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THERMODYNAMIC FORMALISM FOR SUBSYSTEMS OF EXPANDING THURSTON MAPS II

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ABSTRACT. Expanding Thurston maps were introduced by M. Bonk and D. Meyer with motivation from complex dynamics and Cannon's conjecture from geometric group theory via Sullivan's dictionary. In this paper, we study subsystems of expanding Thurston maps motivated via Sullivan's dictionary as analogs of some subgroups of Kleinian groups. We prove the uniqueness and various ergodic properties of the equilibrium states for strongly primitive subsystems and real-valued Hölder continuous potentials, and establish the equidistribution of preimages of subsystems with respect to the equilibrium states. Here, the sphere S^2 is equipped with a natural metric, called a visual metric, introduced by M. Bonk and D. Meyer. As a result, for strongly primitive subsystems of expanding Thurston maps without periodic critical point, we obtain a level-2 large deviation principle for Birkhoff averages and iterated preimages.

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1. Introduction

A Thurston map is a (non-homeomorphic) branched covering map on a topological 2-sphere S^2 such that each of its critical points has a finite orbit (postcritically-finite). The most important examples are given by postcritically-finite rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. While Thurston maps are purely topological objects, a deep theorem due to W. P. Thurston characterizes Thurston maps that are, in a suitable sense, described in the language of topology and combinatorics, equivalent to postcritically-finite rational maps (see [DH93]). This suggests that for the relevant rational maps, an explicit analytic expression is not so important, but rather a geometric-combinatorial description. This viewpoint is natural and fruitful for considering more general dynamics that are not necessarily conformal.

In the early 1980s, D. P. Sullivan introduced a "dictionary" that is now known as *Sullivan's dictionary*, which connects two branches of conformal dynamics, iterations of rational maps and actions of Kleinian groups. Under Sullivan's dictionary, the counterpart to Thurston's theorem in geometric group theory is Cannon's Conjecture [Can94]. An equivalent formulation of Cannon's Conjecture, viewed from a quasisymmetric uniformization perspective ([Bon06, Conjecture 5.2]), predicts that if the boundary at infinity $\partial_\infty G$ of a Gromov hyperbolic group G is homeomorphic to S^2, then $\partial_\infty G$ equipped with a visual metric is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$.

Inspired by Sullivan's dictionary and their interest in Cannon's Conjecture, M. Bonk and D. Meyer [BM10, BM17], as well as P. Haïssinsky and K. M. Pilgrim [HP09], studied a subclass of Thurston maps, called *expanding Thurston maps*, by imposing some additional condition of expansion. These maps are characterized by a contraction property for inverse images (see Subsection 3.2 for the precise definition). In particular, a postcritically-finite rational map on $\widehat{\mathbb{C}}$ is expanding if and only if its Julia set is equal to $\widehat{\mathbb{C}}$. For an expanding Thurston map on S^2, we can equip S^2 with a natural class of metrics d, called *visual metrics*, that are quasisymmetrically equivalent to each other and are constructed in a similar way as the visual metrics on the boundary $\partial_\infty G$ of a Gromov hyperbolic group G (see [BM17, Chapter 8] for details, and see [HP09] for a related construction). In the language above, the following theorem was obtained in [BM10, BM17, HP09], which can be seen as an analog of Cannon's conjecture for expanding Thurston maps.

Theorem (M. Bonk & D. Meyer [BM10, BM17]; P. Haïssinky & K. M. Pilgrim [HP09])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with no periodic critical points and d be a visual metric for f. Then f is topologically conjugate to a rational map if and only (S^2, d) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$.*

The dynamical systems that we study in this paper are called *subsystems* of expanding Thurston maps (see Subsections 3.3 for a precise definition), inspired by a translation of the notion of subgroups from geometric group theory via Sullivan's dictionary. To reveal the connections between subsystems and subgroups, we first quickly review some backgrounds in Gromov hyperbolic groups and recall the notion of *tile graphs* for expanding Thurston maps (see [BM17, Chapters 4 and 10] for details).

Let G be a Gromov hyperbolic group and S a finite generating set of G. Then the *Cayley graph* $\mathcal{G}(G,S)$ of G is Gromov hyperbolic with respect to the word-metric. The boundary at infinity of G is defined as $\partial_\infty G \coloneqq \partial_\infty \mathcal{G}(G,S)$, which is well-defined since a change of the generating set induces a quasi-isometry of the Cayley graphs.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ a Jordan curve with $\operatorname{post} f \subseteq S^2$. An associated *tile graph* $\mathcal{G}(f,\mathcal{C})$ is defined as follows. Its vertices are given by the tiles in the cell decompositions $\mathcal{D}^n(f,\mathcal{C})$ on all levels $n \in \mathbb{N}_0$. We consider $X^{-1} \coloneqq S^2$ as a tile of level -1 and add it as a vertex. One joins two vertices by an edge if the corresponding tiles intersect and have levels differing by at most 1 (see [BM17, Chapter 10] for details). Then the graph $\mathcal{G}(f,\mathcal{C})$

is Gromov hyperbolic and its boundary at infinity $\partial_\infty \mathcal{G} \coloneqq \partial_\infty \mathcal{G}(f, \mathcal{C})$ is well-defined and can be naturally identified with S^2. Moreover, under this identification, a metric on $\partial_\infty \mathcal{G} \cong S^2$ is visual in the sense of Gromov hyperbolic spaces if and only if it is visual in the sense of expanding Thurston maps.

From the point of view of tile graphs and Cayley graphs, roughly speaking, for expanding Thurston maps, 1-tiles together with the maps restricted to those tiles play the role of generators for Gromov hyperbolic groups. For example, one can construct the original expanding Thurston map f from all its 1-tiles and the maps restricted to those tiles. If we start with all n-tiles for some $n \in \mathbb{N}$, then we get an iteration f^n of f, which corresponds to a finite index subgroup of the original group in the group setting. Inspired by this similarity, it is natural to investigate more general cases, for example, a map generated by some 1-tiles, which leads to our study of subsystems. Moreover, the notion of tile graphs can be easily generalized to subsystems, and the similar identifications hold with S^2 generalized to the tile maximal invariant sets associated with subsystems.

Under Sullivan's dictionary, an expanding Thurston map corresponds to a Gromov hyperbolic group whose boundary at infinity is S^2. In this sense, a subsystem corresponds to a Gromov hyperbolic group whose boundary at infinity is a subset of S^2. In particular, for Gromov hyperbolic groups whose boundary at infinity is a Sierpiński carpet, there is an analog of Cannon's conjecture—the Kapovich–Kleiner conjecture. It predicts that these groups arise from some standard situation in hyperbolic geometry. Similar to Cannon's conjecture, one can reformulate the Kapovich–Kleiner conjecture in an equivalent way as a question related to quasisymmetric uniformization. For subsystems, it is easy to find examples where the tile maximal invariant set is homeomorphic to the standard Sierpiński carpet (see Subsection 3.3 for examples of subsystems). In this case, an analog of the Kapovich–Kleiner conjecture for subsystems is established in [BLL24].

In this paper, we study the dynamics of subsystems of expanding Thurston maps from the point of view of ergodic theory. Ergodic theory has been an essential tool in the study of dynamical systems. The investigation of the existence and uniqueness of invariant measures and their properties has been a central part of ergodic theory. However, a dynamical system may possess a large class of invariant measures, some of which may be more interesting than others. It is, therefore, crucial to examine the relevant invariant measures.

The *thermodynamic formalism* serves as a viable mechanism for generating invariant measures endowed with desirable properties. More precisely, for a continuous transformation on a compact metric space, we can consider the *topological pressure* as a weighted version of the *topological entropy*, with the weight induced by a real-valued continuous function, called *potential*. The Variational Principle identifies the topological pressure with the supremum of its measure-theoretic counterpart, the *measure-theoretic pressure*, over all invariant Borel probability measures [Bow75, Wal82]. Under additional regularity assumptions on the transformation and the potential, one gets the existence and uniqueness of an invariant Borel probability measure maximizing the measure-theoretic pressure, called the *equilibrium state* for the given transformation and the potential. The study of the existence and uniqueness of the equilibrium states and their various other properties, such as ergodic properties, equidistribution, fractal dimensions, etc., has been the primary motivation for much research in the area.

The ergodic theory for expanding Thurston maps has been investigated in [Li17] by the first-named author of the current paper. In [Li18], the first-named author of the current paper works out the thermodynamic formalism and investigates the existence, uniqueness, and other properties of equilibrium states for expanding Thurston maps. In particular, for each expanding Thurston map without periodic critical points, by using a general framework devised by Y. Kifer [Kif90], the first-named author of the current paper establishes level-2 large deviation principles for iterated preimages and periodic points in [Li15].

The current paper is the second in a series of two papers (together with [LSZ24]) investigating the ergodic theory of subsystems of expanding Thurston maps. In the previous paper [LSZ24], we investigated the thermodynamic formalism and demonstrated the existence of equilibrium states for subsystems and real-valued Hölder continuous potentials. In the present paper, we study the uniqueness and ergodic properties of equilibrium states for subsystems and real-valued Hölder continuous potentials. Based on the existence and uniqueness of equilibrium states, for subsystems of expanding Thurston maps without periodic critical point, we establish level-2 large deviation principles for iterated preimages.

1.1. **Main results.** In order to state our results more precisely, we quickly review some key concepts. We refer the reader to Section 3 for more details.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. We say that a map $F\colon \operatorname{dom}(F) \to S^2$ is a *subsystem of f with respect to $\mathcal{C}$* if $\operatorname{dom}(F) = \bigcup \mathfrak{X}$ for some non-empty subset $\mathfrak{X} \subseteq \mathbf{X}^1(f, \mathcal{C})$ and $F = f|_{\operatorname{dom}(F)}$. We denote by $\operatorname{Sub}(f, \mathcal{C})$ the set of all subsystems of f with respect to $\mathcal{C}$.

Consider a subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. For each $n \in \mathbb{N}_0$, we define the *set of n-tiles of F* to be

$$\mathfrak{X}^n(F, \mathcal{C}) \coloneqq \{X^n \in \mathbf{X}^n(f, \mathcal{C}) : X^n \subseteq F^{-n}(F(\operatorname{dom}(F)))\},$$

where we set $F^0 \coloneqq \operatorname{id}_{S^2}$ when $n = 0$. We call each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ an *n-tile* of F. We define the *tile maximal invariant set* associated with F with respect to $\mathcal{C}$ to be

$$\Omega(F, \mathcal{C}) \coloneqq \bigcap_{n\in\mathbb{N}} \Bigl(\bigcup \mathfrak{X}^n(F, \mathcal{C})\Bigr),$$

which is a compact subset of S^2.

One of the key properties of $\Omega(F, \mathcal{C})$ is $F(\Omega(F, \mathcal{C})) \subseteq \Omega(F, \mathcal{C})$ (see Proposition 3.15 (ii)). Therefore, we can restrict F to $\Omega(F, \mathcal{C})$ and consider the map $F|_{\Omega(F,\mathcal{C})}\colon \Omega(F, \mathcal{C}) \to \Omega(F, \mathcal{C})$ and its iterations.

In the following theorem, we investigate the uniqueness and various ergodic properties of the equilibrium states for strongly primitive subsystems (see Definition 3.17 in Subsection 3.3) and Hölder continuous potentials, and establish the equidistribution of preimages of subsystems with respect to the equilibrium states.

Theorem 1.1. *Let $f\colon X \to X$ be either an expanding Thurston map on a topological 2-sphere $X = S^2$ equipped with a visual metric or a postcritically-finite rational map with no periodic critical points on the Riemann sphere $X = \widehat{\mathbb{C}}$ equipped with the chordal metric. Let $\phi\colon X \to \mathbb{R}$ be Hölder continuous. Let $\mathcal{C} \subseteq X$ be a Jordan curve containing $\operatorname{post} f$ with the property that $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider a strongly primitive subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$.*

Then there exists a unique equilibrium state $\mu_{F,\phi}$ for $F|_\Omega$ and $\phi|_\Omega$. Moreover, $\mu_{F,\phi}$ is non-atomic and the measure-preserving transformation $F|_\Omega$ of the probability space $(\Omega, \mu_{F,\phi})$ is forward quasi-invariant, exact, and in particular, mixing and ergodic.

In addition, the preimages points of F are equidistributed with respect to $\mu_{F,\phi}$, i.e., for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X and each sequence $\{c_n\}_{n\in\mathbb{N}}$ of colors in $\{b, w\}$ satisfying $x_n \in X^0_{c_n}$ for each $n \in \mathbb{N}$, we have

$$\frac{1}{Z_n(\phi)} \sum_{y\in F^{-n}(x_n)} \deg_{c_n}(F^n, y) \exp\bigl(S^F_n \phi(y)\bigr) \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^i(y)} \xrightarrow{w^*} \mu_{F,\phi} \qquad \text{as } n \to +\infty,$$

where $S^F_n \phi(y) \coloneqq \sum_{i=0}^{n-1} \phi(F^i(y))$ and $Z_n(\phi) \coloneqq \sum_{y\in F^{-n}(x_n)} \deg_{c_n}(F^n, y) \exp\bigl(S^F_n \phi(y)\bigr)$.

Here X^0_b, $X^0_w \in \mathbf{X}^0(f, \mathcal{C})$ are the black 0-tile and the white 0-tile (see Subsection 3.2), respectively, $\deg_b(F^n, x)$ and $\deg_w(F^n, x)$ are the black degree and white degree of F^n at x (see

Definition 3.16 in Subsection 3.3), respectively, and the symbol w^* indicates convergence in the weak* topology.

Theorem 1.1 follows immediately from Remark 3.8, Theorems 5.1, 6.3, Corollaries 6.4, 6.6, and Theorem 7.1.

Remark. The existence of equilibrium states in Theorem 1.1 has been established in [LSZ24, Theorem 1.1].

Based on Theorem 1.1, for strongly primitive subsystems of expanding Thurston maps without periodic critical point, we obtain a level-2 large deviation principle (see Subsection 8.1 for definitions) for Birkhoff averages and iterated preimages.

Theorem 1.2. *Let $f\colon X \to X$ be either an expanding Thurston map with no periodic critical points on a topological 2-sphere $X = S^2$ equipped with a visual metric or a postcritically-finite rational map with no periodic critical points on the Riemann sphere $X = \widehat{\mathbb{C}}$ equipped with the chordal metric. Let $\phi\colon X \to \mathbb{R}$ be Hölder continuous. Let $\mathcal{C} \subseteq X$ be a Jordan curve containing* $\operatorname{post} f$ *with the property that $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider a strongly primitive subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$. Let $\mathcal{P}(\Omega)$ denote the space of Borel probability measures on Ω equipped with the weak*-topology. Let $\mu_{F,\phi}$ be the unique equilibrium state for $F|_\Omega$ and $\phi|_\Omega$.*

For each $n \in \mathbb{N}$, let $V_n\colon \Omega \to \mathcal{P}(\Omega)$ be the continuous function defined by

$$V_n(x) \coloneqq \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^i(x)}, \tag{1.1}$$

and denote $S_n^F \phi(x) \coloneqq \sum_{i=0}^{n-1} \phi(F^i(x))$ for each $x \in \Omega$. For each $n \in \mathbb{N}$, we consider the following Borel probability measures on $\mathcal{P}(\Omega)$.

Birkhoff averages. *$\Sigma_n \coloneqq (V_n)_*(\mu_{F,\phi})$ (i.e., Σ_n is the push-forward of $\mu_{F,\phi}$ by $V_n\colon \Omega \to \mathcal{P}(\Omega)$).*

Iterated preimages. *Given a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in $\Omega \setminus \mathcal{C}$, put*

$$\Omega_n(x_n) \coloneqq \sum_{y \in (F|_\Omega)^{-n}(x_n)} \frac{\exp\bigl(S_n^F \phi(y)\bigr)}{\sum_{y' \in (F|_\Omega)^{-n}(x_n)} \exp\bigl(S_n^F \phi(y')\bigr)} \delta_{V_n(y)}. \tag{1.2}$$

Then each of the sequences $\{\Sigma_n\}_{n\in\mathbb{N}}$ and $\{\Omega_n(x_n)\}_{n\in\mathbb{N}}$ converges to $\delta_{\mu_{F,\phi}}$ in the weak topology, and satisfies a large deviation principle with the rate function $I_\phi\colon \mathcal{P}(\Omega) \to [0, +\infty]$ given by*

$$I_\phi(\mu) \coloneqq \begin{cases} P(F,\phi) - h_\mu(F|_\Omega) - \int \phi \,\mathrm{d}\mu & \text{if } \mu \in \mathcal{M}(\Omega, F|_\Omega); \\ +\infty & \text{if } \mu \in \mathcal{P}(\Omega) \setminus \mathcal{M}(\Omega, F|_\Omega). \end{cases} \tag{1.3}$$

Furthermore, for each convex open subset $\mathcal{G}$ of $\mathcal{P}(\Omega)$ containing some invariant measure, we have $\inf_{\mathcal{G}} I_\phi = \inf_{\overline{\mathcal{G}}} I_\phi$, and

$$\lim_{n\to+\infty} \frac{1}{n} \log \Sigma_n(\mathcal{G}) = \lim_{n\to+\infty} \frac{1}{n} \log \Omega_n(x_n)(\mathcal{G}) = -\inf_{\mathcal{G}} I_\phi, \tag{1.4}$$

and (1.4) remains true with $\mathcal{G}$ replaced by its closure $\overline{\mathcal{G}}$.

We will prove Theorem 1.2 in Subsection 8.3. As an immediate consequence of Theorem 1.2, we get the following corollary. See Subsection 8.3 for the proof.

Corollary 1.3. *Let $f\colon X \to X$ be either an expanding Thurston map with no periodic critical points on a topological 2-sphere $X = S^2$ equipped with a visual metric or a postcritically-finite rational map with no periodic critical points on the Riemann sphere $X = \widehat{\mathbb{C}}$ equipped with the chordal metric. Let $\phi\colon X \to \mathbb{R}$ be Hölder continuous. Let $\mathcal{C} \subseteq X$ be a Jordan curve containing* $\operatorname{post} f$ *with the property that $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider a strongly primitive subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$. Let $\mu_{F,\phi}$ be the unique equilibrium state for $F|_\Omega$ and $\phi|_\Omega$.*

Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in $\Omega\backslash\mathcal{C}$. Then for each $\mu\in\mathcal{M}(\Omega, F|_\Omega)$ and each convex local basis G_μ of $\mathcal{P}(\Omega)$ at μ, we have

$$\begin{aligned} h_\mu(F|_\Omega)+\int\phi\,\mathrm{d}\mu &= \inf_{\mathcal{G}\in G_\mu}\biggl\{\lim_{n\to+\infty}\frac{1}{n}\log\mu_{F,\phi}(\{x\in\Omega : V_n(x)\in\mathcal{G}\})\biggr\}+P(F,\phi) \\ &= \inf_{\mathcal{G}\in G_\mu}\biggl\{\lim_{n\to+\infty}\frac{1}{n}\log\sum_{y\in(F|_\Omega)^{-n}(x_n),V_n(y)\in\mathcal{G}}\exp\bigl(S_n^F\phi(y)\bigr)\biggr\}. \end{aligned} \tag{1.5}$$

Here V_n and $S_n^F\phi$ are as defined in Theorem 1.2.

1.2. **Strategy and organization of the paper.** We now discuss the strategy of the proofs of our main results and describe the organization of the paper.

