Differentiation

Let f be real-valued function defined on an open interval containing a point a. We say f is differentiable at a, or f has a derivative at a, if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite. Then, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

$dom(f') \subseteq dom(f)$

Q $f(x) = x^n \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x) - f(a) = x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-1})$

$\therefore \frac{f(x)-f(a)}{x-a} = x^{n-1} + ax^{n-2} + \dots + a^{n-2}x + a^{n-1}$ for $x \neq a$

Then $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = a^{n-1} + a^{n-1} + \dots a^{n-1}$ (n times)

$= n a^{n-1}$

Theorem: If f is differentiable at point a, then f is cont at a

Proof: $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ (Given)

To prove: $\lim_{x\to a} f(x) = f(a)$

We have: $f(x) = (x-a)\cdot \frac{f(x)-f(a)}{x-a} + f(a)$

$\forall x \in dom(f),\ x \neq a.$

Now, $\lim_{x\to a} f(x) = \lim_{x\to a}(x-a) \cdot \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} f(a)$

$= 0 + f(a) = f(a)$

Theorem: Let f and g be function differentiable at a. Then cf, $f+g$, fg, f/g is also differentiable at a, except f/g if $g(a)=0$. The formulas are:

(1) $(cf)'(a) = c\cdot f'(a)$

(2) $(f \pm g)'(a) = f'(a) \pm g'(a)$

(3) [product rule] $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$

(4) [quotient rule] $(f/g)'(a) = \frac{[g(a)f'(a) - f(a)g'(a)]}{g^2(a)}$

Proof: (1) $(cf)'(a) = \lim_{x\to a} \frac{(cf)(x) - (cf)(a)}{x-a} = \lim_{x\to a} c\cdot \frac{f(x)-f(a)}{x-a} = c\cdot f'(a)$

(2) $\frac{(f+g)(x)-(f+g)(a)}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}$

$\therefore \lim_{x\to a} \frac{(f+g)(x)-(f+g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$

$= f'(x) + g'(x)$

$\therefore (f+g)'(a) = f'(a) + g'(a)$

(3) $\frac{(fg)(x)-(fg)(a)}{x-a} = \frac{f(x)\cdot g(x) - f(a)\cdot g(a) - g(x)f(a) + f(a)g(x)}{x-a}$

$= g(x)\cdot \frac{f(x)-f(a)}{x-a} + f(a)\cdot \frac{g(x)-g(a)}{x-a}$

$\therefore$ Applying limit $x\to a$, we get $g(a)\cdot f'(a) + f(a)\cdot g'(a)$

(4) $\frac{(f/g)(x)-(f/g)(a)}{x-a} = \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \frac{f(x)g(a) - g(x)\cdot f(a)}{g(x)\cdot g(a)\cdot(x-a)}$

$= \frac{f(x)g(a) - g(x)\cdot f(a) - g(a)\cdot f(a) + g(a)\cdot f(a)}{g(x)\cdot g(a)\cdot(x-a)}$

$= \frac{g(a)\cdot[f(x)-f(a)]}{(x-a)} - \frac{f(a)[g(x)-g(a)]}{(x-a)} \cdot \frac{1}{g(x)\cdot g(a)}$

Applying limits. $(f/g)'(a) = \frac{g(a)\cdot f'(a) - f(a)\cdot g'(a)}{g^2(a)}$

Chain Rule: If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $(g\circ f)'(a) = g'(f(a))\cdot f'(a)$.

faulty Proof: $\frac{(g\circ f)(x) - (g\circ f)(a)}{x-a} = \frac{g(f(x)) - g(f(a))}{f(x)-f(a)} \cdot \frac{f(x)-f(a)}{x-a}$

$\Rightarrow$ ~~$(g\circ f)'(a) = g'(f(a))$~~ Applying limit $x\to a$

$\therefore (g\circ f)'(a) = g'(f(a))\cdot f'(a)$

Proof:

Q. $h(x) = \sin(x^3+7x)\ \forall x \in \mathbb{R}$

$\Rightarrow$ Let $f(x) = x^3+7x$ and $g(y) = \sin y$.

