The Art Gallery Theorem For Polygons With Holes

Frank Hoffmann[1] Michael Kaufmann[2] Klaus Kriegel[1]

Abstract

Art Gallery Problems which have been extensively studied over the last decade ask how to station a small (minimum) set of guards in a polygon such that every point of the polygon is watched by at least one guard. Here we give an answer to one of the main open questions in this field: Any polygon, possibly with holes, can be watched by at most $\lfloor \frac{n+h}{3} \rfloor$ *point guards where* n *is the total number of vertices and* h *the number of its holes.*

1 Introduction

The original Art Gallery Problem raised by V. Klee asks how many guards (stars) are sufficient to watch (equiv. to cover) an n-sided simple polygon. V. Chvatal [2] gave the answer proving that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary. Since then many results have been published studying variants of the problem or analyzing algorithmic aspects. Art gallery problems can be seen as a special class of covering problems, where a given polygon has to be covered by a small (minimum) number of star polygons. A star polygon is a polygon with such a simple shape that one guard inside of the polygon is sufficient to watch the whole polygon.

Most of the material on Art Gallery Problems is contained in the excellent monograph of J. O'Rourke [7] and a recent survey of T. Shermer [9]. One of the most challenging open problems in this field asks about polygons with holes. There is a lower bound, compare Fig.1, which states there are polygons with h holes that need $\lfloor \frac{n+h}{3} \rfloor$ point guards to be watched. Guards are called point guards if they are placed somewhere inside of the polygon while vertex guards are placed on the corners of the polygon. On the other hand, we have a trivial upper bound of $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards by reducing the problem to a simply connected one, see [7] for details. A. Aggarwal [1] and, independently, T.Shermer [8] showed that in the case of $h = 1$ the lower bound is tight. The aim of our paper here is to prove that $\lfloor \frac{n+h}{3} \rfloor$ point guards are sufficient for any number h of holes, compare with Conjecture 5.1. in [7]. Recently, it has been shown [4] that in case of rectilinear polygons $\lfloor n/4 \rfloor$ point guards is the exact bound independent of the number of holes whereas at least $\lfloor \frac{n+h}{4} \rfloor$ vertex guards are necessary, see also [6], [5].

Some ideas from that work on rectilinear galleries now prove to be useful also for dealing with general polygons. Especially it is possible to classify all local configurations. More precisely, we consider a covering of the gallery by convex regions (rooms). Guards will be stationed in the entrances to rooms, that is in their intersections. Any point will belong to at most 3 rooms. Then, the gallery problem can be formulated as a graph-theoretical covering problem for a weighted hypergraph with nodes being rooms and edges given by entrances. We feel that the visibility structure which is based on polygon edge prolongations is more appropriate to gallery-type problems than a rather arbitrarily chosen triangulation graph. (Remark: However, using a clever triangulation graph approach, Z. Füredi and D. Kleitman [3] have recently solved the Prison Yard Problem, i.e. how to watch simultaneously the interior and the exterior of a simple polygon.)

The paper is organized as follows.Section 2 introduces the visibility structure we use and presents a transformation of arbitrary polygons into standard form polygons composed of 3 local situations (entrances) only. Next, in section 3 we give the graph-theoretic formulation and solution to the gallery problem for polygons in standard form, and we conclude with giving a complexity analysis and discussing open problems.

[1]Karl-Weierstraß-Institut für Mathematik, Mohrenstr. 39, O-1086 Berlin, Germany, *hoffmann@db0adw11.bitnet*

[2]Max-Planck-Institut für Informatik, Im Stadtwald, W-6600 Saarbrücken, Germany, *mk@cs.uni-sb.de*

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2 Preliminaries and Standard Form Polygons

Let P be an n-sided polygon with h holes, i.e., a finite, connected and closed region in the plane bounded by a finite collection of $h+1$ pairwise disjoint simple polygons with a total number of n vertices denoted by $size(P) = n$. One of the simple polygons, say P_0, forms the outer periphery; the other ones describe holes which are cut out of P_0.

A star in P is the union of a family of convex subsets of P with a non-empty common intersection (the so-called kernel). We say, that point $x \in P$ sees point $y \in P$, written as $x \sim y$, if the segment $\langle x, y \rangle$ is completely contained in P. Observe, two neighboring boundary vertices see each other, since P is closed. Clearly, the set $Watch(x)$ consisting of all points seen from $x \in P$ forms a star. Let $w(P)$ denote the minimal number of points such that the union of the corresponding stars $Watch(x)$ covers P completely. Or, equivalently, we ask for the minimal number of point guards in the art gallery P.

The 'Art Gallery Problem' for polygons with holes can now be formulated as follows:

Determine the function $w(n,h) = max\ w(P)$ where the maximum is taken over all polygons P with h holes and $size(P) = n$.

