Dynamic Regression

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Motivation

Consider the following 'dynamic' model

$$Y_t = \beta_1 + \beta_2 X_t + \gamma Y_{t-1} + u_t,$$

where $t = 1, \ldots, T$ indicates time-series observations and all other classical assumptions hold, and $|\gamma| < 0$.

- Dynamic model: observations in one period are linked to those in some other period.
- For example, this may be a model for cigarettes consumption: consumption (Y_t) today is explicitly linked to past consumption Y_{t-1}, even after controlling for other determinants X_t (current income, for example).

The ADL Model

A very simple dynamic structure that cointains many interesting dynamic models as particular examples is the *autorregresive distributed lag (ADL)* model:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \gamma Y_{t-1} + \epsilon_t, \qquad |\gamma| < 1, \ \ t = 1, \ldots, T$$

where all other classical assumptions hold.

The model can be estimated by standard OLS regressing Y_t on X_t, X_{t-1} and Y_{t-1}.

Steady state

A steady state is a stable situation where all variables take constant values and ϵ_t is set to its expected value.

$$\tilde{Y} = \alpha + \beta_0 \tilde{X} + \beta_1 \tilde{X} + \gamma \tilde{Y}$$

Hence

$$\tilde{Y} = \frac{\alpha}{1-\gamma} + \frac{\beta_0 + \beta_1}{1-\gamma} \tilde{X}$$

$$\tilde{Y} = \tilde{\alpha} + \tilde{\beta} \tilde{X}$$

is the long run relationship. And

$$\frac{\partial \tilde{Y}}{\partial \tilde{X}} = \tilde{\beta} = \frac{\beta_0 + \beta_1}{1-\gamma}$$

is the *long run* effect of X

An alternative approach to the long run relationship is as follows. Consider the effect of increasing X marginally at t. The effects will be

At t: β_0.
At $t+1$: $\beta_1 + \beta_0\gamma$.
At $t+2$: $(\beta_1 + \beta_0\gamma)\gamma$.
At $t+3$: $(\beta_1 + \beta_0\gamma)\gamma^2$.
$\ldots$

Then the sum of all effects is

$$\frac{\beta_0 + \beta_1}{1-\gamma},$$

the long run effect.

Particular cases

Static regression ($\beta_1 = \gamma = 0$): The simple model introduced in Chapter 2 is an extreme case with no dynamics.

The simple model with AR(1) autocorrelation ($\beta_0 = \beta_1/\gamma$): It is interesting to see that the simple linear model with AR(1) autocorrelation also arises as a particular case of the ADL model. Let us rewrite it using the following notation:

$$\begin{aligned} Y_t &= b_0 + b_1 X_t + u_t, \qquad t = 1, \ldots, T \qquad (1) \\ u_t &= \phi u_{t-1} + \epsilon_t, \qquad |\phi| < 1 \qquad (2) \end{aligned}$$

Using a trick from the previous class

$$Y_t = (b_0 - \phi b_0) + b_1 X_t - \phi b_1 + \phi Y_{t-1} + \epsilon_t$$

This can be written as:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \gamma Y_{t-1} + \epsilon_t$$

where $\alpha \equiv b_0 - \phi b_0$, $\beta_0 \equiv b_1$, $\beta_1 \equiv -\phi b_1$ and $\gamma \equiv \phi$. But according to the definitions, for the specification to hold, we must impose the restriction $\beta_0 = -\beta_1/\gamma$. Hence the linear model with serial correlation actually corresponds to a particular configuration of the ADL model.

Partial adjustment ($\beta_1 = 0$): It is better to understand this specification through and example.

Y_t^* is the desired demand for money in real terms, and consider a money demand function:

$$Y_t^* = b_0 + b_1 X_t + u_t \tag{3}$$

where X_t is real income.

Now assume that due to market imperfections or adjustment costs agents cannot realize their desired holdings immediately, instead, the observed demand for money (Y_t) is determined as:

$$Y_t - Y_{t-1} = \delta\,(Y_t^* - Y_{t-1}), \qquad 0 < \delta \leq 1 \tag{4}$$

This is the partial adjustment hypothesis.

- $\delta = 1$ means *observed* changes are equal to *desired* changes in money demand: full adjustment.
- $\delta < 1$ agents are able to fulfill they desired change only partially. δ is usually called the adjustment coefficient.

Estimation: We cannot estimate the demand model directly: Y^* is not observable.
Replace Y_t^* and solve for Y_t (we will leave the simple algebraic steps for you to check) to get:

$$
\begin{aligned}
Y_t &= \delta b_0 + \delta b_1 X_t + (1-\delta) Y_{t-1} + \delta u_t && (5) \\
&= \alpha + \beta_0 X_t + \gamma Y_{t-1} + \epsilon_t && (6)
\end{aligned}
$$

where $\alpha \equiv \delta b_0$, $\beta_0 \equiv \delta b_1$, $\gamma \equiv 1-\delta$ and $\epsilon_t \equiv \delta u_t$ This is the ADL model with $\beta_1 = 0$. Can be estimated by OLS.

