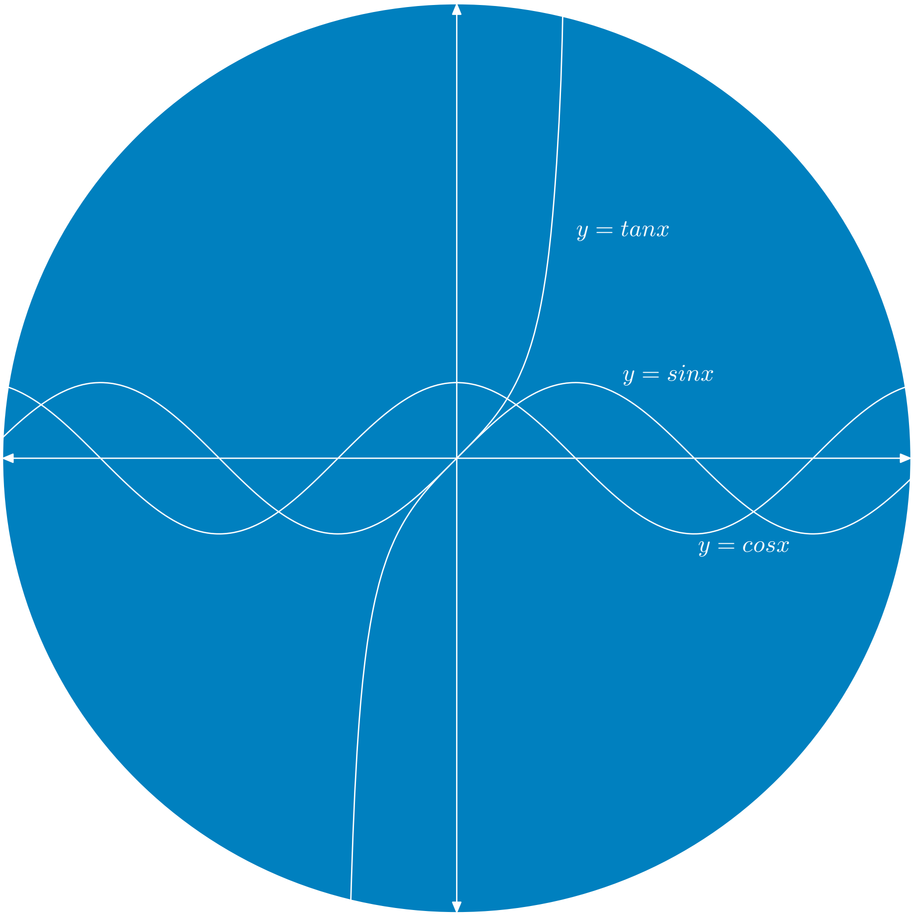
Shiv Shankar Dayal



An Angle in Trigonometry

A problem-oriented approach

An Angle in Trigonometry
Early Draft [May 20, 2025]

Dedicated to my wife, Binita

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Preface

This is a book on trigonometry, which, covers basics of trigonometry till high school level. It covers the most essential topics to take up a bachelor's course where knowledge of trigonometry is required. I will try to cover as much as I can and will keep adding new material over a long period.

Trigonomtery is probably one of the most fundamental subjects in Mathematics as further study of subjects like coordinate geometry, 3D and 2D geometry, engineerring and rest all depend on it. It is very important to understand trigonometry for the readers if they want to advance further in mathematics.

How to Read This Book?

Every chapter will have theory. Read that first. Make sure you understand that. Of course, you have to meet the prerequisites for the book. Then, go on and try to solve the problems. In this book, there are no pure problems. Almost all have answers except those which are of similar kind and repetitive in nature for the sake of practice. If you can solve the problem then all good else look at the answer and try to understand that. Then, few days later take on the problem again. If you fail to understand the answer you can always email me with your work and I will try to answer to the best of my ability. However, if you have a local expert seek his/her advice first.

Note that mathematics is not only about solving problems. If you understand the theory well, then you will be able to solve problems easily. However, problems do help enforce with the enforcement of theory in your mind.

I am a big fan of old MIR publisher's problem books, so I emphasize less on theory and more on problems. I hope that you find this style much more fun as a lot of theory is boring. Mathematics is about problem solving as that is the only way to enforce theory and find innovtive techniques for problem solving.

Some of the problems in certain chapters rely on other chapters which you should look ahead or you can skip those problems and come back to it later. Since this books is meant for self study answers of most of the problems have been given which you can make use of. However, do not use for just copying but rather to develop understanding.

Who Should Read This Book?

Since this book is written for self study anyone with interest in trigonometry can read it. That does not mean that school or college students cannot read it. You need to be selective as to what you need for your particular requirements. This is mostly high school course with a little bit of lower classes' course thrown in with a bit of detail here and there.

Prerequisite

You should have knowledge till grade 8th course. Attempt has been made to keep it simple and give as much as background to the topic which is reasonable and required. However, not everything will be covered below grade 10.

Goals for Readers

The goal of for reading this book is becoming proficient in solving simple and basic problems of trigonometry. Another goal would be to be able to study other subjects which require this knowledge like trigonometry or calculus or physics or chemistry or other subjects. If you can solve 95% problems after 2 years of reading this book then you have achieved this goal.

All of us possess a certain level of intelligence. At average any person can read this book. But what is most important is you have to have interest in the subject. Your interest gets multiplied with your intelligence and thus you will be more capable than you think you can be. One more point is focus and effort. It is not something new which I am telling but I am saying it again just to emphasize the point. Trust me if you are reading this book for just scoring a nice grade in your course then I have failed in my purpose of explaining my ideas.

Also, if you find this book useful feel free to share it with others without hesitation as it is free as in freedom. There are no conditions to share it.

Confession

I feel like an absolute thief while writing this book for nothing given in this book is mine. All of it belongs to others who did the original work and I have just copied shamelessly. I have nothing new to put in the book. This book is just the result of the pain I feel when I see young children wasting their life for they are poor. And therefore, this book is licensed under GNU FDL. Even if I manage to create few new problems it is still based on knowledge of other pioneers of the subject but perhaps that is how we are supposed to progress bit-by-bit.

Acknowledgements

I am in great debt of my family and free software community because both of these groups have been integral part of my life. Family has prvided direct support while free software community has provided the freedom and freed me from the slavery which comes as a package with commercial software. I am especially grateful to my wife, son and parents because it is their time which I have borrowed to put in the book. To pay my thanks from free software community I will take one name and that is Richard Stallman who started all this and is still fighting this never-ending war. When I was doing the Algebra book then I realized how difficult it is to put Math on web in HTML format and why Donald Knuth wrote TEX. Also, TEX was one of the first softwares to be released as a free software.

Now as this book is being written using ConTEXt so obviously Hans Hagen and all the people involved with it have my thanks along with Donald Knuth. I use Emacs with Auctex and hope that someday I will use it in a much more productive way someday.

I have used Asymptote and tikz for drawing all the diagrams. Both are wonderful packages and work very nicely. Asymptote in particular is very nice for 3d-drawings and linear equation solving. I have yet to learn Metafun which comes with ConTEXt.

I would like to thank my parents, wife, son and daughter for taking out their fair share of time and the support which they have extended to me during my bad times. After that I would like to pay my most sincere gratitude to my teachers particularly H. N. Singh, Yogendra Yadav, Satyanand Satyarthi, Kumar Shailesh and Prof. T. K. Basu. Now is the turn of people from software community. I must thank the entire free software community for all the resources they have developed to make computing better. However, few names I know and here they go. Richard Stallman is the first, Donald Knuth, Edger Dijkstra, John von Neumann after that as their lives have strong influence in how I think and base my life on. Cover graphics has been done by Koustav Halder so much thanks to him. I am not a native English speaker and this book has just gone through one pair of eyes therefore chances are high that it will have lots of errors(particularly with commas and spelling mistakes). At the same time it may contain lots of technical errors. Please feel free to drop me an email at shivshankar.dayal@gmail.com where I will try to respond to each mail as much as possible. Please use your real names in email not something like coolguy. If you have more problems which you want to add it to the book please send those by email or create a PR on github. The github url is https://github.com/shivshankardayal/Trigonometry-Context.

Shiv Shankar Dayal
Nalanda, 2023

I
Theory and Problems

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Chapter 1
Measurement of Angles

The word trigonometry comes from means measurement of triangles. The word originally comes from Greek language. measurement. The objective of studying plane trigonometry is to develop a method of solving plane triangles. However, as time changes everything it has changed the scope of trigonometry to include polygons and circles as well. A lot of concepts in this book will come from your geometry classes in lower classes. It is a good idea to review the concepts which you have studied till now without which you are going to struggle while studying trigonometry in this book.

1.1 Angles in Geometry

If we consider a line extending to infinity in both directions, and a point OO which divides this line in two parts one on each side of the point then each part is called a ray or half-line. Thus O divides the line into two rays OA and OA'.

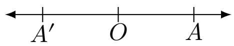


Figure 1.1

The point O is called vertex or origin for these days. An angle is a figure formed by two rays or half lies meeting at a common vertex. These half lines are called *sides of the angle.*

An angle is denoted by the symbol $\angle$ followed by three capital letters of which the middle one represents the vertex and remaining two points point to two sides. Otherwise the angle is simply written as one letter representing the vertex of the angle.

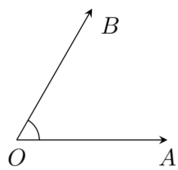


Figure 1.2 An angle

The angle in above image is written as $\angle AOB$ or $\angle BOA$ or $\angle O$.

Each angle can be measured and there are different units for the measurement. In Geometry, an angle always lie between 0° and 360° and negative angles are meaningless. Measure of an angle is the smallest amount of rotation from the direction of one ray of the angle to the direction of the other.

1.2 Angles in Trigonometry

Angles are more generalized in Trigonometry. They can have positive or negative values. As was the case in gerometry, similarly angles are measured in Trigonometry. The starting and ending positions of revolving rays are called initial side and terminal side respectively. The revolving half line is called the generating line or the radius vector. For example, if OA and OB are the initial and final position of the radius vector then angle formed will be $\angle AOB$.

1.3 Angles Exceeding 360°

In Geometry, angles are limited to 0° to 360°. However, when multiple revolutions are involved angles are more than 360°. For example, the revolving line starts from the initial position and makes n complete revolutions in anticlockwise direction and also further angle α in the same direction. We then have a certain angle β_n given by $\beta_n = x \times 360^\circ + \alpha$, where $0^\circ < \alpha < 360^\circ$ and n is zero or positive integer. Thus, there are infinite possible angles.

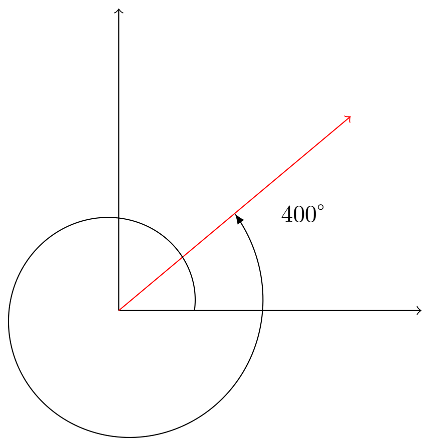


Figure 1.3 An angle

Angles formed by anticlockwise rotation of the radius vector are taken as positive and angles formed by clockwise rotation of the radius vector are taken as negative.

1.4 Quadrants

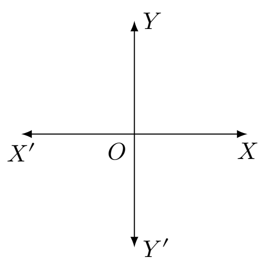


Figure 1.4 Quadrants

Let XOX' and YOY' be two mutually perpendicular lines in a plane and OX be the initial half line. The lines divide the whole reason in quadrants. $XOY, YOX', X'OY$' and $Y'OX$ are respectively called 1st, 2nd, 3rd and 4th quadrants. According to terminal side lying in 1st, 2nd, 3rd and 4th quadrants the angles are said to be in 1st, 2nd, 3rd and 4th quadrants respectively. A *quandrant angle* is an angle formed if terminal side coincides with one of the axes.

For any angle $\angle$ which is not a quadrant angle and when number of revolutions is zero and radius vector rotates in anticlockwise directions:

- $0^\circ < \alpha < 90^\circ$ if α lies in first quadrant
- $90^\circ < \alpha < 180^\circ$ if α lies in second quadrant
- $180^\circ < \alpha < 270^\circ$ if α lies in third quadrant
- $270^\circ < \alpha < 360^\circ$ if α lies in fourth quadrant
- when terminal side lies on OY, angle formed $= 90^\circ$
- when terminal side lies on OX', angle formed $= 180^\circ$
- when terminal side lies on OY', angle formed $= 270^\circ$
- when terminal side lies on OX, angle formed $= 360^\circ$

1.5 Units of Measurement

In Geometry, angles are usually measured in terms of right angles, however, that is an inconvenient system for smaller angles. So we introduce different systems of measurements. There are three system of units for this:

1. Sexagecimal or British system: In British system, a right angle is divided into 90 equal parts called degrees. Each degree is then divided into 60 equal parts called minutes and each minute is further is divided into 60 parts called seconds.

   A degree, a minute and a second are denoted by $1^\circ, 1"$, and 1 respectively.

2. Centesimal or French System: In French system, a right angle is divided into 100 equal parts called grades. Each grade is then divided into 100 equal parts called minutes and each minute is further is divided into 100 parts called seconds.

   A degree, a grade and a second are denoted by $1^g, 1"$, and 1 respectively.

3. Radian or Circular Measure: An arc equal to radius of a circle when subtends an anngle on the center then that angle is 1 radian and is denoted by 1^c. The angle made by half of perimeter is π radians. Also, from Geometry we know that angle subtended is the ratio between length of cord and radius. This ratio is in radians. Since both length or chord and radius have same unit radian is a constant.

1.5.1 Relationship between Systems of Measurements

If measure of an angle if D degrees, G grades and C radians then upon elementary manipulation we find that $\frac{D}{180} = \frac{G}{200} = \frac{C}{\pi}$.

1.5.2 Meaning of π

The ratio of circumference and diameter of a circle is always constant and this constant is denoted by gree letter π.

π is an irrational number. In general, we use the value of $\frac{22}{7}$ but $\frac{355}{113}$ is more accurate though not exact. If r be the radius of a circle and c be the circumference then $\frac{c}{2r} = \pi$ leading circumference to be $c = 2\pi r$.

1.6 Problems

1. Reduce $63°14'51''$ to centisimal measure.
2. Reduce $45°20'10''$ to centisimal and radian measure.
3. Reduce $94^g23'27''$ to Sexagecimal measure.
4. Reduce 1.2 radians in Sexageciaml measure.

Express in terms of right angle; the angles

5. $60°$
6. $75°15'$
7. $63°17'25''$
8. $130°30'$
9. $210°30'30''$
10. $370°20'48''$

Express in grades, minutes and degrees

11. $30°$
12. $81°$
13. $138°30'$
14. $35°47'15''$
15. $235°12'36''s$
16. $475°13'48''$

Express in terms of right angles and also in degrees, minutes and seconds; the angles

17. 120^g
18. $45^g35'24''$
19. $39^g45'36''$
20. $255^g8'9''$
21. $759^g0'5''$
22. Reduce $55°12'36''$ to centisimal measure.
23. Reduce $18°33'45''$ to circular measure.
24. Reduce $196^g35'24''$ to sexagecimal measure.
25. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hand of a watch.

26. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both angles in degrees.

27. Prove that the number of Sexagecimal minutes in any angle is to the number of Centisimal minutes in the same angle as 27 : 50.

28. Divide $44°8'$ into two parts such that the number of Sexagecimal seconds in one part may be equal to number of Centisimal seconds in the other part.

29. The angles of a triangle are in the ratio of 3 : 4 : 5, find the smallest angle in degrees and greatest angle in radians.

30. Find the angle between the hour hand and the minute hand in circular measure at half past four.

31. If p, q and r denote the grade measure, degree measure and the radian measure of the same angle, prove that

    i. $\frac{p}{10} = \frac{q}{9} = \frac{20r}{\pi}$

    ii. $p - q = \frac{20r}{\pi}$

32. Two angles of a triangle are $72°53'51''$ and $41°22'50''$ respectively. Find the third angle in radians.

33. The angles of triangle are in A.P. and the number of radians in the greatest angle is to the number of degrees in the least one as $\pi : 60$; find the angles in degrees.

34. The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest is $40 : \pi$; find the angles in degrees.

35. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least is $800 : \pi$; and the sum of angles is $126°$. Find the angles in grades.

36. Find the angle between the hour-hand and minute-hand in circular measure at 4 o'clock.

37. Express in sexagecimal system the angle between the minute-hand and hour-hand of a clock at quarter to twelve.

38. The diamter of a wheel is 28 cm; through what distance does its center move during one rotation of wheel along the ground?

39. What must be the radius of a circular running path, round which an athlete must run 5 time in order to describe 1760 meters?

40. The wheel of a railway carriage is 90 cm in diameter and it makes 3 revolutions per second; how fast is the train going?

41. A mill sail whose length is 540 cm makes 10 revolutions per minute. What distance does its end travel in one hour?

42. Assuming that the earth describes in one year a circle, or $149,700,000$ km. radius, whose center is the sun, how many miles does earth travel in a year?

43. The radius of a carriage wheel is 50 cm, and in $\frac{1}{9}$ th of a second it turns through 80° about its center, which is fixed; how many km. does a point on the rim of the wheel travel in one hour?

44. Express in terms of three systems of angular measurements the magnitude of an angle of a regular decagon.

45. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, while the third is $\frac{\pi x}{75}$ radians; express them all in degrees.

46. The circular measure of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. What is the number of degrees of the third angle?

47. The angles of a triangle are in A.P. The number of radians in the least angle is to the number of degree in the mean angle is 1 : 120. Find the angles in radians.

48. Find the magnitude, in radians and degrees, of the interior angle of 1. a regular pentagon 2. a regular heptagon 3. a regualr octagon 4. a regular duodecagon 5. a polygon with 17 sides

49. The angle in one regular polygon is to that in another is 3 : 2, also the number of sides in the first is twice that in the second. How many sides are there in the polygons?

50. The number of sides in two regular polygons are as 5 : 4, and the difference between their angles is 9°; find the number of sides in the polygons.

51. Find two regular polygons such that the number of their sides may be 3 to 4 and the number of degrees of an angle of the first to the number of grades of the second as 4 to 5.

52. The angles of a qadrilateral are in A.P. and the greatest is double the least; express the least angle in radians.

53. Find in radians, degrees, and grades the angle between hour-hand and minute-hand of a clock at 1. half-past three 2. twenty minutes to six 3. a quarter past eleven.

54. Find the times 1. between fours and five o'clock when the angle between the minute hand and the hour-hand is 78°, 2. between seven and eight o'clock when the angle is 54°

55. The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.

56. The angles of quadrilateral are in A.P. and the number of grades in the least angle is to the number of radians in the greatest is $100 : \pi$. Find the angles in degrees.

57. The anlges of a polygons are in A.P. The least angle is $\frac{5\pi}{12}$ common difference is 10°, find the number of sides in the polygon.

58. Find the angle subtended at the center of a circle of radius 3 cm. by an arc of length 1 cm.

59. In a circle of radius 5 cm., what is the length of the arc which subtends an angle of $33^\circ 15'$ at the center.

60. Assuming the average distance of sun from the earth to be $149,700,000$ km., and the angle subtended by the sun at the eye of a person on the earth is $32'$, find the sun's diameter.

61. Assuming that a person of normal sight can read print at such a distance that the letter subtends and angle of $5'$ at his eye, find what is the height of the letters he can read at a distance of 1. 12 meters 2. 1320 meters.

62. Find the number of degrees subtended at the center of a circle by an arc whose length is 0.357 times the radius.

63. Express in radians and degrees the angle subtended at the center of a circle by an arc whose length is 15 cm., the radius of the circle being 25 cm.

64. The value of the divisions on the outer rim of a graduated cicle is $5'$ and the distance between successive graduations is .1 cm. Find the radius of the circle.

65. The diamter of a graduated circle is 72 cm., and the gradiuations on the rim are $5'$ apart; find the distance of one graduation to to another.

66. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by $1°10'$ may be 0.5 cm.

67. Taking the radius of earth to as 6400 km., find the difference in latitude of two places, one of which is 100 km. north of another.

68. Assuming the earth to be a sphere and the difference between two parallels of latitude, which subtends an angle of $1°$ at the eath's center, to be $69\frac{1}{2}$ km., find the radius of the earth.

69. What is the ratio of radii of the circles at the center of which two arcs of same length subtend angles of $60°$ and $75°$?

70. If an arc, of length 10 cm., on a circle of 8 cm. diameter subtend at the center of circle an angle of $143°14'22''$, find the value of π to 4 places of decimals.

71. If the circumference of a circle be divided into five parts which are in A.P., and if the greatest part be six times the least find in radians the the magnitude of the angles the parts subtend at the center of the circle.

72. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius; express the angle of the sector in degrees.

73. At what distance a man, whose height is 2 m., subtend an angle of $10'$.

74. Find the length which at a distance of 5280 m., will subtend an angle of $1'$ at the eye.

75. Assuming the distance of the earth from the moon to be 38400 km., and the angle subtended by the moon at the eye of a person on earth to be $31'$, find the diameter of the moon.

76. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second; how fast is the train going?

77. Assuming that moon subtends an angle of $30'$ at the eye of an observer, find how far from the eye a coin of one inch diameter must be held so as just to hide the moon.

78. A wheel make 30 revolutions per minute. Find the circular measure of the angle described by spoke in half a second.

79. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends an angle of 56° at the center. Find the diamter of the circle.

Chapter 2
Trigonometric Ratios

From Geometry, we know that an acute angle is an angle whose measure is between 0° and 90°. Consider the following figure:

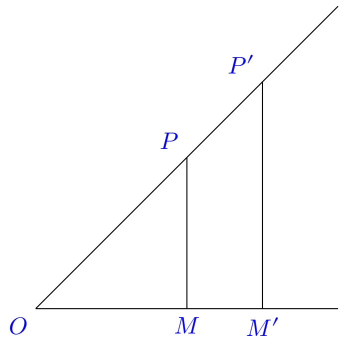


Figure 2.1 Trigonometric ratios

This picture contains two similar triangles $\triangle OMP$ and $\triangle OM'P'$. We are interested in $\angle MOP$ or $\angle M'OP'0$. In the $\triangle MOP$ and $\triangle M'OP'$, OP, OP' are called the hypotenuses i.e. sides opposite to the right angle, $PM, P'M'$ are called perpendiculars i.e. sides opposite to the angle of interest and OM, OM' are called bases i.e. the third angle.

Hypotenuses are usually denoted by h, perpendiculars by p and bases by b. Let $OM = b, OM' = b', PM = p, P'M' = p', OP = h, OP' = h'$. Since the two triangles are similar $\therefore \frac{p}{p'} = \frac{b}{b'} = \frac{h}{h'}$. Thus rhe ratio of any two sides is dependent purely on $\angle O$ or $\angle MOP$ or $\angle M'OP'$.

Since there are three sides, we can choose 2 in ${}^3C_2$ i.e. 3 ways and for each combination there will be two permutations where a side can be in either numerator or denominator. From this we can conclude that there will be six ratios(these are called trigonometric ratios), These six trigonometric ratios or functions are given below:

$\dfrac{MP}{OP}$ or $\dfrac{p}{h}$ is called the **Sine** of the $\angle MOP$.

$\dfrac{OM}{OP}$ or $\dfrac{b}{h}$ is called the **Cosine** of the $\angle MOP$.

$\dfrac{MP}{OM}$ or $\dfrac{p}{b}$ is called the **Tangent** of the $\angle MOP$.

$\dfrac{OP}{MP}$ or $\dfrac{h}{p}$ is called the **Cosecant** of the $\angle MOP$.

$\dfrac{OP}{OM}$ or $\dfrac{h}{b}$ is called the **Secant** of the $\angle MOP$.

$\dfrac{OM}{MP}$ or $\dfrac{b}{p}$ is called the **Cotangent** of the $\angle MOP$.

$1 - \cos MOP$ is called the **Versed Sine** of $\angle MOP$ and $1 - \sin MOP$ is called the **Coversed Sine** of $\angle MOP$. These two are rarely used in trigonometry. It should be noted that the trigonometric ratios are all numbers. The name of the trigonometric ratios are written for brevity $\sin MOP$, $\cos MOP$, $\tan MOP$, $\cot MOP$, $\sec MOP$, $\text{cosec}MOP$, $\text{vers}MP$, $\text{coverse}MOP$.

2.1 Relationship betweeen Trigonometric Functions or Ratios

Let us represent the $\angle MOP$ with θ, we observe from previous section that

$$\sin\theta = \frac{1}{\csc\theta}, \cos\theta = \frac{1}{\sec\theta}, \tan\theta = \frac{1}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{1}{\tan\theta}$$

We also observe that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

From Pythagora theorem in geometry, we know that $\text{hypotenuse}^2 = \text{perpendicular}^2 + \text{base}^2$ or $h^2 = p^2 + b^2$

1. Dividing both side by h^2, we get

$$\frac{p^2}{h^2} + \frac{b^2}{h^2} = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

   We can rewrite this as $\sin^2\theta = 1 - \cos^2\theta$, $\cos^2\theta = 1 - \sin^2\theta$, $\sin\theta = \sqrt{1 - \cos^2\theta}$, $\cos\theta = \sqrt{1 - \sin^2\theta}$.

2. If we divide both sides by b^2, then we get

$$\frac{h^2}{b^2} = \frac{p^2}{b^2} + 1$$

$$\sec^2\theta = \tan^2\theta + 1$$

   We can rewrite this as $\sec^2\theta - \tan^2\theta = 1$, $\tan^\theta = \sec^2\theta - 1$, $\sec\theta = \sqrt{1 + \tan^2\theta}$, $\tan\theta = \sqrt{sec^2\theta - 1}$

3. Similalry, if we divide by p^2, then we get

$$\frac{h^2}{p^2} = 1 + \frac{b^2}{p^2}$$

$$\text{cosec}^2\theta = 1 + \cos^2\theta$$

   We can rewrite this as $\text{cosec}^2\theta - \cot^2\theta = 1$, $\cot^2\theta = \text{cosec}^2\theta - 1$, $\text{cosec}\theta = \sqrt{1 + \cot^2\theta}$, $\cot\theta = \sqrt{\text{cosec}^2\theta - 1}$

2.2 Problems

Prove the following:

1. $\sqrt{\frac{1-\cos A}{1+\cos A}} = \text{cosec}A - \cot A.$

2. $\sqrt{\sec^2 A + \text{cosec}^2 A} = \tan A + \cot A.$

3. $(\text{cosec}A - \sin A)\,(\sec A - \cos A)\,(\tan A + \cot A) = 1.$

4. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A.$

5. $(\sin A + \cos A)\,(1 - \sin A \cos A) = \sin^3 A + \cos^3 A.$

6. $\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\text{cosec}A.$

7. $\sin^6 A - cos^6 A = 1 - 3\cos^2 A \sin^2 A.$

8. $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A.$

9. $\frac{\text{cosec}A}{\text{cosec}A-1} + \frac{\text{cosec}A}{\text{cosec}A+1} = 2\sec^2 A.$

10. $\frac{\text{cosec}A}{\tan A+\cot A} = \cos A.$

11. $(\sec A + \cos A)\,(\sec A - \cos A) = \tan^2 A + \sin^2 A.$

12. $\frac{1}{\tan A+\cot A} = \sin A \cos A.$

13. $\frac{1-\tan A}{1+\tan A} = \frac{\cot A-1}{\cot A+1}.$

14. $\frac{1+\tan^2 A}{1+\cot^2 A} = \frac{\sin^2 A}{\cos^2 A}.$

15. $\frac{\sec A-\tan A}{\sec A+\tan A} = 1 - 2\sec A \tan A + 2\tan^2 A.$

16. $\frac{1}{\sec A-\tan A} = \sec A + \tan A.$

17. $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \sec A\text{cosec}A + 1.$

18. $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A.$

19. $(\sin A + \cos A)\,(\tan A + \cot A) = \sec A + \text{cosec}A.$

20. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$

21. $\cot^4 A + \cot^2 A = \text{cosec}^4 A - \text{cosec}^2 A.$

22. $\sqrt{\text{cosec}^2 A - 1} = \cos A\text{cosec}A.$

23. $\sec^2 A\text{cosec}^2 A = \tan^2 A + \cot^2 A + 2.$

24. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

25. $(1+\cot A-\text{cosec}A)(1+\tan A+\sec A)=2.$

26. $\frac{\cot A\cos A}{\cot A+\cos A}=\frac{\cot A-\cos A}{\cot A\cos A}.$

27. $\frac{\cot A+\tan B}{\cot B+\tan A}=\cot A\tan B.$

28. $\left(\frac{1}{\sec^2 A-\cos^2 A}+\frac{1}{\text{cosec}^2 A-\sin^2 A}\right)\cos^2 A\sin^2 A=\frac{1-\cos^2 A\sin^2 A}{2+\cos^2 A\sin^2 A}.$

29. $\sin^8 A-\cos^8 A=(\sin^2 A-\cos^2 A)(1-2\sin^2 A\cos^2 A).$

30. $\frac{\cos A\text{cosec}A-\sin A\sec A}{\cos A+\sin A}=\text{cosec}A-\sec A.$

31. $\frac{1}{\text{cosec}A-\cot A}-\frac{1}{\sin A}=\frac{1}{\sin A}-\frac{1}{\text{cosec}A+\cot A}.$

32. $\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}.$

33. $(\tan A+\text{cosec}B)^2-(\cot B-\sec A)^2=2\tan A\cot B(\text{cosec}A+\sec B).$

34. $2\sec^2 A-\sec^4 A-2\text{cosec}^2A+\text{cosec}^4A=\cot^4 A-\tan^4 A.$

35. $(\sin A+\text{cosec}A)^2+(\cos A+\sec A)^2=\tan^2 A+\cot^2 A+7.$

36. $(\text{cosec}A+\cot A)(1-\sin A)-(\sec A+\tan A)(1-\cos A)=(\text{cosec}A-\sec A)[2-(1-\cos A)(1-\sin A)].$

37. $(1+\cot A+\tan A)(\sin A-\cos A)=\frac{\sec A}{\text{cosec}^2A}-\frac{\text{cosec}A}{\sec^2 A}.$

38. $\frac{1}{\sec A-\tan A}-\frac{1}{\cos A}=\frac{1}{\cos A}-\frac{1}{\sec A+\tan A}.$

39. $3(\sin A-\cos A)^4+4(\sin^6 A+\cos^6 A)+6(\sin A+\cos A)^2=13.$

40. $\sqrt{\frac{1+\cos A}{1-\cos A}}=\text{cosec}A+\cot A.$

41. $\frac{\cos A}{1+\sin A}+\frac{\cos A}{1-\sin A}=2\sec A.$

42. $\frac{\tan A}{\sec A-1}+\frac{\tan A}{\sec A+1}=2\text{cosec}A.$

43. $\frac{1}{1-\sin A}-\frac{1}{1+\sin A}=2\sec A\tan A.$

44. $\frac{1+\tan^2 A}{1+\cot^2 A}=\left(\frac{1-\tan A}{1-\cot A}\right)^2.$

45. $1+\frac{2\tan^2 A}{\cos^2 A}=\tan^4 A+sec^4A.$

46. $(1-\sin A-\cos A)^2=2(1-\sin A)(1-\cos A).$

47. $\frac{\cot A+\text{cosec}A-1}{\cot A-\text{cosec}A+1}=\frac{1+\cos A}{\sin A}.$

48. $(\sin A + \sec A)^2 + (\cos A + \text{cosec}A)^2 = (1 + \sec A\text{cosec}A)^2$.

49. $\frac{2\sin A\tan A(1-\tan A)+2\sin A\sec^2 A}{(1+\tan A)^2} = \frac{2\sin A}{1+\tan A}$.

50. If $2\sin A = 2 - \cos A$, find $\sin A$.

51. If $8\sin A = 4 + \cos A$, find $\sin A$.

52. If $\tan A + \sec A = 1.5$, find $\sin A$.

53. If $\cot A + \text{cosec}A = 5$, find $\cos A$.

54. If $3\sec^4 A + 8 = 10\sec^2 A$, find the value of $\tan A$.

55. If $\tan^2 A + \sec A = 5$, find $\cos A$.

56. If $\tan A + \cot A = 2$, find $\sin A$.

57. If $\sec^2 A = 2 + 2\tan A$, find $\tan A$.

58. If $\tan A = \frac{2x(x+1)}{2x+1}$, find $\sin A$ and $\cos A$.

59. If $3\sin A + 5\cos A = 5$, show that $5\sin A - 3\cos A = \pm 3$.

60. If $\sec A + \tan A = \sec A - \tan A$ prove that each side is ± 1.

61. If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, prove that

   i. $\sin^4 A + \sin^4 B = 2\sin^2 A\sin^2 B$,

   ii. $\frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1$.