We divide the proof of Theorem 1.1 into three parts: uniqueness, ergodic properties, and equidistribution results. Note that in Theorem 1.1 the existence of equilibrium states and the property that the measure-preserving transformation $F|_\Omega$ of the probability space $(\Omega,\mu_{F,\phi})$ is forward quasi-invariant have been established in [LSZ24] (see Theorem 3.36).

To prove the uniqueness of equilibrium states, we investigate the (Gâteaux) differentiability of the topological pressure function and apply some techniques from functional analysis. More precisely, a general fact from functional analysis (record in Theorem 5.3) states that for an arbitrary convex continuous function $Q\colon V\to\mathbb{R}$ on a separable Banach space V, there exists a unique continuous linear functional $L\colon V\to\mathbb{R}$ tangent to Q at $x\in V$ if and only if the functional $t\mapsto Q(x+ty)$ is differentiable at 0 for all y in a subset U of V that is dense in the weak topology on V. One then observes that for each continuous map $g\colon X\to X$ on a compact metric space X, the topological pressure function $P(g,\cdot)\colon C(X)\to\mathbb{R}$ is continuous and convex (see for example, [PU10, Theorems 3.6.1 and 3.6.2]), and if μ is an equilibrium state for g and $\varphi\in C(X)$, then the continuous linear functional $u\mapsto\int u\,\mathrm{d}\mu$, for $u\in C(X)$, is tangent to $P(g,\cdot)$ at φ (see for example, [PU10, Theorem 3.6.6]). Thus, in order to verify the uniqueness of the equilibrium state associated with a subsystem F and a real-valued Hölder continuous potential ϕ, it suffices to prove the function $t\mapsto P(F,\phi+t\gamma)$ is differentiable at 0, for all γ in a dense subspace of $C(S^2)$. This is established in Theorem 5.15.

To prove Theorem 5.15, we introduce *normalized split Ruelle operators* induced by split Ruelle operators, and establish some uniform bounds in Proposition 5.9 and Lemma 5.10, which are then used to show uniform convergence results in Theorem 5.11 and Lemma 5.14. Unlike the case of expanding Thurston maps or uniformly expanding maps, for subsystems, one cannot define a normalized Ruelle operator just by normalizing potentials. More specifically, since the combinatorial structure of tiles of a subsystem is inadequate (for example, the number of white 1-tiles may not equal the number of black 1-tiles), the eigenfunctions of the split Ruelle operator may not be continuous on the sphere. However, that the eigenfunctions are always continuous in the interior of 0-tiles, i.e., discontinuities can only occur at the boundary of 0-tiles. To overcome such difficulties, we introduce the notion of the split sphere, which is defined as the disjoint union of two 0-tiles (see Definition 3.23), instead of the original topological 2-sphere. Then we can define the normalized split Ruelle operators on the space of continuous functions on the split sphere.

We next prove that the measure-preserving transformation $F|_\Omega$ of the probability space $(\Omega,\mu_{F,\phi})$ is exact (Theorem 6.3), where we use the Jacobian function and the Gibbs property of the equilibrium state $\mu_{F,\phi}$ established in [LSZ24, Proposition 6.29]. It follows in particular that the equilibrium state $\mu_{F,\phi}$ is non-atomic (Corollary 6.4) and the transformation $F|_\Omega$ is mixing and ergodic (Corollary 6.6).

Finally, we prove the equidistribution results for preimages (Theorem 7.1) by applying the uniform convergence results (Theorem 5.11 and Lemma 5.14) established in the proof of the uniqueness.

To prove Theorem 1.2, we use a variant of Y. Kifer's result [Kif90] formulated by H. Comman and J. Rivera-Letelier [CRL11], recorded in Theorem 8.2. In order to apply Theorem 8.2, we just need to verify three conditions: (1) The existence and uniqueness of the equilibrium state. (2) The upper semi-continuity of the measure-theoretic entropy. (3) Some characterization of the topological pressure (see Propositions 8.3 and 8.4). The first condition has been established in Theorem 1.1. The second condition is known for expanding Thurston maps without periodic critical points (see [LS24, Theorem 1.1]) and is satisfied in our setting by Lemma 8.5. The last condition can be verified by using a characterization of topological pressure established in [LSZ24] (see (3.34) in Theorem 3.35).

We now give a brief description of the structure of this paper.

In Section 2, we fix some notation that will be used throughout the paper. In Section 3, we first review some notions from ergodic theory and dynamical systems and go over some key concepts and results on Thurston maps. Then we review some concepts and results on subsystems of expanding Thurston maps. In Section 4, we state the assumptions on some of the objects in this paper, which we will repeatedly refer to later as the *Assumptions in Section 4*. In Section 5, we prove the uniqueness of the equilibrium states for subsystems. We introduce normalized split Ruelle operators and prove the uniform convergence for functions under iterations of the normalized split Ruelle operators. In Section 6, we prove some ergodic properties of the unique equilibrium state for subsystem. In Section 7, we establish equidistribution results for preimages for subsystems of expanding Thurston maps. In Section 8, we prove level-2 large deviation principles for iterated preimages for subsystems of expanding Thurston maps without periodic critical point.

2. Notation

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. Let S^2 denote an oriented topological 2-sphere. We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1 and write $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$. The symbol log denotes the logarithm to the base e. For $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the greatest integer $\leqslant x$, and $\lceil x \rceil$ the smallest integer $\geqslant x$. We denote by $\operatorname{sgn}(x)$ the sign function for each $x \in \mathbb{R}$. The cardinality of a set A is denoted by $\operatorname{card}(A)$.

Let $g\colon X \to Y$ be a map between two sets X and Y. We denote the restriction of g to a subset Z of X by $g|_Z$.

Consider a map $f\colon X \to X$ on a set X. The inverse map of f is denoted by f^{-1}. We write f^n for the n-th iterate of f, and $f^{-n} \coloneqq (f^n)^{-1}$, for each $n \in \mathbb{N}$. We set $f^0 \coloneqq \mathrm{id}_X$, the identity map on X. For a real-valued function $\varphi\colon X \to \mathbb{R}$, we write

$$S_n\varphi(x) = S_n^f\varphi(x) \coloneqq \sum_{j=0}^{n-1} \varphi\bigl(f^j(x)\bigr) \tag{2.1}$$

for each $x \in X$ and each $n \in \mathbb{N}_0$. We omit the superscript f when the map f is clear from the context. Note that when $n = 0$, by definition we always have $S_0\varphi = 0$.

Let (X, d) be a metric space. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}_d(Y) \coloneqq \sup\{d(x, y) : x, y \in Y\}$, the interior of Y by $\operatorname{int}(Y)$, and the characteristic function of Y by $\mathbb{1}_Y$, which maps each $x \in Y$ to $1 \in \mathbb{R}$ and vanishes otherwise. For each $r > 0$ and each $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x, r)$ (resp. $\overline{B_d}(x, r)$). We often omit the metric d in the subscript when it is clear from the context.

For a compact metrizable topological space X, we denote by $C(X)$ (resp. $B(X)$) the space of continuous (resp. bounded Borel) functions from X to $\mathbb{R}$, by $\mathcal{M}(X)$ the set of finite signed Borel measures, and $\mathcal{P}(X)$ the set of Borel probability measures on X. By the Riesz representation theorem (see for example, [Fol13, Theorems 7.17 and 7.8]), we identify the dual of $C(X)$ with the space $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$, we use $\|\mu\|$ to denote the total variation norm of μ, $\operatorname{supp}\mu$ the support of μ (the smallest closed set $A \subseteq X$ such that $|\mu|(X \setminus A) = 0$), and

$$\langle \mu, u \rangle \coloneqq \int u \, \mathrm{d}\mu$$

for each $u \in C(X)$. For a point $x \in X$, we define δ_x as the Dirac measure supported on $\{x\}$. For a continuous map $g \colon X \to X$, we set $\mathcal{M}(X, g)$ to be the set of g-invariant Borel probability measures on X. If we do not specify otherwise, we equip $C(X)$ with the uniform norm $\|\cdot\|_{C(X)} \coloneqq \|\cdot\|_\infty$, and equip $\mathcal{M}(X)$, $\mathcal{P}(X)$, and $\mathcal{M}(X, g)$ with the weak* topology.

The space of real-valued Hölder continuous functions with an exponent $\beta \in (0, 1]$ on a metric space (X, d) is denoted as $C^{0,\beta}(X, d)$. For each $\phi \in C^{0,\beta}(X, d)$,

$$|\phi|_{\beta,\,(X,d)} \coloneqq \sup \left\{ \frac{|\phi(x) - \phi(y)|}{d(x, y)^\beta} : x,\, y \in X,\, x \neq y \right\},$$

and the Hölder norm is defined as $\|\phi\|_{C^{0,\beta}(X,d)} \coloneqq |\phi|_{\beta,\,(X,d)} + \|\phi\|_{C(X)}$.

3. Preliminaries

3.1. **Thermodynamic formalism.** We first review some basic concepts from ergodic theory and dynamical systems. We refer the reader to [PU10, Chapter 3], [Wal82, Chapter 9], or [KH95, Chapter 20] for more detailed studies of these concepts.

Let (X, d) be a compact metric space and $g \colon X \to X$ a continuous map. Given $n \in \mathbb{N}$,

$$d_g^n(x, y) \coloneqq \max\bigl\{ d\bigl(g^k(x), g^k(y)\bigr) : k \in \{0,\, 1,\, \dots,\, n-1\} \bigr\}, \qquad \text{for } x,\, y \in X,$$

defines a metric on X. A set $F \subseteq X$ is *(n, ϵ)-separated* (with respect to g), for some $n \in \mathbb{N}$ and $\epsilon > 0$, if for each pair of distinct points $x,\, y \in F$, we have $d_g^n(x, y) \geqslant \epsilon$. Given $\epsilon > 0$ and $n \in \mathbb{N}$, let $F_n(\epsilon)$ be a maximal (in the sense of inclusion) (n, ϵ)-separated set in X.

For each real-valued continuous function $\psi \in C(X)$, the following limits exist and are equal, and we denote these limits by $P(g, \psi)$ (see for example, [PU10, Theorem 3.3.2]):

$$P(g, \psi) \coloneqq \lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{x \in F_n(\epsilon)} \exp(S_n \psi(x)) = \lim_{\epsilon \to 0^+} \liminf_{n \to +\infty} \frac{1}{n} \log \sum_{x \in F_n(\epsilon)} \exp(S_n \psi(x)), \tag{3.1}$$

where $S_n \psi(x) = \sum_{j=0}^{n-1} \psi(g^j(x))$ is defined in (2.1). We call $P(g, \psi)$ the *topological pressure* of g with respect to the *potential* ψ. Note that $P(g, \psi)$ is independent of d as long as the topology on X defined by d remains the same (see for example, [PU10, Section 3.2]). The quantity $h_{\mathrm{top}}(g) \coloneqq P(g, 0)$ is called the *topological entropy* of g.

We denote by $\mathcal{M}(X, g)$ the set of all g-invariant Borel probability measures on X.

Let $\mu \in \mathcal{M}(X, g)$. Then we say that g is *ergodic* for μ (or μ is *ergodic* for g) if for each set $A \in \mathcal{B}$ with $g^{-1}(A) = A$ we have $\mu(A) = 0$ or $\mu(A) = 1$. The map g is called *mixing* for μ if

$$\lim_{n \to +\infty} \mu(g^{-n}(A) \cap B) = \mu(A)\mu(B) \tag{3.2}$$

for all $A,\, B \in \mathcal{B}$. It is easy to see that if g is mixing for μ, then g is also ergodic.

For each real-valued continuous function $\psi \in C(X)$, the *measure-theoretic pressure* $P_\mu(g, \psi)$ of g for the measure $\mu \in \mathcal{M}(X, g)$ and the potential ψ is

$$P_\mu(g, \psi) \coloneqq h_\mu(g) + \int \psi \, \mathrm{d}\mu, \tag{3.3}$$

where $h_\mu(g)$ is the measure-theoretic entropy of g for μ.

The topological pressure is related to the measure-theoretic pressure by the so-called *Variational Principle*. It states that (see for example, [PU10, Theorem 3.4.1])

$$P(g, \psi) = \sup\{ P_\mu(g, \psi) : \mu \in \mathcal{M}(X, g) \} \tag{3.4}$$

for each $\psi \in C(X)$. In particular, when ψ is the constant function 0,

$$h_{\mathrm{top}}(g) = \sup\{h_\mu(g) : \mu \in \mathcal{M}(X, g)\}. \tag{3.5}$$

A measure μ that attains the supremum in (3.4) is called an *equilibrium state* for the map g and the potential ψ. A measure μ that attains the supremum in (3.5) is called a *measure of maximal entropy* of g.

Let $\widetilde{X}$ be another compact metric space. If μ is a measure on X and the map $\pi\colon X \to \widetilde{X}$ is continuous, then the *push-forward* $\pi_*\mu$ of μ by π is the measure given by $\pi_*\mu(A) := \mu\bigl(\pi^{-1}(A)\bigr)$ for all Borel sets $A \subseteq \widetilde{X}$.

3.2. **Thurston maps.** In this subsection, we go over some key concepts and results on Thurston maps, and expanding Thurston maps in particular. For a more thorough treatment of the subject, we refer to [BM17].

Let S^2 denote an oriented topological 2-sphere. A continuous map $f\colon S^2 \to S^2$ is called a *branched covering map* on S^2 if for each point $x \in S^2$, there exists a positive integer $d \in \mathbb{N}$, open neighborhoods U of x and V of $y := f(x)$, open neighborhoods U' and V' of 0 in $\widehat{\mathbb{C}}$, and orientation-preserving homeomorphisms $\varphi\colon U \to U'$ and $\eta\colon V \to V'$ such that $\varphi(x) = 0, \eta(y) = 0$, and $\bigl(\eta \circ f \circ \varphi^{-1}\bigr)(z) = z^d$ for each $z \in U'$. The positive integer d above is called the *local degree* of f at x and is denoted by $\deg_f(x)$ or $\deg(f, x)$.

The *degree* of f is $\deg f = \sum_{x \in f^{-1}(y)} \deg_f(x)$ for $y \in S^2$ and is independent of y. If $f\colon S^2 \to S^2$ and $g\colon S^2 \to S^2$ are two branched covering maps on S^2, then so is $f \circ g$, and $\deg(f \circ g, x) = \deg(g, x) \deg(f, g(x))$ for each $x \in S^2$, and moreover, $\deg(f \circ g) = (\deg f)(\deg g)$.

A point $x \in S^2$ is a *critical point* of f if $\deg_f(x) \geqslant 2$. The set of critical points of f is denoted by $\operatorname{crit} f$. A point $y \in S^2$ is a *postcritical point* of f if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of f is denoted by $\operatorname{post} f$. Note that $\operatorname{post} f = \operatorname{post} f^n$ for all $n \in \mathbb{N}$.

Definition 3.1 (Thurston maps)**.** A Thurston map is a branched covering map $f\colon S^2 \to S^2$ on S^2 with $\deg f \geqslant 2$ and $\operatorname{card}(\operatorname{post} f) < +\infty$.

We now recall the notation for cell decompositions of S^2 used in [BM17] and [Li17]. A *cell of dimension* n in S^2, $n \in \{1, 2\}$, is a subset $c \subseteq S^2$ that is homeomorphic to the closed unit ball $\overline{\mathbb{B}^n}$ in $\mathbb{R}^n$, where $\mathbb{B}^n$ is the open unit ball in $\mathbb{R}^n$. We define the *boundary of* c, denoted by ∂c, to be the set of points corresponding to $\partial \mathbb{B}^n$ under such a homeomorphism between c and $\overline{\mathbb{B}^n}$. The *interior of* c is defined to be $\operatorname{inte}(c) = c \setminus \partial c$. For each point $x \in S^2$, the set $\{x\}$ is considered as a *cell of dimension* 0 in S^2. For a cell c of dimension 0, we adopt the convention that $\partial c = \emptyset$ and $\operatorname{inte}(c) = c$.

We record the following definition of cell decompositions from [BM17, Definition 3.2].

Definition 3.2 (Cell decompositions)**.** Let $\mathbf{D}$ be a collection of cells in S^2. We say that $\mathbf{D}$ is a *cell decomposition of* S^2 if the following conditions are satisfied:

(i) the union of all cells in $\mathbf{D}$ is equal to S^2,
(ii) if $c \in \mathbf{D}$, then ∂c is a union of cells in $\mathbf{D}$,
(iii) for $c_1, c_2 \in \mathbf{D}$ with $c_1 \neq c_2$, we have $\operatorname{inte}(c_1) \cap \operatorname{inte}(c_2) = \emptyset$,
(iv) every point in S^2 has a neighborhood that meets only finitely many cells in $\mathbf{D}$.

Definition 3.3 (Refinements)**.** Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of S^2. We say that $\mathbf{D}'$ is a *refinement* of $\mathbf{D}$ if the following conditions are satisfied:

(i) every cell $c \in \mathbf{D}$ is the union of all cells $c' \in \mathbf{D}'$ with $c' \subseteq c$.
(ii) for every cell $c' \in \mathbf{D}'$ there exists a cell $c \in \mathbf{D}$ with $c' \subseteq c$.

Definition 3.4 (Cellular maps and cellular Markov partitions)**.** Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of S^2. We say that a continuous map $f\colon S^2 \to S^2$ is *cellular* for $(\mathbf{D}', \mathbf{D})$ if for every cell $c \in \mathbf{D}'$, the restriction $f|_c$ of f to c is a homeomorphism of c onto a cell in $\mathbf{D}$. We say that $(\mathbf{D}', \mathbf{D})$ is a *cellular Markov partition* for f if f is cellular for $(\mathbf{D}', \mathbf{D})$ and $\mathbf{D}'$ is a refinement of $\mathbf{D}$.

Let $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing $\operatorname{post} f$. Then the pair f and $\mathcal{C}$ induces natural cell decompositions $\mathbf{D}^n(f, \mathcal{C})$ of S^2, for each $n \in \mathbb{N}_0$, in the following way:

By the Jordan curve theorem, the set $S^2 \setminus \mathcal{C}$ has two connected components. We call the closure of one of them the *white* 0*-tile* for $(f, \mathcal{C})$, denoted by X^0_w, and the closure of the other one the *black* 0*-tile* for $(f, \mathcal{C})$, denoted be X^0_b. The set of 0*-tiles* is $\mathbf{X}^0(f, \mathcal{C}) \coloneqq \{X^0_b, X^0_w\}$. The set of 0*-vertices* is $\mathbf{V}^0(f, \mathcal{C}) \coloneqq \operatorname{post} f$. We set $\overline{\mathbf{V}}^0(f, \mathcal{C}) \coloneqq \{\{x\} : x \in \mathbf{V}^0(f, \mathcal{C})\}$. The set of 0*-edges* $\mathbf{E}^0(f, \mathcal{C})$ is the set of the closures of the connected components of $\mathcal{C} \setminus \operatorname{post} f$. Then we get a cell decomposition

$$\mathbf{D}^0(f, \mathcal{C}) \coloneqq \mathbf{X}^0(f, \mathcal{C}) \cup \mathbf{E}^0(f, \mathcal{C}) \cup \overline{\mathbf{V}}^0(f, \mathcal{C})$$

of S^2 consisting of *cells of level* 0, or 0*-cells*.