The main result we are going to prove in this paper now reads

Main Theorem: $w(n,h) = \lfloor \frac{n+h}{3} \rfloor$

First of all, we can assume w.l.o.g. P to be in general position, there are no 3 collinear vertices and no parallel edges. Let c be a concave corner formed by polygon edges s_1, s_2. Consider the prolongations $\overline{s_1}$, $\overline{s_2}$ of these edges into P (in direction c) until they hit the boundary of P. We define $cone(x)$ to be the intersection of the cone spanned by $\overline{s_1}$, $\overline{s_2}$ with $Watch(x)$. Denote by $\overline{P}$ the partition of P induced by all edges and edge prolongations (p-edges for short). $\overline{P}$ is the basic visibility structure we are working with. More precisely, we restrict the visibility in the following sense. Let $A(\overline{P})$ be the set of all atoms in $\overline{P}$. We say that a point x strictly sees y denoted by $x \approx y$ iff there are atoms $A, B \in A(\overline{P})$ such that $x \in A, y \in B$ and for all $x' \in A, y' \in B$ holds $x' \sim y'$. Subsequently we use only this stronger visibility definition, especially for $Watch(x)$.

The standard form polygons we are aiming at are described as unions of standard form convex regions, so-called 'rooms'.

We start with defining 'entrances' to rooms, compare with Fig. 2.

V-shape:
A concave vertex x defines an V-shape iff

- $cone(x)$ does not contain a convex vertex and
- $cone(x) \in A(\overline{P})$.

L-shape:

1. A concave vertex x and a set of convex vertices $\{y_i\}$ define an L-shape iff
   - $y_i \in cone(x)$ for all i and
   - $cone(x) \in A(\overline{P})$.
2. Two concave vertices x, y define an L-shape iff
   - $\{x, y\} \subset cone(x) \cap cone(y)$ and
   - $cone(x) \cap cone(y) \in A(\overline{P})$.

T-shape:
Two concave vertices x,y define a T-shape iff

- $cone(x) \cap cone(y)$ is non-empty and in $(A\overline{P})$,
- one of but not both x and y are elements of $cone(x) \cap cone(y)$,
- there is no z such that $cone(x) \cap cone(y) \subset cone(z)$.

Observe, it is possible that a vertex defines simultaneously a V-shape and is part of a T-shape. We make the following observation. If x and y form a T-shape and neither x nor y defines a V-shape and, say, $x \in cone(y)$, then there is a vertex z such that x and z also define a T-shape.In this situation we call y and z T-partners of x. Let S be the set of formal symbols $\{C, V, L, CL, T, VT\}$.
A room r consists of a cyclically ordered set $X = (x_1, x_2, \ldots, x_k)$ of polygon vertices and a sequence $shape(r) = x_1 s_1 x_2 s_2 x_3 \ldots x_k s_k$ with all $s_i \in S$ such that
(i) X spans a convex polygon on k vertices denoted by $conv(X)$.
(ii)

$$s_i = \begin{cases} C & \text{if } x_i \text{ is convex} \\ V & \text{if } x_i \text{ defines a } V\text{-shape} \\ L & \text{if } x_i, x_{i+1} \text{ define an } L\text{-shape} \\ CL & \text{if } x_i \text{ is convex and} \\ & x_i, x_{i+1} \text{ are part of an } L\text{-shape} \\ T & \text{if } x_i, x_{i+1} \text{ define a } T\text{-shape} \\ VT & \text{if } x_i \text{ defines a } V\text{- and} \\ & x_i, x_{i+1} \text{ define a } T\text{-shape} \end{cases}$$

(iii) The set $\overline{r} = conv(X) \cup \bigcup_{i \in I'} cone(x_i)$ is convex where I' denotes the set of all indices i for which $x_i \notin cone(x_j)$ for all $j \neq i$. Moreover, $\overline{r}$ has only intersections with p-edges arising from X or from a T-partner of some $x \in X$.
(iv) X is maximal with respect to (i),(ii),(iii).
Fig. 3 shows two examples. The first one with shape $x_1Tx_2Tx_3T$, which we call a wheel of rotating T's, the second is described by $x_1Tx_2Tx_3V$.
The room r will be identified with the convex set $\overline{r}$.We say that a polygon has standard form if it has a representation as a covering by rooms. It is worth mentioning that we will treat standard form polygons as combinatorial objects rather than geometrical ones. We state it as an open problem to characterize those standard forms which have a geometrical realization. That is, one wants to realize all visibilities described in the standard form but not more. The main aim of this paragraph is to sketch the proof of the following proposition.

Proposition 2.1: *For any polygon P there is an equal sized standard form polygon P' with $h(P) = h(P')$ such that a watchman solution for P' implies a solution of equal size for P.*

We start with some comments. The idea is very simple. If a vertex x is locally a V-shape but in fact its cone has some intersections with other cones then treating x like a V-shape makes the watchman problem harder since we ignore certain visibilities between atoms in $\overline{P}$. There is a simple geometrical property we have to pay attention to. Namely, there are no rooms on two vertices both defining a V-shape.
We say that two concave vertices x, y define an $\infty -$ *pair* if the intersection of their cones is nonempty in P but infinite in the plane. There is one special local configuration which has to be treated separately whenever it occurs during the transformation process given in the proof below. This situation is shown in Fig. 4. It consists of a V-shape with 4 neighboring convex regions each of them having only a V-shape as a second entrance. Then we apply a reduction technique to P which is very similar to that in [4] ,[5]. It means that we cut out from P a star decreasing its size by 3.

Proof: (sketch). First we define an auxiliary graph $F(P)$. A vertex in $F(P)$ either corresponds to the cone of a V-shape or an L-shape or is one of the remaining 'free' vertices of P. The edges of $F(P)$ are induced from the edges of P where we assume that a V-shape implies a subdivision of the opposite polygon edge. Observe that $F(P)$ can be disconnected.

