Complex Numbers Problems 191-200

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Problem 191

191. If $z - 6 - 8i \leq 4$, then find the least and greatest value of z.

Solution of Problem 191

Solution: $|z-6-8i| \leq |4| \Rightarrow -4 \leq |z|-|6+8i| \leq 4$

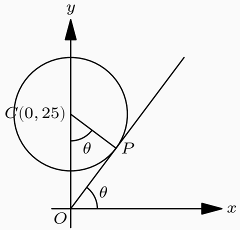
$\Rightarrow -4 \leq |z|-10 \leq 4 \Rightarrow 6 \leq |z| \leq 14$

Problem 192

192. If $z - 25i \leq 15$ then find the least positive value of $\arg(z)$.

Solution of Problem 192

Solution: The diagram is given below:



Given $z - 25i \leq 15$, which represents a circle having center $(0, 25)$ and a radius 15.

Let OP be tangent to the circle at point P, then $\angle XOP$ will represent least value of $\arg(z)$.

Let $\angle XOP = \theta$ then $\angle OCP = \theta$. Now $OC = 25, CP = 15 \therefore OP = 20$

$\therefore \tan\theta = \frac{OP}{CP} = \frac{4}{3}$. $\therefore$ Least value of $\arg(z) = \theta = \tan^{-1}\frac{4}{3}$

Problem 193

193. Show that the equation $|z - z_1|^2 + |z - z_2|^2 = k$ where $k \in R$ will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$.

Solution of Problem 193

Solution: Given, $|z - z_1|^2 + |z - z_2|^2 = k$

$\Rightarrow |z|^2 + |z_1|^2 - 2z\overline{z_1} + |z|^2 + |z_2|^2 - 2z\overline{z_2} = k$

$\Rightarrow 2|z|^2 - 2z(\overline{z_1} + \overline{z_2}) = k - (|z_1|^2 + |z_2|^2)$

$\Rightarrow |z|^2 - 2z\left(\overline{\frac{z_1+z_2}{2}}\right) + \frac{1}{4}|z_1 + z_2|^2 = \frac{k}{2} + \frac{1}{4}[|z_1 + z_2|^2 - 2|z_1|^2 - 2|z_2|^2]$

$\Rightarrow \left|z - \frac{z_1+z_2}{2}\right|^2 = \frac{1}{2}\left[k - \frac{1}{2}|z_1 - z_2|^2\right]$

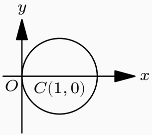
The above equation represents a circle with center at $\frac{z_1+z_2}{2}$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{|z_1-z_2|^2}{2}$.

Problem 194

194. If $|z-1|=1$, prove that $\frac{z-2}{z}=i\tan[\arg(z)]$.

Solution of Problem 194

Solution: Since $|z-1|=1, z$ represents a circle with center $(1,0)$ and a radius of of 1. It is shown below:



Now $|z-1|=1$. Let $z=x+iy$ then $x^2+y^2=2x$. Also,

$$\frac{z-2}{z}=\frac{x-2+iy}{x+iy}=\frac{x^2-2x+y^2+2iy}{x^2+y^2}=i\frac{y}{x}$$

Case I. When z lies in the first quadrant. This implies $\arg(z)=\theta$, where $\tan\theta=\frac{y}{x}\therefore i\tan[\arg(z)]=i\tan\theta=i\frac{y}{x}$.

Case II. When z lies in the fourth quadrant. Thus, $\arg(z)=2\pi-\theta$, where $\tan\theta=\frac{-y}{x}$

$\therefore i\tan[\arg(z)]=i\tan(2\pi-\theta)=i\frac{y}{x}$.

Problem 195

195. Find the locus of z if $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$.

Solution of Problem 195

Solution: Let $z = x + iy$. Now we have $\frac{z-1}{z+1} = \frac{(x^2-1)+y^2}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2}$

$\therefore \arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \tan\left(\arg\left(\frac{z-1}{z+1}\right)\right) = \frac{2y}{x^2-1+y^2}$

$\Rightarrow x^2 + y^2 - 1 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 2$, which is equation of a circle having center at $(0, 1)$ and radius $\sqrt{2}$.

Probem 196

196. If α is real and z is a complex number and u and v be the real and imaginary parts of $(z-1)(\cos\alpha - i\sin\alpha) + (z-1)^{-1}(\cos\alpha + i\sin\alpha)$. Prove that the locus of the points represneting the complex numbers such that $v = 0$ is a circle of unit radius with center at a point $(1, 0)$ and a straight line passing through the center of the circle.