We can recursively define the unique cell decomposition $\mathbf{D}^n(f, \mathcal{C})$, $n \in \mathbb{N}$, consisting of n*-cells* such that f is cellular for $(\mathbf{D}^{n+1}(f, \mathcal{C}), \mathbf{D}^n(f, \mathcal{C}))$. We refer to [BM17, Lemma 5.12] for more details. We denote by $\mathbf{X}^n(f, \mathcal{C})$ the set of n-cells of dimension 2, called n*-tiles*; by $\mathbf{E}^n(f, \mathcal{C})$ the set of n-cells of dimension 1, called n*-edges*; by $\overline{\mathbf{V}}^n(f, \mathcal{C})$ the set of n-cells of dimension 0; and by $\mathbf{V}^n(f, \mathcal{C})$ the set $\{x : \{x\} \in \overline{\mathbf{V}}^n(f, \mathcal{C})\}$, called the set of n*-vertices*. The k*-skeleton*, for $k \in \{0, 1, 2\}$, of $\mathbf{D}^n(f, \mathcal{C})$ is the union of all n-cells of dimension k in this cell decomposition.

We record [BM17, Lemma 5.17].

Lemma 3.5 (M. Bonk & D. Meyer [BM17])**.** *Let $k, n \in \mathbb{N}_0$, $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve with $\operatorname{post} f \subseteq \mathcal{C}$.*

(i) *If $c \subseteq S^2$ is a topological cell such that $f^k|_c$ is a homeomorphism onto its image and $f^k(c)$ is an n-cell, then c is an $(n+k)$-cell.*

(ii) *If X is an n-tile and $p \in S^2$ is a point with $f^k(p) \in \operatorname{inte}(X)$, then there exists a unique $(n+k)$-tile X' with $p \in X'$ and $f^k(X') = X$.*

For $n \in \mathbb{N}_0$, we define the *set of black n-tiles* as

$$\mathbf{X}^n_b(f, \mathcal{C}) \coloneqq \{X \in \mathbf{X}^n(f, \mathcal{C}) : f^n(X) = X^0_b\},$$

and the *set of white n-tiles* as

$$\mathbf{X}^n_w(f, \mathcal{C}) \coloneqq \{X \in \mathbf{X}^n(f, \mathcal{C}) : f^n(X) = X^0_w\}.$$

From now on, if the map f and the Jordan curve $\mathcal{C}$ are clear from the context, we will sometimes omit $(f, \mathcal{C})$ in the notation above.

Definition 3.6 (Expansion)**.** A Thurston map $f\colon S^2 \to S^2$ is called *expanding* if there exists a metric d on S^2 that induces the standard topology on S^2 and a Jordan curve $\mathcal{C} \subseteq S^2$ containing $\operatorname{post} f$ such that

$$\lim_{n \to +\infty} \max\{\operatorname{diam}_d(X) : X \in \mathbf{X}^n(f, \mathcal{C})\} = 0. \tag{3.6}$$

Remark 3.7. It is clear that if f is an expanding Thurston map, so is f^n for each $n \in \mathbb{N}$. We observe that being expanding is a topological property of a Thurston map and independent of the choice of the metric d that generates the standard topology on S^2. By Lemma 6.2 in [BM17], it is also independent of the choice of the Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$. More precisely, if f is an expanding Thurston map, then

$$\lim_{n\to+\infty} \max\{\operatorname{diam}_{\widetilde{d}}(X) : X \in \mathbf{X}^n(f,\widetilde{\mathcal{C}})\} = 0,$$

for each metric $\widetilde{d}$ that generates the standard topology on S^2 and each Jordan curve $\widetilde{\mathcal{C}} \subseteq S^2$ that contains $\operatorname{post} f$.

For an expanding Thurston map f, we can fix a particular metric d on S^2 called a *visual metric for f*. For the existence and properties of such metrics, see [BM17, Chapter 8]. For a visual metric d for f, there exists a unique constant $\Lambda > 1$ called the *expansion factor* of d (see [BM17, Chapter 8] for more details). One major advantage of a visual metric d is that in (S^2, d) we have good quantitative control over the sizes of the cells in the cell decompositions discussed above.

Remark 3.8. If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational expanding Thurston map, then a visual metric is quasisymmetrically equivalent to the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$ (see [BM17, Theorem 18.1 (ii)]). Here the chordal metric σ on $\widehat{\mathbb{C}}$ is given by $\sigma(z,w) \coloneqq \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$, and $\sigma(\infty, z) = \sigma(z,\infty) \coloneqq \frac{2}{\sqrt{1+|z|^2}}$ for all $z \in \mathbb{C}$. We also note that quasisymmetric embeddings of bounded connected metric spaces are Hölder continuous (see [Hei01, Section 11.1 and Corollary 11.5]). Accordingly, the classes of Hölder continuous functions on $\widehat{\mathbb{C}}$ equipped with the chordal metric and on $S^2 = \widehat{\mathbb{C}}$ equipped with any visual metric for f are the same (up to a change of the Hölder exponent).

A Jordan curve $\mathcal{C} \subseteq S^2$ is *f-invariant* if $f(\mathcal{C}) \subseteq \mathcal{C}$. If $\mathcal{C}$ is f-invariant with $\operatorname{post} f \subseteq \mathcal{C}$, then the cell decompositions $\mathbf{D}^n(f,\mathcal{C})$ have nice compatibility properties. In particular, $\mathbf{D}^{n+k}(f,\mathcal{C})$ is a refinement of $\mathbf{D}^n(f,\mathcal{C})$, whenever $n, k \in \mathbb{N}_0$. Intuitively, this means that each cell $\mathbf{D}^n(f,\mathcal{C})$ is "subdivided" by the cells in $\mathbf{D}^{n+k}(f,\mathcal{C})$. A cell $c \in \mathbf{D}^n(f,\mathcal{C})$ is actually subdivided by the cells in $\mathbf{D}^{n+k}(f,\mathcal{C})$ "in the same way" as the cell $f^n(c) \in \mathbf{D}^0(f,\mathcal{C})$ by the cells in $\mathbf{D}^k(f,\mathcal{C})$.

For convenience we record Proposition 12.5 (ii) of [BM17] here, which is easy to check but useful.

Proposition 3.9 (M. Bonk & D. Meyer [BM17])**.** *Let $k, n \in \mathbb{N}_0$, $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be an f-invariant Jordan curve with* $\operatorname{post} f \subseteq \mathcal{C}$*. Then every $(n+k)$-tile X^{n+k} is contained in a unique k-tile X^k.*

M. Bonk and D. Meyer [BM17, Theorem 15.1] proved that there exists an f^n-invariant Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$ for each sufficiently large n depending on f.

Lemma 3.10 (M. Bonk & D. Meyer [BM17])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\widetilde{\mathcal{C}} \subseteq S^2$ be a Jordan curve with* $\operatorname{post} f \subseteq \widetilde{\mathcal{C}}$*. Then there exists an integer $N(f,\widetilde{\mathcal{C}}) \in \mathbb{N}$ such that for each $n \geqslant N(f,\widetilde{\mathcal{C}})$ there exists an f^n-invariant Jordan curve $\mathcal{C}$ isotopic to $\widetilde{\mathcal{C}}$ rel.* $\operatorname{post} f$*.*

We record the following lemma from [Li18, Lemma 3.13], which generalizes [BM17, Lemma 15.25].

Lemma 3.11 (M. Bonk & D. Meyer [BM17]; Z. Li [Li18])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve that satisfies* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Then there exists a constant $C_0 > 1$, depending only on f, $\mathcal{C}$, $n_{\mathcal{C}}$, and d, with the following property:*

If $n, k \in \mathbb{N}_0$, $X^{n+k} \in \mathbf{X}^{n+k}(f,\mathcal{C})$, and $x, y \in X^{n+k}$, then

$$C_0^{-1} d(x,y) \leqslant d(f^n(x), f^n(y))/\Lambda^n \leqslant C_0 d(x,y). \tag{3.7}$$

The next distortion lemma follows immediately from [Li18, Lemma 5.1].

Lemma 3.12. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve that satisfies* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\beta}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0, 1]$. Then there exists a constant $C_1 \geqslant 0$ depending only on f, $\mathcal{C}$, d, ϕ, and β such that for all $n \in \mathbb{N}_0$, $X^n \in \mathbf{X}^n(f, \mathcal{C})$, and $x, y \in X^n$,*

$$|S_n\phi(x) - S_n\phi(y)| \leqslant C_1 d(f^n(x), f^n(y))^\beta \leqslant C_1(\operatorname{diam}_d(S^2))^\beta. \tag{3.8}$$

Quantitatively, we choose

$$C_1 \coloneqq C_0 |\phi|_{\beta,(S^2,d)} \big/ \big(1 - \Lambda^{-\beta}\big), \tag{3.9}$$

where $C_0 > 1$ is the constant depending only on f, $\mathcal{C}$, and d from Lemma 3.11.

3.3. **Subsystems of expanding Thurston maps.** In this subsection, we review some concepts and results on subsystems of expanding Thurston maps. We refer the reader to [LSZ24, Section 5] for details.

We first introduce the definition of subsystems along with relevant concepts and notations that will be used frequently throughout this paper. Additionally, we will provide examples to illustrate these ideas.

Definition 3.13. Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. We say that a map $F\colon \operatorname{dom}(F) \to S^2$ is a *subsystem of f with respect to $\mathcal{C}$* if $\operatorname{dom}(F) = \bigcup \mathfrak{X}$ for some non-empty subset $\mathfrak{X} \subseteq \mathbf{X}^1(f, \mathcal{C})$ and $F = f|_{\operatorname{dom}(F)}$. We denote by $\operatorname{Sub}(f, \mathcal{C})$ the set of all subsystems of f with respect to $\mathcal{C}$. Define

$$\operatorname{Sub}_*(f, \mathcal{C}) \coloneqq \{F \in \operatorname{Sub}(f, \mathcal{C}) : \operatorname{dom}(F) \subseteq F(\operatorname{dom}(F))\}.$$

Consider a subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. For each $n \in \mathbb{N}_0$, we define the *set of n-tiles of F* to be

$$\mathfrak{X}^n(F, \mathcal{C}) \coloneqq \{X^n \in \mathbf{X}^n(f, \mathcal{C}) : X^n \subseteq F^{-n}(F(\operatorname{dom}(F)))\}, \tag{3.10}$$

where we set $F^0 \coloneqq \operatorname{id}_{S^2}$ when $n = 0$. We call each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ an *n-tile* of F. We define the *tile maximal invariant set* associated with F with respect to $\mathcal{C}$ to be

$$\Omega(F, \mathcal{C}) \coloneqq \bigcap_{n \in \mathbb{N}} \Big(\bigcup \mathfrak{X}^n(F, \mathcal{C}) \Big), \tag{3.11}$$

which is a compact subset of S^2. Indeed, $\Omega(F, \mathcal{C})$ is forward invariant with respect to F, namely, $F(\Omega(F, \mathcal{C})) \subseteq \Omega(F, \mathcal{C})$ (see Proposition 3.15 (ii)). We denote by F_Ω the map $F|_{\Omega(F,\mathcal{C})}\colon \Omega(F, \mathcal{C}) \to \Omega(F, \mathcal{C})$.

Let X^0_b, $X^0_w \in \mathbf{X}^0(f, \mathcal{C})$ be the black 0-tile and the white 0-tile, respectively. We define the *color set of F* as

$$\mathfrak{C}(F, \mathcal{C}) \coloneqq \big\{c \in \{b, w\} : X^0_c \in \mathfrak{X}^0(F, \mathcal{C})\big\}.$$

For each $n \in \mathbb{N}_0$, we define the *set of black n-tiles of F* as

$$\mathfrak{X}^n_b(F, \mathcal{C}) \coloneqq \big\{X \in \mathfrak{X}^n(F, \mathcal{C}) : F^n(X) = X^0_b\big\},$$

and the *set of white n-tiles of F* as

$$\mathfrak{X}^n_w(F, \mathcal{C}) \coloneqq \big\{X \in \mathfrak{X}^n(F, \mathcal{C}) : F^n(X) = X^0_w\big\}.$$

Moreover, for each $n \in \mathbb{N}_0$ and each pair of c, $c' \in \{b, w\}$ we define

$$\mathfrak{X}^n_{cc'}(F, \mathcal{C}) \coloneqq \big\{X \in \mathfrak{X}^n_c(F, \mathcal{C}) : X \subseteq X^0_{c'}\big\}.$$

In other words, for example, a tile $X \in \mathfrak{X}^n_{bw}(F, \mathcal{C})$ is a *black n-tile of F contained in X^0_w*, i.e., an n-tile of F that is contained in the white 0-tile X^0_w as a set, and is mapped by F^n onto the black 0-tile X^0_b.

By abuse of notation, we often omit $(F, \mathcal{C})$ in the notations above when it is clear from the context.

We discuss three examples below and refer the reader to [LSZ24, Subsection 5.1] for more examples.

Example 3.14. Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$.

(i) The map F satisfies $\operatorname{dom}(F) = X_b^1 \cup X_w^1$ for some $X_b^1 \in \mathbf{X}_b^1(f, \mathcal{C})$ and $X_w^1 \in \mathbf{X}_w^1(f, \mathcal{C})$ satisfying $X_b^1 \subseteq \operatorname{inte}(X_w^0)$ and $X_w^1 \subseteq \operatorname{inte}(X_b^0)$. In this case, F is surjective and $\Omega = \{p, q\}$ for some $p \in X_b^1$ and $q \in X_w^1$. One sees that $F(\Omega) = \Omega$ since $F(p) = q$ and $F(q) = p$.

(ii) The map $F\colon \operatorname{dom}(F) \to S^2$ is represented by Figure 3.1. Here S^2 is identified with a pillow that is obtained by gluing two squares together along their boundaries. Moreover, each square is subdivided into 3×3 subsquares, and $\operatorname{dom}(F)$ is obtained from S^2 by removing the interior of the middle subsquare $X_w^1 \in \mathbf{X}_w^1(f, \mathcal{C})$ and $X_b^1 \in \mathbf{X}_b^1(f, \mathcal{C})$ of the respective squares. In this case, Ω is a Sierpiński carpet. It consists of two copies of the standard square Sierpiński carpet glued together along the boundaries of the squares.

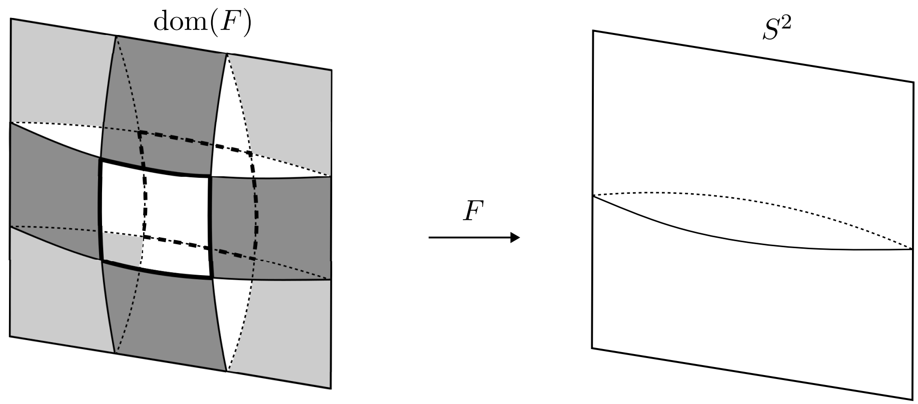

Figure 3.1. A Sierpiński carpet subsystem.

(iii) The map $F\colon \operatorname{dom}(F) \to S^2$ is represented by Figure 3.2. Here S^2 is identified with a pillow that is obtained by gluing two equilateral triangles together along their boundaries. Moreover, each triangle is subdivided into 4 small equilateral triangles, and $\operatorname{dom}(F)$ is obtained from S^2 by removing the interior of the middle small triangle $X_b^1 \in \mathbf{X}_b^1(f, \mathcal{C})$ and $X_w^1 \in \mathbf{X}_w^1(f, \mathcal{C})$ of the respective triangle. In this case, Ω is a Sierpiński gasket. It consists of two copies of the standard Sierpiński gasket glued together along the boundaries of the triangles.

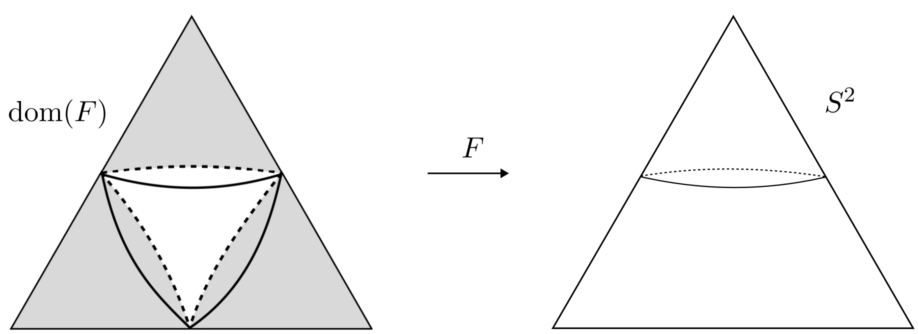

Figure 3.2. A Sierpiński gasket subsystem.

We summarize some preliminary results for subsystems in the following proposition.

Proposition 3.15 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$. Consider arbitrary $n, k \in \mathbb{N}_0$. Then the following statements hold:*

(i) *If $X \in \mathfrak{X}^{n+k}(F, \mathcal{C})$ is any $(n+k)$-tile of F, then $F^k(X)$ is an n-tile of F, and $F^k|_X$ is a homeomorphism of X onto $F^k(X)$. As a consequence we have $\bigl\{F^k(X) : X \in \mathfrak{X}^{n+k}(F, \mathcal{C})\bigr\} \subseteq \mathfrak{X}^n(F, \mathcal{C})$.*

(ii) *The tile maximal invariant set Ω is forward invariant with respect to F, i.e., $F(\Omega) \subseteq \Omega$.*

(iii) *If $f(\mathcal{C}) \subseteq \mathcal{C}$, then $\bigcup \mathfrak{X}^{n+k}(F, \mathcal{C}) \subseteq \bigcup \mathfrak{X}^n(F, \mathcal{C}) \subseteq \bigcup \mathfrak{X}^1(F, \mathcal{C}) = \operatorname{dom}(F)$ for all $n, k \in \mathbb{N}$.*

(iv) *If $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}_*(f, \mathcal{C})$, then $F(\Omega) = \Omega \neq \emptyset$.*

Proposition 3.15 (i) and (ii) are from [LSZ24, Proposition 5.4 (i) and (ii)]. Proposition 3.15 (iii) is from [LSZ24, Proposition 5.5 (i)]. Proposition 3.15 (iv) is from [LSZ24, Proposition 5.6 (ii)].

We record the following notions of degrees and local degrees for subsystems from [LSZ24, Subsection 5.3].

