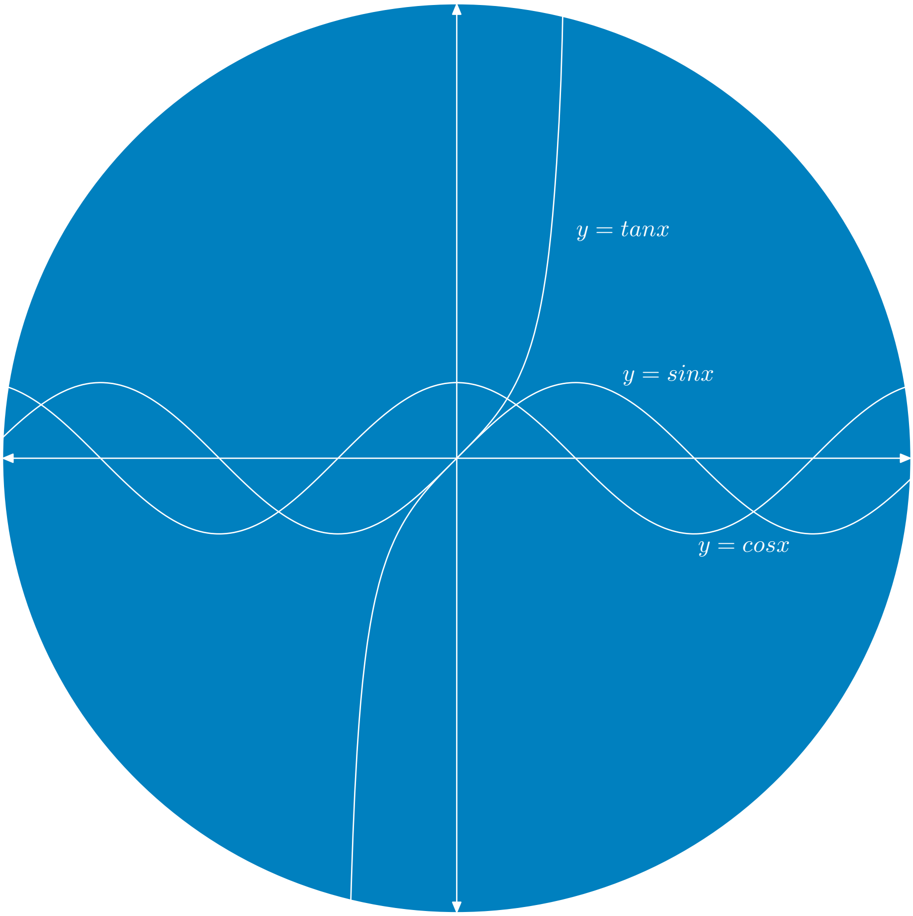
Shiv Shankar Dayal



An Angle in Trigonometry

A problem-oriented approach

An Angle in Trigonometry
Early Draft [May 20, 2025]

Dedicated to my wife, Binita

Table of Contents

Preface

This is a book on trigonometry, which, covers basics of trigonometry till high school level. It covers the most essential topics to take up a bachelor's course where knowledge of trigonometry is required. I will try to cover as much as I can and will keep adding new material over a long period.

Trigonomtery is probably one of the most fundamental subjects in Mathematics as further study of subjects like coordinate geometry, 3D and 2D geometry, engineerring and rest all depend on it. It is very important to understand trigonometry for the readers if they want to advance further in mathematics.

How to Read This Book?

Every chapter will have theory. Read that first. Make sure you understand that. Of course, you have to meet the prerequisites for the book. Then, go on and try to solve the problems. In this book, there are no pure problems. Almost all have answers except those which are of similar kind and repetitive in nature for the sake of practice. If you can solve the problem then all good else look at the answer and try to understand that. Then, few days later take on the problem again. If you fail to understand the answer you can always email me with your work and I will try to answer to the best of my ability. However, if you have a local expert seek his/her advice first.

Note that mathematics is not only about solving problems. If you understand the theory well, then you will be able to solve problems easily. However, problems do help enforce with the enforcement of theory in your mind.

I am a big fan of old MIR publisher's problem books, so I emphasize less on theory and more on problems. I hope that you find this style much more fun as a lot of theory is boring. Mathematics is about problem solving as that is the only way to enforce theory and find innovtive techniques for problem solving.

Some of the problems in certain chapters rely on other chapters which you should look ahead or you can skip those problems and come back to it later. Since this books is meant for self study answers of most of the problems have been given which you can make use of. However, do not use for just copying but rather to develop understanding.

Who Should Read This Book?

Since this book is written for self study anyone with interest in trigonometry can read it. That does not mean that school or college students cannot read it. You need to be selective as to what you need for your particular requirements. This is mostly high school course with a little bit of lower classes' course thrown in with a bit of detail here and there.

Prerequisite

You should have knowledge till grade 8th course. Attempt has been made to keep it simple and give as much as background to the topic which is reasonable and required. However, not everything will be covered below grade 10.

Goals for Readers

The goal of for reading this book is becoming proficient in solving simple and basic problems of trigonometry. Another goal would be to be able to study other subjects which require this knowledge like trigonometry or calculus or physics or chemistry or other subjects. If you can solve 95% problems after 2 years of reading this book then you have achieved this goal.

All of us possess a certain level of intelligence. At average any person can read this book. But what is most important is you have to have interest in the subject. Your interest gets multiplied with your intelligence and thus you will be more capable than you think you can be. One more point is focus and effort. It is not something new which I am telling but I am saying it again just to emphasize the point. Trust me if you are reading this book for just scoring a nice grade in your course then I have failed in my purpose of explaining my ideas.

Also, if you find this book useful feel free to share it with others without hesitation as it is free as in freedom. There are no conditions to share it.

Confession

I feel like an absolute thief while writing this book for nothing given in this book is mine. All of it belongs to others who did the original work and I have just copied shamelessly. I have nothing new to put in the book. This book is just the result of the pain I feel when I see young children wasting their life for they are poor. And therefore, this book is licensed under GNU FDL. Even if I manage to create few new problems it is still based on knowledge of other pioneers of the subject but perhaps that is how we are supposed to progress bit-by-bit.

Acknowledgements

I am in great debt of my family and free software community because both of these groups have been integral part of my life. Family has prvided direct support while free software community has provided the freedom and freed me from the slavery which comes as a package with commercial software. I am especially grateful to my wife, son and parents because it is their time which I have borrowed to put in the book. To pay my thanks from free software community I will take one name and that is Richard Stallman who started all this and is still fighting this never-ending war. When I was doing the Algebra book then I realized how difficult it is to put Math on web in HTML format and why Donald Knuth wrote TEX. Also, TEX was one of the first softwares to be released as a free software.

Now as this book is being written using ConTEXt so obviously Hans Hagen and all the people involved with it have my thanks along with Donald Knuth. I use Emacs with Auctex and hope that someday I will use it in a much more productive way someday.

I have used Asymptote and tikz for drawing all the diagrams. Both are wonderful packages and work very nicely. Asymptote in particular is very nice for 3d-drawings and linear equation solving. I have yet to learn Metafun which comes with ConTEXt.

I would like to thank my parents, wife, son and daughter for taking out their fair share of time and the support which they have extended to me during my bad times. After that I would like to pay my most sincere gratitude to my teachers particularly H. N. Singh, Yogendra Yadav, Satyanand Satyarthi, Kumar Shailesh and Prof. T. K. Basu. Now is the turn of people from software community. I must thank the entire free software community for all the resources they have developed to make computing better. However, few names I know and here they go. Richard Stallman is the first, Donald Knuth, Edger Dijkstra, John von Neumann after that as their lives have strong influence in how I think and base my life on. Cover graphics has been done by Koustav Halder so much thanks to him. I am not a native English speaker and this book has just gone through one pair of eyes therefore chances are high that it will have lots of errors(particularly with commas and spelling mistakes). At the same time it may contain lots of technical errors. Please feel free to drop me an email at shivshankar.dayal@gmail.com where I will try to respond to each mail as much as possible. Please use your real names in email not something like coolguy. If you have more problems which you want to add it to the book please send those by email or create a PR on github. The github url is https://github.com/shivshankardayal/Trigonometry-Context.

Shiv Shankar Dayal
Nalanda, 2023

I
Theory and Problems

Chapter 1
Measurement of Angles

The word trigonometry comes from means measurement of triangles. The word originally comes from Greek language. measurement. The objective of studying plane trigonometry is to develop a method of solving plane triangles. However, as time changes everything it has changed the scope of trigonometry to include polygons and circles as well. A lot of concepts in this book will come from your geometry classes in lower classes. It is a good idea to review the concepts which you have studied till now without which you are going to struggle while studying trigonometry in this book.

1.1 Angles in Geometry

If we consider a line extending to infinity in both directions, and a point OO which divides this line in two parts one on each side of the point then each part is called a ray or half-line. Thus O divides the line into two rays OA and OA'.

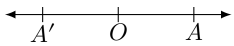


Figure 1.1

The point O is called vertex or origin for these days. An angle is a figure formed by two rays or half lies meeting at a common vertex. These half lines are called *sides of the angle.*

An angle is denoted by the symbol $\angle$ followed by three capital letters of which the middle one represents the vertex and remaining two points point to two sides. Otherwise the angle is simply written as one letter representing the vertex of the angle.

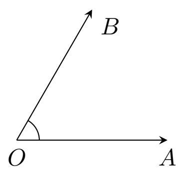


Figure 1.2 An angle

The angle in above image is written as $\angle AOB$ or $\angle BOA$ or $\angle O$.

Each angle can be measured and there are different units for the measurement. In Geometry, an angle always lie between 0° and 360° and negative angles are meaningless. Measure of an angle is the smallest amount of rotation from the direction of one ray of the angle to the direction of the other.

1.2 Angles in Trigonometry

Angles are more generalized in Trigonometry. They can have positive or negative values. As was the case in gerometry, similarly angles are measured in Trigonometry. The starting and ending positions of revolving rays are called initial side and terminal side respectively. The revolving half line is called the generating line or the radius vector. For example, if OA and OB are the initial and final position of the radius vector then angle formed will be $\angle AOB$.

1.3 Angles Exceeding 360°

In Geometry, angles are limited to 0° to 360°. However, when multiple revolutions are involved angles are more than 360°. For example, the revolving line starts from the initial position and makes n complete revolutions in anticlockwise direction and also further angle α in the same direction. We then have a certain angle β_n given by $\beta_n = x \times 360^\circ + \alpha$, where $0^\circ < \alpha < 360^\circ$ and n is zero or positive integer. Thus, there are infinite possible angles.

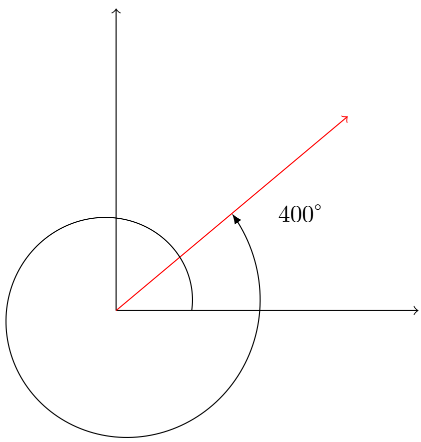


Figure 1.3 An angle

Angles formed by anticlockwise rotation of the radius vector are taken as positive and angles formed by clockwise rotation of the radius vector are taken as negative.

1.4 Quadrants

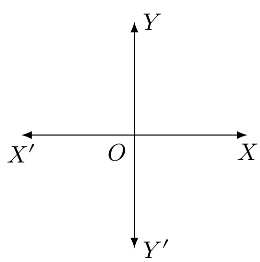


Figure 1.4 Quadrants

Let XOX' and YOY' be two mutually perpendicular lines in a plane and OX be the initial half line. The lines divide the whole reason in quadrants. $XOY, YOX', X'OY'$ and $Y'OX$ are respectively called 1st, 2nd, 3rd and 4th quadrants. According to terminal side lying in 1st, 2nd, 3rd and 4th quadrants the angles are said to be in 1st, 2nd, 3rd and 4th quadrants respectively. A *quandrant angle* is an angle formed if terminal side coincides with one of the axes.

For any angle $\angle$ which is not a quadrant angle and when number of revolutions is zero and radius vector rotates in anticlockwise directions:

- $0^\circ < \alpha < 90^\circ$ if α lies in first quadrant
- $90^\circ < \alpha < 180^\circ$ if α lies in second quadrant
- $180^\circ < \alpha < 270^\circ$ if α lies in third quadrant
- $270^\circ < \alpha < 360^\circ$ if α lies in fourth quadrant
- when terminal side lies on OY, angle formed $= 90^\circ$
- when terminal side lies on OX', angle formed $= 180^\circ$
- when terminal side lies on OY', angle formed $= 270^\circ$
- when terminal side lies on OX, angle formed $= 360^\circ$

1.5 Units of Measurement

In Geometry, angles are usually measured in terms of right angles, however, that is an inconvenient system for smaller angles. So we introduce different systems of measurements. There are three system of units for this:

1. Sexagecimal or British system: In British system, a right angle is divided into 90 equal parts called degrees. Each degree is then divided into 60 equal parts called minutes and each minute is further is divided into 60 parts called seconds.

   A degree, a minute and a second are denoted by 1°, 1", and 1 respectively.

2. Centesimal or French System: In French system, a right angle is divided into 100 equal parts called grades. Each grade is then divided into 100 equal parts called minutes and each minute is further is divided into 100 parts called seconds.

   A degree, a grade and a second are denoted by 1^g, 1", and 1 respectively.

3. Radian or Circular Measure: An arc equal to radius of a circle when subtends an anngle on the center then that angle is 1 radian and is denoted by 1^c. The angle made by half of perimeter is π radians. Also, from Geometry we know that angle subtended is the ratio between length of cord and radius. This ratio is in radians. Since both length or chord and radius have same unit radian is a constant.

1.5.1 Relationship between Systems of Measurements

If measure of an angle if D degrees, G grades and C radians then upon elementary manipulation we find that $\frac{D}{180} = \frac{G}{200} = \frac{C}{\pi}$.

1.5.2 Meaning of π

The ratio of circumference and diameter of a circle is always constant and this constant is denoted by gree letter π.

π is an irrational number. In general, we use the value of $\frac{22}{7}$ but $\frac{355}{113}$ is more accurate though not exact. If r be the radius of a circle and c be the circumference then $\frac{c}{2r} = \pi$ leading circumference to be $c = 2\pi r$.

1.6 Problems

1. Reduce $63^\circ 14' 51''$ to centisimal measure.
2. Reduce $45^\circ 20' 10''$ to centisimal and radian measure.
3. Reduce $94^g 23' 27''$ to Sexagecimal measure.
4. Reduce 1.2 radians in Sexageciaml measure.

Express in terms of right angle; the angles

5. 60°
6. $75^\circ 15'$
7. $63^\circ 17' 25''$
8. $130^\circ 30'$
9. $210^\circ 30' 30''$
10. $370^\circ 20' 48''$

Express in grades, minutes and degrees

11. 30°
12. 81°
13. $138^\circ 30'$
14. $35^\circ 47' 15''$
15. $235^\circ 12' 36'' s$
16. $475^\circ 13' 48''$

Express in terms of right angles and also in degrees, minutes and seconds; the angles

17. 120^g
18. $45^g 35' 24''$
19. $39^g 45' 36''$
20. $255^g 8' 9''$
21. $759^g 0' 5''$
22. Reduce $55^\circ 12' 36''$ to centisimal measure.
23. Reduce $18^\circ 33' 45''$ to circular measure.
24. Reduce $196^g 35' 24''$ to sexagecimal measure.
25. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hand of a watch.

26. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both angles in degrees.

27. Prove that the number of Sexagecimal minutes in any angle is to the number of Centisimal minutes in the same angle as 27 : 50.

28. Divide $44°8'$ into two parts such that the number of Sexagecimal seconds in one part may be equal to number of Centisimal seconds in the other part.

29. The angles of a triangle are in the ratio of 3 : 4 : 5, find the smallest angle in degrees and greatest angle in radians.

30. Find the angle between the hour hand and the minute hand in circular measure at half past four.

31. If p, q and r denote the grade measure, degree measure and the radian measure of the same angle, prove that

    i. $\frac{p}{10} = \frac{q}{9} = \frac{20r}{\pi}$

    ii. $p - q = \frac{20r}{\pi}$

32. Two angles of a triangle are $72°53'51''$ and $41°22'50''$ respectively. Find the third angle in radians.

33. The angles of triangle are in A.P. and the number of radians in the greatest angle is to the number of degrees in the least one as $\pi : 60$; find the angles in degrees.

34. The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest is $40 : \pi$; find the angles in degrees.

35. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least is $800 : \pi$; and the sum of angles is $126°$. Find the angles in grades.

36. Find the angle between the hour-hand and minute-hand in circular measure at 4 o'clock.

37. Express in sexagecimal system the angle between the minute-hand and hour-hand of a clock at quarter to twelve.

38. The diamter of a wheel is 28 cm; through what distance does its center move during one rotation of wheel along the ground?

39. What must be the radius of a circular running path, round which an athlete must run 5 time in order to describe 1760 meters?

40. The wheel of a railway carriage is 90 cm in diameter and it makes 3 revolutions per second; how fast is the train going?

41. A mill sail whose length is 540 cm makes 10 revolutions per minute. What distance does its end travel in one hour?

42. Assuming that the earth describes in one year a circle, or $149,700,000$ km. radius, whose center is the sun, how many miles does earth travel in a year?

43. The radius of a carriage wheel is 50 cm, and in $\frac{1}{9}$ th of a second it turns through 80° about its center, which is fixed; how many km. does a point on the rim of the wheel travel in one hour?

44. Express in terms of three systems of angular measurements the magnitude of an angle of a regular decagon.

45. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, while the third is $\frac{\pi x}{75}$ radians; express them all in degrees.

46. The circular measure of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$. What is the number of degrees of the third angle?

47. The angles of a triangle are in A.P. The number of radians in the least angle is to the number of degree in the mean angle is $1 : 120$. Find the angles in radians.

48. Find the magnitude, in radians and degrees, of the interior angle of 1. a regular pentagon 2. a regular heptagon 3. a regualr octagon 4. a regular duodecagon 5. a polygon with 17 sides

49. The angle in one regular polygon is to that in another is $3 : 2$, also the number of sides in the first is twice that in the second. How many sides are there in the polygons?

50. The number of sides in two regular polygons are as $5 : 4$, and the difference between their angles is 9°; find the number of sides in the polygons.

51. Find two regular polygons such that the number of their sides may be 3 to 4 and the number of degrees of an angle of the first to the number of grades of the second as 4 to 5.

52. The angles of a qadrilateral are in A.P. and the greatest is double the least; express the least angle in radians.

53. Find in radians, degrees, and grades the angle between hour-hand and minute-hand of a clock at 1. half-past three 2. twenty minutes to six 3. a quarter past eleven.

54. Find the times 1. between fours and five o'clock when the angle between the minute hand and the hour-hand is 78°, 2. between seven and eight o'clock when the angle is 54°

55. The interior angles of a polygon are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.

56. The angles of quadrilateral are in A.P. and the number of grades in the least angle is to the number of radians in the greatest is $100 : \pi$. Find the angles in degrees.

57. The anlges of a polygons are in A.P. The least angle is $\frac{5\pi}{12}$ common difference is 10°, find the number of sides in the polygon.

58. Find the angle subtended at the center of a circle of radius 3 cm. by an arc of length 1 cm.

59. In a circle of radius 5 cm., what is the length of the arc which subtends an angle of $33^{\circ}15'$ at the center.

60. Assuming the average distance of sun from the earth to be $149,700,000$ km., and the angle subtended by the sun at the eye of a person on the earth is $32'$, find the sun's diameter.

61. Assuming that a person of normal sight can read print at such a distance that the letter subtends and angle of $5'$ at his eye, find what is the height of the letters he can read at a distance of 1. 12 meters 2. 1320 meters.

62. Find the number of degrees subtended at the center of a circle by an arc whose length is 0.357 times the radius.

63. Express in radians and degrees the angle subtended at the center of a circle by an arc whose length is 15 cm., the radius of the circle being 25 cm.

64. The value of the divisions on the outer rim of a graduated cicle is $5'$ and the distance between successive graduations is .1 cm. Find the radius of the circle.

65. The diamter of a graduated circle is 72 cm., and the gradiuations on the rim are $5'$ apart; find the distance of one graduation to to another.

66. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by $1°10'$ may be 0.5 cm.

67. Taking the radius of earth to as 6400 km., find the difference in latitude of two places, one of which is 100 km. north of another.

68. Assuming the earth to be a sphere and the difference between two parallels of latitude, which subtends an angle of $1°$ at the eath's center, to be $69\frac{1}{2}$ km., find the radius of the earth.

69. What is the ratio of radii of the circles at the center of which two arcs of same length subtend angles of $60°$ and $75°$?

70. If an arc, of length 10 cm., on a circle of 8 cm. diameter subtend at the center of circle an angle of $143°14'22''$, find the value of π to 4 places of decimals.

71. If the circumference of a circle be divided into five parts which are in A.P., and if the greatest part be six times the least find in radians the the magnitude of the angles the parts subtend at the center of the circle.

72. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius; express the angle of the sector in degrees.

73. At what distance a man, whose height is 2 m., subtend an angle of $10'$.

74. Find the length which at a distance of 5280 m., will subtend an angle of $1'$ at the eye.

75. Assuming the distance of the earth from the moon to be 38400 km., and the angle subtended by the moon at the eye of a person on earth to be $31'$, find the diameter of the moon.

76. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second; how fast is the train going?

77. Assuming that moon subtends an angle of $30'$ at the eye of an observer, find how far from the eye a coin of one inch diameter must be held so as just to hide the moon.