$\therefore h'(x) = g(f(x))' \Rightarrow h'(x) = g'(f(x))\cdot f'(x)$

$\Rightarrow h'(x) = \cos[f(x)]\cdot(3x^2+7) = \cos(x^3+7x)(3x^2+7)$

A function f is differentiable on domain of f'. Hence, f is not differentiable in every point that is not in domain of f'. So, if we were given to prove, where a function is not differentiable, then use chain rule to show where $f'(x)$ is not defined at which point.

28.1 (a) $e^{|x|}$ Let $h(x) = e^{|x|}$, $f(y) = |y|$, $g(z) = e^z$

Using chain rule, $h(x) = g(f(x))$

Then $h'(x) = g'(f(x))\cdot f'(x) = g'(|x|)\cdot \frac{|x|}{x} = e^{|x|}\cdot\frac{|x|}{x}$

[Please note: if $f(x) = |x|$, then $f'(x) = \frac{|x|}{x}$]

Hence, $h'(x) = e^{|x|}\cdot \operatorname{Sgn}(x)$.

Clearly $h'(x)$ is defined at $x \neq 0$. $\therefore S = \{0\}$

(b) $\sin|x|$ $h(x) = \sin|x|$, $f(y) = |y|$, $g(z) = \sin z$

Then $h'(x) = g'(f(x))\cdot f'(x) = \sin|x| \cdot \frac{|x|}{x}$

[Please note: $\frac{|x|}{x}$ is not defined at 0]

(c) $|\sin x|$ $h(x) = |\sin(x)|$, $f(y) = \sin y$, $g(z) = |z|$

$h'(x) = g'(f(x))\cdot f'(x) = \frac{|\sin x|}{\sin x}\cdot \cos x = \frac{|\sin x|}{\sin x}\cdot\cos x$

So, $|\sin x|$ is defined at every point and $\cos x$ also.

But $\frac{1}{\sin x}$ is not defined at $x=0$

$\therefore S = \{n\pi : n \in \mathbb{Z}\}$

(d) $|x| + |x-1|$ Let $h(x) = |x| + |x-1|$, $f(y) = |y|$, $g(z) = |z-1|$

Then $h(x) = f(x) + g(x) \Rightarrow h'(x) = f'(x) + g'(x) = \frac{|x|}{x} + \frac{|x-1|}{x-1}$

$\therefore S = \{0, 1\}$

(e) $|x^2-1|$ Let $h(x) = |x^2-1|$, $f(x) = x^2-1$, $g(x) = |x|$

$h'(x) = g'(f(x))\cdot f'(x) = g'(x^2-1)\cdot f'(x) = \frac{|x^2-1|}{(x^2-1)}\cdot 2x$

$\therefore S = \{-1, 1\}$

(f) $|x^3-8|$ Let $f(x) = x^3-8$, $g(x) = |x|$, then $h(x) = g\circ f(x)$

$\therefore h'(x) = g'(f(x))\cdot f'(x) = g'(x^3-8)\cdot f'(x) = \frac{|x^3-8|}{x^3-8}\cdot 3x^2$ $\therefore S = \{2\}$

Interior extremum Theorem: If f is defined on an open interval containing x_0, if f assumes its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0)=0$.

$\Rightarrow$ Let f defined on (a,b) where $a<x_0<b$.
Let f attains its maximum at x_0.
Assume first that $f'(x_0)>0$.
$\because f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$, $\exists\ \delta>0 : a<x_0-\delta<x_0+\delta<b$

and $0<|x-x_0|<\delta$ implies $\frac{f(x)-f(x_0)}{x-x_0}>0$ —— ①

If we slect $x : x_0<x<x_0+\delta$, then from ① $f(x)>f(x_0)$, which is contradiction, since $f(x_0)$ is maximum.