Definition 3.16 (Degrees)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$. The *degree* of F is defined as

$$\deg(F) \coloneqq \sup\bigl\{\operatorname{card}\bigl(F^{-1}(\{y\})\bigr) : y \in S^2\bigr\}.$$

Fix arbitrary $x \in S^2$ and $n \in \mathbb{N}$. We define the *black degree* of F^n at x as

$$\deg_b(F^n, x) \coloneqq \operatorname{card}(\mathfrak{X}^n_b(F, \mathcal{C}, x)),$$

where $\mathfrak{X}^n_b(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}^n_b(F, \mathcal{C}) : x \in X\}$ is the *set of black n-tiles of F at x*. Similarly, we define the *white degree* of F^n at x as

$$\deg_w(F^n, x) \coloneqq \operatorname{card}(\mathfrak{X}^n_w(F, \mathcal{C}, x)),$$

where $\mathfrak{X}^n_w(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}^n_w(F, \mathcal{C}) : x \in X\}$ is the *set of white n-tiles of F at x*. Moreover, the *local degree* of F^n at x is defined as

$$\deg(F^n, x) \coloneqq \max\{\deg_b(F^n, x), \deg_w(F^n, x)\}, \tag{3.12}$$

and the *set of n-tiles of F at x* is $\mathfrak{X}^n(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}^n(F, \mathcal{C}) : x \in X\}$. Furthermore, for each pair of $c, c' \in \{b, w\}$ we define

$$\begin{aligned} \mathfrak{X}^n_{cc'}(F, \mathcal{C}, x) &\coloneqq \{X \in \mathfrak{X}^n_{cc'}(F, \mathcal{C}) : x \in X\}, \\ \deg_{cc'}(F^n, x) &\coloneqq \operatorname{card}(\mathfrak{X}^n_{cc'}(F, \mathcal{C}, x)), \end{aligned}$$

and the *local degree matrix of F^n at x* is

$$\operatorname{Deg}(F^n, x) \coloneqq \begin{bmatrix} \deg_{bb}(F^n, x) & \deg_{wb}(F^n, x) \\ \deg_{bw}(F^n, x) & \deg_{ww}(F^n, x) \end{bmatrix}.$$

We record the following two definitions from [LSZ24, Subsection 5.5].

Definition 3.17 (Irreducibility)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$. We say F is an *irreducible* (resp. a *strongly irreducible*) subsystem (of f with respect to $\mathcal{C}$) if for each pair of $c, c' \in \{b, w\}$, there exists an integer $n_{cc'} \in \mathbb{N}$ and $X^{n_{cc'}} \in \mathfrak{X}^{n_{cc'}}_c(F, \mathcal{C})$ satisfying $X^{n_{cc'}} \subseteq X^0_{c'}$ (resp. $X^{n_{cc'}} \subseteq \operatorname{inte}\bigl(X^0_{c'}\bigr)$). We denote by n_F the constant $\max_{c, c' \in \{b, w\}} n_{cc'}$, which depends only F and $\mathcal{C}$.

Obviously, if F is irreducible then $\mathfrak{C}(F, \mathcal{C}) = \{b, w\}$ and $F(\operatorname{dom}(F)) = S^2$.

Definition 3.18 (Primitivity)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$. We say that F is a *primitive* (resp. *strongly primitive*) subsystem (of f with respect to $\mathcal{C}$) if there exists an integer $n_F \in \mathbb{N}$ such that for each pair of c, $c' \in \{b, w\}$ and each integer $n \geqslant n_F$, there exists $X^n \in \mathfrak{X}^n_c(F, \mathcal{C})$ satisfying $X^n \subseteq X^0_{c'}$ (resp. $X^n \subseteq \operatorname{inte}\bigl(X^0_{c'}\bigr)$).

We record [LSZ24, Lemmas 5.21 and 5.22] below.

Lemma 3.19 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$. *Let $F \in \operatorname{Sub}(f, \mathcal{C})$ be irreducible (resp. strongly irreducible). Let $n_F \in \mathbb{N}$ be the constant from Definition 3.18, which depends only on F and $\mathcal{C}$. Then for each $k \in \mathbb{N}_0$, each $c \in \{b, w\}$, and each k-tile $X^k \in \mathfrak{X}^k(F, \mathcal{C})$, there exists an integer $n \in \mathbb{N}$ with $n \leqslant n_F$ and $X^{k+n}_c \in \mathfrak{X}^{k+n}_c(F, \mathcal{C})$ satisfying $X^{k+n}_c \subseteq X^k$ (resp. $X^{k+n}_c \subseteq \operatorname{inte}(X^k)$).*

Lemma 3.20 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$. *Let $F \in \operatorname{Sub}(f, \mathcal{C})$ be primitive (resp. strongly primitive). Let $n_F \in \mathbb{N}$ be the constant from Definition 3.18, which depends only on F and $\mathcal{C}$. Then for each $n \in \mathbb{N}$ with $n \geqslant n_F$, each $m \in \mathbb{N}_0$, each $c \in \{b, w\}$, and each m-tile $X^m \in \mathfrak{X}^m(F, \mathcal{C})$, there exists an $(n+m)$-tile $X^{n+m}_c \in \mathfrak{X}^{n+m}_c(F, \mathcal{C})$ such that $X^{n+m}_c \subseteq X^m$ (resp. $X^{n+m}_c \subseteq \operatorname{inte}(X^m)$).*

The following distortion lemma serves as cornerstones in the development of thermodynamic formalism for subsystems of expanding Thurston maps (see [LSZ24, Lemma 5.25]).

Lemma 3.21 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve that satisfies* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\beta}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0, 1]$. Then the following statements hold:*

(i) *For each $n \in \mathbb{N}_0$, each $c \in \mathfrak{C}(F, \mathcal{C})$, and each pair of x, $y \in X^0_c$, we have*

$$\frac{\sum_{X^n \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\bigl(S^F_n \phi\bigl((F^n|_{X^n})^{-1}(x)\bigr)\bigr)}{\sum_{X^n \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\bigl(S^F_n \phi\bigl((F^n|_{X^n})^{-1}(x)\bigr)\bigr)} \leqslant \exp\bigl(C_1 d(x,y)^\beta\bigr) \leqslant \exp\bigl(C_1 (\operatorname{diam}_d(S^2))^\beta\bigr), \tag{3.13}$$

where $C_1 \geqslant 0$ is the constant defined in (3.9) *in Lemma 3.12 and depends only on f, $\mathcal{C}$, d, ϕ, and β.*

(ii) *If F is irreducible, then there exists a constant $\widetilde{C} \geqslant 1$ depending only on F, $\mathcal{C}$, d, ϕ, and β such that for each $n \in \mathbb{N}_0$, each pair of c, $c' \in \mathfrak{C}(F, \mathcal{C})$, each $x \in X^0_c$, and each $y \in X^0_{c'}$, we have*

$$\frac{\sum_{X^n_c \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\bigl(S^F_n \phi\bigl((F^n|_{X^n_c})^{-1}(x)\bigr)\bigr)}{\sum_{X^n_{c'} \in \mathfrak{X}^n_{c'}(F,\mathcal{C})} \exp\bigl(S^F_n \phi\bigl((F^n|_{X^n_{c'}})^{-1}(y)\bigr)\bigr)} \leqslant \widetilde{C}. \tag{3.14}$$

Quantitatively, we choose

$$\widetilde{C} \coloneqq (\deg f)^{n_F} \exp\bigl(2 n_F \|\phi\|_\infty + C_1 (\operatorname{diam}_d(S^2))^\beta\bigr), \tag{3.15}$$

where $n_F \in \mathbb{N}$ is the constant in Definition 3.17 and depends only on F and $\mathcal{C}$, and $C_1 \geqslant 0$ is the constant defined in (3.9) *in Lemma 3.12 and depends only on f, $\mathcal{C}$, d, ϕ, and β.*

3.4. **Ergodic theory of subsystems.** In this subsection, we review some concepts and results on ergodic theory of subsystems of expanding Thurston maps. We refer the reader to [LSZ24, Section 6] for details and proofs.

We first recall the topological pressure for subsystems.

Definition 3.22 (Topological pressure)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$. For a real-valued function $\varphi\colon S^2 \to \mathbb{R}$, we denote

$$Z_n(F, \varphi) \coloneqq \sum_{X^n \in \mathfrak{X}^n(F, \mathcal{C})} \exp\bigl(\sup\bigl\{S_n^F \varphi(x) : x \in X^n\bigr\}\bigr)$$

for each $n \in \mathbb{N}$. We define the *topological pressure* of F with respect to the *potential* φ by

$$P(F, \varphi) \coloneqq \liminf_{n \to +\infty} \frac{1}{n} \log(Z_n(F, \varphi)). \tag{3.16}$$

We denote

$$\overline{\varphi} \coloneqq \varphi - P(F, \varphi). \tag{3.17}$$

We introduce the notion of the split sphere (see Definition 3.23), and set up some identifications and conventions (see Remarks 3.25 and 3.26), which will be used frequently in this paper.

Definition 3.23. Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. We define the *split sphere* $\widetilde{S}$ to be the disjoint union of X_b^0 and X_w^0, i.e.,

$$\widetilde{S} \coloneqq X_b^0 \sqcup X_w^0 = \bigl\{(x, c) : c \in \{b, w\},\, x \in X_c^0\bigr\}.$$

For each $c \in \{b, w\}$, let

$$i_c\colon X_c^0 \to \widetilde{S} \tag{3.18}$$

be the natural injection (defined by $i_c(x) \coloneqq (x, c)$). Recall that the topology on $\widetilde{S}$ is defined as the finest topology on $\widetilde{S}$ for which both the natural injections i_b and i_w are continuous. In particular, $\widetilde{S}$ is compact and metrizable.

Let X and Y be normed vector spaces. Recall that a bounded linear map T from X to Y is said to be an *isomorphism* if T is bijective and T^{-1} is bounded (in other words, $\|T(x)\| \geqslant C\|x\|$ for some $C > 0$), and T is called an *isometry* if $\|T(x)\| = \|x\|$ for all $x \in X$.

Proposition 3.24 (Dual of the product space is isometric to the product of the dual spaces)**.** *Let X and Y be normed vector spaces and define $T\colon X^* \times Y^* \to (X \times Y)^*$ by $T(u, v)(x, y) = u(x) + v(y)$. Then T is an isomorphism which is an isometry with respect to the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ on $X \times Y$, the corresponding operator norm on $(X \times Y)^*$, and the norm $\|(u, v)\| \coloneqq \|u\| + \|v\|$ on $X^* \times Y^*$.*

See [LSZ24, Proposition 6.12] for a proof of Proposition 3.24.

By Proposition 3.24 and the Riesz representation theorem (see [Fol13, Theorems 7.17 and 7.8]), we can identify $\bigl(C\bigl(X_b^0\bigr) \times C\bigl(X_w^0\bigr)\bigr)^*$ with the product of spaces of finite signed Borel measures $\mathcal{M}\bigl(X_b^0\bigr) \times \mathcal{M}\bigl(X_w^0\bigr)$, where we use the norm $\|(u_b, u_w)\| \coloneqq \max\{\|u_b\|, \|u_w\|\}$ on $C\bigl(X_b^0\bigr) \times C\bigl(X_w^0\bigr)$, the corresponding operator norm on $\bigl(C\bigl(X_b^0\bigr) \times C\bigl(X_w^0\bigr)\bigr)^*$, and the norm $\|(\mu_b, \mu_w)\| \coloneqq \|\mu_b\| + \|\mu_w\|$ on $\mathcal{M}\bigl(X_b^0\bigr) \times \mathcal{M}\bigl(X_w^0\bigr)$.

Notational Remark. From now on, we write

$$\begin{aligned}(\mu_b, \mu_w)(A_b, A_w) &\coloneqq \mu_b(A_b) + \mu_w(A_w),\\ \langle(\mu_b, \mu_w), (u_b, u_w)\rangle &\coloneqq \langle\mu_b, u_b\rangle + \langle\mu_w, u_w\rangle = \int_{X_b^0} u_b \,\mathrm{d}\mu_b + \int_{X_w^0} u_w \,\mathrm{d}\mu_w,\end{aligned}$$

whenever $(\mu_b, \mu_w) \in \mathcal{M}\bigl(X_b^0\bigr) \times \mathcal{M}\bigl(X_w^0\bigr)$, $(u_b, u_w) \in B\bigl(X_b^0\bigr) \times B\bigl(X_w^0\bigr)$, and A_b and A_w are Borel subset of X_b^0 and X_w^0, respectively. In particular, for each Borel set $A \subseteq S^2$, we define

$$(\mu_b, \mu_w)(A) \coloneqq (\mu_b, \mu_w)\bigl(A \cap X_b^0, A \cap X_w^0\bigr) = \mu_b\bigl(A \cap X_b^0\bigr) + \mu_w\bigl(A \cap X_w^0\bigr). \tag{3.19}$$

Remark 3.25. In the natural way, the product space $C(X_b^0) \times C(X_w^0)$ (resp. $B(X_b^0) \times B(X_w^0)$) can be identified with $C(\widetilde{S})$ (resp. $B(\widetilde{S})$). Similarly, the product space $\mathcal{M}(X_b^0) \times \mathcal{M}(X_w^0)$ can be identified with $\mathcal{M}(\widetilde{S})$. Under such identifications, we write

$$\int (u_b, u_w) \,\mathrm{d}(\mu_b, \mu_w) := \langle (\mu_b, \mu_w), (u_b, u_w) \rangle \qquad \text{and} \qquad (u_b, u_w)(\mu_b, \mu_w) := (u_b \mu_b, u_w \mu_w)$$

whenever $(\mu_b, \mu_w) \in \mathcal{M}(X_b^0) \times \mathcal{M}(X_w^0)$ and $(u_b, u_w) \in B(X_b^0) \times B(X_w^0)$.

Moreover, we have the following natural identification of $\mathcal{P}(\widetilde{S})$:

$$\mathcal{P}(\widetilde{S}) = \left\{ (\mu_b, \mu_w) \in \mathcal{M}(X_b^0) \times \mathcal{M}(X_w^0) : \mu_b \text{ and } \mu_w \text{ are positive measures, } \mu_b(X_b^0) + \mu_w(X_w^0) = 1 \right\}.$$

Here we follow the terminology in [Fol13, Section 3.1] that a *positive measure* is a signed measure that takes values in $[0, +\infty]$.

Remark 3.26. It is easy to see that (3.19) defines a finite signed Borel measure $\mu := (\mu_b, \mu_w)$ on S^2. Here we use the notation μ (resp. (μ_b, μ_w)) when we view the measure as a measure on S^2 (resp. $\widetilde{S}$), and we will always use these conventions in this paper. In this sense, for each $u \in B(S^2)$ we have

$$\langle \mu, u \rangle = \int u \,\mathrm{d}\mu = \int (u_b, u_w) \,\mathrm{d}(\mu_b, \mu_w) = \int_{X_b^0} u \,\mathrm{d}\mu_b + \int_{X_w^0} u \,\mathrm{d}\mu_w, \tag{3.20}$$

where $u_b := u|_{X_b^0}$ and $u_w := u|_{X_w^0}$. Moreover, if both μ_b and μ_w are positive measures and $\mu_b(X_b^0) + \mu_w(X_w^0) = 1$, then $\mu = (\mu_b, \mu_w)$ defined by (3.19) is a Borel probability measure on S^2. In view of the identifications in Remark 3.25, this means that if $(\mu_b, \mu_w) \in \mathcal{P}(\widetilde{S})$, then $\mu \in \mathcal{P}(S^2)$.

We next introduce the split Ruelle operator and its adjoint operator, which are the main tools in [LSZ24] and this paper to develop the thermodynamic formalism for subsystems of expanding Thurston maps. We summarize relevant definitions and facts about the split Ruelle operator and its adjoint operator, and refer the reader to [LSZ24, Subsections 6.2 and 6.4] for a detailed discussion.

Definition 3.27 (Partial split Ruelle operators)**.** Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$ and $\varphi \in C(S^2)$. We define a map $\mathcal{L}_{F,\varphi,c,c'}^{(n)} \colon B(X_{c'}^0) \to B(X_c^0)$, for c, $c' \in \{b, w\}$, and $n \in \mathbb{N}_0$, by

$$\begin{aligned} \mathcal{L}_{F,\varphi,c,c'}^{(n)}(u)(y) &:= \sum_{x \in F^{-n}(y)} \deg_{cc'}(F^n, x) u(x) \exp\left(S_n^F \varphi(x)\right) \\ &= \sum_{X^n \in \mathfrak{X}_{cc'}^n(F,\mathcal{C})} u\left((F^n|_{X^n})^{-1}(y)\right) \exp\left(S_n^F \varphi\left((F^n|_{X^n})^{-1}(y)\right)\right) \end{aligned} \tag{3.21}$$

for each real-valued bounded Borel function $u \in B(X_{c'}^0)$ and each point $y \in X_c^0$.

The following lemma proved in [LSZ24, Lemma 6.8] shows that the partial split Ruelle operators are well-defined and well-behaved under iterations.

Lemma 3.28 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$ and $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}_*(f, \mathcal{C})$ and $\varphi \in C(S^2)$. Then for all n, $k \in \mathbb{N}_0$, c, $c' \in \{b, w\}$, and $u \in C(X_{c'}^0)$, we have*

$$\mathcal{L}_{F,\varphi,c,c'}^{(n)}(u) \in C(X_c^0) \qquad \textit{and} \tag{3.22}$$

$$\mathcal{L}_{F,\varphi,c,c'}^{(n+k)}(u) = \sum_{c'' \in \{b,w\}} \mathcal{L}_{F,\varphi,c,c''}^{(n)}\left(\mathcal{L}_{F,\varphi,c'',c'}^{(k)}(u)\right). \tag{3.23}$$

Definition 3.29 (Split Ruelle operators)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$ and $\varphi \in C(S^2)$. The *split Ruelle operator* for the subsystem F and the potential φ

$$\mathbb{L}_{F,\varphi}\colon C\big(X_b^0\big) \times C\big(X_w^0\big) \to C\big(X_b^0\big) \times C\big(X_w^0\big)$$

on the product space $C\big(X_b^0\big) \times C\big(X_w^0\big)$ is defined by

$$\mathbb{L}_{F,\varphi}(u_b, u_w) \coloneqq \big(\mathcal{L}^{(1)}_{F,\varphi,b,b}(u_b) + \mathcal{L}^{(1)}_{F,\varphi,b,w}(u_w), \mathcal{L}^{(1)}_{F,\varphi,w,b}(u_b) + \mathcal{L}^{(1)}_{F,\varphi,w,w}(u_w)\big) \tag{3.24}$$

for each $u_b \in C\big(X_b^0\big)$ and each $u_w \in C\big(X_w^0\big)$.

The following lemma proved in [LSZ24, Lemma 6.10] shows that the split Ruelle operator is well-behaved under iterations.

Lemma 3.30 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$ and $\varphi \in C(S^2)$. We assume in addition that $F(\operatorname{dom}(F)) = S^2$. Then for all $n \in \mathbb{N}_0$, $u_b \in C\big(X_b^0\big)$, and $u_w \in C\big(X_w^0\big)$,*

$$\mathbb{L}^n_{F,\varphi}(u_b, u_w) = \big(\mathcal{L}^{(n)}_{F,\varphi,b,b}(u_b) + \mathcal{L}^{(n)}_{F,\varphi,b,w}(u_w),\ \mathcal{L}^{(n)}_{F,\varphi,w,b}(u_b) + \mathcal{L}^{(n)}_{F,\varphi,w,w}(u_w)\big). \tag{3.25}$$

Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$ and $\varphi \in C(S^2)$. We assume in addition that $F(\operatorname{dom}(F)) = S^2$. Note that the split Ruelle operator $\mathbb{L}_{F,\varphi}$ (see Definition 3.29) is a positive, continuous operator on $C\big(X_b^0\big) \times C\big(X_w^0\big)$. Thus, the adjoint operator

$$\mathbb{L}^*_{F,\varphi}\colon \big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^* \to \big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$$

of $\mathbb{L}_{F,\varphi}$ acts on the dual space $\big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$ of the Banach space $C\big(X_b^0\big) \times C\big(X_w^0\big)$. Recall that we identify $\big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$ with the product of spaces of finite signed Borel measures $\mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$, where we use the norm $\|(u_b, u_w)\| = \max\{\|u_b\|, \|u_w\|\}$ on $C\big(X_b^0\big) \times C\big(X_w^0\big)$, the corresponding operator norm on $\big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$, and the norm $\|(\mu_b, \mu_w)\| = \|\mu_b\| + \|\mu_w\|$ on $\mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$. Then by Remark 3.25, we can also view $\mathbb{L}_{F,\varphi}$ (resp. $\mathbb{L}^*_{F,\varphi}$) as an operator on $C(\widetilde{S})$ (resp. $\mathcal{M}(\widetilde{S})$).

