proof) Let $(x_n) \to c$, then

$f(x_n) \to l$ (given) $\left[\text{by defn. criteria}\right]$

$g(x_n) \to m$ (given)

Now (T.S) If $x_n \to c$ then $(f+g)(x_n) \to l+m$

$(f+g)(x_n) = f(x_n) + g(x_n)$

as $x_n$ is defn

$\Rightarrow \lim\limits_{n\to\infty} (f+g)(x_n) = \lim\limits_{n\to\infty} \left[ f(x_n) + g(x_n) \right]$

$= \lim\limits_{n\to\infty} f(x_n) + \lim\limits_{n\to\infty} g(x_n)$

$$\boxed{\lim_{n\to\infty} (f+g)(x_n) = l+m}$$

Def

(2) $\lim\limits_{x\to c} (f-g)(x) = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)$

(3) $\lim\limits_{x\to c} (f\cdot g)(x) = \left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right]$

Th (4) $\lim\limits_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$ $\left[g(x) \neq 0 \text{ and } \lim g(x) \neq 0\right]$

$a <$

$\boxed{Ex\ 4.3}$

Q5(a) $\lim_{x \to 1^+} \frac{x}{x-1}$ $x \neq 1$

## Infinite limit

$\hookrightarrow \lim_{x \to c} f(x) = \infty$

**Definition** $\forall\ \alpha > 0$ (where $\alpha$ is any large real no.)

Then $\exists\ \delta > 0 :$ $f(x) > \alpha$ whenever $0 < |x - c| < \delta$.

$\boxed{Ex\ 4.3}$ Q5(a) $\lim_{x \to 1^+} \frac{x}{x-1}$ $x \neq 1$

Definition change acc to Question

Sol: To S $\rightarrow$ $\forall\ \alpha > 0$ $\exists\ \delta > 0 :$ $f(x) > \alpha$ if $1 < x < 1 + \delta$

Now $f(x) = \frac{x}{x-1} > \alpha$. Let $x > 1$ & $\alpha > 1$

If $x > \alpha(x-1) = x\alpha - \alpha$

$\Rightarrow \alpha > x\alpha - x = x(\alpha - 1)$

$\Rightarrow \frac{\alpha}{\alpha - 1} > x$

Now $x < \frac{\alpha}{\alpha - 1} = \frac{\alpha - 1 + 1}{\alpha - 1}$ $\boxed{\pm 1}$

$\therefore x < \frac{\alpha - 1 + 1}{\alpha - 1}$

$$\boxed{x < 1 + \frac{1}{\alpha - 1}}$$

$f(y_n) = 0 \ \forall n$ [$\because$ $y_n$ is irrational $\forall n$]

$\Rightarrow f(y_n) \longrightarrow 0$

Now, we have two seqn [$x_n$ & $y_n$] converging to same limit but their image seqn [$f(x_n)$ & $f(y_n)$] converges to diff. limit.

$\therefore \lim_{x \to c} f(x)$ doesn't exist. [when $c \neq 0$]

---

# Section – 4.2 (Bartle)

## $\boxed{\text{Algebra of limit}}$ and $\boxed{\text{Squeeze theorem}}$

Defn→ Bounded function on a neighbourhood of the point $x = c$

$f: A \longrightarrow \mathbb{R}$ $\quad A \subseteq \mathbb{R}$

$f$ is said to be bounded in some nbd of $c$ if $\exists$ $\delta$ nbd of $c$ ~~such that~~ & $M > 0$ such that $|f(x)| \le M \quad \forall x \in \boxed{A} \cap \boxed{V_\delta(c)}$ ← [$\delta$ nbd of $c$]

(If limit of function exist, then it is bounded)

**Theorem 4.2.1** If $f: A \to \mathbb{R}$ and $c$ is a cluster point of $A$, if $\lim_{x \to c} f(x) = l$

Then $f$ is bounded in some nbd of $c$

proof) T.S. $\exists V_\delta(c)$, $|f(x)| \le M$ $\quad \forall x \in A \cap V_\delta(c)$

$M > 0$

y is real
but when $x \in (-\frac{1}{2}, \frac{1}{2})$
y is imaginary

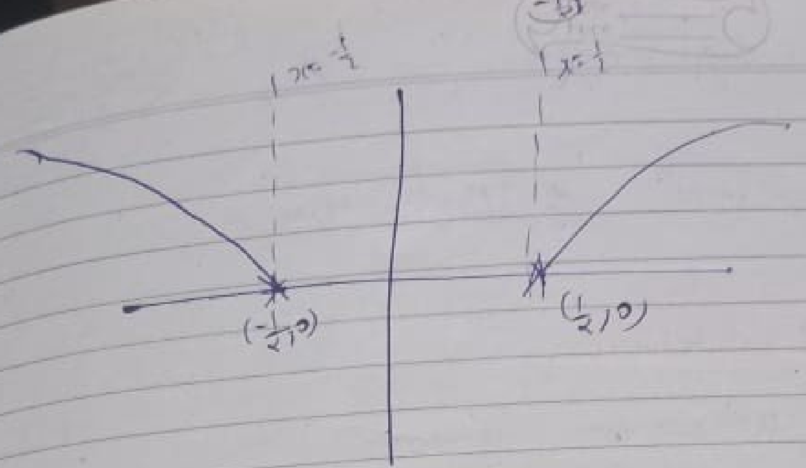
imaginary point

0), $(-\frac{1}{2}, 0)$

t]

upward]

-D




∴ ~~conca~~ both side concave down.

Vertical tangent at $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$

Critical point → $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, 0)$

and no point of Inflection

---

Q33 $y = 2x + 3x^{2/3}$

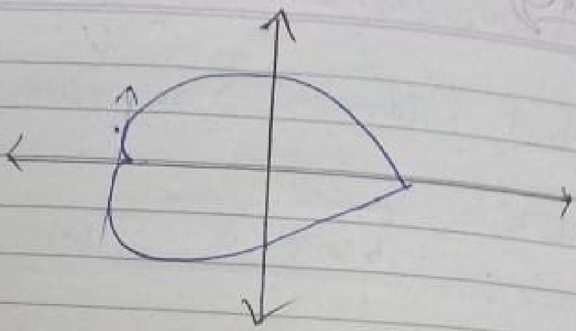
$\therefore y = x^{2/3}(2x^{1/3} + 3)$.

Sym → No.

→ (0,0) lies on curve ⟩
tangent at (0,0) is $x = 0$
(y-axis)

⇒ [put $y = 0$, $x = 0, -\frac{27}{8}$]

[put $x = 0$ ⇒ $y = 0$]

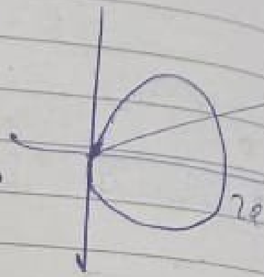


Q] $r = 2a\cos\theta$

$[0, \pi]$ Sym about polar axis

$r = 0$ $\quad\theta = \frac{\pi}{2}$

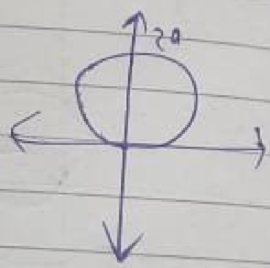
| | | | |
|---|---|---|---|
| $\theta$ | $0$ | $\pi/2$ | $\pi$ |
| $r$ | $2a$ | $0$ | $-2a$ |




Q] $r = 2a\sin\theta$

$[-\pi, \frac{\pi}{2}]$ . $r = 0 \Rightarrow \theta = 0$

| | | | |
|---|---|---|---|
| $\theta$ | $0$ | $\pi/2$ | $-\pi$ |
| $r$ | $0$ | $2a$ | $-2a$ |




Q39 $r^2 = 16\sin 2\theta$

Sym) $\theta \to \pi + \theta$, $\quad \sin 2\theta = \sin(2\pi + 2\theta) = \sin 2\theta$

Symm about origin/pole.

So 1st & 3rd Quad will have same shape.

put $r = 0$, $\sin 2\theta = 0$

$2\theta = 0$, $2\theta = \pi$, $2\theta = 2\pi$

$2\theta = 3\pi$, $2\theta = 4\pi$

| | | | |
|---|---|---|---|
| $\theta$ | $0$ | $\pi/4$ | $\pi/2$ |
| $r$ | $0$ | $4$ | $0$ |

for $\frac{\pi}{2} < \theta <$

for $3\frac{\pi}{2}$

Q $y = \frac{(x-2)^3}{x^2}$

Soln :) $y = \frac{x^3 - 8 - 6x^2 + 12x}{x^2}$

$y = x - 6 + \frac{12\cancel{x}}{x} - \frac{8}{x^2}$ — (1)

$\Rightarrow \cancel{y = x^2} \lim_{x \to \infty} y - (x-6) = 0$

$\Rightarrow \boxed{y = x - 6}$ ⟵ Oblique Asymptote

(1) No symmetry

(2) Not passing (0,0)

(3) x intercept ⟶ $\boxed{x = 2}$ [2,0]

y intercept ⟶ None

(4) Vert asy, $\boxed{x=0}$

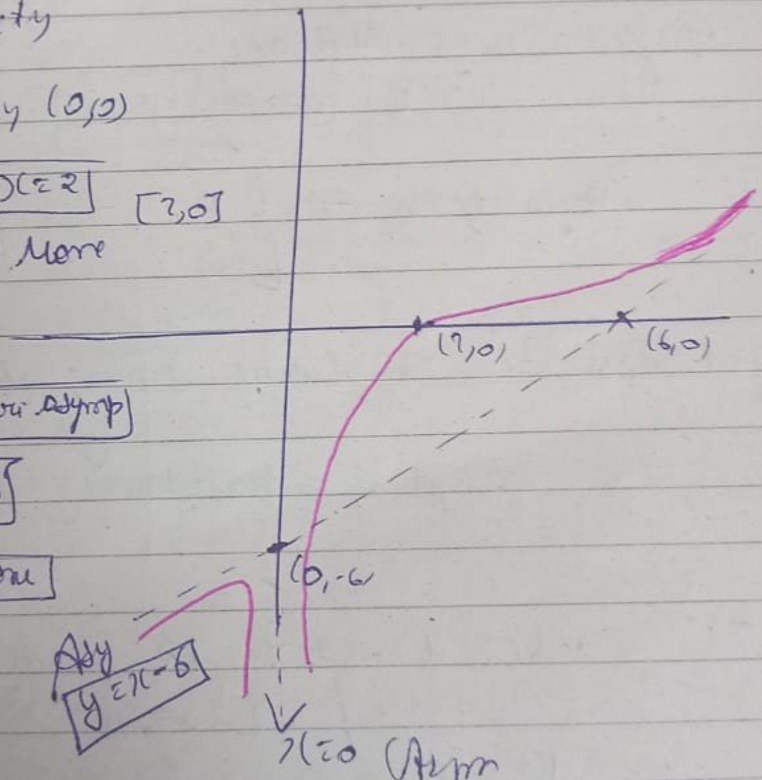
horizontal

↳ $\boxed{\text{No horiz asymp}}$

oblique, $\boxed{y = x-6}$

Curvilinear $\boxed{\text{None}}$

(5) sign analysis




$$\begin{array}{ccccccc} -ve & 0 & -ve & 2 & +ve & 6 & +ve \end{array}$$

(6) $\frac{dy}{dx} = 1 - \frac{12}{x^2} + \frac{8}{x^3} \times 2 = \frac{x^3 + 16 - 12x}{x^3}$

$= 8 + 16 - 12(2)$ $\quad -64 + 16 + 48$

$x^{-2}$ $\quad -2x^{-2-1}$ $\quad -2x^{-3}$ $\quad -\frac{2}{x^3}$

$\boxed{x = 2}$ ⟵ Stationary

$\boxed{x = -4}$ ⟵ point

# UNIT-I

$S = \{1,2,3\}$

Every open int. must contain point diff from $S$

**Cluster point :-** A point $c \in \mathbb{R}$ is said to be a cluster point of the set $S \subseteq \mathbb{R}$ if every nbd of $c$ contains a point of $S$ diff from $c$ i.e. $\forall x \neq c$ & $x \in (c-\delta, c+\delta) \subseteq S$

Eg. $N = \{1, 2, 3, \ldots\} \rightarrow$ No cluster pt.