78. A wheel make 30 revolutions per minute. Find the circular measure of the angle described by spoke in half a second.

79. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends an angle of 56° at the center. Find the diamter of the circle.

Chapter 2
Trigonometric Ratios

From Geometry, we know that an acute angle is an angle whose measure is between $0°$ and $90°$. Consider the following figure:

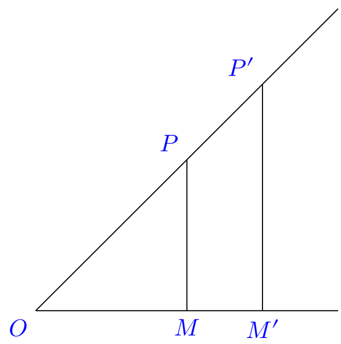


Figure 2.1 Trigonometric ratios

This picture contains two similar triangles $\triangle OMP$ and $\triangle OM'P'$. We are interested in $\angle MOP$ or $\angle M'OP'0$. In the $\triangle MOP$ and $\triangle M'OP'$, OP, OP' are called the hypotenuses i.e. sides opposite to the right angle, $PM, P'M'$ are called perpendiculars i.e. sides opposite to the angle of interest and OM, OM' are called bases i.e. the third angle.

Hypotenuses are usually denoted by h, perpendiculars by p and bases by b. Let $OM = b, OM' = b', PM = p, P'M' = p', OP = h, OP' = h'$. Since the two triangles are similar $\therefore \frac{p}{p'} = \frac{b}{b'} = \frac{h}{h'}$. Thus rhe ratio of any two sides is dependent purely on $\angle O$ or $\angle MOP$ or $\angle M'OP'$.

Since there are three sides, we can choose 2 in ${}^3C_2$ i.e. 3 ways and for each combination there will be two permutations where a side can be in either numerator or denominator. From this we can conclude that there will be six ratios(these are called trigonometric ratios), These six trigonometric ratios or functions are given below:

$\frac{MP}{OP}$ or $\frac{p}{h}$ is called the **Sine** of the $\angle MOP$.

$\frac{OM}{OP}$ or $\frac{b}{h}$ is called the **Cosine** of the $\angle MOP$.

$\frac{MP}{OM}$ or $\frac{p}{b}$ is called the **Tangent** of the $\angle MOP$.

$\frac{OP}{MP}$ or $\frac{h}{p}$ is called the **Cosecant** of the $\angle MOP$.

$\frac{OP}{OM}$ or $\frac{h}{b}$ is called the **Secant** of the $\angle MOP$.

$\frac{OM}{MP}$ or $\frac{b}{p}$ is called the **Cotangent** of the $\angle MOP$.

$1 - \cos MOP$ is called the **Versed Sine** of $\angle MOP$ and $1 - \sin MOP$ is called the **Coversed Sine** of $\angle MOP$. These two are rarely used in trigonometry. It should be noted that the trigonometric ratios are all numbers. The name of the trigonometric ratios are written for brevity $\sin MOP$, $\cos MOP$, $\tan MOP$, $\cot MOP$, $\sec MOP$, $\mathrm{cosec}MOP$, $\mathrm{vers}MP$, $\mathrm{coverse}MOP$.

2.1 Relationship betweeen Trigonometric Functions or Ratios

Let us represent the $\angle MOP$ with θ, we observe from previous section that

$$\sin\theta = \frac{1}{\csc\theta}, \cos\theta = \frac{1}{\sec\theta}, \tan\theta = \frac{1}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{1}{\tan\theta}$$

We also observe that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

From Pythagora theorem in geometry, we know that $\text{hypotenuse}^2 = \text{perpendicular}^2 + \text{base}^2$ or $h^2 = p^2 + b^2$

1. Dividing both side by h^2, we get

$$\frac{p^2}{h^2} + \frac{b^2}{h^2} = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

We can rewrite this as $\sin^2\theta = 1 - \cos^2\theta$, $\cos^2\theta = 1 - \sin^2\theta$, $\sin\theta = \sqrt{1-\cos^2\theta}$, $\cos\theta = \sqrt{1-\sin^2\theta}$.

2. If we divide both sides by b^2, then we get

$$\frac{h^2}{b^2} = \frac{p^2}{b^2} + 1$$

$$\sec^2\theta = \tan^2\theta + 1$$

We can rewrite this as $\sec^2\theta - \tan^2\theta = 1$, $\tan^\theta = \sec^2\theta - 1$, $\sec\theta = \sqrt{1+\tan^2\theta}$, $\tan\theta = \sqrt{sec^2\theta - 1}$

3. Similalry, if we divide by p^2, then we get

$$\frac{h^2}{p^2} = 1 + \frac{b^2}{p^2}$$

$$\mathrm{cosec}^2\theta = 1 + \cos^2\theta$$

We can rewrite this as $\mathrm{cosec}^2\theta - \cot^2\theta = 1$, $\cot^2\theta = \mathrm{cosec}^2\theta - 1$, $\mathrm{cosec}\theta = \sqrt{1+\cot^2\theta}$, $\cot\theta = \sqrt{\mathrm{cosec}^2\theta - 1}$

2.2 Problems

Prove the following:

1. $\sqrt{\frac{1-\cos A}{1+\cos A}} = \text{cosec} A - \cot A.$

2. $\sqrt{\sec^2 A + \text{cosec}^2 A} = \tan A + \cot A.$

3. $(\text{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1.$

4. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A.$

5. $(\sin A + \cos A)(1 - \sin A \cos A) = \sin^3 A + \cos^3 A.$

6. $\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\text{cosec} A.$

7. $\sin^6 A - cos^6 A = 1 - 3\cos^2 A \sin^2 A.$

8. $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A.$

9. $\frac{\text{cosec} A}{\text{cosec} A-1} + \frac{\text{cosec} A}{\text{cosec} A+1} = 2\sec^2 A.$

10. $\frac{\text{cosec} A}{\tan A+\cot A} = \cos A.$

11. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A.$

12. $\frac{1}{\tan A+\cot A} = \sin A \cos A.$

13. $\frac{1-\tan A}{1+\tan A} = \frac{\cot A-1}{\cot A+1}.$

14. $\frac{1+\tan^2 A}{1+\cot^2 A} = \frac{\sin^2 A}{\cos^2 A}.$

15. $\frac{\sec A-\tan A}{\sec A+\tan A} = 1 - 2\sec A \tan A + 2\tan^2 A.$

16. $\frac{1}{\sec A-\tan A} = \sec A + \tan A.$

17. $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \sec A\text{cosec} A + 1.$

18. $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A.$

19. $(\sin A + \cos A)(\tan A + \cot A) = \sec A + \text{cosec} A.$

20. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$

21. $\cot^4 A + \cot^2 A = \text{cosec}^4 A - \text{cosec}^2 A.$

22. $\sqrt{\text{cosec}^2 A - 1} = \cos A\text{cosec} A.$

23. $\sec^2 A\text{cosec}^2 A = \tan^2 A + \cot^2 A + 2.$

24. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

25. $(1+\cot A-\operatorname{cosec} A)(1+\tan A+\sec A)=2.$

26. $\frac{\cot A \cos A}{\cot A+\cos A}=\frac{\cot A-\cos A}{\cot A \cos A}.$

27. $\frac{\cot A+\tan B}{\cot B+\tan A}=\cot A \tan B.$

28. $\left(\frac{1}{\sec^2 A-\cos^2 A}+\frac{1}{\operatorname{cosec}^2 A-\sin^2 A}\right)\cos^2 A \sin^2 A=\frac{1-\cos^2 A \sin^2 A}{2+\cos^2 A \sin^2 A}.$

29. $\sin^8 A-\cos^8 A=(\sin^2 A-\cos^2 A)(1-2\sin^2 A\cos^2 A).$

30. $\frac{\cos A \operatorname{cosec} A-\sin A \sec A}{\cos A+\sin A}=\operatorname{cosec} A-\sec A.$

31. $\frac{1}{\operatorname{cosec} A-\cot A}-\frac{1}{\sin A}=\frac{1}{\sin A}-\frac{1}{\operatorname{cosec} A+\cot A}.$

32. $\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}.$

33. $(\tan A+\operatorname{cosec} B)^2-(\cot B-\sec A)^2=2\tan A\cot B(\operatorname{cosec} A+\sec B).$

34. $2\sec^2 A-\sec^4 A-2\operatorname{cosec}^2 A+\operatorname{cosec}^4 A=\cot^4 A-\tan^4 A.$

35. $(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2=\tan^2 A+\cot^2 A+7.$

36. $(\operatorname{cosec} A+\cot A)(1-\sin A)-(\sec A+\tan A)(1-\cos A)=(\operatorname{cosec} A-\sec A)[2-(1-\cos A)(1-\sin A)].$

37. $(1+\cot A+\tan A)(\sin A-\cos A)=\frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}.$

38. $\frac{1}{\sec A-\tan A}-\frac{1}{\cos A}=\frac{1}{\cos A}-\frac{1}{\sec A+\tan A}.$

39. $3(\sin A-\cos A)^4+4(\sin^6 A+\cos^6 A)+6(\sin A+\cos A)^2=13.$

40. $\sqrt{\frac{1+\cos A}{1-\cos A}}=\operatorname{cosec} A+\cot A.$

41. $\frac{\cos A}{1+\sin A}+\frac{\cos A}{1-\sin A}=2\sec A.$

42. $\frac{\tan A}{\sec A-1}+\frac{\tan A}{\sec A+1}=2\operatorname{cosec} A.$

43. $\frac{1}{1-\sin A}-\frac{1}{1+\sin A}=2\sec A\tan A.$

44. $\frac{1+\tan^2 A}{1+\cot^2 A}=\left(\frac{1-\tan A}{1-\cot A}\right)^2.$

45. $1+\frac{2\tan^2 A}{\cos^2 A}=\tan^4 A+sec^4 A.$

46. $(1-\sin A-\cos A)^2=2(1-\sin A)(1-\cos A).$

47. $\frac{\cot A+\operatorname{cosec} A-1}{\cot A-\operatorname{cosec} A+1}=\frac{1+\cos A}{\sin A}.$

48. $(\sin A + \sec A)^2 + (\cos A + \mathrm{cosec} A)^2 = (1 + \sec A \mathrm{cosec} A)^2$.

49. $\frac{2\sin A\tan A(1-\tan A)+2\sin A\sec^2 A}{(1+\tan A)^2} = \frac{2\sin A}{1+\tan A}$.

50. If $2\sin A = 2 - \cos A$, find $\sin A$.

51. If $8\sin A = 4 + \cos A$, find $\sin A$.

52. If $\tan A + \sec A = 1.5$, find $\sin A$.

53. If $\cot A + \mathrm{cosec} A = 5$, find $\cos A$.

54. If $3\sec^4 A + 8 = 10\sec^2 A$, find the value of $\tan A$.

55. If $\tan^2 A + \sec A = 5$, find $\cos A$.

56. If $\tan A + \cot A = 2$, find $\sin A$.

57. If $\sec^2 A = 2 + 2\tan A$, find $\tan A$.

58. If $\tan A = \frac{2x(x+1)}{2x+1}$, find $\sin A$ and $\cos A$.

59. If $3\sin A + 5\cos A = 5$, show that $5\sin A - 3\cos A = \pm 3$.

60. If $\sec A + \tan A = \sec A - \tan A$ prove that each side is ± 1.

61. If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, prove that

    i. $\sin^4 A + \sin^4 B = 2\sin^2 A \sin^2 B$,

    ii. $\frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1$.

62. If $\cos A + \sin A = \sqrt{2}\cos A$, prove that $\cos A - \sin A = \pm\sqrt{2}\sin A$.

63. If $a\cos A - b\sin A = c$, prove that $a\sin A + b\cos A = \sqrt{a^2 + b^2 - c^2}$.

64. If $1 - \sin A = 1 + \sin A$, then prove that value of each side is $\pm\cos A$.

65. If $\sin^4 A + \sin^2 A = 1$, prove that

    i. $\frac{1}{\tan^4 A} + \frac{1}{\tan^2 A} = 1$,

    ii. $\tan^4 A - \tan^2 = 1$.

66. If $\cos^2 A - \sin^2 A = \tan^2 B$, prove that $2\cos^2 B - 1 = \cos^2 B - \sin^2 B = \tan^2 A$.

67. If $\sin A + \mathrm{cosec} A = 2$, then prove that $\sin^n A + \mathrm{cosec}^n A = 2$.

68. If $\tan^2 A = 1 - e^2$, prove that $\sec A + \tan^3 A \mathrm{cosec} A = (2 - e^2)^{\frac{3}{2}}$.

69. Eliminate A between the equations $a\sec A + b\tan A + c = 0$ and $p\sec A + q\tan A + r = 0$.

70. If cosec$A - \sin A = m$ and $\sec A - \cos A = n$, elimiate A.

71. Is the equation $\sec^2 A = \frac{4xy}{(x+y)^2}$ possible for real values of x and y?.

72. Show that the equation $\sin A = x + \frac{1}{x}$ is imossible for real values of x.

73. If $\sec A - \tan A = p, p \neq 0$, find $\tan A, \sec A$ and $\sin A$.

74. If $\sec A = p + \frac{1}{4p}$, show that $\sec A + \tan A = 2p$ or $\frac{1}{2p}$.

75. If $\frac{\sin A}{\sin B} = p, \frac{\cos A}{\cos B} = q$, find $\tan A$ and $\tan B$.

76. If $\frac{\sin A}{\sin B} = \sqrt{2}, \frac{\tan A}{\tan B} = \sqrt{3}$, find A and B.

77. If $\tan A + \cot A = 2$, find $\sin A$.

78. If $m = \tan A + \sin A$ and $n = \tan A - \sin A$, prove that $m^2 - n^2 = 4\sqrt{mn}$.

79. If $\sin A + \cos A = m$ and $\sec A + \text{cosec}A = n$, prove that $n(m^2 - 1) = 2m$.

80. If $x\sin^3 A + y\cos^3 A = \sin A\cos A$ and $x\sin A - y\cos A = 0$, prove that $x^2 + y^2 = 1$.

81. Prove that $\sin^2 A = \frac{(x+y)^2}{4xy}$ is possible for real values of x and y only when $x = y$ and $x, y \neq 0$.

Chapter 3
Trigonometric Ratios of Any Angle and Sign

3.1 Angle of 45°

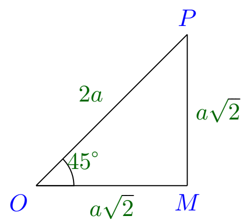


Figure 3.1

Consider the above figure, which is a right-angle triangle, drawn so that $\angle OMP = 90^\circ$ and $\angle MOP = 45^\circ$. We know that the sum of all angles of a triangle is 180°. Thus,

$\angle OPM = 180^\circ - \angle MOP - \angle OMP = 180^\circ - 90^\circ - 45^\circ = 45^\circ$

$\therefore OM = MP$. Let $OP = 2a$, then from Pythagora theorem, we can write

$4a^2 = OP^2 = OM^2 + MP^2 = 2OM^2 \Rightarrow Om = a\sqrt{2} = MP$

$\sin 45^\circ = \frac{MP}{OP} = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}}$.

Other trigonometric ratios can be deduced similarly for this angle.

3.2 Angles of 30° and 60°

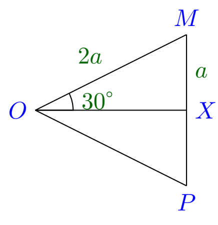


Figure 3.2

Consider an equilateral $\triangle OMP$. Let the sides OM, OP, MP be each $2a$. We draw a bisector of $\angle MOP$, which will be a perpendicular bisector of MP at X because the triangle is equilateral. Thus, $MX = a$. In $\triangle OMX, OM = 2a, \angle MOX = 30^\circ, \angle OXM = 90^\circ$ because each angle in an equilateral triangle is 60°.

$\sin MOX = \frac{MX}{OM} = \frac{1}{2} \Rightarrow \sin 30^\circ = \frac{1}{2}$

Similarly, $\angle OMX = 60^\circ$ because the sum of all angles of a triangle is 180°.

$\cos OMX = \frac{MX}{OM} = \frac{1}{2} \Rightarrow \cos 60^\circ = \frac{1}{2}$

All other trigonometric ratios can be found from these two.

3.3 Angle of 0°

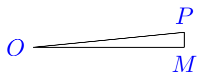


Figure 3.3

Consider the $\triangle MOP$ such that side MP is smaller than any quantity we can assign i.e. what we denote by 0. Thus, $\angle MOP$ is what is called approaching 0 or $\lim_{x\to 0}$ in terms of calculus. Why we take such a value is because if any angle of a triangle is equal to 0° then the triangle won't exist. Thus these values are limiting values as you will learn in calculus.

However, in this case, $\sin 0^\circ = \dfrac{MP}{OP} = \frac{0}{OP} = 0$. Other trigonometric ratios can be found from this easily.

3.4 Angle of 90°

In the previous figure, as $\angle OMP$ will approach 0°, the $\angle OPM$ will approach 90°. Also, OP will approach the length of OM. Similar to previous case, in right-angle trianglee if one angle (other than right angle) approaches 0° the other one will appraoch 90° and at that value the triangle will cease to exist.

Thus, $\sin 90^\circ = \dfrac{OM}{OP} = \frac{OP}{OP} = 1$. Now other angles can be found easily from this.

Given below is a table of most useful angles:

Angle	**0°**	**30°**	**45°**	**60°**	**90°**
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Table 3.1 Values of useful angles

3.5 Complementary Angles

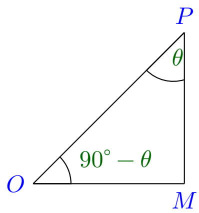


Figure 3.4

Angles are said to be complementary if their sum is equal to one right angle i.e. 90°. Thus, if measure of one angle is θ the other will automatically be $90^\circ - \theta$.

Consider the figure. $\triangle OMP$ is a right-angle triangle, whose $\angle OMP$ is a right angle. Since the sum of all angles is 180°, therefore sum of $\angle MOP$ and $\angle MPO$ will be equalto one right angle or 90° i.e. they are complementary angles.

Let $\angle MPO = \theta$ then $\angle MOP = 90^\circ - \theta$. When $\angle MPO$ is considered MP becomes the base and OM becomes the perpendicular.

Thus, $\sin(90^\circ - \theta) = \sin MOP = \frac{MP}{OP} = \cos MPO = \cos\theta$

$\cos(90^\circ - \theta) = \sin MPO = \frac{MO}{OP} = \sin\theta$

$\tan(90^\circ - \theta) = \tan MOP = \frac{PM}{OM} = \cot MPO = \cot\theta$

Similarly, $\cot(90^\circ - \theta) = \tan\theta$, $\operatorname{cosec}(90^\circ - \theta) = \sec\theta$, $\sec(90^\circ - \theta) = \operatorname{cosec}\theta$.

3.6 Supplementary Angles

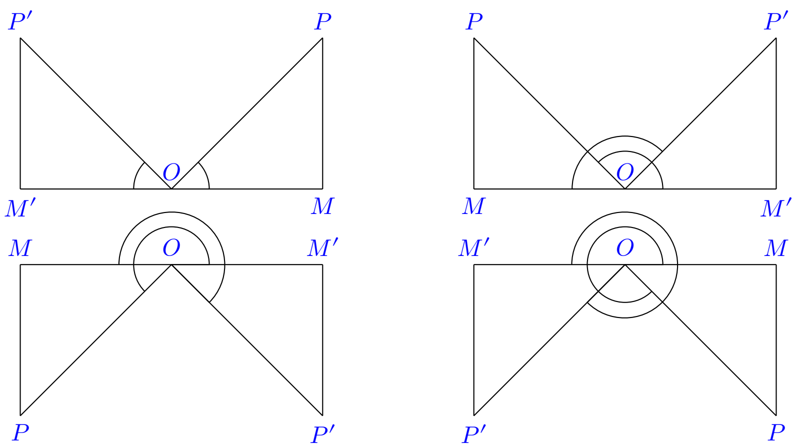


Figure 3.5

Angles are said to be supplementary if their sum is equal to two right angles i.e. 180°. Thus, if measure of one angle is θ, the other will automatically be $180^\circ - \theta$.

Consider the above figure which includes the angles of $180^\circ - \theta$. In each figure OM and OM' are drawn in different directions, while MP and $M'P'$ are drawn in the same direction so that $OM' = -OM$ and $M'P' = MP$. Hence we can say that

$\sin(180^\circ - \theta) = \sin MOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta$

$\cos(180^\circ - \theta) = \cos MOP' = \frac{OM'}{OP'} = -\frac{OM}{OP} = -\cos\theta$

$\tan(180^\circ - theta) = \tan MOP' = \frac{OM'}{M'P'} = -\frac{OM}{MP} = -\tan\theta$

Similarly, $\cot(180^\circ - \theta) = -\cot\theta, \sec(180^\circ - \theta) = -\sec\theta, \mathrm{cosec}(180^\circ - \theta) = \mathrm{cosec}\theta$

3.7 Angles of $-\theta$

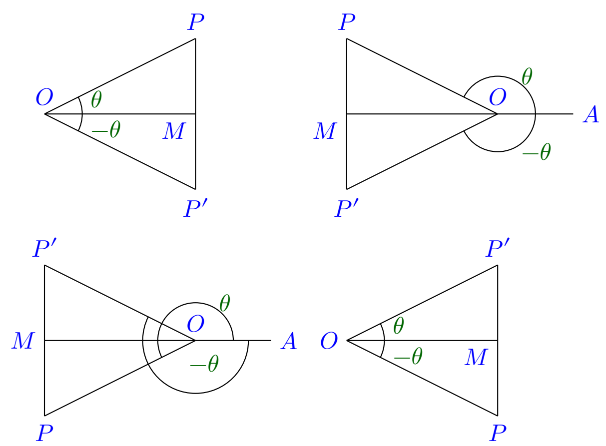


Figure 3.6

Consider the above diagram which plots the angles of θ and $-\theta$. Note that MP and MP' are equal in magnitude but opposite in sign. Thus, we have

$\sin(-\theta) = \frac{MP'}{OP'} = -\frac{MP}{OP} = -\sin\theta.$

$\cos(-\theta) = \frac{OM}{MP'} = \frac{OM}{OP} = \cos\theta.$

$tan(-\theta) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\theta.$

Similarly, $\cot(-\theta) = -\cot\theta, \sec(-\theta) = sec\theta, \mathrm{cosec}(-\theta) = -\mathrm{cosec}\theta.$

3.8 Angles of $90^\circ + \theta$

The diagram has been left as an exercise. Similarly, it can be proven that $\sin(90^\circ+\theta) = \cos\theta, \cos(90^\circ+\theta) = -\sin\theta, \tan(90^\circ+\theta) = -\cot\theta, \cot(90^\circ+\theta) = -\tan\theta, \sec(90^\circ+\theta) = -\text{cosec}\theta, \text{cosec}(90^\circ+\theta) = \sec\theta$.