We record [LSZ24, Proposition 6.15 (iv)] in the following. Roughly speaking, the following proposition says that measures will be concentrated on the limit set under iteration of the adjoint operator $\mathbb{L}^*_{F,\varphi}$.

Proposition 3.31 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f, \mathcal{C})$ and $\varphi \in C(S^2)$. We assume in addition that $F(\operatorname{dom}(F)) = S^2$. Consider arbitrary $n \in \mathbb{N}$ and $(\mu_b, \mu_w) \in \mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$. Then we have*

$$\big(\mathbb{L}^*_{F,\varphi}\big)^n(\mu_b, \mu_w)\Big(\bigcup \widetilde{\mathfrak{X}}^{n-1}(F, \mathcal{C})\Big) = \big(\mathbb{L}^*_{F,\varphi}\big)^n(\mu_b, \mu_w)\Big(\bigcup \widetilde{\mathfrak{X}}^{n}(F, \mathcal{C})\Big),$$

where $\widetilde{\mathfrak{X}}^k(F, \mathcal{C}) \coloneqq \bigcup_{c \in \{b,w\}} \big\{i_c(X^k) : X^k \in \mathfrak{X}^k(F, \mathcal{C}),\, X^k \subseteq X_c^0\big\}$ for each $k \in \mathbb{N}_0$ (here i_c is defined in (3.18)*).*

We record the following three results (Lemma 3.32, Theorems 3.34, and 3.35) on the split Ruelle operators and their adjoint operators in our context.

Lemma 3.32 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $F \in \operatorname{Sub}(f, \mathcal{C})$ be irreducible. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\beta}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0, 1]$. Then there exists a constant*

$\widetilde{C}_1 \geqslant 0$ depending only on F, $\mathcal{C}$, d, ϕ, and β such that for each $n \in \mathbb{N}$, each $c \in \{b, w\}$, and each pair of x, $y \in X^0_c$, the following inequalities holds:

$$\mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{x}) / \mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{y}) \leqslant \exp\big(C_1 d(x,y)^\beta\big) \leqslant \widetilde{C}, \tag{3.26}$$

$$\widetilde{C}^{-1} \leqslant \mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{x}) \leqslant \widetilde{C}, \tag{3.27}$$

$$\big|\mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{x}) - \mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{y})\big| \leqslant \widetilde{C}\big(\exp\big(C_1 d(x,y)^\beta\big) - 1\big) \leqslant \widetilde{C}_1 d(x,y)^\beta, \tag{3.28}$$

where $\widetilde{x} \coloneqq i_c(x) = (x,c) \in \widetilde{S}$, $\widetilde{y} \coloneqq i_c(y) = (y,c) \in \widetilde{S}$ (recall Remark 3.25), $C_1 \geqslant 0$ is the constant in Lemma 3.12 depending only on f, $\mathcal{C}$, d, ϕ, and β, and $\widetilde{C} \geqslant 1$ is the constant in Lemma 3.21 (ii) *depending only on F, $\mathcal{C}$, d, ϕ, and β.*

Recall from (3.17) that $\overline{\phi} = \phi - P(F,\phi)$. Lemma 3.32 was proved in [LSZ24, Lemma 6.22].

Definition 3.33 (Gibbs measures for subsystems)**.** Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f,\mathcal{C})$. Let d be a visual metric on S^2 for f and ϕ be a real-valued Hölder continuous function on S^2 with respect to the metric d. A Borel probability measure $\mu \in \mathcal{P}(S^2)$ is called a *Gibbs measure* with respect to F, $\mathcal{C}$, and ϕ if there exist constants $P_\mu \in \mathbb{R}$ and $C_\mu \geqslant 1$ such that for each $n \in \mathbb{N}_0$, each n-tile $X^n \in \mathfrak{X}^n(F,\mathcal{C})$, and each $x \in X^n$, we have

$$\frac{1}{C_\mu} \leqslant \frac{\mu(X^n)}{\exp(S^F_n\phi(x) - nP_\mu)} \leqslant C_\mu. \tag{3.29}$$

One observes that for each Gibbs measure μ with respect to F, $\mathcal{C}$, and ϕ, the constant P_μ is unique.

Theorem 3.34 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}(f,\mathcal{C})$. We assume in addition that $F(\operatorname{dom}(F)) = S^2$. Let d be a visual metric on S^2 for f and ϕ be a real-valued Hölder continuous function on S^2 with respect to the metric d. Then there exists a Borel probability measure $m_{F,\phi} = (m_b, m_w) \in \mathcal{P}(\widetilde{S})$ such that*

$$\mathbb{L}^*_{F,\phi}(m_b, m_w) = \kappa(m_b, m_w), \tag{3.30}$$

where $\kappa = \big\langle \mathbb{L}^*_{F,\phi}(m_b, m_w), \mathbb{1}_{\widetilde{S}} \big\rangle$. Moreover, if F is strongly irreducible, then any $m_{F,\phi} = (m_b, m_w) \in \mathcal{P}(\widetilde{S})$ that satisfies (3.30) *for some $\kappa > 0$ has the following properties:*

(i) *$m_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = 0$.*
(ii) *For each Borel set $A \subseteq \operatorname{dom}(F)$ on which F is injective, $m_{F,\phi}(F(A)) = \int_A \kappa \exp(-\phi)\,\mathrm{d}\mu$.*
(iii) *The measure $m_{F,\phi}$ is a Gibbs measure with respect to F, $\mathcal{C}$, and ϕ with $P_\mu = P(F,\phi) = \log\kappa$. Here $P(F,\phi)$ is defined in* (3.16)*.*

We follow the conventions discussed in Remarks 3.25 and 3.26. In particular, we use the notation (m_b, m_w) (resp. $m_{F,\phi}$) to emphasize that we treat the eigenmeasure as a Borel probability measure on $\widetilde{S}$ (resp. S^2). The existence of eigenmeasure in Theorem 3.34 was established in [LSZ24, Theorem 6.16]. Theorem 3.34 (i) is part of [LSZ24, Proposition 6.27]. Theorem 3.34 (ii) and (iii) follow immediately from [LSZ24, Proposition 6.29].

The next theorem was established in [LSZ24, Theorem 6.25].

Theorem 3.35 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying* $\operatorname{post} f \subseteq \mathcal{C}$ *and $f(\mathcal{C}) \subseteq \mathcal{C}$. Let $F \in \operatorname{Sub}(f,\mathcal{C})$ be strongly irreducible. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Let $\phi \in C^{0,\beta}(S^2,d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0,1]$.*

Then the sequence $\big\{\frac{1}{n}\sum_{j=0}^{n-1}\mathbb{L}^{j}_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)\big\}_{n\in\mathbb{N}}$ *converges uniformly to a function* $\widetilde{u}_{F,\phi}=(u_b,u_w)\in C^{0,\beta}\big(X^0_b,d\big)\times C^{0,\beta}\big(X^0_w,d\big)$*, which satisfies*

$$\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi})=\widetilde{u}_{F,\phi}\qquad \textit{and} \tag{3.31}$$

$$\widetilde{C}^{-1}\leqslant\widetilde{u}_{F,\phi}(\widetilde{x})\leqslant\widetilde{C},\qquad \textit{for each } \widetilde{x}\in\widetilde{S}, \tag{3.32}$$

where $\widetilde{C}\geqslant 1$ *is the constant from Lemma 3.21* (ii) *depending only on* f, $\mathcal{C}$, d, ϕ, *and* β. *Moreover, if we let* $m_{F,\phi}=(m_b,m_w)$ *be an eigenmeasure from Theorem 3.34, then*

$$\int_{\widetilde{S}}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b,m_w)=1, \tag{3.33}$$

and $\mu_{F,\phi}=(\mu_b,\mu_w):=\widetilde{u}_{F,\phi}(m_b,m_w)$ *is an* f*-invariant Gibbs measure with respect to* F, $\mathcal{C}$, *and* ϕ, *with* $\mu_{F,\phi}(\Omega(F,\mathcal{C}))=1$ *and*

$$P_{\mu_{F,\phi}}=P_{m_{F,\phi}}=P(F,\phi)=\lim_{n\to+\infty}\frac{1}{n}\log\big(\mathbb{L}^{n}_{F,\phi}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{y})\big), \tag{3.34}$$

for each $\widetilde{y}\in\widetilde{S}$. *In particular,* $\mu_{F,\phi}(U)\neq 0$ *for each open set* $U\subseteq S^2$ *with* $U\cap\Omega(F,\mathcal{C})\neq\emptyset$.

We record the following Variational Principle and the existence of equilibrium states for subsystems established in [LSZ24, Theorems 6.30 and 6.31]. Recall that $F(\Omega)\subseteq\Omega$ by Proposition 3.15 (ii).

Theorem 3.36 (Z. Li, X. Shi, Y. Zhang [LSZ24])**.** *Let* $f\colon S^2\to S^2$ *be an expanding Thurston map with a Jordan curve* $\mathcal{C}\subseteq S^2$ *satisfying* $\operatorname{post} f\subseteq\mathcal{C}$ *and* $f(\mathcal{C})\subseteq\mathcal{C}$. *Let* $F\in\operatorname{Sub}(f,\mathcal{C})$ *be strongly irreducible. Let* d *be a visual metric on* S^2 *for* f *and* ϕ *be a real-valued Hölder continuous function on* S^2 *with respect to the metric* d. *Then we have*

$$P(F,\phi)=P(F|_\Omega,\phi|_\Omega)=\sup\Big\{h_\mu(F|_\Omega)+\int_\Omega\phi\,\mathrm{d}\mu:\mu\in\mathcal{M}(\Omega,F|_\Omega)\Big\}, \tag{3.35}$$

and there exists an equilibrium state for $F|_\Omega$ *and* $\phi|_\Omega$, *where* $P(F,\phi)$ *is defined by* (3.16) *and* $P(F|_\Omega,\phi|_\Omega)$ *is defined by* (3.1).

Moreover, any measure $\mu_{F,\phi}\in\mathcal{M}(S^2,f)$ *defined in Theorem 3.35 is an equilibrium state for* $F|_\Omega$ *and* $\phi|_\Omega$, *and the map* $F|_\Omega$ *with respect to such* $\mu_{F,\phi}$ *is forward quasi-invariant (i.e., for each Borel set* $A\subseteq\Omega$, *if* $\mu_{F,\phi}(A)=0$, *then* $\mu_{F,\phi}((F|_\Omega)(A))=0$*) and non-singular (i.e., for each Borel set* $A\subseteq\Omega$, *if* $\mu_{F,\phi}(A)=0$, *then* $\mu_{F,\phi}((F|_\Omega)^{-1}(A))=0$*).*

4. The Assumptions

We state below the hypotheses under which we will develop our theory in most parts of this paper. We will selectively use some of those assumptions in the later sections.

The Assumptions.

(1) $f\colon S^2\to S^2$ is an expanding Thurston map.
(2) $\mathcal{C}\subseteq S^2$ is a Jordan curve containing $\operatorname{post} f$ with the property that there exists an integer $n_{\mathcal{C}}\in\mathbb{N}$ such that $f^{n_{\mathcal{C}}}(\mathcal{C})\subseteq\mathcal{C}$ and $f^{m}(\mathcal{C})\nsubseteq\mathcal{C}$ for each $m\in\{1,\,\dots,\,n_{\mathcal{C}}-1\}$.
(3) $F\in\operatorname{Sub}(f,\mathcal{C})$ is a subsystem of f with respect to $\mathcal{C}$.
(4) d is a visual metric on S^2 for f with expansion factor $\Lambda>1$.
(5) $\beta\in(0,1]$.
(6) $\phi\in C^{0,\beta}(S^2,d)$ is a real-valued Hölder continuous function with an exponent β.

Observe that by Lemma 3.10, for each f in (1), there exists at least one Jordan curve $\mathcal{C}$ that satisfies (2). Since for a fixed f, the number $n_{\mathcal{C}}$ is uniquely determined by $\mathcal{C}$ in (2), in the remaining part of the paper, we will say that a quantity depends on $\mathcal{C}$ even if it also depends on $n_{\mathcal{C}}$.

Recall that the expansion factor Λ of a visual metric d on S^2 for f is uniquely determined by d and f. We will say that a quantity depends on f and d if it depends on Λ.

Note that even though the value of L is not uniquely determined by the metric d, in the remainder of this paper, for each visual metric d on S^2 for f, we will fix a choice of linear local connectivity constant L. We will say that a quantity depends on the visual metric d without mentioning the dependence on L, even though if we had not fixed a choice of L, it would have depended on L as well.

In the discussion below, depending on the conditions we will need, we will sometimes say "Let f, $\mathcal{C}$, d, ϕ satisfy the Assumptions.", and sometimes say "Let f and $\mathcal{C}$ satisfy the Assumptions.", etc.

5. Uniqueness of the equilibrium states for subsystems

This section is devoted to the uniqueness of the equilibrium states for subsystems, with the main result being Theorem 5.1. We first define normalized split Ruelle operators and obtain some basic properties in Subsection 5.1. Then in Subsection 5.2 we introduce the notion of abstract modulus of continuity and prove the uniform convergence for functions under iterations of the normalized split Ruelle operators. Finally, in Subsection 5.3 we establish Theorem 5.1.

Theorem 5.1. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$ and $F_\Omega \coloneqq F|_\Omega$. Then there exists a unique equilibrium state $\mu_{F,\phi}$ for F_Ω and $\phi|_\Omega$. Moreover, the map F_Ω with respect to $\mu_{F,\phi}$ is forward quasi-invariant (i.e., for each Borel set $A \subseteq \Omega$, if $\mu_{F,\phi}(A) = 0$, then $\mu_{F,\phi}(F_\Omega(A)) = 0$) and non-singular (i.e., for each Borel set $A \subseteq \Omega$, if $\mu_{F,\phi}(A) = 0$, then $\mu_{F,\phi}(F_\Omega^{-1}(A)) = 0$).*

We adopt the following convention.

Remark 5.2. Let X be a non-empty Borel subset of S^2. Given a Borel probability measure $\mu \in \mathcal{P}(X)$, by abuse of notation, we can view μ as a Borel probability measure on S^2 by setting $\mu(A) \coloneqq \mu(A \cap X)$ for all Borel subsets $A \subseteq S^2$. Conversely, for each measure $\nu \in \mathcal{P}(S^2)$ supported on X, we can view ν as a Borel probability measure on X.

To prove the uniqueness of the equilibrium state of a continuous map g on a compact metric space X, one of the techniques is to prove the (Gâteaux) differentiability of the topological pressure function $P(g, \cdot)\colon C(X) \to \mathbb{R}$. We summarize the general ideas below, but refer the reader to [PU10, Section 3.6] for a detailed treatment.

For a continuous map $g\colon X \to X$ on a compact metric space X, the topological pressure function $P(g, \cdot)\colon C(X) \to \mathbb{R}$ is Lipschitz continuous [PU10, Theorem 3.6.1] and convex [PU10, Theorem 3.6.2]. For an arbitrary convex continuous function $Q\colon V \to \mathbb{R}$ on a real topological vector space V, we call a continuous linear functional $L\colon V \to \mathbb{R}$ *tangent to Q at $x \in V$* if

$$Q(x) + L(y) \leqslant Q(x + y), \qquad \text{for each } y \in V. \tag{5.1}$$

We denote the set of all continuous linear functionals tangent to Q at $x \in V$ by $V^*_{x,Q}$. It is know (see for example, [PU10, Proposition 3.6.6]) that if $\mu \in \mathcal{M}(X, g)$ is an equilibrium state for g and $\varphi \in C(X)$, then the continuous linear functional $u \mapsto \int u \,\mathrm{d}\mu$ for $u \in C(X)$ is tangent to the topological pressure function $P(g, \cdot)$ at φ. Indeed, let $\varphi, \gamma \in C(X)$ and $\mu \in \mathcal{M}(X, g)$ be an equilibrium state for g and φ. Then $P(g, \varphi + \gamma) \geqslant h_\mu(g) + \int \varphi + \gamma \,\mathrm{d}\mu$ by the Variational Principle (3.4) in Subsection 3.1, and $P(g, \varphi) = h_\mu(g) + \int \varphi \,\mathrm{d}\mu$. It follows that $P(g, \varphi) + \int \gamma \,\mathrm{d}\mu \leqslant P(g, \varphi + \gamma)$.

Thus to prove the uniqueness of the equilibrium state for a continuous map $g\colon X \to X$ and a continuous potential φ, it suffices to show that $\operatorname{card}\bigl(C(X)^*_{\varphi,P(g,\cdot)}\bigr) = 1$. Then we can apply the following fact from functional analysis (see [PU10, Theorem 3.6.5] for a proof):

Theorem 5.3 (F. Przytycki & M. Urbański [PU10])**.** *Let V be a separable Banach space and $Q\colon V \to \mathbb{R}$ be a convex continuous function. Then for each $x \in V$, the following statements are equivalent:*

(i) $\operatorname{card}\bigl(V^*_{x,Q}\bigr) = 1$.

(ii) *The function $t \mapsto Q(x+ty)$ is differentiable at 0 for each $y \in V$.*

(iii) *There exists a subset $U \subseteq V$ that is dense in the weak topology on V such that the function $t \mapsto Q(x+ty)$ is differentiable at 0 for each $y \in U$.*

Now the problem of the uniqueness of equilibrium states transforms to the problem of (Gâteaux) differentiability of the topological pressure function. To investigate the latter, we need a closer study of the fine properties of split Ruelle operators.

5.1. **Normalized split Ruelle operator.** In this subsection, we define normalized split Ruelle operators and establish some basic properties which will be frequently used later.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$ and $f(\mathcal{C}) \subseteq \mathcal{C}$. Let d be a visual metric for f on S^2, and $\phi \in C^{0,\beta}(S^2,d)$ a real-valued Hölder continuous function with an exponent $\beta \in (0,1]$. Let X^0_b, $X^0_w \in \mathbf{X}^0(f,\mathcal{C})$ be the black 0-tile and the white 0-tile, respectively. Recall from Definition 3.23 and Remark 3.25 that $\widetilde{S} = X^0_b \sqcup X^0_w$ is the disjoint union of X^0_b and X^0_w, and the product spaces $C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$, $B\bigl(X^0_b\bigr) \times B\bigl(X^0_w\bigr)$, and $\mathcal{M}\bigl(X^0_b\bigr) \times \mathcal{M}\bigl(X^0_w\bigr)$ are identified with $C(\widetilde{S})$, $B(\widetilde{S})$, and $\mathcal{M}(\widetilde{S})$, respectively.