62. If $\cos A + \sin A = \sqrt{2}\cos A$, prove that $\cos A - \sin A = \pm\sqrt{2}\sin A$.

63. If $a\cos A - b\sin A = c$, prove that $a\sin A + b\cos A = \sqrt{a^2 + b^2 - c^2}$.

64. If $1 - \sin A = 1 + \sin A$, then prove that value of each side is $\pm\cos A$.

65. If $\sin^4 A + \sin^2 A = 1$, prove that

   i. $\frac{1}{\tan^4 A} + \frac{1}{\tan^2 A} = 1$,

   ii. $\tan^4 A - \tan^2 = 1$.

66. If $\cos^2 A - \sin^2 A = \tan^2 B$, prove that $2\cos^2 B - 1 = \cos^2 B - \sin^2 B = \tan^2 A$.

67. If $\sin A + \text{cosec}A = 2$, then prove that $\sin^n A + \text{cosec}^n A = 2$.

68. If $\tan^2 A = 1 - e^2$, prove that $\sec A + \tan^3 A\text{cosec}A = (2 - e^2)^{\frac{3}{2}}$.

69. Eliminate A between the equations $a\sec A + b\tan A + c = 0$ and $p\sec A + q\tan A + r = 0$.

70. If cosec$A - \sin A = m$ and $\sec A - \cos A = n$, elimiate A.

71. Is the equation $\sec^2 A = \frac{4xy}{(x+y)^2}$ possible for real values of x and y?.

72. Show that the equation $\sin A = x + \frac{1}{x}$ is imossible for real values of x.

73. If $\sec A - \tan A = p, p \neq 0$, find $\tan A, \sec A$ and $\sin A$.

74. If $\sec A = p + \frac{1}{4p}$, show that $\sec A + \tan A = 2p$ or $\frac{1}{2p}$.

75. If $\frac{\sin A}{\sin B} = p, \frac{\cos A}{\cos B} = q$, find $\tan A$ and $\tan B$.

76. If $\frac{\sin A}{\sin B} = \sqrt{2}, \frac{\tan A}{\tan B} = \sqrt{3}$, find A and B.

77. If $\tan A + \cot A = 2$, find $\sin A$.

78. If $m = \tan A + \sin A$ and $n = \tan A - \sin A$, prove that $m^2 - n^2 = 4\sqrt{mn}$.

79. If $\sin A + \cos A = m$ and $\sec A +$ cosec$A = n$, prove that $n(m^2 - 1) = 2m$.

80. If $x\sin^3 A + y\cos^3 A = \sin A\cos A$ and $x\sin A - y\cos A = 0$, prove that $x^2 + y^2 = 1$.

81. Prove that $\sin^2 A = \frac{(x+y)^2}{4xy}$ is possible for real values of x and y only when $x = y$ and $x, y \neq 0$.

Chapter 3
Trigonometric Ratios of Any Angle and Sign

3.1 Angle of 45°

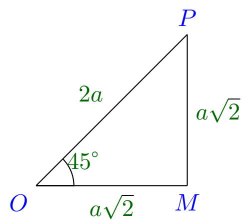


Figure 3.1

Consider the above figure, which is a right-angle triangle, drawn so that $\angle OMP = 90^\circ$ and $\angle MOP = 45^\circ$. We know that the sum of all angles of a triangle is 180°. Thus,

$\angle OPM = 180^\circ - \angle MOP - \angle OMP = 180^\circ - 90^\circ - 45^\circ = 45^\circ$

$\therefore OM = MP$. Let $OP = 2a$, then from Pythagora theorem, we can write

$4a^2 = OP^2 = OM^2 + MP^2 = 2OM^2 \Rightarrow Om = a\sqrt{2} = MP$

$\sin 45^\circ = \frac{MP}{OP} = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}}$.

Other trigonometric ratios can be deduced similarly for this angle.

3.2 Angles of 30° and 60°

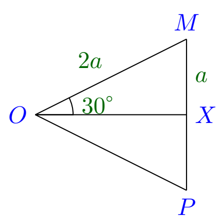


Figure 3.2

Consider an equilateral $\triangle OMP$. Let the sides OM, OP, MP be each $2a$. We draw a bisector of $\angle MOP$, which will be a perpendicular bisector of MP at X because the triangle is equilateral. Thus, $MX = a$. In $\triangle OMX$, $OM = 2a$, $\angle MOX = 30^\circ$, $\angle OXM = 90^\circ$ because each angle in an equilateral triangle is 60°.

$\sin MOX = \frac{MX}{OM} = \frac{1}{2} \Rightarrow \sin 30^\circ = \frac{1}{2}$

Similarly, $\angle OMX = 60^\circ$ because the sum of all angles of a triangle is 180°.

$\cos OMX = \frac{MX}{OM} = \frac{1}{2} \Rightarrow \cos 60^\circ = \frac{1}{2}$

All other trigonometric ratios can be found from these two.

3.3 Angle of 0°

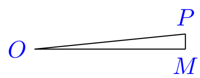


Figure 3.3

Consider the $\triangle MOP$ such that side MP is smaller than any quantity we can assign i.e. what we denote by 0. Thus, $\angle MOP$ is what is called approaching 0 or $\lim_{x\to 0}$ in terms of calculus. Why we take such a value is because if any angle of a triangle is equal to 0° then the triangle won't exist. Thus these values are limiting values as you will learn in calculus.

However, in this case, $\sin 0^\circ = \dfrac{MP}{OP} = \frac{0}{OP} = 0$. Other trigonometric ratios can be found from this easily.

3.4 Angle of 90°

In the previous figure, as $\angle OMP$ will approach 0°, the $\angle OPM$ will approach 90°. Also, OP will approach the length of OM. Similar to previous case, in right-angle trianglee if one angle (other than right angle) approaches 0° the other one will appraoch 90° and at that value the triangle will cease to exist.

Thus, $\sin 90^\circ = \dfrac{OM}{OP} = \frac{OP}{OP} = 1$. Now other angles can be found easily from this.

Given below is a table of most useful angles:

Angle	**0°**	**30°**	**45°**	**60°**	**90°**
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Table 3.1 Values of useful angles

3.5 Complementary Angles

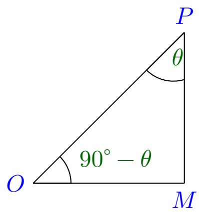


Figure 3.4

Angles are said to be complementary if their sum is equal to one right angle i.e. 90°. Thus, if measure of one angle is θ the other will automatically be $90^\circ - \theta$.

Consider the figure. $\triangle OMP$ is a right-angle triangle, whose $\angle OMP$ is a right angle. Since the sum of all angles is 180°, therefore sum of $\angle MOP$ and $\angle MPO$ will be equalto one right angle or 90° i.e. they are complementary angles.

Let $\angle MPO = \theta$ then $\angle MOP = 90^\circ - \theta$. When $\angle MPO$ is considered MP becomes the base and OM becomes the perpendicular.

Thus, $\sin(90^\circ - \theta) = \sin MOP = \frac{MP}{OP} = \cos MPO = \cos\theta$

$\cos(90^\circ - \theta) = \sin MPO = \frac{MO}{OP} = \sin\theta$

$\tan(90^\circ - \theta) = \tan MOP = \frac{PM}{OM} = \cot MPO = \cot\theta$

Similarly, $\cot(90^\circ - \theta) = \tan\theta$, $\text{cosec}(90^\circ - \theta) = \sec\theta$, $\sec(90^\circ - \theta) = \text{cosec}\theta$.

3.6 Supplementary Angles

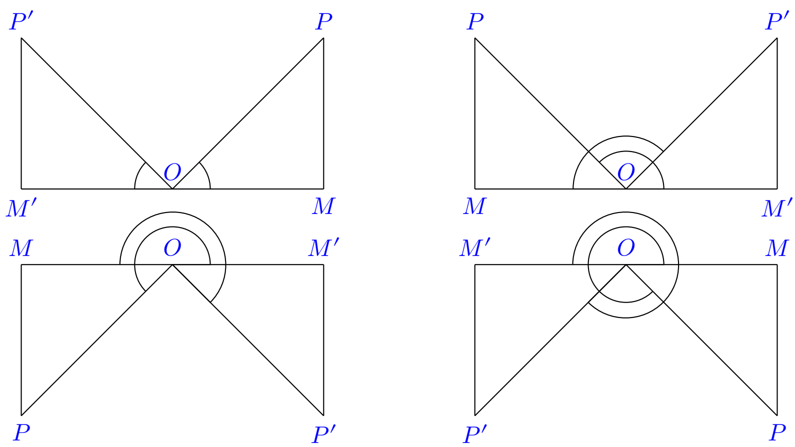


Figure 3.5

Angles are said to be supplementary if their sum is equal to two right angles i.e. 180°. Thus, if measure of one angle is θ, the other will automatically be $180^\circ - \theta$.

Consider the above figure which includes the angles of $180^\circ - \theta$. In each figure OM and OM' are drawn in different directions, while MP and $M'P'$ are drawn in the same direction so that $OM' = -OM$ and $M'P' = MP$. Hence we can say that

$\sin(180^\circ - \theta) = \sin MOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta$

$\cos(180^\circ - \theta) = \cos MOP' = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$

$\tan(180^\circ - theta) = \tan MOP' = \frac{OM'}{M'P'} = -\frac{OM}{MP} = -\tan\theta$

Similarly, $\cot(180^\circ - \theta) = -\cot\theta, \sec(180^\circ - \theta) = -\sec\theta, \text{cosec}(180^\circ - \theta) = \text{cosec}\theta$

3.7 Angles of $-\theta$

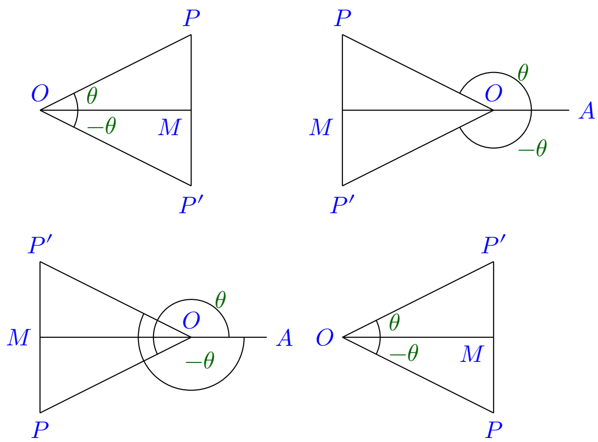


Figure 3.6

Consider the above diagram which plots the angles of θ and $-\theta$. Note that MP and MP' are equal in magnitude but opposite in sign. Thus, we have

$\sin(-\theta) = \frac{MP'}{OP'} = -\frac{MP}{OP} = -\sin\theta.$

$\cos(-\theta) = \frac{OM}{MP'} = \frac{OM}{OP} = \cos\theta.$

$tan(-\theta) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\theta.$

Similarly, $\cot(-\theta) = -\cot\theta, \sec(-\theta) = sec\theta, \text{cosec}(-\theta) = -\text{cosec}\theta.$

3.8 Angles of $90^\circ + \theta$

The diagram has been left as an exercise. Similarly, it can be proven that $\sin(90^\circ+\theta) = \cos\theta, \cos(90^\circ+\theta) = -\sin\theta, \tan(90^\circ+\theta) = -\cot\theta, \cot(90^\circ+\theta) = -\tan\theta, \sec(90^\circ+\theta) = -\text{cosec}\theta, \text{cosec}(90^\circ+\theta) = \sec\theta$.

Angles of $180^\circ+\theta, 270^\circ-\theta, 270^\circ+\theta$ can be found using previous relations.

3.9 Angles of $360^\circ + \theta$

For angles of θ the radius vector makes an angle of θ with initial side. For angles of $360^\circ+\theta$ it will complete a full revolution and then make an angle of θ with initial side. Thus, the trigonometrical ratios for an angle of $360^\circ+\theta$ are the same as those for θ.

It is clear that angle will remain θ for any multiple of 360°.

3.10 Problems

1. If $A = 30^\circ$, verify that
    i. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
    ii. $\sin 2A = 2\sin A\cos A$
    iii. $\cos 3A = 4\cos^3 A - 3\cos A$
    iv. $\sin 3A = 3\sin A - 4\sin^3 A$
    v. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
2. If $A = 45^\circ$, verify that
    i. $\sin 2A = 2\sin A\cos A$
    ii. $\cos 2A = 1 - 2\sin^2 A$
    iii. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Verify that

3. $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ = \frac{3}{2}$
4. $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ = 4\frac{1}{3}$
5. $\sin 30^\circ \cos 60^\circ + \sin 60^\circ \cos 30^\circ = 1$
6. $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ = -\frac{\sqrt{3}-1}{2\sqrt{2}}$

7. $\text{cosec}^2 45^\circ . \sec^2 30^\circ . \sin^2 90^\circ . \cos 60^\circ = 1\frac{1}{3}$

8. $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 30^\circ = \frac{1}{4}$

Prove that

9. $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ) = 1$

10. $\cos 570^\circ \sin 510^\circ - \sin 330^\circ \cos 390^\circ = 0$

What are the values of $\cos A - \sin A$ and $\tan A + \cot A$ when A has the values

11. $\frac{\pi}{3}$

12. $\frac{2\pi}{3}$

13. $\frac{5\pi}{4}$

14. $\frac{7\pi}{4}$

15. $\frac{11\pi}{3}$

What values between 0° and 360° may A have when

16. $\sin A = \frac{1}{\sqrt{2}}$

17. $\cos A = -\frac{1}{2}$

18. $\tan A = -1$

19. $\cot A = -\sqrt{3}$

20. $\sec A = -\frac{2}{\sqrt{3}}$

21. $\text{cosec} A = -2$

Express in terms of the ratios of a positive angle, which is less than 45°, the quantities

22. $\sin(-65^\circ)$

23. $\cos(-84^\circ)$

24. $\tan 137^\circ$

25. $\sin 168^\circ$

26. $\cos 287^\circ$

27. $\tan(-246^\circ)$

28. $\sin 843^\circ$

29. $\cos(-928^\circ)$

30. $\tan 1145^\circ$

31. $\cos 1410^\circ$

32. $\cot(-1054^\circ)$

33. $\sec 1327^\circ$

34. $\operatorname{cosec}(-756^\circ)$

What sign has $\sin A + \cos A$ for the following values of A?

35. 140°
36. 278°
37. -356°
38. -1125°

What sign has $\sin A - \cos A$ for the following values of A?

39. 215°
40. 825°
41. -634°
42. -457°
43. Find the sine and cosine of all angles in the first four quadrants whose tangents are equal to $\cos 135^\circ$.

Prove that

44. $\sin(270^\circ + A) = -\cos A$ and $\tan(270^\circ + A) = -\cot A$
45. $\cos(270^\circ - A) = -\sin A$ and $\cot(270^\circ - A) = \tan A$
46. $\cos A + \sin(270^\circ + A) - \sin(270^\circ - A) + \cos(180^\circ + A) = 0$
47. $\sec(270^\circ - A)\sec(90^\circ - A) - \tan(270^\circ - A)\tan(90^\circ + A) + 1 = 0$
48. $\cot A + \tan(180^\circ + A) + \tan(90^\circ + A) + \tan(360^\circ - A) = 0$
49. Find the value of $3\tan^2 45^\circ - \sin^2 60^\circ - \frac{1}{2}\cot^2 30^\circ + \frac{1}{8}\sec^2 45^\circ$
50. Simplify $\frac{\sin 300^\circ.\tan 330^\circ.\sec 420^\circ}{\tan 135^\circ.\sin 210^\circ.\sec 315^\circ}$
51. Show that $\tan 1^\circ \tan 2^\circ \ldots \tan 89^\circ = 1$
52. Show that $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \ldots + \sin^2 90^\circ = 9\frac{1}{2}$
53. Find the value of $\cos^2\frac{\pi}{16} + \cos^2\frac{3\pi}{16} + \cos^2\frac{5\pi}{16} + \cos^2\frac{7\pi}{16}$

Find the value of the following:

54. $\sec^2\frac{\pi}{6}\sec^2\frac{\pi}{4} + \tan^2\frac{\pi}{3}\sin^2\frac{\pi}{2}$
55. $\cot^2 30^\circ - 2\cos^2 60^\circ - \frac{3}{4}\sin^2 45^\circ - 4\sin^2 30^\circ$
56. $\frac{\sec 480^\circ \operatorname{cosec} 570^\circ.\tan 330^\circ}{\sin 600^\circ.\cos 660^\circ.\cot 405^\circ}$
57. If $A = 30^\circ$, show that $\cos^6 A + \sin^6 A = 1 - \sin^2 A\cos^2 A$
58. Show that $\left(\tan\frac{\pi}{4} + \cot\frac{\pi}{4} + \sec\frac{\pi}{4}\right)\left(\tan\frac{\pi}{4} + \cot\frac{\pi}{4} - \sec\frac{\pi}{4}\right) = \operatorname{cosec}^2\frac{\pi}{4}$

59. Show that $\sin^2 6^\circ + \sin 6^2 12^\circ + \sin^2 18^\circ + ... + \sin^2 84^\circ + \sin^2 90^\circ = 8$

60. Show that $\tan 9^\circ . \tan 27^\circ . \tan 45^\circ . \tan 63^\circ . \tan 81^\circ = 1$

61. Show that $\sum_{r=1}^{9} \sin^2 \frac{r\pi}{18} = 5$

62. If $4n\alpha = \pi$, show that $\tan\alpha \tan 2\alpha \tan 3\alpha. .... \tan(2n-2)\alpha \tan(2n-1)\alpha = 1$

Chapter 4
Compound Angles

Algebraic sum of two or more angles is called a *compound angle*. If A, B, C are any angle then $A+B$, $A-B$, $A-B+C$, $A+B+C$, $A-B-C$, $A+B-C$ etc. are all compound angles.

4.1 The Addition Formula

$\sin(A+B) = \sin A\cos B + \sin B\cos A$ $\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$

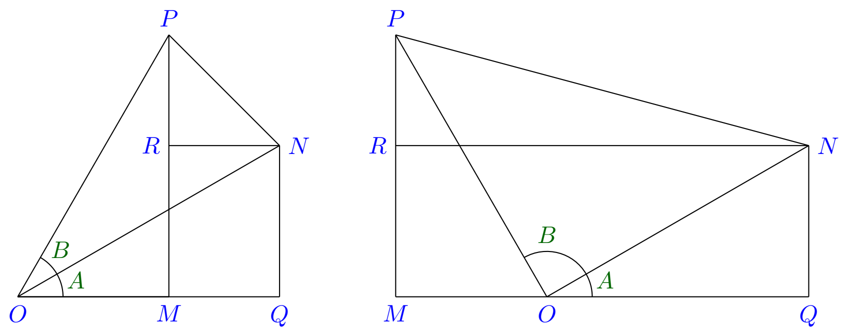


Figure 4.1

Consider the diagram above. PM and PN are perpendicualr to OQ and ON. RN is parallel to OQ and NQ is perpendicular to OQ. The left diagram represents the case when sum of angles is an acute angle while the right diagram represents the case when sum of angles is an obtuse angle.

$\angle RPN = 90^\circ - \angle PNR = \angle RNO = \angle NOQ = \angle A$

Now we can write, $\sin(A+B) = \sin QOP = \frac{MP}{OP} = \frac{MR+RP}{OP} = \frac{QN}{OP} + \frac{RP}{OP}$

$= \frac{QN}{ON}\frac{ON}{OP} + \frac{RP}{NP}\frac{NP}{OP} = \sin A\cos B + \cos A\sin B$

Also, $\cos(A+B) = \cos QOP = \frac{OM}{OP} = \frac{OQ-MQ}{OP} = \frac{OQ}{ON}\frac{ON}{OP} - \frac{RN}{NP}\frac{NP}{OP}$

$= \cos A\cos B - \sin A\sin B$

These two results lead to $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

We have shown that addition formula is true when angles involved are acute angles. The same proof can be applied to prove the results for all values of A and B.

Consider $A' = 90^\circ + A \therefore \sin A' = \cos A$ and $\cos A' = \sin A$

$\sin(A'+B) = \cos(A+B) = \cos A\cos B - \sin A\sin B = \sin A'\cos B + \cos A'\sin B$

Similarly $\cos(A'+B) = -\sin(A+B) = -\sin A\cos B - \sin B\cos A = \cos A'\cos B - \sin A'\sin B$

We can prove it again for $B' = 90^\circ + B$ and so on by increasing the values of A and B. Then we can again increase values by 90° and proceeding this way we see that the formula holds true for all values of A and B.

4.2 The Subtraction Formula

$\sin(A-B) = \sin A\cos B - \sin B\cos A$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$ $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

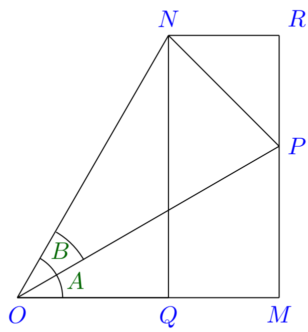


Figure 4.2

Conside the diagram above. The angle MOP is $A-B$. We take a point P, and draw PM and PN perpendicular to OM and ON respectively. From N we draw NQ and NR perpendicular to OQ and MP respectively.

$\angle RPN = 90^\circ - \angle PNR = \angle QON = A$

Thus, we can write $\sin(A-B) = \sin MOP = \frac{MP}{OP} = \frac{MR-PR}{OP} = \frac{QN}{ON}\frac{ON}{OP} - \frac{PR}{PN}\frac{PN}{OP}$

Thus, $\sin(A-B) = \sin A\cos B - \cos A\sin B$

Also, $\cos(A-B) = \frac{OM}{OP} = \frac{OQ+QM}{OP} = \frac{OQ}{ON}\frac{ON}{OP} + \frac{RN}{NP}\frac{NP}{OP}$

$= \cos A\cos B + \sin A\sin B$

We have shown that subtraction formula is true when angles involved are acute angles. The same proof can be applied to prove the results for all values of A and B.

From the results obtained we find upon division that $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$

4.3 Important Deductions

1. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

   L.H.S. $= (\sin A\cos B + \sin B\cos A)(\sin A\cos B - \sin B\cos A)$

$= \sin^2 A \cos^2 B - \sin^2 B \cos^2 A = \sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 B \sin^2 A$

$= \sin^2 A - \sin^2 B = (1 - \cos^2 A) - (1 - \cos^2 B)$

$= \cos^2 B - \cos^2 A$

2. $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

L.H.S. $= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B$

$= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

3. $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

L.H.S. $= \cot(A + B) = \frac{\cos(A+B)}{\sin(A+B)}$

$= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

Dividing numerator and denominator by $\sin A \sin B$

$= \frac{\cot A \cot B - 1}{\cot B + \cot A}$

4. $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

L.H.S. $= \cot(A - B) = \frac{\cos(A-B)}{\sin(A-B)}$

$= \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$

Dividing numerator and denominator by $\sin A \sin B$

$= \frac{\cot A \cot B + 1}{\cot B - \cot A}$

5. $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

L.H.S. $= \tan[(A + B) + C] = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$

$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B}\tan C}$

$= \frac{\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B}}{\frac{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}{1 - \tan A \tan B}}$

$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

4.4 To express $a\cos\theta + b\sin\theta$ in the form of $k\cos\phi$ or $k\sin\phi$

$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$

Let $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ then $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

Thus, $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$

$= \sqrt{a^2+b^2}\cos(\theta-\alpha) = k\cos\phi$ where $k = \sqrt{a^2+b^2}$ and $\phi = \theta - \alpha$

Alternatively, if $\frac{a}{\sqrt{a^2+b^2}} = \sin\alpha$ then $\frac{b}{\sqrt{a^2+b^2}} = \cos\alpha$

Thus, $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}(\sin\alpha\cos\theta + \cos\alpha + \sin\theta)$

$= \sqrt{a^2+b^2}\sin(\theta+\alpha) = k\sin\phi$ where $k = \sqrt{a^2+b^2}$ and $\phi = \theta + \alpha$

4.5 Problems

1. If $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{9}{41}$, find the values of $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$.
2. If $\sin\alpha = \frac{45}{53}$ and $\sin\beta = \frac{33}{65}$, find the values of $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$.
3. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha+beta)$.

Prove the following:

4. $\cos(45^\circ - A)\cos(45^\circ - B) - \sin(45^\circ - A)\sin(45^\circ - B) = \sin(A+B)$.
5. $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A-B)$.
6. $\frac{\sin(A-B)}{\cos A\cos B} + \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0$.
7. $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.
8. $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.
9. $\cos\alpha\cos(\gamma-\alpha) - \sin\alpha\sin(\gamma-\alpha) = \cos\gamma$.
10. $\cos(\alpha+\beta)\cos\gamma - \cos(\beta+\gamma)\cos\alpha = \sin\beta\sin(\gamma-\alpha)$.
11. $\sin(n+1)A\sin(n-1)A + \cos(n+1)A\cos(n-1)A = \cos 2A$.
12. $\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A = \cos A$.
13. Find the value of $\cos 15^\circ$ and $\sin 105^\circ$.

14. Find the value of $\tan 105^\circ$.

15. Find the value of $\frac{\tan 495^\circ}{\cot 855^\circ}$.

16. Evaluate $\sin\left(n\pi + (-1)^n \frac{\pi}{4}\right)$, where n is an integer.

Prove the following:

17. $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

18. $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

19. $\tan 75^\circ = 2 + \sqrt{3}$.

20. $\tan 15^\circ = 2 - \sqrt{3}$.

Find the value of following:

21. $\cos 1395^\circ$.

22. $\tan(-330^\circ)$.

23. $\sin 300^\circ \text{cosec} 1050^\circ - \tan(-120^\circ)$.

24. $\tan\left(\frac{11\pi}{12}\right)$.

25. $\tan\left((-1)^n \frac{\pi}{4}\right)$.

Prove the following:

26. $\cos 18^\circ - \sin 18^\circ = \sqrt{2} \sin 27^\circ$.

27. $\tan 70^\circ = 2 \tan 50^\circ + \tan 20^\circ$.

28. $\cot\left(\frac{\pi}{4} + x\right) \cot\left(\frac{\pi}{4} - x\right) = 1$.

29. $\cos(m+n)\theta . \cos(m-n)\theta - \sin(m+n)\theta \sin(m-n)\theta = \cos 2m\theta$.

30. $\frac{\tan(\theta+\phi)+\tan(\theta-\phi)}{1-\tan(\theta+\phi)\tan(\theta-\phi)} = \tan 2\theta$.

31. $\cos 9^\circ + \sin 9^\circ = \sqrt{2} \sin 54^\circ$.

32. $\frac{\cos 20^\circ - \sin 20^\circ}{\cos 20^\circ + \sin 20^\circ} = \tan 25^\circ$.

33. $\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}$.

34. $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$.

35. $\frac{1}{\tan 3A + \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 4A$.

36. $\frac{\sin 3\alpha}{\sin\alpha}+\frac{\cos 3\alpha}{cos\alpha}=4\cos 2\alpha.$

37. $\frac{\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)}=\sin 2A.$

38. $\tan 40^{\circ}+2\tan 10^{\circ}=\tan 50^{\circ}.$

39. $\tan(\alpha+\beta)\tan(\alpha-\beta)=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha-\sin^2\beta}.$

40. $\tan^2\alpha-\tan^2\beta=\frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta}.$

41. $\tan[(2n+1)\pi+\theta]+\tan[(2n+1)\pi-\theta]=0.$

42. $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{3\pi}{4}+\theta\right)+1=0.$

43. If $\tan\alpha=p$ and $\tan\beta=q$ prove that $\cos(\alpha+\beta)=\frac{1-pq}{\sqrt{(1+p^2)(1+q^2)}}.$

44. if $\tan\beta=\frac{2\sin\alpha\sin\gamma}{\sin(\alpha+\gamma)}$, show that $\cot\alpha,\cot\beta,\cot\gamma$ are in A.P.

45. Eliminate θ if $\tan(\theta-\alpha)=a$ and $\tan(\theta+\alpha)=b$.

46. Eliminate α and β if $\tan\alpha+\tan\beta=b$, $\cot\alpha+\cot\beta=a$ and $\alpha+\beta=\gamma$.

47. If $A+B=45^{\circ}$, show that $(1+\tan A)(1+\tan B)=2$.

48. If $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$, prove that $1+\cot\alpha\tan\beta=0$.

49. If $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, prove that $\tan(\alpha-\beta)=(1-n)\alpha$.

50. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, prove that $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$.

51. If $\tan\alpha=\frac{m}{m+1}$, $\tan\beta=\frac{1}{2m+1}$, prove that $\alpha+\beta=\frac{\pi}{4}$.

52. If $A+B=45^{\circ}$, show that $(\cot A-1)(\cot B-1)=2$.

53. If $\tan\alpha-\tan\beta=x$ and $\cot\beta-\cot\alpha=y$, prove that $\cot(\alpha-\beta)=\frac{x+y}{xy}$.

54. If a right angle be divided into three pats α,β and γ, prove that $\cot\alpha=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}$.

55. If $2\tan\beta+\cot\beta=\tan\alpha$, show that $\cot\beta=2\tan(\alpha-\beta)$.

56. If in any $\triangle ABC$, $C=90^{\circ}$, prove that $\operatorname{cosec}(A-B)=\frac{a^2+b^2}{a^2-b^2}$ and $\sec(A-B)=\frac{c^2}{2ab}$.

57. If $\cot A=\sqrt{ac}$, $\cot B=\sqrt{\frac{c}{a}}$, $\tan C=\sqrt{\frac{c}{a^3}}$ and $c=a^2+a+1$, prove that $A=B+C$.

58. If $\frac{\tan(A-B)}{\tan A}+\frac{\sin^2 C}{\sin^2 A}=1$, prove that $\tan A\tan B=\tan^2 C$.

59. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta = 1$ show that $\tan\alpha + \tan\beta = 0$.

60. If $\sin\theta = 3\sin(\theta + 2\alpha)$, prove that $\tan(\theta + \alpha)$, prove that $\tan(\theta + \alpha) + 2\tan\alpha = 0$.

61. If $3\tan\theta\tan\phi = 1$, prove that $2\cos(\theta + \phi) = \cos(\theta - \alpha)$.

62. Find the sign of the expression $\sin\theta + \cos\theta$ when $\theta = 100^\circ$.

63. Prove that the value of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ lies between -4 and 10.

64. If $m\tan(\theta - 30^\circ) = n\tan(\theta + 120^\circ)$, show that $\cos 2\theta = \frac{m+n}{2(m-n)}$.

65. if $\alpha + \beta = \theta$ and $\tan\alpha : \tan\beta = x : y$, prove that $\sin(\alpha - \beta) = \frac{x-y}{x+y}\sin\theta$.

66. Find the maximum and minimum value of $7\cos\theta + 24\sin\theta$.

67. Show that $\sin 100^\circ - \sin 10^\circ$ is positive.

Chapter 5
Transformation Formulae

5.1 Transformation of products into sums or differences

We know that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Adding these, we get $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

Subtracting, we get $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

We also know that $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Adding, we get $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

Subtrating we get $2\sin \sin B = \cos(A-B) - \cos(A+B)$

5.2 Transformation of sums or differences into products

We have $2\sin A \cos B = \sin(A+B)\sin(A-B)$

Substituting for $A+B = C$, $A-B = D$ so that $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

We also have $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

Following similarly $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

For $2\cos A \cos B = \cos(A+B) + \cos(A-B)$, we get $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

For $2\sin \sin B = \cos(A-B) - \cos(A+B)$, we get $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

5.3 Problems

1. Find the value of $\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ}$.
2. Simplify the expression $\frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)}$.

Prove that

3. $\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$.
4. $\frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\tan\theta$.
5. $\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$.

6. $\frac{\sin 7A-\sin A}{\sin 8A-\sin 2A}=\cos 4A\sec 5A$.

7. $\frac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cot(A+B)\cot(A-B)$.

8. $\frac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\frac{\tan(A+B)}{\tan(A-B)}$.

9. $\frac{\sin A+\sin 2A}{\cos A-\cos 2A}=\cot\frac{A}{2}$.

10. $\frac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\tan A$.

11. $\frac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B)$.

12. $\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A)\cos(45^\circ+B)$.

13. $\frac{\cos 3A-\cos A}{\sin 3A-\sin A}+\frac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=\frac{\sin A}{\cos 2A\cos 3A}$.

14. $\frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B)$.

15. $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$.

16. $\frac{\cos 3\theta+2\cos 5\theta+\cos 7\theta}{\cos\theta+2\cos 3\theta+\cos 5\theta}=\cos 2\theta-\sin 2\theta\tan 3\theta$.

17. $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$.

18. $\frac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}=\tan\theta$.

19. $\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\frac{\sin 3A}{\sin 5A}$.

20. $\frac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\frac{\sin A}{\sin B}$.

21. $\frac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A$.

22. $\frac{\sin A+\sin B}{\sin A-\sin B}=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$.

23. $\frac{\cos A+\cos B}{\cos B-\cos A}=\cot\frac{A+B}{2}\cot\frac{A-B}{2}$.

24. $\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\frac{A+B}{2}$.

25. $\frac{\sin A-\sin B}{\cos B-\cos A}=\cot\frac{A+B}{2}$.

26. $\frac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)-\sin(A-B+C)+\sin(A+B-C)}=\cot B$.

27. $\cos 3A+\cos 5A+\cos 7A+\cos 15A=4\cos 4A\cos 5A\cos 6A$.

28. $\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)+\cos(A+B+C)=4\cos A\cos B\cos C$.

29. $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$.

30. $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$.

31. $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha = 4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha$.