$Q = \{$ All real no are cluster point $\}$

**Limits :-**

Let $x = c$ be a cluster point in $A$ then function may not be defined at $x = c$.

$$\boxed{\lim_{x \to c} f(x) = l}$$

We say a function $f$ has the limit $l$, if we approach $c$ from both sides of domain of fun $f$

$|f(x) - l| < \varepsilon$ or $|x - c| < \delta \cdot \forall x \neq c$

or $0 < |x - c| < \delta$

★ $\lim_{x \to c} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - c| < \delta$

**Exercise 4.1**

Q5 $I = (0, a)$, $a > 0$ $f(x) = x^2$, $x \in I$ Show that $\boxed{\lim_{x \to c} f(x) = x^2 = c^2}$

Solution :- We have,

$|f(x) - l| = |x^2 - c^2| = |x + c||x - c|$

But $x, c \in I$. $\therefore x, c < a$.

$\therefore |x + c||x - c| < 2a|x - c|$

$\therefore$ ~~by definition~~, $|f(x) - l| < \varepsilon \Rightarrow |f(x) - l| < 2a|x - c| < \varepsilon$

whenever $\Rightarrow |x - c| < \delta$ [If $\delta = \frac{\varepsilon}{2a}$]

$\Rightarrow |f(x) - l^2| < \varepsilon$ whenever $|x - c| < \delta$

$$\therefore \boxed{\lim_{x \to c} f(x) = c^2}$$

Q10 (a) Show that lim, use def. of limit to show that

(a) $\lim_{x \to 2} (x^2 + 4x) = 12$

Soln

To show that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - 12| < \varepsilon$ whenever $0 < |x - 2| < \delta$

Now $|f(x)-l| < \varepsilon$

$\Rightarrow -\varepsilon < f(x)-l < \varepsilon$, where $\boxed{0<|x-c|<\delta}$

Or $l-\varepsilon < f(x) < l+\varepsilon$ where $0<|x-c|<\delta$

Let $\varepsilon = l$

$\therefore 0 < f(x) < 2l$ where. $0<|x-c|<\delta$

$\Rightarrow f(x)$ is +ve as $l$ is +ve $\forall x \neq c$

$x \in V_\delta(c)$

Or ~~$\varepsilon = \frac{l}{2}$~~

$\therefore$ ~~$\frac{l}{2} < f(x) < \frac{3l}{2}$~~

---

Q9 Let $f, g$ be defined on $A$ to $\mathbb{R}$ and let $c$ be cluster point of $A$

(a) Show that if both $\lim_{x\to c} f$ and $\lim_{x\to c}(f+g)$ exist, then $\boxed{\lim_{x\to c} g}$ exist.

apply algebra of limit

Soln $\lim_{x\to c} f$ & $\lim_{x\to c}(f+g)$ exists, let they be

$l$ & $m$. & ~~$\lim_{x\to c} g = p$~~

Clearly $\lim_{x\to c} f(x) = l$ — (1)

$\lim_{x\to c}(f+g) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ [Algebra of lim]

$\therefore \lim_{x\to c} g(x) = \boxed{m-l}$ $\leftarrow$ which is again a constant let it be $p$

$\therefore \boxed{\lim_{x\to c} g(x) = p}$

Hence $\lim g\,[g(x)]$ also exists

$x^2-3x-2x-6$

$\therefore \boxed{\lim_{x\to 3} \frac{2x+3}{4x-3} = 3}$ Hence, proved

Q11 (b) $\lim_{x\to 6} \frac{x^2-3x}{x+3} = 2$

Sol: $|f(x)-l| = \left|\frac{x^2-3x}{x+3} - 2\right| = \left|\frac{x^2-3x-2x-6}{x+3}\right|$

$= \left|\frac{x^2-5x-6}{x+3}\right| = \left|\frac{(x+1)(x-6)}{x+3}\right| = |x-6|\left|\frac{x+1}{x+3}\right|$

$=$ Clearly $\boxed{\left|\frac{x+1}{x+3}\right| < 1}$

$\Rightarrow |x-6|\left|\frac{x+1}{x+3}\right| < |x-6| < \varepsilon$

$\therefore |x-6| < \varepsilon$

Choosing $\delta = \min\{1, \varepsilon\}$

$\Rightarrow \boxed{|x-6| < \delta}$

$\Rightarrow |f(x)-2| < \varepsilon$ whenever $|x-6| < \delta$

Hence, $\lim_{x\to 6} f(x) = 2$

$\Rightarrow \boxed{\lim_{x\to 6}\left(\frac{x^2-3x}{x+3}\right) = 2}$

Hence, proved

— x — x —

(C) $r^2 = a^2 \cos 2\theta$

Soln $\boxed{\text{Step 1}}$ Symmetry

$\theta \to -\theta$ $\therefore$ eq remain unchanged

$\therefore$ $\boxed{\text{Sym about polar axis}}$.

$(\theta \to \pi + \theta)$ $r^2 = a^2 \cos(2\pi + 2\theta) \to \boxed{r^2 = a^2 \cos 2\theta}$

remain unchanged

$\hookrightarrow$ $\boxed{\text{Symm about pol}}$.

$(\theta \to \pi - \theta)$ $r^2 = a^2 \cos 2(\pi - \theta) = a^2 \cos(2\pi - 2\theta)$

$= a^2 \cos 2\theta$

eq remain unchanged

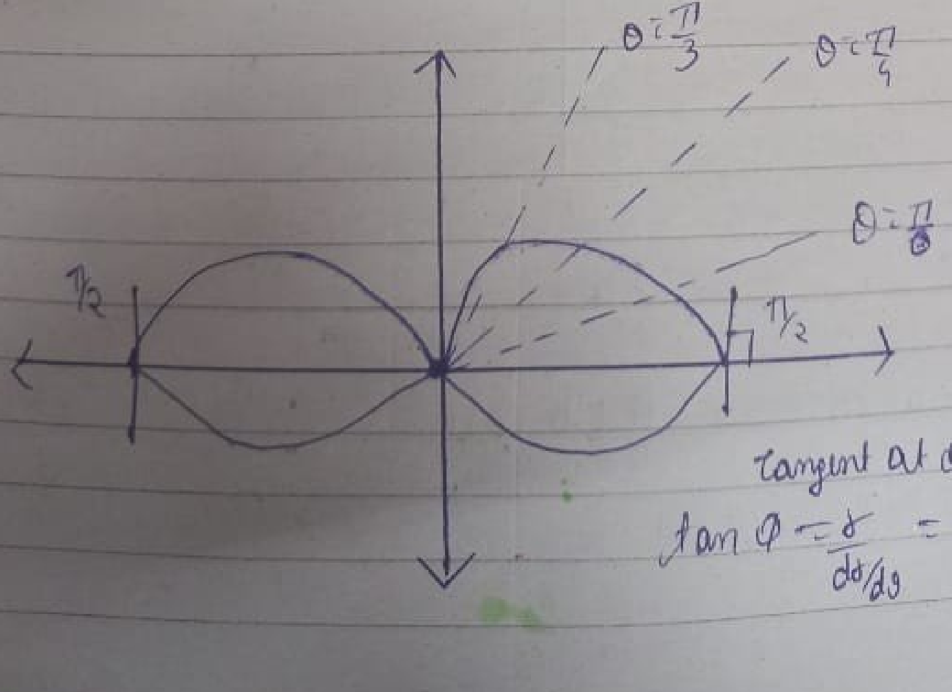
$\boxed{\text{Symm about y axis}}$.

$\boxed{\text{Step 2}}$ put $r = 0$

$\Rightarrow 0 = a^2 \cos 2\theta$ or $\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

Now

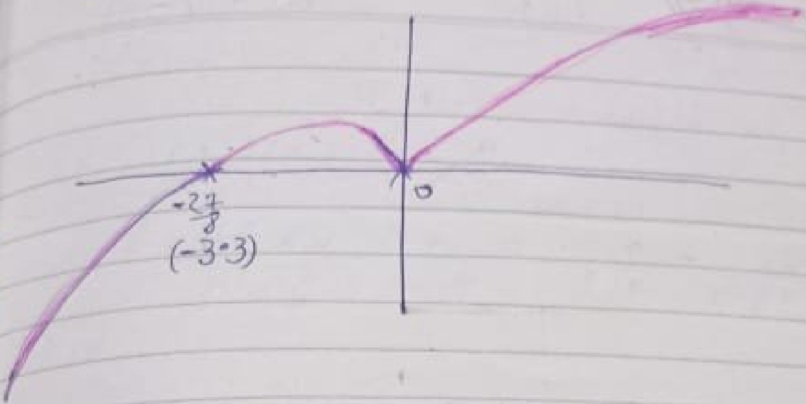
| $\theta$ | 0 | 30 | 45 | 60 | 90 |
|---|---|---|---|---|---|
| $r$ | 0 | $\frac{a}{\sqrt{2}}$ | 0 | Imag. $(\sqrt{-ve})$ | Imagn $(\sqrt{-ve})$ |




Tangent at other point

$\tan\phi = \frac{r}{dr/d\theta} = \frac{a^2 \cos 2\theta}{2a^2 \sin 2\theta}$

$x^{-\frac{1}{3}-1}$




$$\frac{d^2y}{dx^2} = 2\left(\frac{-1}{3}\right)x^{-4/3} = \frac{-2}{3x^{4/3}} \leftarrow \text{always +ve}$$

↓

always concave down

Hence Answers → Inflection point – none

Critical point (-1, 1) and (0, 0)

---

Q35) $y = 4x^{1/3} - x^{4/3}$ ⇒ $y = x^{1/3}[4 - x]$

Solution] No symmetry

(2) → (0, 0) lies, Tangent = $\boxed{x = 0}$ [y axis]

(3) Intercepts

$x = 0$, $\boxed{y = 0}$ (0, 0)

$y = 0$ (x = 0), (x = 4) [4, 0]

(4) Sign analysis →

[-ve] 0 [+ve] 4 [-ve]

(5) End behaviour →

as $x \to +\infty$, $y \to -\infty$

$x \to -\infty$, $y \to -\infty$

[Numerator]

denom]

★ No to discuss Asymptote in these type of Question

Q31) $y = (4x^2-1)^{1/2}$ — ① Method

Sol) $\boxed{y^2 = 4x^2 - 1}$

$y = (4x^2-1)^{1/2}$
↓
as $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$
y become Imaginary
∴ $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$
Can't possible

① Sym. about both axis
② Not passes.
③ x-intersection. $\boxed{x = \pm \frac{1}{2}}$

y-intersection → None

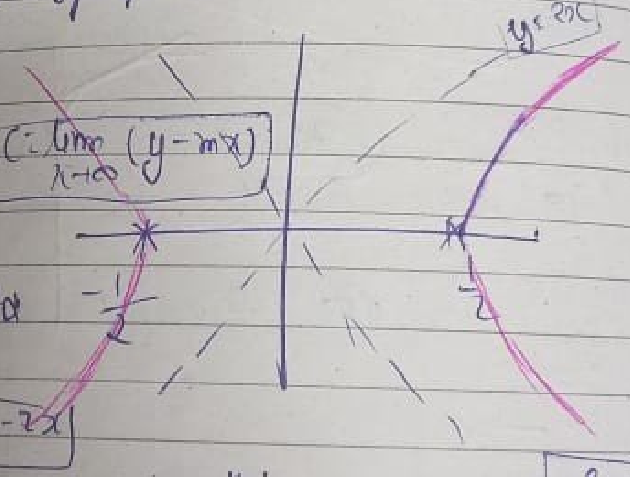
④ ~~x~~ Oblique asymptote.

