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MATHEMATICAL MODELLING

1. Farming fish

» Compartmental Diagram:

Birth → [Fish population] → Natural Death
→ Death due to harvesting

Assumptions:

1. The fish population is large enough to ignore random differences between individuals.
2. Per capita birth rate, death rate and harvesting rate are constant per day.
3. There are no other factor affecting population size.

Balance law:

$$\begin{Bmatrix}\text{rate of change of} \\ \text{no. of fish}\end{Bmatrix} = \begin{Bmatrix}\text{rate of} \\ \text{birth}\end{Bmatrix} - \begin{Bmatrix}\text{rate of} \\ \text{natural} \\ \text{death}\end{Bmatrix} - \begin{Bmatrix}\text{rate of deaths} \\ \text{by harvesting}\end{Bmatrix}$$

This balance law translates into: $\frac{dX}{dt} = 0.7X - 0.2X - 300$,

where X is population of fish at time t.

(b) $\frac{dX}{dt} = 0.5X - 300 \Rightarrow \frac{\ln(X-600)}{0.5} = t + c_1$

where c_1 is constant

$\Rightarrow X = e^{(t+c_1)\times 0.5} + 600 = e^{0.5t} \cdot C + 600$

$\therefore X = C \cdot e^{0.5t} + 600$ where C is constant.

We have to find fish born in 1 week, if present population is 240000.

$\because$ per capita birth rate per day per fish is 0.7,

$\therefore$ Total fish born $= 0.7 \times 7 \times 240000 = 1176000$

(c) Now, for equilibrium position:

$\frac{dX}{dt} = 0 \Rightarrow 0.5X = 300 \Rightarrow X = 600$

$\therefore X_e = 600$

2. Modelling spread of technology

$\Rightarrow \dfrac{dN}{dt} = aN\left(1 - \dfrac{N}{N^*}\right)$ [Given]

a) Here N/N^* represent fraction of population which adopted technology. Hence, fraction which not yet adopted is $1 - N/N^*$

b) $N^* = 17015$, $a = 0.490$, $N_0 = 141$, Year $= 1997$

Now, $\dfrac{dN}{dt} = 0.490N\left(1 - \dfrac{N}{17015}\right) = 0.490N - \dfrac{0.490}{17015}N^2$

$$\Rightarrow \frac{dN}{-8337.5N + 0.490N^2} = \frac{dt}{-17015} \Rightarrow \int \frac{dN}{N^2 - 17015.31N} = -\int \frac{dt}{17015}$$

$$\Rightarrow \int \frac{dN}{N(N-17015.31)} = -\int \frac{dt}{17015} \Rightarrow -\frac{\ln N}{17015.31} + \frac{\ln(N-17015.31)}{17015.31} = -\frac{t}{17015} + C_1$$

$$\Rightarrow -\ln|N| + \ln|N - 17015| = -t + C_2$$

$$\Rightarrow \left|\frac{N - 17015}{N}\right| = C \cdot e^{-0.49t}$$

At $t=0$, $N_0 = 141$ $\therefore$ $C = 120$

Now 80% of $N^* = \dfrac{80}{100} \times 17015 = \dfrac{4}{5} \times 17015 = 4 \times 3403 = 13612$

$\therefore$ Time to reach 80% is : 12.599

3. Density dependent births:

$\Rightarrow$ Given: $B(X) = \beta - (\beta - \alpha)\delta\dfrac{X}{K}$ $\quad A(X) = \alpha + (\beta - \alpha)(1 - \delta)\dfrac{X}{K}$

where $B(X)$ and $A(X)$ are density dependent birth rate and death rate respectively.

$K =$ population carrying capacity

α, β are intrinsic death rate and birth rate.