Solution of Problem 196

Solution: Let $z = x + iy$. Now, $u + iv = (z-1)(\cos\alpha - i\sin\alpha) + \frac{1}{z-1}(\cos\alpha + i\sin\alpha)$

$= (x-1)\cos\alpha + y\sin\alpha + i[y\cos\alpha - (x-1)\sin\alpha] + \frac{x-1-iy}{(x-1)^2+y^2}(\cos\alpha + i\sin\alpha) = 0$

Equating imaginary parts, we get

$v = y\cos\alpha - (x-1)\sin\alpha + \frac{(x-1)\sin\alpha - y\cos\alpha}{(x-1)^2+y^2} = 0 \Rightarrow [y\cos\alpha - (x-1)\sin\alpha][(x-1)^2 + y^2] = 0$

$\therefore$ Either $y\cos\alpha - (x-1)\sin\alpha = 0 \Rightarrow y = \tan\alpha(x-1)$, which is a straight line passing through $(1,0)$ or $(x-1)^2 + y^2 - 1 = 0$ which is a circle with center $(1,0)$ and unit radius.

Problem 197

197. If $|a_n| < 2$ for $n = 1, 2, 3, ...$ and $1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, show that z does not lie in the interior of the circle $|z| = \frac{1}{3}$.

Solution of Problem 197

Solution: Given, $1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0 \Rightarrow |a_1 z| + |a_2 z^2| + \cdots + |a_n z^b| \geq 1$ and

L.H.S. $< 2|z| + 2|z|^2 + \cdots$ to $\infty [\because |a_n| < 2]$.

Let $|z| < 1$ then $\frac{2|z|}{1-|z|} < 1 \Rightarrow |z| > \frac{1}{3}$

When $|z| > 1$, clearly $|z| > \frac{1}{3}$; hence, z does not lie in the interior of the circle with radius $\frac{1}{3}$.

Problem 198

198. Show that all the roots of the equation $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + \cos\theta_n = 2$, where $\theta_0, \theta_1, \dots, \theta_n \in R$ lie outside the circle $|z| = \frac{1}{2}$.

Solution of Problem 198

Solution: Given, $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + \cos\theta_n = 2$

$\Rightarrow 2 = |z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \cdots + \cos\theta_n|$

$< |z^n \cos\theta_0| + |z^{n-1}\cos\theta_1| + \cdots + |\cos\theta_n|$

$= |z^n||\cos\theta_0| + |z^{n-1}||\cos\theta_1| + \cdots + |\cos\theta_n|$

$\leq |z|^n + |z|^{n-1} + \cdots + 1 < 1 + |z| + |z|^2 + \cdots$ to ∞

$\Rightarrow 2 < \frac{1}{1-|z|} \Rightarrow |z| > \frac{1}{2}$[when $|z| < 1$]

Hence z lies outside the circle $|z| = \frac{1}{2}$.

Thus all roots of the given equation lie outside the circle $|z| = \frac{1}{2}$.

Problem 199

199. z_1, z_2, z_3 are non-zero, non-collinear complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$. Show that z_1, z_2, z_3 lie on a circle passing through origin.

Solution of Problem 199

Solution: Recall that points z_1, z_2, z_3 are concyclic if $\left(\frac{z_2-z_4}{z_1-z_4}\right)\left(\frac{z_1-z_3}{z_2-z_3}\right)$ is real. We assume that z_4 is origin.

Given, $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} = \frac{z_2+z_3}{z_2z_3} \therefore z_1 = \frac{2z_2z_3}{z_1+z_3}$.

Putting the value of z_1 and z_4 in the concyclic condition expression we obtain

$\left(\frac{z_2-z_4}{z_1-z_4}\right)\left(\frac{z_1-z_3}{z_2-z_3}\right) = \frac{1}{2}$.

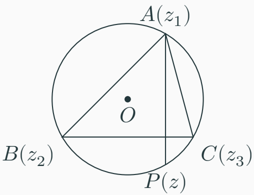
Thus, z_1, z_2, z_3 lie on a circle passing through origin.

Problem 200

200. A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane and the circumcenter of the $\triangle ABC$ lies on the origin. If the altitude of the triangle through vertex A meets the circle again at P, prove that P represents the complex number $\frac{z_2 z_3}{z_1}$.

Solution of Problem 200

Solution: The origin O is the circumcenter of $\triangle ABC$ and AP is perpendiculse to BC. Let $P = z$.



We have $OP = OA = OB = OC \therefore |z| = |z_1| = |z_2| = |z_3| \Rightarrow |z|^2 = |z_1|^2 = |z_2|^2 = |z_3|^2 \Rightarrow z\overline{z} = z_1\overline{z_1} = z\overline{z_2} = z\overline{z_3}$.

Since AP is perpendicular to BC, therefore

$\arg\left(\frac{z_1-z}{z_2-z_3}\right) = \frac{\pi}{2}$ or $\frac{-\pi}{2} \Rightarrow \frac{z_1-z}{z_2-z_3}$ is purely imaginary.

$\Rightarrow \overline{\left(\frac{z_1-z}{z_2-z_3}\right)} = -\frac{z_1-z}{z_2-z_3}$

Solving the above equation gives $z = \frac{z_2 z_3}{z_1}$.