Simplify:

32. $\cos\left[\theta + \left(n - \frac{3}{2}\right)\phi\right] - \cos\left[\theta + \left(n + \frac{3}{2}\right)\phi\right]$.

33. $\sin\left[\theta + \left(n - \frac{3}{2}\right)\phi\right] + \sin\left[\theta + \left(n + \frac{3}{2}\right)\phi\right]$.

Express as a sum or difference the following:

34. $2\sin 5\theta \sin 7\theta$.

35. $2\cos 7\theta \sin 5\theta$.

36. $2\cos 11\theta \cos 3\theta$.

37. $2\sin 54^\circ \sin 66^\circ$.

Prove that

38. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$.

39. $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$.

40. $\sin A \sin(A + 2B) - \sin B \sin(B + 2A) = \sin(A - B)\sin(A + B)$.

41. $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$.

42. $\frac{2\sin(A-C)\cos C - \sin(A-2C)}{2\sin(B-C)\cos C - \sin(B-2C)} = \frac{\sin A}{\sin B}$.

43. $\frac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$.

44. $\frac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A$.

45. $\cos(36^\circ - A)\cos(36^\circ + A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A$.

46. $\cos A \sin(B - C) + \cos B \sin(C - A) + \cos C \sin(A - B) = 0$.

47. $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A$.

48. $\sin(\beta - \gamma)\cos(\alpha - \delta) + \sin(\gamma - \alpha)\cos(\beta - \delta) + \sin(\alpha - \beta)\cos(\gamma - \delta) = 0$.

49. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$.

50. $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$.

51. $\cos 18^\circ - \sin 18^\circ = \sqrt{2}\sin 27^\circ$.

52. $\frac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A} = \tan 3A$.

53. $\left(\frac{\cos A+\cos B}{\sin A-\sin A}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = 2\cot^n \frac{A-B}{2}$ or 0 accordingh as n is even or odd.

54. If α, β, γ are in A.P., show that $\cos\beta = \frac{\sin\alpha-\sin\gamma}{\cos\gamma-\cos\alpha}$.

55. If $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$ prove that $\sin 3\theta + \sin 3\phi = 0$.

56. $\sin 65^\circ + cos 65^\circ = \sqrt{2}\cos 20^\circ$.

57. $\sin 47^\circ + \cos 77^\circ = \cos 17^\circ$.

58. $\frac{\cos 10^\circ - \sin 10^\circ}{\cos 10^\circ + \sin 10^\circ} = \tan 35^\circ$.

59. $\cos 80^\circ + \cos 40^\circ - cos 20^\circ = 0$.

60. $\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{7\pi}{5} = 0$.

61. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.

62. If $\sin\alpha - \sin\beta = \frac{1}{3}$ and $\cos\beta - \cos\alpha = \frac{1}{2}$, prove that $\cot\frac{\alpha+\beta}{2} = \frac{2}{3}$.

63. If $\mathrm{cosec} A + secA = \mathrm{cosec} B + \sec B$, prove that $\tan A\tan B = \cot\frac{A+B}{2}$.

64. If $\sec(\theta+\alpha) + \sec(\theta-\alpha) = 2\sec\theta$, show that $\cos^2\theta = 1 + \cos\alpha$.

65. Show that $\sin 50^\circ \cos 85^\circ = \frac{1-\sqrt{2}\sin 35^\circ}{2\sqrt{2}}$.

66. Prove that $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8}$.

67. Prove that $\sin A\sin(60^\circ - A)\sin(60^\circ + A) = \frac{1}{4}\sin 3A$.

68. If $\alpha + \beta = 90^\circ$, find the maximum value of $\sin\alpha\sin\beta$.

69. Prove that $\sin 25^\circ \cos 115^\circ = \frac{1}{2}(\sin 40^\circ - 1)$.

70. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$.

71. Prove that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$.

72. Prove that $\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ = 3$.

73. Prove that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$.

74. Prove that $4\cos\theta\cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{\pi}{3}-\theta\right)=\cos 3\theta$.

75. Prove that $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta)=\tan 3\theta$.

76. If $\alpha+\beta=90^\circ$, show that the maximum value of $\cos\alpha\cos\beta$ is $\frac{1}{2}$.

77. If $\cos\alpha=\frac{1}{\sqrt{2}}$, $\sin\beta=\frac{1}{\sqrt{3}}$, show that $\tan\frac{\alpha+\beta}{2}\cot\frac{\alpha-\beta}{2}=5+2\sqrt{6}$ or $5-2\sqrt{6}$.

78. If $x\cos\theta=y\cos\left(\theta+\frac{2\pi}{3}\right)=z\cos\left(\theta+\frac{4\pi}{3}\right)$, prove that $xy+yz+xz=0$.

79. If $\sin\theta=n\sin(\theta+2\alpha)$, prove that $\tan(\theta+\alpha)=\frac{1+n}{1-n}\tan\alpha$.

80. If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)}=\frac{1-m}{1+m}$, prove that $\tan\left(\frac{\pi}{4}-\theta\right)\tan\left(\frac{\pi}{4}-\alpha\right)=m$.

81. If $y\sin\phi=x\sin(2\theta+\phi)$, show that $(x+y)\cot(\theta+\phi)=(y-x)\cot\theta$.

82. If $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-beta)\sin(\gamma-\delta)$, prove that $\cot\alpha\cot\beta\cot\gamma=cot\delta$.

83. If $\frac{\cos(A-B)}{\cos(A+B)}+\frac{\cos(C+D)}{\cos(C-D)}=0$, prove that $\tan A\tan B\tan C\tan D=-1$.

84. If $\tan(\theta+\phi)=3\tan\theta$, prove that $\sin(2\theta+\phi)=2\sin\phi$.

85. If $\tan(\theta+\phi)=3\tan\theta$, prove that $\sin 2(\theta+\phi)+\sin 2\theta=2\sin 2\phi$.

Chapter 6
Multiple and Submultiple Angles

6.1 Multiple Angles

An angle of the form nA, where n is an integer is called a *multiple angle*. For example, $2A, 3A, 4A, \ldots$ are multiple angles of A.

6.1.1 Trigonometrical Ratios of $2A$

From previous chapter we know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Substituting $B = A$, we get $\sin 2A = 2\sin A\cos A$

Similarly, $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$ (recall formula from previous chapter and substitute $B = A$ $\cos^2 A = 1 - \sin^2 A$ and $\sin^2 A = 1 - \cos^2 a$)

Also, $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ (recall formula from previous chapter and put $B = A$)

6.1.2 $\sin 2A$ and $\cos 2A$ in terms of $\tan A$

$\sin 2A = \frac{2\sin A\cos A}{\sin^2 A+\cos^2 A}[\because \sin^2 A + \cos^2 A = 1]$

Dividing both numerator and denominator by $\cos^2 A$, we get

$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

$\cos A = \cos^2 A - \sin^2 A = \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}[\because \sin^2 A + \cos^2 A = 1]$

Dividing both numerator and denominator by $\cos^2 A$, we get

$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} = \frac{\cot^2 A-1}{\cot^2 A+1}$

6.1.3 Trigonometrical Ratios of $3A$

$\sin 3A = \sin 2A\cos A + \cos 2A\sin A = 2\sin A\cos^2 A + \cos^2 A\sin A - \sin^3 A$

$= 2\sin A(1-\sin^2 A) + (1-2\sin^2 A)\sin A - \sin^3 A = 3\sin A - 4\sin^3 A$

$\cos 3A = \cos 2A\cos A - \sin 2A\sin A = (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - 2(1-\cos^2 A)\cos A = 4\cos^3 A - 3\cos A$

We know that $\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

Putting $B = A$ and $C = A$, we get $\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Similarly, $\cot 3A = \frac{\cot^3 A-3\cot A}{3\cot^2 A-1}$

6.2 Some Important Formulae

1. $\cos 2A = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \frac{1}{2}(1 - \cos 2A)$
2. $\cos 2A = 2\cos^2 - 1 \Rightarrow \cos^2 A = \frac{1}{2}(1 + \cos 2A)$
3. $\sin 3A = 3\sin A - 4\sin^3 A \Rightarrow \sin^3 A = \frac{1}{2}(3\sin A - \sin 3A)$
4. $\cos 3A = 4\cos^3 A - 3\cos A \Rightarrow \cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

6.3 Submultiple Angles

An angle of the form $\frac{A}{n}$, where n is an integer is called a *submultiple angle*. For exmaple, $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \ldots$ are submultiple angles of A.

6.3.1 Trigonometrical Ratios of $A/2$

We know that, $\sin 2A = 2\sin A\cos A$. Putting $A = A/2$, we get $\sin A = 2\sin A/2\cos A/2$

$\cos 2A = \cos^2 A - \sin^A$. Putting $A = A/2$, we get $\cos A - \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$\cos 2A = 2\cos^2 A - 1$. Putting $A = A/2$, we get $\cos A = 2\cos^2\frac{A}{2} - 1$

$\cos 2A = 1 - 2\sin^2 A$. Putting $A = A/2$, we get $\cos A = 1 - 2\sin^2\frac{A}{2}$

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$. Putting $A = A/2$, we get $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$\sin 2A = \frac{2\tan A}{1+\tan^2 A} \therefore \sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} \therefore \cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

$\cot 2A = \frac{\cot^2 A-1}{2\cot A} \therefore \cot A = \frac{\cot^2\frac{A}{2}-1}{2\cot\frac{A}{2}}$

6.3.2 Trigonometrical Ratios of $A/3$

$\sin 3A = 3\sin A - 4\sin^3 A$. Putting $A = \frac{A}{3}$, we get $\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$

$\cos 3A = 4\cos^3 A - 3\cos A$. Putting $A = \frac{A}{3}$, we get $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$

$\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A} \Rightarrow \tan A = \frac{3\tan\frac{A}{3}-\tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$

6.3.3 Values of $cos A/2$, $\sin A/2$ and $\tan A/2$ in terms of $\cos A$

$\cos^2\frac{A}{2} = \frac{1+\cos A}{2} \therefore \cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

$\sin^2\frac{A}{2} = \frac{1-\cos A}{}2 \therefore \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$

$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} \therefore \tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

6.3.4 Values of $\sin A/2$ and $\cos A/2$ in terms of $\sin A$

$\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)^2 = \cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\cos\frac{A}{2}\sin\frac{A}{2}$

$= 1 + \sin A \Rightarrow \cos\frac{A}{2} + \sin\frac{A}{2} = \sqrt{1+\sin a}$

Similarly, $\cos\frac{A}{2} - \sin\frac{A}{2} = \sqrt{1-\sin a}$

Adding, we get $\cos\frac{A}{2} = \pm\frac{1}{2}\sqrt{1+\sin A} \pm \frac{1}{2}\sqrt{1-\sin a}$

Subtracting, we get $\cos\frac{A}{2} = \pm\frac{1}{2}\sqrt{1+\sin A} \mp \frac{1}{2}\sqrt{1-\sin A}$

6.3.5 Value of $\sin 18^\circ$ and $\cos 72^\circ$

Let $A = 18^\circ$, then $\sin 5A = 90^\circ \therefore 2A + 3A = 90^\circ \Rightarrow \sin 2A = \sin(90^\circ - \sin 3A) \therefore 2\sin A\cos A = 4\cos^3 A - 3\cos A$

Dividing both sides by $\cos A$, we get $2\sin A = 4\cos^2 A - 3 = 4(1-\sin^2 A) - 3 \Rightarrow 4\sin^2 A + 2\sin A - 1 = 0 \Rightarrow \sin A = \frac{-1\pm\sqrt{5}}{4}$

However, since $A = 18^\circ \therefore \sin A > 0 \therefore \sin 18^\circ = \frac{-1+\sqrt{5}}{4} \therefore \sin(90^\circ - 18^\circ) = \cos 72^\circ = \frac{\sqrt{5}-1}{4}$

6.3.6 Value of $\cos 18^\circ$ and $\sin 72^\circ$

$\cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{10+2\sqrt{5}}{16} \therefore \cos 18^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}[\because \cos 18^\circ > 0]$

$\cos(90^\circ - 18^\circ) = \sin 72^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}$

6.3.7 Value of $\tan 18^\circ$ and $\tan 72^\circ$

$\tan 18^\circ = \frac{\sin 18^\circ}{\cos 18^\circ} = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$

$\tan 18^\circ \cot 18^\circ = 1 \Rightarrow \tan 72^\circ = \frac{1}{\tan 18^\circ} = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$

6.3.8 Value of $\cos 36^\circ$ and $\sin 54^\circ$

$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$

$\sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$

6.3.9 Value of $\sin 36^\circ$ and $\cos 54^\circ$

$\sin 36^\circ = 1 - \cos^2 36^\circ = 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

$\cos 54^\circ = \cos(90^\circ - 36^\circ) = \sin 36^\circ = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

Several other angles like, 9°, 15°, $22\frac{1}{2}^\circ$, $7\frac{1}{2}^\circ$ etc can be found similarrly.

6.4 Problems

1. Find the value of $\sin 2A$, when
   a. $\cos A = \frac{3}{5}$.
   b. $\sin A = \frac{12}{13}$.
   c. $\tan A = \frac{16}{63}$.
2. Find the value of $\cos 2A$, when
   a. $\cos A = \frac{15}{17}$.
   b. $\sin A = \frac{4}{5}$.
   c. $\tan A = \frac{5}{12}$.
3. If $\tan A = \frac{b}{a}$, find the value of $a\cos 2A + b\sin 2A$.

Prove that

4. $\frac{\sin 2A}{1+\cos 2A} = \tan A$.
5. $\frac{\sin 2A}{1-\cos 2A} = \cot A$.
6. $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.
7. $\tan A + \cot A = 2\text{cosec}2A$.
8. $\tan A - \cot A = -2\cot 2A$.
9. $\text{cosec}2A + \cot 2A = \cot A$.
10. $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\frac{A}{2}\cot\frac{B}{2}$.
11. $\frac{\cos A}{1\mp\sin A} = \tan\left(45^\circ \pm \frac{A}{2}\right)$.

12. $\frac{\sec 8A-1}{\sec 4A-1} = \frac{\tan 8A}{\tan 2A}$.

13. $\frac{1+\tan^2(45^\circ-A)}{1-\tan^2(45^\circ-A)} = \text{cosec}2A$.

14. $\frac{\sin A+\sin B}{\sin A-\sin B} = \frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$.

15. $\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B} = \tan(A+B)$.

16. $\tan\left(\frac{\pi}{4}+A\right) - \tan\left(\frac{\pi}{4}-A\right) = 2\tan 2A$.

17. $\frac{\cos A+\sin A}{\cos A-\sin A} - \frac{\cos A-\sin A}{\cos A+\sin A} = 2\tan 2A$.

18. $\cot(A+15^\circ) - \tan(A-15^\circ) = \frac{4\cos 2A}{1+2\sin 2A}$.

19. $\frac{\sin A+\sin 2A}{1+\cos A+\cos 2A} = \tan A$.

20. $\frac{1+\sin A-\cos A}{1+\sin A+cos A} = \tan\frac{A}{2}$.

21. $\frac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A} = \tan\frac{A}{2}$.

22. $\frac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)-\cos(n+1)A} = \cot\frac{A}{2}$.

23. $\sin(2n+1)A\sin A = \sin^2(n+1)A - \sin^2 nA$.

24. $\frac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B} = 2\cos(A+B)$.

25. $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\frac{A}{2}\cos\frac{3A}{2}$.

26. $\tan 2A = (\sec 2A+1)\sqrt{\sec^2 A - 1}$.

27. $\cos^3 2A + 3\cos 2A = 4(\cos^6 A - \sin^6 A)$.

28. $1+\cos^2 2A = 2(\cos^4 A + \sin^4 A)$.

29. $\sec^2 A(1+\sec 2A) = 2\sec 2A$.

30. $\text{cosec}A - 2\cot 2A\cos A = 2\sin A$.

31. $\cot A = \frac{1}{2}\left(\cot\frac{A}{2} - \tan\frac{A}{2}\right)$.

32. $\sin A\sin(60^\circ-A)\sin(60^\circ+A) = \frac{1}{4}\sin 3A$.

33. $\cos A\cos(60^\circ-A)\cos(60^\circ+A) = \frac{1}{4}\cos 3A$.

34. $\cot A + \cot(60^\circ+A) - \cot(60^\circ-A) = 3\cot 3A$.

35. $\cos 4A = 1 - 8\cos^2 A + 8\cos^4 A$.

36. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

37. $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$.

38. $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$.

39. $\frac{2\cos 2^n A+1}{2\cos A+1} = (2\cos A - 1)(2\cos 2A - 1)(2\cos 2^2 A - 1)\ldots(2\cos 2^{n-1} - 1)$.

40. If $\tan A = \frac{1}{7}$, $\sin B = \frac{1}{\sqrt{10}}$, prove that $A + 2B = \frac{\pi}{4}$, where $0 < A < \frac{\pi}{4}$ and $0 < B < \frac{\pi}{4}$.

Prove that

41. $\tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right) = 2\sec 2A$.

42. $\sqrt{3}\text{cosec}20^\circ - \sec 20^\circ = 4$.

43. $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A = \cot A$.

44. $\cos^2 A + \cos^2\left(\frac{2\pi}{3} - A\right) + \cos^2\left(\frac{2\pi}{3} + A\right) = \frac{3}{2}$.

45. $2\sin^2 A + 4\cos(A + B)\sin A\sin B + \cos 2(A + B)$ is idnependent of A.

46. If $\cos A = \frac{1}{2}\left(a + \frac{1}{a}\right)$, show that $\cos 2A = \frac{1}{2}\left(a^2 + \frac{1}{a^2}\right)$.

Prove that

47. $\cos^2 A + \sin^2 A\cos 2B = \cos^2 B + \sin^2 B\cos 2A$.

48. $1 + \tan A\tan 2A = \sec 2A$.

49. $\frac{1+\sin 2A}{1-\sin 2A} = \left(\frac{1+\tan A}{1-\tan A}\right)^2$.

50. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.

51. $\cot^2 A - \tan^2 A = 4\cot 2A\text{cosec}2A$.

52. $\frac{1+\sin 2A}{\cos 2A} = \frac{\cos A+\sin A}{\cos A-\sin A} = \tan\left(\frac{\pi}{4} + A\right)$.

53. $\cos^6 A - \sin^6 A = \cos 2A\left(1 - \frac{1}{4}\sin^2 2A\right)$.

54. $\cos^2 A + \cos^2\left(\frac{\pi}{3} + A\right) + \cos^2\left(\frac{\pi}{3} - A\right) = \frac{3}{2}$.

55. $(1 + \sec 2A)(1 + \sec 2^2 A)(1 + \sec 2^3 A)\ldots(1 + \sec 2^n A) = \frac{\tan 2^n A}{\tan A}$.

56. $\frac{\sin 2^n A}{\sin A} = 2^n\cos A\cos 2A\cos 2^2 A\ldots\cos 2^{n-1} A$.

57. $3(\sin A-\cos A)^4+6(\sin A+\cos A)^2+4(\sin^6 A+\cos^6 A)=13$.

58. $2(\sin^6 A+\cos^6 A)-3(\sin^4 A+\cos^4 A)+1=0$.

59. $\cos^2 A+\cos^2(A+B)-2\cos A\cos B\cos(A+B)$ if independent of A.

60. $\cos^3 A\cos 3A+\sin^3 A\sin 3A=\cos^3 2A$.

61. $\tan A\tan(60^\circ-A)\tan(60^\circ+A)=\tan 3A$.

62. $\sin^2 A+\sin^3\left(\frac{2\pi}{3}+A\right)+\sin^3\left(\frac{4\pi}{3}+A\right)=-\frac{3}{4}\sin 3A$.

63. $4(\cos^3 10^\circ+\sin^3 20^\circ)=3(\cos 10\circ+\sin 20^\circ)$.

64. $\sin A\cos^3 A-\cos A\sin^3 A=\frac{1}{4}\sin 4A$.

65. $\cos^3 A\sin 3A+\sin^3 A\cos 3A=\frac{3}{4}\sin 4A$.

66. $\sin A\sin(60^\circ+A)\sin(A+120^\circ)=\sin 3A$.

67. $\cot A+\cot(60^\circ+A)+\cot(120^\circ+A)=3\cot 3A$.

68. $\cos 5A=16\cos^5 A-20\cos^3 A+5\cos A$.

69. $\sin 5A=5\sin A-20\sin^3 A+16\sin^5 A$.

70. $\cos 4A-\cos 4B=8(\cos A-\cos B)(\cos A+\cos B)(\cos A-\sin B)(\cos A+\sin B)$.

71. $\tan 4A=\frac{4\tan A-4\tan^3 A}{1-6\tan^2 A+\tan^4 A}$.

72. If $2\tan A=3\tan B$, prove that $\tan(A-B)=\frac{\sin 2B}{5-\cos 2B}$.

73. If $\sin A+\sin B=x$ and $\cos A+\cos B=y$, show that $\sin(A+B)=\frac{2xy}{x^2+y^2}$.

74. If $A=\frac{\pi}{2^n+1}$, prove that $\cos A.\cos 2A.\cos 2^2A.\dots.\cos 2^{n-1}A=\frac{1}{2^n}$.

75. If $\tan A=\frac{y}{x}$, prove that $x\cos 2A+y\sin 2A=x$.

76. If $\tan^2 A=1+2\tan^2 B$, prove that $\cos 2B=1+2\cos 2A$.

77. If A and B lie between 0 and $\frac{\pi}{2}$ and $\cos 2A=\frac{3\cos 2B-1}{3-\cos 2B}$, prove that $\tan A=\sqrt{2}\tan B$.

78. If $\tan B=3\tan A$, prove that $\tan(A+B)=\frac{2\sin 2B}{1+\cos 2B}$.

79. If $x\sin A=y\cos A$, prove that $\frac{x}{\sec 2A}+\frac{y}{\text{cosec}2A}=x$.

80. If $\tan A=\sec 2B$, prove that $\sin 2A=\frac{1-\tan^4 B}{1+\tan^4 B}$.

81. If $A = \frac{\pi}{3}$, prove that $\cos A.\cos 2A.\cos 3A.\cos 4A.\cos 5A.\cos 6A = -\frac{1}{16}$.

82. If $A = \frac{\pi}{15}$, prove that $\cos 2A.\cos 4A.\cos 8A.\cos 14A = \frac{1}{16}$.

83. If $\tan A \tan B = \sqrt{\frac{a-b}{a+b}}$, prove that $(a - b\cos 2A)(a - b\cos 2B) = a^2 - b^2$.

84. If $\sin A = \frac{1}{2}$ and $\sin B = \frac{1}{3}$, find the value of $\sin(A+B)$ and $\sin(2A+2B)$.

85. If $\cos A = \frac{11}{61}$ and $\sin B = \frac{4}{5}$, find the value of $\sin^2\frac{A-B}{2}$ and $cos^2\frac{A+B}{2}$, the angle of A and B being positive acute angles.

86. Given $\sec A = \frac{5}{4}$, find $\tan\frac{A}{2}$ and $\tan A$.

87. If $\cos A = .3$, find the value of $\tan\frac{A}{2}$, and explain the resulting ambiguity.

88. If $\sin A + \sin B = x$ and $\cos A + \cos B = y$, find the value of $\tan\frac{A-B}{2}$.

Prove that

89. $(\cos A + \cos B)^2 + (\sin A - \sin B)^2 = 4\cos^2\frac{A+B}{2}$.

90. $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\frac{A-B}{2}$.

91. $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\frac{A-B}{2}$.

92. $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$.

93. $(\tan 4A + \tan 2A)(1 - \tan^2 3A\tan^2 A) = 2\tan 3A\sec^2 A$.

94. $\left(1 + \tan\frac{A}{2} - \sec\frac{A}{2}\right)\left(1 + \tan\frac{A}{2} + \sec\frac{A}{2}\right) = \sin A\sec^2\frac{A}{2}$.

95. $\frac{1+\sin A - \cos A}{1 + \sin A + \cos A} = \tan\frac{A}{2}$.

96. $\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}} = \frac{1+\sin A}{\cos A} = \tan\left(\frac{\pi}{4} + \frac{A}{2}\right)$.

97. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$.

98. $\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \tan^2\frac{A}{2}$.

99. $\cot\frac{A}{2} - \tan\frac{A}{2} = 2\cot A$.

100. $\frac{1+\sin A}{1-\sin A} = \tan^2\left(\frac{\pi}{4} + \frac{A}{2}\right)$.

101. $\sec A + \tan A = \tan\left(\frac{\pi}{4} + \frac{A}{2}\right)$.

102. $\frac{\sin A+\sin B-\sin(A+B)}{\sin A+\sin B+\sin(A+B)} = \tan\frac{A}{2}\tan\frac{B}{2}$.

103. $\tan\left(\frac{\pi}{4}-\frac{A}{2}\right) = \sec A - \tan A = \sqrt{\frac{1-\sin A}{1+\sin A}}$.

104. $\operatorname{cosec}\left(\frac{\pi}{4}+\frac{A}{2}\right)\operatorname{cosec}\left(\frac{\pi}{4}-\frac{A}{2}\right) = 2\sec A$.

105. $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2$.

106. $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$.

107. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$.

108. Find the value of $\sin\frac{23\pi}{24}$.

109. If $A = 112^\circ 30'$, find the value of $\sin A$ and $\cos A$.

Prove that

110. $\sin^2 24^\circ - \sin^2 6^\circ = \frac{1}{8}(\sqrt{5}-1)$.

111. $\tan 6^\circ . \tan 42^\circ . \tan 66^\circ . \tan 78^\circ = 1$.

112. $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ = \cos 7^\circ$.

113. $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \frac{1}{8}$.

114. $\cot 142\frac{1}{2}^\circ = \sqrt{2}+\sqrt{3}-2-\sqrt{6}$.

115. $\sin^2 48^\circ - \cos^2 12^\circ = -\frac{\sqrt{5}+1}{8}$.

116. $4(\sin 24^\circ + \cos 6^\circ) = \sqrt{3}+\sqrt{15}$.

117. $\cot 6^\circ \cot 42^\circ \cot 66^\circ \cot 78^\circ = 1$.

118. $\tan 12^\circ \tan 24^\circ \tan 48^\circ \tan 84^\circ = 1$.

119. $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \frac{1}{16}$.

120. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$.

121. $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$.

122. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^7}$.

123. $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65} = \frac{1}{64}$.

124. If $\tan\frac{A}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{B}{2}$, prove that, $\cos A = \frac{a\cos B+b}{a+b\cos B}$.

125. If $\tan\frac{A}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{B}{2}$, prove that $\cos B = \frac{\cos A - e}{1 - e\cos A}$.

126. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, prove that $\sin(A+B) = \frac{2ab}{a^2+b^2}$.

127. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, prove that $\cos(A-B) = \frac{1}{2}(a^2+b^2-2)$.

128. If A and B be two different roots of equation $a\cos\theta + b\sin\theta = c$, prove that

   a. $\tan(A+B) = \frac{2ab}{a^2-b^2}$.

   b. $\cos(A+B) = \frac{a^2-b^2}{a^2+b^2}$.

129. If $\cos A + \cos B = \frac{1}{3}$ and $\sin A + \sin B = \frac{1}{4}$, prove that $\cos\frac{A-B}{2} = \pm\frac{5}{24}$.

130. If $2\tan\frac{A}{2} = \tan\frac{B}{2}$, prove that $\cos A = \frac{3+5\cos B}{5+3\cos B}$.

131. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, prove that one value of $\cos\frac{A-B}{2} = \frac{8}{\sqrt{65}}$.

132. If $\sec(A+B) + \sec(A-B) = 2\sec A$, prove that $\cos B = \pm\sqrt{2}\cos\frac{B}{2}$.

133. If $\cos\theta = \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$, prove that one of the values of $\tan\frac{\theta}{2}$ is $\frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$.

134. If $\tan\alpha = \frac{\sin\theta\sin\phi}{\cos\theta+\cos\phi}$, prove that one of the values of $\tan\frac{\alpha}{2}$ is $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$.

135. If $\cos\theta = \frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$, prove that one of the values of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$.

Chapter 7
Trigonometric Identities

We will use the theory learned so far to solve following trigonometric identities.

7.1 Problems

1. If $A+B+C=\pi$, prove that $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\sin C$.

2. If $A+B+C=180^\circ$, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

3. Show that $\sin^2 A+\sin^2 B+2\sin A\sin B\cos(A+B)=\sin^2(A+B)$.

4. If $A+B+C=180^\circ$, prove that $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$.

5. If $A+B+C=180^\circ$, prove that $\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C)$.

6. If $A+B+C=180^\circ$, prove that $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\sin C$.

7. If $A+B+C=180^\circ$, prove that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

8. If $A+B+C=180^\circ$, prove that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

9. If $A+B+C=\frac{\pi}{2}$, prove that $\sin^2 A+\sin^2 B+\sin^2 C=1-2\sin A\sin B\sin C$.

10. If $A+B+C=\frac{\pi}{2}$, prove that $\cos^2 A+\cos^2 B+\cos^2 C=2+2\sin A\sin B\sin C$.

11. If $A+B+C=2\pi$, prove that $\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C=1$.

12. If $A+B=C$, prove that $\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C=1$.

13. If $A+B=\frac{\pi}{3}$, prove that $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$.

14. Show that $\cos^2 B+\cos^2(A+B)-2\cos A\cos B\cos(A+B)$ is independent of B..

15. If $A+B+C=\pi$ and $A+B=2C$, prove that $4(\sin^2 A+\sin^2 B-\sin A\sin B)=3$.

16. If $A+B+C=2\pi$, prove that $\cos^2 B+\cos^2 C-\sin^2 A-2\cos A\cos B\cos C=0$.

17. If $A+B+C=0$, prove that $\cos^2 A+\cos^2 B+\cos^2 C=1+2\cos A\cos B\cos C$.

18. Prove that $\cos^2(B-C)+\cos^2(C-A)+\cos^2(A-B)=1+2\cos(B-C)\cos(C-A)\cos(A-B)$.

19. If $A+B+C=\pi$, prove that $\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B=\sin A\sin B\sin C$.

20. If $A+B+C=\pi$, prove that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

21. If $A+B+C=\pi$, prove that $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$.

22. If $A+B+C=\pi$, prove that $\tan(B+C-A)+\tan(C+A-B)+\tan(A+B-C)=\tan(B+C-A)\tan(C+A-B)\tan(A+B-C)$.

23. If $A+B+C=\pi$, prove that $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$.

24. In a $\triangle ABC$, if $\cot A+\cot B+\cot C=\sqrt{3}$, prove that the triangle is equilateral.

25. If A, B, C, D are angles of a quadrilateral, prove that $\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}=\tan A\tan B\tan C\tan D$.

26. If $A+B+C=\frac{\pi}{2}$, show that $\cot A+\cot B+\cot C=\cot A\cot B\cot C$.

27. If $A+B+C=\frac{\pi}{2}$, show that $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$.

28. If $A+B+C=\pi$, prove that $\tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C$.

29. If $A+B+C=\pi$, prove that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

30. If $A+B+C=\pi$, prove that $\frac{\cot A+\cot B}{\tan A+\tan B}+\frac{\cot B+\cot C}{\tan B+\tan C}+\frac{\cot C+\cot A}{\tan C+\tan A}=1$.

31. Prove that $\tan(A-B)+\tan(B-C)+\tan(C-A)=\tan(A-B)\tan(B-C)\tan(C-A)$.

32. If $x+y+z=0$, show that $\cot(x+y-z)\cot(z+x-y)+\cot(x+y-z)\cot(y+z-x)+\cot(y+z-x)\cot(z+x-y)=1$.

33. If $A+B+C=n\pi$(n being zero or an integer), show that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

34. If $A+B+C=\pi$, prove that $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

35. If $A+B+C=\pi$, prove that $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

36. Prove that $\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}=8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

37. If $A+B+C=\pi$, prove that $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}$

$\cos\frac{\pi-C}{4}$.

38. If $A+B+C=\pi$, prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}$

$\sin\frac{A+B}{4}$.

39. If $A+B+C=\pi$, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

40. Prove that $1+\cos 56^\circ+\cos 58^\circ-\cos 66^\circ=4\cos 28^\circ\cos 29^\circ\sin 33^\circ$.

41. If $A+B+C=\pi$, prove that $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$.

42. If $A+B+C=\pi$, prove that $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$.

43. If $A+B+C=\pi$, prove that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

44. If $A+B+C=\pi$, prove that $\cos A+\cos B-\cos C=4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-1$.

45. If $A+B+C=\pi$, prove that $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$.

46. If $A+B+C=\pi$, prove that $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2$.

47. If $A+B+C=\pi$, prove that $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\ \sin\frac{B}{2}\sin\frac{C}{2}$.

48. If $x+y+z=\frac{\pi}{2}$, prove that $\cos(x-y-z)+\cos(y-z-x)+\cos(z-x-y)-4\cos x\cos y\cos z=0$.

49. Show that $\sin(x-y)+\sin(y-z)+\sin(z-x)+4\sin\frac{x-y}{2}\sin\frac{y-z}{2}\sin\frac{z-x}{2}=0$.

50. If $A+B+C=180^\circ$, prove that $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

51. If $A+B+C=\pi$, prove that $\sin\frac{B+C}{2}+\sin\frac{C+A}{2}+\sin\frac{A+B}{2}=4\cos\frac{\pi-A}{4}$

$\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$.