Angles of $180^\circ+\theta, 270^\circ-\theta, 270^\circ+\theta$ can be found using previous relations.

3.9 Angles of $360^\circ + \theta$

For angles of θ the radius vector makes an angle of θ with initial side. For angles of $360^\circ+\theta$ it will complete a full revolution and then make an angle of θ with initial side. Thus, the trigonometrical ratios for an angle of $360^\circ+\theta$ are the same as those for θ.

It is clear that angle will remain θ for any multiple of 360°.

3.10 Problems

1. If $A = 30^\circ$, verify that
   i. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
   ii. $\sin 2A = 2\sin A\cos A$
   iii. $\cos 3A = 4\cos^3 A - 3\cos A$
   iv. $\sin 3A = 3\sin A - 4\sin^3 A$
   v. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
2. If $A = 45^\circ$, verify that
   i. $\sin 2A = 2\sin A\cos A$
   ii. $\cos 2A = 1 - 2\sin^2 A$
   iii. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Verify that

3. $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ = \frac{3}{2}$
4. $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ = 4\frac{1}{3}$
5. $\sin 30^\circ\cos 60^\circ + \sin 60^\circ\cos 30^\circ = 1$
6. $\cos 45^\circ\cos 60^\circ - \sin 45^\circ\sin 60^\circ = -\frac{\sqrt{3}-1}{2\sqrt{2}}$

7. $\operatorname{cosec}^2 45^\circ . \sec^2 30^\circ . \sin^2 90^\circ . \cos 60^\circ = 1\frac{1}{3}$

8. $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 30^\circ = \frac{1}{4}$

Prove that

9. $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ) = 1$

10. $\cos 570^\circ \sin 510^\circ - \sin 330^\circ \cos 390^\circ = 0$

What are the values of $\cos A - \sin A$ and $\tan A + \cot A$ when A has the values

11. $\frac{\pi}{3}$

12. $\frac{2\pi}{3}$

13. $\frac{5\pi}{4}$

14. $\frac{7\pi}{4}$

15. $\frac{11\pi}{3}$

What values between 0° and 360° may A have when

16. $\sin A = \frac{1}{\sqrt{2}}$

17. $\cos A = -\frac{1}{2}$

18. $\tan A = -1$

19. $\cot A = -\sqrt{3}$

20. $\sec A = -\frac{2}{\sqrt{3}}$

21. $\operatorname{cosec} A = -2$

Express in terms of the ratios of a positive angle, which is less than 45°, the quantities

22. $\sin(-65^\circ)$

23. $\cos(-84^\circ)$

24. $\tan 137^\circ$

25. $\sin 168^\circ$

26. $\cos 287^\circ$

27. $\tan(-246^\circ)$

28. $\sin 843^\circ$

29. $\cos(-928^\circ)$

30. $\tan 1145^\circ$

31. $\cos 1410^\circ$

32. $\cot(-1054^\circ)$

33. $\sec 1327^\circ$

34. $\operatorname{cosec}(-756^\circ)$

What sign has $\sin A + \cos A$ for the following values of A?

35. 140°
36. 278°
37. -356°
38. -1125°

What sign has $\sin A - \cos A$ for the following values of A?

39. 215°
40. 825°
41. -634°
42. -457°

43. Find the sine and cosine of all angles in the first four quadrants whose tangents are equal to $\cos 135^\circ$.

Prove that

44. $\sin(270^\circ + A) = -\cos A$ and $\tan(270^\circ + A) = -\cot A$
45. $\cos(270^\circ - A) = -\sin A$ and $\cot(270^\circ - A) = \tan A$
46. $\cos A + \sin(270^\circ + A) - \sin(270^\circ - A) + \cos(180^\circ + A) = 0$
47. $\sec(270^\circ - A)\sec(90^\circ - A) - \tan(270^\circ - A)\tan(90^\circ + A) + 1 = 0$
48. $\cot A + \tan(180^\circ + A) + \tan(90^\circ + A) + \tan(360^\circ - A) = 0$
49. Find the value of $3\tan^2 45^\circ - \sin^2 60^\circ - \frac{1}{2}\cot^2 30^\circ + \frac{1}{8}\sec^2 45^\circ$
50. Simplify $\frac{\sin 300^\circ.\tan 330^\circ.\sec 420^\circ}{\tan 135^\circ.\sin 210^\circ.\sec 315^\circ}$
51. Show that $\tan 1^\circ \tan 2^\circ \ldots \tan 89^\circ = 1$
52. Show that $\sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \ldots + \sin^2 90^\circ = 9\frac{1}{2}$
53. Find the value of $\cos^2\frac{\pi}{16} + \cos^2\frac{3\pi}{16} + \cos^2\frac{5\pi}{16} + \cos^2\frac{7\pi}{16}$

Find the value of the following:

54. $\sec^2\frac{\pi}{6}\sec^2\frac{\pi}{4} + \tan^2\frac{\pi}{3}\sin^2\frac{\pi}{2}$
55. $\cot^2 30^\circ - 2\cos^2 60^\circ - \frac{3}{4}\sin^2 45^\circ - 4\sin^2 30^\circ$
56. $\frac{\sec 480^\circ \operatorname{cosec} 570^\circ.\tan 330^\circ}{\sin 600^\circ.\cos 660^\circ.\cot 405^\circ}$
57. If $A = 30^\circ$, show that $\cos^6 A + \sin^6 A = 1 - \sin^2 A\cos^2 A$
58. Show that $\left(\tan\frac{\pi}{4} + \cot\frac{\pi}{4} + \sec\frac{\pi}{4}\right)\left(\tan\frac{\pi}{4} + \cot\frac{\pi}{4} - \sec\frac{\pi}{4}\right) = \operatorname{cosec}^2\frac{\pi}{4}$

59. Show that $\sin^2 6^\circ + \sin 6^2 12^\circ + \sin^2 18^\circ + ... + \sin^2 84^\circ + \sin^2 90^\circ = 8$

60. Show that $\tan 9^\circ . \tan 27^\circ . \tan 45^\circ . \tan 63^\circ . \tan 81^\circ = 1$

61. Show that $\sum_{r=1}^{9} \sin^2 \frac{r\pi}{18} = 5$

62. If $4n\alpha = \pi$, show that $\tan \alpha \tan 2\alpha \tan 3\alpha . .... \tan(2n-2)\alpha \tan(2n-1)\alpha = 1$

Chapter 4
Compound Angles

Algebraic sum of two or more angles is called a *compound angle*. If A, B, C are any angle then $A+B$, $A-B$, $A-B+C$, $A+B+C$, $A-B-C$, $A+B-C$ etc. are all compound angles.

4.1 The Addition Formula

$\sin(A+B) = \sin A\cos B + \sin B\cos A$ $\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$

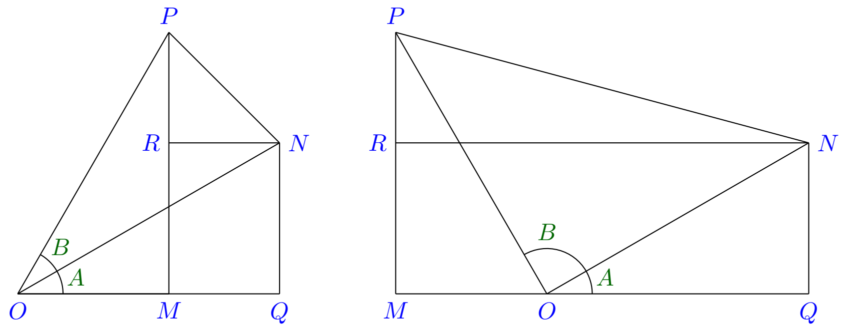


Figure 4.1

Consider the diagram above. PM and PN are perpendicualr to OQ and ON. RN is parallel to OQ and NQ is perpendicular to OQ. The left diagram represents the case when sum of angles is an acute angle while the right diagram represents the case when sum of angles is an obtuse angle.

$\angle RPN = 90^\circ - \angle PNR = \angle RNO = \angle NOQ = \angle A$

Now we can write, $\sin(A+B) = \sin QOP = \frac{MP}{OP} = \frac{MR+RP}{OP} = \frac{QN}{OP} + \frac{RP}{OP}$

$= \frac{QN}{ON}\frac{ON}{OP} + \frac{RP}{NP}\frac{NP}{OP} = \sin A\cos B + \cos A\sin B$

Also, $\cos(A+B) = \cos QOP = \frac{OM}{OP} = \frac{OQ-MQ}{OP} = \frac{OQ}{ON}\frac{ON}{OP} - \frac{RN}{NP}\frac{NP}{OP}$

$= \cos A\cos B - \sin A\sin B$

These two results lead to $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

We have shown that addition formula is true when angles involved are acute angles. The same proof can be applied to prove the results for all values of A and B.

Consider $A' = 90^\circ + A \therefore \sin A' = \cos A$ and $\cos A' = \sin A$

$\sin(A'+B) = \cos(A+B) = \cos A\cos B - \sin A\sin B = \sin A'\cos B + \cos A'\sin B$

Similarly $\cos(A'+B) = -\sin(A+B) = -\sin A\cos B - \sin B\cos A = \cos A'\cos B - \sin A'\sin B$

We can prove it again for $B' = 90^\circ + B$ and so on by increasing the values of A and B. Then we can again increase values by 90° and proceeding this way we see that the formula holds true for all values of A and B.

4.2 The Subtraction Formula

$\sin(A - B) = \sin A \cos B - \sin B \cos A$ $\cos(A - B) = \cos A \cos B + \sin A \sin B$ $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

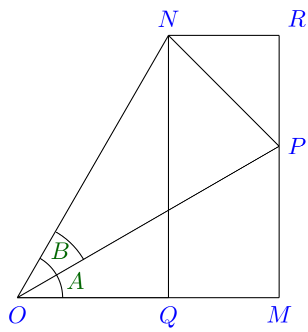


Figure 4.2

Conside the diagram above. The angle MOP is $A - B$. We take a point P, and draw PM and PN perpendicular to OM and ON respectively. From N we draw NQ and NR perpendicular to OQ and MP respectively.

$\angle RPN = 90^\circ - \angle PNR = \angle QON = A$

Thus, we can write $\sin(A - B) = \sin MOP = \frac{MP}{OP} = \frac{MR-PR}{OP} = \frac{QN}{ON}\frac{ON}{OP} - \frac{PR}{PN}\frac{PN}{OP}$

Thus, $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Also, $\cos(A - B) = \frac{OM}{OP} = \frac{OQ+QM}{OP} = \frac{OQ}{ON}\frac{ON}{OP} + \frac{RN}{NP}\frac{NP}{OP}$

$= \cos A \cos B + \sin A \sin B$

We have shown that subtraction formula is true when angles involved are acute angles. The same proof can be applied to prove the results for all values of A and B.

From the results obtained we find upon division that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

4.3 Important Deductions

1. $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

   L.H.S. $= (\sin A \cos B + \sin B \cos A)(\sin A \cos B - \sin B \cos A)$

$= \sin^2 A \cos^2 B - \sin^2 B \cos^2 A = \sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 B \sin^2 A$

$= \sin^2 A - \sin^2 B = (1 - \cos^2 A) - (1 - \cos^2 B)$

$= \cos^2 B - \cos^2 A$

2. $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

L.H.S. $= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B$

$= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

3. $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

L.H.S. $= \cot(A + B) = \frac{\cos(A+B)}{\sin(A+B)}$

$= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

Dividing numerator and denominator by $\sin A \sin B$

$= \frac{\cot A \cot B - 1}{\cot B + \cot A}$

4. $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

L.H.S. $= \cot(A - B) = \frac{\cos(A-B)}{\sin(A-B)}$

$= \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$

Dividing numerator and denominator by $\sin A \sin B$

$= \frac{\cot A \cot B + 1}{\cot B - \cot A}$

5. $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

L.H.S. $= \tan[(A + B) + C] = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$

$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B}\tan C}$

$= \frac{\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B}}{\frac{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}{1 - \tan A \tan B}}$

$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

4.4 To express $a\cos\theta+b\sin\theta$ in the form of $k\cos\phi$ or $k\sin\phi$

$a\cos\theta+b\sin\theta = \sqrt{a^2+b^2}\Big(\frac{a}{\sqrt{a^2+b^2}}\cos\theta+\frac{b}{\sqrt{a^2+b^2}}\sin\theta\Big)$

Let $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ then $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

Thus, $a\cos\theta+b\sin\theta = \sqrt{a^2+b^2}(\cos\alpha\cos\theta+\sin\alpha\sin\theta)$

$= \sqrt{a^2+b^2}\cos(\theta-\alpha) = k\cos\phi$ where $k=\sqrt{a^2+b^2}$ and $\phi=\theta-\alpha$

Alternatively, if $\frac{a}{\sqrt{a^2+b^2}} = \sin\alpha$ then $\frac{b}{\sqrt{a^2+b^2}} = \cos\alpha$

Thus, $a\cos\theta+b\sin\theta = \sqrt{a^2+b^2}(\sin\alpha\cos\theta+\cos\alpha+\sin\theta)$

$= \sqrt{a^2+b^2}\sin(\theta+\alpha) = k\sin\phi$ where $k=\sqrt{a^2+b^2}$ and $\phi=\theta+\alpha$

4.5 Problems

1. If $\sin\alpha=\frac{3}{5}$ and $\cos\beta=\frac{9}{41}$, find the values of $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$.
2. If $\sin\alpha=\frac{45}{53}$ and $\sin\beta=\frac{33}{65}$, find the values of $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$.
3. If $\sin\alpha=\frac{15}{17}$ and $\cos\beta=\frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha+beta)$.

Prove the following:

4. $\cos(45^\circ-A)\cos(45^\circ-B)-\sin(45^\circ-A)\sin(45^\circ-B)=\sin(A+B)$.
5. $\sin(45^\circ+A)\cos(45^\circ-B)+\cos(45^\circ+A)\sin(45^\circ-B)=\cos(A-B)$.
6. $\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0$.
7. $\sin 105^\circ+\cos 105^\circ=\cos 45^\circ$.
8. $\sin 75^\circ-\sin 15^\circ=\cos 105^\circ+\cos 15^\circ$.
9. $\cos\alpha\cos(\gamma-\alpha)-\sin\alpha\sin(\gamma-\alpha)=\cos\gamma$.
10. $\cos(\alpha+\beta)\cos\gamma-\cos(\beta+\gamma)\cos\alpha=\sin\beta\sin(\gamma-\alpha)$.
11. $\sin(n+1)A\sin(n-1)A+\cos(n+1)A\cos(n-1)A=\cos 2A$.
12. $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$.
13. Find the value of $\cos 15^\circ$ and $\sin 105^\circ$.

14. Find the value of $\tan 105^\circ$.

15. Find the value of $\frac{\tan 495^\circ}{\cot 855^\circ}$.

16. Evaluate $\sin\left(n\pi + (-1)^n \frac{\pi}{4}\right)$, where n is an integer.

Prove the following:

17. $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

18. $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

19. $\tan 75^\circ = 2 + \sqrt{3}$.

20. $\tan 15^\circ = 2 - \sqrt{3}$.

Find the value of following:

21. $\cos 1395^\circ$.

22. $\tan(-330^\circ)$.

23. $\sin 300^\circ \mathrm{cosec} 1050^\circ - \tan(-120^\circ)$.

24. $\tan\left(\frac{11\pi}{12}\right)$.

25. $\tan\left((-1)^n \frac{\pi}{4}\right)$.

Prove the following:

26. $\cos 18^\circ - \sin 18^\circ = \sqrt{2} \sin 27^\circ$.

27. $\tan 70^\circ = 2\tan 50^\circ + \tan 20^\circ$.

28. $\cot\left(\frac{\pi}{4} + x\right)\cot\left(\frac{\pi}{4} - x\right) = 1$.

29. $\cos(m+n)\theta . \cos(m-n)\theta - \sin(m+n)\theta \sin(m-n)\theta = \cos 2m\theta$.

30. $\frac{\tan(\theta+\phi)+\tan(\theta-\phi)}{1-\tan(\theta+\phi)\tan(\theta-\phi)} = \tan 2\theta$.

31. $\cos 9^\circ + \sin 9^\circ = \sqrt{2} \sin 54^\circ$.

32. $\frac{\cos 20^\circ - \sin 20^\circ}{\cos 20^\circ + \sin 20^\circ} = \tan 25^\circ$.

33. $\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}$.

34. $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$.

35. $\frac{1}{\tan 3A + \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 4A$.

36. $\frac{\sin 3\alpha}{\sin \alpha} + \frac{\cos 3\alpha}{cos\alpha} = 4\cos 2\alpha$.

37. $\frac{\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)} = \sin 2A$.

38. $\tan 40^\circ + 2\tan 10^\circ = \tan 50^\circ$.

39. $\tan(\alpha+\beta)\tan(\alpha-\beta) = \frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha-\sin^2\beta}$.

40. $\tan^2\alpha - \tan^2\beta = \frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta}$.

41. $\tan[(2n+1)\pi+\theta] + \tan[(2n+1)\pi-\theta] = 0$.

42. $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{3\pi}{4}+\theta\right) + 1 = 0$.

43. If $\tan\alpha = p$ and $\tan\beta = q$ prove that $\cos(\alpha+\beta) = \frac{1-pq}{\sqrt{(1+p^2)(1+q^2)}}$.

44. if $\tan\beta = \frac{2\sin\alpha\sin\gamma}{\sin(\alpha+\gamma)}$, show that $\cot\alpha, \cot\beta, \cot\gamma$ are in A.P.

45. Eliminate θ if $\tan(\theta-\alpha) = a$ and $\tan(\theta+\alpha) = b$.

46. Eliminate α and β if $\tan\alpha + \tan\beta = b, \cot\alpha + \cot\beta = a$ and $\alpha+\beta = \gamma$.

47. If $A+B = 45^\circ$, show that $(1+\tan A)(1+\tan B) = 2$.

48. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, prove that $1 + \cot\alpha\tan\beta = 0$.

49. If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, prove that $\tan(\alpha-\beta) = (1-n)\alpha$.

50. If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, prove that $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$.

51. If $\tan\alpha = \frac{m}{m+1}, \tan\beta = \frac{1}{2m+1}$, prove that $\alpha+\beta = \frac{\pi}{4}$.

52. If $A+B = 45^\circ$, show that $(\cot A - 1)(\cot B - 1) = 2$.

53. If $\tan\alpha - \tan\beta = x$ and $\cot\beta - \cot\alpha = y$, prove that $\cot(\alpha-\beta) = \frac{x+y}{xy}$.

54. If a right angle be divided into three pats α, β and γ, prove that $\cot\alpha = \frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}$.

55. If $2\tan\beta + \cot\beta = \tan\alpha$, show that $\cot\beta = 2\tan(\alpha-\beta)$.

56. If in any $\triangle ABC, C = 90^\circ$, prove that $\text{cosec}(A-B) = \frac{a^2+b^2}{a^2-b^2}$ and $\sec(A-B) = \frac{c^2}{2ab}$.

57. If $\cot A = \sqrt{ac}, \cot B = \sqrt{\frac{c}{a}}, \tan C = \sqrt{\frac{c}{a^3}}$ and $c = a^2+a+1$, prove that $A = B+C$.

58. If $\frac{\tan(A-B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$, prove that $\tan A\tan B = \tan^2 C$.

59. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta = 1$ show that $\tan\alpha + \tan\beta = 0$.

60. If $\sin\theta = 3\sin(\theta + 2\alpha)$, prove that $\tan(\theta + \alpha)$, prove that $\tan(\theta + \alpha) + 2\tan\alpha = 0$.

61. If $3\tan\theta\tan\phi = 1$, prove that $2\cos(\theta + \phi) = \cos(\theta - \alpha)$.

62. Find the sign of the expression $\sin\theta + \cos\theta$ when $\theta = 100^\circ$.

63. Prove that the value of $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ lies between -4 and 10.

64. If $m\tan(\theta - 30^\circ) = n\tan(\theta + 120^\circ)$, show that $\cos 2\theta = \frac{m+n}{2(m-n)}$.

65. if $\alpha + \beta = \theta$ and $\tan\alpha : \tan\beta = x : y$, prove that $\sin(\alpha - \beta) = \frac{x-y}{x+y}\sin\theta$.