$\boxed{m = \lim_{x \to \infty} \left(\frac{y}{x}\right)}$

$\boxed{c = \lim_{x \to \infty} (y - mx)}$

$= \lim_{x \to \infty}$ ~~...~~

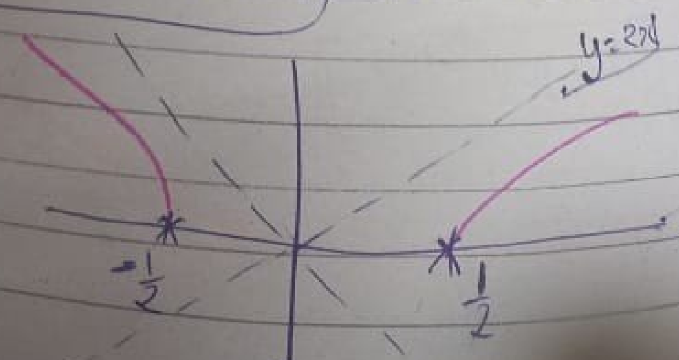
⇒ Oblique asymptote
↓
$\boxed{y = 2x}$, $\boxed{y = -2x}$




[Process of obtaining this is written on next page]

$\boxed{y^2 = 4x^2 - 1}$ ① for

for $y = (4x^2-1)^{1/2}$ ← as This will be always +ve

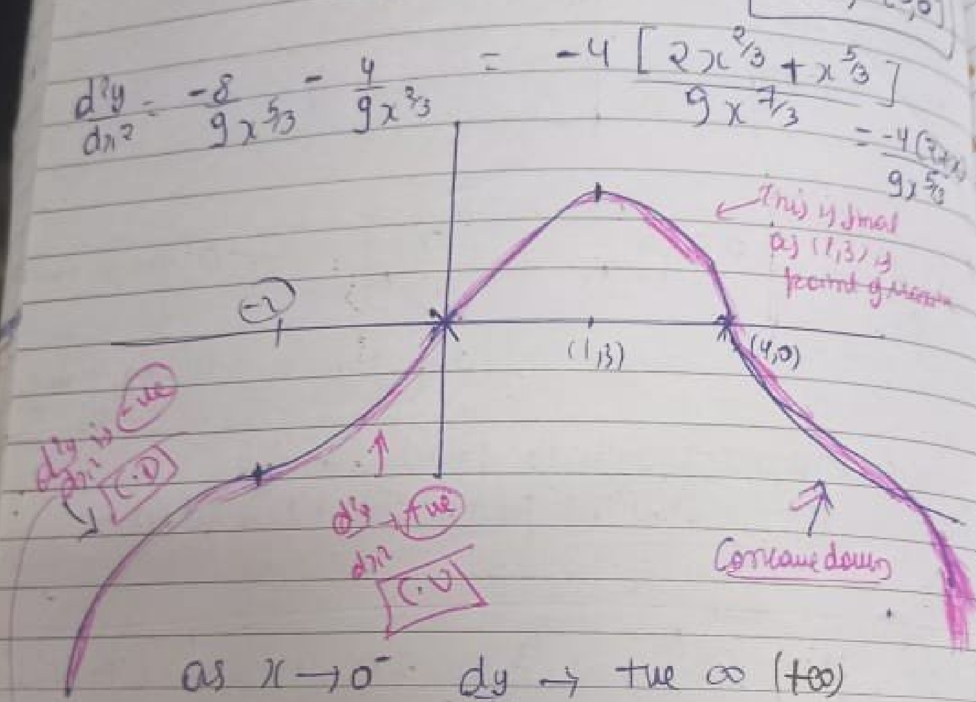


$\frac{4}{3}x^{\frac{1}{3}}$ $\frac{1}{3}-1$ $\frac{4}{3}(\frac{1}{3})x^{-\frac{2}{3}}$

$\frac{dy}{dx} = \frac{4}{3x^{2/3}} - \frac{4}{3}x^{1/3} = \frac{4(1-x)}{3(x^{2/3})}$

$\therefore$ Critical points, $x = 1, 0$
$\therefore$ $y = 3, 0$ $\Rightarrow$ $\boxed{[1,3], [0,0]}$

$\frac{d^2y}{dx^2} = \frac{-8}{9x^{5/3}} - \frac{4}{9x^{2/3}} = \frac{-4[2x^{2/3} + x^{5/3}]}{9x^{7/3}} = \frac{-4(2+x)}{9x^{5/3}}$




as $x \to 0^-$ $\frac{dy}{dx} \to$ +ve $\infty$ $(+\infty)$

as $x \to 0^+$ $\frac{dy}{dx} \to$ +ve $\infty$ $(+\infty)$

as $x \to -2^{-}$ ; $\frac{d^2y}{dx^2} \to \boxed{-ve}$

$x < -2$

$\boxed{x+2<0}$

Also, $\frac{d^2y}{dx^2} = \frac{-4(2+x)}{9x^{5/3}}$

as $x \to 4^+$ $\boxed{\frac{d^2y}{dx^2} \to -ve}$ $\leftarrow$ Concave down after (4,0)

$\frac{4}{3}(-\frac{2}{3})x^{-\frac{5}{3}}$

Hence, Critical inflect

polar coordinate

$|\vec{OP}| = r$

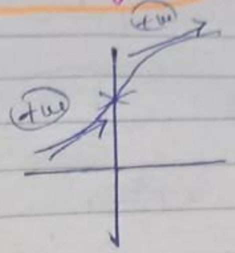
Relationship

# Graphs with vertical tangents

Case I

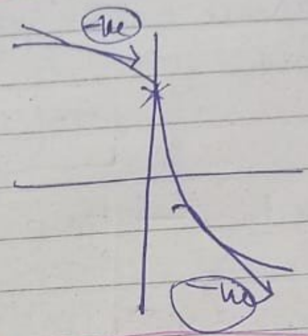
$\lim_{x \to 0^-} f(x) = +\infty$

$\lim_{x \to 0^+} f(x) = +\infty$




$\boxed{\text{Tangent } + \ +}$

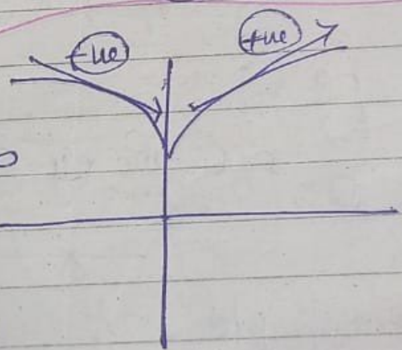
Case II




$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = -\infty$

$\boxed{\text{Tangent } - \ -}$



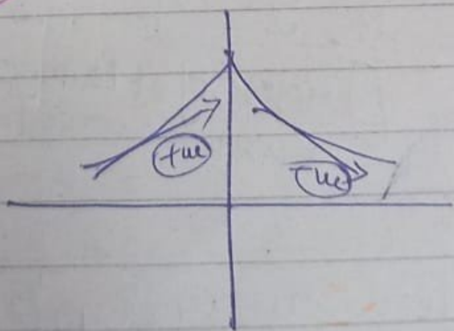

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = +\infty$

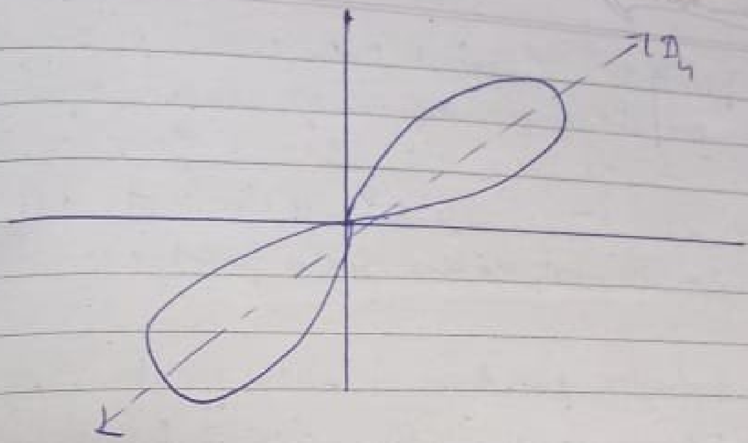
$\boxed{\text{Tangent } - \ +}$

$\lim_{x \to 0^-} f(x) = +\infty$

$\lim_{x \to 0^+} f(x) = -\infty$




$\boxed{\text{Tangent } + \ -}$




for $\frac{\pi}{2} < \theta < \pi \Rightarrow \pi < 2\theta < 2\pi$

$\Rightarrow r^2 < 0$ Imaginary.

for $\frac{3\pi}{2} < \theta < 2\pi \Rightarrow 3\pi < 2\theta < 4\pi$

$\Rightarrow r^2 < 0$ Imagnory

---X---X---

(9) $\lim_{x\to 2} \frac{1}{1-x} = -1$

Now $|f(x) - l| = \left|\frac{1}{1-x} + 1\right| = \left(\frac{1+1-x}{1-x}\right) = \left|\frac{2-x}{1-x}\right| = \frac{|x-2|}{|x-1|}$

Let $|x-2| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-2 < \frac{1}{2}$ or $\frac{3}{2} < x < \frac{5}{2}$

or $\frac{1}{2} < x-1 < \frac{3}{2}$

$\Rightarrow \boxed{\frac{1}{x-1} < 2}$

$\therefore \frac{|x-2|}{|x-1|} < 2|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{2}$

Choosing $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$ we have

$|f(x) - l| < \varepsilon \quad \forall \quad 0 < |x-2| < \delta \quad \Rightarrow \boxed{\lim_{x\to 2} f(x) = -1}$

Q11 (a) $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$

Soln $\left|\frac{2x+3}{4x-9} - 3\right| < \varepsilon$

$\left|\frac{2x+3-12x+27}{4x-9}\right| < \varepsilon \Rightarrow \left|\frac{-10x+30}{4x-9}\right| < \varepsilon$

$\frac{10|x-3|}{|4x-9|}$ let $|x-3| < 1$ or $2 < x < 4$

$8 < 4x < 16$

$-1 < 4x-9 < 7$

$\frac{1}{4x-9} > -1$ } will not work

$\Rightarrow \boxed{\frac{1}{4x-9} < 1}$

$10|x-3| < \varepsilon$

$\boxed{|x-3| < \frac{\varepsilon}{10}}$

Choosing $\delta = \min\left\{\frac{1}{1}, \frac{\varepsilon}{10}\right\}$

$\therefore \boxed{|x-3| < \delta}$ So $\lim_{x\to 3}$

Step 3 Making table $\theta$ vs $r$

| $\theta$ | | | | |
|---|---|---|---|---|
| $r$ | | | | |

—X—X—X—

Example. $r = a(1+\cos\theta) = f(\theta)$

This curve is known as cardioid.

Step 1 · Curve is symm about polar axis. As on changing $\theta \to -\theta$, eq remains unchanged.

∴ now we trace graph from $[0, \pi]$, the rest of graph will be traced by symmetry.

Step 2 put $r=0$.

$\Rightarrow 1+\cos\theta = 0$

$\cos\theta = -1$

$\Rightarrow \boxed{\theta = \pi}$

$a\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)$

$a\left(\frac{\sqrt{3}}{2}+1\right)$ $a(2.14)$

$a\left(\frac{2+\sqrt{3}}{2}\right)$ 1.8

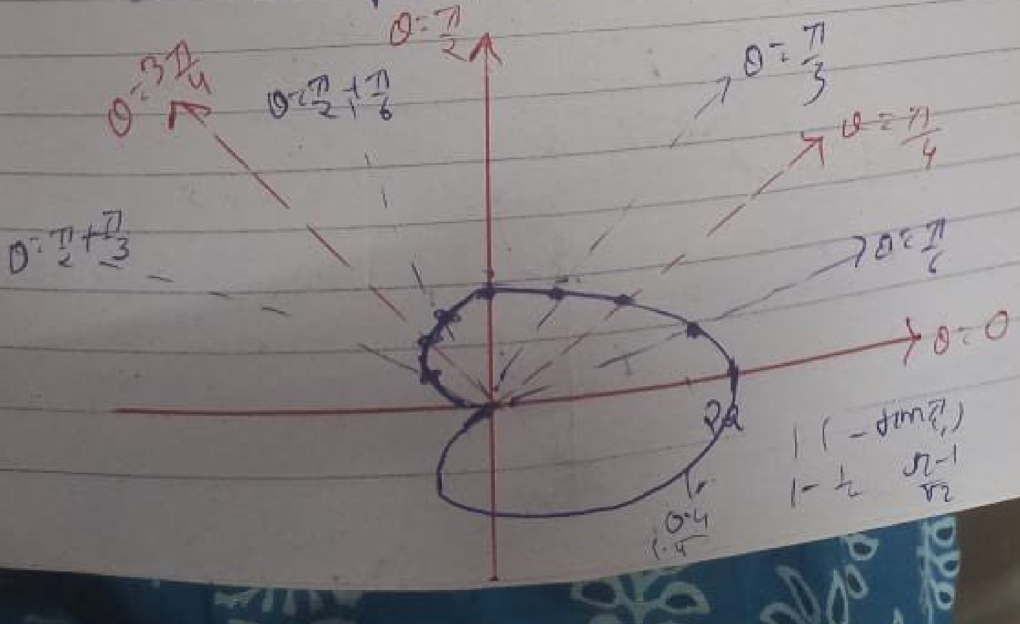
1.7

$\frac{3.7}{2}$

1.8

Step 3

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | |
|---|---|---|---|---|---|---|
| $r$ | $2a$ | $1.87a$ | $1.7a$ | $1.5a$ | $a$ | |