$\delta \in [0, 1]$ is parameter describing extent

$$\therefore \frac{dX}{dt} = B(X) - A(X)$$

$$= \beta - (\beta - \alpha)\delta\frac{X}{K} - \alpha - (\beta - \alpha)(1 - \delta)\frac{X}{K}$$

$$= \beta - \cancel{\frac{\beta\delta X}{K}} + \cancel{\frac{\delta\alpha X}{K}} - \alpha - \frac{\beta X}{K} + \cancel{\frac{\delta\beta X}{K}} + \frac{\alpha X}{K} - \cancel{\frac{\delta\alpha X}{K}}$$

$$= \beta - \alpha - \frac{\beta X}{K} + \frac{\alpha X}{K} = (\beta - \alpha) - \frac{X}{K}(\beta - \alpha)$$

$$= (\beta - \alpha)\left[1 - \frac{X}{K}\right]$$

. Here $r = (\beta - \alpha)$

4. Mouse population model

$\Rightarrow$ Birth $\rightarrow$ $\boxed{\text{Mouse}}$ $\rightarrow$ Natural deaths / Death due to trap

Assumptions: The population is large enough to ignore random differences.

- Per capita birth and death rate are constant.
- 4 weeks per months.
- No other factor affecting population size.

Balance law:

{rate of change of no. of mice} = {rate of births} - {rate of natural death} - {rate of death by trap}

This translates into,

$$\frac{dM}{dt} = 8M - 2M - 80 \qquad M_0 = 1000$$

5. Fishing with quotas

$\Rightarrow$ Given, $\dfrac{dX}{dt} = rX\left(1 - \dfrac{X}{K}\right) - h_0X$

a) $\dfrac{dX}{dt} = 0 \Rightarrow X\left(r - \dfrac{rX}{K} - h_0\right) = 0$ $\Rightarrow X_e = 0$ or $X_e = K\left(1 - \dfrac{h_0}{r}\right)$

b) The population will become extinct when $X_e = \left(1 - \dfrac{h}{r}\right) = 0$

This occurs when $h = r$.

6. Predicting population size

⇒ Given, $\frac{dX}{dt} = 0.2X - 0.001X^2$, $x_0 = 100$

To find: $X(2)$

Now, Solving DE gives,

$$\frac{dX}{0.001X^2 - 0.2X} = -dt \Rightarrow \frac{dX}{X^2 - 200X} = \frac{-dt}{1000} \Rightarrow \frac{dX}{X(X-200)} = \frac{-dt}{1000}$$

$$\Rightarrow \int \left(\frac{-1}{200X} + \frac{1}{200(X-200)}\right) dX = -\int \frac{dt}{1000} \Rightarrow \frac{1}{200} \ln\left|\frac{X-200}{X}\right| = \frac{-t}{1000} + C_1$$

Now, $X(0) = 100$ $\therefore \frac{1}{200}\ln(1) = C_1 \Rightarrow C_1 = 0$

Hence, $X(2)$: $\frac{1}{200}\ln\left|\frac{X-200}{X}\right| = \frac{-2}{1000} \Rightarrow \ln\left|\frac{X-200}{X}\right| = \frac{-400}{1000} = \frac{-2}{5}$

$$\Rightarrow \left|\frac{X-200}{X}\right| = e^{\left(\frac{-2}{5}\right)} = 0.67 \Rightarrow 200 - X = 0.67X$$

$$\Rightarrow 1.67X = 200 \Rightarrow \underline{X = 120}$$

7. Modelling the population of a country

⇒ (a) Compartmental Diagram:

Births → [Population of a country] → Deaths
Immigration → [Population of a country] → Emmigration

Assumptions:

1. The population is large enough to ignore random differences among themselves.
2. Per capita birth rate and death rate be constant.
3. Overall immigration and emigration rate is constant.
4. There are no other factor affecting population.

