Shiv Shankar Dayal

An Angle in Trigonometry
A problem-oriented approach

A Variable in Algebra
Version 1.0 [March 12, 2023]

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Dedicated to my family
and Free Software Community

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Preface

This is a book on trigonometry, which, covers basics of trigonometry till high school level. It covers the most essential topics to take up a bachelor's course where knowledge of trigonometry is required. There is no specific purpose for writing this book. This is a book for self study and is not recommended for courses in schools and universities. I will try to cover as much as I can and will keep adding new material over a long period. I have no interest in writing a book in a fixed way which serves a university or college course as I have always loved freedom. Life, freedom and honor in that order are important. That was one of the reasons why I never did a masters course. Enough of personal rant. Let us get on with the actual work.

Trigonomtery is probably one of the most fundamental subjects in Mathematics as further study of subjects like coordinate geometry, 3D and 2D geometry, engineerring and rest all depend on it. It is very important to understand trigonometry for the readers if they want to advance further in mathematics.

Why I Wrote This Book?

I wrote this book for myself! I did not write it for anybody else. My knowledge, which I have acquired by reading books written by many great mathematicians and authors and interactions with many intelligent people, is what has been put in the book. I have just tried to add my flavor to it. Think of it as notes for me. Just that I like to organize my notes so it has taken form of a book and nothing more. If you benefit from this then that is a pure coincidence and not intentional at all.

How to Read This Book?

No I will not simply tell that you must solve the problems. My advice would be more detailed. Every chapter will have theory. Read that first. Make sure you understand that. Of course, you have to meet the prerequisites for the book. Then, go on and try to solve the problems. In this book, there are no pure problems. Almost all have answers except those which are of similar kind and repetitive in nature for the sake of practice. If you can solve the problem then all good else look at the answer and try to understand that. Then, few days later take on the problem again. If you fail to understand the answer you can always email me with your work and I will try to answer to the best of my ability. However, if you have a local expert seek his/her advice first.

Note that mathematics is not only about solving problems. If you understand the theory well, then you will be able to solve problems easily. However, problems do help enforce with the enforcement of theory in your mind.

I am a big fan of old MIR publisher's problem books, so I emphasize less on theory and more on problems. I hope that you find this style much more fun as a lot of theory is boring. Mathematics is about problem solving as that is the only way to enforce theory and find innovtive techniques for problem solving.

Some of the problems in certain chapters rely on other chapters which you should look ahead or you can skip those problems and come back to it later. Since this books is meant for self study answers of most of the problems have been given which you can make use of. However, do not use for just copying but rather to develop understanding.

Who Should Read This Book?

Since this book is written for self study anyone with interest in trigonometry can read it. That does not mean that school or college students cannot read it. You need to be selective as to what you need for your particular requirements. This is mostly high school course with a little bit of lower classes' course thrown in with a bit of detail here and there.

Prerequisite

You should have knowledge till grade 8th course. Attempt has been made to keep it simple and give as much as background to the topic which is reasonable and required. However, not everything will be covered below grade 10.

Goals for Readers

The goal of for reading this book is becoming proficient in solving simple and basic problems of trigonometry. Another goal would be to be able to study other subjects which require this knowledge like trigonometry or calculus or physics or chemistry or other subjects. If you can solve 95% problems after 2 years of reading this book then you have achieved this goal.

All of us possess a certain level of intelligence. At average any person can read this book. But what is most important is you have to have interest in the subject. Your interest gets multiplied with your intelligence and thus you will be more capable than you think you can be. One more point is focus and effort. It is not something new which I am telling but I am saying it again just to emphasize the point. Trust me if you are reading this book for just scoring a nice grade in your course then I have failed in my purpose of explaining my ideas.

Also, if you find this book useful feel free to share it with others without hesitation as it is free as in freedom. There are no conditions to share it.

Acknowledgements

I am in great debt of my family and free software community because both of these groups have been integral part of my life. Family has prvided direct support while free software community has provided the freedom and freed me from the slavery which comes as a package with commercial software. I am especially grateful to my wife, son and parents because it is their time which I have borrowed to put in the book. To pay my thanks from free software community I will take one name

and that is Richard Stallman who started all this and is still fighting this never-ending war. When I was doing the Algebra book then I realized how difficult it is to put Math on web in HTML format and why Donald Knuth wrote TEX. Also, TEX was one of the first softwares to be released as a free software.