66. Find the maximum and minimum value of $7\cos\theta + 24\sin\theta$.

67. Show that $\sin 100^\circ - \sin 10^\circ$ is positive.

Chapter 5
Transformation Formulae

5.1 Transformation of products into sums or differences

We know that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Adding these, we get $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

Subtracting, we get $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

We also know that $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Adding, we get $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

Subtrating we get $2 \sin \sin B = \cos(A-B) - \cos(A+B)$

5.2 Transformation of sums or differences into products

We have $2 \sin A \cos B = \sin(A+B) \sin(A-B)$

Substituting for $A+B=C$, $A-B=D$ so that $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$

$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

We also have $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

Following similarly $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

For $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$, we get $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

For $2 \sin \sin B = \cos(A-B) - \cos(A+B)$, we get $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

5.3 Problems

1. Find the value of $\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ}$.

2. Simplify the expression $\frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)}$.

Prove that

3. $\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$.

4. $\frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\tan\theta$.

5. $\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$.

6. $\frac{\sin 7A-\sin A}{\sin 8A-\sin 2A}=\cos 4A \sec 5A$.

7. $\frac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cot(A+B)\cot(A-B)$.

8. $\frac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\frac{\tan(A+B)}{\tan(A-B)}$.

9. $\frac{\sin A+\sin 2A}{\cos A-\cos 2A}=\cot\frac{A}{2}$.

10. $\frac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\tan A$.

11. $\frac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B)$.

12. $\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A)\cos(45^\circ+B)$.

13. $\frac{\cos 3A-\cos A}{\sin 3A-\sin A}+\frac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=\frac{\sin A}{\cos 2A\cos 3A}$.

14. $\frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B)$.

15. $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$.

16. $\frac{\cos 3\theta+2\cos 5\theta+\cos 7\theta}{\cos\theta+2\cos 3\theta+\cos 5\theta}=\cos 2\theta-\sin 2\theta\tan 3\theta$.

17. $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$.

18. $\frac{\sin(\theta+\phi)-2\sin\theta+\sin(\theta-\phi)}{\cos(\theta+\phi)-2\cos\theta+\cos(\theta-\phi)}=\tan\theta$.

19. $\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\frac{\sin 3A}{\sin 5A}$.

20. $\frac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\frac{\sin A}{\sin B}$.

21. $\frac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A$.

22. $\frac{\sin A+\sin B}{\sin A-\sin B}=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$.

23. $\frac{\cos A+\cos B}{\cos B-\cos A}=\cot\frac{A+B}{2}\cot\frac{A-B}{2}$.

24. $\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\frac{A+B}{2}$.

25. $\frac{\sin A-\sin B}{\cos B-\cos A}=\cot\frac{A+B}{2}$.

26. $\frac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)-\sin(A-B+C)+\sin(A+B-C)}=\cot B$.

27. $\cos 3A+\cos 5A+\cos 7A+\cos 15A=4\cos 4A\cos 5A\cos 6A$.

28. $\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)+\cos(A+B+C)=4\cos A\cos B\cos C$.

29. $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$.

30. $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$.

31. $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha = 4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha$.

Simplify:

32. $\cos\left[\theta + \left(n - \frac{3}{2}\right)\phi\right] - \cos\left[\theta + \left(n + \frac{3}{2}\right)\phi\right]$.

33. $\sin\left[\theta + \left(n - \frac{3}{2}\right)\phi\right] + \sin\left[\theta + \left(n + \frac{3}{2}\right)\phi\right]$.

Express as a sum or difference the following:

34. $2\sin 5\theta \sin 7\theta$.

35. $2\cos 7\theta \sin 5\theta$.

36. $2\cos 11\theta \cos 3\theta$.

37. $2\sin 54^\circ \sin 66^\circ$.

Prove that

38. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$.

39. $\cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2} = \sin 5\theta\sin\frac{5\theta}{2}$.

40. $\sin A\sin(A + 2B) - \sin B\sin(B + 2A) = \sin(A - B)\sin(A + B)$.

41. $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$.

42. $\frac{2\sin(A-C)\cos C - \sin(A-2C)}{2\sin(B-C)\cos C - \sin(B-2C)} = \frac{\sin A}{\sin B}$.

43. $\frac{\sin A\sin 2A + \sin 3A\sin 6A + \sin 4A\sin 13A}{\sin A\cos 2A + \sin 3A\cos 6A + \sin 4A\cos 13A} = \tan 9A$.

44. $\frac{\cos 2A\cos 3A - \cos 2A\cos 7A + \cos A\cos 10A}{\sin 4A\sin 3A - \sin 2A\sin 5A + \sin 4A\sin 7A} = \cot 6A\cot 5A$.

45. $\cos(36^\circ - A)\cos(36^\circ + A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A$.

46. $\cos A\sin(B - C) + \cos B\sin(C - A) + \cos C\sin(A - B) = 0$.

47. $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A$.

48. $\sin(\beta - \gamma)\cos(\alpha - \delta) + \sin(\gamma - \alpha)\cos(\beta - \delta) + \sin(\alpha - \beta)\cos(\gamma - \delta) = 0$.

49. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$.

50. $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$.

51. $\cos 18^\circ - \sin 18^\circ = \sqrt{2}\sin 27^\circ$.

52. $\frac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A} = \tan 3A$.

53. $\left(\frac{\cos A+\cos B}{\sin A-\sin A}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = 2\cot^n\frac{A-B}{2}$ or 0 accordingh as n is even or odd.

54. If α, β, γ are in A.P., show that $\cos\beta = \frac{\sin\alpha-\sin\gamma}{\cos\gamma-\cos\alpha}$.

55. If $\sin\theta + \sin\phi = \sqrt{3}(\cos\phi - \cos\theta)$ prove that $\sin 3\theta + \sin 3\phi = 0$.

56. $\sin 65^\circ + cos 65^\circ = \sqrt{2}\cos 20^\circ$.

57. $\sin 47^\circ + \cos 77^\circ = \cos 17^\circ$.

58. $\frac{\cos 10^\circ-\sin 10^\circ}{\cos 10^\circ+\sin 10^\circ} = \tan 35^\circ$.

59. $\cos 80^\circ + \cos 40^\circ - cos 20^\circ = 0$.

60. $\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{7\pi}{5} = 0$.

61. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.

62. If $\sin\alpha - \sin\beta = \frac{1}{3}$ and $\cos\beta - \cos\alpha = \frac{1}{2}$, prove that $\cot\frac{\alpha+\beta}{2} = \frac{2}{3}$.

63. If $\operatorname{cosec} A + secA = \operatorname{cosec} B + \sec B$, prove that $\tan A\tan B = \cot\frac{A+B}{2}$.

64. If $\sec(\theta+\alpha) + \sec(\theta-\alpha) = 2\sec\theta$, show that $\cos^2\theta = 1 + \cos\alpha$.

65. Show that $\sin 50^\circ\cos 85^\circ = \frac{1-\sqrt{2}\sin 35^\circ}{2\sqrt{2}}$.

66. Prove that $\sin 20^\circ\sin 40^\circ\sin 80^\circ = \frac{\sqrt{3}}{8}$.

67. Prove that $\sin A\sin(60^\circ - A)\sin(60^\circ + A) = \frac{1}{4}\sin 3A$.

68. If $\alpha + \beta = 90^\circ$, find the maximum value of $\sin\alpha\sin\beta$.

69. Prove that $\sin 25^\circ\cos 115^\circ = \frac{1}{2}(\sin 40^\circ - 1)$.

70. Prove that $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ = \frac{3}{16}$.

71. Prove that $\cos 20^\circ\cos 40^\circ\cos 80^\circ = \frac{1}{8}$.

72. Prove that $\tan 20^\circ\tan 40^\circ\tan 60^\circ\tan 80^\circ = 3$.

73. Prove that $\cos 10^\circ\cos 30^\circ\cos 50^\circ\cos 70^\circ = \frac{3}{16}$.

74. Prove that $4\cos\theta\cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{\pi}{3}-\theta\right)=\cos 3\theta$.

75. Prove that $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta)=\tan 3\theta$.

76. If $\alpha+\beta=90^\circ$, show that the maximum value of $\cos\alpha\cos\beta$ is $\frac{1}{2}$.

77. If $\cos\alpha=\frac{1}{\sqrt{2}}, \sin\beta=\frac{1}{\sqrt{3}}$, show that $\tan\frac{\alpha+\beta}{2}\cot\frac{\alpha-\beta}{2}=5+2\sqrt{6}$ or $5-2\sqrt{6}$.

78. If $x\cos\theta=y\cos\left(\theta+\frac{2\pi}{3}\right)=z\cos\left(\theta+\frac{4\pi}{3}\right)$, prove that $xy+yz+xz=0$.

79. If $\sin\theta=n\sin(\theta+2\alpha)$, prove that $\tan(\theta+\alpha)=\frac{1+n}{1-n}\tan\alpha$.

80. If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)}=\frac{1-m}{1+m}$, prove that $\tan\left(\frac{\pi}{4}-\theta\right)\tan\left(\frac{\pi}{4}-\alpha\right)=m$.

81. If $y\sin\phi=x\sin(2\theta+\phi)$, show that $(x+y)\cot(\theta+\phi)=(y-x)\cot\theta$.

82. If $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-beta)\sin(\gamma-\delta)$, prove that $\cot\alpha\cot\beta\cot\gamma=cot\delta$.

83. If $\frac{\cos(A-B)}{\cos(A+B)}+\frac{\cos(C+D)}{\cos(C-D)}=0$, prove that $\tan A\tan B\tan C\tan D=-1$.

84. If $\tan(\theta+\phi)=3\tan\theta$, prove that $\sin(2\theta+\phi)=2\sin\phi$.

85. If $\tan(\theta+\phi)=3\tan\theta$, prove that $\sin 2(\theta+\phi)+\sin 2\theta=2\sin 2\phi$.

Chapter 6
Multiple and Submultiple Angles

6.1 Multiple Angles

An angle of the form nA, where n is an integer is called a *multiple angle*. For example, $2A, 3A, 4A, \ldots$ are multiple angles of A.

6.1.1 Trigonometrical Ratios of $2A$

From previous chapter we know that $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Substituting $B = A$, we get $\sin 2A = 2\sin A\cos A$

Similarly, $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$ (recall formula from previous chapter and substitute $B = A$ $\cos^2 A = 1 - \sin^2 A$ and $\sin^2 A = 1 - \cos^2 a$)

Also, $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ (recall formula from previous chapter and put $B = A$)

6.1.2 $\sin 2A$ and $\cos 2A$ in terms of $\tan A$

$\sin 2A = \frac{2\sin A\cos A}{\sin^2 A+\cos^2 A}[\because \sin^2 A + \cos^2 A = 1]$

Dividing both numerator and denominator by $\cos^2 A$, we get

$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

$\cos A = \cos^2 A - \sin^2 A = \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}[\because \sin^2 A + \cos^2 A = 1]$

Dividing both numerator and denominator by $\cos^2 A$, we get

$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} = \frac{\cot^2 A-1}{\cot^2 A+1}$

6.1.3 Trigonometrical Ratios of $3A$

$\sin 3A = \sin 2A\cos A + \cos 2A\sin A = 2\sin A\cos^2 A + \cos^2 A\sin A - \sin^3 A$

$= 2\sin A(1-\sin^2 A) + (1-2\sin^2 A)\sin A - \sin^3 A = 3\sin A - 4\sin^3 A$

$\cos 3A = \cos 2A\cos A - \sin 2A\sin A = (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - 2(1-\cos^2 A)\cos A = 4\cos^3 A - 3\cos A$

We know that $\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

Putting $B = A$ and $C = A$, we get $\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Similarly, $\cot 3A = \frac{\cot^3 A-3\cot A}{3\cot^2 A-1}$

6.2 Some Important Formulae

1. $\cos 2A = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \frac{1}{2}(1 - \cos 2A)$
2. $\cos 2A = 2\cos^2 - 1 \Rightarrow \cos^2 A = \frac{1}{2}(1 + \cos 2A)$
3. $\sin 3A = 3\sin A - 4\sin^3 A \Rightarrow \sin^3 A = \frac{1}{2}(3\sin A - \sin 3A)$
4. $\cos 3A = 4\cos^3 A - 3\cos A \Rightarrow \cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

6.3 Submultiple Angles

An angle of the form $\frac{A}{n}$, where n is an integer is called a *submultiple angle*. For exmaple, $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \ldots$ are submultiple angles of A.

6.3.1 Trigonometrical Ratios of $A/2$

We know that, $\sin 2A = 2\sin A\cos A$. Putting $A = A/2$, we get $\sin A = 2\sin A/2\cos A/2$

$\cos 2A = \cos^2 A - \sin^A$. Putting $A = A/2$, we get $\cos A - \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$\cos 2A = 2\cos^2 A - 1$. Putting $A = A/2$, we get $\cos A = 2\cos^2\frac{A}{2} - 1$

$\cos 2A = 1 - 2\sin^2 A$. Putting $A = A/2$, we get $\cos A = 1 - 2\sin^2\frac{A}{2}$

$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$. Putting $A = A/2$, we get $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$\sin 2A = \frac{2\tan A}{1+\tan^2 A} \therefore \sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} \therefore \cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

$\cot 2A = \frac{\cot^2 A - 1}{2\cot A} \therefore \cot A = \frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}}$

6.3.2 Trigonometrical Ratios of $A/3$

$\sin 3A = 3\sin A - 4\sin^3 A$. Putting $A = \frac{A}{3}$, we get $\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3}$

$\cos 3A = 4\cos^3 A - 3\cos A$. Putting $A = \frac{A}{3}$, we get $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$

$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A} \Rightarrow \tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1-3\tan^2\frac{A}{3}}$

6.3.3 Values of $cos A/2$, $\sin A/2$ and $\tan A/2$ in terms of $\cos A$

$\cos^2\frac{A}{2} = \frac{1+\cos A}{2} \therefore \cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

$\sin^2\frac{A}{2}=\frac{1-\cos A}{}2 \therefore \sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}}$

$\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A} \therefore \tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}}$

6.3.4 Values of $\sin A/2$ and $\cos A/2$ in terms of $\sin A$

$\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2=\cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\cos\frac{A}{2}\sin\frac{A}{2}$

$=1+\sin A \Rightarrow \cos\frac{A}{2}+\sin\frac{A}{2}=\sqrt{1+\sin a}$

Similarly, $\cos\frac{A}{2}-\sin\frac{A}{2}=\sqrt{1-\sin a}$

Adding, we get $\cos\frac{A}{2}=\pm\frac{1}{2}\sqrt{1+\sin A}\pm\frac{1}{2}\sqrt{1-\sin a}$

Subtracting, we get $\cos\frac{A}{2}=\pm\frac{1}{2}\sqrt{1+\sin A}\mp\frac{1}{2}\sqrt{1-\sin A}$

6.3.5 Value of $\sin 18^\circ$ and $\cos 72^\circ$

Let $A=18^\circ$, then $\sin 5A=90^\circ \therefore 2A+3A=90^\circ \Rightarrow \sin 2A=\sin(90^\circ-\sin 3A) \therefore 2\sin A\cos A=4\cos^3 A-3\cos A$

Dividing both sides by $\cos A$, we get $2\sin A=4\cos^2 A-3=4(1-\sin^2 A)-3 \Rightarrow 4\sin^2 A+2\sin A-1=0 \Rightarrow \sin A=\frac{-1\pm\sqrt{5}}{4}$

However, since $A=18^\circ \therefore \sin A>0 \therefore \sin 18^\circ=\frac{-1+\sqrt{5}}{4} \therefore \sin(90^\circ-18^\circ)=\cos 72^\circ=\frac{\sqrt{5}-1}{4}$

6.3.6 Value of $\cos 18^\circ$ and $\sin 72^\circ$

$\cos^2 18^\circ=1-\sin^2 18^\circ=1-\left(\frac{\sqrt{5}-1}{4}\right)^2=\frac{10+2\sqrt{5}}{16} \therefore \cos 18^\circ=\frac{1}{4}\sqrt{10+2\sqrt{5}}[\because \cos 18^\circ>0]$

$\cos(90^\circ-18^\circ)=\sin 72^\circ=\frac{1}{4}\sqrt{10+2\sqrt{5}}$

6.3.7 Value of $\tan 18^\circ$ and $\tan 72^\circ$

$\tan 18^\circ=\frac{\sin 18^\circ}{\cos 18^\circ}=\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$

$\tan 18^\circ\cot 18^\circ=1 \Rightarrow \tan 72^\circ=\frac{1}{\tan 18^\circ}=\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$

6.3.8 Value of $\cos 36^\circ$ and $\sin 54^\circ$

$\cos 36^\circ=1-2\sin^2 18^\circ=1-2\left(\frac{\sqrt{5}-1}{4}\right)^2=\frac{\sqrt{5}+1}{4}$

$\sin 54^\circ=\sin(90^\circ-36^\circ)=\cos 36^\circ=\frac{\sqrt{5}+1}{4}$

6.3.9 Value of $\sin 36^\circ$ and $\cos 54^\circ$

$\sin 36^\circ = 1 - \cos^2 36^\circ = 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$

$\cos 54^\circ = \cos(90^\circ - 36^\circ) = \sin 36^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$

Several other angles like, 9°, 15°, $22\frac{1}{2}^\circ$, $7\frac{1}{2}^\circ$ etc can be found similarrly.

6.4 Problems

1. Find the value of $\sin 2A$, when
   a. $\cos A = \frac{3}{5}$.
   b. $\sin A = \frac{12}{13}$.
   c. $\tan A = \frac{16}{63}$.
2. Find the value of $\cos 2A$, when
   a. $\cos A = \frac{15}{17}$.
   b. $\sin A = \frac{4}{5}$.
   c. $\tan A = \frac{5}{12}$.
3. If $\tan A = \frac{b}{a}$, find the value of $a\cos 2A + b\sin 2A$.

Prove that

4. $\frac{\sin 2A}{1+\cos 2A} = \tan A$.
5. $\frac{\sin 2A}{1-\cos 2A} = \cot A$.
6. $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.
7. $\tan A + \cot A = 2\mathrm{cosec}2A$.
8. $\tan A - \cot A = -2\cot 2A$.
9. $\mathrm{cosec}2A + \cot 2A = \cot A$.
10. $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\frac{A}{2}\cot\frac{B}{2}$.
11. $\frac{\cos A}{1\mp\sin A} = \tan\left(45^\circ \pm \frac{A}{2}\right)$.

12. $\frac{\sec 8A-1}{\sec 4A-1}=\frac{\tan 8A}{\tan 2A}$.

13. $\frac{1+\tan^2(45^\circ-A)}{1-\tan^2(45^\circ-A)}=\mathrm{cosec}2A$.

14. $\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$.

15. $\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B)$.

16. $\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)=2\tan 2A$.

17. $\frac{\cos A+\sin A}{\cos A-\sin A}-\frac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A$.

18. $\cot(A+15^\circ)-\tan(A-15^\circ)=\frac{4\cos 2A}{1+2\sin 2A}$.

19. $\frac{\sin A+\sin 2A}{1+\cos A+\cos 2A}=\tan A$.

20. $\frac{1+\sin A-\cos A}{1+\sin A+cosA}=\tan\frac{A}{2}$.

21. $\frac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A}=\tan\frac{A}{2}$.

22. $\frac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)-\cos(n+1)A}=\cot\frac{A}{2}$.

23. $\sin(2n+1)A\sin A=\sin^2(n+1)A-\sin^2 nA$.

24. $\frac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B}=2\cos(A+B)$.

25. $\sin 3A+\sin 2A-\sin A=4\sin A\cos\frac{A}{2}\cos\frac{3A}{2}$.

26. $\tan 2A=(\sec 2A+1)\sqrt{\sec^2 A-1}$.

27. $\cos^3 2A+3\cos 2A=4(\cos^6 A-\sin^6 A)$.

28. $1+\cos^2 2A=2(\cos^4 A+\sin^4 A)$.

29. $\sec^2 A(1+\sec 2A)=2\sec 2A$.

30. $\mathrm{cosec}A-2\cot 2A\cos A=2\sin A$.

31. $\cot A=\frac{1}{2}\left(\cot\frac{A}{2}-\tan\frac{A}{2}\right)$.

32. $\sin A\sin(60^\circ-A)\sin(60^\circ+A)=\frac{1}{4}\sin 3A$.

33. $\cos A\cos(60^\circ-A)\cos(60^\circ+A)=\frac{1}{4}\cos 3A$.

34. $\cot A+\cot(60^\circ+A)-\cot(60^\circ-A)=3\cot 3A$.

35. $\cos 4A = 1 - 8\cos^2 A + 8\cos^4 A$.

36. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

37. $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$.

38. $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$.

39. $\frac{2\cos 2^n A+1}{2\cos A+1} = (2\cos A - 1)(2\cos 2A - 1)(2\cos 2^2 A - 1)\ldots(2\cos 2^{n-1} - 1)$.

40. If $\tan A = \frac{1}{7}$, $\sin B = \frac{1}{\sqrt{10}}$, prove that $A + 2B = \frac{\pi}{4}$, where $0 < A < \frac{\pi}{4}$ and $0 < B < \frac{\pi}{4}$.

Prove that

41. $\tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right) = 2\sec 2A$.

42. $\sqrt{3}\text{cosec}20^\circ - \sec 20^\circ = 4$.

43. $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A = \cot A$.

44. $\cos^2 A + \cos^2\left(\frac{2\pi}{3} - A\right) + \cos^2\left(\frac{2\pi}{3} + A\right) = \frac{3}{2}$.

45. $2\sin^2 A + 4\cos(A + B)\sin A\sin B + \cos 2(A + B)$ is idnependent of A.

46. If $\cos A = \frac{1}{2}\left(a + \frac{1}{a}\right)$, show that $\cos 2A = \frac{1}{2}\left(a^2 + \frac{1}{a^2}\right)$.

Prove that

47. $\cos^2 A + \sin^2 A\cos 2B = \cos^2 B + \sin^2 B\cos 2A$.

48. $1 + \tan A\tan 2A = \sec 2A$.