$1\left(-\sin\frac{\pi}{4}\right)$

$1-\frac{1}{2}$ $\frac{\sqrt{2}-1}{\sqrt{2}}$

$\frac{0.4}{1.4}$

Sign analysis

$-ve \quad -\frac{27}{8} \quad +ve \quad 0 \quad +ve$

$\frac{2}{3}-1$

$-\frac{1}{3}$ $-3$

$(-4)$ $-8+3$

$(-8)$

$((-2)^4)(2(-2)$

$-4+1$

$(-1)$

$1(-1+3)$

End behaviour

as $x \to +\infty$, $y \to \infty$

$x \to -\infty$, $y \to -\infty$

Now

$\frac{dy}{dx} = 2 + 3 \times \frac{2}{3} x^{-\frac{1}{3}} = 2 + \frac{2}{x^{1/3}}$

$\frac{dy}{dx} \Rightarrow \frac{2[x^{1/3}+1]}{x^{1/3}}$

Critical pts, $x^{1/3}+1=0$ [Numerator]

$x^{1/3}=-1$

$\boxed{x=-1}$

For $x^{1/3}=0$ [denom]

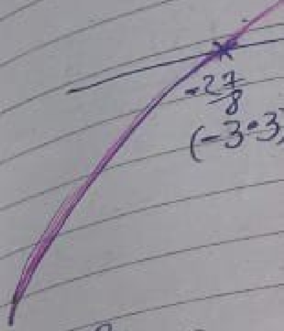
$\Rightarrow \boxed{x=0}$

when $x=-1$, $y=1$

$x=0$, $y=0$

as $x \to 0^-$ $\frac{dy}{dx} \to -\infty$

$x \to 0^+$ $\frac{dy}{dx} \to +\infty$



$-\frac{27}{8}$

$(-3\cdot3$

$\frac{d^2y}{dx^2} =$

Hence An

Q35 y

Solution

(2)

(3) Inter

$x=$

$y=$

(4)

(5)

End b

$\frac{4}{3}\left(\frac{1}{3}\right)x^{-\frac{7}{3}}$

$= \frac{4(1-x)}{3(x^{2/3})}$

$[(1,3)], [(0,0)]$

$\frac{2}{3} + x^{\frac{5}{3}}$

$= \frac{-4(2+x)}{9x^{5/3}}$

This is final as (1,3) is point of maxima

(4,0)

concave down

$(-\infty)$

$(+\infty)$

concave down (4,0)

$\frac{4}{3}\left(-\frac{2}{3}\right)x^{\frac{5}{3}}$ $\quad -\frac{2}{3}-1$ $\quad \frac{4}{3}\left(\frac{1}{3}\right)x^{\frac{2}{3}}$ $\quad x^{\frac{3}{3}}[2+x]$

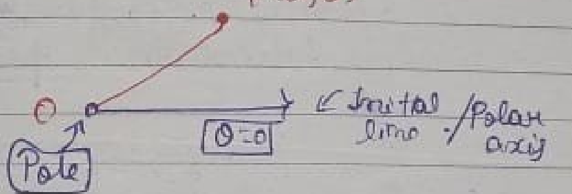
up

Hence, Critical point $[(0,0), (1,3)]$

inflection point $[(0,0)$ & $(-2, -6(2^{1/3}))]$

# POLAR CURVE

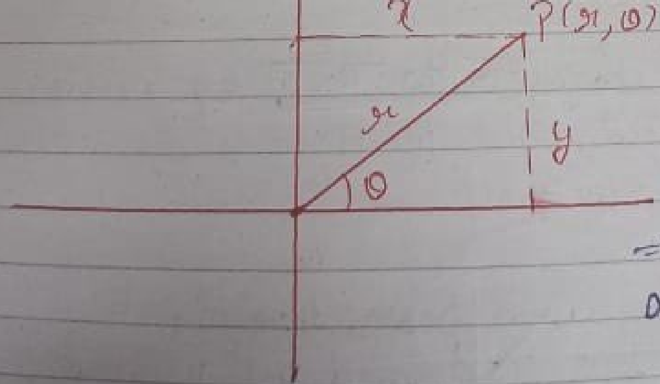
polar coordinate




$|\vec{OP}| = r$ and $\theta$ = angle OP makes with polar axis

☆ $\boxed{\text{Clockwise} = -ve}$ $\boxed{\text{anti-clockwise} = +ve}$

Relationship B/w Polar Coordinate with Rectangular coord.




$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$

and $\tan\theta = \frac{y}{x}$

for $\theta = 0$ $\quad \tan\phi = \infty$

$\theta = \pi$ $\quad \tan\phi = \infty$ $\qquad \phi = \frac{\pi}{2}$

Q $r = a(1 + \sin\theta)$

Solution ① Symmetry. $\theta \to \pi - \theta$

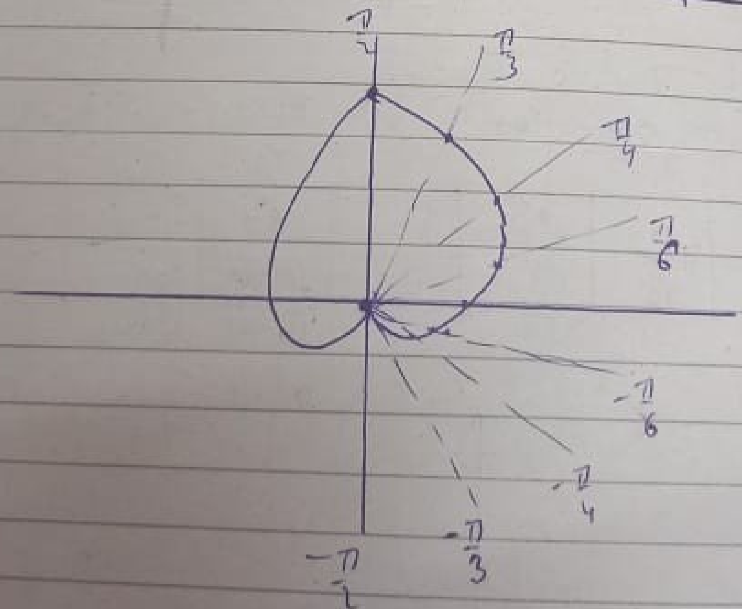
↳ eq remains unchanged

∴ curve sym w.r.t. y axis

So tracing it $[-\frac{\pi}{2}, \frac{\pi}{2}]$ & rest will be obtained by symmetry.

② pole put $r = 0 \Rightarrow \sin\theta = -1 \Rightarrow \boxed{\theta = -\frac{\pi}{2}}$

| $\theta$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{3}$ | $-\frac{\pi}{4}$ | $-\frac{\pi}{6}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | 0 | 0.15a | 0.37a | 0.5a | a | 1.5a | 1.7a | 1.85a | 2a |




b) Limacon:- $r = a + b\cos\theta$ $\qquad$ $\boxed{\text{If } a = b, \text{ cardioids}}$

$\Rightarrow a, b > 0$

| θ | $\frac{\pi}{2}+\frac{\pi}{6}$ | $\frac{\pi}{2}+\frac{\pi}{4}$ | $\frac{\pi}{2}+\frac{\pi}{3}$ | $\pi$ |
|---|---|---|---|---|
| r | 0.5a | 0.3a | 0.13a | 0 |

(II) $r = f(\theta) = a(1-\cos\theta)$

Step 1 (1) Symmetry about polar axis ($\therefore$ we need to draw from $[0,\pi]$ as $[\pi, 2\pi]$ is symmetric)

(2) Step 2 . put $r=0$

$0 = a(1-\cos\theta)$

$1-\cos\theta = 0$

$\cos\theta = 1$

$\Rightarrow \boxed{\theta = 0}$

(3) Step 3

| θ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| r | 0 | 0.15a | 0.29a | 0.5a | a |

$\left(1-\frac{\sqrt{3}}{2}\right)a$

$\left(\frac{2-\sqrt{3}}{2}\right)a$

$2-1.7$

$\frac{0.3}{2}$

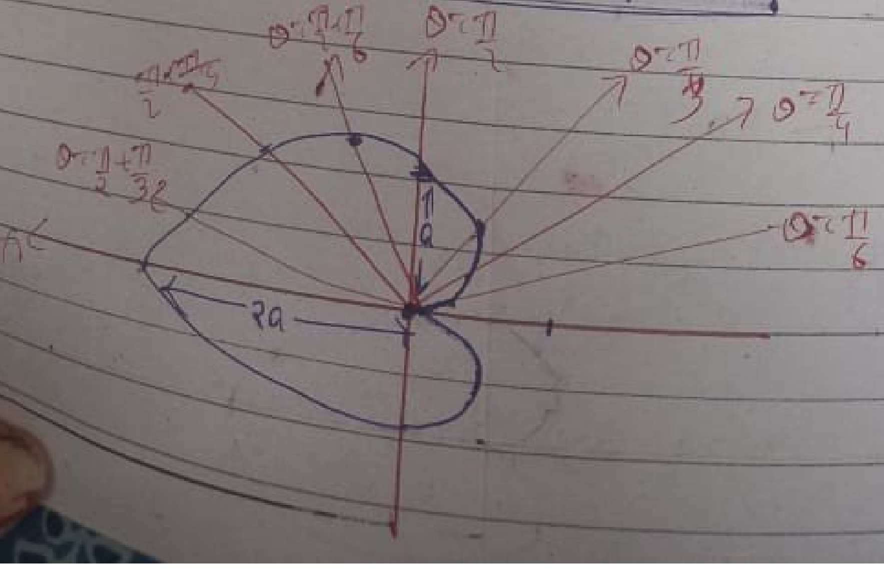
$0.15$

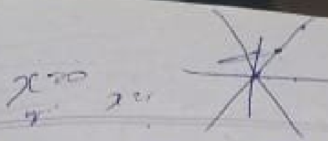
| θ | $\frac{\pi}{2}+\frac{\pi}{6}$ | $\frac{\pi}{2}+\frac{\pi}{4}$ | $\frac{\pi}{2}+\frac{\pi}{3}$ | $\pi$ |
|---|---|---|---|---|
| r | 1.5a | 1.7a | 1.87a | 2a |

$\left(1-\frac{1}{\sqrt{2}}\right)a$

$\frac{1.41-1}{\sqrt{2}}$

$\boxed{\frac{0.41}{1.41}}$






Oblique asymptote. $y^2 = 4x^2 - 1$

$\cancel{f(x,y) = 4x^2 - y^2 - 1}$

$f_x = 8x$

$f_y = -2y$

$\phi_2(x,y) = 4x^2 - y^2$

$\phi_1(x,y) = 0$

$\phi_0(x,y) = -1$

Now put $x = 1$, $y = m$

$\phi_2 m = 4 - m^2$

$\phi_1 m = 0$

$\phi_0 m = -1$

$\therefore \boxed{m = \pm 2}$

Now $c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)} = \frac{-(0)}{-2m} = 0$

∴ Oblique asymptote are

$y = mx + c$

$\Rightarrow \boxed{y = 2x}$ $[m = +2, c = 0]$

$\Rightarrow \boxed{y = -2x}$ $[m = -2, c = 0]$

Second Method todo $\boxed{y = \sqrt{(4x^2 - 1)}}$

Sol, ① Sym about y-axis

② ~~pass~~ (0,0) doesn't lie

③ x intercept $x = \pm\frac{1}{2}$ $(-\frac{1}{2}, 0), (\frac{1}{2}, 0)$

y intercept y is imaginary

④ Sign analysis

Now, $\frac{d^2y}{dx^2} = \frac{12\times 2}{x^3} - \frac{16\times 3}{x^4}$

$= \frac{24x - 16\times 3}{x^4}$

$\boxed{x>2}$

$24 - 16$

$24\times 3 - 16(3)$

$72 - 48$

$x^4$

(+ve)

Hence, Stationary point $\left[-4, -\frac{27}{2}\right], [2, 0]$

Inflection point $(2, 0)$

Asy $\boxed{x=0}$ ~~y=x+5~~ $\boxed{y=x-6}$

Asym. Crossing → $\left[\frac{2}{3}, \frac{-16}{3}\right]$

Asym. Crossing → The curve crosses oblique asym $y = x-6$.