Consider a concave vertex $x \in F(P)$ with neighbors y, z and the following condition (V):

> There is a segment s of an edge in $F(P)$ incident neither with y nor z such that $s \subset cone(x)$.
> If x and s are on one face of $F(P)$ then for any x' defining together with x an $\infty - pair$ such that $x' \in cone(x)$ holds : $s \not\subset cone(x')$.

Whenever condition (V) is applicable we update $F(P)$ accordingly by setting x to be a V-shape and ignoring visibilities. We end up with a connected graph. Its convex faces correspond to rooms consisting of V- and L-shapes only. The remaining vertices (which are on the non-convex faces) can be interpreted as parts of T-shapes. □

We remark that the standard form is not uniquely determined. Fig.5 shows an example of a polygon and one of its standard form representations.

3 Solving the Problem in Standard-form by a Graph Covering Approach

Let P be a polygon with n points and h holes, and let R denote the set of rooms of P. Our aim is to define a hypergraph $\mathbf{R}$ with vertex set R and an edge set corresponding to a subset of the set of entrances, such that one can find a covering of R by $\lfloor \frac{n+h}{3} \rfloor$ edges which corresponds to a guard placement. Since any entrance connects at most three rooms we can restrict ourselfs to hypergraphs of the form (R, E) where $E = E_2 \cup E_3$, $E_2 \subseteq \binom{R}{2}$, $E_3 \subseteq \binom{R}{3}$ and $\binom{R}{i}$ denotes the set of all i-subsets of R. We denote by $deg(r)$ the cardinality of the set $\{e \in E | r \in e\}$. The notion 'edge' will be used for all elements of E whereas 'hyperedge' will be used for elements of E_3 only. Furthermore a planar embedding of such hypergraphs is given in a natural way and we denote the number of inner faces by $h(\mathbf{R})$.
A subset $C \subseteq E$ is called a k-covering of $\mathbf{R} = (R, E)$ if any vertex $r \in R$ is contained in at least one $e \in C$ and $card(C) = k$. The main techniques for constructing a covering are based on a weight function ω defined on R with the property that the total weight $\omega(R) = \sum_{r \in R} \omega(r)$ is equal to n. Roughly speaking, we will define auxiliary weight functions ν and c indicating for any edge the number of polygon corners forming the corresponding entrance and for any vertex of the graph the number of convex polygon corners contained in the corresponding room exclusively. The

weight function ω will be defined by uniformly distributing the edge weights among the incident rooms. However, this can be done only if any corner belongs to at most one entrance. Unfortunately, the hypergraph formed by R and the set of all entrances is not suitable for our approach and hence we have to modify it. Especially wheels of rotating T's (see Fig. 3a) play an exceptional role and we introduce two subsets of R. These are R_0 for the rooms in the center of such a wheel and R_{rot} for neighbors of the center. First, we give a combinatorial characterization of the hypergraphs we aim at and then we sketch how to obtain them from a polygon.

Definition *A planar embedded hypergraph* $\mathbf{R} = (R, E_2 \cup E_3)$ *with a weight function* ω *on* R *is called admissible if there are (possibly empty) subsets* $R_0, R_{rot} \subseteq R$, $E_{rot} \subseteq E_3$ *and auxiliary functions* $\nu : E \longrightarrow \mathbf{N}$, $c : R \longrightarrow \mathbf{N}$ ($\mathbf{N}$ *denotes the set of nonnegative integers) with the following properties:*

(A1): $R_0 \cap R_{rot} = \emptyset$

(A2): *Any hyperedge* $e \in E_{rot}$ *is incident with some* $r \in R_0$.

(A3): *For any* $r \in R_0$ *all neighbors are contained in* R_{rot} *and* $\omega(r) = 0$. *Furthermore the number* j *of neighbors is odd and there is a cyclic ordering* $(r_1, r_2, \ldots, r_j)$ *of them such that* $\{r, r_i, r_{i+1}\} \in E_{rot}$ *for* $1 \leq i \leq j$ $(r_{j+1} := r_1)$.

(A4): *Any* $r \in R_{rot}$ *has three neighbors* r_-, r_+, r' *such that* $r_-, r_+ \in R_{rot}, r' \in R_0$ *and* $\{r, r_-, r'\}, \{r, r_+, r'\} \in E_{rot}$.

Furthermore, either there are no other edges incident with r *and* $\omega(r) = 2$ *or there is exactly one other edge incident with* r *and* $\omega(r) = 3/2$.

(A5): *For the function* ν *the following holds:*

- *if* $e \in E_{rot}$ *then* $\nu(e) = 1$,
- *if* $e \in E_3 \setminus E_{rot}$ *then* $\nu(e) \geq 2$,
- *if* $e \in E_2$ *then* $\nu(e) \geq 1$.