49. $\frac{1+\sin 2A}{1-\sin 2A} = \left(\frac{1+\tan A}{1-\tan A}\right)^2$.

50. $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.

51. $\cot^2 A - \tan^2 A = 4\cot 2A\text{cosec}2A$.

52. $\frac{1+\sin 2A}{\cos 2A} = \frac{\cos A+\sin A}{\cos A-\sin A} = \tan\left(\frac{\pi}{4} + A\right)$.

53. $\cos^6 A - \sin^6 A = \cos 2A\left(1 - \frac{1}{4}\sin^2 2A\right)$.

54. $\cos^2 A + \cos^2\left(\frac{\pi}{3} + A\right) + \cos^2\left(\frac{\pi}{3} - A\right) = \frac{3}{2}$.

55. $(1 + \sec 2A)(1 + \sec 2^2 A)(1 + \sec 2^3 A)\ldots(1 + \sec 2^n A) = \frac{\tan 2^n A}{\tan A}$.

56. $\frac{\sin 2^n A}{\sin A} = 2^n\cos A\cos 2A\cos 2^2 A\ldots\cos 2^{n-1} A$.

57. $3(\sin A - \cos A)^4 + 6(\sin A + \cos A)^2 + 4(\sin^6 A + \cos^6 A) = 13.$

58. $2(\sin^6 A + \cos^6 A) - 3(\sin^4 A + \cos^4 A) + 1 = 0.$

59. $\cos^2 A + \cos^2(A + B) - 2\cos A \cos B \cos(A + B)$ if independent of A.

60. $\cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A.$

61. $\tan A \tan(60^\circ - A) \tan(60^\circ + A) = \tan 3A.$

62. $\sin^2 A + \sin^3\left(\frac{2\pi}{3} + A\right) + \sin^3\left(\frac{4\pi}{3} + A\right) = -\frac{3}{4}\sin 3A.$

63. $4(\cos^3 10^\circ + \sin^3 20^\circ) = 3(\cos 10\circ + \sin 20^\circ).$

64. $\sin A \cos^3 A - \cos A \sin^3 A = \frac{1}{4}\sin 4A.$

65. $\cos^3 A \sin 3A + \sin^3 A \cos 3A = \frac{3}{4}\sin 4A.$

66. $\sin A \sin(60^\circ + A) \sin(A + 120^\circ) = \sin 3A.$

67. $\cot A + \cot(60^\circ + A) + \cot(120^\circ + A) = 3\cot 3A.$

68. $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A.$

69. $\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A.$

70. $\cos 4A - \cos 4B = 8(\cos A - \cos B)(\cos A + \cos B)(\cos A - \sin B)(\cos A + \sin B).$

71. $\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}.$

72. If $2\tan A = 3\tan B$, prove that $\tan(A - B) = \frac{\sin 2B}{5 - \cos 2B}.$

73. If $\sin A + \sin B = x$ and $\cos A + \cos B = y$, show that $\sin(A + B) = \frac{2xy}{x^2 + y^2}.$

74. If $A = \frac{\pi}{2^n + 1}$, prove that $\cos A.\cos 2A.\cos 2^2 A.\dots.\cos 2^{n-1} A = \frac{1}{2^n}.$

75. If $\tan A = \frac{y}{x}$, prove that $x\cos 2A + y\sin 2A = x.$

76. If $\tan^2 A = 1 + 2\tan^2 B$, prove that $\cos 2B = 1 + 2\cos 2A.$

77. If A and B lie between 0 and $\frac{\pi}{2}$ and $\cos 2A = \frac{3\cos 2B - 1}{3 - \cos 2B}$, prove that $\tan A = \sqrt{2}\tan B.$

78. If $\tan B = 3\tan A$, prove that $\tan(A + B) = \frac{2\sin 2B}{1 + \cos 2B}.$

79. If $x\sin A = y\cos A$, prove that $\frac{x}{\sec 2A} + \frac{y}{\mathrm{cosec} 2A} = x.$

80. If $\tan A = \sec 2B$, prove that $\sin 2A = \frac{1 - \tan^4 B}{1 + \tan^4 B}.$

81. If $A = \frac{\pi}{3}$, prove that $\cos A.\cos 2A.\cos 3A.\cos 4A.\cos 5A.\cos 6A = -\frac{1}{16}$.

82. If $A = \frac{\pi}{15}$, prove that $\cos 2A.\cos 4A.\cos 8A.\cos 14A = \frac{1}{16}$.

83. If $\tan A \tan B = \sqrt{\frac{a-b}{a+b}}$, prove that $(a - b\cos 2A)(a - b\cos 2B) = a^2 - b^2$.

84. If $\sin A = \frac{1}{2}$ and $\sin B = \frac{1}{3}$, find the value of $\sin(A+B)$ and $\sin(2A+2B)$.

85. If $\cos A = \frac{11}{61}$ and $\sin B = \frac{4}{5}$, find the value of $\sin^2\frac{A-B}{2}$ and $cos^2\frac{A+B}{2}$, the angle of A and B being positive acute angles.

86. Given $\sec A = \frac{5}{4}$, find $\tan\frac{A}{2}$ and $\tan A$.

87. If $\cos A = .3$, find the value of $\tan\frac{A}{2}$, and explain the resulting ambiguity.

88. If $\sin A + \sin B = x$ and $\cos A + \cos B = y$, find the value of $\tan\frac{A-B}{2}$.

Prove that

89. $(\cos A + \cos B)^2 + (\sin A - \sin B)^2 = 4\cos^2\frac{A+B}{2}$.

90. $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\frac{A-B}{2}$.

91. $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\frac{A-B}{2}$.

92. $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$.

93. $(\tan 4A + \tan 2A)(1 - \tan^2 3A\tan^2 A) = 2\tan 3A\sec^2 A$.

94. $\left(1 + \tan\frac{A}{2} - \sec\frac{A}{2}\right)\left(1 + \tan\frac{A}{2} + \sec\frac{A}{2}\right) = \sin A\sec^2\frac{A}{2}$.

95. $\frac{1+\sin A-\cos A}{1+\sin A+\cos A} = \tan\frac{A}{2}$.

96. $\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}} = \frac{1+\sin A}{\cos A} = \tan\left(\frac{\pi}{4} + \frac{A}{2}\right)$.

97. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$.

98. $\frac{2\sin A-\sin 2A}{2\sin A+\sin 2A} = \tan^2\frac{A}{2}$.

99. $\cot\frac{A}{2} - \tan\frac{A}{2} = 2\cot A$.

100. $\frac{1+\sin A}{1-\sin A} = \tan^2\left(\frac{\pi}{4} + \frac{A}{2}\right)$.

101. $\sec A + \tan A = \tan\left(\frac{\pi}{4} + \frac{A}{2}\right)$.

102. $\frac{\sin A+\sin B-\sin(A+B)}{\sin A+\sin B+\sin(A+B)} = \tan\frac{A}{2}\tan\frac{B}{2}$.

103. $\tan\left(\frac{\pi}{4}-\frac{A}{2}\right) = \sec A - \tan A = \sqrt{\frac{1-\sin A}{1+\sin A}}$.

104. $\operatorname{cosec}\left(\frac{\pi}{4}+\frac{A}{2}\right)\operatorname{cosec}\left(\frac{\pi}{4}-\frac{A}{2}\right) = 2\sec A$.

105. $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8} = 2$.

106. $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8} = \frac{3}{2}$.

107. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$.

108. Find the value of $\sin\frac{23\pi}{24}$.

109. If $A = 112^\circ 30'$, find the value of $\sin A$ and $\cos A$.

Prove that

110. $\sin^2 24^\circ - \sin^2 6^\circ = \frac{1}{8}(\sqrt{5}-1)$.

111. $\tan 6^\circ . \tan 42^\circ . \tan 66^\circ . \tan 78^\circ = 1$.

112. $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ = \cos 7^\circ$.

113. $\sin 12^\circ \sin 48^\circ \sin 54^\circ = \frac{1}{8}$.

114. $\cot 142\frac{1}{2}^\circ = \sqrt{2}+\sqrt{3}-2-\sqrt{6}$.

115. $\sin^2 48^\circ - \cos^2 12^\circ = -\frac{\sqrt{5}+1}{8}$.

116. $4(\sin 24^\circ + \cos 6^\circ) = \sqrt{3}+\sqrt{15}$.

117. $\cot 6^\circ \cot 42^\circ \cot 66^\circ \cot 78^\circ = 1$.

118. $\tan 12^\circ \tan 24^\circ \tan 48^\circ \tan 84^\circ = 1$.

119. $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \frac{1}{16}$.

120. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$.

121. $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$.

122. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^7}$.

123. $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65} = \frac{1}{64}$.

124. If $\tan\frac{A}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{B}{2}$, prove that, $\cos A = \frac{a\cos B+b}{a+b\cos B}$.

125. If $\tan\frac{A}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{B}{2}$, prove that $\cos B = \frac{\cos A - e}{1 - e\cos A}$.

126. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, prove that $\sin(A+B) = \frac{2ab}{a^2+b^2}$.

127. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, prove that $\cos(A-B) = \frac{1}{2}(a^2+b^2-2)$.

128. If A and B be two different roots of equation $a\cos\theta + b\sin\theta = c$, prove that

   a. $\tan(A+B) = \frac{2ab}{a^2-b^2}$.

   b. $\cos(A+B) = \frac{a^2-b^2}{a^2+b^2}$.

129. If $\cos A + \cos B = \frac{1}{3}$ and $\sin A + \sin B = \frac{1}{4}$, prove that $\cos\frac{A-B}{2} = \pm\frac{5}{24}$.

130. If $2\tan\frac{A}{2} = \tan\frac{B}{2}$, prove that $\cos A = \frac{3+5\cos B}{5+3\cos B}$.

131. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$, prove that one value of $\cos\frac{A-B}{2} = \frac{8}{\sqrt{65}}$.

132. If $\sec(A+B) + \sec(A-B) = 2\sec A$, prove that $\cos B = \pm\sqrt{2}\cos\frac{B}{2}$.

133. If $\cos\theta = \frac{\cos\alpha\cos\beta}{1-\sin\alpha\sin\beta}$, prove that one of the values of $\tan\frac{\theta}{2}$ is $\frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$.

134. If $\tan\alpha = \frac{\sin\theta\sin\phi}{\cos\theta+\cos\phi}$, prove that one of the values of $\tan\frac{\alpha}{2}$ is $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$.

135. If $\cos\theta = \frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$, prove that one of the values of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$.

Chapter 7
Trigonometric Identities

We will use the theory learned so far to solve following trigonometric identities.

7.1 Problems

1. If $A+B+C=\pi$, prove that $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\sin C$.
2. If $A+B+C=180^\circ$, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.
3. Show that $\sin^2 A+\sin^2 B+2\sin A\sin B\cos(A+B)=\sin^2(A+B)$.
4. If $A+B+C=180^\circ$, prove that $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$.
5. If $A+B+C=180^\circ$, prove that $\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C)$.
6. If $A+B+C=180^\circ$, prove that $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\sin C$.
7. If $A+B+C=180^\circ$, prove that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.
8. If $A+B+C=180^\circ$, prove that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.
9. If $A+B+C=\frac{\pi}{2}$, prove that $\sin^2 A+\sin^2 B+\sin^2 C=1-2\sin A\sin B\sin C$.
10. If $A+B+C=\frac{\pi}{2}$, prove that $\cos^2 A+\cos^2 B+\cos^2 C=2+2\sin A\sin B\sin C$.
11. If $A+B+C=2\pi$, prove that $\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C=1$.
12. If $A+B=C$, prove that $\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C=1$.
13. If $A+B=\frac{\pi}{3}$, prove that $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$.
14. Show that $\cos^2 B+\cos^2(A+B)-2\cos A\cos B\cos(A+B)$ is independent of B..
15. If $A+B+C=\pi$ and $A+B=2C$, prove that $4(\sin^2 A+\sin^2 B-\sin A\sin B)=3$.
16. If $A+B+C=2\pi$, prove that $\cos^2 B+\cos^2 C-\sin^2 A-2\cos A\cos B\cos C=0$.
17. If $A+B+C=0$, prove that $\cos^2 A+\cos^2 B+\cos^2 C=1+2\cos A\cos B\cos C$.
18. Prove that $\cos^2(B-C)+\cos^2(C-A)+\cos^2(A-B)=1+2\cos(B-C)\cos(C-A)\cos(A-B)$.
19. If $A+B+C=\pi$, prove that $\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B=\sin A\sin B\sin C$.
20. If $A+B+C=\pi$, prove that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

21. If $A+B+C=\pi$, prove that $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$.

22. If $A+B+C=\pi$, prove that $\tan(B+C-A)+\tan(C+A-B)+\tan(A+B-C)=\tan(B+C-A)\tan(C+A-B)\tan(A+B-C)$.

23. If $A+B+C=\pi$, prove that $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$.

24. In a $\triangle ABC$, if $\cot A+\cot B+\cot C=\sqrt{3}$, prove that the triangle is equilateral.

25. If A, B, C, D are angles of a quadrilateral, prove that $\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}=\tan A\tan B\tan C\tan D$.

26. If $A+B+C=\frac{\pi}{2}$, show that $\cot A+\cot B+\cot C=\cot A\cot B\cot C$.

27. If $A+B+C=\frac{\pi}{2}$, show that $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$.

28. If $A+B+C=\pi$, prove that $\tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C$.

29. If $A+B+C=\pi$, prove that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

30. If $A+B+C=\pi$, prove that $\frac{\cot A+\cot B}{\tan A+\tan B}+\frac{\cot B+\cot C}{\tan B+\tan C}+\frac{\cot C+\cot A}{\tan C+\tan A}=1$.

31. Prove that $\tan(A-B)+\tan(B-C)+\tan(C-A)=\tan(A-B)\tan(B-C)\tan(C-A)$.

32. If $x+y+z=0$, show that $\cot(x+y-z)\cot(z+x-y)+\cot(x+y-z)\cot(y+z-x)+\cot(y+z-x)\cot(z+x-y)=1$.

33. If $A+B+C=n\pi$(n being zero or an integer), show that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

34. If $A+B+C=\pi$, prove that $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

35. If $A+B+C=\pi$, prove that $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

36. Prove that $\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}=8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

37. If $A+B+C=\pi$, prove that $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}$

$\cos\frac{\pi-C}{4}$.

38. If $A+B+C=\pi$, prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}$

$\sin\frac{A+B}{4}$.

39. If $A+B+C=\pi$, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

40. Prove that $1+\cos 56^\circ+\cos 58^\circ-\cos 66^\circ=4\cos 28^\circ\cos 29^\circ\sin 33^\circ$.

41. If $A+B+C=\pi$, prove that $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$.

42. If $A+B+C=\pi$, prove that $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$.

43. If $A+B+C=\pi$, prove that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

44. If $A+B+C=\pi$, prove that $\cos A+\cos B-\cos C=4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-1$.

45. If $A+B+C=\pi$, prove that $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$.

46. If $A+B+C=\pi$, prove that $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2$.

47. If $A+B+C=\pi$, prove that $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\ \sin\frac{B}{2}\sin\frac{C}{2}$.

48. If $x+y+z=\frac{\pi}{2}$, prove that $\cos(x-y-z)+\cos(y-z-x)+\cos(z-x-y)-4\cos x\cos y\cos z=0$.

49. Show that $\sin(x-y)+\sin(y-z)+\sin(z-x)+4\sin\frac{x-y}{2}\sin\frac{y-z}{2}\sin\frac{z-x}{2}=0$.

50. If $A+B+C=180^\circ$, prove that $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

51. If $A+B+C=\pi$, prove that $\sin\frac{B+C}{2}+\sin\frac{C+A}{2}+\sin\frac{A+B}{2}=4\cos\frac{\pi-A}{4}$

$\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$.

52. If $xy+yz+zx=1$, prove that $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$.

53. If $x+y+z=xyz$, show that $\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$.

54. If $x+y+z=xyz$, prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$.

55. If $x+y+z=xyz$, prove that $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$.

56. If $A+B+C+D=2\pi$, prove that $\cos A+\cos B+\cos C+\cos D=4\cos\frac{A+B}{2}$

$\cos\frac{B+C}{2}\cos\frac{C+A}{2}$.

57. If $A+B+C=2S$, prove that $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)=2+2\cos A\cos B\cos C$.

58. If $A+B+C=\pi$, prove that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1$.

59. If $A+B+C=\pi$, prove that $(\tan A+\tan B+\tan C)(\cot A+\cot B+\cot C)=1+\sec A\sec B\sec C$.

60. If $A+B+C=\pi$, prove that $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot C)=\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$.

61. If $A+B+C=\pi$, prove that $\frac{1}{2}\sum \sin^2 A(\sin 2B+\sin 2C)=3\sin A\sin B\sin C$.

62. If $A+B+C+D=2\pi$, prove that $\cos A-\cos B+\cos C-\cos D=4\sin\frac{A+B}{2}$

$\sin\frac{A+D}{2}\cos\frac{A+C}{2}$.

63. If A, B, C, D be the angles of a cyclic quadrilateral, prove that $\cos A+\cos B+\cos C+\cos D=0$.

64. If $A+B+C=\pi$, prove that $\cot^2 A+\cot^2 B+\cot^2 C\geq 1$.

65. If $A+B+C=\pi$, prove that $\cos\frac{A}{2}\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}\cos\frac{A-B}{2}=\sin A+\sin B+\sin C$.

66. In a $\triangle ABC$, prove that $\sin 3A\sin(B-C)+\sin 3B\sin(C-A)+\sin 3C\sin(A-B)=0$.

Chapter 8
Properties of Triangles

In this chapter we will study the relations between the sides and trigonometrical ratios of the angles of a triangle. We already know that a triangle has three sides and three angles. In a $\triangle ABC$ we will denote the angles BAC, CBA, ACB by A, B, C and the corresponsing sides i.e. sides opposite to them by a, b, c respectively.

Thus, $BC = a$, $AC = b$, $AB = c$

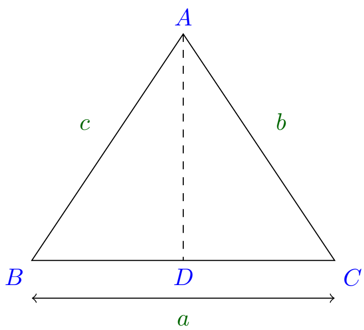
We will also denote the radius of the circumcircle of the $\triangle ABC$ by R and the area by $\triangle$. We also know some basic properties of a triangle for example, $A + B + C = 180°$ and $a + b > c, b + c > a, c + a > b$.

8.1 Sine Formula or Sine Rule or Law of Sines

Theorem 1

In $\triangle ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Proof



Case I: When $\angle C$ is accute. □

From A draw $AD \perp BC$. From $\triangle ABD$,

$\sin B = \frac{AD}{AB} = \frac{AD}{c} \Rightarrow AD = c \sin B$

From $\triangle ACD$,

$\sin C = \frac{AD}{AC} = \frac{AD}{b} \Rightarrow AD = b \sin C$

Thus, $c \sin B = b \sin C$

Case II: When $\angle C$ is obtuse:

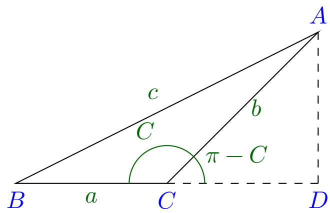


Figure 8.1

From A draw $AD \perp BC$. From $\triangle ABD$,

$\sin B = \frac{AD}{AB} = \frac{AD}{c} \Rightarrow AD = c \sin B$

From $\triangle ACD$,

$\sin(\pi - C) = \frac{AD}{AC} = \frac{AD}{b} \Rightarrow AD = b \sin C$

Thus, $c \sin B = b \sin C$

Case III: When $\angle C$ is 90°:

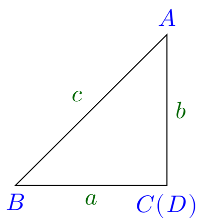


Figure 8.2

From A draw $AD \perp BC$. From $\triangle ABD$,

$\sin B = \frac{AD}{AB} = \frac{AD}{c} \Rightarrow AD = c \sin B \Rightarrow AC = c \sin B [\because C$ and D are same points $]$

$b = c \sin B \Rightarrow b \sin 90^\circ = c \sin B \Rightarrow b \sin C = c \sin B$

Thus, from all cases we have established that $\frac{b}{\sin B} = \frac{c}{\sin C}$

Similarly by drawing perpendicular from C to AB, we can prove that

$\frac{a}{\sin A} = \frac{b}{\sin B}$ and thus $\triangle ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Theorem 2

In a $\triangle ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$*, where* R *is the radius of the circumcircle of* $\triangle ABC$*.*

Proof

Case I: When $\angle A$ is acute.

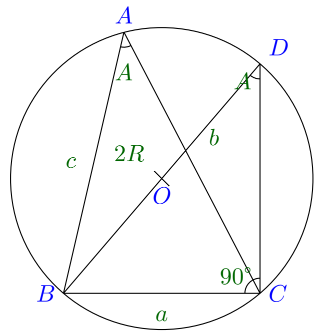


Figure 8.3

From $\triangle BDC, \sin A = \frac{BC}{BD} = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R.$

Case II: When $\angle A$ is obtuse.

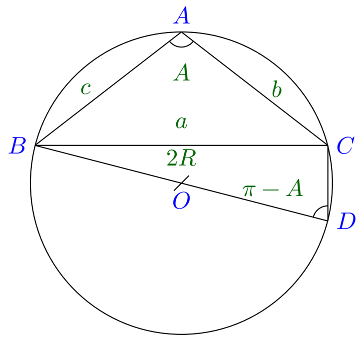


Figure 8.4

From $\triangle BDC, \sin(\pi - A) = \frac{BC}{BD} = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R.$

Case III: When $\angle A$ is 90°.