Similarly, if we assumes $f'(x_0)<0$, $\exists\ \delta>0$:
$0<|x-x_0|<\delta \Rightarrow \frac{f(x)-f(x_0)}{x-x_0}<0$ —— ②
If we $x : x_0-\delta<x<x_0$, then ② implies $f(x)>f(x_0)$
which contradiction
$\therefore f'(x_0)=0$

Q.29.1 (a) x^2 on $[-1,2]$: clearly x^2 is cont and dif at $[-1,2]$
$\therefore$ MVT holds. Now $f'(x_0)=\frac{f(2)-f(-1)}{2-(-1)}=\frac{4-1}{3}=1$.
$\because f'(x_0)=2x_0$ and we $f'(x_0)=1$
$\therefore x_0=\frac{1}{2}$

(b) $\sin x$ on $[0,\pi]$: cont and diff.
$\therefore f'(x_0)=\frac{f(\pi)-f(0)}{\pi}=0$ $\quad\therefore f'(x_0)=\cos x_0$ $\quad\therefore x_0=\pi/2$.
(c) $|x|$ on $[-1,2]$. cont but not diff at 0.
(d) $\frac{1}{x}$ on $[-1,1]$. not cont and dif at 0.
(e) $\frac{1}{x}$ on $[1,3]$. $f'(x_0)=\frac{f(3)-f(1)}{2}=\frac{\frac{1}{3}-1}{2}=\frac{\frac{-2}{3}}{2}=\frac{-2}{3}\times\frac{1}{2}=\frac{-1}{3}$
Also $f'(x_0)=-1x^{-2}=\frac{-1}{x^2}$ $\Rightarrow x_0=\sqrt{3}$
(f) $\mathrm{sgn}(x)$ on $[-2,2]$: Not cont and dif at 0.

Impressions PAGE

Rolle's theorem: If f is a f^n defined on closed interval $[a,b]$ such that (i) f is cont on $[a,b]$
(ii) f is diff on (a,b)
(iii) $f(a)=f(b)$
Then $\exists$ a pt $c\in(a,b) : f'(c)=0$

Proof: Given f is cont on $[a,b]$
$\therefore f$ is bdd and f attains its bounds
So, $\exists\ x_0, y_0 \in [a,b] : f(x_0)=\min f = m$
& $f(y_0)=\max f = M$

Case I: If x_0, y_0 are end pts (a or b), then because $f(a)=f(b)$ $\therefore m=M$
$\Rightarrow f$ is constant on $[a,b]$ $\therefore f'(x)=0\ \forall x\in[a,b]$
Case II: when m or M are not end points.
In this case $f'(x_0)=f'(y_0)=0$ (by interior extremum theorem)
*Rolle's theorem is special case of Mean value theorem

Lagrange's Mean value theorem: If f is a f^n defined on $[a,b]$ st (i) f is cont on $[a,b]$
(ii) f is diff on (a,b)
then $\exists$ a pt $c\in(a,b) : f'(c)=\frac{f(b)-f(a)}{b-a}$

Proof: If $L(x)$ stands for equation of line joining pts $(a,f(a))$ & $(b,f(b))$, then
$(L(x)-f(a))=\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$

$\Rightarrow L(x)=\left[\frac{f(b)-f(a)}{b-a}\right](x-a)+f(a)$

there $\left[\begin{array}{l} L(a)=f(a) \text{ — ①} \\ L(b)=f(b) \text{ — ②} \end{array} \quad L'(x)=\frac{f(b)-f(a)}{b-a}\right]$ — ③
observe

Impressions PAGE

Increasing and decreasing function in interval I

Def^n (Inc): A f^n f defined on $\mathbb{R}$ is said to be an increasing f^n if for $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

Strictly inc: $f(x_1) < f(x_2)$. Similarly in dec.

Corollary: Let f be diff f^n on interval (a,b). Then,

(i) f is strictly inc or dec if $f'(x) > 0$ or $f'(x) < 0$ $\forall x \in (a,b)$

(ii) f is inc or dec if $f'(x) \geq 0$ or $f'(x) \leq 0$ $\forall x \in (a,b)$

$\Rightarrow$ (i) Consider $x_1, x_2 \in (a,b)$ st $a < x_1 < x_2 < b$.

