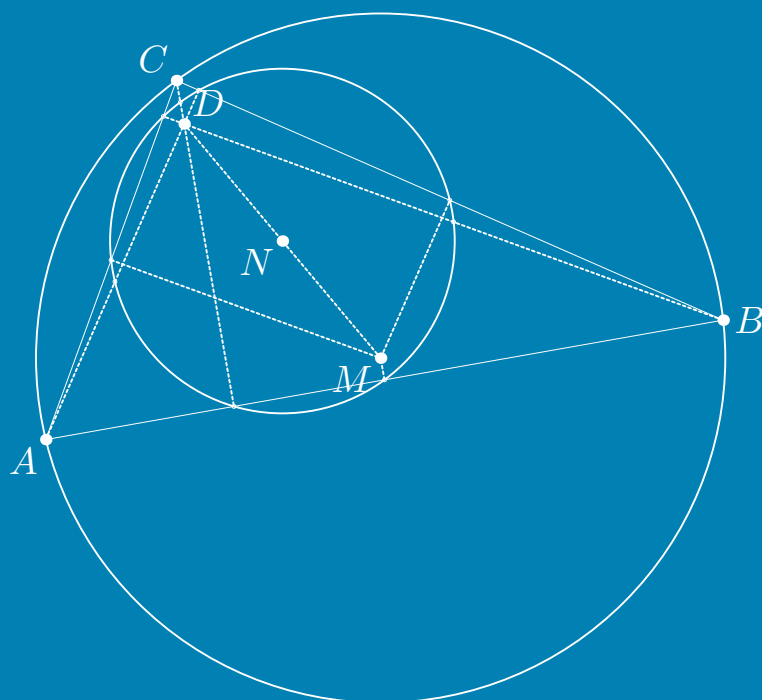


Geometry with Trigonometry

A problem-oriented approach

Shiv Shankar Dayashru



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Dedicated to my family.

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Preface

This is a book on Geometry and Trigonometry for high-school students. Usually the books on geometry and trigonometry are two separate books. However, because the concepts are closely related, I decided to merge both of them in a single book. Before this I wrote the book on high-school algebra, so it is natural to cover trigonometry as the next book. Algebra and trigonometry form bulk of the mathematics for high-school syllabus, therefore, it is necessary that students get good grasp on the basic concepts of both of these subjects.

Usually, geometry is spread over 3-4 years of syllabus in schools, however, this book presents all the material of those years in one book. It covers lines, triangles, quadrilaterals, convex polygons and circles. Trigonometry will cover plane trigonometry with some information on circles. This is a book for self study and is not recommended for courses in schools and universities.

Trigonometry and geometry are most fundamental subjects in Mathematics as further study of subjects like coordinate geometry, calculus, engineering and rest all depend on it. It is very important to understand these subjects for the readers if they want to advance further in mathematics.

Who should read this book?

Anyone with basic knowledge of class 10 maths should be able to read this book without much difficulty. Other than that there are no other prerequisite. Since this book is written for self study anyone with interest in trigonometry and geometry can read it. That does not mean that school or college students cannot read it. You need to be selective as to what you need for your particular requirements. This is mostly high-school course with a little bit of lower classes course thrown in with a bit of detail here and there

How to read this book?

The usual advice about learning math is to solve problems, however, I will not simply tell that you must solve the problems. My advice is more detailed. Every chapter will have theory. Read that first. Make sure you understand that. Of course, you have to meet the prerequisites for the book. Then, go on and try to solve the problems. In

this book, there are no pure problems. Almost all have answers except those which are of similar kind and repetitive in nature for the sake of practice.

If you can solve the problem then all good else look at the answer and try to understand that. Then, few days later take on the problem again. If you fail to understand the answer you can always email me with your work and I will try to answer to the best of my ability. However, if you have a local expert seek his/her advice first.

Note that mathematics is not only about solving problems. If you understand the theory well, then you will be able to solve problems easily. However, problems do help enforce the theory in our mind.

I am a big fan of old MIR publishers problem books, so I emphasize less on theory and more on problems. I hope that you find this style much more fun as a lot of theory is boring. Mathematics is about problem solving as that is the only way to enforce theory and find innovative techniques for problem solving.

Some of the problems in certain chapters rely on other chapters which you should look ahead or you can skip those problems and come back to it later. Since this book is meant for self study answers of most of the problems have been given which you can make use of. However, do not use for just copying but rather to develop understanding of the subject.

Goals for readers

The goal of for reading this book is becoming proficient in solving simple and basic problems of trigonometry and geometry. Another goal would be to be able to study other subjects which require this knowledge like trigonometry or calculus or physics or chemistry or other subjects. If you can solve 95% problems after 2 years of reading this book then you have achieved this goal.