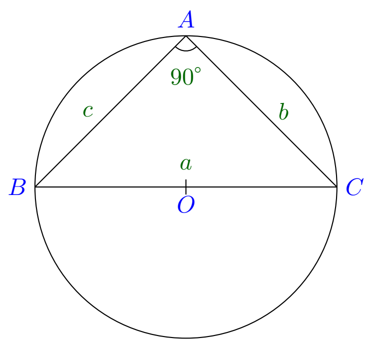


Figure 8.5

From $\triangle BDC$, $a = BC = 2R \Rightarrow \frac{a}{\sin A} = 2R$.

Similarly, by joining the diameter through A and O and through C and O, we can show that $\frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

8.2 Tangent Rule

Theorem 3

In any $\triangle ABC$*,* $\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}, \tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2},$ *and* $\tan \dfrac{C-A}{2} = \dfrac{c-a}{c+a} \cot \dfrac{B}{2}$.

Proof

By sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$ (say)

$\therefore b = K \sin B, c = k \sin C \therefore \dfrac{b-c}{b+c} = \dfrac{K(\sin B - \sin C)}{K(\sin B + \sin C)} = \dfrac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} = \cot \dfrac{B+C}{2} \tan \dfrac{B-C}{2} = \tan \dfrac{A}{2} \tan \dfrac{B-C}{2} \Rightarrow \tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \cot \dfrac{A}{2}$. □

Similarly, we can prove the two other equations.

8.3 Cosine Formula or Cosine Rule

Theorem 4

In any $\triangle ABC$, $\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{c^2+a^2-b^2}{2ca}, \cos C = \frac{a^2+b^2-c^2}{2ab}$.

Proof

Case I: When $\angle C$ is acute.

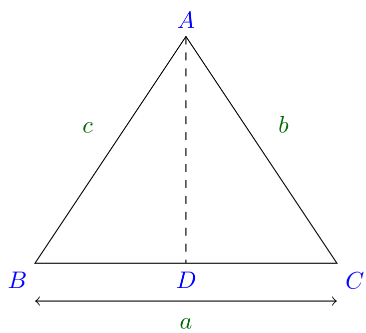


Figure 8.6

$AD = b\sin C$, $\cos C = \frac{CD}{AC} \Rightarrow CD = b\cos C \Rightarrow BD = BC - CD = a - b\cos C$.

Case II: When $\angle C$ is obtuse.

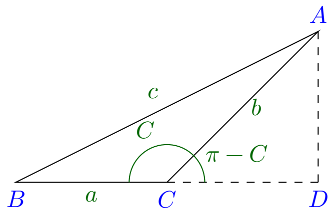


Figure 8.7

$AD = b\sin(\pi - C) = b\sin C$, $\cos(\pi - C) = \frac{CD}{AC} \Rightarrow CD = -\cos C \Rightarrow BC = BC + CD = a - b\cos C$.

Case III: When $\angle C$ is 90°.

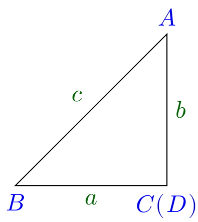


Figure 8.8

Here, C and D are same points. $AD = AC = b = b\sin C$, $CD = 0 = b\cos C[\because \cos C = \cos 90^\circ = 0]$

$BD = BC - CD = a - b\cos C$, thus, in all cases $AD = b\sin C$ and $BD = a - b\cos C$

Now, $AB^2 = AD^2 + BD^2 \Rightarrow c^2 = b^2\sin^2 C + (a - b\cos C)^2 \Rightarrow \cos C = \frac{a^2+b^2-c^2}{2ab}$.

Similarly, we can prove it for $\angle A$ and $\angle B$.

8.4 Projection Formulae

Theorem 5

In any $\triangle ABC$, $c = a\cos B + b\cos A$, $b = c\cos A + a\cos C$, $a = b\cos C + c\cos B$.

Proof

Case I: When $\angle C$ is acute.

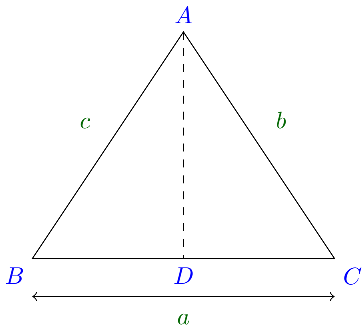


Figure 8.9

$BC = a = BD + CD = c\cos B + b\cos C$.

Case II: When $\angle C$ is obtuse.

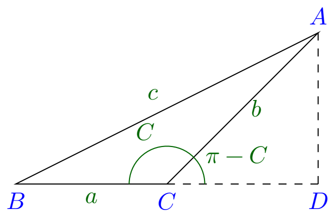


Figure 8.10

$BC = a = BD - CD = c\cos B - b\cos(\pi - C) = c\cos B + b\cos C$ **Case III:** When $\angle C$ is 90°.

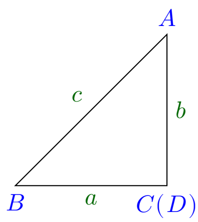


Figure 8.11

$BD = a = BC + CD = c\cos B + b\cos C[\because C = 90^\circ \therefore \cos C = 0]$

Thus, in all cases $a = b\cos C + c\cos B$. Similarly, we can prove for other sides. □

8.5 Sub-Angle Rules

Theorem 6

In any $\triangle ABC, \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, *where* $2s = a + b + c$.

Proof

$$2\sin^2\frac{A}{2} = 1-\cos A = 1-\frac{b^2+c^2-a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a+c-b)}{2bc}$$

$$= \frac{(2s-2c)(2s-2b)}{2bc} \Rightarrow \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\Rightarrow \sin\frac{A}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{bc}}$$

But $\frac{A}{2}$ is an acute angle so $\sin\frac{A}{2} > 0 \therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$$2\cos^2\frac{A}{2} = 1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} = \frac{(a+b+c)(b+c-a)}{2bc}$$

$$= \frac{(2s)(2s-2a)}{2bc} \Rightarrow \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\Rightarrow \cos\frac{A}{2} = \pm\sqrt{\frac{s(s-a)}{bc}}$$

But $\frac{A}{2}$ is an acute angle is $\cos\frac{A}{2} > 0 \therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

From the two equation which we have found it follows that $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. Similarly, we can prove the relations for other angles. □

8.6 Sines of Angles in Terms of Sides

Theorem 7

In any $\triangle ABC$, $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b(s-c)}$, $\sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b(s-c)}$, $\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b(s-c)}$.

Proof

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} = \frac{2}{bc}\sqrt{s(s-a)(s-b(s-c)}$$

Similarly, we can prove it for other angles. □

8.7 Area of a Triangle

Theorem 8

If Δ *denotes the area of* $\triangle ABC$, *then* $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$.

Proof

Case I: When $\angle C$ is acute.

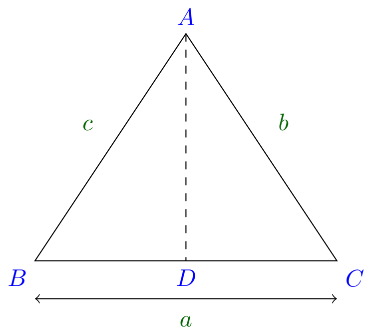


Figure 8.12

$\sin C = \frac{AD}{AC} \Rightarrow AD = b \sin C \therefore \Delta = \frac{1}{2} BC \times AD = \frac{1}{2} ab \sin C.$

Case II: When $\angle C$ is obtuse.

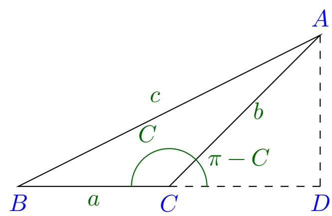


Figure 8.13

$\sin(\pi - C) = \frac{AD}{AC} \Rightarrow AD = b \sin C \therefore \Delta = \frac{1}{2} BC \times AD = \frac{1}{2} ab \sin C.$

Case III: When $\angle C$ is 90°.

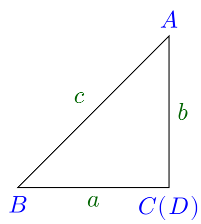


Figure 8.14

$\Delta = \frac{1}{2} BC \times AD = \frac{1}{2} ab \sin C [\because C = 90^\circ \therefore \sin C = 1].$

Thus in all cases $\Delta = \frac{1}{2} ab \sin C$. Similarly, we can prove two other formulae. □

8.8 Area in Terms of Sides

Theorem 9

If Δ be the area of any $\triangle ABC$, when $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Proof

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab.2\sin\frac{C}{2}\cos\frac{C}{2} = ab\sqrt{\frac{(s-a)(s-b)}{ab}}.\sqrt{\frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)(s-c)}.$$

8.8.1 Area in Terms of Radius of Circumcircle

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab.\frac{c}{2R} = \frac{abc}{4R}.$$

8.9 Tangent and Cotangent of Sub-angles of a Triangle

Theorem 10

In any $\triangle ABC$, $\tan\frac{A}{2} = \frac{(s-b)(s-c)}{\Delta}$, $\tan\frac{B}{2} = \frac{(s-a)(s-c)}{\Delta}$, $\tan\frac{C}{2} = \frac{(s-a)(s-c)}{\Delta}$, $\cos\frac{A}{2} = \frac{s(s-a)}{\Delta}$, $\cot\frac{B}{2} = \frac{s(s-b)}{\Delta}$, $\cot\frac{C}{2} = \frac{s(s-c)}{\Delta}$.

Proof

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-b)^2(s-c)^2}{s(s-a)(s-b)(s-c)}} = \frac{(s-b)(s-c)}{\Delta}.$$

Similarly, we can prove for other angles and cotangents. □

8.10 Dividing a Side in a Ratio

Theorem 11

If D be a point on the side BC of a $\triangle ABC$ such that $BD : dC = m : n$ and $\angle ADC = \theta, \angle BAD = \alpha$ and $\angle DAC = \beta$, then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$, $(m+n)\cot\theta = n\cot B - m\cot C$.

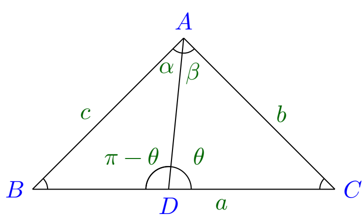


Figure 8.15

Proof

$\angle ADB = \pi - \theta$, $\angle ABD = \pi - (\alpha + \pi - \theta) = \theta - \alpha$, $\angle ACD = \pi - (\theta + \beta)$. From $\triangle ABC$, $\frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta-\alpha)}$. Form $\triangle ADC$, $\frac{DC}{\sin\beta} = \frac{AB}{\sin[\pi-(\theta+\beta)]}$.

Dividing, we get $\dfrac{BD\sin\beta}{DC\sin\alpha} = \dfrac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$

$\Rightarrow \dfrac{m\sin\beta}{n\sin\alpha} = \dfrac{\sin\theta\cos\beta + \cos\theta\cos\beta}{\sin\theta\cos\alpha - \cos\theta\sin\alpha}$

$\Rightarrow m\sin\theta\sin\beta\cos\alpha - m\cos\theta\sin\alpha\sin\beta = n\sin\alpha\sin\theta\cos\beta + n\sin\alpha\cos\theta\sin\beta$

Dividing boths ides by $\sin\alpha\sin\beta\sin\theta$, we get

$m\cot\alpha - m\cot\theta = n\cot\beta + n\cot\theta$

$\Rightarrow (m+n)\cot\theta = n\cot\beta + n\cot\theta$.

Thus, first part is proved and now we will prove the second part.

$\angle BAD = 180^\circ - (180^\circ - \theta + B) = \theta - B$, $\angle DAC = 180^\circ - (\theta + C)$

From $\triangle BAD$, $\frac{BD}{\sin(\theta-B)} = \frac{AD}{\sin B}$. From $\triangle ADC$, $\frac{DC}{\sin[180^\circ-(\theta+C)]} = \frac{AD}{\sin C}$

$\Rightarrow \dfrac{DC}{\sin(\theta+c)} = \dfrac{AD}{\sin C}$

Dividing, we get

$\dfrac{BD}{DC}.\dfrac{\sin(\theta+C)}{\sin(\theta-B)} = \dfrac{\sin C}{\sin B}$

$\Rightarrow \dfrac{m}{n}.\dfrac{\sin\theta\cos C + \cos\theta\sin C}{\sin\theta\cos B - \cos\theta\sin B} = 1$

Proceeding like previous proof, we have

$(m+n)\cot\theta = n\cot B - m\cot C$. □

8.11 Results Related with Circumcircle

A circle passing through the vertices of a triangle is called a circumcircle. Its radius is called the circumradius.

Theorem 12

Let O be the center of the circumscribing circle of $\triangle ABC$. Then, $R = \frac{abc}{4\Delta}$.

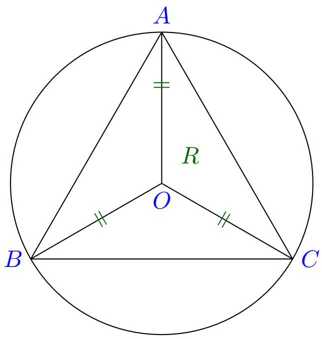


Figure 8.16

Proof

By sine rule, $\dfrac{a}{\sin A} = 2R \Rightarrow R = \dfrac{a}{2\sin A} \because \Delta = \dfrac{1}{2}bc\sin A \therefore \sin A = \dfrac{2\Delta}{bc} \Rightarrow R = \dfrac{a}{\frac{2.2\Delta}{bc}} = \dfrac{abc}{4\Delta}$. □

8.12 Results Related with Incircle

The circle touching all the three sides of a triangle internally is called the inscribed circle or in-circle. Its radius is called in-radius and denoted by r. In the figure I is the incenter of the $\triangle ABC$.

Clearly, it is the point of intersection of internal bisector of angles of the $\triangle ABC$.

Theorem 13

In $\triangle ABC, r = \frac{\Delta}{s}$.

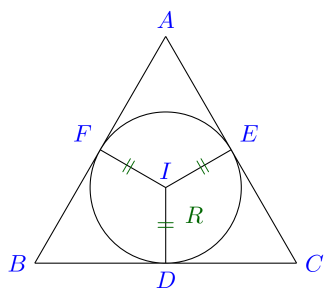


Figure 8.17

Proof

Area of $\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB \Rightarrow \Delta = \dfrac{1}{2}ar + \dfrac{1}{2}br + \dfrac{1}{2}cr \Rightarrow r = \dfrac{\Delta}{s}$ □

8.12.1 Other Forms

1. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

   R.H.S. $= 4.\frac{abc}{4\Delta}.\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{(s-a)(s-c)}{ca}}.\sqrt{\frac{(s-a)(s-b)}{ab}}$

   $= \frac{abc}{\Delta}.\frac{(s-a)(s-b)(s-c)}{abc}.\frac{s}{s} = \frac{abc}{\Delta}.\frac{\Delta}{s} = \frac{\Delta}{s} = r.$

2. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

   $r = \frac{\Delta}{s} = \frac{\Delta}{s}.\frac{s-a}{s-a} = (s-a)\tan\frac{A}{2}.$

   Similarly, we can prove for other angles.

8.13 Results Related with Escribed Circles

Let ABC be a triangle. Let the bisectors of exterior angles B and C meet at I_1. Let $I_1D \perp BC$. If we take I_1 as the center aand draw a circle it will touch all the three sides(two extended) of the triangle. We can draw three such circles, one opposite to each side. We denote these radii by r_1, r_2 and r_3 for angle sA, B and C respectively.

Theorem 14

In such a $\triangle ABC, r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}.$

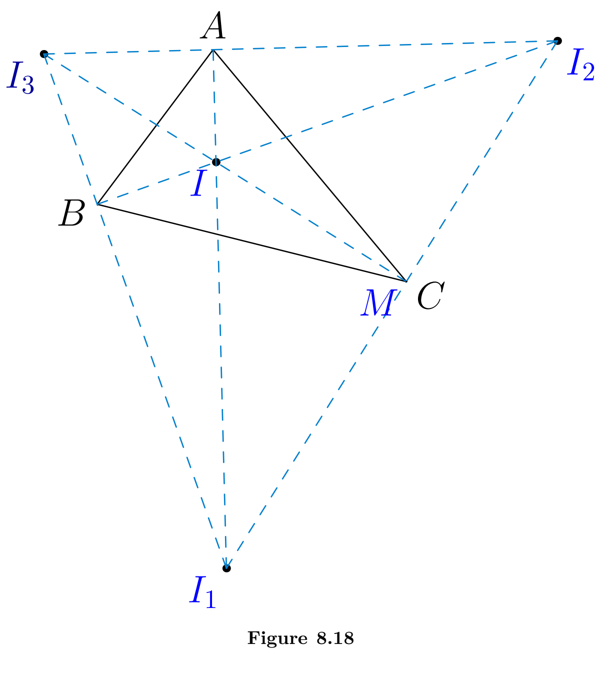


Figure 8.18

Proof

$\Delta ABC = \Delta I_1AB + \Delta I_1AC - \Delta I_1BC = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 = \frac{1}{2}(2s-2a)r_1 = (s-a)r_1 \Rightarrow r_1 = \frac{\Delta}{s-a}$. Similarly, it can be proven for r_2 and r_3. □

8.13.1 Other Forms

1. $r_1 = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
2. $r_2 = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
3. $r_3 = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

8.14 Distannces of Centers from Vertices

We have already shown that for circumcenter distance is equal to circum-radius i.e. R.

Referring to the image of incircle, $IF = r, \angle FAI = \frac{A}{2}$. From right-angle $\triangle FIA$, $\sin\frac{A}{2} = \frac{r}{AI} \Rightarrow AI = r\text{cosec}\frac{A}{2}$.

Similarly, $BI = r\text{cosec}\frac{B}{2}$ and $CI = r\text{cosec}\frac{C}{2}$.

8.14.1 Orthocenter

Orthocenter is the point of intersection of perpendiculars from a vertex to opposite side.

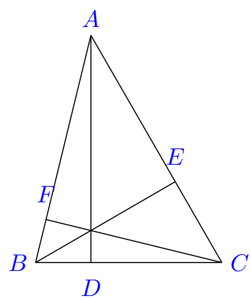


Figure 8.19

Let the orthocenter be H which is intersection of perpendiculars from any vertex to opposite side. From right-angle $\triangle AEB$, $\cos A = \frac{AE}{AB} \Rightarrow AE = c\cos A$

From right-angle $\triangle ACD$, $\angle DAC = 90^\circ - C$. From right-angle $\triangle AEH$, $\cos(90^\circ - C) = \frac{AE}{AH}$

$\Rightarrow AH = \frac{c\cos A}{\sin C} = 2R\cos A$. Similarly, $BH = 2R\cos B$ and $CH = 2R\cos C$.

8.14.2 Centroid

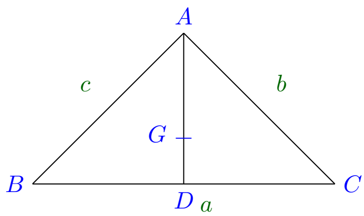


Figure 8.20

Let G be the centroid. Since, it is the point of intersection of medians, it will lie on median AD.

From geometry, $AB^2 + AC^2 = 2BD^2 + 2AD^2 \Rightarrow c^2 + b^2 = 2.\frac{a^2}{4} + 2AD^2$

$\Rightarrow 2AD^2 = \frac{2b^2+2c^2-a^2}{2}$

$\because AG : GD = 2 : 1$ [property of centeroid that it divides median in the ratio 2 : 1]

$AG = \frac{2}{3}AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$. Similarly, $BG = \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$ and $CG = \frac{1}{3}\sqrt{2a^2 + 2b^2 - c^2}$.

A Angles Made by Medians with Sides

If $\angle BAD = \beta$ and $\angle CAD = \gamma$, then we have $\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} \Rightarrow \sin\gamma = \frac{DC.\sin C}{AD}$

$= \frac{a\sin C}{\sqrt{2b^2+2c^2-a^2}}$. Similarly, $\sin\beta = \frac{a\sin B}{\sqrt{2b^2+2c^2-a^2}}$.

If $\angle ADC$ be *theta* then we have $\sin\theta = \frac{2b\sin C}{2b^2+2c^2-a^2}$.

8.15 Escribed Triangles

Refer to Figure fig:esc, in which I is the incenter and I_1, I_2 and I_3 are the centers of the excircles opposite to vertices A, B and C respectively. We know that IC will bisect the $\angle ACB$, I_1C will bisect the external angles at C and I_1B will bisect the angle at B produces by extending the sides i.e. $\angle BCM$ as shown in the figure.

$\therefore \angle ICI_1 = \angle ICM + \angle ICM = \frac{1}{2}\angle ACB + \frac{1}{2}\angle BCM = 90°$.

Similarly, $\angle ICI_1$ and $\angle ICI_3$ will be right angles.

Hence I_1CI_2 is perpendicular to IC. Similarly, I_2AI_3 is perpendicular to IA, and I_1BI_3 is perpendicular to IB.

We also see that IA and I_1A bot bisect $\angle A$ so AII_1 is a straight line. Similarly I_2IB and I_3IC are straight lines. The $\triangle I_1I_2I_3$ is called the *excentric* triangle of $\triangle ABC$.

8.16 Distance between Orthocenter and Circumcenter

Let O be the circumcenter. $OF \perp AAB$ and H be orthocenter. Then $\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$.

Let $BL \perp AC$ so it will pass through H. $\angle HAL = 90^\circ - C$, $\angle OAH = A - \angle OAF - \angle HAL = A - (180^circ - 2C) = C - B$

Also, $OA = R$ and $HA = 2R\cos A . OH^2 = OA^2 + HA^2 - 2OA.HA.\cos OAH = R^2 + 4R^2\cos^2 A - 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)] = R^2 - 8R^2\cos A\cos B\cos C$

$\Rightarrow OH = R\sqrt{1 - 8\cos A\cos B\cos C}$.