Now as this book is being written using ConTEXt so obviously Hans Hagen and all the people involved with it have my thanks along with Donald Knuth. I use Emacs with Auctex and hope that someday I will use it in a much more productive way someday.

I have used Asymptote and tikz for drawing all the diagrams. Both are wonderful packages and work very nicely. Asymptote in particular is very nice for 3d-drawings and linear equation solving. I have yet to learn Metafun which comes with ConTEXt.

I would like to thank my parents, wife and son for taking out their fair share of time and the support which they have extended to me during my bad times. After that I would like to pay my most sincere gratitude to my teachers particularly H. N. Singh, Yogendra Yadav, Satyanand Satyarthi, Kumar Shailesh and Prof. T. K. Basu. Now is the turn of people from software community. I must thank the entire free software community for all the resources they have developed to make computing better. However, few names I know and here they go. Richard Stallman is the first, Donald Knuth, Edger Dijkstra, John von Neumann after that as their lives have strong influence in how I think and base my life on. Cover graphics has been done by Koustav Halder so much thanks to him. I am not a native English speaker and this book has just gone through one pair of eyes therefore chances are high that it will have lots of errors(particularly with commas and spelling mistakes). At the same time it may contain lots of technical errors. Please feel free to drop me an email at shivshankar.dayal@gmail.com where I will try to respond to each mail as much as possible. Please use your real names in email not something like coolguy. If you have more problems which you want to add it to the book please send those by email or create a PR on github. The github url is https://github.com/shivshankardayal/Trigonometry-Context.

Shiv Shankar Dayal
Nalanda, 2023

I

Theory and Problems

Chapter 1
Measurement of Angles

The word trigonometry comes from means measurement of triangles. The word originally comes from Greek language. measurement. The objective of studying plane trigonometry is to develop a method of solving plane triangles. However, as time changes everything it has changed the scope of trigonometry to include polygons and circles as well. A lot of concepts in this book will come from your geometry classes in lower classes. It is a good idea to review the concepts which you have studied till now without which you are going to struggle while studying trigonometry in this book.

1.1 Angles in Geometry

If we consider a line extending to infinity in both directions, and a point OO which divides this line in two parts one on each side of the point then each part is called a ray or half-line. Thus O divides the line into two rays OA and OA'.

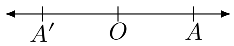

Figure 1.1

The point O is called vertex or origin for these days. An angle is a figure formed by two rays or half lies meeting at a common vertex. These half lines are called *sides of the angle*.

An angle is denoted by the symbol $\angle$ followed by three capital letters of which the middle one represents the vertex and remaining two points point to two sides. Otherwise the angle is simply written as one letter representing the vertex of the angle.

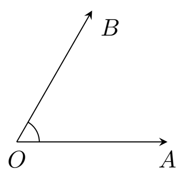

Figure 1.2 An angle

The angle in above image is written as $\angle AOB$ or $\angle BOA$ or $\angle O$.

Each angle can be measured and there are different units for the measurement. In Geometry, an angle always lie between 0° and 360° and negative angles are meaningless. Measure of an angle is the smallest amount of rotation from the direction of one ray of the angle to the direction of the other.

1.2 Angles in Trigonometry

Angles are more generalized in Trigonometry. They can have positive or negative values. As was the case in gerometry, similarly angles are measured in Trigonometry. The starting and ending positions of revolving rays are called initial side and terminal side respectively. The revolving half line is called the generating line or the radius vector. For example, if OA and OB are the initial and final position of the radius vector then angle formed will be $\angle AOB$.

1.3 Angles Exceeding $360°$

In Geometry, angles are limited to $0°$ to $360°$. However, when multiple revolutions are involved angles are more than $360°$. For example, the revolving line starts from the initial position and makes n complete revolutions in anticlockwise direction and also further angle α in the same direction. We then have a certain angle β_n given by $\beta_n = x \times 360° + \alpha$, where $0° < \alpha < 360°$ and n is zero or positive integer. Thus, there are infinite possible angles.

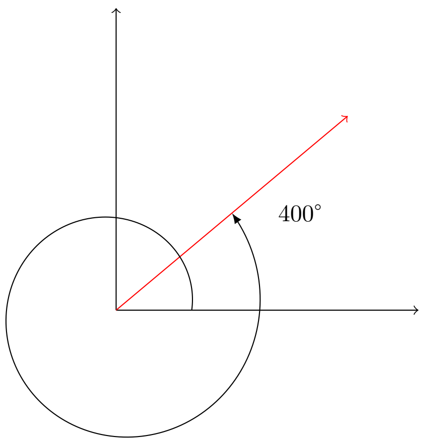

Figure 1.3 An angle

Angles formed by anticlockwise rotation of the radius vector are taken as positive and angles formed by clockwise rotation of the radius vector are taken as negative.