All of us possess a certain level of intelligence. At average any person can read this book. But what is most important is you have to have interest in the subject. Your interest gets multiplied with your intelligence and thus you will be more capable than you think you can be. One more point is focus and effort. It is not something new which I am telling but I am saying it again just to emphasize the point. Trust me if you are reading this book for just scoring a nice grade in your course then I have failed in my purpose of explaining my ideas.

Also, if you find this book useful feel free to share it with others without hesitation as it is free as in freedom. There are no conditions to share it.

Acknowledgements

I am in great debt of my family and free software community because both of these groups have been integral part of my life. Family has provided direct support while

free software community has provided the freedom and freed me from the slavery which comes as a package with commercial software. I am especially grateful to my wife, son, daughter, and parents because it is their time which I have borrowed to put in the book.

To pay my thanks from free software community I will take one name and that is Richard Stallman who started all this and is still fighting this never-ending war. When I was doing the algebra book then I realized how difficult it is to put Math on web in HTML format and why Donald Knuth wrote T_EX. Also, T_EX was one of the first softwares to be released as a free software. This book has been typeset with T_EX using **Emacs**. I have used macros from the book “TeX for Impatient” and modified them as well as the **eplain**’s macros.

I have used **Asymptote** for drawing all the diagrams. It is a wonderful packages and work very nicely particularly for 3D.

I would like to pay my most sincere gratitude to my teachers particularly H. N. Singh, Yogendra Yadav, Satyanand Satyarthi, Kumar Shailesh and Prof. T. K. Basu. Now is the turn of people from software community. I must thank the entire free software community for all the resources they have developed to make computing better. However, few names I know and here they go. Richard Stallman is the first, Donald Knuth, Edger Dijkstra, John von Neumann after that as their lives have strong influence in how I think and base my life on.

I am not a native English speaker and this book has just gone through one pair of eyes therefore chances are high that it will have lots of errors(particularly with commas and spelling mistakes). At the same time it may contain lots of technical errors. Please feel free to drop me an email at shivshankar.dayal@gmail.com, where I will try to respond to each mail as much as possible. Please use your real names in email not something like coolguy. If you have more problems which you want to add it to the book please send those by email or create a PR on [github](https://github.com).

Part I

Geometry

1 | Geometry

Geometry is one of the oldest branches of math along with arithmetic, and one of the most fundamental. The *Elements* by Euclid, a Greek mathematician c. 300 BC, which is a series of 13 books, is one of the first profound works on Geometry. Most of what we will study about geometry in this book is based on the book from Euclid. Some scholars believe that the *Elements* is largely a compilation of work of earlier Greek mathematicians. Geometry comes from an ancient Greek word meaning ‘land measurement’. What we will study in this book lays the fundamental principles of basic geometry, which will be useful to you when you study more advanced branches of mathematics.

Points, Lines and Planes

A *point* is a dimensionless figure. This means that it has no length, width and depth. Essentially, a point is invisible. However, we usually denote it with a .(dot). If we join two points such that resulting figure has only one dimension of length, and the length of line is equal to the shortest distance between the two points then the resulting figure would be a *line*.

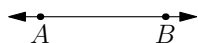
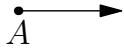


Fig. 1. A Line

The Fig. 1 shown is a figure of a straight line. A line in general is also called a straight line, which is the shortest distance between the two points.

On any straight line there are infinite points(or between any two points there are infinite points). A line stretches from $+\infty$ to $-\infty$. Usually a line is drawn with arrows at both ends, which denote that the line stretches to infinity. If there are two points on the line A and B then the line is also written as \overleftrightarrow{AB} , and read as line AB . A *line segment* on the other hand is of finite length, and is drawn without arrows. A line segment, for example, in the figure can start at A and end at B . Such a line segment is written as \overline{AB} or just AB . A and B are called *endpoints* of the line segment AB .



The Fig. 1 shown is a figure of a ray. A *ray* has one endpoint and it stretches to infinity on other side. Thus, you can split a line into two rays. A ray is written as \overrightarrow{AB} .

Note that \overrightarrow{AB} is not same as \overrightarrow{BA} . However, a line segment AB is same as line segment BA .

If you rotate a line by 180° (rotate in such a manner that ends occupy other end's place), then you will get a *plane*. A plane has two dimensions, length and width, both of which are infinite. For example, top surface of a table or a page of this book (assuming the page has zero thickness) are planes. In this book we will restrict ourselves to planar geometry i.e. 2D or two-dimensional geometry.

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