Let $F \in \operatorname{Sub}(f,\mathcal{C})$ be strongly irreducible and $\widetilde{u}_{F,\phi} = (u_b,u_w) \in C(\widetilde{S})$ be the function given by Theorem 3.35. Note that $\widetilde{u}_{F,\phi}(\widetilde{x}) > 0$ for each $\widetilde{x} \in \widetilde{S} = X^0_b \sqcup X^0_w$ by (3.32) in Theorem 3.35. Recall from (3.17) that $\overline{\phi} = \phi - P(F,\phi)$.

Definition 5.4 (Partial normalized split Ruelle operator)**.** Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. For each $n \in \mathbb{N}_0$ and each pair of c, $c' \in \{b,w\}$, we define a map $\widetilde{\mathcal{L}}^{(n)}_{F,\phi,c,c'}\colon B\bigl(X^0_{c'}\bigr) \to B(X^0_c)$ by

$$
\begin{aligned}
&\widetilde{\mathcal{L}}^{(n)}_{F,\phi,c,c'}(v)(x)\\
&\qquad := \sum_{y \in F^{-n}(x)} \deg_{cc'}(F^n,y)v(y)\exp\bigl(S^F_n\phi(y) - nP(F,\phi) + \log u_{c'}(y) - \log u_c(x)\bigr)\\
&\qquad = \sum_{\substack{y = (F^n|_{X^n})^{-1}(x)\\ X^n \in \mathfrak{X}^n_{cc'}(F,\mathcal{C})}} v(y)\exp\bigl(S^F_n\phi(y) - nP(F,\phi) + \log u_{c'}(y) - \log u_c(x)\bigr)
\end{aligned} \tag{5.2}
$$

for each $v \in B\bigl(X^0_{c'}\bigr)$ and each point $x \in X^0_c$.

Remark. By (3.21) in Definition 3.27, we can write the right hand side of (5.2) as

$$
\begin{aligned}
&\widetilde{\mathcal{L}}^{(n)}_{F,\phi,c,c'}(v)(x)\\
&\qquad = \frac{1}{u_c(x)} \sum_{X^n \in \mathfrak{X}^n_{cc'}(F,\mathcal{C})} (u_{c'} \cdot v)\bigl((F^n|_{X^n})^{-1}(x)\bigr)\exp\bigl(S^F_n\phi\bigl((F^n|_{X^n})^{-1}(x)\bigr) - nP(F,\phi)\bigr)\\
&\qquad = \frac{1}{u_c(x)}\mathcal{L}^{(n)}_{F,\overline{\phi},c,c'}(u_{c'}v)(x).
\end{aligned} \tag{5.3}
$$

Thus, it follows immediately from Lemma 3.28 that for all n, $k \in \mathbb{N}_0$, c, $c' \in \{b, w\}$, and $v \in C\bigl(X^0_{c'}\bigr)$,

$$\widetilde{\mathcal{L}}^{(n)}_{F,\phi,c,c'}(v) \in C\bigl(X^0_c\bigr) \qquad \text{and} \tag{5.4}$$

$$\widetilde{\mathcal{L}}^{(n+k)}_{F,\phi,c,c'}(v) = \sum_{c'' \in \{b,w\}} \widetilde{\mathcal{L}}^{(n)}_{F,\phi,c,c''}\bigl(\widetilde{\mathcal{L}}^{(k)}_{F,\phi,c'',c'}(v)\bigr). \tag{5.5}$$

Definition 5.5 (Normalized split Ruelle operators)**.** Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Let $\widetilde{u}_{F,\phi} = (u_b, u_w) \in C(\widetilde{S})$ be the continuous function given by Theorem 3.35. The *normalized split Ruelle operator* $\widetilde{\mathbb{L}}_{F,\phi} \colon C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr) \to C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$ for the subsystem F and the potential ϕ is defined by

$$\widetilde{\mathbb{L}}_{F,\phi}(v_b, v_w) := \bigl(\widetilde{\mathcal{L}}^{(1)}_{F,\phi,b,b}(u_b) + \widetilde{\mathcal{L}}^{(1)}_{F,\phi,b,w}(u_w), \widetilde{\mathcal{L}}^{(1)}_{F,\phi,w,b}(u_b) + \widetilde{\mathcal{L}}^{(1)}_{F,\phi,w,w}(u_w)\bigr) \tag{5.6}$$

for each $v_b \in C\bigl(X^0_b\bigr)$ and each $v_w \in C\bigl(X^0_w\bigr)$, or equivalently, $\widetilde{\mathbb{L}}_{F,\phi} \colon C(\widetilde{S}) \to C(\widetilde{S})$ is defined by

$$\widetilde{\mathbb{L}}_{F,\phi}(\widetilde{v}) := \frac{1}{\widetilde{u}_{F,\phi}} \mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}\widetilde{v}) \tag{5.7}$$

for each $\widetilde{v} \in C(\widetilde{S})$.

Note that by (5.2) and (5.4), the normalized split Ruelle operator $\widetilde{\mathbb{L}}_{F,\phi}$ is well-defined, and the equivalence of the two definitions (5.6) and (5.7) follows from (5.3) and Definition 3.29. By (5.2), $\widetilde{\mathbb{L}}^0_{F,\phi}$ is the identity map on $C(\widetilde{S})$. Moreover, one sees that $\widetilde{\mathbb{L}}_{F,\phi} \colon C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr) \to C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$ has a natural extension to the space $B\bigl(X^0_b\bigr) \times B\bigl(X^0_w\bigr)$ given by (5.6) for each $v_b \in B\bigl(X^0_b\bigr)$ and each $v_w \in B\bigl(X^0_w\bigr)$.

We show that the normalized split Ruelle operator $\widetilde{\mathbb{L}}_{F,\phi}$ is well-behaved under iterations.

Lemma 5.6. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Then for all $n \in \mathbb{N}_0$ and $\widetilde{v} = (v_b, v_w) \in C(\widetilde{S})$, we have*

$$\widetilde{\mathbb{L}}^n_{F,\phi}(\widetilde{v}) = \frac{1}{\widetilde{u}_{F,\phi}} \mathbb{L}^n_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}\widetilde{v}) \qquad \textit{and} \tag{5.8}$$

$$\widetilde{\mathbb{L}}^n_{F,\phi}(v_b, v_w) = \bigl(\widetilde{\mathcal{L}}^{(n)}_{F,\phi,b,b}(v_b) + \widetilde{\mathcal{L}}^{(n)}_{F,\phi,b,w}(v_w),\ \widetilde{\mathcal{L}}^{(n)}_{F,\phi,w,b}(v_b) + \widetilde{\mathcal{L}}^{(n)}_{F,\phi,w,w}(v_w)\bigr). \tag{5.9}$$

Proof. It follows immediately from (5.7) and Definition 3.29 that (5.8) holds for all $n \in \mathbb{N}$. Since F is surjective, i.e., $F(\operatorname{dom}(F)) = S^2$, we know that $\mathbb{L}^0_{F,\phi}$ is the identity map on $C(\widetilde{S})$ by Definition 3.29. Thus (5.8) also holds for $n = 0$.

The case where $n = 0$ and the case where $n = 1$ both hold by definition. Assume now (5.9) holds for $n = k$ for some $k \in \mathbb{N}$. Then by Definition 5.5 and (5.5), for each $c \in \{b, w\}$ we have

$$\begin{aligned}
\pi_c\bigl(\widetilde{\mathbb{L}}^{k+1}_{F,\phi}(v_b, v_w)\bigr) &= \pi_c\bigl(\widetilde{\mathbb{L}}_{F,\phi}\bigl(\widetilde{\mathcal{L}}^{(k)}_{F,\phi,b,b}(v_b) + \widetilde{\mathcal{L}}^{(k)}_{F,\phi,b,w}(v_w),\ \widetilde{\mathcal{L}}^{(k)}_{F,\phi,w,b}(v_b) + \widetilde{\mathcal{L}}^{(k)}_{F,\phi,w,w}(v_w)\bigr)\bigr) \\
&= \sum_{c' \in \{b,w\}} \widetilde{\mathcal{L}}^{(1)}_{F,\phi,c,c'}\bigl(\widetilde{\mathcal{L}}^{(k)}_{F,\phi,c',b}(v_b) + \widetilde{\mathcal{L}}^{(k)}_{F,\phi,c',w}(v_w)\bigr) \\
&= \sum_{c'' \in \{b,w\}} \sum_{c' \in \{b,w\}} \widetilde{\mathcal{L}}^{(1)}_{F,\phi,c,c'}\bigl(\widetilde{\mathcal{L}}^{(k)}_{F,\phi,c',c''}(v_{c''})\bigr) \\
&= \sum_{c'' \in \{b,w\}} \widetilde{\mathcal{L}}^{(k+1)}_{F,\phi,c,c''}(v_{c''}),
\end{aligned}$$

for $v_b \in C\bigl(X^0_b\bigr)$ and $v_w \in C\bigl(X^0_w\bigr)$. This completes the inductive step, establishing (5.9). $\square$

Remark. Similarly, one can show that (5.8) and (5.9) holds for $\widetilde{v} = (v_b, v_w) \in B\bigl(X_b^0\bigr) \times B\bigl(X_w^0\bigr)$.

Recall $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ from Theorem 3.34. By Theorem 3.34 (iii), $\mathbb{L}^*_{F,\overline{\phi}}(m_b, m_w) = (m_b, m_w)$. Then we can show that $(\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ defined in Theorem 3.35 satisfies

$$\widetilde{\mathbb{L}}^*_{F,\phi}(\mu_b, \mu_w) = (\mu_b, \mu_w), \tag{5.10}$$

where $\widetilde{\mathbb{L}}^*_{F,\phi}$ is the adjoint operator of $\widetilde{\mathbb{L}}_{F,\phi}$ on the space $\mathcal{M}\bigl(X_b^0\bigr) \times \mathcal{M}\bigl(X_w^0\bigr)$. Indeed, by (5.7), for every $\widetilde{v} \in C(\widetilde{S})$,

$$\begin{aligned}\bigl\langle \widetilde{\mathbb{L}}^*_{F,\phi}(\mu_b, \mu_w), \widetilde{v} \bigr\rangle &= \bigl\langle \widetilde{u}_{F,\phi}(m_b, m_w), \widetilde{\mathbb{L}}_{F,\phi}(\widetilde{v}) \bigr\rangle = \bigl\langle (m_b, m_w), \mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}\widetilde{v}) \bigr\rangle \\ &= \bigl\langle \mathbb{L}^*_{F,\overline{\phi}}(m_b, m_w), \widetilde{u}_{F,\phi}\widetilde{v} \bigr\rangle = \langle (m_b, m_w), \widetilde{u}_{F,\phi}\widetilde{v} \rangle = \langle (\mu_b, \mu_w), \widetilde{v} \rangle.\end{aligned}$$

Recall that we equip the spaces $C(\widetilde{S})$ and $C\bigl(X_b^0\bigr) \times C\bigl(X_w^0\bigr)$ with the uniform norm given by

$$\|\widetilde{v}\|_{C(\widetilde{S})} = \|(v_b, v_w)\| = \max\bigl\{ \|v_b\|_{C(X_b^0)}, \|v_w\|_{C(X_w^0)} \bigr\}$$

for $\widetilde{v} = (v_b, v_w) \in C\bigl(X_b^0\bigr) \times C\bigl(X_w^0\bigr)$.

Lemma 5.7. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Then the operator norm of $\widetilde{\mathbb{L}}_{F,\phi}$ is $\bigl\|\widetilde{\mathbb{L}}_{F,\phi}\bigr\|_{C(\widetilde{S})} = 1$. In addition, $\widetilde{\mathbb{L}}_{F,\phi}\bigl(\mathbb{1}_{\widetilde{S}}\bigr) = \mathbb{1}_{\widetilde{S}}$.*

Proof. By Definition 5.5, (3.31) in Theorem 3.35, and (5.2) in Definition 5.4, we have

$$\widetilde{\mathbb{L}}_{F,\phi}\bigl(\mathbb{1}_{\widetilde{S}}\bigr) = \frac{1}{\widetilde{u}_{F,\phi}} \mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}) = \mathbb{1}_{\widetilde{S}},$$

i.e., for each $\widetilde{x} = (x, c) \in \widetilde{S}$,

$$1 = \sum_{c' \in \{b,w\}} \sum_{X^1 \in \mathfrak{X}^1_{cc'}(F,\mathcal{C})} \exp\bigl(\phi(x_{X^1}) - P(F,\phi) + \log u_{c'}(x_{X^1}) - \log u_c(x)\bigr),$$

where we write $x_{X^1} \coloneqq (F|_{X^1})^{-1}(x)$ for $X^1 \in \mathfrak{X}^1_{cc'}(F,\mathcal{C})$. Then for each $\widetilde{x} = (x, c) \in \widetilde{S}$ and each $\widetilde{v} = (v_b, v_w) \in C\bigl(X_b^0\bigr) \times C\bigl(X_w^0\bigr)$, we have

$$\begin{aligned}\Bigl|\widetilde{\mathbb{L}}_{F,\phi}(\widetilde{v})(\widetilde{x})\Bigr| &= \Biggl| \sum_{c' \in \{b,w\}} \sum_{X^1 \in \mathfrak{X}^1_{cc'}(F,\mathcal{C})} v_{c'}(x_{X^1}) \exp\bigl(\phi(x_{X^1}) - P(F,\phi) + \log u_{c'}(x_{X^1}) - \log u_c(x)\bigr) \Biggr| \\ &\leqslant \|\widetilde{v}\|_{C(\widetilde{S})} \Biggl| \sum_{c' \in \{b,w\}} \sum_{X^1 \in \mathfrak{X}^1_{cc'}(F,\mathcal{C})} \exp\bigl(\phi(x_{X^1}) - P(F,\phi) + \log u_{c'}(x_{X^1}) - \log u_c(x)\bigr) \Biggr| \\ &= \|\widetilde{v}\|_{C(\widetilde{S})}.\end{aligned}$$

Thus, $\bigl\|\widetilde{\mathbb{L}}_{F,\phi}\bigr\|_{C(\widetilde{S})} \leqslant 1$. Since $\widetilde{\mathbb{L}}_{F,\phi}\bigl(\mathbb{1}_{\widetilde{S}}\bigr) = \mathbb{1}_{\widetilde{S}}$, we get $\bigl\|\widetilde{\mathbb{L}}_{F,\phi}\bigr\|_{C(\widetilde{S})} = 1$. □

5.2. **Uniform convergence.** In this subsection, we prove the uniform convergence for functions under iterations of the normalized split Ruelle operators.

Let (X, d) be a metric space. A function $\eta\colon [0, +\infty) \to [0, +\infty)$ is an *abstract modulus of continuity* if it is continuous at 0, non-decreasing, and $\eta(0) = 0$. Given any constant $\tau \in [0, +\infty]$ and any abstract modulus of continuity η, we define the subclass $C^\tau_\eta(X, d)$ of $C(X)$ as

$$C^\tau_\eta(X, d) \coloneqq \bigl\{ u \in C(X) : \|u\|_\infty \leqslant \tau \text{ and for } x, y \in S^2, |u(x) - u(y)| \leqslant \eta(d(x, y)) \bigr\}.$$

Assume now that (X,d) is compact. Then by the Arzelà–Ascoli Theorem, each $C^{\tau}_{\eta}(X,d)$ is precompact in $C(X)$ equipped with the uniform norm. It is easy to see that each $C^{\tau}_{\eta}(X,d)$ is compact. On the other hand, for $v\in C(X)$, we can define an abstract modulus of continuity by

$$\eta(t)=\sup\{|v(x)-v(y)| : x,\, y\in X,\, d(x,y)\leqslant t\} \tag{5.11}$$

for $t\in[0,+\infty)$, so that $v\in C^{\iota}_{\eta}(X,d)$, where $\iota=\|v\|_{\infty}$.

The following lemma is easy to check (see also [Li17, Lemma 5.24]).

Lemma 5.8. *Let (X,d) be a metric space. For all constants $\tau>0$, $\tau_1,\,\tau_2\geqslant 0$ and abstract moduli of continuity $\eta_1,\,\eta_2$, we have*

$$\big\{v_1v_2 : v_1\in C^{\tau_1}_{\eta_1}(X,d),\, v_2\in C^{\tau_2}_{\eta_2}(X,d)\big\}\subseteq C^{\tau_1\tau_2}_{\tau_1\eta_2+\tau_2\eta_1}(X,d),$$
$$\big\{1/v : v\in C^{\tau_1}_{\eta_1}(X,d),\, v(x)\geqslant\tau \text{ for each } x\in X\big\}\subseteq C^{\tau^{-1}}_{\tau^{-2}\eta_1}(X,d).$$

Proposition 5.9. *Let f, $\mathcal{C}$, F, d, Λ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C})\subseteq\mathcal{C}$ and $F\in\operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. Then for each $\beta\in(0,1]$, each $\tau\geqslant 0$, and each $K\geqslant 0$, there exist constants $\widehat{\tau}\geqslant 0$ and $\widehat{C}\geqslant 0$ with the following property:*

For each abstract modulus of continuity η, there exists an abstract modulus of continuity $\widehat{\eta}$ such that for each $\phi\in C^{0,\beta}(S^2,d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)}\leqslant K$, we have

$$\big\{\mathbb{L}^n_{F,\overline{\phi}}(\widetilde{v}) : \widetilde{v}\in C^{\tau}_{\eta}\big(X^0_b,d\big)\times C^{\tau}_{\eta}\big(X^0_w,d\big),\, n\in\mathbb{N}_0\big\}\subseteq C^{\widehat{\tau}}_{\widehat{\eta}}\big(X^0_b,d\big)\times C^{\widehat{\tau}}_{\widehat{\eta}}\big(X^0_w,d\big), \tag{5.12}$$

$$\big\{\widetilde{\mathbb{L}}^n_{F,\phi}(\widetilde{v}) : \widetilde{v}\in C^{\tau}_{\eta}\big(X^0_b,d\big)\times C^{\tau}_{\eta}\big(X^0_w,d\big),\, n\in\mathbb{N}_0\big\}\subseteq C^{\widehat{\tau}}_{\widehat{\eta}}\big(X^0_b,d\big)\times C^{\widehat{\tau}}_{\widehat{\eta}}\big(X^0_w,d\big), \tag{5.13}$$

where $\widehat{\eta}(t)\coloneqq\widehat{C}\big(t^{\beta}+\eta(C_0t)\big)$ is an abstract modulus of continuity, and $C_0>1$ is the constant depending only on f, $\mathcal{C}$, and d from Lemma 3.11.

Proof. We write $\widetilde{C}^{\tau}_{\eta}(\widetilde{S},d)\coloneqq C^{\tau}_{\eta}\big(X^0_b,d\big)\times C^{\tau}_{\eta}\big(X^0_w,d\big)$ for each $\tau>0$ and each abstract modulus of continuity η in this proof. For each $\widetilde{v}\in\widetilde{C}^{\tau}_{\eta}(\widetilde{S},d)$, write $\widetilde{v}=(v_b,v_w)$.