Empirical example: dynamic demand for cigarettes

- Explained variable: lsales (log of sales)
- Explanatory variables:

  lt = log of tax rate
  lprice = log of net price
  lsales1 = lsales in previous period
  d1, d2,d2 = seasonal dummies
  lwindup = log of industrial wages
  d882, d893, d901 = dummies for "anomalous" periods.

We fitted a partial adjustment model: addictive behavior

Coefficient	Std. Err	t- stat	p-value	
lt	-0.370	0.059	-6.300	0.000
Lprice	0.026	-2.530	0.015	0.000
Lsales1	0.406	0.065	6.250	0.000
d1	-0.084	0.009	-9.740	0.000
d2	-0.105	0.007	-14.900	0.000
d3	-0.061	0.007	-8.500	0.000
lwindup	0.195	0.038	5.210	0.000
d882	-0.087	0.021	-4.180	0.000
d893	-0.157	0.022	-7.260	0.000
d901	-0.088	0.020	-4.430	0.000
cons	11.043	1.205	9.170	0.000

Comments

- Sales are strongly related to its past (adictive behavior).
- The short run effect of price (short run price elasticity) is -0.066.
- The long-run effect of price (long run price elasticity) is -0.066/(1-0.406) $=$ 0.111.
- The short run effect of income (short run price elasticity) is 0.195
- The long-run effect of income (long run price elasticity) is 0.195/(1-0.406) $=$0.328
- The adjustment coefficient of the partial adjustmen process is: 1-0.406 $=$0.594

Error correction

Consider again the ADL model:

$$Y_t = \alpha + \gamma Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + u_t$$

Substract Y_{t-1} in both sides:

$$\Delta Y_t = \alpha - (1-\gamma) Y_{t-1} + + \beta_0 X_t + \beta_1 X_{t-1} + u_t$$

Now add and substract $\beta_0 X_{t-1}$ in both sides:

$$
\begin{aligned}
\Delta Y_t &= \alpha - (1-\gamma)Y_{t-1} + \beta_0 X_t - \beta_0 X_{t-1} + \beta_0 X_{t-1} + \beta_1 X_{t-1} + u_t \\
&= \alpha - (1-\gamma)Y_{t-1} + \beta_0 \Delta X_t + (\beta_0 + \beta_1) X_{t-1} + u_t \\
&= \beta_0 \Delta X_t - (1-\gamma)Y_{t-1} + \alpha + (\beta_0 + \beta_1) X_{t-1} + u_t \\
&= \beta_0 \Delta X_t - (1-\gamma)\left[Y_{t-1} - \frac{\alpha}{1-\gamma} - \frac{(\beta_0 + \beta_1)}{1-\gamma} X_{t-1}\right] + u_t \\
&= \beta_0 \Delta X_t + \delta Z_{t-1} + u_t
\end{aligned}
$$

$Z_{t-1} \equiv Y_{t-1} - \alpha/(1-\gamma) - [(\beta_0 + \beta_1)/(1-\gamma)]X_{t-1}$, and $\delta \equiv -(1-\gamma)$.

This is the error correction representation of the ADL model.

$$\Delta Y_t = \beta_0 \Delta X_t + \delta Z_{t-1} + u_t$$

Intuition:

- Look at Z_{t-1}. Recalling the long-run expression of the ADL model we get:

$$Z_{t-1} = Y_{t-1} - \tilde{\alpha} - \tilde{\beta} X_{t-1}$$

- Z_{t-1} measures how far was the system from its long run equilibrium value.
- Error correction: Y moves due to changes in X (short run) and according to how far it is from the long-run target (error correction).

Error correction and cointegration

Consider the EC representation of the ADL model

$$\Delta Y_t = \beta_0 \Delta X_t + \delta Z_{t-1} + u_t$$

and suppose both Y_t and X_t are $I(1)$.

- ΔY_t is $I(0)$.
- ΔX_t and u_t are $I(0)$.
- The model is not coherent unless....

The Granger representation theorem: if Y_t and X_t are $I(1)$, the ADL representation is coherent (both sides are $I(0)$) *if and only if* Y_t and X_t are cointegrated.

Intuition

- Cointegration: Z_t is $I(0)$ (by cointegration) and the system is balanced.
- Coherent ADL: Z_t is $I(0)$ then Y_t and X_t are cointegrated.

The ADL provides an estimable model for cointegrated variables, that captures long and short run dynamics.

Estimation strategies

Under cointegration

1. Direct: Estimate the ADL model directly and derive all relevant coefficients (long run, short run) using the error correction representation.
2. Two-stage: First estimate the long run 'cointegrating' relationship by OLS
$$\tilde{Y} = \tilde{\alpha} + \tilde{\beta}\tilde{X}$$
and obtain $\hat{Z}_t = Y_t - \hat{\alpha} - \hat{\beta}X_t$. Then estimate the error correction model
$$\Delta Y_t = \beta_0 \Delta X_t + \delta Z_{t-1} + u_t$$
replacing Z_{t-1} by $\hat{Z}_{t-1}$.