8.17 Distance between Incenter and Circumcenter

Let O be the orthocenter and $OF \perp AB$. Let I be the incenter and $IC \perp AB$.

$\angle OAF = 90^\circ - C \therefore \angle OAI = \angle IAF - \angle OAF = \frac{A}{2} - 90^\circ + C = \frac{C-B}{2}$.

Also, $AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$.

$\therefore OI^2 = OA^2 + AI^2 - 2.OA.AI.\cos OAI$

$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$

$OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2 - 2Rr}$.

8.18 Area of a Cyclic Quadrilateral

Theorem 15

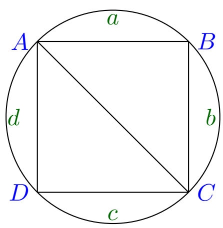


Figure 8.21

If a, b, c, d be the sides and s be the subperimeter of a cyclic quadrilateral, then its area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Proof

Let $ABCD$ be a cyclic quadrilateral having sides $AB = a$, $BC = b$, $CD = c$ and $AD = d$. Since opposing angles of a quadrilateral are complementary, therefore $B + D = A + C = \pi$.

Applying cosine law in $\triangle ABC$, $\cos B = \frac{a^2+b^2-AC^2}{2ab} \Rightarrow AC^2 = a^2+b^2-2ab\cos B$.

Similarly in $\triangle ACD$, $AC^2 = c^2+d^2+2cd\cos B$. Thus, $\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$.

Area of quadrilaterl $ABCD = \Delta ABC + \Delta ACD = \frac{1}{2}ad\sin B + \frac{1}{2}cd\sin B$

Solving last two equations, we get area of quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$. □

8.19 Problems

1. The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double the smallest angle.
2. In a $\triangle ABC$, if $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$
3. If $\triangle = a^2 - (b-c)^2$, where $\triangle$ is the area of the $\triangle ABC$, then prove that $\tan A = \frac{8}{15}$
4. In a triangle ABC, the angles A, B, C are in A.P. Prove that $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2-ac+c^2}}$
5. If p_1, p_2, p_3 be the altitudes of a triangle ABC from the vertices A, B, C respectively and Δ be the area of the triangle, prove that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{2ab\cos^2\frac{C}{2}}{\Delta(a+b+c)}$
6. In any $\triangle ABC$, if $\tan\theta = \frac{2\sqrt{ab}}{a-b}\sin\frac{C}{2}$, prove that $c = (a-b)\sec\theta$
7. In a $\triangle ABC$, $a = 6$, $b = 3$ and $\cos(A-B) = \frac{4}{5}$, then find its area.
8. In a $\triangle ABC$, $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that area of $\triangle BAD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find $\angle ABD$
9. If the sides of a triangle are 3, 5 and 7, prove that the triangle is obtuse angled triangle and find the obtuse angle.
10. In a triangle ABC, if $\angle A = 45^\circ$, $\angle B = 75^\circ$, prove that $a + c\sqrt{2} = 2b$
11. In a triangle ABC, $\angle C = 90^\circ$, $a = 3$, $b = 4$ and D is a point on AB, so that $\angle BCD = 30^\circ$, find the length of CD.
12. The sides of a triangle are $4cm$, $5cm$ and $6cm$. Show that the smallest angle is half of the greatest angle.
13. In an isosceles triangle with base a, the vertical angle is 10 times any of the base angles. Find the length of equal sides of the triangle.
14. The angles of a triangle are in the ratio of $2:3:7$, then prove that the sides are in the ratio of $\sqrt{2}:2:(\sqrt{3}+1)$
15. In a triangle ABC, if $\frac{\sin A}{7} = \frac{\sin B}{6} = \frac{\sin C}{5}$, show that $\cos A : \cos B : \cos C = 7 : 19 : 25$

16. In any triangle ABC if $\tan\frac{A}{2} = \frac{5}{6}$, $\tan\frac{B}{2} = \frac{20}{37}$, find $\tan\frac{C}{2}$ and prove that in this triangle $a + c = 2b$.

17. In a triangle ABC if $\angle C = 60°$, prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

18. If α, β, γ be the lengths of the altitudes of a triangle ABC, prove that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\cot A+\cot B+\cot C}{\Delta}$, where Δ is the area of the triangle.

19. In a triangle ABC, if $\frac{a}{b} = 2 + \sqrt{3}$ and $\angle C = 60°$, show that $\angle A = 105°$ and $\angle B = 15°$.

20. If two sides of a triangle and the included angle are given by $a = (1 + \sqrt{3})$, $b = 2$ and $C = 60°$, find the other two angles and the third side.

21. The sides of a triangle are x, y and $\sqrt{x^2 + xy + y^2}$. prove that the greatest angle is $120°$.

22. The sides of a triangle are $2x + 3$, $x^2 + 3x + 3$ and $x^2 + 2x$, prove that greatest amgle is $120°$.

23. In a triangle ABC, if $3a = b + c$, prove that $\cot\frac{B}{2}\cot\frac{C}{2} = 2$

24. In a triangle ABC, prove that $a\sin\left(\frac{A}{2} + B\right) = (b + c)\sin\frac{A}{2}$

25. In a triangle ABC, prove that $\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$

26. In a triangle ABC, prove that $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C = 0$

27. In a trianlge ABC, prove that $a^3\cos(B - C) + b^3\cos(C - A) + c^3\cos(A - B) = 3abc$

28. In a triangle ABC, prove that $\frac{\cos^2\frac{B-C}{2}}{(b+c)^2} + \frac{\sin^2\frac{B-C}{2}}{(b-c)^2} = \frac{1}{a^2}$

29. In a triangle ABC, prove that $\frac{a}{\cos B\cos C} + \frac{b}{\cos C\cos A} + \frac{c}{\cos A\cos B} = 2a\tan B\tan C\sec A$

30. In a triangle ABC, prove that $(b - c)\cos\frac{A}{2} = a\sin\frac{B-C}{2}$

31. In a triangle ABC, prove that $\tan\left(\frac{A}{2} + B\right) = \frac{c+b}{c-b}\tan\frac{A}{2}$

32. In a triangle ABC, prove that $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

33. In a triangle ABC, prove that $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$

34. In a triangle ABC, prove that $\frac{\cos^2 B-\cos^2 C}{b+c} + \frac{\cos^2 C-\cos^2 A}{c+a} + \frac{\cos^2 A-\cos^2 B}{a+b} = 0$

35. In a triangle ABC, prove that $a^3\sin(B - C) + b^3\sin(C - A) + c^3\sin(A - B) = 0$

36. In a triangle ABC, prove that $(b + c - a)\tan\frac{A}{2} = (c + a - b)\tan\frac{B}{2} = (a + b - c)\tan\frac{C}{2}$

37. In a triangle ABC, prove that $1 - \tan\frac{A}{2}\tan\frac{B}{2} = \frac{2c}{a+b+c}$

38. In a triangle ABC, prove that $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$

39. In a triangle ABC, prove that $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

40. In a triangle ABC, prove that $\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$

41. In a triangle ABC, prove that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

42. In a triangle ABC, prove that $\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}$

43. In a triangle ABC, prove that $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$

44. In a triangle ABC, prove that $(b - c)\cot\frac{A}{2} + (c - a)\cot\frac{B}{2} + (a - b)\cot\frac{C}{2} = 0$

45. In a triangle ABC, prove that $(a - b)^2\cos^2\frac{C}{2} + (a + b)^2\sin^2\frac{C}{2} = c^2$

46. In a triangle ABC, prove that $\frac{a-b}{a+b} = \cot\frac{A+B}{2}\tan\frac{A-B}{2}$

47. In a triangle ABC, D is the middle point of BC. If AD is perpendicular to AC, prove that $\cos A\cos C = \frac{2(c^2-a^2)}{3ac}$

48. If D be the middle point of the side BC of the triangle ABC where area is Δ and $\angle ADB = \theta$, prove that $\frac{AC^2-AB^2}{4\Delta} = \cot\theta$

49. $ABCD$ is a trapezium such that AB and DC are parallel and BC is perpendicular to the. If $\angle ADB = \theta$, $BC = p$, $CD = q$, show that $AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$

50. Let O be a point inside a triangle ABC such that $\angle OAB = \angle OBC = \angle OCA = \theta$, show that $\cot\theta = \cot A + \cot B + \cot C$.

51. The median AD of a triangle ABC is perpendicular to AB. Prove that $\tan A + 2\tan B = 0$.

52. In a triangle ABC, if $\cot A + \cot B + \cot C = \sqrt{3}$

53. In a triangle ABC, if $(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A + B)$

54. In a triangle ABC, if θ be any angle, show that $b\cos\theta = c\cos(A - \theta) + a\cos(C + \theta)$

55. In a triangle ABC, AD is the median. If $\angle BAD = \theta$, prove that $\cos\theta = 2\cot A + \cot B$

56. The bisector of angle A of a triangle ABC meets BC in D, show that $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

57. Let A and B be two points on one bank of a straight river and C and D be two points on the other bank, the direction from A to B along the river being the same as from C to D. If $AB = a$, $\angle CAD = \alpha$, $\angle DAB = \beta$, $\angle CBA = \gamma$, prove that $CD = \frac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$

58. In a triangle ABC, if $2\cos A = \frac{\sin B}{\sin C}$, prove that the triangle is isosceles.

59. If the cosines of two angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right angled.

60. In a triangle ABC, if $a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}$, prove that the triangle is isosceles.

61. In a triangle ABC, if $\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c-b}{c}$, prove that $A = 60^\circ$

62. In a triangle ABC, if $c^4 - 2(a^2+b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$, prove that $C = 60^\circ$ or 120°

63. In a triangle ABC, if $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$, prove that the triangle is either isosceles or right angled.

64. If A, B, C are angles of a $\triangle ABC$ and if $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in A.P., prove that $\cos A, \cos B, \cos C$ are in A.P.

65. In a triangle ABC, if $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, show that $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

66. If a^2, b^2, c^2 are in A.P., then prove that $\cot A, \cot B, \cot C$ are in A.P.

67. The angles A, B and C of a triangle ABC are in A.P. If $2b^2 = 3c^2$, determine the angle A.

68. If in a triangle ABC, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then show that the sides a, b, c are in A.P.

69. In a triangle ABC, if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in A.P.

70. In a triangle ABC, $\sin A, \sin B, \sin C$ are in A.P. show that $3\tan\frac{A}{2}\tan\frac{C}{2} = 1$.

71. In a triangle ABC, if a^2, b^2, c^2 are in A.P., show that $\tan A, \tan B, \tan C$ are in H.P.

72. In a triangle ABC, if a^2, b^2, c^2 are in A.P., show that $\cot A, \cot B, \cot C$ are in A.P.

73. If the angles A, B, C of a triangle ABC be in A.P. and $b : c = \sqrt{3} : \sqrt{2}$, find the angle A.

74. The sides of a triangle are in A.P. and the greatest angle exceeds the least angle by 90°. Prove that the sides are in the ratio $\sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1$.

75. If the sides a, b, c of a triangle are in A.P. and if a is the least side, prove that $\cos A = \frac{4c-3b}{2c}$

76. The two adjacent sides of a cyclic quadrilateral are 2 and 5 nad the angle between them is 60°. If the third side is 3, find the fourth side.

77. Find the angle A of triangle ABC, in which $(a+b+c)(b+c-a) = 3bc$

78. If in a triangle ABC, $\angle A = \frac{\pi}{3}$ and AD is a median, then prove that $4AD^2 = b^2 + bc + c^2$

79. Prove that the median AD and BE of a ΔABC intersect at right angle if $a^2 + b^2 = 5c^2$

80. If in a triangle ABC, $\frac{\tan A}{1} = \frac{\tan B}{2} = \frac{\tan C}{3}$, then prove that $6\sqrt{2}a = 3\sqrt{5}b = 2\sqrt{10}c$

81. The sides of a triangle are $x^2 + x + 1, 2x + 1$ and $x^2 - 1$, prove that the greatest anngle is $120°$.

82. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

83. For a triangle ABC having area 12 sq. cm. and base is 6 cm. The difference of base angles is $60°$. Show that angle A opposite to the base is given by $8\sin A - 6\cos A = 3$.

84. In any triangle ABC, if $\cos\theta = \frac{a}{b+c}, \cos\phi = \frac{b}{a+c}, \cos\psi = \frac{c}{a+b}$ where θ, ϕ and ψ lie between 0 and π, prove that $\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = 1$.

85. In a triangle ABC, if $\cos A\cos B + \sin A\sin B\sin C = 1$, show that the sides are in the proportion $1 : 1 : \sqrt{2}$.

86. The product of the sines of the angles of a triangle is p and the product of their cosines is q. Show that the tangents of the angles are the roots of the equation $qx^3 - px^2 + (1 + q)x - p = 0$

87. In a $\triangle ANC$, if $\sin^3\theta = \sin(A - \theta)\sin(B - \theta)\sin(C - \theta)$, prove that $\cot\theta = \cot A + \cot B + \cot C$.

88. In a triangle of base a, the ratio of the other two sides is $r(< 1)$, show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$

89. Given the base a of a triangle, the opposite angle A, and the product k^2 of the other two sides. Solve the triangle and show that there is such triangle if $a < 2k\sin\frac{A}{2}$, k being positive.

90. A ring 10 cm in diameter, is suspended from a point 12 cm above its center by 6 equal strings, attached at equal intervals. Find the cosine of the angle between consecutive strings.

91. If $2b = 3a$ and $\tan^2\frac{A}{2} = \frac{3}{5}$, prove that there are two values of third side, one of which is double the other.

92. The angles of a triangle are in the ratio $1 : 2 : 7$, prove that the ratio of the greater side to the least side is $\sqrt{5} + 1 : \sqrt{5} - 1$.

93. If f, g, h are internal bisectors of the angles of a triangle ABC, show that $\frac{1}{f}\cos\frac{A}{2} + \frac{1}{g}\cos\frac{B}{2} + \frac{1}{h}\cos\frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

94. If in a triangle ABC, $BC = 5, CA = 4, AB = 3$ and D and E are points on BC scuh that $BD = DE = EC$. If $\angle CAB = \theta$, then prove that $\tan\theta = \frac{3}{8}$.

95. In a triangle ABC, median AD and CE are drawn. If $AD = 5, \angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$, find the area of the triangle ABC.

96. The sides of a triangle are $7, 4\sqrt{3}$ and $\sqrt{13}$ cm. Then prove that the smallest angle is $30°$.

97. In an isosceles, right angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle in two parts whose cotangents are 2 and 3.

98. The sides of a triangle are such that $\frac{a}{1+m^2n^2} = \frac{b}{m^2+n^2} = \frac{c}{(1-m^2)(1+n^2)}$, prove that $A = 2\tan^{-1}\frac{m}{n}$, $B = 2\tan^{-1} mn$ and $\Delta = \frac{mnbc}{m^2+n^2}$.

99. The sides a, b, c if a triangle ABC are the roots of the equation $x^3 - px^2 + qx - r = 0$, prove that its area is $\frac{1}{4}\sqrt{p(4pq - p^3 - 8r)}$

100. Two sides of a triangle are of lengths $\sqrt{6}$ cm and 4 cm and the angle opposite to the smaller side is 30°. How many such triangles are possible? Fine the length of their third side and area.

101. The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right) = 4\left(1+\frac{1}{t_2^2}\right)$

102. The three medians of a triangle ABC make angles α, β, γ with each other, prove that $\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0$.

103. Perpendiculars are drawn from the angles A, B, C of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle. If these produced parts be α, β, γ respectively, show that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$

104. In a triangle ABC, the vertices A, B, C are at distance p, q, r from the orthocenter respectively. Show that $aqr + brp + cpq = abc$

105. The area of a circular plot of land in the form of a unit circle is to be divided into two equal parts by the arc of a circle whose center is on the circumference of the plot. Show that the radius of the circular arc is given by $\cos\theta$ where θ is given by $\frac{\pi}{2} = \sin 2\theta - 2\theta\cos 2\theta$

106. BC is a side of a square, on the perpendicular bisector of BC, two points P, Q are taken, equidistant from the center of square. BP and CQ are joined and cut in A. Prove that in the trangle ABC, $\tan A(\tan B - \tan C)^2 + 8 = 0$

107. If the bisector of the angle C of a triangle ABC cuts AB in D and the circum-circle in E, prove that $CE : DE = (a+b)^2 : c^2$.

108. The internal bisectors of the angles of a triangle ABC meet the sides at D, E and F. Show that the area of the triangle DEF is equal to $\frac{2\Delta abc}{(b+c)(c+a)(a+b)}$

109. In a triangle ABC, the measures of the angles A, B and C are $3\alpha, 3\beta$ and 3γ respectively. P, Q and R are the points within the triangle such that $\angle BAR = \angle RAQ = \angle QAC = \alpha$, $\angle CBP = \angle PBR = \angle RBA = \beta$ and $\angle ACQ = \angle QCP = \angle PCB = \gamma$. Show that $AR = 8R\sin\beta\sin\gamma\cos(30^\circ - \gamma)$

110. A circle touches the x axis at O (origin) and intersects the y axis above origin at $B.A$ is a point on that part of cirlce which lies to the right of OB, and the tangents at A and B meet at T. If $\angle AOB = \theta$, find the angles which the directed line OA, AT and OB makes with OX.

If lengths of these lines are c, t and d respectively, show that $c\sin\theta - t(1+\cos 2\theta) = 0$ and $c\cos\theta + t\sin 2\theta = d$.

111. If in a triangle ABC, the median AD and the perpendicular AE from the vertex A to the side BC divides the angle A into three equal parts, show that $\cos\frac{A}{3}.\sin^2\frac{A}{3} = \frac{3a^2}{32bc}$

112. In a triangle ABC, if $\cos A + \cos B + \cos C = \frac{3}{2}$, prove that the triangle is equilateral.

113. Prove that a triangle ABC is equilateral if and only if $\tan A + \tan B + \tan C = 3\sqrt{3}$.

114. In a triangle ABC, prove that $(a+b+c)\tan\frac{C}{2} = a\cot\frac{A}{2} + b\cot\frac{B}{2} - c\cot\frac{C}{2}$

115. In a triangle ABC, prove that $\sin^4 A + \sin^4 B + \sin^4 C = \frac{3}{2} + 2\cos A\cos B\cos C + \frac{1}{2}\cos 2A + \cos 2B + \cos 2C$

116. In a triangle ABC prove that $\cos^4 A + \cos^4 B + \cos^4 C = \frac{1}{2} - 2\cos A\cos B\cos C + \frac{1}{2}\cos 2A\cos 2B\cos 2C$

117. In a triangle ABC, prove that $\cot B + \frac{\cos C}{\cos A\sin B} = \cot C + \frac{\cos B}{\cos A\sin C}$

118. In a triangle ABC, prove that $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

119. In a triangle ABC, prove that $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$

120. In a triangle ABC, prove that $\sin^3 A\cos(B-C) + \sin^3 B\cos(C-A) + \sin^3 C\cos(A-B) = 3\sin A\sin B\sin C$

121. In a triangle ABC, prove that $\sin^3 A + \sin^3 B + \sin^3 C = 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}$

122. In a triangle ABC, prove that $\sin 3A\sin^3(B-C) + \sin 3B\sin^3(C-A) + \sin 3C\sin^3(A-B) = 0$

123. In a triangle ABC, prove that $\sin 3A\cos^3(B-C) + \sin 3B\cos^3(C-A) + \sin 3C\cos^3(A-B) = \sin 3A\sin 3B\sin 3C$

124. In a triangle ABC, prove that $\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2} + b\sin^2\frac{A}{2}\right) = c\cot\frac{C}{2}$

125. The sides of a triangle ABC are in A.P. If the angles A and C are the greatest and the smallest angles respectively, prove that $4(1-\cos A)(1-\cos C) = \cos A + \cos C$

126. In a triangle ABC, if a, b, c are in H.P., prove that $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are also in H.P.

127. If the sides a, b, c of a triangle ABC be in A.P., prove that $\cos A\cot\frac{A}{2}, \cos B\cot\frac{B}{2}, \cos C\cot\frac{C}{2}$ are in A.P.

128. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ th of an equilateral triangle of the same perimieter. Prove that the sides are in the ratio $3:5:7$.

129. If the tangents of the angles of a triangle are in A.P., prove that the squares of the sides are in the proportion $x^2(x^2+9):(3+x^2)^2:9(1+x^2)$, where x is the least or the greatest tangent.

130. If the sides of a triangle are in A.P. and if its greatest angle exceeds the least angle by α, show that the sides are in the ratio $1-x:1:1+x$ where $x=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$

131. If the sides of triangle ABC are in G.P. with common ratio $r(r>1)$, show that $r<\frac{1}{2}(\sqrt{5}+1)$ and $A<B<\frac{\pi}{3}<C$

132. If p and q be the perpendiculars from the vertices A and B on any line passing through the vertex C of the triangle ABC but not passing through the interior of the angle ABC, prove that $a^2p^2+b^2q^2-2abpq\cos C=a^2b^2\sin^2 C$

133. ABC is a triangle, O is a point inside the triangle such that $\angle OAB=\angle OBC=\angle OCA=\theta$, then show that $\operatorname{cosec}^2\theta=\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C$

134. If x, y, z be the lengths of perpendiculars from the circumcenter on the sides BC, CA, AB of a triangle ABC, prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$

135. In any triangle ABC if D is any point on the base BC such that $BD:DC=m:n$ and if $AD=x$, prove that $(m+n)^2x^2=(m+n)(mb^2+nc^2)-mna^2$

136. In a triangle ABC, if $\sin A+\sin B+\sin C=\frac{3\sqrt{3}}{2}$, prove that the triangle is equilateral.

137. In a triangle ABC, if $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{8}$, prove that the triangle is equilateral.