∴ pt of intersection $y = x-6$ with curve (1)

$\Rightarrow (x-6) = (x-6) + \frac{12}{x} - \frac{8}{x^2}$

$\Rightarrow \frac{12}{x} - \frac{8}{x^2} = 0 \Rightarrow \boxed{x = \frac{2}{3}}$

$\Rightarrow y = x - 6 = \frac{2}{3} - 6 = \frac{-16}{3}$

∴ Asym. Crossing → $\left[\frac{2}{3}, \frac{-16}{3}\right]$

# Graphs w

$\lim_{x\to 0^-} f(x) = +$

$\lim_{x\to 0^+} f(x) = +$

Tangent +

$\lim_{x\to 0^-} f(x) =$

$\lim_{x\to 0^+} f(x)$

Tangent

$\lim_{x\to 0^-} f(x)$

$\lim_{x\to 0^+} f(x)$

Tangent –

$\lim_{x\to 0^-} f$

$\lim_{x\to 0} f$

Ca

$x < \frac{1}{2}$ $\qquad$ $x < -\frac{1}{2}$

$\boxed{2x-1<0}$ $\qquad$ $\boxed{2x+1<0}$

$(-)\cdot(-) > 0$

$\Rightarrow (2x-1)(2x+1) > 0$

$\Rightarrow \boxed{4x^2-1 > 0}$ $\therefore$ y is real.

but when $x \in (-\frac{1}{2}, \frac{1}{2})$

y is imaginary

$-\frac{1}{2}$ → imaginary ← $\frac{1}{2}$

(6) End behaviour

as $x \to +\infty$ $\quad y \to \infty$

$x \to -\infty$ $\quad y \to \infty$

(7) $\frac{dy}{dx} = \frac{4x}{\sqrt{4x^2-1}}$

for critical point, $\quad 4x = 0$

$\Rightarrow \boxed{x=0}$

then y is imaginary

$\Rightarrow$ This is imaginary point

for this we take critical point of

$\sqrt{4x^2-1} = 0$ $\quad \Rightarrow \boxed{x = \pm\frac{1}{2}}$

$\therefore$ Critical points → $(\frac{1}{2}, 0), (-\frac{1}{2}, 0)$

(8) $x \to -\frac{1}{2}$, $\frac{dy}{dx} \to -\infty$ → [downward]

$x \to +\frac{1}{2}$ $\quad \frac{dy}{dx} \to +\infty$ → [upward]

$\therefore$ Conc…

bot…

Vertical tange…

Critical poi…

and no po…

Q33 $\quad y =$

$y =$

Sym → …

→ (99 …

⇒ [put

[ pu…

Doen Case I when $a < b \Rightarrow \frac{a}{b} < 1$

① Symm about polar axis. → Tracing curve in $[0, \pi]$

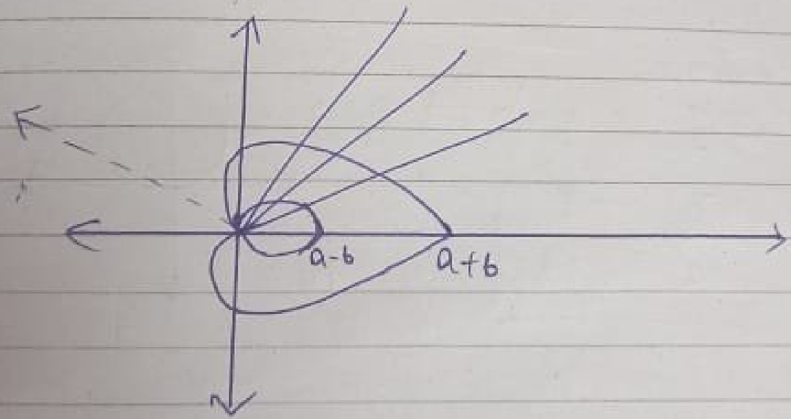
pole, put $r = 0$

$\Rightarrow a + b\cos\theta = 0 \Rightarrow \cos\theta = -\frac{a}{b}$

$\boxed{\frac{a}{b} < 1}$ $\boxed{\theta = \pi - \cos^{-1}\left(\frac{a}{b}\right)}$

$\boxed{\theta = \pi - \alpha}$

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\pi/3$ | $\pi/2$ | $\frac{\pi}{2}+\frac{\pi}{6}$ | $\frac{\pi}{2}+\frac{\pi}{4}$ | $\pi/2 + \pi/3$ | $\pi - \alpha$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $r$ | $a+b$ | $a + 0.86b$ | $a + 0.7b$ | $a + 0.5b$ | $a$ | $a - 0.5b$ | $a - 0.7b$ | $a - 0.8b$ | 0 | $a-b$ |




Case II when $a > b \Rightarrow \frac{a}{b} > 1$

① Sym → about polar axis, tracing it $[0, \pi]$

② pole. $\boxed{r = 0} \Rightarrow a + b\cos\theta = 0$

$\theta = \cos^{-1}\left(-\frac{a}{b}\right)$

$\therefore \frac{a}{b} > 0$ so $r \neq 0$ at any angle

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}+\frac{\pi}{6}$ | $\frac{\pi}{2}+\frac{\pi}{4}$ | $\frac{\pi}{2}+\frac{\pi}{3}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | $a+b$ | decrease → | | | $a$ | | | | $a-b$ |

$$\therefore \boxed{\lim_{x \to c} f(x) = l}$$

## Divergent criteria

① $\lim_{x \to c} f(x) \neq l$ if $\exists$ a seqn $(x_n)$ in $A$ · $x_n \to c$ but $f(x_n) \not\to l$

② $\exists$ two seqn $(x_n)$ & $(y_n)$ in $A$ : $\left[\begin{matrix}(x_n) \to c \\ (y_n) \to c\end{matrix}\right]$

but $f(x_n)$ and $f(y_n)$ will converge to different limit $\Leftrightarrow$ $f(x_n)$ is not conv. seqn or $\lim_{x \to c} f(x)$ doesn't exist.

Example ① $f(x) = \frac{1}{x}$

$\lim_{x \to 0} f(x) \to$ This doesn't exist.

let $x_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$.

then $(x_n) \to 0$ $[c = 0]$ $\left\{ \begin{matrix} \text{Clearly } \boxed{x_n \to c} \\ x_n = \frac{1}{n},\ c = 0 \end{matrix} \right\}$

but $f(x_n) = \frac{1}{x_n} = n$

$\therefore \lim_{x_n \to \infty} \to f(x_n) \to \infty$ $\left\{ \begin{matrix} \text{but } f(x_n) \not\to l \\ \therefore l = \infty \text{ here} \end{matrix} \right\}$

Ex(4.1)

Q12 Show that the follows limit do not exist.

① (a) $f(x) = \frac{1}{x^2}$

Sol: let $x_n = \frac{1}{n}$

Clearly $\boxed{x_n}$

Now $f(x_n)$

$\therefore$

(b) $\lim_{x \to 0} \frac{1}{\sqrt{x}}$

Here also we

(c) $\lim_{x \to 0} f(x +$

$f(x) =$

(c) $f(x$

let $x$

$f(x_n) =$

$\therefore \boxed{x_n}$

but $f$

to dif

$\therefore$ by div

from (1) & (2)

Choose $\delta = \varepsilon$, and $x = x_n$ in eq (1)

We have, $0 < |x_n - c| < \delta = \varepsilon$ or $\boxed{0 < |x_n - c| < \varepsilon}$

$\Rightarrow |f(x_n) - l| < \varepsilon \quad \forall\, n \geq k$

$\therefore f(x_n) \to l$

$\boxed{\text{Conversely}}$, To show → If every seq $(x_n)$ which converge to $c$, then $[f(x_n)] \longrightarrow l$

Then, we should have $\lim\limits_{x \to c} f(x) = l$

Let us assume that $\lim\limits_{x \to c} f(x) \neq l$

but every seq $(x_n) \longrightarrow c$

$\Rightarrow [f(x_n)] \longrightarrow l$

$\lim\limits_{x \to c} f(x) = l \Rightarrow \forall\, \varepsilon > 0,\ \exists\, \delta > 0$

$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$

$\because$ We are assuming that $\lim\limits_{x \to c} f(x) \neq l$.

$\therefore$ Whatever $\delta > 0$ is given we can find at least one $\varepsilon = \varepsilon_0$. $0 < |x - c| < \delta$

but $|f(x) - l| \geq \varepsilon_0$. ——— (★)

Let $\delta = \dfrac{1}{n}$ and getting a seqn $x_n \in \left[c - \dfrac{1}{n},\ c + \dfrac{1}{n}\right]$

these both seqn tends to $c$.

then $c - \dfrac{1}{n} < x_n < c + \dfrac{1}{n} \quad \forall\, n \in \mathbb{N}$

$\boxed{\text{By squeeze Theorem}}$

Clearly $x_n \to c$ but by ★ $|f(x_n) - l| \geq \varepsilon_0$

$\Rightarrow f(x_n) \not\to l$.

Which is a contradiction.

$\therefore$ Our assumption that $\lim\limits_{x \to c} f(x) \neq l$ is wrong

# Sequential criterion for getting limit of function

for the function, $f : A \to \mathbb{R}$ to have limit at a point $x = c$ [where $c$ is Cluster point]

$[x_n \to c,\ f(x_n) \to l]$

then, $\lim_{x \to c} f(x) = l$

iff for every sequence $(x_n)$ in A converging to $c$ then $f(x_n)$ converge to $l$.

$\boxed{\text{proof}}$ let $\lim_{x \to c} f(x) = l$ [given]

$\boxed{\text{to show}}$

~~$\because x_n \to c$~~

~~$\therefore$ for every $\varepsilon > 0$ $\exists\, k \in \mathbb{N}$~~ ~~$|x_n - c| < \varepsilon \ \forall\ n \geq k$~~ — (1)

~~$f(x_n) \to l$~~

~~$\hookrightarrow$ for every $\varepsilon > 0, \exists\, k \in \mathbb{N}$~~

~~$|f(x_n) - l| < \varepsilon \quad \forall\, n \geq k$~~

~~since $\lim_{x \to c} f(x) = l$~~

~~$\therefore \forall\, \varepsilon > 0\ \exists$~~

$\because x_n \to c$, $\forall\, \varepsilon > 0\ \exists\, k \in \mathbb{N}$ such that

$|x_n - c| < \varepsilon \quad \forall\, n \geq k$ — (1)

also $f(x) \to l$ [given] $\therefore |f(x) - l| < \varepsilon\ \forall\, \varepsilon > 0, \exists\, \delta > 0$

if $0 < |x - c| < \delta$ — (2)

Based on Squeeze Th

$\boxed{Ex-4.2}$

**Q8** Let $n \in \mathbb{N}$ be such that $n \geq 3$. derive the inequality $-x^2 \leq x^n \leq x^2$ for $-1 < x < 1$. Then use the fact that $\lim_{x\to 0} x^2 = 0$ to show that $\boxed{\lim_{x\to 0} x^n = 0}$

**Solution**

Given :- $|x| < 1$,

$\Rightarrow |x|^k < 1$ or $|x^k| < 1$

$\therefore -1 < x^k < 1$

$-x^2 < x^{k+2} < x^2$ $\boxed{k \geq 1}$

$\therefore -x^2 < x^n < x^2$ $\forall \boxed{n \geq 3}$

$$\underset{\downarrow\, f(x)}{-x^2} < \underset{\downarrow\, g(x)}{x^n} < \underset{\downarrow\, h(x)}{x^2}$$

$\therefore \lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x)$

$\therefore$ By Squeeze Theorem:-

$\boxed{\lim_{x\to 0} g(x) = 0}$

(when lim is +ve, then function is +ve around that point)

☆ **Theorem:-** [4.2.9] If $\lim_{x\to c} f(x) = l$ and if $l > 0$

Then $\exists$ a $\delta$-nbd of $c$ ; $V_\delta(c)$, $x \in A$ $[x \neq c]$ such that $f(x) > 0$. $\forall x \in V_\delta(c)$, $x \in A$, $x \neq 0$

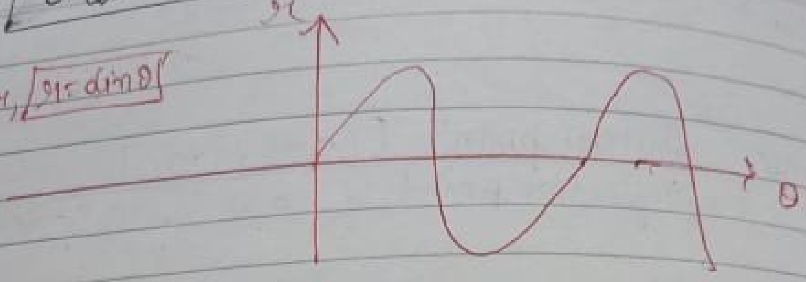
**Proof** :- Given: $\lim_{x\to c} f(x) = l$ where $l > 0$

$\therefore \forall \varepsilon > 0, \exists \delta > 0 :$ $0 < |x-c| < \delta$

$\Rightarrow |f(x) - l| < \varepsilon$

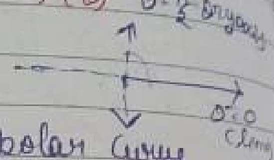
## θ & r-coordinate system.

for, $\boxed{r = \sin\theta}$




## Curve Tracing of polar Curves

$r = f(\theta)$




**Step 1** Symmetry :-

(i) If $\theta \to -\theta$, eq of polar curve remains unchanged then the curve is symmetrical about polar axis.