52. If $xy+yz+zx=1$, prove that $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$.

53. If $x+y+z=xyz$, show that $\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$.

54. If $x+y+z=xyz$, prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$.

55. If $x+y+z=xyz$, prove that $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$.

56. If $A+B+C+D=2\pi$, prove that $\cos A+\cos B+\cos C+\cos D=4\cos\frac{A+B}{2}$

$\cos\frac{B+C}{2}\cos\frac{C+A}{2}$.

57. If $A+B+C=2S$, prove that $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)=2+2\cos A\cos B\cos C$.

58. If $A+B+C=\pi$, prove that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1$.

59. If $A+B+C=\pi$, prove that $(\tan A+\tan B+\tan C)(\cot A+\cot B+\cot C)=1+\sec A\sec B\sec C$.

60. If $A+B+C=\pi$, prove that $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot C) = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

61. If $A+B+C=\pi$, prove that $\frac{1}{2}\sum \sin^2 A(\sin 2B+\sin 2C) = 3\sin A\sin B\sin C$.

62. If $A+B+C+D=2\pi$, prove that $\cos A-\cos B+\cos C-\cos D = 4\sin\frac{A+B}{2}$

$\sin\frac{A+D}{2}\cos\frac{A+C}{2}$.

63. If A, B, C, D be the angles of a cyclic quadrilateral, prove that $\cos A+\cos B+\cos C+\cos D=0$.

64. If $A+B+C=\pi$, prove that $\cot^2 A+\cot^2 B+\cot^2 C \geq 1$.

65. If $A+B+C=\pi$, prove that $\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}\cos\frac{A-B}{2} = \sin A+\sin B+\sin C$.

66. In a $\triangle ABC$, prove that $\sin 3A\sin(B-C)+\sin 3B\sin(C-A)+\sin 3C\sin(A-B)=0$.

Chapter 8
Properties of Triangles

In this chapter we will study the relations between the sides and trigonometrical ratios of the angles of a triangle. We already know that a triangle has three sides and three angles. In a $\triangle ABC$ we will denote the angles BAC, CBA, ACB by A, B, C and the corresponsing sides i.e. sides opposite to them by a, b, c respectively.

Thus, $BC = a$, $AC = b$, $AB = c$

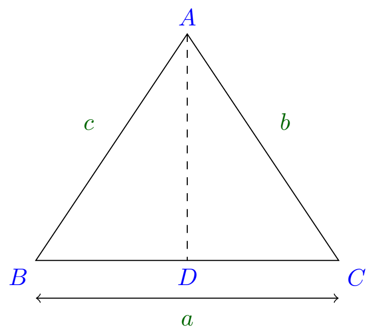
We will also denote the radius of the circumcircle of the $\triangle ABC$ by R and the area by $\triangle$. We also know some basic properties of a triangle for example, $A + B + C = 180^\circ$ and $a + b > c, b + c > a, c + a > b$.

8.1 Sine Formula or Sine Rule or Law of Sines

Theorem 1

In $\triangle ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Proof:



Case I: When $\angle C$ is accute. $\square$

From A draw $AD \perp BC$. From $\triangle ABD$,

$\sin B = \frac{AD}{AB} = \frac{AD}{c} \Rightarrow AD = c \sin B$

From $\triangle ACD$,

$\sin C = \frac{AD}{AC} = \frac{AD}{b} \Rightarrow AD = b \sin C$

Thus, $c \sin B = b \sin C$

Case II: When $\angle C$ is obtuse:

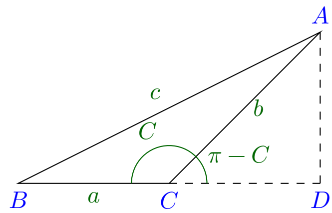


Figure 8.1

From A draw $AD \perp BC$. From $\triangle ABD$,

$\sin B = \frac{AD}{AB} = \frac{AD}{c} \Rightarrow AD = c \sin B$

From $\triangle ACD$,

$\sin(\pi - C) = \frac{AD}{AC} = \frac{AD}{b} \Rightarrow AD = b \sin C$

Thus, $c \sin B = b \sin C$

Case III: When $\angle C$ is 90°:

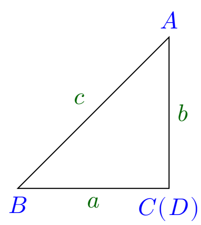


Figure 8.2

From A draw $AD \perp BC$. From $\triangle ABD$,

$\sin B = \frac{AD}{AB} = \frac{AD}{c} \Rightarrow AD = c \sin B \Rightarrow AC = c \sin B [\because C$ and D are same points $]$

$b = c \sin B \Rightarrow b \sin 90^\circ = c \sin B \Rightarrow b \sin C = c \sin B$

Thus, from all cases we have established that $\frac{b}{\sin B} = \frac{c}{\sin C}$

Similarly by drawing perpendicular from C to AB, we can prove that

$\frac{a}{\sin A} = \frac{b}{\sin B}$ and thus $\triangle ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Theorem 2

In a $\triangle ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circumcircle of $\triangle ABC$.

Proof:

Case I: When $\angle A$ is acute.

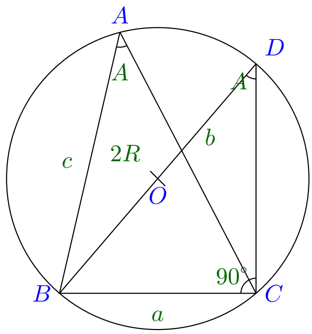


Figure 8.3

From $\triangle BDC, \sin A = \frac{BC}{BD} = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R.$

Case II: When $\angle A$ is obtuse.

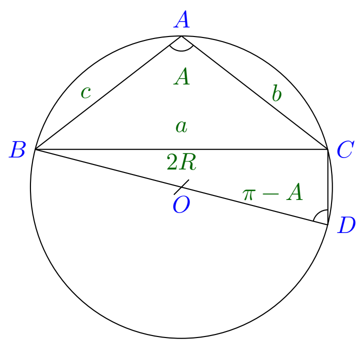


Figure 8.4

From $\triangle BDC, \sin(\pi - A) = \frac{BC}{BD} = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R.$

Case III: When $\angle A$ is 90°.

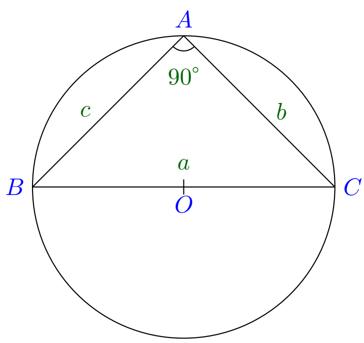


Figure 8.5

From $\triangle BDC$, $a = BC = 2R \Rightarrow \frac{a}{\sin A} = 2R$.

Similarly, by joining the diameter through A and O and through C and O, we can show that $\frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

8.2 Tangent Rule

Theorem 3

In any $\triangle ABC$, $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}, \tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$, *and* $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$.

Proof:

By sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$ (say)

$\therefore b = K\sin B, c = k\sin C \therefore \dfrac{b-c}{b+c} = \dfrac{K(\sin B - \sin C)}{K(\sin B + \sin C)} = \dfrac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \cot\dfrac{B+C}{2}\tan\dfrac{B-C}{2} = \tan\dfrac{A}{2}\tan\dfrac{B-C}{2} \Rightarrow \tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$. □

Similarly, we can prove the two other equations.

8.3 Cosine Formula or Cosine Rule

Theorem 4

In any $\triangle ABC$, $\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{c^2+a^2-b^2}{2ca}, \cos C = \frac{a^2+b^2-c^2}{2ab}$.

Proof:

Case I: When $\angle C$ is acute.

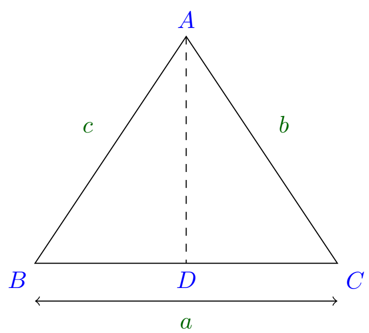


Figure 8.6

$AD = b\sin C$, $\cos C = \frac{CD}{AC} \Rightarrow CD = b\cos C \Rightarrow BD = BC - CD = a - b\cos C$.

Case II: When $\angle C$ is obtuse.

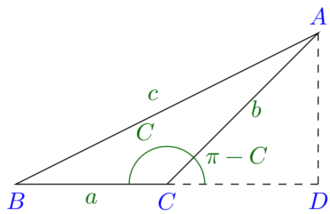


Figure 8.7

$AD = b\sin(\pi - C) = b\sin C$, $\cos(\pi - C) = \frac{CD}{AC} \Rightarrow CD = -\cos C \Rightarrow BC = BC + CD = a - b\cos C$.

Case III: When $\angle C$ is 90°.

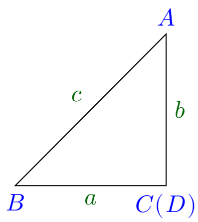


Figure 8.8

Here, C and D are same points. $AD = AC = b = b\sin C$, $CD = 0 = b\cos C[\because \cos C = \cos 90^\circ = 0]$

$BD = BC - CD = a - b\cos C$, thus, in all cases $AD = b\sin C$ and $BD = a - b\cos C$

Now, $AB^2 = AD^2 + BD^2 \Rightarrow c^2 = b^2\sin^2 C + (a - b\cos C)^2 \Rightarrow \cos C = \frac{a^2+b^2-c^2}{2ab}$.

Similarly, we can prove it for $\angle A$ and $\angle B$.

8.4 Projection Formulae

Theorem 5

In any $\triangle ABC$, $c = a\cos B + b\cos A$, $b = c\cos A + a\cos C$, $a = b\cos C + c\cos B$.

Proof:

Case I: When $\angle C$ is acute.

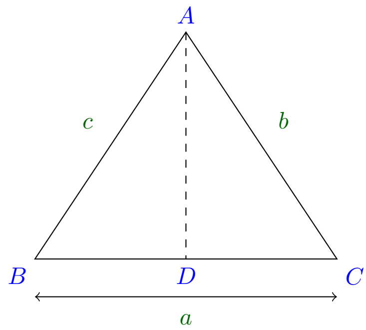


Figure 8.9

$BC = a = BD + CD = c\cos B + b\cos C$.

Case II: When $\angle C$ is obtuse.

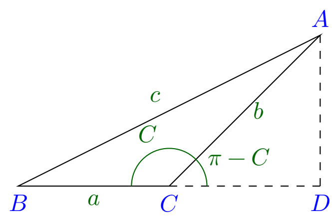


Figure 8.10

$BC = a = BD - CD = c\cos B - b\cos(\pi - C) = c\cos B + b\cos C$ **Case III:** When $\angle C$ is 90°.

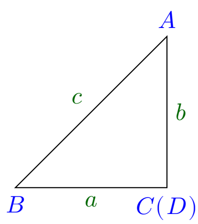


Figure 8.11

$BD = a = BC + CD = c\cos B + b\cos C[\because C = 90^\circ \therefore \cos C = 0]$

Thus, in all cases $a = b\cos C + c\cos B$. Similarly, we can prove for other sides. □

8.5 Sub-Angle Rules

Theorem 6

In any $\triangle ABC, \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, *where* $2s = a + b + c$.

Proof:

$$2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a+c-b)}{2bc}$$

$$= \frac{(2s-2c)(2s-2b)}{2bc} \Rightarrow \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\Rightarrow \sin\frac{A}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{bc}}$$

But $\frac{A}{2}$ is an acute angle so $\sin\frac{A}{2} > 0 \therefore \sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

$$2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} = \frac{(a+b+c)(b+c-a)}{2bc}$$

$$= \frac{(2s)(2s-2a)}{2bc} \Rightarrow \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\Rightarrow \cos\frac{A}{2} = \pm\sqrt{\frac{s(s-a)}{bc}}$$

But $\frac{A}{2}$ is an acute angle is $\cos\frac{A}{2} > 0 \therefore \cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

From the two equation which we have found it follows that $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. Similarly, we can prove the relations for other angles. □

8.6 Sines of Angles in Terms of Sides

Theorem 7

In any $\triangle ABC$, $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b(s-c)}$, $\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b(s-c)}$, $\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b(s-c)}$.

Proof:

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} = \frac{2}{bc}\sqrt{s(s-a)(s-b(s-c)}$$

Similarly, we can prove it for other angles. □

8.7 Area of a Triangle

Theorem 8

If Δ *denotes the area of* $\triangle ABC$, *then* $\Delta = \dfrac{1}{2}ab\sin C = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B$.

Proof:

Case I: When $\angle C$ is acute.

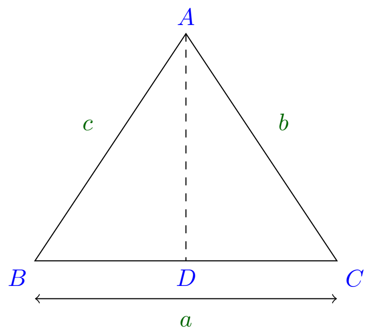


Figure 8.12

$\sin C = \frac{AD}{AC} \Rightarrow AD = b\sin C \therefore \Delta = \frac{1}{2}BC \times AD = \frac{1}{2}ab\sin C.$

Case II: When $\angle C$ is obtuse.

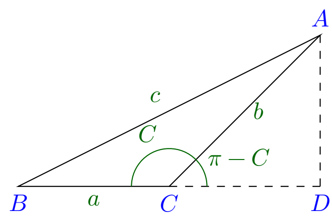


Figure 8.13

$\sin(\pi - C) = \frac{AD}{AC} \Rightarrow AD = b\sin C \therefore \Delta = \frac{1}{2}BC \times AD = \frac{1}{2}ab\sin C.$

Case III: When $\angle C$ is 90°.

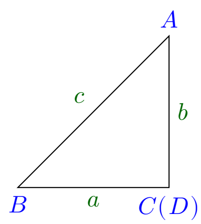


Figure 8.14

$\Delta = \frac{1}{2}BC \times AD = \frac{1}{2}ab\sin C[\because C = 90^\circ \therefore \sin C = 1].$

Thus in all cases $\Delta = \frac{1}{2}ab\sin C$. Similarly, we can prove two other formulae. $\square$

8.8 Area in Terms of Sides

Theorem 9

If Δ be the area of any $\triangle ABC$, when $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Proof:

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab.2\sin\frac{C}{2}\cos\frac{C}{2} = ab\sqrt{\frac{(s-a)(s-b)}{ab}}.\sqrt{\frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)(s-c)}.$$

8.8.1 Area in Terms of Radius of Circumcircle

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab.\frac{c}{2R} = \frac{abc}{4R}.$$

8.9 Tangent and Cotangent of Sub-angles of a Triangle

Theorem 10

In any $\triangle ABC$, $\tan\frac{A}{2} = \frac{(s-b)(s-c)}{\Delta}$, $\tan\frac{B}{2} = \frac{(s-a)(s-c)}{\Delta}$, $\tan\frac{C}{2} = \frac{(s-a)(s-c)}{\Delta}$, $\cos\frac{A}{2} = \frac{s(s-a)}{\Delta}$, $\cot\frac{B}{2} = \frac{s(s-b)}{\Delta}$, $\cot\frac{C}{2} = \frac{s(s-c)}{\Delta}$.

Proof:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-b)^2(s-c)^2}{s(s-a)(s-b)(s-c)}} = \frac{(s-b)(s-c)}{\Delta}.$$

Similarly, we can prove for other angles and cotangents. $\square$

8.10 Dividing a Side in a Ratio

Theorem 11

If D be a point on the side BC of a $\triangle ABC$ such that $BD : dC = m : n$ and $\angle ADC = \theta, \angle BAD = \alpha$ and $\angle DAC = \beta$, then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$, $(m+n)\cot\theta = n\cot B - m\cot C$.

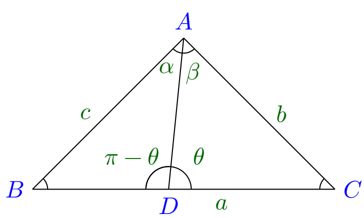


Figure 8.15

Proof:

$\angle ADB = \pi - \theta$, $\angle ABD = \pi - (\alpha + \pi - \theta) = \theta - \alpha$, $\angle ACD = \pi - (\theta + \beta)$. From $\triangle ABC$, $\frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta-\alpha)}$. Form $\triangle ADC$, $\frac{DC}{\sin\beta} = \frac{AB}{\sin[\pi-(\theta+\beta)]}$.

Dividing, we get $\dfrac{BD\sin\beta}{DC\sin\alpha} = \dfrac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$

$\Rightarrow \dfrac{m\sin\beta}{n\sin\alpha} = \dfrac{\sin\theta\cos\beta + \cos\theta\cos\beta}{\sin\theta\cos\alpha - \cos\theta\sin\alpha}$

$\Rightarrow m\sin\theta\sin\beta\cos\alpha - m\cos\theta\sin\alpha\sin\beta = n\sin\alpha\sin\theta\cos\beta + n\sin\alpha\cos\theta\sin\beta$

Dividing boths ides by $\sin\alpha\sin\beta\sin\theta$, we get

$m\cot\alpha - m\cot\theta = n\cot\beta + n\cot\theta$

$\Rightarrow (m+n)\cot\theta = n\cot\beta + n\cot\theta$.

Thus, first part is proved and now we will prove the second part.

$\angle BAD = 180^\circ - (180^\circ - \theta + B) = \theta - B$, $\angle DAC = 180^\circ - (\theta + C)$

From $\triangle BAD$, $\frac{BD}{\sin(\theta-B)} = \frac{AD}{\sin B}$. From $\triangle ADC$, $\frac{DC}{\sin[180^\circ-(\theta+C)]} = \frac{AD}{\sin C}$

$\Rightarrow \dfrac{DC}{\sin(\theta+c)} = \dfrac{AD}{\sin C}$

Dividing, we get

$\dfrac{BD}{DC}\cdot\dfrac{\sin(\theta+C)}{\sin(\theta-B)} = \dfrac{\sin C}{\sin B}$

$\Rightarrow \dfrac{m}{n}\cdot\dfrac{\sin\theta\cos C + \cos\theta\sin C}{\sin\theta\cos B - \cos\theta\sin B} = 1$

Proceeding like previous proof, we have

$(m+n)\cot\theta = n\cot B - m\cot C$. □

8.11 Results Related with Circumcircle

A circle passing through the vertices of a triangle is called a circumcircle. Its radius is called the circumradius.

Theorem 12

Let O be the center of the circumscribing circle of $\triangle ABC$. Then, $R = \frac{abc}{4\Delta}$.

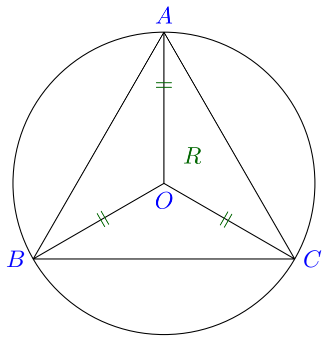


Figure 8.16

Proof:

By sine rule, $\dfrac{a}{\sin A} = 2R \Rightarrow R = \dfrac{a}{2\sin A} \because \Delta = \dfrac{1}{2}bc\sin A \therefore \sin A = \dfrac{2\Delta}{bc} \Rightarrow R = \dfrac{a}{\frac{2.2\Delta}{bc}} = \dfrac{abc}{4\Delta}$. □

8.12 Results Related with Incircle

The circle touching all the three sides of a triangle internally is called the inscribed circle or in-circle. Its radius is called in-radius and denoted by r. In the figure I is the incenter of the $\triangle ABC$.

Clearly, it is the point of intersection of internal bisector of angles of the $\triangle ABC$.

Theorem 13

In $\triangle ABC, r = \frac{\Delta}{s}$.

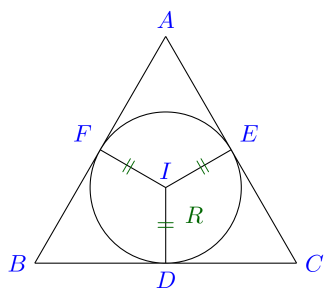


Figure 8.17

Proof:

Area of $\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB \Rightarrow \Delta = \dfrac{1}{2}ar + \dfrac{1}{2}br + \dfrac{1}{2}cr \Rightarrow r = \dfrac{\Delta}{s}$ □

8.12.1 Other Forms

1. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

   R.H.S. $= 4.\frac{abc}{4\Delta}.\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{(s-a)(s-c)}{ca}}.\sqrt{\frac{(s-a)(s-b)}{ab}}$

   $= \frac{abc}{\Delta}.\frac{(s-a)(s-b)(s-c)}{abc}.\frac{s}{s} = \frac{abc}{\Delta}.\frac{\Delta}{s} = \frac{\Delta}{s} = r.$

2. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

   $r = \frac{\Delta}{s} = \frac{\Delta}{s}.\frac{s-a}{s-a} = (s-a)\tan\frac{A}{2}.$

   Similarly, we can prove for other angles.

8.13 Results Related with Escribed Circles

Let ABC be a triangle. Let the bisectors of exterior angles B and C meet at I_1. Let $I_1D \perp BC$. If we take I_1 as the center aand draw a circle it will touch all the three sides(two extended) of the triangle. We can draw three such circles, one opposite to each side. We denote these radii by r_1, r_2 and r_3 for angle sA, B and C respectively.

Theorem 14

In such a $\triangle ABC, r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$.

Proof:

$\Delta ABC = \Delta I_1AB + \Delta I_1AC - \Delta I_1BC = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 = \frac{1}{2}(2s-2a)r_1 = (s-a)r_1 \Rightarrow r_1 = \frac{\Delta}{s-a}$. Similarly, it can be proven for r_2 and r_3. □

8.13.1 Other Forms

1. $r_1 = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

2. $r_2 = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

3. $r_3 = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

8.14 Distannces of Centers from Vertices

We have already shown that for circumcenter distance is equal to circum-radius i.e. R.

Referring to the image of incircle, $IF = r, \angle FAI = \frac{A}{2}$. From right-angle $\triangle FIA, \sin\frac{A}{2} = \frac{r}{AI} \Rightarrow AI = r\mathrm{cosec}\frac{A}{2}$.

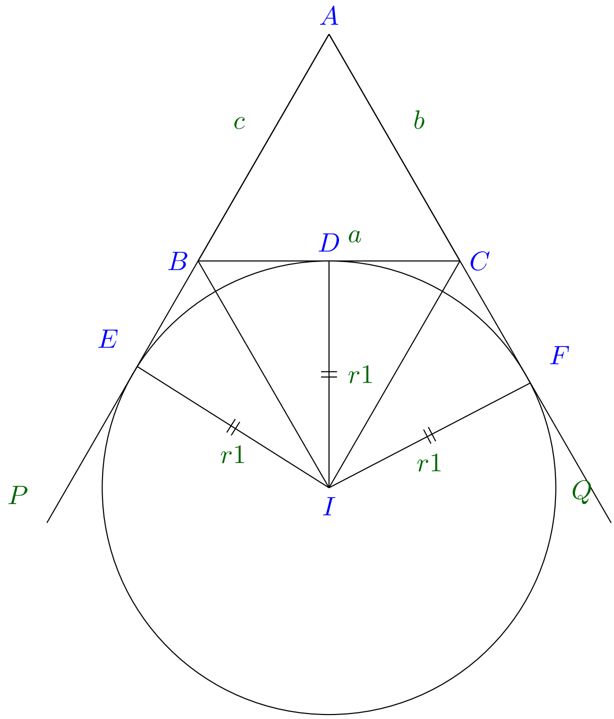


Figure 8.18

Similarly, $BI = r\text{cosec}\frac{B}{2}$ and $CI = r\text{cosec}\frac{C}{2}$.

8.14.1 Orthocenter

Orthocenter is the point of intersection of perpendiculars from a vertex to opposite side.

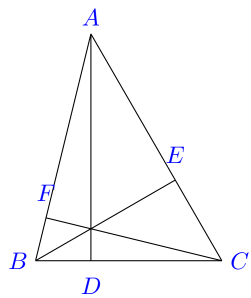


Figure 8.19

Let the orthocenter be H which is intersection of perpendiculars from any vertex to opposite side.
From right-angle $\triangle AEB$, $\cos A = \frac{AE}{AB} \Rightarrow AE = c\cos A$

From right-angle $\triangle ACD$, $\angle DAC = 90^\circ - C$. From right-angle $\triangle AEH$, $\cos(90^\circ - C) = \frac{AE}{AH}$

$\Rightarrow AH = \frac{c\cos A}{\sin C} = 2R\cos A$. Similarly, $BH = 2R\cos B$ and $CH = 2R\cos C$.

8.14.2 Centroid

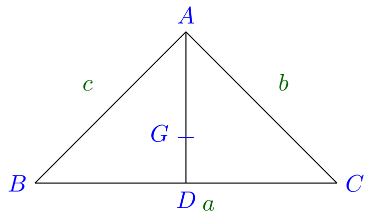


Figure 8.20

Let G be the centroid. Since, it is the point of intersection of medians, it will lie on median AD.

From geometry, $AB^2 + AC^2 = 2BD^2 + 2AD^2 \Rightarrow c^2 + b^2 = 2.\frac{a^2}{4} + 2AD^2$

$\Rightarrow 2AD^2 = \frac{2b^2+2c^2-a^2}{2}$

$\because AG : GD = 2 : 1$ [property of centeroid that it divides median in the ratio $2 : 1$]

$AG = \frac{2}{3}AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$. Similarly, $BG = \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$ and $CG = \frac{1}{3}\sqrt{2a^2 + 2b^2 - c^2}$.

A Angles Made by Medians with Sides

If $\angle BAD = \beta$ and $\angle CAD = \gamma$, then we have $\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} \Rightarrow \sin\gamma = \frac{DC.\sin C}{AD}$

$= \frac{a\sin C}{\sqrt{2b^2+2c^2-a^2}}$. Similarly, $\sin\beta = \frac{a\sin B}{\sqrt{2b^2+2c^2-a^2}}$.

If $\angle ADC$ be *theta* then we have $\sin\theta = \frac{2b\sin C}{2b^2+2c^2-a^2}$.

8.15 Escribed Triangles

Refer to Figure fig:esc, in which I is the incenter and I_1, I_2 and I_3 are the centers of the excircles opposite to vertices A, B and C respectively. We know that IC will bisect the $\angle ACB$, I_1C will bisect the external angles at C and I_1B will bisect the angle at B produces by extending the sides i.e. $\angle BCM$ as shown in the figure.

$\therefore \angle ICI_1 = \angle ICM + \angle ICM = \frac{1}{2}\angle ACB + \frac{1}{2}\angle BCM = 90^\circ$.

Similarly, $\angle ICI_1$ and $\angle ICI_3$ will be right angles.

Hence I_1CI_2 is perpendicular to IC. Similarly, I_2AI_3 is perpendicular to IA, and I_1BI_3 is perpendicular to IB.

We also see that IA and I_1A bot bisect $\angle A$ so AII_1 is a straight line. Similarly I_2IB and I_3IC are straight lines. The $\triangle I_1I_2I_3$ is called the *excentric* triangle of $\triangle ABC$.

8.16 Distance between Orthocenter and Circumcenter

Let O be the circumcenter. $OF \perp AAB$ and H be orthocenter. Then $\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$.

Let $BL \perp AC$ so it will pass through H. $\angle HAL = 90^\circ - C$, $\angle OAH = A - \angle OAF - \angle HAL = A - (180^circ - 2C) = C - B$

Also, $OA = R$ and $HA = 2R\cos A.OH^2 = OA^2 + HA^2 - 2OA.HA.\cos OAH = R^2 + 4R^2\cos^2 A - 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C-B)] = R^2 - 8R^2\cos A\cos B\cos C$

$\Rightarrow OH = R\sqrt{1 - 8\cos A\cos B\cos C}$.

8.17 Distance between Incenter and Circumcenter

Let O be the orthocenter and $OF \perp AB$. Let I be the incenter and $IC \perp AB$.

$\angle OAF = 90^\circ - C \therefore \angle OAI = \angle IAF - \angle OAF = \frac{A}{2} - 90^\circ + C = \frac{C-B}{2}$.

Also, $AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$.

$\therefore OI^2 = OA^2 + AI^2 - 2.OA.AI.\cos OAI$

$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$

$OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2 - 2Rr}$.

8.18 Area of a Cyclic Quadrilateral

Theorem 15

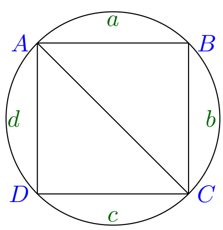


Figure 8.21

If a, b, c, d be the sides and s be the subperimeter of a cyclic quadrilateral, then its area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Proof:

Let $ABCD$ be a cyclic quadrilateral having sides $AB = a$, $BC = b$, $CD = c$ and $AD = d$. Since opposing angles of a quadrilateral are complementary, therefore $B + D = A + C = \pi$.

Applying cosine law in $\triangle ABC$, $\cos B = \frac{a^2+b^2-AC^2}{2ab} \Rightarrow AC^2 = a^2 + b^2 - 2ab\cos B$.

Similarly in $\triangle ACD$, $AC^2 = c^2 + d^2 + 2cd\cos B$. Thus, $\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$.

Area of quadrilaterl $ABCD = \Delta ABC + \Delta ACD = \frac{1}{2}ad\sin B + \frac{1}{2}cd\sin B$

Solving last two equations, we get area of quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$. □

8.19 Problems

1. The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double the smallest angle.
2. In a $\triangle ABC$, if $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$
3. If $\triangle = a^2 - (b-c)^2$, where $\triangle$ is the area of the $\triangle ABC$, then prove that $\tan A = \frac{8}{15}$
4. In a triangle ABC, the angles A, B, C are in A.P. Prove that $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2-ac+c^2}}$
5. If p_1, p_2, p_3 be the altitudes of a triangle ABC from the vertices A, B, C respectively and Δ be the area of the triangle, prove that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{2ab\cos^2\frac{C}{2}}{\Delta(a+b+c)}$
6. In any $\triangle ABC$, if $\tan\theta = \frac{2\sqrt{ab}}{a-b}\sin\frac{C}{2}$, prove that $c = (a-b)\sec\theta$
7. In a $\triangle ABC$, $a = 6$, $b = 3$ and $\cos(A-B) = \frac{4}{5}$, then find its area.
8. In a $\triangle ABC$, $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that area of $\triangle BAD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find $\angle ABD$
9. If the sides of a triangle are 3, 5 and 7, prove that the triangle is obtuse angled triangle and find the obtuse angle.
10. In a triangle ABC, if $\angle A = 45^\circ$, $\angle B = 75^\circ$, prove that $a + c\sqrt{2} = 2b$
11. In a triangle ABC, $\angle C = 90^\circ$, $a = 3$, $b = 4$ and D is a point on AB, so that $\angle BCD = 30^\circ$, find the length of CD.
12. The sides of a triangle are $4cm, 5cm$ and $6cm$. Show that the smallest angle is half of the greatest angle.
13. In an isosceles triangle with base a, the vertical angle is 10 times any of the base angles. Find the length of equal sides of the triangle.

14. The angles of a triangle are in the ratio of $2:3:7$, then prove that the sides are in the ratio of $\sqrt{2}:2:(\sqrt{3}+1)$

15. In a triangle ABC, if $\frac{\sin A}{7}=\frac{\sin B}{6}=\frac{\sin C}{5}$, show that $\cos A:\cos B:\cos C=7:19:25$

16. In any triangle ABC if $\tan\frac{A}{2}=\frac{5}{6}$, $\tan\frac{B}{2}=\frac{20}{37}$, find $\tan\frac{C}{2}$ and prove that in this triangle $a+c=2b$.

17. In a triangle ABC if $\angle C=60^\circ$, prove that $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$

18. If α, β, γ be the lengths of the altitudes of a triangle ABC, prove that $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\cot A+\cot B+\cot C}{\Delta}$, where Δ is the area of the triangle.

19. In a triangle ABC, if $\frac{a}{b}=2+\sqrt{3}$ and $\angle C=60^\circ$, show that $\angle A=105^\circ$ and $\angle B=15^\circ$.

20. If two sides of a triangle and the included angle are given by $a=(1+\sqrt{3})$, $b=2$ and $C=60^\circ$, find the other two angles and the third side.