(A6): *For any* $r \in R \setminus (R_0 \cup R_{rot})$ *we have*

$$\omega(r) = \sum_{r \in e, e \in E_3} \frac{\nu(e)}{3} d + \sum_{r \in e, e \in E_2} \frac{\nu(e)}{2} + c(r).$$

(A7): *Defining for any* $r \in R \setminus R_0$ *the completed weight* ω^* *by*

$\omega^*(r) := c(r) + \sum_{r \in e, e \in E} \nu(e)$ *then we have*

$$\omega^*(r) \geq 3$$

Proposition 3.1: *For any polygon in standard form with* n *vertices and* h *holes there is an admissible hypergraph* $\mathbf{R} = (R, E)$ *of total weight* n *and with* $h(\mathbf{R}) = h$ *such that a covering of* R *by* k *edges corresponds to a placement of* k *guards watching the polygon.*

Proof: (sketch). We start with the hypergraph (R, E) consisting of all rooms and all entrances. No changes are necessary if for any polygon corner there is only one entrance or only one room it belongs to. Then, for any entrance e let $\nu(e)$ be the number of corners defining the local shape and for any room r let $c(r)$ be the number of convex corners which only belong to r. The sets R_0 and R_{rot} are empty and the weights of the other rooms will be defined as in point **(A6)** of the definition. It is easy to verify that **(A7)** also holds.

Suppose now, that there are polygon corners belonging to two entrances. This can happen only if for some room r there is a subsequence $xTyTz$ in *shape*(r). Suppose that $\sigma = x_i T x_{i+1} T \ldots T x_j$ is a maximal subsequence of this type and w.l.o.g. $i = 1$. Denote by $r_1, r_2, \ldots, r_j$ the neighboring rooms of r such that $x_k T x_{k+1}$ corresponds to the hyperedge $\{r, r_k, r_{k+1}\}$ for $1 \leq k \leq j-1$. If j is even then keep $\{r, r_{2i-1}, r_{2i}\}$ in E_3 for $1 \leq i \leq j/2$ and delete $\{r, r_{2i}, r_{2i+1}\}$ for $1 \leq i < j/2$. Now any polygon vertex x_k uniquely corresponds to one entrance and hence we can set $\nu(e) = 2$ for the nondeleted hyperedges.

If j is odd then the above approach can fail because it is not clear how to define the entrance between r and the last neighbor r_j. Consider the examples in Fig. 3a and 3b. In both we have the hyperedges $\{r, r_1, r_2\}, \{r, r_2, r_3\} \in E_3$. If we deleted the second one replacing it by an edge $\{r, r_3\}$ then we would have no other choise than defining $\nu(\{r, r_1, r_2\}) = 2$ and $\nu(\{r, r_3\}) = 1$. This would imply a contradiction with condition **(A7)** for the room r_3 in example 3a $(\omega^*(r_3) = 1 + 1 = 2)$, whereas there is no contradiction in example 3b $(\omega^*(r_3) = 1 + 1 + 1 = 3)$. The reason is that in example 3b x_3 forms simultaneously a T-shape and a V-shape. So, there is enough weight additional to $\nu(\{r, r_3\})$ in r_3.

In general, if the last corner x_j in σ forms simultaneously a T-shape and a V-shape (resp. a T-shape and an L-shape) then we can replace $\{r, r_{j-1}, r_j\}$ by $\{r, r_j\}$ defining $\nu(\{r, r_j\}) = 1$ (resp. $= 2$). For the remaining subsequence $x_1 T x_2 T \ldots T x_{j-1}$ we proceed as for even j. An analogous procedure can be applied if this holds for x_1 instead of x_j.

If none of the above possibilities hold then by maximality of σ this sequence represents the whole *shape*(r) and we have $\{r, r_j, r_1\} \in E_3$ (see Fig. 3a).

There is a last chance to solve the problem similarly as above, namely if for some room r_i there is enough weight additional to the unit weight from the entrance between r and r_j such that we do not get a contradiction with property **(A7)** for r_i. For example this holds if r_i has two additional entrances or one additional entrance and one convex corner. Then we delete both hyperedges containing r and r_i, replace them by $\{r, r_i\}$ with $\nu(\{r, r_i\}) = 1$ and proceed with the rest of σ as for even j. Finally, if nothing of the above was possible we insert r into R_0 and $r_1, \ldots, r_j$ into R_{rot} and all considered hyperedges to E_{rot}. It is easy to check that conditions **(A1) - (A4)** of the definition hold. Repeat the above procedure until there are no further conflicts when mapping polygon corners either to one entrance or exclusively to one room. This concludes the description how to construct an admissible hypergraph. Note that all performed modifications do not change the topological structure of the graph and hence $h(\mathbf{R}) = h$. Since the deletion of edges makes the visibility problem harder it is obvious that a covering of the vertex set by edges induces a guard placement in the polygon. □

Now, we are going to construct a covering for given hypergraphs. Mainly this will be done by several reduction techniques.

Definition: *Let* $\mathbf{R}$ *and* $\mathbf{R}'$ *be (admissible—) hypergraphs with total weight* n *(resp.* n'*) and with* h *(resp.* h'*) inner faces then* $\mathbf{R}$ *is said to be reducible to* $\mathbf{R}'$ *if any* k*-covering of* $\mathbf{R}'$ *induces a* $k + \left\lfloor \frac{n+h-n'-h'}{3} \right\rfloor$*-covering of* $\mathbf{R}$.

The first reduction type we present simplifies the structure of the graph. It is somehow strange because one must pay for this simplification with an increasing weight of the graph. We remark that this type of reductions is not necessary. However we make use of it because it supports the further investigations remarkably.