138. In a triangle ABC, if $\cos A+2\cos B+\cos C=2$, prove that the sides of the triangle are in A.P.

139. The sides a, b, c of a triangle ABC of a triangle are in A.P., then find the value of $\tan\frac{A}{2}+\tan\frac{C}{2}$ in terms of $\cot\frac{B}{2}$.

140. In a triangle ABC, if $\frac{a-b}{b-c}=\frac{s-a}{s-c}$, prove that r_1, r_2, r_3 are in A.P.

141. If the sides a, b, c of a triangle ABC are in G.P., then prove that x, y, z are also in G.P., where $x=(b^2-c^2)\frac{\tan B+\tan C}{\tan B-\tan C}$, $y=(c^2-a^2)\frac{\tan C+\tan A}{\tan C-\tan A}$, $z=(a^2-b^2)\frac{\tan A+\tan B}{\tan A-\tan B}$

142. The ex-radii r_1, r_2, r_3 of a triangle ABC are in H.P. Show that its sides a, b, c are in A.P.

143. In usual notation, $r_1=r_2+r_3+r$, prove that the triangle is right-angled.

144. If A, B, C are the angles of a triangle, prove that $\cos A+\cos B+\cos C=1+\frac{r}{R}$

145. Show that the radii of the three escribed circles of a triangle are the roots of the equation $x^3-x^2(4R+r)+xs^2-rs^2=0$

146. The radii r_1, r_2, r_3 of escribed circle of a triangle ABC are in H.P. If its area if 24 sq. cm. and its perimeter is 24 cm., find the length of its sides.

147. In a triangle ABC, $8R^2 = a^2 + b^2 + c^2$, prove that the triangle is right-angled.

148. The radius of the circle passing through the center of the inscribed circle and through the point of the base BC is $\frac{a}{2} \sec \frac{A}{2}$

149. Three circles touch each other externally. The tangents at their point of connect meet at a point whose distance from the point of contact is 4. Find the ratio of the product of radii to the sum of of radii of all the circles.

150. In a triangle ABC, if O be the circumcenter and H, the orthocenter, show that $OH = R\sqrt{1 - 8\cos A \cos B \cos C}$

151. Let ABC be a triangle having O and I as its circumcenter an in-center respectively. If R and r be the circumradius and in-radius respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right angled triangle if and only if b is the arithmetic means of a and c.

152. In any triangle ABC, prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

153. Let ABC be a triangle with in-center I and in-radius r. Let D, E and F be the feet of perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respectively, prove that $\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}$

154. Show that the line joining the orthocenter to the circumference of a triangle ABC is inclined to BC at an angle $\tan^{-1}\left(\frac{3-\tan B \tan C}{\tan B - \tan C}\right)$

155. If a circle be drawn touching the inscribed and circumscribed circle of a triangle and BC externally, prove that its radius is $\frac{\Delta}{a} \tan^2 \frac{A}{2}$.

156. The bisectors of the angles of a triangle ABC meet its circumcenter in the position D, E, F. Show that the area of the triangle DEF is to that of ABC is $R : 2r$.

157. If the bisectors of the angles of a triangle ABC meet the opposite sides in A', B', C', prove that the ratio of the areas of the triangles $A'B'C'$ and ABC is $2\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} : \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{C-A}{2}$.

158. If a, b, c are the sides of a triangle $\lambda a, \lambda b, \lambda c$ the sides of a similar triangle inscribed in the former and θ the angle between the sides of a and λa, prove that $2\lambda \cos \theta = 1$.

159. If r be the radius of in-circle and r_1, r_2, r_3 be the ex-radii of a triangle ABC, prove that $r_1 + r_2 + r_3 - r = 4R$

160. If r be the radius of in-circle and r_1, r_2, r_3 be the ex-radii of a triangle ABC, prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

161. If r be the radius of in-circle and r_1, r_2, r_3 be the ex-radii of a triangle ABC, prove that $\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{\Delta^2}$ where Δ denotes the area of the triangle ABC.

162. If r is the radius of in-circle of a triangle ABC, prove that $r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$.

163. If A, A_1, A_2 and A_3 be respectively the areas of the inscribed and escribed circles of a triangle, prove that $\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}$

164. In a triangle ABC, prove that $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$.

165. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB=AC$ and h is the altitude from A to BC then the triangle ABC has perimeter $P=2(\sqrt{2rh-h^2}+\sqrt{2rh})$. Find its area.

166. If p_1, p_2, p_3 are the altitudes of the triangle ABC from the vertices A, B, C respectively, prove that $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=\frac{1}{R}$.

167. Three circles whose radii are a, b, c touch one another externally and the tangents at their point of contact meet in a point. Prove that the distance of this point from either of their points of contact is $\sqrt{\frac{abc}{a+b+c}}$

168. In a triangle ABC, prove that $r_1r_2r_3=r^3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$.

169. In a triangle ABC, prove that $a(rr_1+r2r_3)=b(rr_2+r_3r_1)=c(rr_3+r_1r_2)=abc$.

170. In a triangle ABC, prove that $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c$.

171. In a triangle ABC, prove that $4R\sin A\sin B\sin C=a\cos A+b\cos B+c\cos C$.

172. In a triangle ABC, prove that $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$

173. In a triangle ABC, prove that $r^2+r_1^2+r_2^2+r_3^2=16R^2-a^2-b^2-c^2$

174. In a triangle ABC, prove that $IA.IB.IC=abc\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

175. In a triangle ABC, prove that $AI_1=r_1\mathrm{cosec}\frac{A}{2}$

176. In a triangle ABC, prove that $II_1=a\sec\frac{A}{2}$

177. In a triangle ABC, prove that $I_2I_3=a\mathrm{cosec}\frac{A}{2}$

178. In a triangle ABC, if I is the in-center and I_1, I_2 and I_3 are the centers of the escribed circles, then prove that $II_1.II_2.II_3=16R^2r$

179. In a triangle ABC, if I is the in-center and I_1, I_2 and I_3 are the centers of the escribed circles, then prove that $II_1^2.I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 = 16R^2$

180. In a triangle ABC, if O is the circumcenter and I, the in-center then prove that $OI^2 = R^2(3 - 2\cos A - 2\cos B - 2\cos C)$.

181. In a triangle ABC, if H is the orthocenter and I the in-center then prove that $IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

182. In a triangle ABC, if O is the circumcenter, G, the cetroid and H, the orthocenter then prove that $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$.

183. Given an isosceles triangle with lateral side of length b, base angle $\alpha < \frac{\pi}{4}$; R, r the radii and O, I the centers of the circumcircle and in-circle respectively, then prove that $R = \frac{1}{2} b \operatorname{cosec} \frac{\alpha}{2}$.

184. Given an isosceles triangle with lateral side of length b, base angle $\alpha < \frac{\pi}{4}$; R, r the radii and O, I the centers of the circumcircle and in-circle respectively, then prove that $r = \frac{b \sin 2\alpha}{2(1+\cos\alpha)}$

185. Given an isosceles triangle with lateral side of length b, base angle $\alpha < \frac{\pi}{4}$; R, r the radii and O, I the centers of the circumcircle and in-circle respectively, then prove that $OI = \left|\frac{b\cos\frac{3\alpha}{2}}{2\sin\alpha\cos\frac{\alpha}{2}}\right|$

186. In a triangle ABC, prove that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$

187. In a triangle ABC, prove that $\frac{r_1}{(s-b)(s-c)} + \frac{r_2}{(s-c)(s-a)} + \frac{r_3}{(s-a)(s-b)} = \frac{3}{r}$.

188. If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the in-circle, prove that $r^2 = \frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}$

189. Tangents are drawn to the in-circle of triangle ABC which are parallel to its sides. If x, y, z be the lengths of the tangents and a, b, c be the sides of triangle then prove that $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

190. If t_1, t_2, t_3 be the length of tangents from the centers of escribed circles to the circumcircle, prove that $\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{2s}{abc}$.

191. If in a triangle ABC, $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is right angled.

192. In a triangle ABC, prove that the area of the in-circle is to the area of the triangle itself is $\pi : \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

193. Let $A_1, A_2, A_3, \ldots, A_n$ be the vertices of polygon having an n sides such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$ then find the value of n.

194. Prove that the sum of radii of the circles, which are respectively inscribed in and circumscibed about a regular polygon of n sides, is $\frac{a}{2}\cot\frac{\pi}{2n}$, where a is the side of the polygon.

195. The sides of a quadrilateral are $3, 4, 5$ and 6 cms. The sum of a pair of opposite angles is 120°. Show that the area of the quadrilateral is $3\sqrt{30}$ sq. cm.

196. The two adjacent sides of a quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the two remaining sides.

197. A cyclic quadrilateral $ABCD$ of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

198. If $ABCD$ be a quadrilateral inscribed in a circle, prove that $\tan\frac{B}{2} = \sqrt{\frac{(S-a)(S-b)}{(S-c)(S-d)}}$.

199. a, b, c and d are the sides of a quadrilateral taken in order and α is the angle between diagonals opposite to b or d, prove that the area of the quadrilateral is $\frac{1}{2}(a^2 - b^2 + c^2 - d^2)\tan\alpha$

200. If a quadrilateral can be inscribed in one circle and circumscribed about another circle, prove that its area is $\sqrt{abcd}$ and the radius of the latter circle is $\frac{2\sqrt{abcd}}{a+b+c+d}$.

201. The sides of a quadrilateral which can be inscribed in a circle are $3, 3, 4$ and 4 cm; find the radii of in-circle and circumcircle.

202. A square whose sides are 2 cm., has its corners cut away so as to form a regular octagon; find its area.

203. If an equilateral triangle and a regular hexagon have the same perimeter, prove that ratio of their areas is $2 : 3$.

204. Given that the area of a polygon of n sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as $3 : 2$, find n.

205. The area of a polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as $3 : 4$. Fine the value of n.

206. There are two regular polygons, the number of sides in one being the double the number in the other, and an angle of one ploygon is to an angle of the other is $9 : 8$; find the number of sides of each polygon.

207. Six equal circles, each of radius a, are placed so that each touches to others, their centers are joined to form a hexagon. Prove that the area which the circles enclose is $2a^2(3\sqrt{3} - \pi)$.

208. A cyclic quadrilateral $ABCD$ of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

209. If $ABCD$ is a cyclic quadrilateral, then prove that $AC.BD = AB.CD + BC.AD$

210. If the number of sides of two regular polygons having the same perimeter be n and $2n$ respectively, prove that their areas are in the ratio $2\cos\frac{\pi}{n} : \left(1 + \cos\frac{\pi}{n}\right)$.

211. In a triangle ABC, prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$

212. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the center. Find the minimum value of the arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right), \cos\left(\beta + \frac{\pi}{2}\right)$ and $\cos\left(\gamma + \frac{\pi}{2}\right)$

213. In a triangle ABC, prove that $\tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} \geq 1$

214. Let $1 < m < 3$. In a triangle ABC if $2b = (m+1)a$ and $\cos A = \frac{1}{2}\sqrt{\frac{(m-1)(m+3)}{m}}$, prove that there are two values of the third side, one of which is m times the other.

215. If Δ denotes the area of any triangle and s its semiperimeter, prove that $\Delta < \frac{s^2}{4}$.

216. Let A, B, C be three angles such that $A = \frac{\pi}{4}$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

217. Through the angular points of a triangle straight lines are drawn, which make the same angle α with the opposite side of the triangle. Prove that the area of the triangle formed by them is to the area of the triangle is $4\cos^2\alpha : 1$

218. Consider the following statements about a triangle ABC

    a. The sides a, b, c and Δ are rational.

    b. $a, \tan\frac{B}{2}, \tan\frac{C}{2}$ are rational

    c. $a, \sin A, \sin B, \sin C$ are rational.

    Prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

219. Two sides of a triangle are of length $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30°. How many such triangles are possible? Find the length of their third side and area.

220. A circle is inscribed in an equilateral triangle of side a. Prove that the area of any square inscribed in this circle is $\frac{a^2}{6}$.

221. In a triangle ABC, AD is the altitude from A. Given $b > c$, $\angle C = 23^\circ$ and $AD = \frac{abc}{b^2 - c^2}$, then find $\angle B$.

222. In a triangle ABC, $a : b : c = 4 : 5 : 6$, then find the ratio of the radius of the circumcircle to that of in-circle.

223. In a triangle ABC, $\angle B = \frac{\pi}{3}, \angle C = \frac{\pi}{4}$ and D divides BC internally in the ratio of $1 : 3$. Prove that $\frac{\sin\angle BAD}{\sin\angle CAD} = \frac{1}{\sqrt{6}}$

224. In a triangle ABC, angle A is greater than angle B. If the measure of angle A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0, 0 < k < 1$, then find the measure of angle C.

225. ABC is a triangle such that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C)$, if A, B, C are in A.P. determine the value of A, B and C.

226. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the length of perpendicular drawn from the opposite vertex on the hypotenuse. Find the two angles.

227. In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of the equation $ax^2 + bx + c = 0 (a \neq 0)$, then prove that $a + b = c$.

228. In a triangle ABC, the medians to the side BC is of length $\frac{1}{\sqrt{1-6\sqrt{3}}}$ and it divides the angle A into angles of 30° and 45°. Find the lngth of side BC.

229. If A, B, C are the anngles of an acute-angled triangle, show that $\tan A + \tan B + \tan C \geq 3\sqrt{3}$.

230. In a triangle ABC, $\cos\frac{A}{2} = \frac{1}{2}\sqrt{\frac{b}{c} + \frac{c}{b}}$, show that the square describe on one side of the is equal to twice the rectangle contained by two other sides.

231. If in a triangle ABC, θ be the angle determined by the relation $\cos\theta = \frac{a-b}{c}$. Prove that $\cos\frac{A-B}{2} = \frac{(a+b)\sin\theta}{2\sqrt{ab}}$ and $\cos\frac{A+B}{2} = \frac{c\cos\theta}{2\sqrt{ab}}$.

232. If R be the circum-radius and r the in-radius of a triangle ABC, show that $R \geq 2r$.

233. If $\cos A = \tan B, \cos B = \tan C$ and $\cos C = \tan A$, show that $\sin A = \sin B = \sin C = 2\sin 18^\circ$, where A, B, C lie between 0 and π.

234. In a triangle ABC, prove that $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

235. In a triangle ABC, prove that $\tan^2 A + \tan^2 B + \tan^2 C \geq 9$

236. In a triangle ABC, prove that $\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2} \geq 6$

237. In a triangle ABC, prove that $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$

238. In a triangle ABC, prove that $\cos A \cos B \cos C \leq \frac{1}{8}$

239. Two circles of radii a and b cut each other at an angle θ. Prove that the length of the common chord is $\frac{2ab\sin\theta}{\sqrt{a^2+b^2 2ab\cos\theta}}$.

240. Three equal circles touch one another; find the radius of the circle which touches all the three circles.

241. In a triangle ABC, prove that $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r}\cos[rB - (n-r)A] = C^n$

242. In a triangle ABC, $\tan A + \tan B + \tan C = k$, then find the interval in which k should lie so that there exists one isosceles triangle ABC.

243. If Δ be the area and s, the semi-perimeter of a triangle, then prove that $\Delta \leq \frac{s^2}{3\sqrt{3}}$.

244. Show that the tirangle having sides $3x + 4y$, $4x + 3y$ and $5x + 5y$ units where $x > 0, y > 0$ is obtuse-angled triangle.

245. Let ABC be a triangle having altitudes h_1, h_2, h_3 from the vertices A, B, C respectively and r be the in-radius, then prove that $\frac{h_1+r}{h_1-r} + \frac{h_2+r}{h_2-r} + \frac{h_3+r}{h_3-r} \geq 0$.

246. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle, whose area is Δ, and Δ_1, Δ_2 and Δ_3 be the corresponding areas for the escribed circles, prove that $\Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta$.

Chapter 9
Inverse Circular Functions

Inverse functions related to trigonometric ratios are called inverse trigonometric functions. The definition of different inverse trigonometric functions is given below:

If $\sin\theta = x$, then $\theta = \sin^{-1} x$, provided $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

If $\cos\theta = x$, then $\theta = \cos^{-1} x$, provided $-1 \le x \le 1$ and $0 \le \theta \le \pi$.

If $\tan\theta = x$, then $\theta = \tan^{-1} x$, provided $-\infty < x < \infty$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

If $\cot\theta = x$, then $\theta = \cot^{-1} x$, provided $-\infty < x < \infty$ and $0 < \theta < \pi$.

If $\sec\theta = x$, then $\theta = \sec^{-1} x$, provided $x \le -1$ or $x \ge 1$ and $0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$.

If $\mathrm{cosec}\theta = x$, then $\theta = \mathrm{cosec}^{-1}x$, provided $x \le -1$ or $x \ge 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$.

Note: In the above definition, restrictions on θ are due to the consideration of principal values of inverse terms. If these restrictions are removed, the terms will represent inverse trigonometric relations and not functions.

Notations: I. $\mathrm{Arc}\sin x$ denotes the sine inverse of x [General value]. $\arcsin x$ denotes the principal value of sine inverse of x.

II. $\sin^{-1} x$ denotes the principal value of sine inverse x. From the above notations three important results follow;

1. $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$ and θ is the principal value.
2. $\sin^{-1} x = \arcsin x, \cos^{-1} x = \arccos x$.
3. From the definition of the inverse functions, we know that if $y = f(x)$ is a function then for f^{-1} to be a function, f must be one-to-one and onto mapping.

When we consider $y = \mathrm{Arc}\sin x$, for any $x \in [-1, 1]$ infinite number of values of y are obtained and hence it does not represent inverse functions. When $y = \arcsin x$ or $\sin^{-1} x$, corresponding to one value of $x \in [-1, 1]$, one value of y is obtained and hence it represents the inverse trigonometric function.

Hence, for inverse trigonometric functions, consideration of principal values is essential.

9.1 Principal Value

Numerically smallest angle is known as the principal value.

Since inverse trigonometric terms are in fact angles, definitions of principal value of inverse trigonometric term is the same as the definition of the principal values of angles.

Suppose we have to find the principal value of $\sin^{-1}\frac{1}{2}$. Let $\sin^{-1}\frac{1}{2} = \theta$, then $\sin\theta = \frac{1}{2} \Rightarrow \theta = \ldots, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$. Among all these angles $\frac{\pi}{6}$ is the numerically smalles angles satisfying $\sin\theta = \frac{1}{2}$ and hence it is the principal value.

9.2 Important Formulae

Theorem 16

1. $\sin\sin^{-1}x = x, -1 \leq x \leq 1$
2. $\cos\cos^{-1}x = x, -1 \leq x \leq 1$
3. $\tan\tan^{-1}x = x, -\infty < x \leq \infty$
4. $\cot\cot^{-1}x = x, -\infty < x \leq \infty$
5. $\sec\sec^{-1}x = x, x \leq -1 \;\; or \; x \geq 1$
6. $\mathrm{cosec}\,\mathrm{cosec}^{-1}x = x, x \leq -1 \;\; or \; x \geq 1$

Proof

Let $\sin^{-1}x = \theta$ then $\sin\theta = x$. Putting the value of θ from first equation in second $\sin\sin^{-1}x = x$. Other formulae can be proved similarly. □

Theorem 17

1. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\; \forall -1 \leq x \leq 1$
2. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\; \forall x \in \mathbb{R}$
3. $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}\; \forall x \leq -1 \;\; or \; x \geq 1$

Proof

Let $\sin^{-1}x = \theta$, then $\sin\theta = x \Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = x \Rightarrow \frac{\pi}{2} - \theta = \cos^{-1}x$

$\Rightarrow \cos^{-1}x + \theta = \frac{\pi}{2} \Rightarrow \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

Similarly other results can be proven. □

Theorem 18

1. $\sin^{-1}x = \mathrm{cosec}^{-1}\frac{1}{x}, -1 \leq x \leq 1$
2. $\mathrm{cosec}^{-1}\mathrm{x} = \sin^{-1}\frac{1}{x}, x \leq -1 \;\; or \; x > 1$

3. $\cos^{-1} x = \sec^{-1} \frac{1}{x}, -1 \leq x \leq 1$

4. $\sec^{-1} x = \cos^{-1} \frac{1}{x}, x \leq -1 \ or \ x \geq 1$

Proof

Let $\sin^{-1} x = \theta$ then $\sin \theta = x \Rightarrow \mathrm{cosec}\theta = \frac{1}{\mathrm{x}}$

$\Rightarrow \theta = \mathrm{cosec}^{-1} \frac{1}{\mathrm{x}} \Rightarrow \sin^{-1} \mathrm{x} = \mathrm{cosec}^{-1} \frac{1}{\mathrm{x}}$

Other results can be proven similarly. □

Theorem 19

1. $\sin^{-1} x = \cos^{-1} \sqrt{1 - x^2},\ \forall\, 0 \leq x \leq 1$

2. $\sin^{-1} x = -\cos^{-1} \sqrt{1 - x^2}\ \forall\ -1 \leq x < 0$

Proof

Let $\sin^{-1} x = \theta$ then $\sin \theta = x$

$\Rightarrow \cos^2 \theta = 1 - x^2 \Rightarrow \cos \theta = \pm\sqrt{1 - x^2}$

Principal values of $\sin^{-1} x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

In this interval $\cos \theta$ is +ve.

$\Rightarrow \sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}$

For $-1 \leq x < 0$ $\sin^{-1} x$ will be negative angle while $\cos^{-1} \sqrt{1 - x^2}$ will be positive angle. Hence to balance that we need to use a negative sign for this. □