1.4 Quadrants

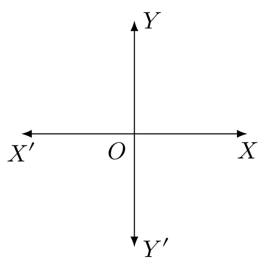

Figure 1.4 Quadrants

Let XOX' and YOY' be two mutually perpendicular lines in a plane and OX be the initial half line. The lines divide the whole reason in quadrants. XOY, YOX', $X'OY$' and $Y'OX$ are respectively called 1st, 2nd, 3rd and 4th quadrants. According to terminal side lying in 1st, 2nd, 3rd and 4th quadrants the angles are said to be in 1st, 2nd, 3rd and 4th quadrants respectively. A *quandrant angle* is an angle formed if terminal side coincides with one of the axes.

For any angle $\angle$ which is not a quadrant angle and when number of revolutions is zero and radius vector rotates in anticlockwise directions:

- $0^\circ < \alpha < 90^\circ$ if α lies in first quadrant
- $90^\circ < \alpha < 180^\circ$ if α lies in second quadrant
- $180^\circ < \alpha < 270^\circ$ if α lies in third quadrant
- $270^\circ < \alpha < 360^\circ$ if α lies in fourth quadrant
- when terminal side lies on OY, angle formed $= 90^\circ$
- when terminal side lies on OX', angle formed $= 180^\circ$
- when terminal side lies on OY', angle formed $= 270^\circ$
- when terminal side lies on OX, angle formed $= 360^\circ$

1.5 Units of Measurement

In Geometry, angles are usually measured in terms of right angles, however, that is an inconvenient system for smaller angles. So we introduce different systems of measurements. There are three system of units for this:

1. Sexagecimal or British system: In British system, a right angle is divided into 90 equal parts called degrees. Each degree is then divided into 60 equal parts called minutes and each minute is further is divided into 60 parts called seconds.

   A degree, a minute and a second are denoted by $1^\circ, 1"$, and 1 respectively.

2. Centesimal or French System: In French system, a right angle is divided into 100 equal parts called grades. Each grade is then divided into 100 equal parts called minutes and each minute is further is divided into 100 parts called seconds.

   A degree, a grade and a second are denoted by $1^g, 1"$, and 1 respectively.

3. Radian or Circular Measure: An arc equal to radius of a circle when subtends an anngle on the center then that angle is 1 radian and is denoted by 1^c. The angle made by half of perimeter is π radians. Also, from Geometry we know that angle subtended is the ratio between length of cord and radius. This ratio is in radians. Since both length or chord and radius have same unit radian is a constant.

1.5.1 Relationship between Systems of Measurements

If measure of an angle if D degrees, G grades and C radians then upon elementary manipulation we find that $\frac{D}{180} = \frac{G}{200} = \frac{C}{\pi}$.

1.5.2 Meaning of π

The ratio of circumference and diameter of a circle is always constant and this constant is denoted by gree letter π.

π is an irrational number. In general, we use the value of $\frac{22}{7}$ but $\frac{355}{113}$ is more accurate though not exact. If r be the radius of a circle and c be the circumference then $\frac{c}{2r} = \pi$ leading circumference to be $c = 2\pi r$.

1.6 Problems

1. Reduce $63^\circ 14' 51''$ to centisimal measure.
2. Reduce $\backslash 45^\circ 20' 10''$ to centisimal and radian measure.
3. Reduce $94^g 23' 27''$ to Sexagecimal measure.
4. Reduce 1.2 radians in Sexageciaml measure.

Express in terms of right angle; the angles

5. 60°
6. $75^\circ 15'$
7. $63^\circ 17' 25''$
8. $130^\circ 30'$
9. $210^\circ 30' 30''$
10. $370^\circ 20' 48''$

Express in grades, minutes and degrees

11. 30°
12. 81°
13. $138^\circ 30'$
14. $35^\circ 47' 15''$
15. $235^\circ 12' 36'' s$
16. $475^\circ 13' 48''$

Express in terms of right angles and also in degrees, minutes and seconds; the angles

17. 120^g
18. $45^g 35' 24''$

19. $39^g 45' 36''$

20. $255^g 8' 9''$

21. $759^g 0' 5''$

22. Reduce $55^\circ 12' 36''$ to centisimal measure.

23. Reduce $18^\circ 33' 45''$ to circular measure.

24. Reduce $196^g 35' 24''$ to sexagecimal measure.

25. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hand of a watch.

26. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both angles in degrees.

27. Prove that the number of Sexagecimal minutes in any angle is to the number of Centisimal minutes in the same angle as $27 : 50$.

28. Divide $44^\circ 8'$ into two parts such that the number of Sexagecimal seconds in one part may be equal to number of Centisimal seconds in the other part.

29. The angles of a triangle are in the ratio of $3 : 4 : 5$, find the smallest angle in degrees and greatest angle in radians.

30. Find the angle between the hour hand and the minute hand in circular measure at half past four.

31. If p, q and r denote the grade measure, degree measure and the radian measure of the same angle, prove that

    i. $\frac{p}{10} = \frac{q}{9} = \frac{20r}{\pi}$

    ii. $p - q = \frac{20r}{\pi}$

32. Two angles of a triangle are $72^\circ 53' 51''$ and $41^\circ 22' 50''$ respectively. Find the third angle in radians.

33. The angles of triangle are in A.P. and the number of radians in the greatest angle is to the number of degrees in the least one as $\pi : 60$; find the angles in degrees.

34. The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest is $40 : \pi$; find the angles in degrees.

35. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least is $800 : \pi$; and the sum of angles is 126°. Find the angles in grades.

36. Find the angle between the hour-hand and minute-hand in circular measure at 4 o'clock.

37. Express in sexagecimal system the angle between the minute-hand and hour-hand of a clock at quarter to twelve.

38. The diamter of a wheel is 28 cm; through what distance does its center move during one rotation of wheel along the ground?

39. What must be the radius of a circular running path, round which an athlete must run 5 time in order to describe 1760 meters?

40. The wheel of a railway carriage is 90 cm in diameter and it makes 3 revolutions per second; how fast is the train going?

41. A mill sail whose length is 540 cm makes 10 revolutions per minute. What distance does its end travel in one hour?

42. Assuming that the earth describes in one year a circle, or $149,700,000$ km. radius, whose center is the sun, how many miles does earth travel in a year?

43. The radius of a carriage wheel is 50 cm, and in $\frac{1}{9}$ th of a second it turns through 80° about its center, which is fixed; how many km. does a point on the rim of the wheel travel in one hour?

44. Express in terms of three systems of angular measurements the magnitude of an angle of a regular decagon.

45. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, while the third is $\frac{\pi x}{75}$ radians; express them all in degrees.

46. The circular measure of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. What is the number of degrees of the third angle?

47. The angles of a triangle are in A.P. The number of radians in the least angle is to the number of degree in the mean angle is 1 : 120. Find the angles in radians.

48. Find the magnitude, in radians and degrees, of the interior angle of 1. a regular pentagon 2. a regular heptagon 3. a regualr octagon 4. a regular duodecagon 5. a polygon with 17 sides

49. The angle in one regular polygon is to that in another is 3 : 2, also the number of sides in the first is twice that in the second. How many sides are there in the polygons?

50. The number of sides in two regular polygons are as 5 : 4, and the difference between their angles is 9°; find the number of sides in the polygons.

51. Find two regular polygons such that the number of their sides may be 3 to 4 and the number of degrees of an angle of the first to the number of grades of the second as 4 to 5.

52. The angles of a qadrilateral are in A.P. and the greatest is double the least; express the least angle in radians.

53. Find in radians, degrees, and grades the angle between hour-hand and minute-hand of a clock at 1. half-past three 2. twenty minutes to six 3. a quarter past eleven.

54. Find the times 1. between fours and five o'clock when the angle between the minute hand and the hour-hand is 78°, 2. between seven and eight o'clock when the angle is 54°

55. The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.

56. The angles of quadrilateral are in A.P. and the number of grades in the least angle is to the number of radians in the greatest is $100 : \pi$. Find the angles in degrees.

57. The anlges of a polygons are in A.P. The least angle is $\frac{5\pi}{12}$ common difference is 10°, find the number of sides in the polygon.

58. Find the angle subtended at the center of a circle of radius 3 cm. by an arc of length 1 cm.

59. In a circle of radius 5 cm., what is the length of the arc which subtends an angle of $33^\circ 15'$ at the center.

60. Assuming the average distance of sun from the earth to be $149, 700, 000$ km., and the angle subtended by the sun at the eye of a person on the earth is $32'$, find the sun's diameter.

