Complex Numbers Problems 211-220

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Problem 211

211. Solve the equation $z^2 + z|z| + |z|^2 = 0$

Solution of Problem 211

Solution: Clearly $z = 0$ is one of the solutions. For other solutions divide both sides by $|z|^2$ which gives us $t^2 + t + 1 = 0$ where $t = \frac{z}{|z|}$.

The equation $t^2 + t + 1 = 0$ has two roots i.e. $t = \omega, \omega^2 \Rightarrow \frac{z}{|z|} = \omega, \omega^2$

$\Rightarrow z = k\omega, k\omega^2$ where $k = |z|$ is a non-negative real number.

Problem 212

212. Solve the equation $2z = |z| + 2i$ in complex numbers.

Solutiion of Problem 212

Solution: Let $z = x + iy$, then $2x + 2iy = \sqrt{x^2 + y^2} + 2i$. Comparing real and imaginary parts, we get

$2y = 2 \Rightarrow y = 1$ and $2x = \sqrt{x^2 + y^2} \Rightarrow 4x^2 = x^2 + 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow z = \pm\frac{1}{\sqrt{3}} + i.$

Problem 213

213. If $a > 0$ and $z|z| + az + 1 = 0$, show that z is a negative real number.

Solution of Problem 213

Solution: Let $z = x + iy$, then $(x + iy)\sqrt{x^2 + y^2} + a(x + iy) + 1 = 0.$

Comparing real and imaginary parts, we get

$y\sqrt{x^2 + y^2} + ay = 0 \Rightarrow y = 0 \ \because \sqrt{x^2 + y^2} + a \neq 0 \ \ [\because a > 0]$ and

$\therefore x\sqrt{x^2 + 0} + ax + 1 = 0 \Rightarrow x^2 + ax + 1 = 0 \Rightarrow x = \frac{-a \pm \sqrt{a^2 - 4}}{2}$

Clearly, both the values of x are negative, so z is a negative real number.

Problem 214

214. For every real number $a > 0$, find all complex numbers z satisfying the equation $z|z| + az + i = 0$.

Solution of Problem 214

Solution: Let $z = x + iy$, then $(x + iy)\sqrt{x^2 + y^2} + ax + aiy + i = 0$

Comparing real and imaginary parts, we get

$x\sqrt{x^2 + y^2} + ax = 0 \Rightarrow x = 0 \ \because \sqrt{x^2 + y^2} + a \neq 0 \ [\because a > 0]$ and

$y\sqrt{x^2 + y^2} + ay + 1 = 0 \Rightarrow y^2 + ay + 1 = 0 \Rightarrow y = \frac{-a \pm \sqrt{a^2 - 4}}{2}$

Problem 215

215. For evert real number $a > 0$, determine the complex numbers z, which will satisfy the equation $|z|^2 - 2iz + 2a(1+i) = 0$

Solution of Problem 215

Solution: Let $z = x + iy$, then $x^2 + y^2 - 2i(x + iy) + 2a(1 + i) = 0$

Comparing real and imaginary parts, we get

$x^2 + y^2 + 2y + 2a = 0 \Rightarrow x^2 + (y - 1)^2 = 1 - 2a$ and $-2x + 2a = 0 \Rightarrow x = a$

$\Rightarrow (y - 1)^2 = 1 - 2a - a^2 \Rightarrow y = 1 \pm \sqrt{1 - 2a - a^2}$

Howoever $1 - 2a - a^2 > 0.$ Roots of equivalent quadratic equation is $a = \frac{2 \pm \sqrt{8}}{-2} \Rightarrow -1 \pm \sqrt{2}$ but $a > 0$ so the range for a is $0 < a < \sqrt{2} - 1.$

Problem 216

216. If α and β are two complex numbers, show that $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2 + \Re(\alpha\overline{\beta}) + \Re(\overline{\alpha}\beta)$.

Solution of Problem 216

Solution: Let $\alpha = a + ib$ and $\beta = c + id$ then $|\alpha + \beta|^2 = (a + c)^2 + (b + d)^2$

$|\alpha|^2 = a^2 + b^2, |\beta|^2 = c^2 + d^2, \Re(\alpha\overline{\beta}) = \Re[(a + ib)(c - id)] = ac + bd, \Re(\overline{\alpha}\beta) = \Re[(a - ib)(c + id)] = ac + bd$

Thus, $|\alpha + \beta|^2 = |\alpha|^2 + |\beta|^2 + \Re(\alpha\overline{\beta}) + \Re(\overline{\alpha}\beta)$.

Problem 217

217: Find the integral solution of the equation $(3+4i)^x = 5^{\frac{x}{2}}$.

Solution of Problem 217

Solution: Let $z = 3 + 4i$ then $|z| = 5$, so the given equation becomes $z^x = |z|^{\frac{x}{2}} \Rightarrow z^{2x} = |z|^x$

$\left(\frac{(3+4i)^2}{5}\right)^x = 1 \Rightarrow \left(\frac{3-16+24i}{5}\right)^x = 1 \Rightarrow \left(\frac{-7+24i}{5}\right)^x = 1$ which is possible only if $x = 0$.

Problem 218

218. Find the integral solution of the equation $(1 - i)^x = 2^x$

Solution of Problem 218

Solution: Given, $(1-x)^x = 2^x \Rightarrow \left(\frac{1-i}{2}\right)^x = 1$ which is possible only if $x = 0$.

219. Find the integral solution of the equation $(1-i)^x = (1+i)^x$.

Solution of Problem 219

Solution: Given, $(1-i)^x = (1+i)^x \Rightarrow \left(\frac{1-i}{1+i}\right)^x = \left(\frac{(1-i)^2}{(1+i)(1-i)}\right)^x = \left(\frac{-2i}{2}\right)^x = (-i)^x = 1 \Rightarrow x = 4n \ \forall n \in I.$

Problem 220

220. Prove that $|1-\overline{z_1}z_2|^2-|z_1-z_2|^2=(1-|z_1|^2)(1-|z_2|^2)$.

Solution of Problem 220

Solution: L.H.S. $= |1 - \overline{z_1} z_2|^2 - |z_1 - z_2|^2 = (1 - \overline{z_1} z_2)(1 - z_1 \overline{z_2}) - (z_1 - z_2)(\overline{z_1} - \overline{z_2}) [\because |z|^2 = z\overline{z}]$

$= (1 - \overline{z_1} z_2 - z_1 \overline{z_2} + |z_1|^2 |z_2|^2) - (|z_1|^2 - \overline{z_1} z_2 - z_1 \overline{z_2} + |z_2|^2)$

$= (1 - |z_1|^2 - |z_2|^2 + |z_1|^2 |z_2|^2) = (1 - |z_1|^2)(1 - |z_2|^2) =$ R.H.S.