1})$$

Adding, we get

$$S = \frac{1}{d}(\sqrt{a_n} - \sqrt{a_1}) = \frac{1}{d}\frac{a_n - a_n}{\sqrt{a_1} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

Problem 17

17. If $a_1, a_2, \dots, a_n$ are in A.P., whose common differemce is d show that $\sum_2^n \tan^{-1} \frac{d}{1+a_{n-1}a_n} = \tan^{-1} \frac{a_n - a_n}{1+a_n a_1}$

Solution of Problem 17

Solution: We have to prove that $\tan^{-1}\frac{d}{1+a_1a_2}+\tan^{-1}\frac{d}{1+a_2a_3}+...+\tan^{-1}\frac{d}{1+a_{n-1}a_n}=\tan^{-1}\frac{a_n-a_n}{1+a_na_1}$

$$t_1=\tan^{-1}\frac{d}{1+a_1a_2}=\tan^{-1}\frac{a_2-a_1}{1+a_1a_2}=\tan^{-1}a_2-\tan^{-1}a_1$$

$$t_2=\tan^{-1}\frac{d}{1+a_2a_3}=\tan^{-1}a_3-\tan^{-1}a_2$$

$$\dots$$

$$t_{n-1}=\tan^{-1}\frac{d}{1+a_{n-1}a_n}=\tan^{-1}a_n-\tan^2-1a_{n-1}$$

Adding, we get

$$\tan^{-1}\frac{d}{1+a_1a_2}+\tan^{-1}\frac{d}{1+a_2a_3}+...+\tan^{-1}\frac{d}{1+a_{n-1}a_n}=\tan^{-1}a_n-\tan^{-1}a_1=\tan^{-1}\frac{a_n-a_1}{1+a_1a_n}$$

Problem 18

18. If $a_1, a_2, \ldots, a_n$ are the first n items of an A.P. with first term a and common difference d such that $ad > 0$. Let $S_n = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} - \ldots + \frac{1}{a_{n-1} a_n}$ Prove that the product $a_1 a_n S_n$ does not depend on a or d.

Solution of Problem 18

Solution:

$$\frac{1}{a_1a_2} = \frac{a_2 - a_1}{da_1a_2} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)$$

$$\frac{1}{a_2a_3} = \frac{1}{d}\left(\frac{1}{a_2} - \frac{1}{a_3}\right)$$

$$\cdots$$

$$\frac{1}{a_{n-1}a_n} = \frac{1}{d}\left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)$$

Adding, we get

$$S_n = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_n}\right) = \frac{n-1}{a_1a_n}$$

$\therefore a_1a_nS_n = n - 1$ which is independent of a and d.

Problem 19

19. If $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$ be in A.P., whose common difference is d and $S_1 = a_1 + a_2 + \ldots + a_n$, $S_2 = a_{n+1} + \ldots + a_{2n}$, $S_3 = a_{2n+1} + \ldots + a_{3n}$ Show that $S_1, S_2, S_3, \ldots$ are in A.P. whose common difference is $n^2 d$

Solution of Problem 19

Solution:

$$S_2 - S_1 = \frac{n}{2}[2a_{n+1} + (n-1)d - 2a_1 - (n-1)d] = \frac{n}{2}[2(a_1 + nd) - 2a_1] = n^2 d$$

Similarly,

$$S_3 - S_2 = S_4 - S_3 = ... = n^2 d$$

Problem 20

20. If a, b, c are three terms of an A.P. such that $a \neq b$, show that $(b - c)/(a - b)$ is a rational number.

Solution of Problem 20

Solution: Let x be the first term and y be the common difference. Also, let a, b, c to be the pth, qth, rth term of the A.P.

$$a = x + (p-1)y, b = x + (q-1)y, c = x + (r-1)y$$
$$(b-c)/(a-b) = \frac{q-r}{p-q}$$

Since p, q, r are integers $\frac{q-r}{p-q}$ will be a rational number.