21. The sides of a triangle are x, y and $\sqrt{x^2+xy+y^2}$. prove that the greatest angle is 120°.

22. The sides of a triangle are $2x+3$, x^2+3x+3 and x^2+2x, prove that greatest amgle is 120°.

23. In a triangle ABC, if $3a=b+c$, prove that $\cot\frac{B}{2}\cot\frac{C}{2}=2$

24. In a triangle ABC, prove that $a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$

25. In a triangle ABC, prove that $\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}=\frac{(a+b+c)^2}{a^2+b^2+c^2}$

26. In a triangle ABC, prove that $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

27. In a trianlge ABC, prove that $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$

28. In a triangle ABC, prove that $\frac{\cos^2\frac{B-C}{2}}{(b+c)^2}+\frac{\sin^2\frac{B-C}{2}}{(b-c)^2}=\frac{1}{a^2}$

29. In a triangle ABC, prove that $\frac{a}{\cos B\cos C}+\frac{b}{\cos C\cos A}+\frac{c}{\cos A\cos B}=2a\tan B\tan C\sec A$

30. In a triangle ABC, prove that $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$

31. In a triangle ABC, prove that $\tan\left(\frac{A}{2}+B\right)=\frac{c+b}{c-b}\tan\frac{A}{2}$

32. In a triangle ABC, prove that $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

33. In a triangle ABC, prove that $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$

34. In a triangle ABC, prove that $\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b}=0$

35. In a triangle ABC, prove that $a^3 \sin(B-C) + b^3 \sin(C-A) + c^3 \sin(A-B) = 0$

36. In a triangle ABC, prove that $(b+c-a)\tan\frac{A}{2} = (c+a-b)\tan\frac{B}{2} = (a+b-c)\tan\frac{C}{2}$

37. In a triangle ABC, prove that $1 - \tan\frac{A}{2}\tan\frac{B}{2} = \frac{2c}{a+b+c}$

38. In a triangle ABC, prove that $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$

39. In a triangle ABC, prove that $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

40. In a triangle ABC, prove that $\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$

41. In a triangle ABC, prove that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

42. In a triangle ABC, prove that $\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}$

43. In a triangle ABC, prove that $(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$

44. In a triangle ABC, prove that $(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0$

45. In a triangle ABC, prove that $(a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2$

46. In a triangle ABC, prove that $\frac{a-b}{a+b} = \cot\frac{A+B}{2}\tan\frac{A-B}{2}$

47. In a triangle ABC, D is the middle point of BC. If AD is perpendicular to AC, prove that $\cos A\cos C = \frac{2(c^2-a^2)}{3ac}$

48. If D be the middle point of the side BC of the triangle ABC where area is Δ and $\angle ADB = \theta$, prove that $\frac{AC^2-AB^2}{4\Delta} = \cot\theta$

49. $ABCD$ is a trapezium such that AB and DC are parallel and BC is perpendicular to the. If $\angle ADB = \theta$, $BC = p$, $CD = q$, show that $AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$

50. Let O be a point inside a triangle ABC such that $\angle OAB = \angle OBC = \angle OCA = \theta$, show that $\cot\theta = \cot A + \cot B + \cot C$.

51. The median AD of a triangle ABC is perpendicular to AB. Prove that $\tan A + 2\tan B = 0$.

52. In a triangle ABC, if $\cot A + \cot B + \cot C = \sqrt{3}$

53. In a triangle ABC, if $(a^2+b^2)\sin(A-B) = (a^2-b^2)\sin(A+B)$

54. In a triangle ABC, if θ be any angle, show that $b\cos\theta = c\cos(A-\theta) + a\cos(C+\theta)$

55. In a triangle ABC, AD is the median. If $\angle BAD = \theta$, prove that $\cos\theta = 2\cot A + \cot B$

56. The bisector of angle A of a triangle ABC meets BC in D, show that $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

57. Let A and B be two points on one bank of a straight river and C and D be two points on the other bank, the direction from A to B along the river being the same as from C to D. If $AB = a$, $\angle CAD = \alpha$, $\angle DAB = \beta$, $\angle CBA = \gamma$, prove that $CD = \frac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$

58. In a triangle ABC, if $2\cos A = \frac{\sin B}{\sin C}$, prove that the triangle is isosceles.

59. If the cosines of two angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right angled.

60. In a triangle ABC, if $a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}$, prove that the triangle is isosceles.

61. In a triangle ABC, if $\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c-b}{c}$, prove that $A = 60^\circ$

62. In a triangle ABC, if $c^4 - 2(a^2+b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$, prove that $C = 60^\circ$ or 120°

63. In a triangle ABC, if $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$, prove that the triangle is either isosceles or right angled.

64. If A, B, C are angles of a $\triangle ABC$ and if $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in A.P., prove that $\cos A, \cos B, \cos C$ are in A.P.

65. In a triangle ABC, if $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, show that $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

66. If a^2, b^2, c^2 are in A.P., then prove that $\cot A, \cot B, \cot C$ are in A.P.

67. The angles A, B and C of a triangle ABC are in A.P. If $2b^2 = 3c^2$, determine the angle A.

68. If in a triangle ABC, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then show that the sides a, b, c are in A.P.

69. In a triangle ABC, if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in A.P.

70. In a triangle ABC, $\sin A, \sin B, \sin C$ are in A.P. show that $3\tan\frac{A}{2}\tan\frac{C}{2} = 1$.

71. In a triangle ABC, if a^2, b^2, c^2 are in A.P., show that $\tan A, \tan B, \tan C$ are in H.P.

72. In a triangle ABC, if a^2, b^2, c^2 are in A.P., show that $\cot A, \cot B, \cot C$ are in A.P.

73. If the angles A, B, C of a triangle ABC be in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, find the angle A.

74. The sides of a triangle are in A.P. and the greatest angle exceeds the least angle by 90°. Prove that the sides are in the ratio $\sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1$.

75. If the sides a, b, c of a triangle are in A.P. and if a is the least side, prove that $\cos A = \frac{4c-3b}{2c}$

76. The two adjacent sides of a cyclic quadrilateral are 2 and 5 nad the angle between them is 60°. If the third side is 3, find the fourth side.

77. Find the angle A of triangle ABC, in which $(a+b+c)(b+c-a) = 3bc$

78. If in a triangle ABC, $\angle A = \frac{\pi}{3}$ and AD is a median, then prove that $4AD^2 = b^2 + bc + c^2$

79. Prove that the median AD and BE of a ΔABC intersect at right angle if $a^2 + b^2 = 5c^2$

80. If in a triangle ABC, $\frac{\tan A}{1} = \frac{\tan B}{2} = \frac{\tan C}{3}$, then prove that $6\sqrt{2}a = 3\sqrt{5}b = 2\sqrt{10}c$

81. The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, prove that the greatest anngle is $120°$.

82. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

83. For a triangle ABC having area 12 sq. cm. and base is 6 cm. The difference of base angles is $60°$. Show that angle A opposite to the base is given by $8\sin A - 6\cos A = 3$.

84. In any triangle ABC, if $\cos\theta = \frac{a}{b+c}$, $\cos\phi = \frac{b}{a+c}$, $\cos\psi = \frac{c}{a+b}$ where θ, ϕ and ψ lie between 0 and π, prove that $\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = 1$.

85. In a triangle ABC, if $\cos A\cos B + \sin A\sin B\sin C = 1$, show that the sides are in the proportion $1 : 1 : \sqrt{2}$.

86. The product of the sines of the angles of a triangle is p and the product of their cosines is q. Show that the tangents of the angles are the roots of the equation $qx^3 - px^2 + (1+q)x - p = 0$

87. In a $\triangle ANC$, if $\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, prove that $\cot\theta = \cot A + \cot B + \cot C$.

88. In a triangle of base a, the ratio of the other two sides is $r(<1)$, show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$

89. Given the base a of a triangle, the opposite angle A, and the product k^2 of the other two sides. Solve the triangle and show that there is such triangle if $a < 2k\sin\frac{A}{2}$, k being positive.

90. A ring 10 cm in diameter, is suspended from a point 12 cm above its center by 6 equal strings, attached at equal intervals. Find the cosine of the angle between consecutive strings.

91. If $2b = 3a$ and $\tan^2\frac{A}{2} = \frac{3}{5}$, prove that there are two values of third side, one of which is double the other.

92. The angles of a triangle are in the ratio $1 : 2 : 7$, prove that the ratio of the greater side to the least side is $\sqrt{5}+1 : \sqrt{5}-1$.

93. If f, g, h are internal bisectors of the angles of a triangle ABC, show that $\frac{1}{f}\cos\frac{A}{2} + \frac{1}{g}\cos\frac{B}{2} + \frac{1}{h}\cos\frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

94. If in a triangle ABC, $BC = 5, CA = 4, AB = 3$ and D and E are points on BC scuh that $BD = DE = EC$. If $\angle CAB = \theta$, then prove that $\tan\theta = \frac{3}{8}$.

95. In a triangle ABC, median AD and CE are drawn. If $AD = 5$, $\angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$, find the area of the triangle ABC.

96. The sides of a triangle are $7, 4\sqrt{3}$ and $\sqrt{13}$ cm. Then prove that the smallest angle is $30°$.

97. In an isosceles, right angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle in two parts whose cotangents are 2 and 3.

98. The sides of a triangle are such that $\frac{a}{1+m^2n^2} = \frac{b}{m^2+n^2} = \frac{c}{(1-m^2)(1+n^2)}$, prove that $A = 2\tan^{-1}\frac{m}{n}, B = 2\tan^{-1}mn$ and $\Delta = \frac{mnbc}{m^2+n^2}$.

99. The sides a, b, c if a triangle ABC are the roots of the equation $x^3 - px^2 + qx - r = 0$, prove that its area is $\frac{1}{4}\sqrt{p(4pq - p^3 - 8r)}$

100. Two sides of a triangle are of lengths $\sqrt{6}$ cm and 4 cm and the angle opposite to the smaller side is $30°$. How many such triangles are possible? Fine the length of their third side and area.

101. The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right) = 4\left(1+\frac{1}{t_2^2}\right)$

102. The three medians of a triangle ABC make angles α, β, γ with each other, prove that $\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0$.

103. Perpendiculars are drawn from the angles A, B, C of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle. If these produced parts be α, β, γ respectively, show that $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$

104. In a triangle ABC, the vertices A, B, C are at distance p, q, r from the orthocenter respectively. Show that $aqr + brp + cpq = abc$

105. The area of a circular plot of land in the form of a unit circle is to be divided into two equal parts by the arc of a circle whose center is on the circumference of the plot. Show that the radius of the circular arc is given by $\cos\theta$ where θ is given by $\frac{\pi}{2} = \sin 2\theta - 2\theta\cos 2\theta$

106. BC is a side of a square, on the perpendicular bisector of BC, two points P, Q are taken, equidistant from the center of square. BP and CQ are joined and cut in A. Prove that in the trangle ABC, $\tan A(\tan B - \tan C)^2 + 8 = 0$

107. If the bisector of the angle C of a triangle ABC cuts AB in D and the circum-circle in E, prove that $CE : DE = (a+b)^2 : c^2$.

108. The internal bisectors of the angles of a triangle ABC meet the sides at D, E and F. Show that the area of the triangle DEF is equal to $\frac{2\Delta abc}{(b+c)(c+a)(a+b)}$

109. In a triangle ABC, the measures of the angles A, B and C are $3\alpha, 3\beta$ and 3γ respectively. P, Q and R are the points within the triangle such that $\angle BAR = \angle RAQ = \angle QAC = \alpha$, $\angle CBP = \angle PBR = \angle RBA = \beta$ and $\angle ACQ = \angle QCP = \angle PCB = \gamma$. Show that $AR = 8R\sin\beta\sin\gamma\cos(30^\circ - \gamma)$

110. A circle touches the x axis at O (origin) and intersects the y axis above origin at $B.A$ is a point on that part of cirlce which lies to the right of OB, and the tangents at A and B meet at T. If $\angle AOB = \theta$, find the angles which the directed line OA, AT and OB makes with OX. If lengths of these lines are c, t and d respectively, show that $c\sin\theta - t(1+\cos 2\theta) = 0$ and $c\cos\theta + t\sin 2\theta = d$.

111. If in a triangle ABC, the median AD and the perpendicular AE from the vertex A to the side BC divides the angle A into three equal parts, show that $\cos\frac{A}{3}.\sin^2\frac{A}{3} = \frac{3a^2}{32bc}$

112. In a triangle ABC, if $\cos A + \cos B + \cos C = \frac{3}{2}$, prove that the triangle is equilateral.

113. Prove that a triangle ABC is equilateral if and only if $\tan A + \tan B + \tan C = 3\sqrt{3}$.

114. In a triangle ABC, prove that $(a+b+c)\tan\frac{C}{2} = a\cot\frac{A}{2} + b\cot\frac{B}{2} - c\cot\frac{C}{2}$

115. In a triangle ABC, prove that $\sin^4 A + \sin^4 B + \sin^4 C = \frac{3}{2} + 2\cos A\cos B\cos C + \frac{1}{2}\cos 2A + \cos 2B + \cos 2C$

116. In a triangle ABC prove that $\cos^4 A + \cos^4 B + \cos^4 C = \frac{1}{2} - 2\cos A\cos B\cos C + \frac{1}{2}\cos 2A\cos 2B\cos 2C$

117. In a triangle ABC, prove that $\cot B + \frac{\cos C}{\cos A\sin B} = \cot C + \frac{\cos B}{\cos A\sin C}$

118. In a triangle ABC, prove that $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

119. In a triangle ABC, prove that $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$

120. In a triangle ABC, prove that $\sin^3 A\cos(B-C) + \sin^3 B\cos(C-A) + \sin^3 C\cos(A-B) = 3\sin A\sin B\sin C$

121. In a triangle ABC, prove that $\sin^3 A + \sin^3 B + \sin^3 C = 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}$

122. In a triangle ABC, prove that $\sin 3A\sin^3(B-C) + \sin 3B\sin^3(C-A) + \sin 3C\sin^3(A-B) = 0$

123. In a triangle ABC, prove that $\sin 3A\cos^3(B-C) + \sin 3B\cos^3(C-A) + \sin 3C\cos^3(A-B) = \sin 3A\sin 3B\sin 3C$

124. In a triangle ABC, prove that $\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2} + b\sin^2\frac{A}{2}\right) = c\cot\frac{C}{2}$

125. The sides of a triangle ABC are in A.P. If the angles A and C are the greatest and the smallest angles respectively, prove that $4(1-\cos A)(1-\cos C) = \cos A + \cos C$

126. In a triangle ABC, if a, b, c are in H.P., prove that $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are also in H.P.

127. If the sides a, b, c of a triangle ABC be in A.P., prove that $\cos A \cot \frac{A}{2}, \cos B \cot \frac{B}{2}, \cos C \cot \frac{C}{2}$ are in A.P.

128. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ th of an equilateral triangle of the same perimieter. Prove that the sides are in the ratio $3 : 5 : 7$.

129. If the tangents of the angles of a triangle are in A.P., prove that the squares of the sides are in the proportion $x^2(x^2+9) : (3+x^2)^2 : 9(1+x^2)$, where x is the least or the greatest tangent.

130. If the sides of a triangle are in A.P. and if its greatest angle exceeds the least angle by α, show that the sides are in the ratio $1-x : 1 : 1+x$ where $x = \sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$

131. If the sides of triangle ABC are in G.P. with common ratio $r(r > 1)$, show that $r < \frac{1}{2}(\sqrt{5}+1)$ and $A < B < \frac{\pi}{3} < C$

132. If p and q be the perpendiculars from the vertices A and B on any line passing through the vertex C of the triangle ABC but not passing through the interior of the angle ABC, prove that $a^2p^2 + b^2q^2 - 2abpq\cos C = a^2b^2\sin^2 C$

133. ABC is a triangle, O is a point inside the triangle such that $\angle OAB = \angle OBC = \angle OCA = \theta$, then show that $\operatorname{cosec}^2\theta = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

134. If x, y, z be the lengths of perpendiculars from the circumcenter on the sides BC, CA, AB of a triangle ABC, prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$

135. In any triangle ABC if D is any point on the base BC such that $BD : DC = m : n$ and if $AD = x$, prove that $(m+n)^2x^2 = (m+n)(mb^2+nc^2) - mna^2$

136. In a triangle ABC, if $\sin A + \sin B + \sin C = \frac{3\sqrt{3}}{2}$, prove that the triangle is equilateral.

137. In a triangle ABC, if $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{8}$, prove that the triangle is equilateral.

138. In a triangle ABC, if $\cos A + 2\cos B + \cos C = 2$, prove that the sides of the triangle are in A.P.

139. The sides a, b, c of a triangle ABC of a triangle are in A.P., then find the value of $\tan\frac{A}{2} + \tan\frac{C}{2}$ in terms of $\cot\frac{B}{2}$.

140. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, prove that r_1, r_2, r_3 are in A.P.

141. If the sides a, b, c of a triangle ABC are in G.P., then prove that x, y, z are also in G.P., where $x = (b^2-c^2)\frac{\tan B+\tan C}{\tan B-\tan C}, y = (c^2-a^2)\frac{\tan C+\tan A}{\tan C-\tan A}, z = (a^2-b^2)\frac{\tan A+\tan B}{\tan A-\tan B}$

142. The ex-radii r_1, r_2, r_3 of a triangle ABC are in H.P. Show that its sides a, b, c are in A.P.

143. In usual notation, $r_1 = r_2 + r_3 + r$, prove that the triangle is right-angled.

144. If A, B, C are the angles of a triangle, prove that $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

145. Show that the radii of the three escribed circles of a triangle are the roots of the equation $x^3 - x^2(4R + r) + xs^2 - rs^2 = 0$

146. The radii r_1, r_2, r_3 of escribed circle of a triangle ABC are in H.P. If its area if 24 sq. cm. and its perimeter is 24 cm., find the length of its sides.

147. In a triangle ABC, $8R^2 = a^2 + b^2 + c^2$, prove that the triangle is right-angled.

148. The radius of the circle passing through the center of the inscribed circle and through the point of the base BC is $\frac{a}{2} \sec \frac{A}{2}$

149. Three circles touch each other externally. The tangents at their point of connect meet at a point whose distance from the point of contact is 4. Find the ratio of the product of radii to the sum of of radii of all the circles.

150. In a triangle ABC, if O be the circumcenter and H, the orthocenter, show that $OH = R\sqrt{1 - 8 \cos A \cos B \cos C}$

151. Let ABC be a triangle having O and I as its circumcenter an in-center respectively. If R and r be the circumradius and in-radius respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right angled triangle if and only if b is the arithmetic means of a and c.

152. In any triangle ABC, prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

153. Let ABC be a triangle with in-center I and in-radius r. Let D, E and F be the feet of perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE, BDIF$ and $CEID$ respectively, prove that $\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$

154. Show that the line joining the orthocenter to the circumference of a triangle ABC is inclined to BC at an angle $\tan^{-1}\left(\frac{3-\tan B \tan C}{\tan B - \tan C}\right)$

155. If a circle be drawn touching the inscribed and circumscribed circle of a triangle and BC externally, prove that its radius is $\frac{\Delta}{a} \tan^2 \frac{A}{2}$.

156. The bisectors of the angles of a triangle ABC meet its circumcenter in the position D, E, F. Show that the area of the triangle DEF is to that of ABC is $R : 2r$.

157. If the bisectors of the angles of a triangle ABC meet the opposite sides in A', B', C', prove that the ratio of the areas of the triangles $A'B'C'$ and ABC is $2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} : \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{C-A}{2}$.

158. If a, b, c are the sides of a triangle $\lambda a, \lambda b, \lambda c$ the sides of a similar triangle inscribed in the former and θ the angle between the sides of a and λa, prove that $2\lambda \cos \theta = 1$.

159. If r be the radius of in-circle and r_1, r_2, r_3 be the ex-radii of a triangle ABC, prove that $r_1 + r_2 + r_3 - r = 4R$

160. If r be the radius of in-circle and r_1, r_2, r_3 be the ex-radii of a triangle ABC, prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

161. If r be the radius of in-circle and r_1, r_2, r_3 be the ex-radii of a triangle ABC, prove that $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2+b^2+c^2}{\Delta^2}$ where Δ denotes the area of the triangle ABC.

162. If r is the radius of in-circle of a triangle ABC, prove that $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$.

163. If A, A_1, A_2 and A_3 be respectively the areas of the inscribed and escribed circles of a triangle, prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$

164. In a triangle ABC, prove that $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

165. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC then the triangle ABC has perimeter $P = 2(\sqrt{2rh - h^2} + \sqrt{2rh})$. Find its area.

166. If p_1, p_2, p_3 are the altitudes of the triangle ABC from the vertices A, B, C respectively, prove that $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$.

167. Three circles whose radii are a, b, c touch one another externally and the tangents at their point of contact meet in a point. Prove that the distance of this point from either of their points of contact is $\sqrt{\frac{abc}{a+b+c}}$

168. In a triangle ABC, prove that $r_1 r_2 r_3 = r^3 \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$.

169. In a triangle ABC, prove that $a(rr_1 + r2r_3) = b(rr_2 + r_3 r_1) = c(rr_3 + r_1 r_2) = abc$.

170. In a triangle ABC, prove that $(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = c$.

171. In a triangle ABC, prove that $4R\sin A\sin B\sin C = a\cos A + b\cos B + c\cos C$.

172. In a triangle ABC, prove that $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$

173. In a triangle ABC, prove that $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$

174. In a triangle ABC, prove that $IA.IB.IC = abc\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

175. In a triangle ABC, prove that $AI_1 = r_1\text{cosec}\frac{A}{2}$

176. In a triangle ABC, prove that $II_1 = a\sec\frac{A}{2}$

177. In a triangle ABC, prove that $I_2I_3 = a\mathrm{cosec}\frac{A}{2}$

178. In a triangle ABC, if I is the in-center and I_1, I_2 and I_3 are the centers of the escribed circles, then prove that $II_1.II_2.II_3 = 16R^2r$

179. In a triangle ABC, if I is the in-center and I_1, I_2 and I_3 are the centers of the escribed circles, then prove that $II_1^2.I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 = 16R^2$

180. In a triangle ABC, if O is the circumcenter and I, the in-center then prove that $OI^2 = R^2(3 - 2\cos A - 2\cos B - 2\cos C)$.

181. In a triangle ABC, if H is the orthocenter and I the in-center then prove that $IH^2 = 2r^2 - 4R^2\cos A\cos B\cos C$.

182. In a triangle ABC, if O is the circumcenter, G, the cetroid and H, the orthocenter then prove that $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$.

183. Given an isosceles triangle with lateral side of length b, base angle $\alpha < \frac{\pi}{4}$; R, r the radii and O, I the centers of the circumcircle and in-circle respectively, then prove that $R = \frac{1}{2}b\mathrm{cosec}\frac{\alpha}{2}$.

184. Given an isosceles triangle with lateral side of length b, base angle $\alpha < \frac{\pi}{4}$; R, r the radii and O, I the centers of the circumcircle and in-circle respectively, then prove that $r = \frac{b\sin 2\alpha}{2(1+\cos\alpha)}$

185. Given an isosceles triangle with lateral side of length b, base angle $\alpha < \frac{\pi}{4}$; R, r the radii and O, I the centers of the circumcircle and in-circle respectively, then prove that $OI = \left|\frac{b\cos\frac{3\alpha}{2}}{2\sin\alpha\cos\frac{\alpha}{2}}\right|$

186. In a triangle ABC, prove that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$

187. In a triangle ABC, prove that $\frac{r_1}{(s-b)(s-c)} + \frac{r_2}{(s-c)(s-a)} + \frac{r_3}{(s-a)(s-b)} = \frac{3}{r}$.

188. If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the in-circle, prove that $r^2 = \frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}$

189. Tangents are drawn to the in-circle of triangle ABC which are parallel to its sides. If x, y, z be the lengths of the tangents and a, b, c be the sides of triangle then prove that $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

190. If t_1, t_2, t_3 be the length of tangents from the centers of escribed circles to the circumcircle, prove that $\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{2s}{abc}$.

191. If in a triangle ABC, $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is right angled.

192. In a triangle ABC, prove that the area of the in-circle is to the area of the triangle itself is $\pi : \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

193. Let $A_1, A_2, A_3, \dots, A_n$ be the vertices of polygon having an n sides such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$ then find the value of n.

194. Prove that the sum of radii of the circles, which are respectively inscribed in and circumscibed about a regular polygon of n sides, is $\frac{a}{2}\cot\frac{\pi}{2n}$, where a is the side of the polygon.

195. The sides of a quadrilateral are $3, 4, 5$ and 6 cms. The sum of a pair of opposite angles is $120°$. Show that the area of the quadrilateral is $3\sqrt{30}$ sq. cm.

196. The two adjacent sides of a quadrilateral are 2 and 5 and the angle between them is $60°$. If the area of the quadrilateral is $4\sqrt{3}$, find the two remaining sides.

197. A cyclic quadrilateral $ABCD$ of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

198. If $ABCD$ be a quadrilateral inscribed in a circle, prove that $\tan\frac{B}{2} = \sqrt{\frac{(S-a)(S-b)}{(S-c)(S-d)}}$.

199. a, b, c and d are the sides of a quadrilateral taken in order and α is the angle between diagonals opposite to b or d, prove that the area of the quadrilateral is $\frac{1}{2}(a^2 - b^2 + c^2 - d^2)\tan\alpha$

200. If a quadrilateral can be inscribed in one circle and circumscribed about another circle, prove that its area is $\sqrt{abcd}$ and the radius of the latter circle is $\frac{2\sqrt{abcd}}{a+b+c+d}$.

201. The sides of a quadrilateral which can be inscribed in a circle are $3, 3, 4$ and 4 cm; find the radii of in-circle and circumcircle.

202. A square whose sides are 2 cm., has its corners cut away so as to form a regular octagon; find its area.

203. If an equilateral triangle and a regular hexagon have the same perimeter, prove that ratio of their areas is $2 : 3$.

204. Given that the area of a polygon of n sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as $3 : 2$, find n.

205. The area of a polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as $3 : 4$. Fine the value of n.

206. There are two regular polygons, the number of sides in one being the double the number in the other, and an angle of one ploygon is to an angle of the other is $9 : 8$; find the number of sides of each polygon.

207. Six equal circles, each of radius a, are placed so that each touches to others, their centers are joined to form a hexagon. Prove that the area which the circles enclose is $2a^2(3\sqrt{3} - \pi)$.

208. A cyclic quadrilateral $ABCD$ of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

209. If $ABCD$ is a cyclic quadrilateral, then prove that $AC.BD = AB.CD + BC.AD$

210. If the number of sides of two regular polygons having the same perimeter be n and $2n$ respectively, prove that their areas are in the ratio $2\cos\frac{\pi}{n} : \left(1+\cos\frac{\pi}{n}\right)$.

211. In a triangle ABC, prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$

212. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the center. Find the minimum value of the arithmetic mean of $\cos\left(\alpha+\frac{\pi}{2}\right)$, $\cos\left(\beta+\frac{\pi}{2}\right)$ and $\cos\left(\gamma+\frac{\pi}{2}\right)$

213. In a triangle ABC, prove that $\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2} \geq 1$

214. Let $1 < m < 3$. In a triangle ABC if $2b = (m+1)a$ and $\cos A = \frac{1}{2}\sqrt{\frac{(m-1)(m+3)}{m}}$, prove that there are two values of the third side, one of which is m times the other.

215. If Δ denotes the area of any triangle and s its semiperimeter, prove that $\Delta < \frac{s^2}{4}$.

216. Let A, B, C be three angles such that $A = \frac{\pi}{4}$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

217. Through the angular points of a triangle straight lines are drawn, which make the same angle α with the opposite side of the triangle. Prove that the area of the triangle formed by them is to the area of the triangle is $4\cos^2\alpha : 1$

218. Consider the following statements about a triangle ABC

    a. The sides a, b, c and Δ are rational.

    b. $a, \tan\frac{B}{2}, \tan\frac{C}{2}$ are rational

    c. $a, \sin A, \sin B, \sin C$ are rational.

    Prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

219. Two sides of a triangle are of length $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30°. How many such triangles are possible? Find the length of their third side and area.

220. A circle is inscribed in an equilateral triangle of side a. Prove that the area of any square inscribed in this circle is $\frac{a^2}{6}$.

221. In a triangle ABC, AD is the altitude from A. Given $b > c$, $\angle C = 23^\circ$ and $AD = \frac{abc}{b^2-c^2}$, then find $\angle B$.

222. In a triangle ABC, $a : b : c = 4 : 5 : 6$, then find the ratio of the radius of the circumcircle to that of in-circle.

223. In a triangle ABC, $\angle B = \frac{\pi}{3}$, $\angle C = \frac{\pi}{4}$ and D divides BC internally in the ratio of $1 : 3$. Prove that $\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{1}{\sqrt{6}}$

224. In a triangle ABC, angle A is greater than angle B. If the measure of angle A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0, 0 < k < 1$, then find the measure of angle C.

225. ABC is a triangle such that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C)$, if A, B, C are in A.P. determine the value of A, B and C.

226. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the length of perpendicular drawn from the opposite vertex on the hypotenuse. Find the two angles.

227. In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2 + bx + c = 0 (a \neq 0)$, then prove that $a + b = c$.

228. In a triangle ABC, the medians to the side BC is of length $\frac{1}{\sqrt{1-6\sqrt{3}}}$ and it divides the angle A into angles of 30° and 45°. Find the lngth of side BC.

229. If A, B, C are the anngles of an acute-angled triangle, show that $\tan A + \tan B + \tan C \geq 3\sqrt{3}$.

230. In a triangle ABC, $\cos\frac{A}{2} = \frac{1}{2}\sqrt{\frac{b}{c} + \frac{c}{b}}$, show that the square describe on one side of the is equal to twice the rectangle contained by two other sides.

231. If in a triangle ABC, θ be the angle determined by the relation $\cos\theta = \frac{a-b}{c}$. Prove that $\cos\frac{A-B}{2} = \frac{(a+b)\sin\theta}{2\sqrt{ab}}$ and $\cos\frac{A+B}{2} = \frac{c\cos\theta}{2\sqrt{ab}}$.

232. If R be the circum-radius and r the in-radius of a triangle ABC, show that $R \geq 2r$.

233. If $\cos A = \tan B, \cos B = \tan C$ and $\cos C = \tan A$, show that $\sin A = \sin B = \sin C = 2\sin 18^\circ$, where A, B, C lie between 0 and π.

234. In a triangle ABC, prove that $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

235. In a triangle ABC, prove that $\tan^2 A + \tan^2 B + \tan^2 C \geq 9$

236. In a triangle ABC, prove that $\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2} \geq 6$

237. In a triangle ABC, prove that $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$

238. In a triangle ABC, prove that $\cos A \cos B \cos C \leq \frac{1}{8}$

239. Two circles of radii a and b cut each other at an angle θ. Prove that the length of the common chord is $\frac{2ab\sin\theta}{\sqrt{a^2+b^2 2ab\cos\theta}}$.

240. Three equal circles touch one another; find the radius of the circle which touches all the three circles.

241. In a triangle ABC, prove that $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos[rB - (n-r)A] = C^n$

242. In a triangle ABC, $\tan A + \tan B + \tan C = k$, then find the interval in which k should lie so that there exists one isosceles triangle ABC.

243. If Δ be the area and s, the semi-perimeter of a triangle, then prove that $\Delta \leq \frac{s^2}{3\sqrt{3}}$.

244. Show that the tirangle having sides $3x + 4y, 4x + 3y$ and $5x + 5y$ units where $x > 0, y > 0$ is obtuse-angled triangle.

245. Let ABC be a triangle having altitudes h_1, h_2, h_3 from the vertices A, B, C respectively and r be the in-radius, then prove that $\frac{h_1+r}{h_1-r} + \frac{h_2+r}{h_2-r} + \frac{h_3+r}{h_3-r} \geq 0$.

246. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle, whose area is Δ, and Δ_1, Δ_2 and Δ_3 be the corresponding areas for the escribed circles, prove that $\Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta$.

Chapter 9
Inverse Circular Functions

Inverse functions related to trigonometric ratios are called inverse trigonometric functions. The definition of different inverse trigonometric functions is given below:

If $\sin\theta = x$, then $\theta = \sin^{-1} x$, provided $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

If $\cos\theta = x$, then $\theta = \cos^{-1} x$, provided $-1 \leq x \leq 1$ and $0 \leq \theta \leq \pi$.

If $\tan\theta = x$, then $\theta = \tan^{-1} x$, provided $-\infty < x < \infty$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

If $\cot\theta = x$, then $\theta = \cot^{-1} x$, provided $-\infty < x < \infty$ and $0 < \theta < \pi$.

If $\sec\theta = x$, then $\theta = \sec^{-1} x$, provided $x \leq -1$ or $x \geq 1$ and $0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$.

If $\text{cosec}\theta = x$, then $\theta = \text{cosec}^{-1}x$, provided $x \leq -1$ or $x \geq 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0$.

Note: In the above definition, restrictions on θ are due to the consideration of principal values of inverse terms. If these restrictions are removed, the terms will represent inverse trigonometric relations and not functions.

Notations: I. $\text{Arc}\sin x$ denotes the sine inverse of x [General value]. $\arcsin x$ denotes the principal value of sine inverse of x.

II. $\sin^{-1} x$ denotes the principal value of sine inverse x. From the above notations three important results follow;

1. $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$ and θ is the principal value.
2. $\sin^{-1} x = \arcsin x, \cos^{-1} x = \arccos x$.
3. From the definition of the inverse functions, we know that if $y = f(x)$ is a function then for f^{-1} to be a function, f must be one-to-one and onto mapping.

When we consider $y = \text{Arc}\sin x$, for any $x \in [-1, 1]$ infinite number of values of y are obtained and hence it does not represent inverse functions. When $y = \arcsin x$ or $\sin^{-1} x$, corresponding to one value of $x \in [-1, 1]$, one value of y is obtained and hence it represents the inverse trigonometric function.

Hence, for inverse trigonometric functions, consideration of principal values is essential.

9.1 Principal Value

Numerically smallest angle is known as the principal value.

Since inverse trigonometric terms are in fact angles, definitions of principal value of inverse trigonometric term is the same as the definition of the principal values of angles.