(ii) If on changing $\theta \to \pi - \theta$, eq remains unchanged then curve is symmetric about $\boxed{y \text{ axis}}$ or ~~also~~ $\boxed{\theta = \frac{\pi}{2} \text{ line}}$

(iii) Symmetry about origin (pole).

If on changing $\boxed{\theta}$ to $\boxed{\pi + \theta}$, eq remains unchanged, then the curve is symmetrical about the pole

**Step 2** whether curve passing through pole.
In this put $r = 0$ and get value of $\theta$

eg → let $r = \sin 3\theta$.
To check whether ~~pass~~ through pole

~~$\Rightarrow r = 0$~~
~~$\Rightarrow 0 = \sin 3\theta. \Rightarrow \theta = \sin^{-1}$~~

**Step 3** Making table

| θ | |
|---|---|
| r | |

Example. $r =$
The curve is ...

**Step 1** · curve is ...
as on ...
unch...
∴ now ...
the rest ...

**Step 2** put ...
⇒ ...

**Step 3**

| θ | |
|---|---|
| r | |

$\theta = \frac{3\pi}{4}$

$\theta = \frac{\pi}{2} + \frac{\pi}{3}$

given $\lim_{x \to c} f(x) = l$

$\Rightarrow$ for $\varepsilon > 0, \exists\ \delta > 0$ such that
$|f(x) - l| < \varepsilon$ if $0 < |x-c| < \delta$.

Now $\cancel{|f(x) - l + l|}$ $|f(x)| = |f(x) - l + l|$
$\leq |f(x) - l| + |l|$
$< \varepsilon + |l|$ if $0 < |x-c| < \delta$

$\boxed{\text{Let } \varepsilon = 1}$

$\Rightarrow \quad |f(x)| < 1 + |l| \quad$ [if $0 < |x-c| < \delta$]

let $\boxed{M = 1 + |l|}$ ← ✱ [Concept] → $l$ is limit means highest value of $f(x)$ for any value of $x$. ∴ for $(1 + l(x))$ is is highest for any value $g(x)$

$\Rightarrow \quad |f(x)| < M$ if $0 < |x-c| < \delta$,
if $c \notin A$

Case II) If $c \in A$,
then $M = \max\{|f(c)|, 1 + |l|\}$

✱ Here. $\boxed{f(x) \leq M}$

X ———— X ————

## [Algebra of limit]

Let $f$ and $g$ be two function having same domain
let $c$ be cluster pt, let $\lim_{x \to c} f(x) = l$
(Limit of the sum = sum of the limit)
and $\lim_{x \to c} g(x) = m$

① $(f+g)(x) = f(x) + g(x)$

$\lim_{x \to c} (f+g)(x) = \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

~~(step)~~ ∴ $f \circ g = f[g(x)] = \begin{cases} 3, & \text{if } x \neq 1 \\ 3 & x = 1 \end{cases}$

~~Now~~ $f\left[\lim_{x \to 1} g(x)\right]$ when $x$

but cluster point are not defined g, ∴ $g(x) = 2$ always

∴ $\lim_{x \to 1} f(g(x)) = \lim_{x \to 1} 3 = 3$

$f\left[\lim_{x \to 1} g(x)\right] = f(2) = 3$

∴ $\boxed{\lim_{x \to 1} f(g(x)) = f\left[\lim_{x \to 1} g(x)\right] = 3}$

# Section 4.3

1. → One sided limit
2. → Infinite limit
3. → Limit at infinity.

Cond => $\boxed{c < x < c+\delta}$

$\boxed{\text{Right } HL}$

$\boxed{LHL}$

Cond = $\boxed{c - \delta < x < c}$

## 4.3.1

**Right hand lim**

$$\lim_{x \to c^+} f(x) = l$$

if for every $\varepsilon > 0$, $\exists\, \delta > 0$ :
$|f(x) - l| < \varepsilon$ such that $c < x < c+\delta$

**Left hand lim**

$$\lim_{x \to c^-} f(x) = l$$

if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ ;
$|f(x) - l| < \varepsilon$ such that $c - \delta < x < c$

(1) (a) $f(x) = \frac{1}{x^2}$ , $x \to 0$

Soln let $x_n = \frac{1}{n}$ as $\boxed{n \to \infty}$

Clearly $\boxed{x_n \to 0}$

Now $f(x_n) = \frac{1}{(\frac{1}{n})^2} = n^2$

$\therefore$ $f(x_n) \longrightarrow \infty$

$\therefore$ $f(x_n)$ is divergent

(b) $\lim_{x \to 0} \frac{1}{\sqrt{x}}$ $(x > 0)$

Here also we will take $x_n = \frac{1}{n}$

(c) ~~$\lim_{x \to 0} f(x)$~~ Signum function $= \operatorname{sgn}(x)$

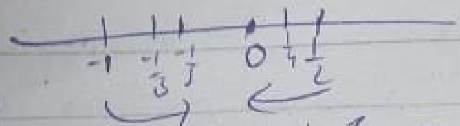
$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(c) $f(x) = \frac{x}{|x|}$

let $x_n = \frac{(-1)^n}{n} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

$f(x_n) = (-1, 1, -1, 1, \ldots)$

$\therefore$ $\boxed{x_n \to 0}$




approaches to 0

but $f(x_n)$ ~~does~~ not converge to single value

$\therefore$ by divergence criteria $\boxed{\lim_{x \to 0} f(x) \text{ doesn't exist}}$

i.e $|x^2+4x-12| = |(x-2)(x+6)| < \varepsilon$

Let $|x-2| < 1$ ∴ $1<x<3$

⇒ $|x-2||x+6| < 9|x-2| < \varepsilon$

⇒ $|x-2| < \frac{\varepsilon}{9}$

⇒ $|x-2| < \delta$ [ if $\delta = \min\{1, \frac{\varepsilon}{9}\}$ ]  ∴ $\boxed{\lim_{x\to 2} f(x) = 12}$

Q8 Show that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$, $c>0$

Soln To show $|\sqrt{x}-\sqrt{c}| < \varepsilon$. or $\left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right| \le \left|\frac{x-c}{\sqrt{c}}\right| < \varepsilon$

$|x-c| < \boxed{\sqrt{c}\,\varepsilon = \delta}$  ∴ $\boxed{\lim_{x\to c} \sqrt{x} = \sqrt{c}}$

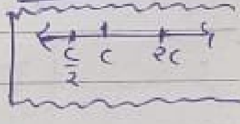
⇒ $\boxed{|x-c| < \delta}$

Q9 Show that $\lim_{x\to c} x^3 = c^3$  $\boxed{\text{to this } c\neq 0}$

Soln $|x^3-c^3| < \varepsilon$

$|(x-c)(x^2+c^2+xc)|$  Taking $|x-c|<1$ ⇒ $\boxed{c-1<x<c+1}$

Case I Let $c\neq 0$ and $x<2c$ ⇒ $|x-c|<c$ ; $x<c+c$




∴ $x^2+cx+c^2 < 7c^2$

∴ $\boxed{|x-c| < \frac{\varepsilon}{7c^2}}$  Let $\delta = \frac{\varepsilon}{7c^2}$ ∴ $\boxed{|x-c|<\delta}$

⇒ $\boxed{\lim_{x\to c} x^3 = c^3}$

Case II If $c=0$ $\lim_{x\to 0} x^3 = 0$

If $|x^3-0| < \varepsilon$ if $|x-0| < \delta$  then $\boxed{|x| < \varepsilon^{1/3} = \delta}$

Q3 $\lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$

Soln $\left|\frac{x}{1+x} - \frac{1}{2}\right| = \left|\frac{2x-1-x}{2(x+1)}\right| = \left|\frac{x-1}{2(x+1)}\right|$

Let $|x-1|<1$ ∴ $0<x<2$  $\boxed{1<x+1<3}$ → $\boxed{\frac{1}{x+1}<1}$

∴ $\left|\frac{x-1}{2(x+1)}\right| < \left|\frac{x-1}{2}\right| < \varepsilon$

If $|x-1| < 2\varepsilon$, choosing $\delta = \min\{1, 2\varepsilon\}$, we have

$|f(x)-\frac{1}{2}| < \varepsilon$ ∀ $0<|x-1|<\varepsilon$

∴ $\boxed{\lim_{x\to 1} f(x) = \frac{1}{2}}$

(Method 1)

Soln let $f(x) = \cos\frac{1}{x}$ & $g(x) = x$

It is bounded function

$\lim_{x \to 0} g(x) = 0$

Method 2 T.S $\left| x\cos\frac{1}{x} - 0 \right| < \varepsilon$ if $0 < |x-0| < \delta$.

$\left| x\cos\frac{1}{x} - 0 \right| = \left| x\cos\frac{1}{x} \right| \leq |x|$ or $|x-0| < \varepsilon$

If $\delta = \varepsilon$

Then we have, $|x| = |x-0| < \delta$

$\Rightarrow |f(x) - l| < \varepsilon$, whenever $|x-0| < \delta$

where $\boxed{\delta = \varepsilon}$, $\boxed{c = 0}$ and $\boxed{l = 0}$

---

Limit Theorem :- [Done]

Squeeze Theorem : let $f(x), g(x)$ & $h(x)$ be three function define on the same domain $A$. let $c$ be a cluster point of $A$.

let $f(x) \leq g(x) \leq h(x) \quad \forall x \in A$

$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$ (say)

then $\boxed{\lim_{x \to c} g(x) = l}$

$\frac{9+6+1}{1+1+2}$

$\frac{x^2+6x+1}{x^2+1-2x}$

point is $\left(-\frac{1}{3}, 0\right)$

$(0, 1)$

$y = 9$

(Oblique asymptote: deg (Num) − deg (denom) = 1)

Case-1 Method to obtain oblique asymptote

$$y = f(x) = \frac{P(x)}{Q(x)} \qquad \deg[P(x)] > \deg[Q(x)]$$

$$= q(x) + \frac{r(x)}{Q(x)}$$

where $q(x) \to$ quotient polynomial

$r(x) \to$ remainder.

Q17 $\boxed{f(x) = y = \frac{x^2}{(x-3)}}$

$$y = (x+3) + \frac{9}{(x-3)}$$

$\boxed{y = x+3}$ is oblique asymptote

$$[y - (x+3)] = \frac{9}{(x-3)}$$

$$\lim_{x \to \pm\infty} [y - (x+3)] = \lim_{x \to \pm\infty} \frac{9}{(x-3)} = 0$$

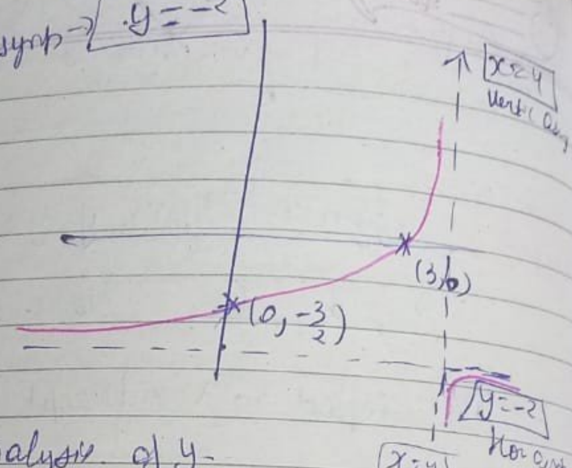
$\Rightarrow \boxed{y = x+3}$ is oblique asymptote.