Let R_T be the set of vertices incident with some hyperedge which is not in R_{rot}.

Proposition 3.2: *Any hypergraph* $\mathbf{R} = (R, E)$ *is reducible to a hypergraph* $\mathbf{R}' = (R', E')$ *such that any* $r \in R'_T$ *is incident with only one hyperedge.*

Proof: Consider some $r \in R_T$ incident with more than one hyperedge. Take two copies r_1 and r_2 of r. Attach e_1 to r_1 and attach all other edges incident with r to r_2. Connect r_1 and r_2 via an additional vertex r^* with $c(r^*) = \nu(\{r_1, r^*\}) = \nu(\{r^*, r_2\}) = 1$ and denote this hypergraph by $\mathbf{R}^*$ (see Fig. 6). Note that $\omega(\mathbf{R}^*) = \omega(\mathbf{R}) + 3$. Consider a k-covering C of $\mathbf{R}^*$. Then at least one of the new edges is in C. Clearly, if we delete this edge from C we obtain a $(k-1)$-covering of $\mathbf{R}$. Repeat this procedure until the claim of the proposition holds. □

Now, we will show that one can reduce admissible hypergraphs step by step until only trivial situations remain. The reductions are based on the following two technical lemmas.

Lemma 3.3: *Let* $\mathbf{R} = (R, E)$ *be an admissible hypergraph,* $e = \{r_1, r_2\} \in E_2$*,* $E' = E \setminus \{e\}$ *and* $\mathbf{R}' = (R, E')$.

> *a) If we define* $c'(r_i) = c(r_i) + \nu(e)$ *for* $i = 1, 2$ *and* $c' \equiv c$ *otherwise,* $\nu' = \nu|_{E'}$ *then* $\mathbf{R}'$ *is admissible and* $\omega'(\mathbf{R}') = \omega(\mathbf{R}) + \nu(e)$
>
> *b) If* $\nu(e) \geq 2$ *and for both* r_1 *and* r_2 *there are incident edges* e_1 *and* e_2 *such that* $\nu(e_1) \geq 2$, $\nu(e_2) \geq 2$ *and if we define* $c'(r_i) = c(r_i) + 1$ *for* $i = 1, 2$*,* $c' \equiv c$ *otherwise and* $\nu' = \nu|_{E'}$ *then* $\mathbf{R}'$ *is admissible and* $\omega'(R') \leq \omega(R)$

Proof: It is sufficient to prove that **(A7)** holds for r_i $(i = 1, 2)$.

a) Note that $\omega^*(r_i) = c(r_i) + \nu(e) + \omega \geq 3$ where ω is a rest term depending on all other edges incident with r_i.

Then we have for $\mathbf{R}'$:

$\omega'^*(r_i) = c'(r_i) + \omega = c(r_i) + \nu(e) + \omega = \omega^*(r_i) \geq 3$.

b) Note that $c'(r_i) = c(r_i) + 1 \geq 1$ and hence $\omega'^*(r_i) \geq c'(r_i) + \nu(e_i) \geq 1 + 2 = 3$. □

Lemma 3.4: *Let* $\mathbf{R} = (R,E)$ *be an admissible hypergraph,* $e = \{r_1, r_2, r_3\} \in E_3$*,* $E' = (E \setminus \{e\}) \cup \{\{r_2, r_3\}\}$ *and* $\mathbf{R}' = (R, E')$*. Moreover define:*

$$c'(r_1) = c(r_1) + \nu(e), c' \equiv c \text{ otherwise}$$

$$\nu'(\{r_2, r_3\}) = \nu(e), \nu' \equiv \nu \text{ otherwise.}$$

Then $\mathbf{R}'$ *is admissible and* $\omega'(\mathbf{R}') = \omega(\mathbf{R}) + \nu(e)$.

The proof is analogous to the previous one.

Remark : Note that $\mathbf{R}'$ is possibly disconnected and that the admissibility of $\mathbf{R}'$ implies the admissibility of all its connected components.

As intuition suggests, we define a path to be a sequence $r_0, e_1, r_1, e_2, \ldots, e_j, r_j$ such that $r_i \in e_{i+1}$ for $0 \leq i < j$ and $r_i \in e_i$ for $0 < i \leq j$. A path is called a cycle if $r_0 = r_j$ and no other vertex as well as no edge occurs twice. Moreover we require that the only nonempty intersections of two edges consist of the vertices between consecutive edges. The last requirement ensures that the closed path around the center of a wheel of rotating T's is not regarded as a cycle. Remark that an admissible hypergraph $\mathbf{R}$ contains a cycle if and only if $h(\mathbf{R}) > 0$.

Proposition 3.5: *Any admissible hypergraph* $\mathbf{R}$ *with* $h(\mathbf{R}) > 0$ *is reducible to some admissible hypergraph* $\mathbf{R}'$ *with* $h(\mathbf{R}') < h(\mathbf{R})$.

Proof: W.l.o.g. we can assume that the reduction of Proposition 3.2 has been already applied to $\mathbf{R}$. Then any cycle in $\mathbf{R}$ contains at least one edge $e \in E_2$. Suppose moreover that $\nu(e) = 1$. Then we can apply claim a) of Lemma 3.3 and get a graph $\mathbf{R}'$ with

$$\omega'(\mathbf{R}') + h(\mathbf{R}') = \omega(\mathbf{R}) + 1 + h(\mathbf{R}) - 1 = \omega(\mathbf{R}) + h(\mathbf{R})$$

Clearly, any covering of $\mathbf{R}'$ is also a covering of $\mathbf{R}$.