Balance law:

$$\begin{Bmatrix}\text{rate of}\\ \text{change of}\\ \text{population}\end{Bmatrix} = \begin{Bmatrix}\text{rate of}\\ \text{births}\end{Bmatrix} - \begin{Bmatrix}\text{rate}\\ \text{of}\\ \text{deaths}\end{Bmatrix} + \begin{Bmatrix}\text{rate}\\ \text{of}\\ \text{immigration}\end{Bmatrix} - \begin{Bmatrix}\text{rate}\\ \text{of}\\ \text{emi-}\\ \text{gration}\end{Bmatrix}$$

This balance law translates into: $\frac{dX}{dt} = (\beta - \alpha)x + i - e$

where α an β are per capita death and birth rate.
i and e be immigration and emigration rate
$X(t)$ be the population at time t.

(b) Let X and Y be two neighbouring country with $X(t)$ and $Y(t)$ be their population at time t. Also we assume that Y is bigger than X.

Compartmental Diagram:

Births → [Country X] → Deaths
Births → [Country Y] → Deaths
[Country X] → Net movement of population → [Country Y]

Assumptions:

1. Per capita birth and death death rate X are constant (say β_x and α_x)
2. Per capita death and birth rate of Y are constant (say α_y and β_y)
3. The population is large enough to ignore random differences and there aren't any factor affecting it.
4. Country Y is bigger than X. So net movement of population is from X to Y.
5. The rate of net movement of people constant. (say γ).

Balance law:

$$\begin{Bmatrix}\text{rate of}\\ \text{change of}\\ \text{pop}^n \text{ of X}\end{Bmatrix} = \begin{Bmatrix}\text{rate of}\\ \text{births}\end{Bmatrix} - \begin{Bmatrix}\text{rate}\\ \text{of}\\ \text{deaths}\end{Bmatrix} - \begin{Bmatrix}\text{net rate of}\\ \text{movement}\\ \text{from X to Y}\end{Bmatrix}$$

$$\begin{Bmatrix}\text{rate of}\\ \text{change of}\\ \text{pop}^n \text{ of Y}\end{Bmatrix} = \begin{Bmatrix}\text{rate of}\\ \text{births}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{deaths}\end{Bmatrix} + \begin{Bmatrix}\text{net rate of}\\ \text{movement}\\ \text{from X to Y}\end{Bmatrix}$$

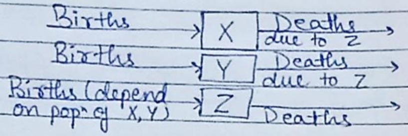
This balance law translates into:

$$\frac{dX}{dt} = \beta_x X - \alpha_x X + \gamma(X - Y) \; ; \; \frac{dY}{dt} = \beta_y Y - \alpha_y Y - \gamma(X - Y)$$

8. Two prey and one predator

⇒ Let X, Y be two prey species and Z be predator.

Compartmental Diagram:



Assumptions: 1. The population are large enough to ignore random differences among people.

2. Only two prey and one predator affect system.

3. Prey grows exponentially in absence of predator and prey do not compete with each other.

4. β_1 and β_2 be the constant per capita birth rate of X and Y. α_1 and α_2 be the death ~~rate~~ constant of X and Y due to Z. α_3 be the interacting constant of Z with X and Y and γ be per capita death rate of Z

Balance law:

$\begin{Bmatrix}\text{rate of change}\\ \text{of prey X}\end{Bmatrix} = \begin{Bmatrix}\text{rate of birth}\\ \text{of prey X}\end{Bmatrix} - \begin{Bmatrix}\text{rate X killed by Z}\end{Bmatrix}$

$\begin{Bmatrix}\text{rate of change}\\ \text{of prey Y}\end{Bmatrix} = \begin{Bmatrix}\text{rate of birth}\\ \text{of prey Y}\end{Bmatrix} - \begin{Bmatrix}\text{rate Y killed by}\\ \text{Z}\end{Bmatrix}$

$\begin{Bmatrix}\text{rate of change}\\ \text{of predator Z}\end{Bmatrix} = \begin{Bmatrix}\text{rate of birth}\\ \text{of Z}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{death of Z}\end{Bmatrix}$

This balance law translates into,

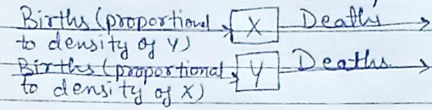
$$\frac{dX}{dt} = \beta_1 X - \alpha_1 XZ$$

$$\frac{dY}{dt} = \beta_2 X - \alpha_2 YZ$$

$$\frac{dZ}{dt} = \alpha_3 XZ - \gamma Z$$

9. Symbiosis

⇒ Let X and Y be two species.