Fix arbitrary $\beta\in(0,1]$, $\tau\geqslant 0$, and $K\geqslant 0$. By Lemma 3.30 and (3.27) in Lemma 3.32, for all $n\in\mathbb{N}_0$, $\widetilde{v}=(v_b,v_w)\in\widetilde{C}^{\tau}_{\eta}(\widetilde{S},d)$, and $\phi\in C^{0,\beta}(S^2,d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)}\leq K$, we have

$$\big\|\mathbb{L}^n_{F,\overline{\phi}}(\widetilde{v})\big\|_{C(\widetilde{S})}\leqslant\|\widetilde{v}\|_{C(\widetilde{S})}\big\|\mathbb{L}^n_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\big\|_{C(\widetilde{S})}\leqslant\widetilde{C}\|\widetilde{v}\|_{C(\widetilde{S})},$$

where $\widetilde{C}\geqslant 1$ is the constant defined in (3.15) in Lemma 3.21 (ii) and depends only on F, $\mathcal{C}$, d, ϕ, and β. By (3.15), quantitatively,

$$\widetilde{C}=(\deg f)^{n_F}\exp\big(2n_F\|\phi\|_{\infty}+C_1(\operatorname{diam}_d(S^2))^{\beta}\big),$$

where $n_F\in\mathbb{N}$ is the constant depending only on F and $\mathcal{C}$ in Definition 3.18 since F is primitive, and $C_1\geqslant 0$ is the constant defined in (3.9) in Lemma 3.12 depends only on F, $\mathcal{C}$, d, ϕ, and β. Quantitatively,

$$C_1=C_0|\phi|_{\beta,\,(S^2,d)}/(1-\Lambda^{-\beta}),$$

where $C_0>1$ is the constant depending only on f, $\mathcal{C}$, and d in Lemma 3.11. Let $C_1'\coloneqq C_0K/(1-\Lambda^{-\beta})$ and

$$\widetilde{C}'\coloneqq(\deg f)^{n_F}\exp\big(2n_FK+C_1'(\operatorname{diam}_d(S^2))^{\beta}\big).$$

Then we have $C_1\leqslant C_1'$ and $\widetilde{C}\leqslant\widetilde{C}'$ for each $\phi\in C^{0,\beta}(S^2,d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)}\leq K$. Note that both C_1' and $\widetilde{C}'$ only depend on F, $\mathcal{C}$, d, K, and β. Thus we can choose $\widehat{\tau}\coloneqq\widetilde{C}'\tau$.

For each $c\in\{b,w\}$ and each pair of $x,\,y\in X^0_c$, we have

$$\begin{aligned}&\Big|\pi_c\big(\mathbb{L}^n_{F,\overline{\phi}}(v_b,v_w)\big)(x)-\pi_c\big(\mathbb{L}^n_{F,\overline{\phi}}(v_b,v_w)\big)(y)\Big|\\&\quad=\Big|\mathcal{L}^{(n)}_{F,\overline{\phi},c,b}(v_b)(x)+\mathcal{L}^{(n)}_{F,\overline{\phi},c,w}(v_w)(x)-\mathcal{L}^{(n)}_{F,\overline{\phi},c,b}(v_b)(y)-\mathcal{L}^{(n)}_{F,\overline{\phi},c,w}(v_w)(y)\Big|\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{c'\in\{b,w\}} \Big| \mathcal{L}^{(n)}_{F,\overline{\phi},c,c'}(v_c)(x) - \mathcal{L}^{(n)}_{F,\overline{\phi},c,c'}(v_c)(y) \Big| \\
&= \sum_{c'\in\{b,w\}} \Bigg| \sum_{X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})} \big(v_{c'} \exp\big(S_n^F\overline{\phi}\big)\big)\big((F^n|_{X^n})^{-1}(x)\big) - \big(v_{c'} \exp\big(S_n^F\overline{\phi}\big)\big)\big((F^n|_{X^n})^{-1}(y)\big) \Bigg| \\
&\leqslant \sum_{c'\in\{b,w\}} \Bigg| \sum_{X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})} v_{c'}\big((F^n|_{X^n})^{-1}(x)\big) \Big(e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(x))} - e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(y))} \Big) \Bigg| \\
&\qquad + \sum_{c'\in\{b,w\}} \Bigg| \sum_{X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})} e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(y))} \Big(v_{c'}\big((F^n|_{X^n})^{-1}(x)\big) - v_{c'}\big((F^n|_{X^n})^{-1}(y)\big) \Big) \Bigg|.
\end{aligned}
$$

The second term above is

$$\leqslant \widetilde{C}\eta(C_0\Lambda^{-n}d(x,y)) \leqslant \widetilde{C}\eta(C_0 d(x,y)) \leqslant \widetilde{C}'\eta(C_0 d(x,y)),$$

due to (3.27) in Lemma 3.32 and the fact that $d\big((F^n|_{X^n})^{-1}(x),(F^n|_{X^n})^{-1}(y)\big) \leqslant C_0\Lambda^{-n}d(x,y)$ by Lemma 3.11, where the constant C_0 comes from.

To estimate the first term, we use the following general inequality for $s,t\in\mathbb{R}$,

$$|\exp(s)-\exp(t)| \leqslant (\exp(s)+\exp(t))(\exp(|s-t|)-1).$$

Then it follows from Lemma 3.12, Lemma 3.30, and (3.27) in Lemma 3.32 that the first term is

$$
\begin{aligned}
&\leqslant \sum_{c'\in\{b,w\}} \sum_{X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})} \|v_{c'}\|_\infty \Big(e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(x))} + e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(y))} \Big) \\
&\qquad\qquad\qquad \cdot \Big(e^{|S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(x)) - S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(y))|} - 1 \Big) \\
&\leqslant \sum_{X^n\in\mathfrak{X}^n_{c}(F,\mathcal{C})} \tau \Big(e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(x))} + e^{S_n^F\overline{\phi}((F^n|_{X^n})^{-1}(y))} \Big) \big(\exp\big(C_1 d(x,y)^\beta\big)-1\big) \\
&\leqslant 2\tau\widetilde{C}\big(\exp\big(C_1 d(x,y)^\beta\big)-1\big) \leqslant 2\tau\widetilde{C}'\big(\exp\big(C_1' d(x,y)^\beta\big)-1\big) \\
&\leqslant 2\tau\widetilde{C}_1 d(x,y)^\beta,
\end{aligned}
$$

for some constant $\widetilde{C}_1>0$ that only depends on C_1', $\widetilde{C}'$, and $\operatorname{diam}_d(S^2)$. Here the justification of the third inequality above is similar to that of (3.28) in Lemma 3.32. Recall that both C_1' and $\widetilde{C}'$ only depend on F, $\mathcal{C}$, d, K, and β, so does $\widetilde{C}_1$.

Hence for each $c\in\{b,w\}$ and each pair of x, $y\in X_c^0$, we have

$$\Big|\pi_c\big(\mathbb{L}^n_{F,\overline{\phi}}(v_b,v_w)\big)(x) - \pi_c\big(\mathbb{L}^n_{F,\overline{\phi}}(v_b,v_w)\big)(y)\Big| \leqslant \widetilde{C}'\eta(C_0 d(x,y)) + 2\tau\widetilde{C}_1 d(x,y)^\beta.$$

By choosing $\widehat{C} := \max\big\{\widetilde{C}', 2\tau\widetilde{C}_1\big\}$, which only depends on F, $\mathcal{C}$, d, K, and β, we complete the proof of (5.12).

We now prove (5.13).

We fix an arbitrary $\phi\in C^{0,\beta}(S^2,d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)} \leq K$. Recall that $\widetilde{u}_{F,\phi}=(u_b,u_w)\in C\big(X_b^0\big)\times C\big(X_w^0\big)$ is a continuous function on $\widetilde{S}$ given by Theorem 3.35. Thus, by (3.32) in Theorem 3.35 we have

$$\|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \leqslant \tau_1,$$

where $\tau_1 \coloneqq \widetilde{C}' = (\deg f)^{n_F} \exp\bigl(2n_F K + C_1'(\operatorname{diam}_d(S^2))^\beta\bigr)$. For each $c \in \{b, w\}$ and each pair of $x, y \in X_c^0$, it follows from Theorem 3.35 and (3.28) in Lemma 3.32 that

$$\begin{aligned} |u_c(x) - u_c(y)| &= \left| \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Bigl(\mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(x,c) - \mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(y,c) \Bigr) \right| \\ &\leqslant \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Bigl| \mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(x,c) - \mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(y,c) \Bigr| \\ &\leqslant \widetilde{C}\bigl(\exp\bigl(C_1 d(x,y)^\beta\bigr) - 1\bigr) \\ &\leqslant \widetilde{C}'\bigl(\exp\bigl(C_1' d(x,y)^\beta\bigr) - 1\bigr). \end{aligned}$$

Thus, $\widetilde{u}_{F,\phi} = (u_b, u_w) \in \widetilde{C}^{\tau_1}_{\eta_1}(\widetilde{S}, d)$, where η_1 is an abstract modulus of continuity defined by

$$\eta_1(t) \coloneqq \widetilde{C}'\bigl(\exp\bigl(C_1' t^\beta\bigr) - 1\bigr), \qquad \text{for } t \in [0, +\infty).$$

Note that it follows from Lemma 5.8 that

$$\bigl\{ \widetilde{v}\widetilde{u}_{F,\phi} : \widetilde{v} = (v_b, v_w) \in \widetilde{C}^\tau_\eta(\widetilde{S}, d),\, \phi \in C^{0,\beta}(S^2, d),\, |\phi|_{\beta,\,(S^2,d)} \leqslant K \bigr\} \subseteq \widetilde{C}^{\tau\tau_1}_{\tau\eta_1 + \tau_1\eta}(\widetilde{S}, d).$$

Then by (5.8) in Lemma 5.6, (5.12), and Lemma 5.8, we get that there exists a constant $\widetilde{\tau} \geqslant 0$ and an abstract modulus of continuity $\widetilde{\eta}$ such that for each $\phi \in C^{0,\beta}(S^2, d)$ with $|\phi|_{\beta,\,(S^2,d)} \leqslant K$,

$$\bigl\{ \widetilde{\mathbb{L}}^n_{F,\phi}(\widetilde{v}) : \widetilde{v} = (v_b, v_w) \in \widetilde{C}^\tau_\eta(\widetilde{S}, d),\, n \in \mathbb{N}_0 \bigr\} \subseteq \widetilde{C}^{\widetilde{\tau}}_{\widetilde{\eta}}(\widetilde{S}, d).$$

On the other hand, by Lemma 5.7, $\bigl\| \widetilde{\mathbb{L}}^n_{F,\phi}(\widetilde{v}) \bigr\|_{C(\widetilde{S})} \leqslant \|\widetilde{v}\|_{C(\widetilde{S})} \leqslant \tau$ for each $\widetilde{v} = (v_b, v_w) \in \widetilde{C}^\tau_\eta(\widetilde{S}, d)$, each $n \in \mathbb{N}_0$, and each $\phi \in C^{0,\beta}(S^2, d)$. Therefore, we have proved (5.13). $\square$

Lemma 5.10. *Let f, $\mathcal{C}$, F, d, Λ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Let η be an abstract modulus of continuity. Then for each $\beta \in (0,1]$, each $K \in [0 + \infty)$, and each $\delta_1 \in (0, +\infty)$, there exist constants $\delta_2 \in (0, +\infty)$ and $N \in \mathbb{N}$ with the following property:*

For each $\widetilde{v} = (v_b, v_w) \in C^{+\infty}_\eta\bigl(X^0_b, d\bigr) \times C^{+\infty}_\eta\bigl(X^0_w, d\bigr)$, each $\phi \in C^{0,\beta}(S^2, d)$, and each choice of $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ from Theorem 3.34, if $\|\phi\|_{C^{0,\beta}(S^2,d)} \leqslant K$, $\|\widetilde{v}\|_{C(\widetilde{S})} \geqslant \delta_1$, and $\int \widetilde{v}\widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = 0$, then

$$\bigl\| \widetilde{\mathbb{L}}^N_{F,\phi}(\widetilde{v}) \bigr\|_{C(\widetilde{S})} \leqslant \|\widetilde{v}\|_{C(\widetilde{S})} - \delta_2.$$

Remark. Note that at this point, we have yet to prove that (m_b, m_w) from Theorem 3.34 is unique. We will prove it in Proposition 5.13. Recall that $\widetilde{u}_{F,\phi} = (u_b, u_w) \in C(\widetilde{S})$ is defined in Theorem 3.35 that depends only on F, $\mathcal{C}$, and ϕ.

Proof. Fix arbitrary constants $\beta \in (0,1]$, $K \in [0, +\infty)$, and $\delta_1 \in (0, +\infty)$. Fix $\epsilon > 0$ sufficiently small such that $\eta(\epsilon) < \delta_1/2$. Then ϵ depends only on η and δ_1. Fix an arbitrary choice of $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ from Theorem 3.34, an arbitrary $\phi \in C^{0,\beta}(S^2, d)$, and an arbitrary $\widetilde{v} = (v_b, v_w) \in C^{+\infty}_\eta\bigl(X^0_b, d\bigr) \times C^{+\infty}_\eta\bigl(X^0_w, d\bigr)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)} \leqslant K$, $\|\widetilde{v}\|_{C(\widetilde{S})} \geqslant \delta_1$, and $\int \widetilde{v}\widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = 0$.

Let $\widetilde{\Omega}$ be the subset of $\widetilde{S}$ defined by

$$\widetilde{\Omega} \coloneqq \bigcap_{n\in\mathbb{N}} \bigcup \widetilde{\mathfrak{X}}^n(F, \mathcal{C}),$$

where $\widetilde{\mathfrak{X}}^n(F, \mathcal{C}) \coloneqq \bigcup_{c\in\{b,w\}} \bigl\{ i_c(X^n) : X^n \in \mathfrak{X}^n(F, \mathcal{C}),\, X^n \subseteq X^0_c \bigr\}$ and i_c is defined by (3.18).

We first show that $(m_b, m_w)\big(\widetilde{\Omega}\big) = 1$. Indeed, since (m_b, m_w) is an eigenmeasure of $\mathbb{L}^*_{F,\phi}$, it follows from Proposition 3.31 and induction on n that for each $n \in \mathbb{N}$,

$$1 \geqslant (m_b, m_w)\Big(\bigcup \widetilde{\mathfrak{X}}^n(F, \mathcal{C})\Big) = (m_b, m_w)\Big(\bigcup \widetilde{\mathfrak{X}}^0(F, \mathcal{C})\Big) = (m_b, m_w)(\widetilde{S}) = 1,$$

where we use the fact that $\bigcup \widetilde{\mathfrak{X}}^0(F, \mathcal{C}) = \widetilde{S}$ since F is irreducible so that $F(\mathrm{dom}(F)) = S^2$. Note that by [LSZ24, Proposition 5.5 (i)], $\big\{\bigcup \widetilde{\mathfrak{X}}^n(F, \mathcal{C})\big\}_{n\in\mathbb{N}}$ is a decreasing sequence of sets. Thus,

$$(m_b, m_w)\big(\widetilde{\Omega}\big) = \lim_{n\to+\infty} (m_b, m_w)\Big(\bigcup \widetilde{\mathfrak{X}}^n(F, \mathcal{C})\Big) = 1.$$

Since $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ is supported on $\widetilde{\Omega}$ and $\int \widetilde{v}\widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = 0$, there exist points $\widetilde{y}_-, \widetilde{y}_+ \in \widetilde{\Omega}$ such that $\widetilde{v}(\widetilde{y}_-) \leqslant 0$ and $\widetilde{v}(\widetilde{y}_+) \geqslant 0$. By Definition 3.23, we have $\widetilde{y}_- = (y_-, c')$ for some $c' \in \{b, w\}$ and $y_- \in X^0_{c'}$, and $\widetilde{y}_+ = (y_+, c'')$ for some $c'' \in \{b, w\}$ and $y_+ \in X^0_{c''}$.

We fix an arbitrary point $\widetilde{x} \in \widetilde{S}$. Then there exist $c \in \{b, w\}$ and $x \in X^0_c$ satisfying $\widetilde{x} = (x, c)$.

Since $\widetilde{y}_- = (y_-, c') \in \widetilde{\Omega}$, it follows from the definition of $\widetilde{\Omega}$ that there exists a sequence of tiles $\{X^n\}_{n\in\mathbb{N}}$ satisfying $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ and $y_- \in X^{n+1} \subseteq X^n \subseteq X^0_{c'}$ for each $n \in \mathbb{N}$. By [BM17, Proposition 8.4 (ii)], there exists an integer $n_\epsilon \in \mathbb{N}$ depending only on F, $\mathcal{C}$, d, η, and δ_1 such that $\mathrm{diam}_d(Y^{n_\epsilon}) < \epsilon$ for each n_ϵ-tile $Y^{n_\epsilon} \in \mathbf{X}^{n_\epsilon}(f, \mathcal{C})$. Thus we have $y_- \in X^{n_\epsilon} \subseteq B_d(y_-, \epsilon) \cap X^0_{c'}$. By Proposition 3.15 (i), we have $X^0 := F^{n_\epsilon}(X^{n_\epsilon}) \in \big\{X^0_b, X^0_w\big\}$. Since $F \in \mathrm{Sub}(f, \mathcal{C})$ is primitive, by Definition 3.18, there exist $n_F \in \mathbb{N}$ and $Y^{n_F} \in \mathfrak{X}^{n_F}(F, \mathcal{C})$ satisfying $Y^{n_F} \subseteq X^0$ and $F^{n_F}(Y^{n_F}) = X^0_c$. Then it follows from [BM17, Lemma 5.17 (i)] and Proposition 3.15 (i) that $Y^{n_\epsilon+n_F} := (F^{n_\epsilon}|_{X^{n_\epsilon}})^{-1}(Y^{n_F}) \in \mathfrak{X}^{n_\epsilon+n_F}(F, \mathcal{C})$. Note that $Y^{n_\epsilon+n_F} \subseteq X^{n_\epsilon} \subseteq X^0_{c'}$ and $F^{n_\epsilon+n_F}(Y^{n_\epsilon+n_F}) = F^{n_F}(Y^{n_F}) = X^0_c$. Thus $Y^{n_\epsilon+n_F} \in \mathfrak{X}^{n_\epsilon+n_F}_{cc'}(F, \mathcal{C})$. Set $y := (F^{n_\epsilon+n_F}|_{Y^{n_\epsilon+n_F}})^{-1}(x)$. Then we have $y \in Y^{n_\epsilon+n_F} \subseteq X^{n_F} \subseteq B_d(y_-, \epsilon) \cap X^0_{c'}$. Thus

$$v_{c'}(y) \leqslant v_{c'}(y_-) + \eta(\epsilon) = \widetilde{v}(\widetilde{y}_-) + \eta(\epsilon) \leqslant \eta(\epsilon) < \delta_1/2 \leqslant \|\widetilde{v}\|_{C(\widetilde{S})} - \delta_1/2.$$