If there is no edge $e \in E_2$ with $\nu(e)$ in the cycle for any $e_i \in E_2$ holds $\nu(e_{i-1}) \geq 2$ and $\nu(e_{i+1}) \geq 2$. Applying claim b) of Lemma 3.3 we obtain the graph $\mathbf{R}'$ we are looking for. □

Corollary 3.6: *Any admissible hypergraph* $\mathbf{R}$ *is reducible to some admissible hypergraph* $\mathbf{R}'$ *with* $h(\mathbf{R}') = 0$.

Now we have two possibilities to finish the proof of our main theorem. The first one is to show that any admissible hypergraph $\mathbf{R}$ with $h(\mathbf{R}) = 0$ represents the standard form of some polygon with $\omega(\mathbf{R})$ corners and without holes (then we are in the classical situation with the known $\lfloor \frac{n}{3} \rfloor$ bound and we are done). Indeed this fact can be proved in contrast to the general case when $h(\mathbf{R}) > 0$. (Consider e.g. a graph representing a tetrahedron such that $\nu(e) = 1$ for any edge and $c(r) = 0$ for any vertex; see also the last problem in paragraph 4). The second way we sketch here consists in reducing these hypergraphs completely.

A vertex r in a hypergraph is called a leaf if $deg(r) = 1$. Remark that a leaf r can have two neighbors if the edge incident with r is a hyperedge.

Proposition 3.7: *Let* $\mathbf{R} = (R, E)$ *be an admissible hypergraph and* r *a leaf of* $\mathbf{R}$. *Then* $\mathbf{R}$ *is reducible to some admissible subgraph* $\mathbf{R}' \subset \mathbf{R}$ *such that* r *is not in* $\mathbf{R}'$.

Proof: (sketch). Note that by definition of leafs $r \notin R_{rot} \cup R_0$ and let e be the unique edge incident with r. We have to distinguish the following three cases:

(a) $e = \{r, r'\} \in E_2$ and $r' \notin R_{rot}$

(b) $e = \{r, r', r"\} \in E_3$ (then clearly $r', r" \notin R_{rot}$)

(c) $e = \{r, r'\} \in E_2$ and $r' \in R_{rot}$

(ad a) Let $\mathbf{R}'$ be the subgraph induced by $R' = R \setminus \{r, r'\}$ and construct its auxiliary weight functions as follows: Start from $\mathbf{R}$ and delete any edge e' incident with r' by claim a) of Lemma 3.3 if $e' \in E_2$ and by Lemma 3.4 if $e' \in E_3$. Finally we delete the isolated vertices r and r'. Clearly $\omega'(\mathbf{R}') = \omega(\mathbf{R}) - c(r) - \nu(e) - c(r') \leq \omega(\mathbf{R}) - 3$ and any k-covering of $\mathbf{R}'$ induces a $k+1$-covering of $\mathbf{R}$ (adding e to the covering set).

(ad b) Analogously for the subgraph induced by $R' = R \setminus \{r, r', r"\}$

(ad c) Consider the general situation characterising a wheel of rotating T's i.e., a center $r_0 \in R_0$ and its neighbors $r_1, r_2, \ldots, r_j \in R_{rot}$ and assume w.l.o.g. that $r' = r_1$. The general difficulty with covering these vertices comes from the necessity to include at least one of the inner hyperedges in the covering set since otherwise the center would remain uncovered. So in contrast to the other cases we construct $\mathbf{R}' = (R', E')$ as follows:

We set $R' = R \setminus \{r, r_0, r_1, r_2, r_3\}$. If r_2 (resp r_3) has a neighbor r_2' (resp r_3') $\in R'$ then by the definition of R_{rot} $\nu(\{r_2, r_2'\} = 1$ and applying claim a) of Lemma 3.3 we update $c'(r_2') = c(r_2') + 1$ (analogously for r_3'). Moreover if $j > 3$ then delete the remaining inner hyperedges replacing them by $\frac{j-3}{2}$ new edges $\{r_4, r_5\}, \{r_6, r_7\}, \ldots, \{r_{j-1}, r_j\}$ and define ν to be 2 for each of them.

It is not hard to check that R' is admissible, $\omega'(\mathbf{R}') \leq \omega(\mathbf{R}) - 6$ and that any k-covering of $\mathbf{R}'$ induces a $k+2$-covering of $\mathbf{R}$ (adding $\{r, r_1\}$ and $\{r_0, r_2, r_3\}$ to the covering set). □

Now we can reduce hypergraphs until there are neither cycles nor leafs left. We observe that there are only three possible structures of the remaining graphs:

- Isolated vertices,
- Isolated wheels of rotating T's,
- Trees (i.e., hypergraphs without cycles) such that instead of leafs there are wheels of rotating T's with one outside edge only (which we will call pseudoleafs).