Assumptions:

1. Let X(t) and Y(t) be the density of population of X and Z.

2. Per capita birth rate of X and Y be proportional to Y and Z population's density. And the constant of proportionality be β_1 and β_2 respectively.

3. Per capita death rate be constant (say α_1, α_2)

4. Population is large enough to ignore random difference between individuals.

5. There are no other factor affecting system.

Balance law: $\begin{Bmatrix}\text{rate of change}\\ \text{of X}\end{Bmatrix} = \begin{Bmatrix}\text{rate of}\\ \text{birth of X}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{death of X}\end{Bmatrix}$

$\begin{Bmatrix}\text{rate of change}\\ \text{of Y}\end{Bmatrix} = \begin{Bmatrix}\text{rate of}\\ \text{birth of X}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{death of Y}\end{Bmatrix}$

Balance law translates into: $\frac{dX}{dt} = \beta_1 XY - \alpha_1 X$; $\frac{dY}{dt} = \beta_2 YX - \alpha_2 Y$

10. Simple age-based model

⇒ $\begin{Bmatrix}\text{rate of change of}\\ \text{juvenile}\end{Bmatrix} = \begin{Bmatrix}\text{rate of juvenile}\\ \text{birth}\end{Bmatrix} - \begin{Bmatrix}\text{rate juvenile}\\ \text{reach maturity}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{juvenile}\\ \text{death}\end{Bmatrix}$

$\begin{Bmatrix}\text{rate of change of}\\ \text{adults}\end{Bmatrix} = \begin{Bmatrix}\text{rate at which}\\ \text{juvenile reach maturity}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{adult death}\end{Bmatrix}$

Assumptions: 1. Let A(t) and J(t) be Adult and juvenile density.

2. Per capita birth and death rate be constant ($\beta_1, \beta_2, \alpha_1, \alpha_2$)

3. Rate at which juvenile turns into adults be constant (σ)

∴ Equations: $\frac{dJ}{dt} = \beta_1 A - \sigma J - \alpha_1 J$; $\frac{dA}{dt} = \sigma J - \alpha_2 A$

11. Predator-prey with protection of young ones.

⇒ Birth → {Young prey} → becomes adult, natural death

Juvenile become Adult → {Adult prey} → natural death, death due to protection of child,

Birth → {Predator} → Natural deaths

Assumptions: 1. Let B_1 be the per capita birth rate of young prey (X_1) α_1 be the natural per capita death rate And Y_1 be the rate at which juvenile become adult.

2. α_2 be the per capita natural death of adult prey (X_2). And Y_2 be the death rate due to protection of child prey.

3. The population is large enough to ignore random difference between individual. And there are no other factor affecting system.

Balance law: {rate of change of juvenile prey} = {rate of birth} - {rate of death} - {rate at which it turns adult}

{rate of change of adult prey} = {rate of change of juvenile into adult} - {rate of N. death} - {rate of death due to predator}

{rate of change of predator} = {rate of birth} - {rate of natural deaths}

Balance law translates into,

$$\frac{dX_1}{dt} = B_1X_2 - \alpha_1X_1 - Y_1X_1 \qquad \frac{dX_2}{dt} = Y_1X_1 - \alpha_2X_2 - Y_2X_1Y$$