Tracing the curve

1. No symmetry
2. pass through origin $(0, 0)$. Tangent at origin $\to \boxed{y=0}$
   $\Rightarrow x^2y - 3y = x^2$
   $\hookrightarrow -3y = 0$
   $\boxed{y = 0}$
3. $\frac{x}{y}$ intersection $\to (0, 0)$
   $(0, 0)$
4. Vert. asy. $\boxed{x = 3}$
   Hor. asym. ~~$y = 1$~~ None
   oblique asymp $\Rightarrow \boxed{y = x+3}$

Step 4 Vertical asymp → $\boxed{x=4}$

Hor. asymp → $\boxed{y=-2}$




Step 5 Sign analysis of y.
points to be considered are ($x = 3$ & $4$).

$-\infty$ — — 3 + + + 4 — — $\infty$

$$\lim_{x\to 4^+} y = \lim_{x\to 4^+}\left(\frac{2x-6}{4-x}\right) = \frac{+\text{finite}}{0(\text{Neg})} = \boxed{-\infty}$$

$$\boxed{\lim_{x\to \pm\infty} y = -2}$$

Step 6 $\frac{dy}{dx}$

$4y - xy = 2x-6$

$4\frac{dy}{dx} - x\frac{dy}{dx} - y = 2$

$\Rightarrow \cancel{\frac{dy}{dx} = \frac{2+y}{4-x}}$

$$\boxed{\frac{dy}{dx} = \frac{2}{(4-x)^2}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{4}{(4-x)^3}}$$

$\boxed{x<4}$ $\frac{d^2y}{dx^2} > 0$ (+ive) ← $\boxed{\text{Concave up}}$

$\boxed{\text{If } x>4}$ $\frac{d^2y}{dx^2} < 0$ (−ve) ← $\boxed{\text{Concave down}}$

So $\boxed{\frac{dy}{dx} = \frac{2}{(4-x)^2}}$

Stationary p
Inflection p

Asymptotes $x$
$y$

No crossing ...

Q $y = \frac{x}{x^2-4}$

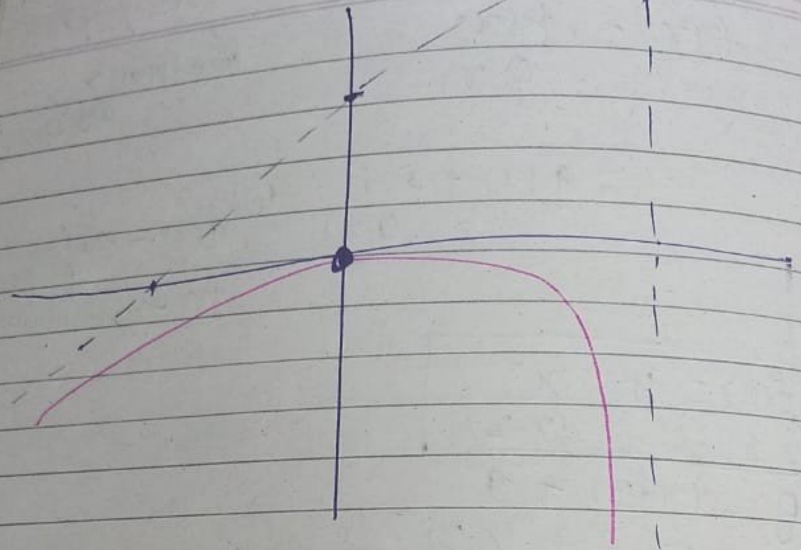
Step 1 ~~Writ~~

Changin
equat

Step 2 yes pa
at (

~~Step 3~~ $f(x,y$

Im




Sign analysis

$$y = \frac{x^2}{x-3}$$

$\boxed{-ve}$ 0 $\boxed{-ve}$ 3 $\boxed{+ve}$

4

(1) No sym

(2) passes through (0,0)
Tangent at (0,0).
$y=0$

(3) $\left.\begin{array}{l} x=0 \Rightarrow y=0 \\ y=0 \Rightarrow x=0 \end{array}\right] \Rightarrow (0,0)$ only

---

Q Curvilinear As

$$y = f(x) = \frac{p(x)}{Q(x)}$$
$$= q(x) + \frac{r(x)}{Q(x)}$$

Take $x \to \pm\infty$

Q19 Stationary
oblique
table the
any asym

Q $y = x^2$

Soln $y - x^2 =$

$\lim_{x \to \pm\infty}$

Curve tracing ste

(1) No symme

(2) Not passing

(3) x-intercep
y-intercep

(4) Vertical
horizo

for oblique asymptote

$$\boxed{\lim_{x \to \pm\infty} \left(\frac{y}{x}\right) = m} \qquad \boxed{y = mx + c}$$

$$\boxed{c = \lim_{x \to \pm\infty} (y - mx)} \quad \text{zero}$$

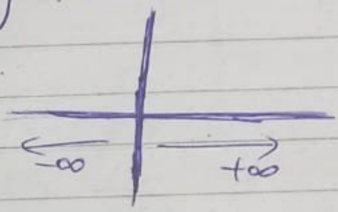
$\frac{-2x-6}{4+x}$ No

**Step 5.** Sign analysis of $y = f(x)$.

Make table

With respect to x-intercept & vertical Asymptote

**Step 6** End behaviour of curve.




**Step 7 / 8** find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

---

Q Give a graph of the rational function.
& label the coordinates of stationary point & inflection point.

p.151 (Readings)

① $y = \frac{2x-6}{4-x}$

$4 - x = 2x - 6$
$3x = 1$

**Step 1** checking symmetry (i) No
(ii) No.

**Step 2** Not passing through origin $(0,0)$

**Step 3** X intercept put $y = 0$, $\boxed{x = 3}$ pass $(3,0)$

Y intercept put $x = 0$, $\boxed{y = -\frac{3}{2}}$ $(0, -\frac{3}{2})$

So $\boxed{\dfrac{dy}{dx} = \dfrac{2}{(4-x)^2}}$

for $x<4$, $\boxed{\dfrac{dy}{dx}>0}$

for $x>4$, $\boxed{\dfrac{dy}{dx}>0}$

at $x=4$ → Not-differentiable

Stationary point (None)

Inflection point (None)

Concave up then concave down → then why? → Inflection point (None)

Asymptote $x=4$, $y=-2$

No crossing with the asymptote

Ans → So curve is breaking, the strategy we know is only applicable to continuous curve.

$\frac{-x}{x-4}$ → $y = \frac{-x}{x^2-4}$

---

Q $y = \dfrac{x}{x^2-4}$

Step 1 ~~w.r.t~~ symm about → y-axis → None; → Origin → yes.

Changing $x \to -x$, $y \to -y$ keeps the equation unchanged.

Step 2 yes passing through origin $(0,0)$. tangent at $(0,0)$ is $\boxed{y = -\frac{x}{4}}$

~~Step 3~~ $f(x,y) \Rightarrow (x^2-4)y - x = 0$

$\Rightarrow x^2y - 4y - x = 0$

Smallest degree → $-4y - x = 0$

$\Rightarrow \boxed{y = \frac{-x}{4}}$ is tangent

⑤ sign analysis. $x^{-1}$ $-1x$.

$\frac{1}{4}-2$

$\frac{1}{2}$

(+ve) 0 (−ve) 1 (+ve)

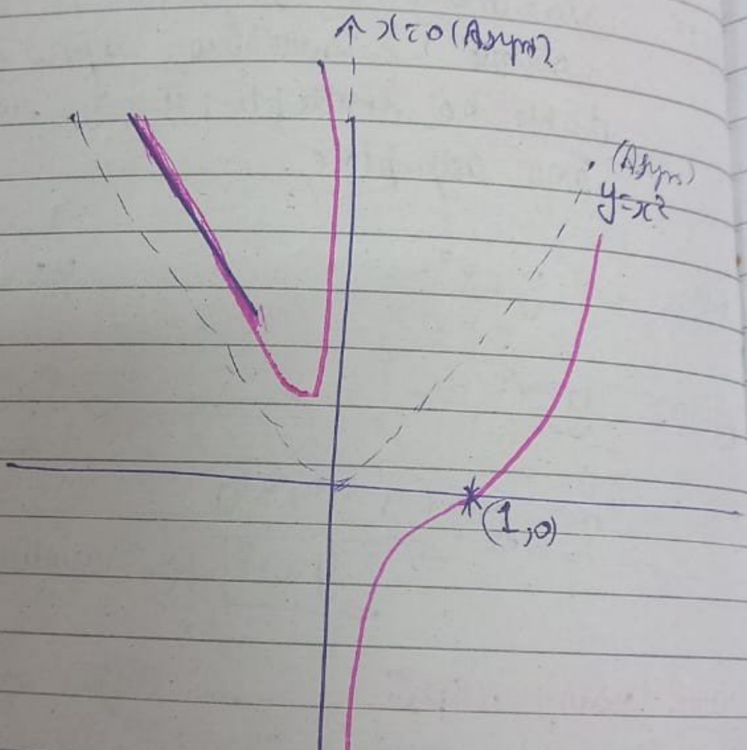
∴ $x<0$ (+ve)
$0<x<1$ (−ve)
$x>1$ +ve

⑥ $\frac{dy}{dx} = \frac{2x+1}{x^2}$

$\boxed{x<0}$ (−ve)

$x>0$ $\boxed{\infty}$ +ve

$\boxed{x>0}$ (+ve)




$\frac{dy}{dx} = \frac{2x^3+1}{x^2}$

$\to x^3 = \left(\frac{-1}{2}\right)^{1/3}$

$(x, y)$

∴ stationary pt $\left[\frac{-1}{(2)^{1/3}}, \frac{3}{2}(2)^{1/3}\right]$

→ Inflection pt (1, 0)
Asym $y=x^2$, $x=0$
Asym (cross) ∴ None

Stationary point → None
Inflection point → (0,0)
Asymptote crossing [at origin (0,0) ]
as $\boxed{y=0}$ is also a asymptote

Q5 $\boxed{y = \frac{x^2}{x^2+4}}$

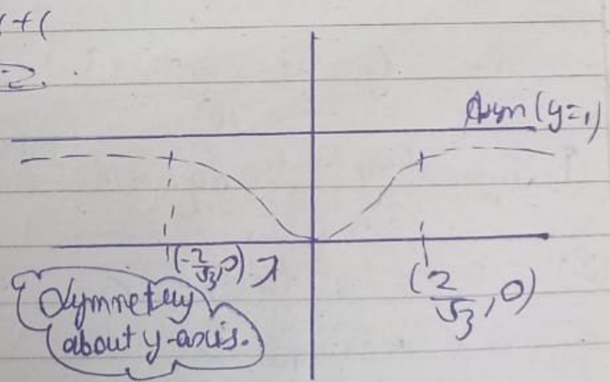
$\boxed{\text{Step 1}}$ Symmetry → about y axis
→ Not about origin.

$\boxed{\text{Step 2}}$ (0, 0) lies on the curve.

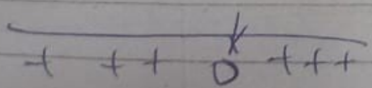
$\boxed{\text{Step 3}}$ ~~$yx^2+4y$~~ · Intercept ⇒ x intercept } (both)
y intercept } (0,0) only

$\boxed{\text{Step 4}}$ Vertical = None
Horiznt (Asy) → $\boxed{y=1}$

Oblique, $y = mx+c$
$\boxed{m=0}$ c=




Step 5 Sign analysis of y




$\boxed{\text{Step 6}}$ $x \to +\infty$, $y \to 1$
$x \to -\infty$, $y \to 1$

## (Curve tracing of rational function)

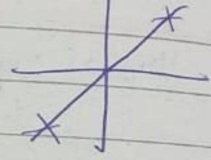
$y = f(x) = \frac{p(x)}{Q(x)}$ where $p(x)$ & $Q(x)$ are polynomial in $x$

$\boxed{Q(x) \neq 0}$

**Step 1** Symmetry of the graph.

(i) Symmetry about y-axis. → If on changing $x$ to $-x$, the eq. remains unchanged. or [all the powers of $x$ are even].