Note that the first and the second case are trivial. Now consider a wheel of rotating T's ($r_0 \in R_0; r_1, \ldots, r_j \in R_{rot}$) which is a pseudoleaf in a tree $\mathbf{R}$ and let r_1 be the only vertex with an outside edge. We construct a subgraph R' by deleting $r_0, r_2, r_3, \ldots, r_j$ and all inner hyperedges of the star and we define $c'(r_1) = 2$ for the vertex r_1 which remains in $\mathbf{R}'$. It is easy to check that $\mathbf{R}'$ is admissible, $\omega'(\mathbf{R}') \leq \omega(\mathbf{R}) - \frac{3(j-1)}{2}$ and that any k-covering of $\mathbf{R}'$ induces a $(k + \frac{j-1}{2})$-covering of $\mathbf{R}$. Note that r_1 is a leaf in $\mathbf{R}'$ and hence we can continue with applying Proposition 3.7.

The construction in Proposition 3.7 provides an appropriate covering of $\mathbf{R}$, which immediately induces a placement for the guards. Hence we can conclude with the main Theorem:

Theorem: *Any polygon with h holes and size n can be watched by at most $\lfloor \frac{n+h}{3} \rfloor$ guards.*

In the final section, we discuss how to realize the placement algorithmically and furthermore some related problems and conjectures, which have been arisen during this research.

4 Discussion and Open Problems

Algorithmic aspects

During the placement algorithm, we have to compute the cone for each vertex, we have to identify V-, L- and T-shapes and thee rooms. Using the method described in proposition 2.1, we derive the standard form from the original polygon P. Then we construct the weights for the appropriate hypergraph and compute the covering.

The first two tasks can be performed by standard operations on line segments. The cones and V-, L- and T-shapes can easily be constructed in time $O(n^3)$. The computation of the rooms is more difficult. We compute them one by one. Starting with a triangle not covered yet by other rooms, we extend the sequence of adjacent vertices more and more until it is maximal. In each extension step a test of all vertices has to be executed, whether a further extension is possible or not. The test takes time $O(n)$. Since the test must be executed only once per vertex and room, the time to construct all rooms is $O(n^3)$.

To bring the problem into standard form, we have to apply a reduction technique as in [4],[5] such that the situation from Fig. 4 will not arise any more (time $O(n^2)$). In the construction of the standard form, we update the graph $F(P)$ by setting vertices x to be V-shapes if x satisfy the condition (V). The corresponding test takes $O(n)$ time such that we can perform the complete transformation to a standard form polygon in time $O(n^2)$.

To bring the corresponding hypergraph in an admissible form, to compute the appropriate weights and to find the covering, we have to execute the reduction techniques from Proposition 3.2 - 3.7, which can be done easily in time $O(n^2)$.

Summing up, we conclude with a total running time of $O(n^3)$. Note that the implementation presented here is quite straightforward and it might be improvable to $O(n^2 \log n)$, although we did not work yet through the details.

Finally, we give some remarks to related aspects which imply some open questions.

Point guards vs. vertex guards

We have seen that in standard form polygons $\lfloor \frac{n+h}{3} \rfloor$ vertex guards are sufficient. In general, however, we can prove the result for point guards only, compare with Fig. 7. Nevertheless, we also conjecture $\lfloor \frac{n+h}{3} \rfloor$, as in [7], vertex guards to be the exact bound for any polygon as well as $\lfloor \frac{n+h}{4} \rfloor$ for rectilinear polygons.

Worst case bounds as a function of n only

If we ask for worst case bounds formulated in the number of vertices only, then our main result easily implies that at most $\lfloor \frac{4n}{9} \rfloor$ point guards are necessary since each hole is defined by at least three vertices. This improves on the previous trivial bound of $\lfloor \frac{5n}{9} \rfloor$ vertex guards. On the other hand, we have the lower bound of $\lfloor \frac{3n}{8} \rfloor$, see Fig.1. Our conjecture is that $min(\lfloor \frac{n+h}{3} \rfloor, \lfloor \frac{3n}{8} \rfloor)$ vertex guards always suffice.

General via rectilinear polygons

One could try to prove the $\lfloor \frac{3n}{8} \rfloor$ bound at least for point guards in the following way. Translate a given gallery problem into a more difficult rectilinear one by adding at most $\lfloor \frac{n}{2} \rfloor$ new vertices. The hypergraph $\mathbf{R}$ seems to be a good candidate to formalize this idea. We were not able to find a counterexample to this approach.

Standard form polygons

Find a combinatorial characterization of those standard form polygons which have a geometrical realization.

Note added in proof

As we have recently learnt I. Bjorling-Sachs and D.L. Souvaine independently proved the main result of our paper (see Technical Report LCSR-TR-165, May 1991, Rutgers University).

References

[1] A. Aggarwal, *The art gallery theorem: its variations, applications, and algorithmic aspects*, Ph.D.thesis, The John Hopkins University, 1984.

[2] V. Chvatal, *A combinatorial theorem in plane geometry.*, J. Combin. Theory Ser. B, 1975, Vol. 18, pp. 39-41.

[3] Z. Füredi and D. Kleitmann, *The Prison Yard Problem*, to appear in Combinatorica.

[4] F. Hoffmann, *On the rectilinear art gallery problem*, Proc. ICALP'90, LNCS 443, 1990, pp. 717-728.

[5] F. Hoffmann and M. Kaufmann, *On the rectilinear art gallery problem - algorithmic aspects*, Proc. WG'90, LNCS 484, 1991, pp. 239-250.