$$\frac{dY}{dt} = B_2Y - \alpha_3Y$$

12. Simple example

⇒ $x'=0 \Rightarrow x=5y$ and $y'=0 \Rightarrow x=y$

$\therefore (x,y)=(0,0)$

13. Finding equilibrium points

⇒(a) $X' = 3X - 2XY$ and $Y' = XY - Y$

$\therefore X'=0 \Rightarrow X(3-2Y)=0 \Rightarrow X=0,\ Y=3/2$

$Y'=0 \Rightarrow Y(X-1)=0 \Rightarrow Y=0,\ X=1$

$\therefore (X,Y)=(0,0),\ (1,3/2)$

(b) $X'=2X-XY$ and $Y'=Y-XY$

$\therefore X'=0 \Rightarrow X(2-Y)=0 \Rightarrow X=0,\ Y=2$

$Y'=0 \Rightarrow Y(1-X)=0 \Rightarrow X=1,\ Y=0$

$\therefore (X,Y)=(0,0),\ (1,2)$

(c) $X'=Y-2XY$ and $Y'=XY-Y^2$

$\therefore X'=0 \Rightarrow Y(1-2X)=0 \Rightarrow Y=0,\ X=1/2$

$Y'=0 \Rightarrow Y(X-Y)=0 \Rightarrow Y=0,\ X=Y$

$\therefore (X,Y)=(1/2,0),\ (1/2,\tfrac{1}{2}),\ (0,0),\ (Y,0) = (X,0),\ (1/2,\tfrac{1}{2})$

14. Using chain rule.

⇒ $\frac{dX}{dt} = -XY$, $\frac{dY}{dt} = -2Y$ $\quad\therefore \frac{dY}{dX} = \frac{dY}{dt}\times\frac{dt}{dX} = \frac{-2Y}{-XY} = \frac{2}{X}$

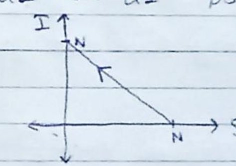
⇒ $\int dY = \int \frac{2}{X}dX \Rightarrow Y = 2\ln|X| + C$

15. Contagious for life

⇒(a) $\frac{dS}{dI} = \frac{dS}{dt}\times\frac{dt}{dI} = \frac{-\beta SI}{\beta SI} = -1 \Rightarrow S=-I+C$

Here C=N, which is popⁿ.

(b)



Here, $\frac{dS}{dt} = -\beta SI < 0$, S↓

$\frac{dI}{dt} = \beta SI > 0$, I↑

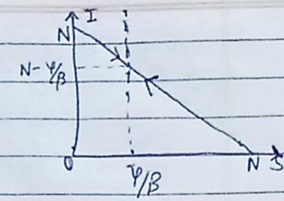
16. Disease with reinfection

⇒a. $\frac{dS}{dI} = \frac{dS}{dt}\times\frac{dt}{dI} = \frac{-\beta SI + \gamma I}{\beta SI - \gamma I} = -1 \Rightarrow S+I=N$

6. The nullclines are $I=0$, $S=\gamma/\beta$

For $S<\gamma/\beta$, $\frac{dS}{dt}>0$; $\frac{dI}{dt}<0$

$S>\gamma/\beta$, $dS/dt<0$; $\frac{dI}{dt}>0$



c. It is not possible to cross nullcline since, as we approach $S=\gamma/\beta$, $S'\to 0$ and $I'\to 0$.

$\therefore$ If we initially have a small no. of susceptible, no. of infectives decreases to $N-\gamma/\beta$ while no. of suspectible ~~decreas~~ increases to γ/β. On the other hand, if initial no. of susceptible is large, no. of infectives increases to $N-\gamma/\beta$ and no. of susceptibles decreases to γ/β.

17. Predator-prey with density dependent growth of prey.

$\Rightarrow X'=0 \Rightarrow \beta_1 X - \frac{\beta_1 X^2}{K} - C_1XY = 0 \Rightarrow X\left(\beta_1 - \frac{\beta_1 X}{K} - C_1 Y\right)=0$

$\Rightarrow X=0$ or $Y=\frac{\beta_1}{C_1}\left(1-\frac{X}{K}\right)$ or $X=\frac{(\beta_1 - C_1 Y)K}{\beta_1}$