61. Assuming that a person of normal sight can read print at such a distance that the letter subtends and angle of $5'$ at his eye, find what is the height of the letters he can read at a distance of 1. 12 meters 2. 1320 meters.

62. Find the number of degrees subtended at the center of a circle by an arc whose length is 0.357 times the radius.

63. Express in radians and degrees the angle subtended at the center of a circle by an arc whose length is 15 cm., the radius of the circle being 25 cm.

64. The value of the divisions on the outer rim of a graduated cicle is $5'$ and the distance between successive graduations is .1 cm. Find the radius of the circle.

65. The diamter of a graduated circle is 72 cm., and the gradiuations on the rim are $5'$ apart; find the distance of one graduation to to another.

66. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by $1^\circ 10'$ may be 0.5 cm.

67. Taking the radius of earth to as 6400 km., find the difference in latitude of two places, one of which is 100 km. north of another.

68. Assuming the earth to be a sphere and the difference between two parallels of latitude, which subtends an angle of 1° at the eath's center, to be $69\frac{1}{2}$ km., find the radius of the earth.

69. What is the ratio of radii of the circles at the center of which two arcs of same length subtend angles of 60° and 75°?

70. If an arc, of length 10 cm., on a circle of 8 cm. diameter subtend at the center of circle an angle of $143^\circ 14' 22''$, find the value of π to 4 places of decimals.

71. If the circumference of a circle be divided into five parts which are in A.P., and if the greatest part be six times the least find in radians the the magnitude of the angles the parts subtend at the center of the circle.

72. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius; express the angle of the sector in degrees.

73. At what distance a man, whose height is 2 m., subtend an angle of $10'$.

74. Find the length which at a distance of 5280 m., will subtend an angle of $1'$ at the eye.

75. Assuming the distance of the earth from the moon to be 38400 km., and the angle subtended by the moon at the eye of a person on earth to be $31'$, find the diameter of the moon.

76. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second; how fast is the train going?

77. Assuming that moon subtends an angle of $30'$ at the eye of an observer, find how far from the eye a coin of one inch diameter must be held so as just to hide the moon.

78. A wheel make 30 revolutions per minute. Find the circular measure of the angle described by spoke in half a second.

79. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends an angle of 56° at the center. Find the diamter of the circle.

Chapter 2

Trigonometric Ratios

From Geometry, we know that an acute angle is an angle whose measure is between 0° and 90°. Consider the following figure:

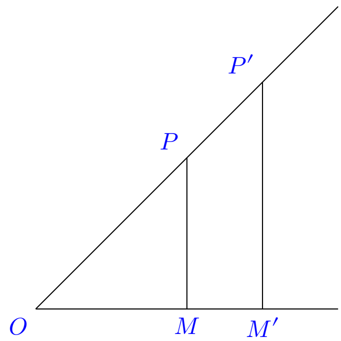

Figure 2.1 Trigonometric ratios

This picture contains two similar triangles $\triangle OMP$ and $\triangle OM'P'$. We are interested in $\angle MOP$ or $\angle M'OP'0$. In the $\triangle MOP$ and $\triangle M'OP'$, OP, OP' are called the hypotenuses i.e. sides opposite to the right angle, $PM, P'M'$ are called perpendiculars i.e. sides opposite to the angle of interest and OM, OM' are called bases i.e. the third angle.

Hypotenuses are usually denoted by h, perpendiculars by p and bases by b. Let $OM = b, OM' = b', PM = p, P'M' = p', OP = h, OP' = h'$. Since the two triangles are similar $\therefore \frac{p}{p'} = \frac{b}{b'} = \frac{h}{h'}$. Thus rhe ratio of any two sides is dependent purely on $\angle O$ or $\angle MOP$ or $\angle M'OP'$.

Since there are three sides, we can choose 2 in $^3C_2$ i.e. 3 ways and for each combination there will be two permutations where a side can be in either numerator or denominator. From this we can conclude that there will be six ratios(these are called trigonometric ratios), These six trigonometric ratios or functions are given below:

$\dfrac{MP}{OP}$ or $\dfrac{p}{h}$ is called the **Sine** of the $\angle MOP$.

$\dfrac{OM}{OP}$ or $\dfrac{b}{h}$ is called the **Cosine** of the $\angle MOP$.