[6] J. Kahn and M. Klawe and D. Kleitman, *Traditional galleries require fewer watchmen*, SIAM Journal of Alg. Disc. Math., 1983, pp. 194-206.

[7] J. O'Rourke, *Art gallery theorems and algorithms*, Oxford University Press, New York, 1987.

[8] T. Shermer, *Triangulation graphs that require extra guards*, NYIT, Computer Graphics tech. report No.3D-13, 1984.

[9] T. Shermer, *Recent results in art galleries*, Simon Fraser University, Technical report 1990, TR 90-10.

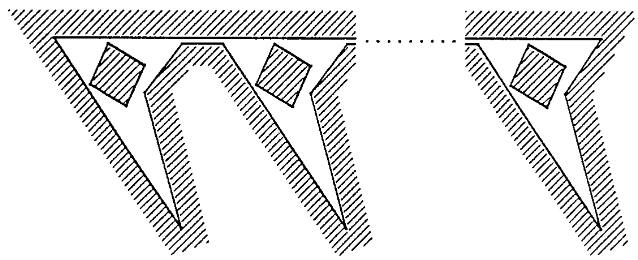


Figure 1: $\lfloor\frac{n+h}{3}\rfloor$ point guards are necessary

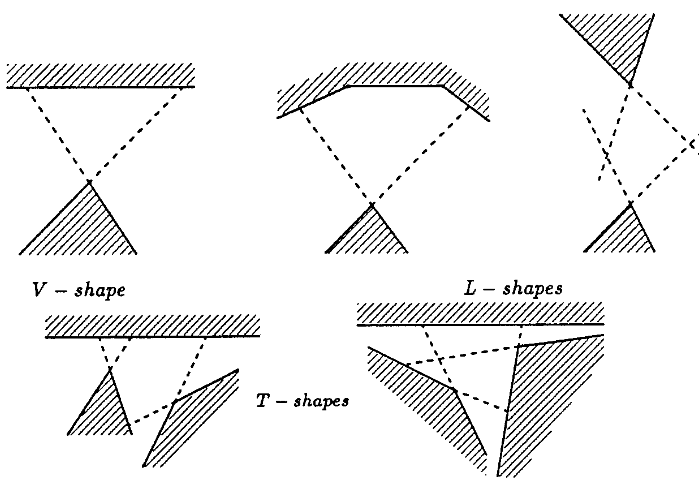
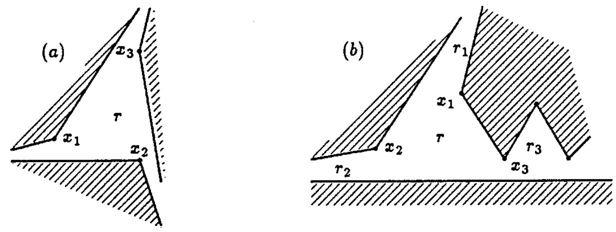


Fig. 2: V-, L- and T-shapes



$shape^*(r) = x_1Tx_2Tx_3T$ $\qquad$ $shape^*(r) = x_1Tx_2Tx_3V$

Fig. 3: Examples of rooms

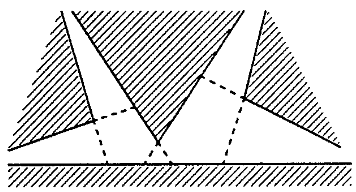


Fig. 4: A special local configuration

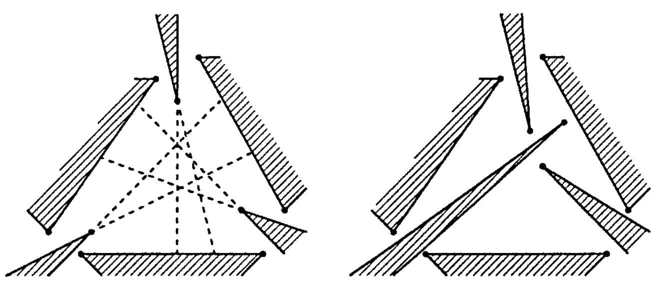


Fig. 5: Example of a local situation and its standard form

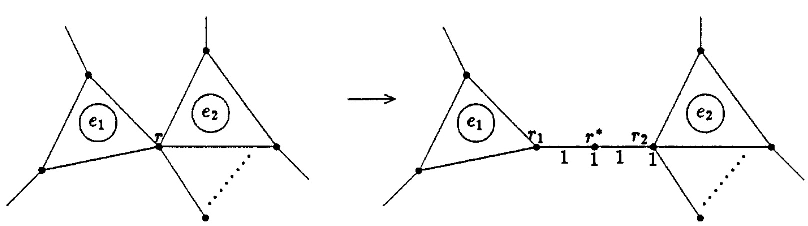
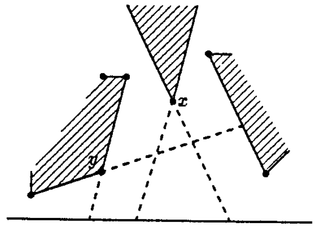


Fig. 6: The reduction such that each vertex r is incident with only one hyperedge



In any standard form x becomes a $V - shape$, (x, y) a $T - shape$. But then it can happen that the algorithm places a watchman in $cone(x) \cap cone(y)$.

Fig. 7: Points guards cannot be avoided.