$Y'=0 \Rightarrow Y(C_2X-\alpha_2)=0 \Rightarrow Y=0$ or $X=\alpha_2/C_2$

$\therefore (X_e,Y_e)=(0,0),\left(\alpha_2/C_2, \frac{\beta_1}{C_1}\left(1-\frac{\alpha_2}{C_2\cdot K}\right)\right), \left(\frac{K(\cancel{\beta_1}\cancel{\alpha X})}{\cancel{\beta_1}}, 0\right)$

It is different from Lotka-Volterra in the sense that $\frac{dX}{dt}$ has an extra term $\left(-\frac{\beta_1 X^2}{K}\right)$ which shows that it has density-dependent growth.

18. Predator-prey with DDT

$\Rightarrow$(a) $X'=0 \Rightarrow X(\beta_1 - C_1Y - P_1)=0$: $X=0$ or $Y=\frac{\beta_1 - P_1}{C_1}$

$Y'=0 \Rightarrow Y(-\alpha_2 + C_2X - P_2)=0$: $Y=0$ or $X=\frac{P_2+\alpha_2}{C_2}$

$\therefore (X_e,Y_e)=(0,0),\left(\frac{P_2+\alpha_2}{C_2}, \frac{\beta_1 - P_1}{C_1}\right)$

(b) In case of no, DDT: $(X_e,Y_e)=\left(\frac{\alpha_2}{C_2}, \beta_1/C_1\right)$

(c) Predator fraction (in equilibrium) $=\frac{Y_e}{\ldots} = \frac{1}{\ldots} = \frac{1}{1+\frac{C_2(\beta_1-P_1)}{\ldots}}$

19. Predator-prey with density dependence and DDT.

$\Rightarrow X'=0 \Rightarrow \beta_1 X - \frac{\beta_1 X^2}{K} - C_1XY - P_1X = 0 \Rightarrow X\left(\beta_1 - \frac{\beta_1 X}{K} - C_1Y - P_1\right)=0$

$\Rightarrow X=0$ or $Y=\frac{\beta_1 - \frac{\beta_1 X}{K} - P_1}{C_1}$ or $X=\frac{(\beta_1 - C_1Y - P_1)\cdot K}{\beta_1}$

$Y'=0 \Rightarrow Y(C_2X-\alpha_2-P_2)=0 \Rightarrow Y=0$ or $X=\frac{P_2+\alpha_2}{C_2}$

$\therefore (X_e,Y_e)=(0,0),\left(\frac{P_2+\alpha_2}{C_2}, \frac{\beta_1\left(1-\frac{P_2+\alpha_2}{KC_2}\right)-P_1}{C_1}\right), \left(\left(\frac{\beta_1 - P_1}{\beta_1}\right)K, 0\right)$

20. One prey and two predator

$\Rightarrow$(a) $X'=0 \Rightarrow X(a_1 - b_1Y - C_1Z)=0 \Rightarrow X=0$ or $Y=\frac{a_1 - C_1Z}{b_1}$ or $Z=\frac{a_1 - Yb_1}{C_1}$

$Y'=0 \Rightarrow Y(a_2X - b_2)=0 \Rightarrow Y=0$ or $X=b_2/a_2$

$Z'=0 \Rightarrow Z(a_3X-b_3)=0 \Rightarrow Z=0$ or $X=b_3/a_3$

~~$\therefore (X_e,Y_e)=(0,0),\left(b_2/a_2, \frac{a_1-C_1Z}{b_1}\right)$~~

$\therefore (X_e,Y_e,Z_e)=(0,0,0),\left(\frac{b_2}{a_2},\frac{a_1}{b_1},0\right),\left(\frac{b_3}{a_3},0,\frac{a_1-Yb_1}{C_1}\right)$

Not possible for all to coexist in equilibrium.

(b) Introducing an extra predator means, either existing or new predator becomes extinct.