Suppose we have to find the principal value of $\sin^{-1}\frac{1}{2}$. Let $\sin^{-1}\frac{1}{2}=\theta$, then $\sin\theta=\frac{1}{2}\Rightarrow\theta=\ldots,-\frac{11\pi}{6},-\frac{7\pi}{6},\frac{\pi}{6},\frac{5\pi}{6},\ldots$. Among all these angles $\frac{\pi}{6}$ is the numerically smalles angles satisfying $\sin\theta=\frac{1}{2}$ and hence it is the principal value.

9.2 Important Formulae

Theorem 16

1. $\sin\sin^{-1}x=x,-1\leq x\leq 1$
2. $\cos\cos^{-1}x=x,-1\leq x\leq 1$
3. $\tan\tan^{-1}x=x,-\infty<x\leq\infty$
4. $\cot\cot^{-1}x=x,-\infty<x\leq\infty$
5. $\sec\sec^{-1}x=x,x\leq -1 \ or\ x\geq 1$
6. $\operatorname{cosec}\operatorname{cosec}^{-1}x=x,x\leq -1 \ or\ x\geq 1$

Proof:

Let $\sin^{-1}x=\theta$ then $\sin\theta=x$. Putting the value of θ from first equation in second $\sin\sin^{-1}x=x$. Other formulae can be proved similarly. □

Theorem 17

1. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\ \forall-1\leq x\leq 1$
2. $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\ \forall x\in\mathbb{R}$
3. $\sec^{-1}x+\operatorname{cosec}^{-1}x=\frac{\pi}{2}\ \forall x\leq -1 \ or\ x\geq 1$

Proof:

Let $\sin^{-1}x=\theta$, then $\sin\theta=x\Rightarrow\cos\left(\frac{\pi}{2}-\theta\right)=x\Rightarrow\frac{\pi}{2}-\theta=\cos^{-1}x$

$\Rightarrow\cos^{-1}x+\theta=\frac{\pi}{2}\Rightarrow\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$.

Similarly other results can be proven. □

Theorem 18

1. $\sin^{-1}x=\operatorname{cosec}^{-1}\frac{1}{x},-1\leq x\leq 1$
2. $\operatorname{cosec}^{-1}\mathrm{x}=\sin^{-1}\frac{1}{x},x\leq -1 \ or\ x>1$

3. $\cos^{-1} x = \sec^{-1} \frac{1}{x}, -1 \leq x \leq 1$

4. $\sec^{-1} x = \cos^{-1} \frac{1}{x}, x \leq -1 \ or \ x \geq 1$

Proof:

Let $\sin^{-1} x = \theta$ then $\sin\theta = x \Rightarrow \mathrm{cosec}\theta = \frac{1}{\mathrm{x}}$

$\Rightarrow \theta = \mathrm{cosec}^{-1} \frac{1}{\mathrm{x}} \Rightarrow \sin^{-1} \mathrm{x} = \mathrm{cosec}^{-1} \frac{1}{\mathrm{x}}$

Other results can be proven similarly. □

Theorem 19

1. $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}, \ \forall\, 0 \leq x \leq 1$

2. $\sin^{-1} x = -\cos^{-1} \sqrt{1-x^2} \ \forall \ -1 \leq x < 0$

Proof:

Let $\sin^{-1} x = \theta$ then $\sin\theta = x$

$\Rightarrow \cos^2\theta = 1 - x^2 \Rightarrow \cos\theta = \pm\sqrt{1-x^2}$

Principal values of $\sin^{-1} x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

In this interval $\cos\theta$ is +ve.

$\Rightarrow \sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$

For $-1 \leq x < 0$ $\sin^{-1} x$ will be negative angle while $\cos^{-1} \sqrt{1-x^2}$ will be positive angle. Hence to balance that we need to use a negative sign for this. □

Theorem 20

1. $\sin^{-1}(-x) = -\sin^{-1} x$

2. $\cos^{-1}(-x) = \pi - \cos^{-1} x$

3. $\tan^{-1}(x) = -\tan^{-1} x$

4. $\cot^{-1} x = \pi - \cot^{-1} x$

Proof:

Let $\cos^{-1}(-x) = \theta$ then $\cos\theta = -x$

$-\cos\theta = x \Rightarrow \cos(\pi - \theta) = x$

$\therefore \theta = \pi - \cos^{-1} x$

Note: $\cos(\pi + \theta)$ is also equal to $-\cos\theta$ but this will make principal value greater than π.

Similarly other results can be proven. □

Theorem 21

1. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$ *where* $x, y > 0$ *and* $xy < 1$

2. $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \frac{x+y}{1-xy}$ *where* $x, y > 0$ *and* $xy > 1$

3. $\tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1} \frac{x+y}{1-xy}$ *where* $x, y, 0$ *and* $xy > 1$

4. $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$ *where* $xy > 1$

Proof:

Let $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$ then

$\tan \alpha = x$ and $\tan \beta = y$

$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{x+y}{1-xy}$

$\Rightarrow \alpha + \beta = \tan^{-1} \frac{x+y}{1=xy}$

$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

Case I. When $x, y > 0$ and $xy < 1$, $\tan^{-1} \frac{x+y}{1-xy} > 0$

therefore $\tan^{-1} \frac{x+y}{1-xy}$ will be a positive angle.

Case II. When $x, y > 0$ and $xy > 1$ $\tan^{-1} \frac{x+y}{1-xy}$ will be a negative angle.

$\therefore \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \frac{x+y}{1-xy}$

Case III. When $x, y < 0$ and $xy > 1$, $\tan^{-1} x + \tan^{-1} y$ will be a negative angle and $\tan^{-1} \frac{x+y}{1-xy}$ will be a positive angle.

To balance it we will need to add $-\pi$

$\therefore \tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1} \frac{x+y}{1-xy}$

Similarly other result can be proven. □

$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{x+y+z-xyz}{1-xy-yz-xz}$ can be proven similarly.

Theorem 22

1. $\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$ *if* $-1 \leq x, y \leq 1$ *and* $x^2 + y^2 \leq 1$ *or if* $xy < 0$ *and* $x^2 + y^2 > 1$

2. $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ *if* $-1 \le x, y \le 1$ *and* $x^2 + y^2 \le 1$ *or if* $xy > 0$ *and* $x^2 + y^2 > 1$

Proof:

Let $\sin^{-1} x = \alpha$ and $\sin^{-1} y = \beta$ then $\sin\alpha = x, \sin\beta = y$.

Now $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\alpha + \beta = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

Similarly we can prove that $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ □

Theorem 23

1. $2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2}$, *where* $|x| < 1$
2. $2\tan^{-1} x = \cos^{-1}\frac{1-x^2}{1+x^2}$, *where* $x \ge 0$
3. $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$, *where* $|x| < 1$

Proof:

1. Let $\tan^{-1} x = \theta$ then $\tan\theta = x$

   $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2 x\theta} = \frac{2x}{1+x^2}$

   $\Rightarrow 2\theta = \sin^{-1}\frac{2x}{1+x^2} \Rightarrow 2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2}$

   Here, $-\frac{\pi}{2} \le \sin^{-1} \le \frac{\pi}{2}$

   $\Rightarrow -\frac{\pi}{2} \le 2\tan^{-1} x \le \frac{\pi}{2}$

   $\Rightarrow -\frac{\pi}{4} \le \tan^{-1} x \le \frac{\pi}{4}$

   $\Rightarrow -1 \le x \le 1 \Rightarrow |x| < 1$

2. $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-x^2}{1+x^2}$

   $\Rightarrow 2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

   $\Rightarrow 2\tan^{-1} x = \cos^{-1}\frac{1-x^2}{1+x^2}$

   For $x \ge 0$ both sides will be balanced.

For $x < 0, 2\tan^{-1} x$ will represent a negative angle where R.H.S. will always lie between 0 and π. Hence two sides cannot be equal.

3. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2x}{1-x^2} \Rightarrow 2\theta = \tan^{-1}\frac{2x}{1-x^2}$

   $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$ which holds good for $|x| < 1$

Theorem 24

1. $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1-x^2}]$ *if* $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

2. $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$ *where* $0 \leq x \leq 1$

These can be proven like $\sin^{-1} x + \sin^{-1} y$

Theorem 25

1. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ *where* $-\frac{1}{2} \leq x \leq \frac{1}{2}$

2. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ *where* $\frac{1}{2} \leq x \leq 1$

3. $3\tan^{-1} x = \tan^{-1}\frac{3x-x^3}{1-3x^2}$ *where* $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

These can be proven like previous proof.

9.3 Graph of Important Inverse Trigonometric Functions

1. $y = \sin^{-1} x, -1 \leq x \leq 1$

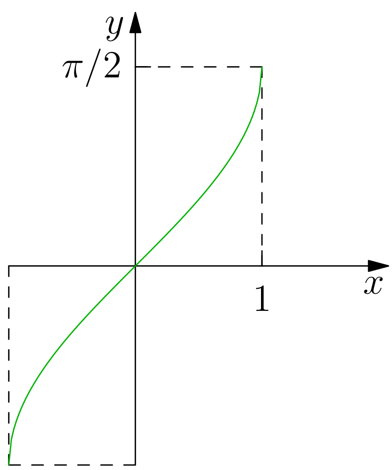


Figure 9.1

From this graph we observer following:

1. Domain is $-1 \leq x \leq 1$
2. Range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
3. $\because \sin^{-1} x = -\sin^{-1} x \therefore y = \sin^{-1} x$ is an odd function.
4. It is a non-periodic function
5. It passes through origin i.e. when $x = 0, y = 0$

2. $y = \cos^{-1} x, -1 \leq x \leq 1$

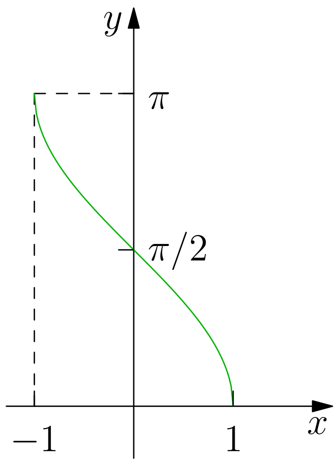


Figure 9.2

Follwing points can be observed from the graph:

1. Domain is $-1 \leq x \leq 1$
2. Range is $0 \leq x \leq \pi$
3. $\because \cos^{-1}(-x) = \pi - \cos^{-1} x$

   $\Rightarrow y = \cos^{-1} x$ is neither odd nor even.
4. It is a non-periodic function

3. $y = \tan^{-1} x, -\infty < x < \infty$

From the graph follwing points can be observed:

1. Domain is $-\infty < x < \infty$
2. Range is $-\frac{\pi}{2} < x < \frac{\pi}{2}$

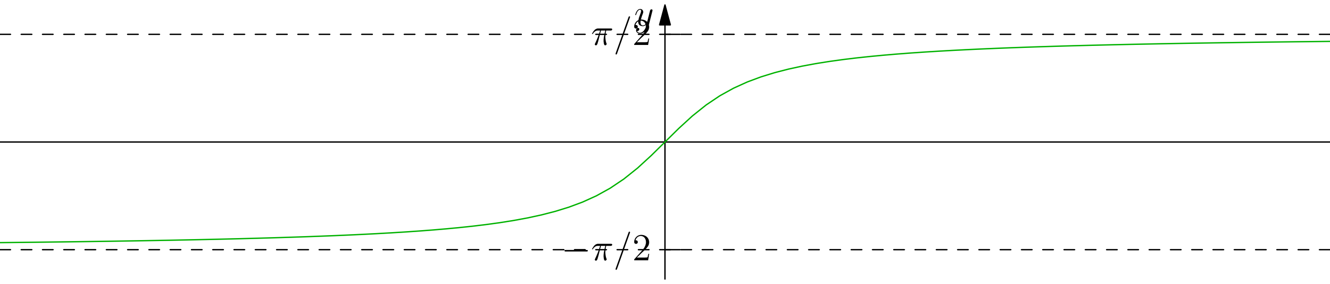


Figure 9.3

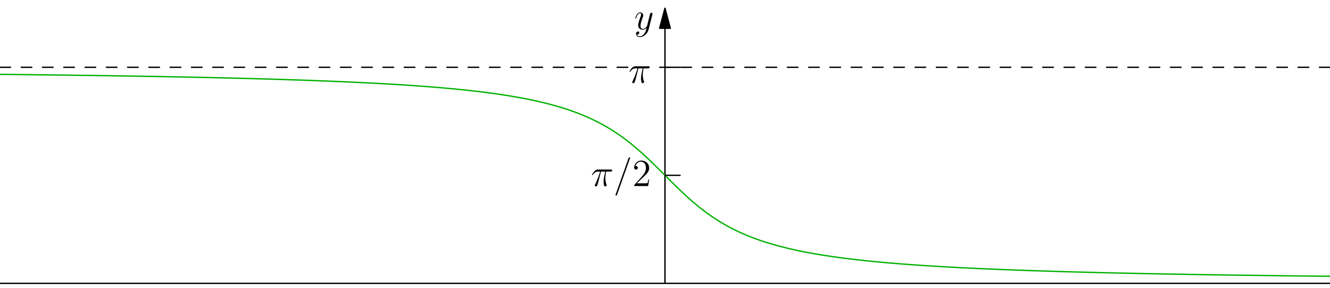


Figure 9.4

3. $y = \tan^{-1} x$ is an odd function

4. It is a non-periodic function.

5. It passes through origin.

4. $y = \cot^{-1} x, -\infty < x < \infty$

From the graph follwing points can be observed:

1. Domain is $-\infty < x < \infty$

2. Range is $0 < y < \pi$

3. The function is neither odd nor even.

4. It is a non-periodic function

9.4 Problems

Evaluate the following:

1. $\tan^{-1}(-1)$

2. $\cot^{-1}(-1)$

3. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Find the value of the following:

4. $\sin\left[\frac{\pi}{3} - \sin^{-1}\frac{-1}{2}\right]$

5. $\sin\left[\cos^{-1}\frac{-1}{2}\right]$

6. $\sin\left[\tan^{-1}(-\sqrt{3}) + \cos^{-1}\frac{-\sqrt{3}}{2}\right]$

7. Evaluate $\tan\left[\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right]$

8. Find the angle $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Find the value of the following:

9. $\sin^{-1}\frac{\sqrt{3}}{2}$

10. $\tan^{-1}\frac{-1}{\sqrt{3}}$

11. $\cot^{-1}(-\sqrt{3})$

12. $\cot^{-1}\cot\frac{5\pi}{4}$

13. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

14. $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}$

15. $\cos\left[\tan^{-1}\left(\frac{3}{4}\right)\right]$

16. $\cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

17. Prove that $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

18. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

19. Prove that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

20. Prove that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$

21. Prove that $\cot^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \frac{\pi}{4}$

22. Prove that $4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$

23. Prove that $\tan^{-1}x = 2\tan^{-1}[\operatorname{cosec}\tan^{-1}\mathrm{x} - \tan\cot^{-1}\mathrm{x}]$

24. Prove that $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right] = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$ for $0 < b \le a$, and $x \ge 0$.

25. Prove that $\tan^{-1}\frac{x-y}{1+xy} + \tan^{-1}\frac{y-z}{1+yz} + \tan^{-1}\frac{z-x}{1+zx} = \tan^{-1}\left(\frac{x^2-y^2}{1+x^2y^2}\right) + \tan^{-1}\left(\frac{y^2-z^2}{1+y^2z^2}\right) + \tan^{-1}\left(\frac{z^2-x^2}{1+z^2x^2}\right)$

26. Prove that $\sin\cot^{-1}\tan\cos^{-1}x = x$

27. Prove that $\tan^{-1}\left(\frac{1}{2}\tan 2x\right) + \tan^{-1}(\cot x) + \tan^{-1}(\cot^3 x) = 0$ if $\frac{\pi}{4} < x < \frac{\pi}{2}, = \pi$ if $0 < x < \pi$

28. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$

29. Prove that $\tan^{-1}\frac{2a-b}{\sqrt{3}b} + \tan^{-1}\frac{2b-a}{\sqrt{3}a} = \frac{\pi}{3}$

30. Prove that $\tan^{-1}\frac{2}{5} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{12} = \frac{\pi}{4}$

31. Prove that $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{32}{43}$

32. Prove that $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi = 2\left(\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$

33. Prove that $\tan^{-1}x + \cot^{-1}y = \tan^{-1}\frac{xy+1}{y-x}$

34. Prove that $\tan^{-1}\frac{1}{x+y} + \tan^{-1}\frac{y}{x^2+xy+1} = \cot^{-1}x$

35. Prove that $2\cot^{-1}5 + \cot^{-1}7 + 2\cot^{-1}8 = \pi/4$

36. Prove that $\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1}\frac{c-a}{1+ca} = 0$

37. Prove that $\tan^{-1}\frac{a^3-b^3}{1+a^3b^3} + \tan^{-1}\frac{b^3-c^3}{1+b^3c^3} + \tan^{-1}\frac{c^3-a^3}{1+c^3a^3} = 0$

38. Prove that $\cot^{-1}\frac{xy+1}{y-x} + \cot^{-1}\frac{yz+1}{z-y} + \cot^{-1}z = \tan^{-1}\frac{1}{x}$

39. Prove that $\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right) = 2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)$

40. Prove that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$

41. Prove that $\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} + \cos^{-1}\frac{63}{65} = \frac{\pi}{2}$

42. Prove that $\sin^{-1}x + \sin^{-1}y = \cos^{-1}\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right)$ where $x, y \in [0, 1]$

43. Prove that $4\left(\sin^{-1}\frac{1}{\sqrt{10}} + \cos^{-1}\frac{2}{\sqrt{5}}\right) = \pi$

44. Prove that $\cos(2\sin^{-1}x) = 1 - 2x^2$

45. Prove that $\frac{1}{2}\cos^{-1}x = \sin^{-1}\sqrt{\frac{1-x}{2}} = \cos^{-1}\sqrt{\frac{1+x}{2}} = \tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$

46. Prove that $\sin^{-1} x + \cos^{-1} y = \tan^{-1} \frac{xy+\sqrt{(1-x^2)(1-y^2)}}{y\sqrt{1-x^2}-x\sqrt{1-y^2}}$

47. Prove that $\tan^{-1} x + \tan^{-1} y = \frac{1}{2} \sin^{-1} \frac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)}$

48. Prove that $2\tan^{-1}(\operatorname{cosec} \tan^{-1} \mathrm{x} - \tan \cot^{-1} \mathrm{x}) = \tan^{-1} \mathrm{x}$

49. Prove that $\cos \tan^{-1} \sin \cot^{-1} x = \sqrt{\frac{x^2+1}{x^2+2}}$

50. In any $\triangle ABC$ if $A = \tan^{-1} 2$ and $B = \tan^{-1} 3$, prove that $C = \frac{\pi}{4}$

51. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$

52. If $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \theta$, prove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

53. If $r = x + y + z$ then prove that $\tan^{-1} \sqrt{\frac{xr}{yz}} + \tan^{-1} \sqrt{\frac{yr}{xz}} + \tan^{-1} \sqrt{\frac{zr}{xy}} = \pi$

54. If $u = \cot^{-1} \sqrt{\cos 2\theta} - \tan^{-1} \sqrt{\cos 2\theta}$ then prove that $\sin u = \tan^2 \theta$

55. Solve $\cos^{-1} x\sqrt{3} + \cos^{-1} x = \frac{\pi}{2}$

56. Solve $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$

57. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, prove that $xy + yz + zx = 1$

58. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, prove that $x + y + z = xyz$

59. If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, prove that $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$

60. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

61. Establish the relationship between $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in A.P. and if further x, y, z are also in A.P. then prove that $x = y = z$.

62. Solve for x, $\cot^{-1} x + \sin^{-1} \frac{1}{\sqrt{5}} = \frac{\pi}{4}$

63. Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

64. Solve $\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \frac{\pi}{3}$

65. Solve $\tan^{-1} \frac{1}{2} = \cot^{-1} x + \tan^{-1} \frac{1}{7}$

66. Solve $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$

67. Solve $\tan^{-1} \frac{x+1}{x-1} + \tan^{-1} \frac{x-1}{x} = \pi + \tan^{-1}(-7)$

68. Solve $\cot^{-1}(a-1)=\cot^{-1}x+\cot^{-1}(a^2-x+1)$

69. Solve $\sin^{-1}\frac{2\alpha}{1+\alpha^2}+\sin^{-1}\frac{2\beta}{1+\beta^2}=2\tan^{-1}x$

70. Solve $\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$

71. Solve $\sin^{-1}\frac{2a}{1+a^2}+\cos^{-1}\frac{1-b^2}{1+b^2}=2\tan^{-1}x$

72. Solve $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$

73. Solve $\tan^{-1}ax+\frac{1}{2}\sec^{-1}bx=\frac{\pi}{4}$

74. Solve $\tan(\cos^{-1}x)=\sin(\tan^{-1}2)$

75. Solve $\tan\left(\sec^{-1}\frac{1}{x}\right)=\sin\cos^{-1}\frac{1}{\sqrt{5}}$

76. Find the values of x and y satisfying $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$ and $\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$

77. Find the angle $\sin^{-1}(\sin 10)$

78. Using principal values, express the following as a single angle $3\tan^{-1}\frac{1}{2}+2\tan^{-1}\frac{1}{5}+\sin^{-1}\frac{142}{65\sqrt{5}}$

79. Find the value of $2\cos^{-1}x+\sin^{-1}x$ at $x=\frac{1}{5}$ where $0\leq\cos^{-1}x\leq\pi$ and $-\frac{\pi}{2}\leq\sin^{-1}x\leq\frac{\pi}{2}$.

80. Show that $\frac{1}{2}\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{1}{2}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{4}{5}$

81. Find the greater angle between $2\tan^{-1}(2\sqrt{2}-1)$ and $3\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{3}{5}$

82. Prove that $\tan^{-1}\left(\frac{a_1x-y}{x+a_1y}\right)+\tan-1\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\tan^{-1}\left(\frac{a3-a_2}{1+a_3a_2}\right)+...+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)+\tan^{-1}\frac{1}{a_n}=\tan^{-1}\frac{x}{y}$

83. Find the sum $\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+...+$ to ∞

84. Show that the function $y=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ is constant for $x\geq 1$. Find the value of this constant.

85. Prove the relations $\cos^{-1}x_0=\frac{\sqrt{1-x_0^2}}{x_1x_2x_3\ldots\text{ to }\infty}$ where the successive quantities x_r are connected by the relation $x_{r+1}=\sqrt{\frac{1+x_r}{2}}$ where $0\leq\cos^{-1}x_0\leq\pi$.

86. If a,b are positive quantities and if $a_1=\frac{a+b}{2}$, $b_1=\sqrt{a_1b}$, $a_2=\frac{a_1+b_1}{2}$, $b_2=\sqrt{a_2b_1}$ and so on then show that $\lim_{n\to\infty}a_n\lim_{n\to\infty}b_n=\frac{\sqrt{b^2-a^2}}{\cos^{-1}\frac{a}{b}}$

87. Using Mathematical Induction prove that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+...+\tan^{-1}\frac{1}{n^2+n+1}=\tan^{-1}\frac{n}{n+2}$

88. If x_1, x_2, x_3, x_4 are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos\beta - \sin\beta = 0$ then prove that $\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta$

89. Find theh value of $\cot^{-1}\left(\cot\frac{5\pi}{4}\right)$

90. Find the value of $\sin^{-1}(\sin 5)$

91. Find the value of $\cos^{-1}\cos\frac{5\pi}{4}$

92. Find the value of $\cos^{-1}(\cos 10)$

93. Evaluate $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\tan^{-1} 2\sqrt{2}$

94. Evaluate $\cot[\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18]$

95. Prove that $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} + \cot^{-1}\frac{56}{33} = \frac{\pi}{2}$

96. Prove that $2\cot^{-1} 5 + \cot^{-1} 7 + 2\cot^{-1} 8 = \frac{\pi}{4}$

97. Prove that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = 2\left(\tan^{-1} 1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$.

98. If $A = \tan^{-1}\frac{1}{7}$ and $B = \tan^{-1}\frac{1}{3}$ then prove that $\cos 2A = \sin 4B$.

99. Find the sum $\tan^{-1}\frac{x}{1+1.2x^2} + \tan^{-1}\frac{x}{1+2.3x^2} + \ldots + \tan^{-1}\frac{1}{1+n(n+1)x^2}$, $x > 0$.

100. Find the sum $\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \ldots + \tan^{-1}\frac{d}{1+a_na_{n+1}}$ if $a_1, a_2, \ldots, a_{n+1}$ form an arithmetic progression with a common difference of d and $d > 0, a_i > 0$ for $i = 1, 2, 3, \ldots, n+1$.

101. For what value of x, the equality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds.

102. If $\tan^{-1} y = 5\tan^{-1} x$, express y as an algebraic function of x and hence show that 18° is a root of $5u^4 - 10u^2 + 1 = 0$.

103. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $x + y + z = \frac{3}{2}$, then prove that $x = y = z$.

104. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$.

105. Prove that $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) = (\alpha+\beta)(\alpha^2+\beta^2)$.

106. Prove that $2\tan^{-1}\left[\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right] = \tan^{-1}\left[\frac{\sin\alpha\cos\beta}{\sin\beta + \cos\alpha}\right]$.

107. Prove that $\tan^{-1}\left[\frac{1}{2}\cos 2\alpha\sec 2\beta + \frac{1}{2}\cos 2\beta\sec 2\alpha\right] = \tan^{-1}[\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)] + \frac{\pi}{4}$.

108. Express $\cot^{-1}\left(\frac{y}{\sqrt{1-x^2-y^2}}\right) = 2\tan^{-1}\sqrt{\frac{3-4x^2}{4x^2}} - \tan^{-1}\sqrt{\frac{3-4x^2}{x^2}}$ as a rational integral equation in x and y.

109. If $\frac{m\tan(\alpha-\theta)}{\cos^2\theta} = \frac{n\tan\theta}{\cos^2(\alpha-\theta)}$ then prove that $\theta = \frac{1}{2}\left[\alpha - \tan^{-1}\left(\frac{n-m}{n+m}\right)\tan\alpha\right]$.

110. If $\sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b} = \sin^{-1}\frac{c^2}{ab}$ then prove that $b^2x^2 + 2xy\sqrt{a^2b^2 - c^4} = c^4 - a^2y^2$.

111. Prove that $\tan^{-1}t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2}$, if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

112. Prove that $\cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}}$ if $a > x > b$ or $a < x < b$.

113. Find all values of p and q such that $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$.

114. Find all positive integral solution of the equation $\tan^{-1}x + \cot^{-1}y = \tan^{-1}3$.

115. Solve $\sin^{-1}\frac{ax}{c} + \sin^{-1}\frac{bx}{c} = \sin^{-1}x$ where $a^2 + b^2 = c^2, c \neq 0$.

116. Convert the trigonometric function $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]$ into an algebraic function $f(x)$. Then from the algebraic function find all the values of x for which $f(x)$ is zero. Express the value of x in the form of $a \pm \sqrt{b}$ where a and b are rational numbers.

117. Solve the equation $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$.

118. Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$.

119. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} + \dots\right) = \frac{\pi}{2} \sim \forall \sim 0 < |x| < \sqrt{2}$ then find x.

120. Find the number of real solutions for $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$.

121. Solve $\sin^{-1}\frac{3x}{5} + \cos^{-1}\frac{4x}{5} = \sin^{-1}x$.

122. Solve $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$.

123. If k be a positive integer, show that the equation $\tan^{-1}x + \tan^{-1}y = \tan^{-1}k$ has no positive integral solution.

124. Solve $\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7)$.

125. Solve $\tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{a^2-x+1}$.

126. Solve $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$.

127. If $\theta = \tan^{-1}\frac{x\sqrt{3}}{2k-x}$ and $\phi = \tan^{-1}\frac{2x-k}{k\sqrt{3}}$, show that one value of $\theta - \phi$ is $\pi/6$.

128. Find all positive integral solutions of the equation $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$.

129. Solve the equation $2\cos^{-1}x = \sin^{-1}2x\sqrt{1-x^2}$

130. Solve $\sin^{-1}\frac{x}{\sqrt{1+x^2}}-\sin^{-1}\frac{1}{\sqrt{1+x^2}}=\sin^{-1}\frac{1+x}{1+x^2}$

131. Show that the function $y=2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right]-\cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$ is a constant for $0<b\leq a$, find the value of this constant for $x\geq 0$.

132. Find the sum $\sum_{i=1}^{n}\tan^{-1}\frac{2i}{2+i^2+i^4}$.

133. Find the sum of infinite terms of the series $\cot^{-1}\left(1^2+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^3+\frac{3}{4}\right)+\ldots$

134. Solve for x the equation $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$

135. Show that the greatest and the least values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ are $\frac{7\pi^3}{8}$ and $\frac{\pi^2}{32}$ respectively.

136. Obtain the integral values of p for which the following system of equations possesses real solution $\cos^{-1}x+(\sin^{-1}y)^2=\frac{p\pi^2}{4}$ and $(\cos^{-1}x)(\sin^{-1}y)^2=\frac{\pi^2}{16}$.

137. If $\tan^{-1}x,\tan^{-1}y,\tan^{-1}z$ be in A.P., find the algebraic relation between x,y and z. If x,y,z be in A.P. prove that $x=y=z$.

138. Show that for $x>0,\tan^{-1}\frac{x}{1+1.2x^2}+\tan^{-1}\frac{x}{1+2.3x^2}+\ldots+\tan^{-1}\frac{x}{1+n(n+1)x^2}=\tan^{-1}\frac{nx}{1+(n+1)x^2}$

Chapter 10
Trigonometrical Equations

An equation involving one or more trigonometrical ratios of unknown angle is called trigonometrical equation.

Ex. $\cos^2 x - 4\sin x = 1$

A trigonometrical identity is satisfied for every value of the unknown angle whereas trigonometrical equation is satisfied for only some values of unknown angle. For example, $1 - \cos^2 x = \sin^2 x$ is a trigonometrical identity because it is satisfied for every value of x.

10.1 Solution of a Trigonometrical Equation

A value of the unknown angle which satisfies the given trigonometrical equation is called a solution or root of the equation.

For example, $2\sin\theta = 1 \Rightarrow \theta = 30^\circ, 150^\circ$ which are two solutions between 0 and 2π.

10.2 General Solution

Some trigonometrical functions are periodic functions, therefore, solutions of trigonometrical equations can be generalized with the help of periodicirty of trigonometrical functions. The solution consisting of all possible solutions of a trigonometrical equation is called its general solution.

For example, $\sin\theta = 0$ has a genral solution which is $n\pi$ where $n \in I$.

Similarly, for $\cos\theta = 0$, the general solution is $(2n+1)\frac{\pi}{2}$, where $n \in I$ and for $\tan\theta = 0$ the solution is again $n\pi$.

10.2.1 General Solution of $\sin\theta = \sin\alpha$

Given, $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

$\Rightarrow 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$

Case I: $\cos\frac{\theta+\alpha}{2} = 0$

$\Rightarrow \theta + \alpha = (2m+1)\pi, m \in I$

Case II: $\sin\frac{\theta-\alpha}{2} = 0$

$\Rightarrow \theta - \alpha = 2m\pi \Rightarrow \theta = 2m\pi + \alpha$

Thus, $\theta = n\pi + (-1)^n\alpha, n \in I$

10.2.2 General Solution of $\cos\theta = \cos\alpha$

Given, $\cos\theta = \cos\alpha \Rightarrow \cos\theta - \cos\alpha = 0$

$\Rightarrow 2\sin\frac{\alpha+\theta}{2}\sin\frac{\theta-\alpha}{2} = 0$

Case I: $\sin\frac{\alpha+\theta}{2} = 0$

$\alpha + \theta = 2n\pi \Rightarrow \theta = 2n\pi - \alpha$

Case II: $\sin\frac{\theta-\alpha}{2} = 0 \Rightarrow \theta = 2n\pi + \alpha$

Thus, $\theta = 2n\pi \pm \alpha$

10.2.3 General Solution of $\tan\theta = \tan\alpha$

Given $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin(\theta - \alpha) = 0 \therefore \theta - \alpha = n\pi$

$\theta = n\pi + \alpha$

10.3 Principal Value

For any equation having multiple solutions, the solution having least numerical value is known as *principal value.*

Example: Let $\sin\theta = \frac{1}{2}$ then $\theta = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6, \ldots, -7\pi/6, -11\pi/6, \ldots$

As $\pi/6$ is the least numerical value so it is the principal value in this case.

10.3.1 Method for Finding Principal Value

For this case we consider $\sin\theta = -\frac{1}{2}$. Since it is negative, θ will be in third or fourth quadrant. We can approach this either using clockwise direction or annticlockwise direction. If we take anticlockwise direction principal value will be greater than π and in case of clockwise direction it will be less than π. For principal value, we have to take numerically smallest angle.

So for principal value:

1. If the angle is in 1st or 2nd quadrant we must select anticlockwise direction i.e. principal value will be positive. If the angle is in 3rd or 4th quadrant we must select clockwise direction i.e. principal value will be negative.
2. Principal value is always numerically smaller than π
3. Principal values always lies in the first circle i.e. first rotation.

10.4 Tips for Finding Complete Solution

1. There should be no extraneous root.
2. There should be no less root.

3. Squaring should be avoided as far as possible. If it is done then check for extraneous roots.
4. Never cancel equal terms containing *unknown* on two sides which are in product. It may cause root loss.
5. The answer should not contain such values of root which may make any of the terms undefined.
6. Domain should not change. If it changes, necessary correction must be made.
7. Check that denominator is not zero at any stage while solving equations.