(ii) Symmetry about origin.



If on changing $x$ to $-x$ and $y$ to $-y$ keeps expression unchanged. eg $\left[y = \frac{1}{x}\right]$

**Step 2** check whether curve passes through origin $(0,0)$.

If $(0,0)$ lies then find the tangent at $(0,0)$.

**Step 3** x-intercepts and y-intercepts

[put $x=0$, get $y$.
[put $y=0$, get $x$ if any

**Step 4** find asymptote of the curve (if any)

If $y = \frac{p(x)}{Q(x)}$ vertical → $Q(x) = 0$

horizontal → $\lim_{x \to \pm\infty}$ get $y = k$ (if exist)

then $\boxed{y = k}$ is hori asymptote

If $y = \pm\infty$ then no asymptote

for oblique ...

$\lim_{x \to \pm\infty} \left(\frac{y}{x}\right) =$

Step 5.

With ...

Step 6

Step 7 / ...

Q Give a ... rational ... & label ... inflect...

① $y =$

Step 1

Step 2

Step 3

with this we need to analyse
$\frac{dy}{dx}\left[\begin{array}{l}+ve \to incr \\ -ve \to decr\end{array}\right.$
$\frac{d^2y}{dx^2}\left[\begin{array}{l}+ve \to concave\ up \\ -ve \to concave\ down\end{array}\right.$

$\boxed{Step 7}$ $\frac{dy}{dx} =$

$$\frac{d^2y}{dx^2} = \frac{32-24x^2}{(x^2+4)^3} = \frac{(x-\alpha)(x-\beta)}{+ve}$$

Ans → Stationary pt

H.W? upto 14

$\frac{d^2y}{dx^2} > 0$   + +   − −

$\frac{d^2y}{dx^2} < 0$   − +   + −

Q. $y = \frac{4}{x^2} - \frac{2}{x} + 3$.

Soln $y = \frac{4-2x+3x^2}{x^2}$

(1) Symmetry → No symmetry {y axis, origin

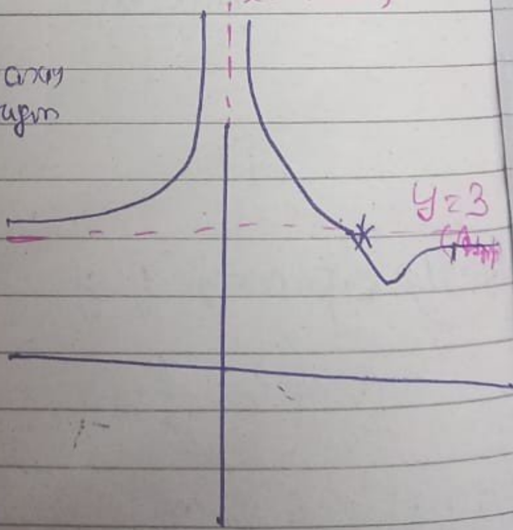
(2) (0,0) → Not passes

(3) X-intercept → No
Y-intercept → No

(4) Vertical asym $x=0$
Hori. asym $y=3$

(6) Sign analysis
+ve 0 +ve
+ve for both.

$4-2+3$ +ve
$4+2+3$




16 − 4 + 3

as $\boxed{\begin{array}{l}x\to\infty \\ y\to 3\end{array}}$

$x\to 0^+$ y
$x\to 0^-$

(7) $\frac{dy}{dx} = x^2$
$= 6x$

$\frac{dy}{dx} =$

at $x=3$
$x=4$

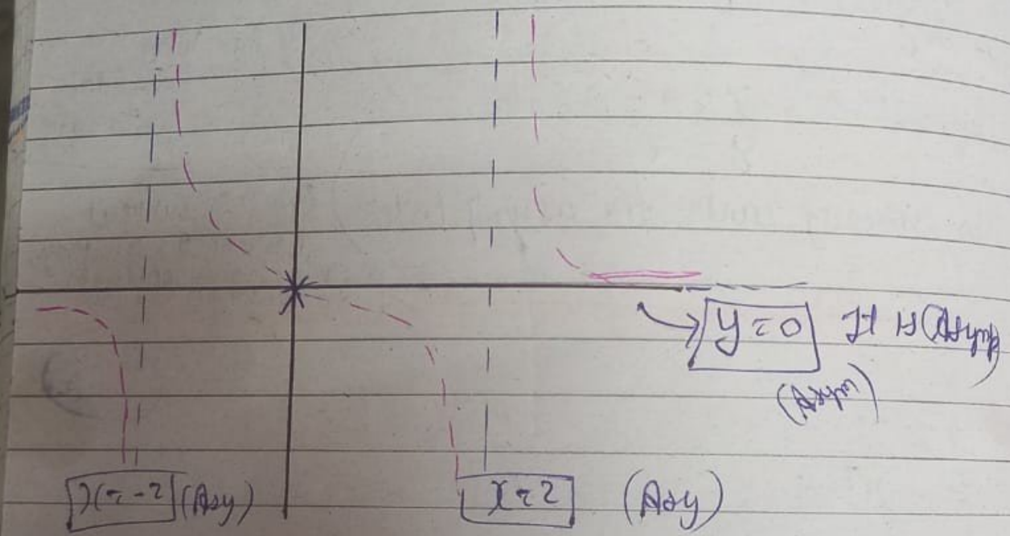
(8) $\frac{d^2y}{dx^2} = x^3$
$=$

$\boxed{\frac{d^2y}{dx^2} = 2}$

$\boxed{y = \frac{x}{x^2-4}}$

**Step3** x-intercept } only (0,0) for both
y-intercept }

**Step4** Asymp:- Vertical Asym → $\boxed{x = \pm 2}$
Hor Asym $\boxed{y = 0}$




**Step5** Sign analysis of y. , $x = 0, \pm 2$

-ve -2 +ve 0 -ve 2 +ve

**Step6**

$\frac{dy}{dx} =$

$\frac{d^2y}{dx} =$

$16-4+3$

$\frac{4-2+3}{5}$ $\frac{4-4+(\,)}{4}$

As $\boxed{\begin{array}{l} x \to \infty \\ y \to 3 \end{array}}$ $\quad$ $x \to -\infty$, $y \to 3$

$x \to 0^+$ $y \to +\infty$

$x \to 0^-$ $y \to +\infty$.

(7) $\frac{dy}{dx} = \frac{x^2[6x-2] - (4-2x+3x^2)[2x}{x^4}$

$= \frac{\cancel{6x^3} - 2x^2 - 8x + 4x^2 - \cancel{6x^3}}{x^4}$

$y = 4 - \frac{8+48}{16}$

$\frac{dy}{dx} = \frac{2x-8}{x^3}$ $\boxed{\text{stationary point } -4} \to$ changes $\to$ U

$(-ve)$ 4 $(+ve)$

at $x=3$ $\quad y = \frac{25}{9} = 2.77$

$x=4$, $y = \frac{11}{4} = 2.75$

$x=3$

$\frac{4-6+27}{9}$ $\frac{25}{9}$

(8) $\frac{d^2y}{dx^2} = \frac{x^3[2] - (2x-8](x^3]}{x^6}$

$= \frac{2x^3 - 2x^4 + 8x^3}{x^6} = \frac{10x^3 - 2x^4}{x^6}$

$\boxed{\frac{d^2y}{dx^2} = \frac{24 - 4x}{x^3}}$ $\quad$ $\boxed{x=5}$

$(+ve)$ $(+ve)$

$(+ve)$ 5 $(+ve)$

up

$\boxed{x=6}$ point of inflection

# Curvilinear Asymptote

24-04-2023

$y = f(x) = \frac{p(x)}{Q(x)}$ , $p(x)$ & $Q(x)$ are polynomial with $Q(x) \neq 0$

$= q(x) + \frac{r(x)}{Q(x)}$ degree of $P(x) >$ degree of $Q(x)$

Take $x \to \pm\infty$

Q19 Stationary pt, inflection pt, hor, verti, oblique & curvilinear asymp, lable their eqn, lable the ~~double~~ pt, (if any) where graph crosses any asymptote.

Q19 $y = x^2 - \frac{1}{x}$ $y = \frac{x^3 - 1}{x}$

Soln $y - x^2 = -\frac{1}{x}$

$\lim_{x \to \pm\infty} (y - x^2) = 0$

$\Rightarrow \boxed{y = x^2}$ is curvilinear asymptote

Curve tracing steps:-

① No symmetry

② Not passing through (0,0)

③ x-intercept , $\boxed{x=1}$

y-intercept + $\boxed{No}$ intersection with y axis.

④ Vertical , $\boxed{x=0}$

horizontally $\boxed{No}$

oblique (None)

Curvilinear $\boxed{y = x^2}$

$\frac{9+6+1}{1+1+2}$

(1) $y = \frac{(3x+1)^2}{(x-1)^2}$ $= \frac{9x^2+6x+1}{x^2+1-2x}$

~~$y = 9x^2 +$~~

1. No symmetry
2. Not pass (0,0)
3. x-intercept $\boxed{x = -\frac{1}{3}}$

   y-intercept $\boxed{y = 1}$

point is $(-\frac{1}{3}, 0)$ → (0,1)

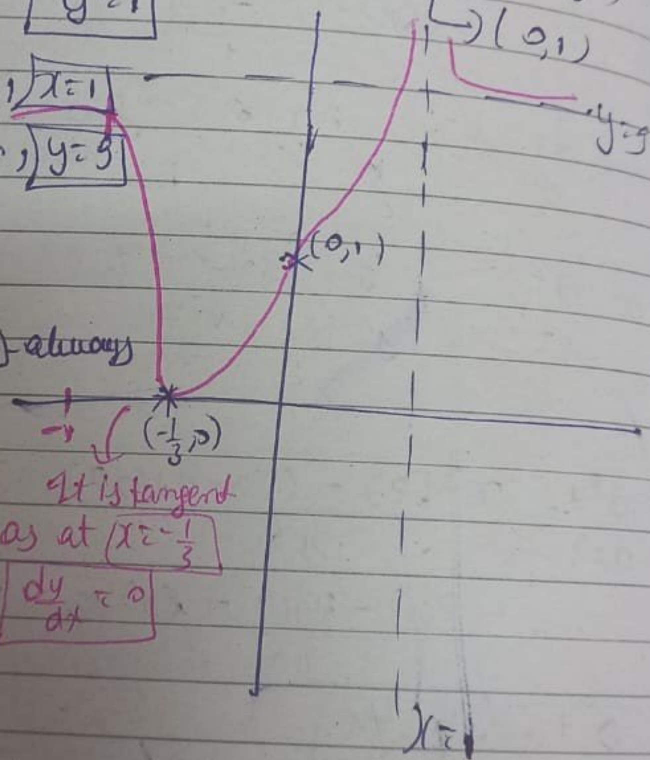
4. Asymptote · V. Asy, $\boxed{x=1}$

   H. Asym, $\boxed{y=9}$

5. Sign area

(+ve) $-\frac{1}{3}$ (+ve) 1 (+ve) $\boxed{y>0}$ always

It means that y is above x axis for any value of x




It is tangent as at $\boxed{x = -\frac{1}{3}}$ $\boxed{\frac{dy}{dx} = 0}$

6. $x \to 1^-$

$\boxed{x = -1}$ $\frac{-9-6+1}{1+1+2} = \frac{-14}{4}$

$\frac{dy}{dx} = \frac{6(x-1)^2(3x+1) - (3x+1)^2(2x-2)}{(x-1)^3}$

Answers → Stationary point $(-\frac{1}{3}, 0)$
→ Inflection point $(-1, 1)$
→ $x = 1$, $y = 9$

Asy. Crossing $(\frac{1}{3}, 9)$

Easiest Meth...

$y = f(x)$ ...

Q17 $f(x) =$ ...

$y = ($ ...

$[y -$ ...

$\lim_{x \to \pm\infty}$ ...

⇒ $y =$ ...

Drawing th...

1. No - dym...
2. pass through...
3. $\frac{x}{y}$ → inter...
4. Vert... hor... obl...