By MVT, for some $x \in (x_1, x_2)$, we have $f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

~~Also since~~ it is inc f^n, If $f'(x) > 0$

It implies $\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \Rightarrow f(x_2) > f(x_1)$

↳ strictly inc

$[\because x_2 - x_1 > 0]$

Intermediate Value theorem for Derivatives

Let f be a differentiable f^n on (a,b). whenever $a < x_1 < x_2 < b$ and c lies bet^n $f'(x_1)$ and $f'(x_2)$, $\exists$ [at least one] x in (x_1, x_2) st $f'(x) = c$.

$\rightarrow$ Proof: Assume $f'(x_1) < c < f'(x_2)$ ——— ①

Let $g(x) = f(x) - cx$ $\forall x \in (a,b)$

[Note: we took $f(x) - cx$ becaus $g'(x) = f'(x) - c$]

Then, we have $g'(x_1) < 0 < g'(x_2)$ from ①

According to Max-min value theorem, g assumes its minimum on $[x_1, x_2]$ at $x_0 \in [x_1, x_2]$

$\therefore g'(x_1) = \lim_{y \to x_1} \frac{g(y) - g(x_1)}{y - x_1} < 0$,

Since $y - x_1 > 0$, $\therefore g(y) - g(x_1)$ must be negative for y closer to and larger than x_1

In particular, $\exists y_1 \in (x_1, x_2) : g(y_1) < g(x_1)$.

$\therefore$ g doesn't take minimum at x_1 so we must have $x_0 \neq x_1$.

Similarly, $\exists y_2 \in (x_1, x_2) : g(y_2) < g(x_2)$, so $x_0 \neq x_2$

$\therefore x_0$ is in (x_1, x_2), so $g'(x_0) = 0$, by Interior extremum.

$\therefore f'(x_0) = g'(x_0) + c = c$.

Theorem: Let f: one-to one cont f^n on I and let J = f(I) If f is diff at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

Q29.1 Let $f(x) = x - \sin x$. We have to show $f(x) \geq 0$ $\forall x \geq 0$ in $[0, \infty)$

$f'(x) = 1 - \cos x$. Here $\cos x \in [-1, 1]$

$\therefore f'(x) \geq 0$

29.2 $f(x) = \tan x - x$ then $f'(x) = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$

We know that $\cos^2 x \in (0, 1)$ $\therefore f'(x) > 0$, $\because f(0) = 0$

(b) Let $f(x) = \frac{x}{\sin x}$

$\therefore f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$. On $(0, \pi/2)$ $\sin x \in (0,1)$

$x \cos x \in (0, \pi/2)$

$\sin^2 x \in (0,1)$

Clearly, $\sin^2 x > 0$

Now, we have to prove $\sin x > x \cos x$

$\Rightarrow \tan x > x$ which is proved in part (a)

$\therefore f'(x) > 0$ on $(0, \pi/2)$

(c) $f(x) = \pi/2 \sin x - x$ $f'(x) = \pi/2 \cos x - 1$

~~We have to prove~~ $\pi/2 \cos x \geq 0$ which is true

Now, $\pi/2 \cos x = 1 \Rightarrow \cos x = 2/\pi \Rightarrow x =$ ~~5.0?5~~ $0.88 = 0.004\pi$

$\therefore x \geq 0.004\pi$, $f'(x) \geq 0$ and vice versa.

$\therefore f(0.004\pi)$ is minimum and $f(0.004\pi) = 0$ (aprox)

$\therefore f(x) \geq 0 \Rightarrow \pi/2 \sin x \geq x$

Clearly $\cos x \in (0,1)$ $\therefore f'(x) \geq 0$

Now $f(0) = 0$ $\therefore f(x) \geq 0 \Rightarrow \pi/2 \sin x \geq x$

13. Let $h(x) = g(x) - f(x)$. Also $-f'(x) + g'(x) \geq 0$

$\therefore h'(x) \geq 0$ Also $h(0) = 0$

$\therefore h(x)$ is inc in $[0, \infty]$ $\Rightarrow g(x) \geq f(x)$ in $[0, \infty)$