10.5 Problems

Find the most general values of θ satisfying the equations:

1. $\sin\theta = -1$
2. $\cos\theta = -\frac{1}{2}$
3. $\tan\theta = \sqrt{3}$
4. $\sec\theta = -\sqrt{2}$

Solve the equations:

5. $\sin 9\theta = \sin\theta$
6. $\sin 5x = \cos 2x$
7. $\sin 3x = \sin x$
8. $\sin 3x = \cos 2x$
9. $\sin ax + \cos bx = 0$
10. $\tan x \tan 4x = 1$
11. $\cos\theta = \sin 105^\circ + \cos 105^\circ$

Solve the following:

12. $7\cos^2\theta + 3\sin^2\theta = 4$
13. $3\tan(\theta - 15^\circ) = \tan(\theta + 15^\circ)$
14. $\tan x + \cot x = 2$
15. $\sin^2\theta = \sin^2\alpha$
16. $\tan^2 x + \cot^2 x = 2$
17. $\tan^2 x = 3\text{cosec}^2\text{x} - 1$

18. $2\sin^2 x + \sin^2 2x = 2$

19. $7\cos^2 x + 3\sin^2 x = 4$

20. $2\cos 2x + \sqrt{2\sin x} = 2$

21. $8\tan^2 \frac{x}{2} = 1 + \sec x$

22. $\cos x \cos 2x \cos 3x = \frac{1}{4}$

23. $\tan x + \tan 2x + \tan 3x = 0$

24. $\cot x - \tan x - \cos x + \sin x = 0$

25. $2\sin^2 x - 5\sin x \cos x - 8\cos^2 x = -2$

26. $(1 - \tan x)(1 + \sin 2x) = 1 + \tan x$

27. Solve for x,$(-\pi \leq x \leq \pi)$, the equation $2(\cos x + \cos 2x) + \sin 2x(1 + 2\cos x) = 2\sin x$

28. Find all the solutions of the equation $4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

29. $2 + 7\tan^2 x = 3.25\sec^2 x$

30. Find all the values of x for which $\cos 2x + \cos 4x = 2\cos x$

31. $3\tan x + \cot x = 5\text{cosecx}$

32. Find the value of x between 0 and 2π for which $2\sin^2 x = 3\cos x$

33. Find the solution of $\sin^2 x - \cos x = \frac{1}{4}$ in the interbal 0 to 2π.

34. Solve $3\tan^2 x - 2\sin x = 0$

35. Find all values of x satisfying the equation $\sin x + \sin 5x = \sin 3x$ between 0 and π.

36. $\sin 6x = \sin 4x - \sin 2x$

37. $\cos 6x + \cos 4x + \cos 2x + 1 = 0$

38. $\cos x + \cos 2x + \cos 3x = 0$

39. Find the values of x between 0 and 2π, for which $\cos 3x + \cos 2x = \sin \frac{3x}{2} + \sin \frac{x}{2}$

40. $\tan x + \tan 2x + \tan 3x = \tan x . \tan 2x . \tan 3x$

41. $\tan x + \tan 2x + \tan x \tan 2x = 1$

42. $\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$

43. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

44. $\cos 6x + \cos 4x = \sin 3x + \sin x$

45. $\sec 4x - \sec 2x = 2$

46. $\cos 2x = (\sqrt{2}+1)\left(\cos x - \frac{1}{\sqrt{2}}\right)$

47. Find all the angles between $-pi$ and π for which $5\cos 2x + 2\cos^2\frac{x}{2} + 1 = 0$

48. $\cot x - \tan x = \sec x$

49. $1 + \sec x = \cot^2\frac{x}{2}$

50. $\cos 3x \cos^3 x + \sin 3x \sin^3 x = 0$

51. $\sin^3 x + \sin x \cos x + \cos^3 x = 1$

52. Find all the value of x between 0 and $\frac{\pi}{2}$, for which $\sin 7x + \sin 4x + \sin x = 0$

53. $\sin x + \sqrt{3}\cos x = \sqrt{2}$

54. Find the values of x for which $27^{\cos 2x}.81^{\sin 2x}$ is minimum. Also, find this minimum value.

55. If $32\tan^8 x = 2\cos^2 y - 3\cos y$ and $3\cos 2x = 1$, then find the general value of y.

56. Find all the values of x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $(1-\tan x)(1+\tan x)sec^2 x + 2^{\tan^2 x} = 0$

57. Solve the equation $e^{\cos x} = e^{-\cos x} + 4$.

58. If $(1+\tan x)(1+\tan y) = 2$. Find all the values of $x+y$.

59. If $\tan(\cot x) = \cot(\tan x)$, prove that $\sin 2x = \frac{4}{(2n+1)\pi}$

60. If x and y are two distinct roots of the equation $a\tan z + b\sec z = c$. Prove that $\tan(x+y) = \frac{2ac}{a^2-c^2}$

61. If $\sin(\pi\cos x) = \cos(\pi\sin x)$, prove that 1. $\cos\left(x \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$ 2. $\sin 2x = -\frac{3}{4}$

62. Determine the smallest positive values of x for which $\tan(x+100^\circ) = \tan(x+50^\circ).\tan x.\tan(x-50^\circ)$

63. Find the general value of x for which $\tan^2 x + \sec 2x = 1$.

64. Solve the equation $\sec x - \text{cosecx} = \frac{4}{3}$

65. Find solutions $x \in [0, 2\pi]$ of equation $\sin 2x - 12(\sin x - \cos x) + 12 = 0$.

66. Find the smallest positive number rp for which the equation $\cos(p\sin x) = \sin(p\cos x)$ has a solution for $x \in [0, 2\pi]$.

67. Solve $\cos x + \sqrt{3}\sin x = 2\cos 2x$

68. Solve $\tan x + \sec x = \sqrt{3}$ for $x \in [0, 2\pi]$.

69. Solve $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$

70. Solve the equation $(2+\sqrt{3})\cos x = 1 - \sin x$

71. Solve the equation $\tan\left(\frac{\pi}{2}\sin x\right) = \cot\left(\frac{\pi}{2}\cos x\right)$

72. Solve $8\cos x\cos 2x\cos 4x = \frac{\sin 6x}{\sin x}$

73. Solve $3 - 2\cos x - 4\sin x - \cos 2x + \sin 2x = 0$

74. Solve $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

75. Solve $\sin^2 x\tan x + \cos^2 x\cot x - \sin 2x = 1 + \tan x + \cot x$

76. Find the most general value of x which satisfies both the equations $\sin x = -\frac{1}{2}$ and $\tan x = \frac{1}{\sqrt{3}}$

77. If $\tan(x-y) = 1$ and $\sec(x+y) = \frac{2}{\sqrt{3}}$, find the smallest positive values of x and y and their most general value.

78. Find the points of intersection of the curves $y = \cos x$ and $y = \sin 3x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

79. Find all values of $x \in [0, 2\pi]$ such that $r\sin x = \sqrt{3}$ and $r + 4\sin x = 2(\sqrt{3}+1)$

80. Find the smallest positive values of x and y satisfying $x - y = \frac{\pi}{4}$ and $\cot x + \cot y = 2$.

81. Find the general values of x and y such that $5\sin x\cos y = 1$ and $4\tan x = \tan y$.

82. Find all values of x lying between 0 and 2π, such that $r\sin x = 3$ and $r = 4(1+\sin x)$

83. If $\sin x = \sin y$ and $\cos x = \cos y$ then prove that either $x = y$ or $x - y = 2n\pi$, where $n \in I$.

84. If $\cos(x-y) = \frac{1}{2}$ and $\sin(x+y) = \frac{1}{2}$ find the smallest positive values of x and y and also their most general values.

85. Find the points of intersection of the curves $y = \cos 2x$ and $y = \sin x$ for, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

86. Find the most general value of x which satisfies the equations $\cos x = \frac{1}{\sqrt{2}}$ and $\tan x = -1$.

87. Find the most general value of x which satisfies the equations $\tan x = \sqrt{3}$ and $\mathrm{cosecx} = -\frac{2}{\sqrt{3}}$

88. If x and y be two distinct values of z lying between 0 and 2π, satisfying the equation $3\cos z + 4\sin z = 2$, find the value of $\sin(x+y)$.

89. Show that the equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2}$ for $0 < x \le \frac{\pi}{2}$ has no real solution.

90. Find the real value of x such that $y = \frac{3+2i\sin x}{1-2i\sin x}$ is either real or purely imaginary.

91. Determine for which values of a the equation $a^2 - 2a + \sec^2 \pi(a + x) = 0$ has solutions and find them.

92. Find the values of x in $(-\pi, \pi)$ which satisfy the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\dots \text{ to } \infty} = 4^3$

93. Solve $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$.

94. Solve $3^{\sin 2x+2\cos^2 x} + 3^{1-\sin 2x+2\sin^2 x} = 28$.

95. If $A = (x/2\cos^2 x + \sin x \leq 2)$ and $B = \left(x/\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}\right)$ find $A \cap B$

96. Solve $\sin x + \cos x = 1 + \sin x \cos x$.

97. Solve $\sin 6x + \cos 4x + 2 = 0$.

98. Prove that the equation $\sin 2x + \sin 3x + \dots + \sin nx = n - 1$ has n solution for any arbitrary integer $n > 2$.

99. Solve $\cos^7 x + \sin^4 x = 1$.

100. Find the number of solutions of the equation $\sin x + 2\sin 2x = 3 + \sin 3x$ in the interval $0 \leq x \leq \pi$.

101. For what value of k the equation $\sin x + \cos(k + x) + \cos(k - x) = 2$ has real solutions.

102. Solve for x and y, the equation $x\cos^3 y + 3x\cos y.\sin^2 y = 14$ and $x\sin^3 y + 3x\cos^2 y\sin y = 13$

103. Find all the values of α for which the equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is valid.

104. Solve $\tan\left(x + \frac{\pi}{4}\right) = 2\cot x - 1$.

105. If x, y be two angles both satisfying the equation $a\cos 2z + b\sin 2z = c$, prove that $\cos^2 x + \cos^2 y = \frac{a^2+ac+b^2}{a^2+b^2}$

106. If x_1, x_2, x_3, x_4 be roots of the equation $\sin(x + y) = k\sin 2x$, no two of which differ by a multiple of 2π, prove that $x_1 + x_2 + x_3 + x_4 = (2n + 1)\pi$.

107. Show that the equation $\sec x +$ cosecx = c has two roots between 0 and π if $c^2 < 8$ and four roots if $c^2 > 8$.

108. Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + y\sin\alpha + z\cos\alpha = 0, x + y\cos\alpha + z\sin\alpha = 0, -x + y\sin\alpha - z\cos\alpha = 0$ has non-trivial solution. For $\lambda = 1$, find all the values of α.

109. Find the values of x and $y, 0 < x, y < \frac{\pi}{2}$, satisfying the equation $\cos x\cos y\cos(x + y) = -\frac{1}{8}$

110. Find the number of distinct real roots of $\begin{bmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x\& & \cos x \\ \cos x & \cos x & \sin x \end{bmatrix} = 0$ in the interval $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

111. Find the number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$.

112. Find the range of y such that the following equation in x, $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 \leq x \leq 2\pi$.

113. Solve $\sum_{r=1}^{n} \sin(rx)\sin(r^2 x) = 1$

114. Show that the equation $\sin x(\sin x + \cos x) = a$ has real solutions if a is a real number lying between $\frac{1}{2}(1 - \sqrt{2})$ and $\frac{1}{2}(1 + \sqrt{2})$.

115. Find the real solutions of the equation $2\cos^2 \frac{x^2+x}{6} = 2^x + 2^{-x}$.

116. Solve the inequality $\sin x \geq \cos 2x$.

117. Find the general solution of the equation $\left(\cos\frac{x}{4} - 2\sin x\right)\sin x + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$

118. Find the general solution of the equation $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$.

119. Solve $\frac{\sin 2x}{\sin\frac{2x+\pi}{3}} = 0$.

120. Solve the equation $3\tan 2x - 4\tan 3x = \tan^2 3x \tan 2x$

121. Solve the equation $\sqrt{1 + \sin 2x} = \sqrt{2}\cos 2x$.

122. Show that $x = 0$ is the only solution satisfying the equation $1 + \sin^2 ax = \cos x$ where a is irrational.

123. Consider the system of linear equarions in x, y and z, $x\sin 3\theta - y + z = 0, x\cos 2\theta + 4y + 3z = 0, 2x + 7y + 7z = 0$. Find the values of θ for which the system has non-trivial solutions.

124. Find all the solutions of the equation $\sin x + \sin\frac{\pi}{8}\sqrt{(1 - \cos x)^2 + \sin^2 x} = 0$ in the interval $\left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$

125. Let $A = \{x : \tan x - \tan^2 x > 0\}$ and $y = \left\{x : |\sin x| < \frac{1}{2}\right\}$. Determine $A \cap B$.

126. If $0 \leq x \leq 2\pi$, then solve $2^{\frac{1}{\sin^2 x}}\sqrt{y^2 - 2y + 2} \leq 2$

127. If $|\tan x| = \tan x + \frac{1}{\cos x}(0 \leq x \leq 2\pi)$ then prove that $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

128. Find the smallest positive solution satisfying $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$

129. Solve the inequality $\sin x \cos x + \frac{1}{2}\tan x \geq 1$

130. Solve $\tan x^{\cos^2 x} = \cot x^{\sin x}$

131. If $0 \leq \alpha, \beta \leq 3$, then $x^2 + 4 + 3\cos(\alpha x + \beta) = 2x$ has at least one solution, then prove thatt $\alpha + \beta = \pi, 3\pi$.

132. Prove that the equation $2\sin x = |x| + a$ has no solution for $a \in \left(\frac{3\sqrt{3}-\pi}{3}, \infty\right)$

Chapter 11
Height and Distance

There are problems where distances between two points are not directly measurable or difficult. Most of such problems can be solved by applying trigonometric ratios with ease. This chapter is dependent on application of what we have studied so far about trigonometric ratios.

1. **Angle of Elevation:**

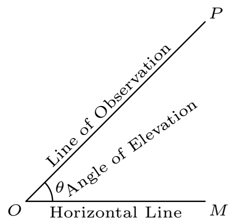


Figure 11.1

Let O and P be two points, where P is at a higher level than O. Also let O be the position of observer and P the position of the object. Draw a horizontal line OM through the point O. OP is called the line of observation or line of sight. Then $\angle POM = \theta$ is called the angle of elevation of P as observed from O.

2. **Angle of Depression**

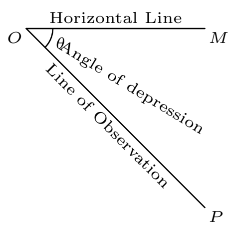


Figure 11.2

In the above example, if P be at a lower level than O, then $\angle MOP = \theta$ is called the angle of depression.

3. **Bearing**

In the above example, if the observer and the object i.e. O and P be on the same level then bearing is defined. Four standard directions; East, West, North and South are taken as cardinal directions for measuring bearing. If $\angle POE = \theta$ is the bearing of point P with respect to O measured from East to North.

North-east means equally inclines to north and east. South-east means equally inclines to south and east. E-N-E means equally inclined to east and north-east.

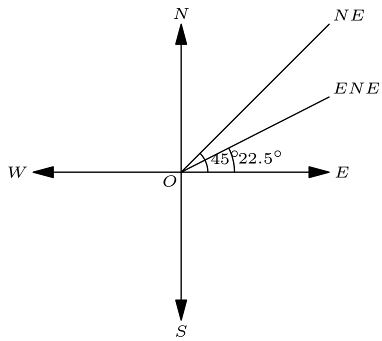


Figure 11.3

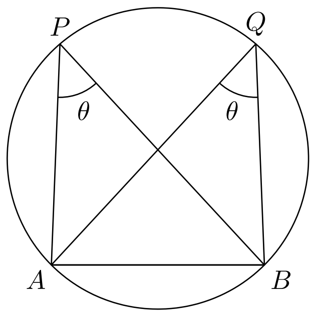


Figure 11.4

11.1 Some Useful Properties of a Circle

Angles on the same segment of a circle are equal. Alternatively, we can say that if the angles APB and AQB subtended on the segment AB are equal, a circle will pass through the points A, B, P and Q i.e. these points are concyclic.

If AR be the tangent to the circle passing through P, Q and R then $\angle PRA = \angle PQR = \theta$

Also, if PQ subtends greatest angle at R which lies on the line AR, then point R will be the point of contact of the tangent to the circle passing through P, Q and R.

11.2 Problems

1. A tower is $100\sqrt{3}$ meters high. Find the angle of elevation of its top point from a point 100 meters away from its foot.

2. The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower is 30°. Find the height of the tower.

3. A kite is flying at a height of 60 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string assuming there is no slack in the string.

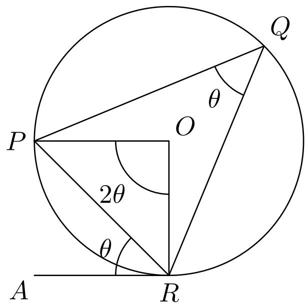


Figure 11.5

4. The string of a kite is 100 m long and it makes an angle of 60° with the horizontal. Find the height of the kite, assuming there is no slack in the string.

5. A circus artist is climbing from the ground a rope stretched from the top of a vertical pole and tied to the ground. The height of the pole is 12 m and the angle made by the rope with the ground level is 30°. Calculate the distance covered by the artist in climbing to the top of the pole.

6. A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole if the angle made by the rope with the ground level is 30°.

7. A bridge across a river makes an angle of 45° with the river banks. If the length of the bridge across the river is 150 m, what is the width of the river?

8. An observer 1.5 m tall is 28.5 m away from a tower. The angle of elevation of the top of the tower from her eyes is 45°. What is the height of the tower?

9. An electician has to repair an electric fault on a pole of a height 4 m. He needs to reach a point 1.3 m below the top of the pole to undertake the repair work. What should be the length of the ladder that he should use when inclined at an angle of 60° to the horizontal would enable him to reach the required position?

10. From a point on the ground 40 m away from the foot of the tower, the angle of elevation of the top of the tower is 30°. The angle of elevation of the top of a water tank(on the top of the tower) is 45°. Find (i) height of the tower (ii) the depth of the tank.

11. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°. When he retreats 20 m from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

12. A tree 12 m high, is broken by the wind in such a way that its top touches the ground and makes an angle of 60° with the ground. At what height from the bottom the tree is broken by the wind?

13. A tree is broken by the wind. The top struck the ground at an angle of 30° and at a distance of 30 m from the root. Find the whole height of the tree.

14. At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent in $5/12$. On walking 192 m towards the tower, the tangent of the angle of elevation is $3/4$. Find the height of the tower.

15. The shadow of a vertical tower on level ground increases by 10 m, when the altitude of sun changes from an angle of elevation 45° to 30°. Find the height of the tower.

16. From the top of a hill, the angle of depression of two consecutive kilometer stones due east are found to be 30° and 45°. Find the height of the hill.

17. Determine the height of a mountain if the elevation of its top at an unknown distance from the base is 30° and at a distance 10 km further off from the mountain, along the same line, the angle of elevation is 15°. (Use $\tan 15^\circ = 0.27$).

18. A person standing on the bank of a river observes that the angle of elevation of the top of a tree standing on the opposite bank is 60°. When he moves 40 m away from the bank, he finds the angle of elevation to be 30°. Find the height of the tree and width of the river.

19. An aeroplane at an altitude of 1200 m finds that two ships are sailing towards it in the same direction. The angles of depression of the ships as observed from the aeroplane are 60° and 30° respectively. Find the distance between two ships.

20. The shadow of a flag-staff is three times as long as the shadow of the flag-staff when the sun rays meet the ground at an angle of 60°. Find the angle between the sun rays and the ground at the time of longer shadow.

21. An aeroplane at an altitude of 200 m observes the angle of depression of opposite sign on the two banks of a river to be 45° and 60°. Find the width of the river.

22. Two pillars of equal height and on either side of a road, which is 100 m wide. The anngle of elevation of the top of the pillars are 60° and 30° at a point on the road between the pillars. Find the position of the point between the pillars and the height of each pillar.

23. As observed from the top of a lighthouse, 100 m above sea level, the angle of depression of a ship, sailing directly towards it, changes from 30° to 45°. Determine the distance travelled by the ship during the period of observation.

24. The angle of elevation of the top Q of a vertical tower PQ from a point X on the ground is 60°. At a point Y, 40 m vertically above X, the angle of elevation is 45°. Find the height of the tower PQ and the distance XQ.

25. From a window 15 m high above the ground in a street, the angles of elevation and depression of the top and the foot of another hourse on the opposite side of the street are 30° and 45° respectively show that the height of the opposite house is 23.66 m. (Use $\sqrt{3} = 1.732$).

26. From the top of a building 60 m high the angles of depression of the top and the bottom of tower are observed to be 30° and 60°. Find the height of the tower.

27. A man standing on the deck of a ship, which is 10 m above the water level. He observes that the angle of elevation of the top of the hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from from the ship and the height of the hill. Given that level of water is in the same line with base of the hill.

28. The angle of elevation of a jet plane from a point A on the ground in 60°. After a flight of 30 seconds the angle of elevation changes to 30°. If the jet plane is flying at a constant height of $3600\sqrt{3}$ m, find the speed of the jet plane.

29. There is a small island in the middle of a 100 m wide river and a tall tree stands on the island. P and Q are points directly opposite to each other on two banks in the line with the tree. If the angle of elevation of the top of the tree from P and Q are respectively 30° and 45°, find the height of the tree.

30. The horizonatal distance between two towers is 140 m. The angle of elevation of the top of the first tower when seen from the second tower is 30°. If the height of the second tower is 60 m, find the height of the first tower.

31. An aeroplane when flying at a height of 4000 m from the ground passes vertically above anohter aeroplane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. Find the vertical distance between the aeroplanes at that instant.

32. A tower stands vertically on the ground. From a point on the ground, 20 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°. What is the height of the tower?

33. The angle of elevation of a ladder leaning against a wall is 60° and the foot of the ladder is 9.5 m away from the wall. Find the length of the ladder.

34. A ladder is placed along the wall of a house such that its upper end is touching the top of the wall. The foot of the ladder is 2 m away from the wall and the ladder is making an angle of 60° with the level of the ground. Determine the height of the wall.

35. An electric pole is 10 m high. A steel wire tied to the top of the pole is affixed at a point on the ground to keep the pole up right. If the wire makes an angle of 45° with the horizontal through the foot of the pole, find the length of the wire.

36. A kite is flying at a height of 75 m from the ground level, attached to a string inclined at 60° to the horizontal. Find the length of the string to the nearest meter.

37. A ladder 15 m long just reaches the top of a vertical wall. If the ladder makes an angle of 60°, find the height of the wall.

38. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff. At a point on the plane 70 m away from the tower, an observer notices that the angle of elevation of the top and the bottom of the flag-staff are 60° and 45° respectively. Find the height of the flag-staff and that of the tower.

39. A vertically straight tree, 15 m high, is broken by the wind in such a way that its top just touches the ground and makes an angle of 60° with the ground. At what height from the ground did it break?

40. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff of height 5 m. At a point on the plane, the angle of elevation of the top and the bottom of the flag-staff are respectively 30° and 60°. Find the height of the tower.

41. A person observed the angle of elevation of the top of the tower as 30°. He walked 50 m towards the foot of the tower along the ground level and found the angle of elevation of the top of the tower to be 60°. Find the height of the tower.

42. The shadow of the tower, when the angle of elevation of the sun is 45°, is found to be 10 m longer than when it was 60°. Find the height of the tower.

43. A skydiver is descending vertically and makes angles of elevation of 45° and 60° at two observing points 100 m apart from each other on the left side. Find the maximum height from which he falls and the distance of the point where he falls on the ground from the nearest observation point.

44. On the same side of a tower, two objects are located. When observed from the top of the tower, their angles of depression are 45° and 60°. If the height of the tower is 150 m, find the diistance between the objects.

45. The angle of elevation of a tower from a point on the same level as the foot of the tower is 30°. On advancing 150 m towards the foot of the tower, the angle of elevation of the tower becomes 60°. Find the height of the tower.

46. The angle of elevation of the top of a tower as observed from a point in the horizontal plane through the foot of the tower is 30°. When the observer moves towards the tower a distance of 100 m, he finds that angle of elevation has become 60°. Find the height of the tower and distance of the initial position from the tower.

47. From the top of a building 15 m high the angle of elevation of the top of a tower is found to be 30°. From the bottom of the same building, the angle of elevation of the same tower is found to be 60°. Find the height of the tower and distance between the tower and the building.

48. On a horizontal plane there is a vertical tower with a flag pole on the top of the tower. At a point 9 m away from the foot of the tower the angle of elevation of the top and bottom of the flag pole are 60° and 30° respectively. Find the height of the tower and the flag pole mounted on it.

49. A tree breaks due to a storm and the broken part bends so that the top of the tree touches the ground making an angle 30° with the ground. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the height of the tree.

50. From a point P on the ground the angle of eleveation of a 10 m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flag from P is 45°. Find the length of flag and the distance of building from point P.

51. A 1.6 m tall girl stands at a distance 3.2 m from a lamp post. The length of the shadow of the girl is 4.8 m on the ground. Find the height of the lamp post by using trigonometric ratios and similar triangles.

52. A 1.5 m tall boy is standing some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walks towards the building.

53. The shadow of a tower standing on level ground is found to be 40 m longer when sun's angle of elevation is 30° than when it is 60°. Find the height of the tower.

54. From a point on the ground the angles of elevation of the bottom and top of a transmission tower fixed at the top of a building 20 m high are 45° and 60° respectively. Find the height of the transmission tower.

55. The angles of depression of the top and bottom of 8 m tall building from the top of a multistoried building are 30° and 45° respectively. Find the height of the multistoried building and the distance between two buildings.

56. A statue 1.6 m tall stands on the top of pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

57. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

58. As observed from the top of a 75 m tall lighthouse, the angle of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between two ships.

59. The angle of elevation of the top of the building from the foot of a tower is 30° and the angle of top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

60. From a point on a bridge across river the angles of depression of the banks on opposite sides of the river are 30° and 45°. If the bridge is at a height of 30 m find the width of the river.

61. Two poles of equal heights are standing opposite to each other on either side of the road which is 80 m wide. From a point between them on the road the angle of elevation of the top of the poles are 60° and 30° respectively. Find the height of the poles and the distance of the point from the poles.

62. A man sitting at a height of 20 m on a tall tree on a small island in middle of a river observes two poles directly opposite to each other on the two banks of the river and in line with the foot of the tree. If the angles of depression of the feet of the poles from a point which the man is sitting on the tree on either side of the river are 60° and 30° respectively. Find the width of the river.

63. A vertical tower stands on a horizontal plane and is surmounted by a flag-staff of height 7 m. From a point on the plane, the angle of elevation of the bottom of the flag-staff is 30° and that of the top of the flag-staff is 45°. Find the height of the tower.

64. The length of the shadow of a tower standing on level plane is found to be $2x$ m longer when the sun's altitude is 30° than when it was 45°. Prove that the height of tower is $x(\sqrt{3}+1)$ m.

65. A tree breaks due to a storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with the ground. The distance from the foot of the tree to the point where the top touches the ground is 10 m. Find the height of the tree.

66. A balloon is connected to a meteorological ground station by a cable of length 215 m inclined at 60° to the horizontal. Determine the height of the balloon from the ground assuming there is no slack in the cable.

67. To men on either side of a cliff 80 m high observe that angle of elevation of the top of the cliff to be 30° and 60° respectively. Find the distance between the two men.

68. Find the angle of the elevation of the sun (sun's altitude) when the length of the shdow of a vertical pole is equal to its height.

69. An aeroplane is flying at a height of 210 m. At some instant the angles of depression of two points in opposite directions on both the banks of the river are 45° and 60°. Find the width of the river.

70. The angle of elevation of the top of a chimney from the top of a tower is 60° and the angle of depression of the foot of the chimney from the top of the tower is 30°. If the height of the tower is 40 m, find the height of the chimney. According to pollution control norms, the minimum height of a smoke emitting chimney should be 100 m. State if the height of the chimney meets the pollution norms.

71. Two ships are in the sea on either side of a lighthouse in such a way that ships and lighthouse are always in the same straight line. The angles of depression of two ships are observed from the top of the lighthouse are 60° and 45° respectively. If the height of the lighthouse is 200 m, find the distance between the two ships.

72. The horizontal distance between two poles is 15 m. The angle of depression of top of the first pole as seen from the top of second pole is 30°. If the height of second pole is 24 m, find the height of the first pole.

73. The angle of depression of two ships from the top of a lighthouse and on the same side of it are found to be 45° and 30° respectively. If the ships are 200 m apart, find the height of lighthouse.

74. The angle of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line are complementary. Prove that the height of the tower is 6 m.

75. The horizontal distance between two trees of different heights is 60 m. The angle of depression of the top of the first tree when seen from the top of the second tree is 45°. If the height of the second tree is 80 m, find the height of the first tree.

76. A flag-staff stands on the top of a 5 m high tower. From a point on the ground, the angle of elevation of the top of the flag-staff is 60° and from the same point, the angle of elevation of the top of the tower is 45°. Find the height of the flag-staff.

77. The angle of elevation of the top of a vertical tower PQ from a point X on the ground is 60°. At a point Y, 40 m vertically above X, the angle of elevation of the top is 45°. Calculate the height of the tower.

78. As observed from the top of a 150 m tall lighthouse, the angle of depressions of two ships approaching it are 30° and 45° respectively. If one ship is directly behind the other, find the distance between two ships.

79. The angle of elevation of the top of a rock from the top and foot of a 100 m high tower are 30° and 45° respectively. Find the height of the rock.

80. A straight highway leads to the foot of the tower of height 50 m. From the top of the tower, the angles of depression of two cars standing on the highway are 30° and 60° respectively. What is distance between the cars and how far is each car from the tower?

81. From the top of a building AB, 60 m high, the angles of depression of the top and bottom of a vertical lamp post CD are observed to be 30° and 60° respectively. Find (i) horizontal distance between AB and CD, (ii) the height of the lamp post, and (iii) the difference between heights of the building and lamp post.

82. Two boats approach a lighthouse mid sea from opposite directions. The angles of elevation of the top of the lighthouse from the two boats are 30° and 45° respectively. If the distance between the ships is 100 m, find the height of the lighthouse.

83. The angle of elevation of a hill from the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of the hill is 30°. If the tower is 50 m high, find the height of the hill.

84. A moving boat is observed from the top of a 150 m high cliff moving away from the cliff. The angle of depression of the boat changes from 60° to 45° in 2 min. Find the speed of the boat.

85. From the top of a 120 m high tower, a man observes two cars on the opposite sides of the tower and in straight line with the base of the tower with angles of depression as 60° and 45°. Find the distance between the cars.

86. Two points A and B are on the same side of a tower and in the same straight line as its base. The angles of depression of these points from the top of tower are 60° and $45\circ$ respectively. If the height of the tower is 15 m, find the distance between the points.

87. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff of height h. At a point on the plane, the angles of elevation of the bottom and the top of the flag-staff are α and β respectively. Prove that the height of the tower is $\frac{h\tan\alpha}{\tan\beta-\tan\alpha}$.

88. The angles of elevation of the top of a tower from two points at distancces a and b meters from the base and in same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ m.

89. Two stations due south of a leaning tower which leans towards north are at distance a and b from its foot. If α, β be the elevations of the top of the tower from these stations, prove that its inclination θ to the horizontal is given by $\cot\theta = \frac{b\cot\alpha - a\cot\beta}{b-a}$.

90. If the angle of elevation of a cloud from a point h meteres above a lake is α and the angle of depression of its reflection in the lake is β, prove that the height of the cloud is $\frac{h(\tan\alpha+\tan\beta)}{\tan\beta-\tan\alpha}$.

91. A round balloon of radius r subtends an angle α at the eye of the observer while the angle of elevation of its center is β. Prove that the height of the center of the balloon is $r\sin\beta\text{cosec}\,\frac{\alpha}{2}$.

92. The angle of elevation of a cliff from a fixed point is θ. After going a distance of k m towards the top of the cliff at an angle of ϕ, it is found that the angle of elevation is α. Show that the height of the cliff is $\frac{k(\cos\phi-\sin\phi\cot\alpha)}{\cot\theta-\cot\alpha}$ m.

93. The angle of elevation of the top of a tower from a point A due south of the tower is α and from B due east of the tower is β. If $AB = d$, show that the height of the tower is $\frac{d}{\sqrt{\cot^2\alpha+\cot^2\beta}}$.

94. The elevation of a tower at a station A due north of it is α and at a station B due west of A is β. Prove that the height of tower is $\frac{AB\sin\alpha\sin\beta}{\sqrt{\sin^2\alpha-\sin^2\beta}}$.

95. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation from the eyes of the girl at any instant is 60°. After some time, the angle of elevation is reduced to 30°. Find the distance travelled by the balloon during the interval.

96. A straight highway leads to the foot of the tower. A man standing on the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of tower with uniform speed. Six seconds later the angle of depression is found to be 60°. Find the further time taken by the car to reach the foot of the tower.