21. Rabbits and foxes

$\Rightarrow$(a) $\frac{dY}{dX}=\frac{dY}{dt}\times\frac{dt}{dX}=\frac{bXY-cY}{-aXY}=\frac{bX-c}{-aX}=\frac{c-bX}{aX}$

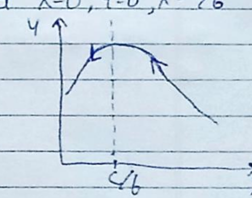
$\Rightarrow Y=\frac{c}{a}\ln|X| - \frac{b}{a}X + C$

(b) The nullclines of this system are at $X=0, Y=0, X=c/b$

$X<c/b$: $Y'<0$, $X>c/b$: $Y'>0$

Also $X'<0$ $\forall$ $X,Y>0$

(c) Yes, intersection with Y axis is possible. Then $X=0$.



22. Microorganism and toxins.

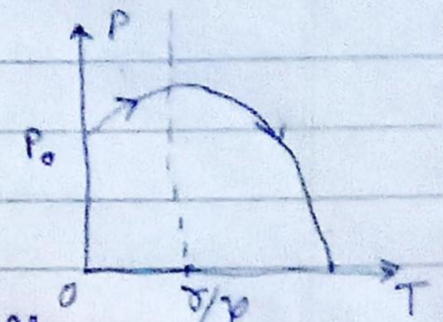
⇒ (a) $\frac{dP}{dT} = \frac{dP}{dt} \times \frac{dt}{dT} = \frac{\gamma P - \varphi PT}{qP} = \frac{\gamma - \varphi T}{q}$

⇒ $P = \frac{\gamma}{q} T - \frac{\varphi}{q} \cdot \frac{T^2}{2} + C$, $C = P_0$

(b) The nullclines are: $P = 0$, $T = \gamma/\varphi$

With $T < \gamma/\varphi$, $P' > 0$; $T > \gamma/\varphi$, $P' < 0$

$P > 0$, $T' > 0$



The amount of toxins always increases.

~~but~~ then decreases as more toxins is produced.

The amount microorganism increases, but then decrease as more toxin is produced.

23. Atmospheric pressure.

⇒ (a) $\frac{dP}{dh} = -kP$ ⇒ $\ln P = -kh + C$, $C = \ln P_0$

⇒ $\ln P = -kh + 6.92$ ⇒ $P = e^{-kh} \cdot 1012.32$

or $P = P_0 e^{-kh}$

At $h = 0$, $P = 1013$ ∴ $P = P_0 e^{-kh}$

At $h = 20$, $P = 50$ ∴ $50 = 1013 e^{-20k}$ ⇒ $k = 0.15$

Hence $P = 1013 . e^{-0.15h}$

(b) At 50 km: $P = 1013 e^{-0.15 \times 50} = 1013 \cdot e^{-7.5} = 0.56$ mb.

(c) $P = 900$ mb, $h = ?$ $900 = 1013 e^{-0.15h}$

⇒ 0.79 km $= h$

Q. Lake Burley Griffin: $F_{in} = F_{out} = 4 \times 10^6$

$V = 28 \times 10^6$ Safety threshold $= 4 \times 10^6$

Now $C(t) = C_{in} - (C_{in} - C_0)e^{-Ft/V}$

As only fresh water enters lake, so $C_{in} = 0$

$\therefore C(t) = C_0 e^{-Ft/V}$

a) Time taken for conc to drop to 5% of C_0 is approx $\frac{3V}{F}$

$= 3 \times 28 \times 10^6 / 4 \times 10^6 = 21$ months.