97. A man on a cliff observes a boat at an angle of depression of 30° which is apporaching the shore to the point immediately beneath the observer with a uniform speed. Six minutes later, the angle of depression of the boat is found to be 60°. Find the time taken by the boat to read the shore.

98. A man on the top of a vertical tower observes a car moving at a uniform speed coming directly towards it. If it takes 12 min for the angle of depression to change from 30° to 45°, find the time taken by the car to reach the foot of the tower.

99. A fire in a building is reported to two fire stations, 20 km apart from each other on a straight road. One fire station observes that the fire is at an angle 60° to the the road and second fire station observes that the fire is at 45° to the road. Which station's fire-fighting team will reach sooner and how much would it have to travel?

100. A man on the deck of a ship is 10 m above the water level. He observes that the angle of elevation of the top of a cliff is 45° and the angle of depression of its base is 30°. Calculate the distance of ship from the cliff and height of the cliff.

101. There are two temples, one on each bank of a river, just opposite to each other. One temple is 50 m high. From the top of this temple, the angle of depression of the top and the bottom of the other temple are 30° and 60° respectively. Find the width of the river and the height of the other temple.

102. The angle of elevation of an aeroplane from a point on the ground is 45°. After a flight of 15 seconds, the elevation changes to 30°. If the aeroplane is flyging at a height of 3000 m, find the speed of the aeroplane.

103. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60°. After 10 seconds, its elevation is observed to be 30°. Find the speed of the aeroplane in km/hr.

104. A tree standing on a horizontal plane is leaning towards east. At two points situated at distance a and b exactly due west of it, with angles of elevation to the top respectively α and β. Prove that the height of of the top from the ground is $\frac{(b-a)\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$.

105. The angle of elevation of a stationary cloud from a point 2500 m above a lake is 15° and the angle of depression of its reflection in the lake is 45°. What is the height of the cloud above the lake level? (Use $\tan 15^\circ = 0.268$).

106. If the angle of elevation of a cloud from a point h meters above a lake is α and the angle of depression of its reflection in the lake is β, prove that the distance of cloud from the point of observation is $\frac{2h\sec\alpha}{\tan\beta-\tan\alpha}$.

107. From an aeroplane vertically above a straight horizontal road, the angles of depression of two consecutive milestones on opposite sides of the aeroplane are observed to be α and β. Show that the height in miles of aeroplane above the rooad is given by $\frac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$.

108. PQ is a post of given height h, and AB is a tower at some distance. If α and β are the angles of elevation of B, at P and Q respectively. Find the height of the tower and its distance from the post.

109. A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a, so that it slides a distance b down the the wall making an angle β with the horizontal. Show that $\frac{a}{b} = \frac{\cos\alpha-\cos\beta}{\sin\beta-\sin\alpha}$.

110. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a point b m just above A is β. Prove that the height of the tower is $b\tan\alpha\cot\beta$.

111. An observer, 1.5 m tall, is 28.5 m away from a tower 30 m high. Determine the angle of elevation of the top of the tower from his eye.

112. From the top of a tower h m high, the angles of depression of two objects, which are in line with the foot of tower are α and $\beta(\beta > \alpha)$. Find the distance between two objects.

113. A window of house is h m above the ground. From the window, the angles of elevation and depression of the top and bottom of the amother house situated on the opposite side of the lane are found to be α and β respectively. Prove that the height of the house is $h(1+\tan\alpha\cot\beta)$ m.

114. The lower windows of a house is at a height of 2 m above the ground and its upper window is 4 m vertically above the lower window. At certain instant the angles of elevation of a balloon from these windows are observed to be 60° and 30° respectively. Find the height of the balloon above the ground.

115. A man standing south of a lamp-post observes his shadow on the horizontal plane to be 24 ft. long. On walking eastward 300 ft. he finds the shadow as 30 ft. If his height is 6 ft., obtain the height of the lamp post above the plane.

116. When the sun's altitude increases from 30° to 60°, the length of the shadow of tower decreases by 5 m. Find the height of the tower.

117. A man observes two objects in a straight line in the west. On walking a distance c to the north, the objects subtend an angle α in front of him. On walking a further distance c to the north, they subtend angle β. Show that distance between the objects is $\frac{3c}{2\cot\beta-\cot\alpha}$.

118. An object is observed from the points A, B, C lying in a horizontal straight line which passes directly underneath the object. The angular elevation at B is twice that at A and at C three times that of A. If $AB = a, BC = b$, show that the height of the object is $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.

119. At the foot of a mountain the elevation of its summit is 45°; after ascending one kilometer towards the mountain upon an incline of 30°, the elevation changes to 60°. Find the height of the mountain.

120. A man observes that when he has walked c m up an inclined plane, the angular depression of an object in a horizontal plane through the foot of the slope is α and when he walked a further distance of c m, the depression is β. Prove that the inclination of the slope to the horizon is the angle whose cotangent is $2\cot\beta - \cot\alpha$.

121. A ladder rests against a vertical wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a so that it slides a distance b down the wall making an angle β with the horizontal. Show that $a = b\tan\frac{\alpha+\beta}{2}$.

122. A balloon moving in a straight line passes vertically above two points A and B on a horizontal plane 1000 m apart. When above A has an altitude 60° as seen from B, and when above B, 30° as seen from A. Find the distance from A of the point at which it will strike the plane.

123. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°. When he retires 40 m from the bank perpendicular to it, he finds the angle to be 30°, find the height of the tree and the breadth of the river.

124. The angles of elevation of a bird flying in a horizontal straight line from a point at four consecutive observations are α, β, γ and δ, the observations being taken at equal intervals of time. Assuming that the speed of the bird is uniform, prove that $\cot^2\alpha - \cot^2\delta = 3(\cot^2\beta - \cot^2\gamma)$.

125. At a point on a level plane a vertical tower subtends an angle α and a pole of height h m at the top of the tower subtends an angle β, show that the height of the tower is $h\sin\alpha\text{cosec}\beta\cos(\alpha+\beta)$ m.

126. AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion CB subtends an angle β at P. If $AP = n.AB$, then show that $\tan\beta = \frac{n}{2n^2+1}$.

127. The angular depression of the top and the foot of a chimney as seen from the top of a second chimney, which is 150 m high and standing on the same level as the first, are θ and ϕ respectively. Find the distance between their tops when $\tan\theta = \frac{4}{3}$ and $\tan\phi = \frac{5}{2}$.

128. The angular elevation of a tower CD at a place A due south of it is 30° and at a place B due west of A, the elevation is 18°. If $AB = a$, show that the height of the tower is $\frac{a}{\sqrt{2+2\sqrt{5}}}$.

129. The elevation of a tower due north of a station at P is θ and at a station Q due west of P is ϕ. Prove that the height of tower is $\frac{PQ.\sin\theta\sin\phi}{\sqrt{\sin^{\theta}-\sin^2\phi}}$.

130. The angle of elevation of a certain peak when observed from each end of a horizontal baseline of length $2a$ is found to be θ. When observed from the mid-point of the base, angle of elevation is ϕ. Prove that the height of the peak is $\frac{a\sin\theta\sin\phi}{\sqrt{\sin(\theta+\phi)\sin(\phi-\theta)}}$.

131. The angles of elevation of the top of a hill as seen from three consecutive milestones of a straight road not passing through the foot of the hill are α, β, γ respectively. Show that the height of the hill is $\frac{\sqrt{2}}{\sqrt{\cot^2\alpha+\cot^2\gamma-2\cot^2\beta}}$.

132. A tower stands in a field whose shape is that of an equilateral triangle and whose sides are 80 ft. It subtends an angle at three corners whose tangents are respectively $\sqrt{3}+1, \sqrt{2}, \sqrt{2}$. Fnd its height.

133. A man on a hill observers that three towers on a horizontal plane subtend equal angles at his eye and that the angles of depression of their bases are α, β, γ. If $a, b, , c$ be the heights of the tower, prove that $\frac{\sin(\beta-\gamma)}{a\sin\alpha}+\frac{\sin(\gamma-\alpha)}{b\sin\beta}+\frac{\sin(\alpha-\beta)}{c\sin\gamma}=0$.

134. A person walking along a canal observes that two objects are in the same line which is inclined at an angle α to the canal. He walks a distnce c further and observes that the objects subtend their greatest angle β. Show that their distance apart is $\frac{2c\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$.

135. A flag-staff is fixed on the top of a tower standing on a horizontal plane. The angles subtended by the flag-staff at two points a m apart, on the same side and on the same horizontal line through the foot of the tower are the same and equal to α. The angle subtended by the tower at the farthest point is β, find the height of the tower and the length of the flag staff.

136. The angle of elevation of a cloud from a point h ft. above the surface of a lake is θ, the anngle of depression of its reflection in the lake is ϕ. Prove that the height of the cloud is $\frac{h\sin(\theta+\phi)}{\sin(\phi-\theta)}$.

137. A road is inclined at an angle 10° to the vertical towards the sun. The height of the shadow on the horizontal ground is 2.05 m. If the elevation of the sun is 38°, find the length of the road.

138. When the sun's altitude increases from 30° to 60°, the length of the shadow of a tower decreases by 30 m. Find the height of the tower.

139. The shadow of a tower standing on a level is found to be 60 m longer when the sun's altitude is 30° than when it is 45°. Find the height of the tower.

140. A man on a cliff observes a boat at an angle of depression of 30°, which is sailing towards the shore to the point immediately beneath him. Three minutes later, the angle of depresssion of the boat is found to be 60°. Assuming that the boat sails at uniform speed, determine how much more time it will take to reach the shore.

141. An aeroplane when 3000 m high passes vertically above another aeroplane at an instant when there angle of elevation at the same observation points are 60° and 45° respectively. How many meters higher is the one than the other.

142. The angles of elevation of an aeroplane at two consecutive milestones respectively are α and β. Find the height of the plane taking it to be between the two milestones and just above the road.

143. The altitude of a certain rock is 47° and after walking towards it 1000 m up a slope inclined at 30° to the horizon an observer finds its altitude to be 77°. Find the height of the rock. ($\sin 47^{\circ} = .73135$.)

144. A man observes that when he moves up a distance c m on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30° and when he moves up further a distance c m, then angle of depression of the point is 45°. Obtain the angle of depression of the slope with the horizontal.

145. On level ground the angle of elevation of the top of the tower is 30°. On moving 20 m nearer the angle of elevation is 60°. What is the height of the tower?

146. An air-pilot at a height h m above the ground observes the angle of depression of the top and bottom of a tower to be 30° and 60°. Find the height of the tower.

147. From the top of a hill 200 m high, the angles of depression of the top and the bottom of a pillar are 30° and 60° respectively. Find the height of the pillar and its distance from the hill.

148. A vertical pole consists of two parts, the lower part being one-third of the whole. The upper part subtends an angle whose tangent is $\frac{1}{2}$ at a point in a horizontal plane through the foot of the pole and 20 m from it. Find the height of the pole.

149. A statue is 8 m high standing on the top of a tower 64 m high on the bank of a river subtends at a point A on the opposite bank facing the tower, the same angle as subtended at the same point A by a man 2 m high standing at the base of of the tower. Show that the breadth of the river is $16\sqrt{6}$ m.

150. A statue a m high placed on a column b m high subtends the same angle as the column to an observer h m high standing on the horizontal plane at a distance d m from the foot of the column. Show that $(a-b)\,d^2 = (a+b)\,b^2 - 2b^2h - (a-b)\,h^2$.

151. The angles of elevation of the top of a tower standing on a horizontal plane from two points on a line passing through the foot of the tower at a distance a and b are complementary angles. Prove that the height of the tower is $\sqrt{ab}$. If the line joining the two points subtend an angle θ at the top of the tower, show that $\sin\theta = \frac{a-b}{a+b}$.

152. A pillar subtends at a point d m apart from its foot the same angle as that subtended at the same point by a statue on the top. If the pillar is h m high, show that the height of the status is $\frac{h(d^2+h^2)}{d^2-h^2}$ m.

153. A vertical tower 50 ft. high stands on a sloping ground. The foot of the tower is at the same level as the middle point of a vertical flag pole. From the top of the tower the angle of depression of the top and the bottom of the pole are 15° and 45° respectively. Find the length of the pole.

154. An observer at an anti-aircraft post A identifies an enemy aircraft due east of his post at an angle of elevation of 60°. At the same instant a detection post D situated 4 km south of A reports the aircraft at an elevation of 30°. Calculate the altitude at which the aircraft is flying.

155. A flag staff PN stands up right on level ground. A base AB is measured at right angled to AN such that the points A, B, N lie in the same horizontal plane. If $\angle PAN = \alpha$ and $\angle PBN = \beta$. Prove that the height of the flag staff is $\frac{AB.\sin\alpha\sin\beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}$.

156. A vertical pole is divided in the ratio $1:9$ by a mark on it. If the two parts subtend equal angle at a distance of 20 m from the base of the pole, find the height of the pole. The lower part is shorter than the upper one.

157. A chimney leans towards north. At equal distances due north and south of it in a horizontal plane, the elevation of the top are α, β. Show that the inclination of the chimney to the vertical is $\tan^{-1}\left[\frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}\right]$.

158. A flag staff 10 m high stands in the center of an equilateral triangle which is horizontal. If each side of the triangle subtends an angle of 60° at the top of flag staff. Prove that the length of the sides are $5\sqrt{6}$ m.

159. Two posts are 120 m apart, and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevation of their tops to be complementary. Find the height of the posts.

160. A pole 100 ft. high stands at the center of an equilateral triangle each side of which subtends and angle of 60° at the top of the pole. Find the side of the triangle.

161. An observer on a carriage moving with a speed v along a straight road observes in one position that two distant trees are in the same line with him which is inclined at an angle θ to the road. After a time t, he observes that the trees subtend their greatest angle ϕ. Show that the distance between the tree is $\frac{2vt\sin\theta\sin\phi}{\cos\theta+\cos\phi}$.

162. A and B are two points on one bank of a straight river and C and D are two points on the other bank. The direction from C to D is the same as from A to B. If $AB = a$, $\angle CAD = \alpha$, $\angle DAB = \beta$, $\angle CBA = \gamma$, prove that $CD = \frac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$.

163. To measure the breadth PQ of a river a man places himself at R in the straight line PQ produced through Q and then walks 100 m at right angles to this line. He then finds PQ and QR subtend angles 15° and 25° at his eye. Find the breadth of the river.

164. A bird is perched on the top of a tree 20 m high and its elevation from a point on the ground is 45°. It flies off horizontally straight away from the observer and in second the elevation of the bird is reduced to 30°. Find its speed.

165. The angles of elevation of a balloon from two stations 2 km apart and from a point halfway between them are observed to be $60^\circ, 30^\circ$ and 45° respectively. Prove that the height of the balloon is $500\sqrt{6}$ m.

166. If the angular elevations of the tops of two spires which appear in a straight line is α and the angular depression of their reflections in a lake, h ft. below the point of observation are β and γ, show that the distance between the two spires is $2h\cos^2\alpha\sin(\gamma-\beta)\operatorname{cosec}(\beta-\alpha)\operatorname{cosec}(\gamma-\alpha)$ ft. where $\gamma > \beta$.

167. A pole stands vertically on the center of a square. When α is the elevation of the sun its shadow just reaches the side of the square and is at a distance x and y from the ends of that side. Show that the height of the pole is $\sqrt{\frac{x^2+y^2}{2}}.\tan\alpha$.

168. A circular plate of radius a touches a vertical wall. The plate is fixed horizontally at a height b above the ground. A lighted candle of length c stands vertically at the center of the plate. Prove that the breadth of the shadow on the wall where it meets the horizontal ground is $\frac{2a}{c}\sqrt{b^2+2bc}$.

169. The extremity of the shadow of a flag-staff which is 6 m high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant x and y ft. respectively from the ends of that side; prove that the height of the pyramid is $\sqrt{\frac{x^2+y^2}{2}}.\tan\alpha-6$, where α is the elevation of the sun.

170. A man observes a tower PQ of height h from a point C on the ground. He moves forward a distance d towards the foot of the tower and finds that the angle of elevation has doubled. He further moves a distance $\frac{3}{4}d$ in the same direction. He finds that the angle of elevation is three times that at P. Prove that $36h^2=35d^2$.

171. A 2 m long object is fired vertically upwards from the mid-point of two locations A and B, 8 m apart. The speed of the object after t seconds is given by $\frac{ds}{dt}=(2t+1)$ m/s. Let α and β be the angles subtended by the object at A and B respectively after one and two seconds. Find the value of $\cos(\alpha-\beta)$.

172. A sign-post in the fom of an isosceles triangle ABC is mounted on a pole of height h fixed to the ground. The base BC of the triangle is parallel to the ground. A man standing on the ground at distance d from the sign-post finds that the top vertex A of the triangle subtends an angle β and either of the two vertices subtends the same angle α at his feet. Find the area of the triangle.

173. A tower is observed from two stations A and B, where B is east of A at a distance 100 m. The tower is due north of A and due north-west of B. The angles of elevations of the tower from A and B are complementary. Find the height of the tower.

174. Two vertical poles whose heights are a and b subtend the samme angles α at a point in the line joining their feet. If they subtend angle β and γ at any point in the horizontal plane at which the line joining their feet subtends a right angle, prove that $(a+b)^2\cot^2\alpha=a^2\cot^2\beta+b^2\cot^2\gamma$.

175. PQ is a vertical tower. P is the foot and Q is the top of the tower. A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal and each is equal to θ. The sides of the $\triangle ABC$ are a, b, c and the area of the $\triangle ABC$ is Δ. Show that the height of the tower is $\frac{abc\tan\theta}{4\Delta}$.

176. An observer at O notices that the angle of elevation of the top of a tower is 90°. The line joining O to the base of the tower makes an angle of $\tan^{-1}\frac{1}{\sqrt{2}}$ with the north and is inclined eastwards. The observer travels a distance of 300 m towards north to a point A and finds

the toewr to his east. The angle of elevation of the top of the tower at A is ϕ. Find ϕ and the height of the tower.

177. A tower AB leans towards west making an angle α with the vertical. The angular elevation of B, the top most point of the tower, is β as observed from a point C due west of A at a distance d from A. If the angular elevation of B from a point D due east of C at a distance $2d$ from C is γ, then prove that $2\tan\alpha = 3\cot\beta - \cot\gamma$.

178. The elevation of the top of a tower at point E due east of the tower is α, and at a point S due south of the tower is β. Prove that it's elevation θ at a point mid-way between E and S is given by $\cot^2\beta + \cot^2\alpha = 4\cot^2\theta$.

179. A vertical tree stands at a point A on a bank of a canal. The angle of elevation of its top from a point B on the other bank of the canal and directly opposite to A is 60°. The angle of elevation of the top from another point C is 30°. If A, B and C are on the same horizontal plane, $\angle ABC = 120°$ and $BC = 20$ m, find the height of the tree and the width of the canal.

180. A person observes the top of a vertical tower of height h from a station S_1 and finds β_1 is the angle of elevation. He moves in a horizontal plane to second station S_2 andd finds that $\angle PS_2S_1$ is γ_1 and the angle subtended by S_2S_1 at P (top of the tower) is δ_1 and the angle of elevation is β_2. He moves again to a third station S_3 such that $S_3S_2 = S_2S_1$, $\angle PS_3S_2 = \gamma_2$ and the angle subtended by S_3S_2 is δ_2. Show that $\frac{\sin\gamma_1\sin\beta_1}{\sin\delta_1} = \frac{\sin\gamma_2\sin\beta_2}{\sin\delta_2} = \frac{h}{S_1S_2}$.

181. A straight pillar PQ stands at a point P. The points A and B are situated due south and east of P respectively. M is mid-point of AB. PAM is an equilateral triangle and N is the foot of the perpendicular from P on AB. Suppose $AN = 20$ m and the angle of elevation of the top of the pillar at N is $\tan^{-1}2$. Find the height of the pillar and the angle of elevation of its top at A and B.

182. ABC is a triangular park with $AB = AC = 100$ m. A television tower stands at the mid point of BC. The angles of elevation of the top of the tower at A, B and C are 45°, 60° and 60° respectively. Find the height of the tower.

183. A square tower stands upon a horizontal plane from which three of the upper corners are visible, their angular elevations are 45°, 60° and 45°. If h be the height of the tower and a is the breadth of its sides, then show that $\frac{h}{a} = \frac{\sqrt{6}(1+\sqrt{5})}{4}$.

184. A right circular cylindrical tower of height h and radius r stands on a horizontal plane. Let A be a point in the horizontal plane and PQR be a semi-circular edge of the top of the tower such that Q is the point in it nearest to A. The angles of elevation of the points P and Q are 45° and 60° respectively. Show that $\frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{5})}{2}$.

185. A is the foot of the vartical pole, B and C are due east of A and D is due south of C. The elevation of the pole at B is double that C and the angle subtended by AB at D is $\tan^{-1}\frac{1}{5}$. Also, $BC = 20$ m, $CD = 30$ m, find the height of the pole.

186. A person wishing to ascertain the height of a tower, stations himself on a horizontal plane through its foot at a point at which the elevation of the top is 30°. On walking a distance a in a certain direction he finds that elevation to the top is same as before, and on walking a

distance $\frac{5}{3}a$ at right angles to his former direction, he finds the elevation of the top to be 60°, prove that the height of the tower is either $\sqrt{\frac{5}{6}}a$ or $\sqrt{\frac{85}{48}}a$.

187. A tower stands in a field whose shape is that of an equilateral triangle and whose side is 80 ft. It subtends angles at three corners whose tangents are respectively $\sqrt{3}+1$, $\sqrt{2}$, $\sqrt{2}$. Find its height.

188. A flag-staff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane, which lie on a line passing through the center of the base of the tower annd whose distance from one another is $2a$, and angle β at a point half way between them. Prove that the heirght of the flag-staff is $a\sin\alpha\sqrt{\frac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}}$.

189. A man standing on a plane observes a row of equal and equidistant pillars, the 10-th and 17-th of which subtend the same angle that they would do if they were in position of the first respectively $\frac{1}{2}$ and $\frac{1}{3}$ of their height. Prove that, neglecting the height of the man's eye, the line of pillars is inclined to be line drawn from his eye to the first at an angle whose secant is nearly 2.6.

190. A tower stands on the edge of the circular lake $ABCD$. The foot of the tower is at D and the angle of elevation of the top from A, B, C are respectively α, β, γ. If $\angle BAC = \angle ACB = \theta$. Show that $2\cos\theta\cot\beta = \cot\alpha + \cot\gamma$.

191. A pole stands at the bank of circular pond. A man walking along the bank finds that angle of elevation of the top of the pole from the points A and B is 30° and from the third point C is 45°. If the distance from A to B and from B to C measured along bank are 40 m and 20 m respectively. Find the radius of the pond and the height of the pole.

192. A man standing on the sea shore observes two buoys in the same direction, the line through them making an angle α with the shore. He then walks a distance along the shore a distance a, when he finds the buoys subtend an angle α at his eye; and on walking a further distance b he finds that they subtend an angle α at his eye. Show that the distance between the buoys is $\left(a+\frac{b}{2}\right)\sec\alpha - \frac{2a(a+b)}{2a+b}\cos\alpha$, assuming the shore to be straight and henglecting the height of the man's eye above the sea.

193. A railway curve in the shape of a quadrant of a circle, has n telegraph posts at its ends and at equal distance along the curve. A man stationed at a point on one of the extreme radii produced sees the p-th and q-th posts from the end nearest him in a straight line. Show that the radius of the curve is $\frac{a}{2}\cos(p+q)\,\phi\text{cosec}p\phi\text{cosec}q\phi$, where $\phi = \frac{\pi}{4(n-1)}$ and a is the distance from the man to the nearest end of curve.

194. A wheel with diameter AB touches the horizontal ground at the point A. There is a rod BC fixed at B such that ABC is vertical. A man from a point P on the ground, in the same plane as that of wheel and at a distance d from A, is watching C and finds its angle of elevation is α. The wheel is then rotated about its fixed center O such that C moves away from the man. The angle of elevation of C when it is about to disappear is β. Find the radius of the wheel and the length of the rod. Also, find distance PC when C is just to disappear.

195. A semi-circular arch AB of length $2L$ and a vertical tower PQ are situated in the same vertical plane. The feet A and B of the arch and the base Q of the tower are on the same horizontal

level, with B between A and Q. A man at A finds the tower hidden from his view due to arch. He starts carwling up the arch and just sees the topmost point P of the tower after covering a distance $\frac{L}{2}$ along the arch. He crawls further to the topmost point of the arch and notes the angle of elevation of P to be θ. Compute the height of the tower in terms of L and θ.

196. A circle passes through three points A, B and C with the line segment AC as its diameter. A line passing through A intersects the chord BC at a point D inside the circle. If angles DAB and CAB are α and β respectively and the distance between point A and the mid-point of the line segment DC is d. Prove that the area of the circle is $\frac{\pi d^2 \cos^2 \alpha}{\cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta \cos(\beta - \alpha)}$.

197. The angle of elevation of a cloud from a point h m above a lake is α, and the angle of depression of its reflection is β. Prove that the distance of the observer from the cloud is $\frac{2h \cos \beta}{\sin(\beta - \alpha)}$.

198. An isosceles triangle of wood is placed in a vertical plane, vertex upwards and faces the the sun. If $2a$ be the base of the triangle, h its height and 30° be the altitude of the sun, prove that the tangent of the angle at the apex of the shadow is $\frac{2ah\sqrt{3}}{3h^2 - a^2}$.

199. A rectangular target faces due south, being vertical and standing on a horizontal plane. Compute the area of the target with that of its shadow on the ground when the sun is β° from the south at an altitude of α°.

200. The extremity of the shadow of a flag staff which is 6 m high and stands on the top of a pyramid on a square base just reaches the side of the base and is distant 56 m and 8 m respectively from the extremeties of that side. Find the sun's altitude if the height of the pyramid is 34 m.

201. The shdadow of a tower is observed to be half the known height of the tower and sometime afterwards is equal to the known height; how much will the sun have gone down in the interval. Given $\log 2 = 0.30103$, $\tan 63^\circ 23' = 10.3009994$ and diff for $1' = 3152$.

202. A man notices two objects in a straight line due west. After walking a distance c due north, he observes that the objects subtend an angle α at his eye; and after walking a further distance $2c$ due norht an angle β. Show that the distance between the objects $\frac{8c}{3 \cot \beta - \cot \alpha}$. Ignore the height of the man.

203. A stationary balloon is observed from three points A, B and C on the plane ground and it is found that its angle of elevation from each of these points is α. If $\angle ABC = \beta$ and $AC = b$, find the height of the balloon.

204. A lighthouse, facing north, sends out a fan-shaped beam of light extending from north-east to north-west. An observer on a steamer, sailing due west first sees the light when he is 5 km away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer?

205. A man walking due north observes that the elevation of a balloon, which is due east of him and is sailing towards the north-west is then 60°; after he has walked 400 yards the balloon is vertically over his head. Find its height, supposing it to have always remained the same.

206. A flag-staff stands on the middle of a square tower. A man on the ground opposite the middle of the face and distant from it 100 m, just sees the flag; on receeding another 100 m the tangents of the elevation of the top of the tower and the top of the flag staff are found to be $\frac{1}{2}$ and $\frac{5}{9}$. Find the dimensions of the tower and the height of the flag staff, the ground being horizontal.

207. A vertical pole stands at a point O on horizontal ground. A and B are points on the ground, d meters apart. The pole subtends angles α and β at A and B respectively. AB subtends an angle γ at O. Find the height of the pole.

208. A vertical tree stands on a hill side that makes an angle α with the horizontal. From a point directly up the hill from the tree, the angle of elevation of the tree top is β. From a point m cm further up the hill the angle of depression of the tree top is γ. If the tree is h meters tall, find h in terms of α, β, γ.

209. A person stands on the diagnal produced of the square base of a church tower, at a distance $2a$ from it and observes the angle of elevation of each of the two outer corners of the top of the tower to be 30°, while that of the nearest corner is 45°. Prove that the breadth of the tower is $a(\sqrt{10}-\sqrt{2})$.

210. The elevation of a steeple at a place due south of it is 45° and at another place due west of the former place is 15°. If the distance between the two places be a, prove that the height of steeple is $\frac{a(\sqrt{3}-1)}{2.\sqrt[4]{3}}$ or $\frac{a}{\sqrt{6+4\sqrt{3}}}$.

211. A tower surmounted by a spire stands on a level plane. A person on the plain observes that when he is at a distance a from the foot of the tower, its top is in line with that of a mountain behind the spire. From a point at a distance b further from the tower, he finds that the spire subtends the same angle as before at his eye and its top is in line with that of the mountain. If the height of the tower above the horizontal plane through the observer's eye is c, prove that the height of the mountain above the plane is $\frac{abc}{c^2-a^2}$.

212. From the bottom of a pole of height h, the angle of elevation of the top of the tower is α. The pole subtends angle β at the top of the tower. Find the height of the tower.

213. A man moves along the bank of a canal and observes a tower on the other bank. He finds that the angle of elevation of the top of the tower from each of the two points A and B, at a distance $6d$ apart is α. From a third point C, between A and B at a distance $2d$ from A, the angle of elevation is found to be β. Find the height of the tower and width of the canal.

214. The angle of elevation of a balloon from two stations 2 km apart and from a point halfway between them are observed to be $60^\circ, 30^\circ$ annd 45° respectively. Prove that the height of the balloon is $500\sqrt{6}$ meters.

215. A flag staff 10 meters high stands in the center of an equilateral triangle which is horizontal. If each side of the triangle subtends an angle of 60° at the top of the flag staff. Prove that the length of the side of the triangle is $5\sqrt{6}$ meters.

216. A tower standing on a cliff subtends an angle β at each of two stations in the same horizontal line passing through the base of the cliff and at a distance of a meters and b meters respectively from the cliff. Prove that the height of the tower is $(a+b)\tan\beta$ meters.

217. A man walking towards a tower AB on which a flag staff is fixed observes that when he is at a point E, distance c meters from the tower, the flag staff subtends its greatest angle. If $\angle BEC = \alpha$, prove that the heights of the tower and flag staff are $c\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$ and $2c\tan\alpha$ meters respectively.

218. Four ships A, B, C and D are at sea in the following positions. B is on a straight line segment AC, B is due north of D and D is due west of C. The distance between B and D is 2 km. If $\angle BDA = 40^\circ, \angle BCD = 25^\circ$, what is the distance between A and D? ($\sin 25^\circ = 0.423$)

219. A train is moving at a constant speed at an angle θ east of north. Observations of the train are made from a fixed point. It is due north at some instant. Ten minutes earlier its bearing was α_1 west of north whereas ten minutes afterwards its nearing is α_2 east of north. Find $\tan\theta$.

220. A man walks in a horizontal circle round the foot of a flag staff, which is inclined to the vertical, the foot of the flag staff being the center of the circle. The greatest and least angles which the flag staff subtends at his eyes are α and β; and when he is mid-way between the corresponding position the angle is θ. If the man's height be neglected, prove that $\tan\theta = \frac{\sqrt{\sin^2(\alpha-\beta)+4\sin^2\alpha\sin^2\beta}}{\sin(\alpha+\beta)}$.

221. A bird flies in a circle on a horizontal plane. An observer stands at a point on the ground. Suppose 60° and 30° are the maximum and the minimum angles of elevation of the bird and that they occur when the bird is at point P and Q respectively on its path. Let θ be the angle of elevation of the bird when it is at a point on the arc of the circle exactly midway between P and Q. Find the numerical value of $\tan^2\theta$. (Assume that the observer is not inside the vertical projection of the path of the bird).

222. A hill on a level plane has the form of a portion of a sphere. At the bottom the surface slopes at an angle α and from a point on the plane distant a from the foot of the hill the elevation of the heighest visible point is β. Prove that the height of the hill above the plane is $\frac{a\sin\beta\sin^2\frac{\alpha}{2}}{\sin^2\frac{\alpha-\beta}{2}}$.

223. A hill standing on a horizontal plane, has a circular base and forms a part of a sphere. At two points on the plane, distant a and b from the base, the angular elevation of the heighest visible points on the hill are θ and ϕ. Prove that the height of the hill is $2\left[\frac{\sqrt{b\cot\frac{\phi}{2}}-\sqrt{a\cot\frac{\theta}{2}}}{\cot\frac{\theta}{2}-\cot\frac{\phi}{2}}\right]^2$.

224. On the top of a hemispherical dome of radius r there stands a flag of height h. From a point on the ground the elevation of the top of the flag is 30°. After moving a distant d towards the dome, when the flag is just visible, the elevation is 45°. Find r and h in terms of d.

225. A man walks on a horizontal plane a distance a, then through a distance a at an angle α with his previous direction. After he has done this n times, the change of his direction being always in the same sense, show that he is distant $\frac{a\sin(n\alpha/2)}{\sin(\alpha/2)}$ from his starting point and that this distance makes an angle $(n-1)\frac{\alpha}{2}$ with his original direction.

226. In order to find the dip of a stream of coal below the surface of the ground, vertical borings are made from the angular point A, B, C of a triangle ABC which is in a horizontal plane; the depths

of a stratum at these points are found to be $x, x+y$ and $x+z$ respectively. Show that the dip θ of the stratum which is assumed to be a plane is given by $\tan\theta\sin A = \sqrt{\frac{y^2}{c^2}+\frac{z^2}{b^2}-\frac{2yz}{bc}\cos A}$.