Now, to fin t: $C(t) = 4 \times 10^6$

$4 \times 10^6 = C_0 e^{-Ft/V} = 10^7 e^{-Ft/V} \Rightarrow 0.4 = e^{-Ft/V} = e^{-\frac{\cancel{4 \times 10^6} \times t}{\cancel{28 \times 10^6}_{7}}} = e^{-\frac{t}{7}}$

$\Rightarrow \ln(0.4) = -t/7 \Rightarrow t = -7\ln(0.4) = 6.414$

Q. North American lake system:

$C_1(t)$ $\qquad\qquad$ $C_2(t)$

$\longrightarrow$ $\boxed{\text{Lake Erie}}$ $\longrightarrow$ $\boxed{\text{Lake Ontaria}}$ $\longrightarrow$

V_1, F_1, C_{1in} $\qquad\qquad$ $V_2, F_2, C_1(t)$

for Erie: $\frac{dC_1}{dt} = \frac{F_1}{V_1} C_{1in} - \frac{F_1}{V_1} C_1(t)$; for Ontario: $\frac{dC_2}{dt} = \frac{F_2}{V_2} C_1(t) - \frac{F_2}{V_2} C_2(t)$

(b) $C_{1in} = 0$. If only unpolluted water enters lake Erie, then $C_{1in} = 0$. The model simplies to $\frac{dC_1}{dt} = -\frac{F_1}{C_1} C_1(t)$

$$\frac{dC_2}{dt} = \frac{F_2}{V_2} C_1(t) - \frac{F_2}{V_2} C_2(t)$$

(c) $\frac{dC_1}{dt} = -\frac{F_1}{V_1} C_1(t) \Rightarrow \int \frac{1}{C_1}\, dC_1 = \int -\frac{F_1}{V_1}\, dt \Rightarrow \ln C_1 = -\frac{F_1}{V_1} t + \ln \alpha$

$\boxed{C_1(t) = \alpha e^{-\frac{F_1}{V_1} t}}$ Let $C_1(0) = C_1(t)$ at $t = 0$. Then $\alpha = C_1(0)$

$\therefore C_1(t) = C_1(0) e^{-F_1/V_1 t}$

d) $\frac{dC_2}{dt} = \frac{F_2}{V_2} C_1(0) e^{-\frac{F_1}{V_1} t} - \frac{F_2}{V_2} C_2(t)$

It is linear

$\therefore I_F = e^{\frac{F_2}{V_2} t}$

So soln: $C_2(t) \cdot e^{F_2/V_2 t} = \int \frac{F_2}{V_2} C_1(0)\, e^{\left(\frac{F_2}{V_2} - \frac{F_1}{V_1}\right) t}\, dt$

$\Rightarrow C_2(t) \cdot e^{\frac{F_2}{V_2} t} = \frac{F_2}{V_2} C_1(0) \int e^{\left(\frac{F_2}{V_2} - \frac{F_1}{V_1}\right) t}\, dt$

$\Rightarrow C_2(t) = \frac{\frac{F_2}{V_2} \cdot C_1(0)\, e^{-F_1/V_1 t}}{\left(\frac{F_2}{V_2} - \frac{F_1}{V_1}\right)} + \beta e^{-\frac{F_2}{V_2} t}$

Let $C_2(t) = C_2(0)$ when $t = 0$

$C_2(0) = \frac{F_2}{V_2} C_1(0) \Big/ \left(\frac{F_2}{V_2} - \frac{F_1}{V_1}\right) + \beta$

Now, $C_2(t) = \frac{\frac{F_2}{V_2} C_1(0)\, e^{-\frac{F_1}{V_1} t}}{\left(\frac{F_2}{V_2} - \frac{F_1}{V_1}\right)} + \left(C_2(0) - \frac{\frac{F_2}{V_2} C_1(0)}{\left(\frac{F_2}{V_2} - \frac{F_1}{V_1}\right)} \right) e^{-\frac{F_2}{V_2} t}$

e) If $C_1(0)$ is more, then peak concentration in 2nd lake is going to be more. If $C_2(0)$ inc, then time taken by lake Ontaria to get rid of pollutant increases.

2. Smoke in bar:

Cigarette smoke → [CO in smoke] Ventilators →

$\frac{dc}{dt} = \cancel{\frac{0.006}{1200}}\ \frac{F_{in}}{V} C_{in} - \frac{F_{out}}{V} C(t)$

Balance law:

Rate of change of CO = Rate at which CO is produce due to smoking − Rate at which CO leaves through ventilators