Denote $N := n_\epsilon + n_F$, which depends only on F, $\mathcal{C}$, d, η, and δ_1. Write $x_{X^N} := \big(F^N|_{X^N}\big)^{-1}(x)$ for $\widetilde{c} \in \{b, w\}$ and $X^N \in \mathfrak{X}^N_{c\widetilde{c}}(F, \mathcal{C})$. By Definition 5.5, (5.2), and Lemma 5.7, we have

$$\begin{aligned}
\widetilde{\mathbb{L}}^N_{F,\phi}(\widetilde{v})(\widetilde{x}) &= \widetilde{\mathcal{L}}^{(N)}_{F,\phi,c,b}(v_b)(x) + \widetilde{\mathcal{L}}^{(N)}_{F,\phi,c,w}(v_w)(x) \\
&= v_{c'}(y) \exp\big(S^F_N\overline{\phi}(y) + \log u_{c'}(y) - \log u_c(x)\big) \\
&\quad + \sum_{\widetilde{c}\in\{b,w\}} \sum_{X^N\in\mathfrak{X}^N_{c\widetilde{c}}(F,\mathcal{C})\setminus\{Y^N\}} v_{\widetilde{c}}(x_{X^N}) \exp\big(S^F_N\overline{\phi}(x_{X^N}) + \log u_{\widetilde{c}}(x_{X^N}) - \log u_c(x)\big) \\
&\leqslant \big(\|\widetilde{v}\|_{C(\widetilde{S})} - \delta_1/2\big) \exp\big(S^F_N\overline{\phi}(y) + \log u_{c'}(y) - \log u_c(x)\big) \\
&\quad + \|\widetilde{v}\|_{C(\widetilde{S})} \sum_{\widetilde{c}\in\{b,w\}} \sum_{X^N\in\mathfrak{X}^N_{c\widetilde{c}}(F,\mathcal{C})\setminus\{Y^N\}} \exp\big(S^F_N\overline{\phi}(x_{X^N}) + \log u_{\widetilde{c}}(x_{X^N}) - \log u_c(x)\big) \\
&\leqslant \|\widetilde{v}\|_{C(\widetilde{S})} \sum_{\widetilde{c}\in\{b,w\}} \sum_{X^N\in\mathfrak{X}^N_{c\widetilde{c}}(F,\mathcal{C})} \exp\big(S^F_N\overline{\phi}(x_{X^N}) + \log u_{\widetilde{c}}(x_{X^N}) - \log u_c(x)\big) \\
&\quad - 2^{-1}\delta_1 \exp\big(S^F_N\overline{\phi}(y) + \log u_{c'}(y) - \log u_c(x)\big) \\
&= \|\widetilde{v}\|_{C(\widetilde{S})} - 2^{-1}\delta_1 \exp\big(S^F_N\overline{\phi}(y) + \log u_{c'}(y) - \log u_c(x)\big).
\end{aligned}$$

Similarly, there exists $z := \big(F^N|_{Z^N}\big)^{-1}(x)$ for some $Z^N \in \mathfrak{X}^N_{cc''}(F, \mathcal{C})$ such that $z \in Z^N \subseteq B_d(y_+, \epsilon) \cap X^0_{c''}$ and

$$\widetilde{\mathbb{L}}^N_{F,\phi}(\widetilde{v})(\widetilde{x}) \geqslant -\|\widetilde{v}\|_{C(\widetilde{S})} + 2^{-1}\delta_1 \exp\big(S^F_N\overline{\phi}(z) + \log u_{c''}(z) - \log u_c(x)\big).$$

Recall that $\widetilde{u}_{F,\phi} = (u_b, u_w) \in C(X_b^0) \times C(X_w^0)$ is a continuous function on $\widetilde{S}$ given by Theorem 3.35. Then by (3.32) in Theorem 3.35 we have

$$\widetilde{C}^{-1} \leqslant \widetilde{u}_{F,\phi}(\widetilde{w}) \leqslant \widetilde{C}, \qquad \text{for each } \widetilde{w} \in \widetilde{S},$$

where $\widetilde{C} \geqslant 1$ is the constant defined in (3.15) in Lemma 3.21 (ii) and depends only on F, $\mathcal{C}$, d, ϕ, and β. Hence we get

$$\begin{aligned} \big\|\widetilde{\mathbb{L}}^N_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})} &\leqslant \|\widetilde{v}\|_{C(\widetilde{S})} - 2^{-1}\delta_1 \widetilde{C}^{-2} \inf_{w \in S^2} \exp\big(S_N^F \overline{\phi}(w)\big) \\ &\leqslant \|\widetilde{v}\|_{C(\widetilde{S})} - 2^{-1}\delta_1 \widetilde{C}^{-2} \exp(-N\|\overline{\phi}\|_\infty). \end{aligned} \tag{5.14}$$

Now we bound $\|\overline{\phi}\|_\infty = \|\phi - P(F,\phi)\|_\infty$. By the definition of Hölder norm in Section 2 and the hypothesis, $\|\phi\|_\infty \leqslant \|\phi\|_{C^{0,\beta}(S^2,d)} \leqslant K$. Recall the definition of topological pressure as given in (3.1) and the Variational Principle (3.4) in Subsection 3.1. Then by (3.35) in Theorem 3.36, (3.3), and the fact that $\|\phi\|_\infty \leqslant K$, we get

$$-K \leqslant P(F,\phi) \leqslant P(f,\phi) \leqslant h_{\mathrm{top}}(f) + K.$$

Then $|P(F,\phi)| \leqslant K + h_{\mathrm{top}}(f) = K + \log(\deg f)$ (see [BM17, Corollary 17.2]). Hence

$$\|\overline{\phi}\|_\infty \leqslant \|\phi\|_\infty + |P(F,\phi)| \leqslant 2K + \log(\deg f).$$

By (3.15) in Lemma 3.21 (ii) and (3.9) in Lemma 3.12, quantitatively, we have

$$\widetilde{C} = (\deg f)^{n_F} \exp\bigg(2n_F\|\phi\|_\infty + C_0 \frac{|\phi|_{\beta,\,(S^2,d)}}{1-\Lambda^{-\beta}} (\operatorname{diam}_d(S^2))^\beta\bigg),$$

where $C_0 > 1$ is the constant depending only on f, $\mathcal{C}$, and d from Lemma 3.11. Set

$$\widetilde{C}' := (\deg f)^{n_F} \exp\bigg(2n_F K + \frac{C_0 K}{1-\Lambda^{-\beta}} (\operatorname{diam}_d(S^2))^\beta\bigg).$$

Then we have $\widetilde{C} \leqslant \widetilde{C}'$ for each $\phi \in C^{0,\beta}(S^2,d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)} \leq K$, and the constant $\widetilde{C}'$ only depends on F, $\mathcal{C}$, d, K, and β.

Therefore, by (5.14), $\big\|\widetilde{\mathbb{L}}^n_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})} \leqslant \|\widetilde{v}\|_{C(\widetilde{S})} - \delta_2$, where

$$\delta_2 := 2^{-1}\delta_1 \big(\widetilde{C}'\big)^{-2} \exp(-2NK - N\log(\deg f)),$$

which depends only on F, $\mathcal{C}$, d, η, β, K, and δ_1. □

Remark. In Lemma 5.10, one cannot reduce the assumption "$F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly primitive" to "$F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible". To see this, let F be as in Example 3.14 (i), which is strongly irreducible but not strongly primitive. In this case, $\Omega = \{p, q\}$ for some points $p \in \operatorname{inte}(X_b^0)$ and $q \in \operatorname{inte}(X_w^0)$ that satisfy $F(p) = q$ and $F(q) = p$. Set $\phi \equiv 0$ on S^2, $v_b \equiv 1$ on X_b^0, and $v_w \equiv -1$ on X_w^0. Then $\widetilde{u}_{F,\phi} = \mathbb{1}_{\widetilde{S}}$ and $(m_b, m_w) = (\delta_p/2, \delta_q/2) \in \mathcal{P}(\widetilde{S})$ is an eigenmeasure of $\mathbb{L}^*_{F,\phi}$ such that $\int \widetilde{v}\widetilde{u}_{F,\phi} \,\mathrm{d}(m_b,m_w) = 0$. However, $\widetilde{\mathbb{L}}_{F,\phi}(\widetilde{v}) = -\widetilde{v}$, which implies that $\big\|\widetilde{\mathbb{L}}^n_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})} = \|\widetilde{v}\|_{C(\widetilde{S})}$ for each $n \in \mathbb{N}$, contradicting the conclusion in Lemma 5.10.

We now establish a generalization of [LZ23, Theorem 5.17].

Theorem 5.11. *Let f, $\mathcal{C}$, F, d, Λ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly primitive. Let $\tau \in (0,+\infty)$ be a constant and $\eta\colon [0,+\infty) \to [0,+\infty)$ an abstract modulus of continuity. Let H be a bounded subset of $C^{0,\beta}(S^2,d)$ for some $\beta \in (0,1]$. Then for each $\widetilde{v} \in C^\tau_\eta(X_b^0,d) \times C^\tau_\eta(X_w^0,d)$, each $\phi \in H$, and each choice of $(m_b,m_w) \in \mathcal{P}(\widetilde{S})$ from Theorem 3.34, we have*

$$\lim_{n\to+\infty} \bigg\| \mathbb{L}^n_{F,\overline{\phi}}(\widetilde{v}) - \widetilde{u}_{F,\phi} \int \widetilde{v} \,\mathrm{d}(m_b,m_w) \bigg\|_{C(\widetilde{S})} = 0. \tag{5.15}$$

If, in addition, $\int \widetilde{v}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = 0$*, then*

$$\lim_{n\to+\infty} \big\| \widetilde{\mathbb{L}}_{F,\phi}^{n}(\widetilde{v}) \big\|_{C(\widetilde{S})} = 0. \tag{5.16}$$

Moreover, the convergence in both (5.15) *and* (5.16) *is uniform in* $\widetilde{v} \in C^{\tau}_{\eta}\big(X^0_b, d\big) \times C^{\tau}_{\eta}\big(X^0_w, d\big)$*,* $\phi \in H$*, and the choice of* (m_b, m_w)*.*

Proof. We write $\widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d) := C^{\tau}_{\eta}\big(X^0_b, d\big) \times C^{\tau}_{\eta}\big(X^0_w, d\big)$ in this proof.

Fix a constant $K \in [0, +\infty)$ such that $\|\phi\|_{C^{0,\beta}(S^2,d)} \leqslant K$ for each $\phi \in H$. Let $\mathcal{M}_{F,\phi}$ be the set of possible choices of $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ from Theorem 3.34, i.e.,

$$\mathcal{M}_{F,\phi} := \big\{ (\nu_b, \nu_w) \in \mathcal{P}(\widetilde{S}) : \mathbb{L}^{*}_{F,\phi}(\nu_b, \nu_w) = \kappa(\nu_b, \nu_w) \text{ for some } c \in \mathbb{R} \big\}. \tag{5.17}$$

We recall that $(\mu_b, \mu_w) \in \mathcal{P}(\widetilde{S})$ defined in Theorem 3.35 by $(\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w)$ depends on the choice of (m_b, m_w).

Define for each $n \in \mathbb{N}_0$,

$$a_n := \sup \Big\{ \big\| \widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v}) \big\|_{C(\widetilde{S})} : \phi \in H,\, \widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d),\, \int \widetilde{v}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = 0,\, (m_b, m_w) \in \mathcal{M}_{F,\phi} \Big\}.$$

By Lemma 5.7, $\big\|\widetilde{\mathbb{L}}_{F,\phi}\big\|_{C(\widetilde{S})} = 1$, so $\big\|\widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})}$ is non-increasing in n for fixed $\phi \in H$ and $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$. Note that $a_0 \leqslant \tau < +\infty$. Thus $\{a_n\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence of non-negative real numbers.

Suppose now that $\lim_{n\to+\infty} a_n = a > 0$. By Proposition 5.9, there exists an abstract modulus of continuity $\widetilde{\eta}$ such that

$$\big\{ \widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v}) : n \in \mathbb{N}_0,\, \phi \in H,\, \widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d) \big\} \subseteq C^{\tau}_{\widetilde{\eta}}\big(X^0_b, d\big) \times C^{\tau}_{\widetilde{\eta}}\big(X^0_w, d\big).$$

Note that for each $\phi \in H$, each $n \in \mathbb{N}_0$, and each $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$ with $\int \widetilde{v}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = 0$, it follows from (5.10) that

$$\int \widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v})\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = \int \widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v})\,\mathrm{d}(\mu_b, \mu_w) = \int \widetilde{v}\,\mathrm{d}(\mu_b, \mu_w) = 0.$$

Then by applying Lemma 5.10 with $\widetilde{\eta}$, β, K, and $\delta_1 = a/2$, we find constants $n_0 \in \mathbb{N}$ and $\delta_2 > 0$ such that

$$\big\| \widetilde{\mathbb{L}}^{n_0}_{F,\phi}\big(\widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v})\big) \big\|_{C(\widetilde{S})} \leqslant \big\| \widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v}) \big\|_{C(\widetilde{S})} - \delta_2,$$

for each $n \in \mathbb{N}_0$, each $\phi \in H$, each $(m_b, m_w) \in \mathcal{M}_{F,\phi}$, and each $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$ with $\int \widetilde{v}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = 0$ and $\big\|\widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})} \geqslant a/2$. Since $\lim_{n\to+\infty} a_n = a$, we can fix integer $m > 1$ sufficiently large such that $a_m \leqslant a + \delta_2/2$. Then for each $\phi \in H$, each $(m_b, m_w) \in \mathcal{M}_{F,\phi}$, and each $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$ with $\int \widetilde{v}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = 0$ and $\big\|\widetilde{\mathbb{L}}^{m}_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})} \geqslant a/2$. we have

$$\big\| \widetilde{\mathbb{L}}^{n_0+m}_{F,\phi}(\widetilde{v}) \big\|_{C(\widetilde{S})} \leqslant \big\| \widetilde{\mathbb{L}}^{m}_{F,\phi}(\widetilde{v}) \big\|_{C(\widetilde{S})} - \delta_2 \leqslant a_m - \delta_2 \leqslant a - \delta_2/2.$$

On the other hand, since $\big\|\widetilde{\mathbb{L}}^{n}_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{S})}$ is non-increasing in n, we have that for each $\phi \in H$, each $(m_b, m_w) \in \mathcal{M}_{F,\phi}$, and each $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$ with $\int \widetilde{v}\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b, m_w) = 0$ and $\big\|\widetilde{\mathbb{L}}^{m}_{F,\phi}(\widetilde{v})\big\|_{C(\widetilde{\Omega})} < a/2$, the following holds:

$$\big\| \widetilde{\mathbb{L}}^{n_0+m}_{F,\phi}(\widetilde{v}) \big\|_{C(\widetilde{S})} \leqslant \big\| \widetilde{\mathbb{L}}^{m}_{F,\phi}(\widetilde{v}) \big\|_{C(\widetilde{S})} < a/2.$$

Thus $a_{n_0+m} \leqslant \max\big\{a - \delta_2/2,\, a/2\big\} < a$, contradicting the fact that $\{a_n\}_{n\in\mathbb{N}_0}$ is a non-increasing sequence and the assumption that $\lim_{n\to+\infty} a_n = a$. This proves the uniform convergence in (5.16).

Next, we prove the uniform convergence in (5.15). By Lemma 5.7 and (5.8) in Lemma 5.6, for each $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$, each $\phi \in H$, and each $(m_b, m_w) \in \mathcal{M}_{F,\phi}$, we have

$$\begin{aligned}
&\left\| \mathbb{L}^n_{F,\overline{\phi}}(\widetilde{v}) - \widetilde{u}_{F,\phi} \int \widetilde{v} \,\mathrm{d}(m_b, m_w) \right\|_{C(\widetilde{S})} \\
&\quad \leqslant \|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \left\| \frac{1}{\widetilde{u}_{F,\phi}} \mathbb{L}^n_{F,\overline{\phi}}(\widetilde{v}) - \int \widetilde{v} \,\mathrm{d}(m_b, m_w) \right\|_{C(\widetilde{S})} \\
&\quad = \|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \left\| \widetilde{\mathbb{L}}^n_{F,\phi}\left(\frac{\widetilde{v}}{\widetilde{u}_{F,\phi}} \right) - \int \frac{\widetilde{v}}{\widetilde{u}_{F,\phi}} \,\mathrm{d}(\mu_b, \mu_w) \right\|_{C(\widetilde{S})} \\
&\quad = \|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \left\| \widetilde{\mathbb{L}}^n_{F,\phi}\left(\frac{\widetilde{v}}{\widetilde{u}_{F,\phi}} - \mathbb{1}_{\widetilde{S}} \int \frac{\widetilde{v}}{\widetilde{u}_{F,\phi}} \,\mathrm{d}(\mu_b, \mu_w) \right) \right\|_{C(\widetilde{S})}.
\end{aligned} \tag{5.18}$$

By (3.31) in Theorem 3.35, we have

$$\widetilde{C}^{-1} \leqslant \|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \leqslant \widetilde{C}, \tag{5.19}$$

where $\widetilde{C} \geqslant 1$ is the constant defined in (3.15) in Lemma 3.21 (ii) and depends only on F, $\mathcal{C}$, d, ϕ, and β. By (3.15) and (3.9), quantitatively,

$$\widetilde{C} = (\deg f)^{n_F} \exp\left(2n_F \|\phi\|_\infty + C_0 \frac{|\phi|_{\beta,\,(S^2,d)}}{1 - \Lambda^{-\beta}} (\mathrm{diam}_d(S^2))^\beta \right),$$

where $C_0 > 1$ is the constant depending only on f, $\mathcal{C}$, and d from Lemma 3.11 and $n_F \in \mathbb{N}$ is the constant depending only on F and $\mathcal{C}$ from Definition 3.18 since F is primitive. Set

$$\widetilde{C}' \coloneqq (\deg f)^{n_F} \exp\left(2n_F K + \frac{C_0 K}{1 - \Lambda^{-\beta}} (\mathrm{diam}_d(S^2))^\beta \right).$$

Then we have $\widetilde{C} \leqslant \widetilde{C}'$ for each $\phi \in C^{0,\beta}(S^2, d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)} \leq K$, and the constant $\widetilde{C}'$ only depends on F, $\mathcal{C}$, d, K, and β. Denote

$$\widetilde{w} \coloneqq \frac{\widetilde{v}}{\widetilde{u}_{F,\phi}} - \mathbb{1}_{\widetilde{S}} \int \frac{\widetilde{v}}{\widetilde{u}_{F,\phi}} \,\mathrm{d}(\mu_b, \mu_w) \in C(\widetilde{S}).$$

Then $\|\widetilde{w}\|_{C(\widetilde{S})} \leqslant 2\|\widetilde{v}/\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \leqslant 2\tau\widetilde{C}'$. Due to the first inequality in (3.32) and the fact that $\widetilde{u}_{F,\phi} \in C^{0,\beta}\big(X^0_b, d\big) \times C^{0,\beta}\big(X^0_w, d\big)$ by Theorem 3.35, we can apply Lemma 5.8 and conclude that there exists an abstract modulus of continuity $\widehat{\eta}$ associated with $\widetilde{v}/\widetilde{u}_{F,\phi}$ such that $\widehat{\eta}$ is independent of the choices of $\widetilde{v} \in \widetilde{C}^{\tau}_{\eta}(\widetilde{S}, d)$, $\phi \in H$, and $(m_b, m_w) \in \mathcal{M}_{F,\phi}$. Thus $\widetilde{w} \in C^{\widehat{\tau}}_{\widehat{\eta}}(X, d)$, where $\widehat{\tau} \coloneqq 2\tau\widetilde{C}'$. Note that $\int \widetilde{w}\widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = \int \widetilde{w} \,\mathrm{d}(\mu_b, \mu_w) = 0$. Finally, we can apply the uniform convergence in (5.16) with $\widetilde{v} = \widetilde{w}$ to conclude the uniform convergence in (5.15) by (5.18) and (5.19). $\square$

The following proposition is an immediate consequence of Theorem 5.11.

Proposition 5.12. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \mathrm{Sub}(f, \mathcal{C})$ is strongly primitive. Let $(\mu_b, \mu_w) \in \mathcal{P}(\widetilde{S})$ be a Borel probability measure defined in Theorem 3.35. Then for each Borel probability measure $(\nu_b, \nu_w) \in \mathcal{P}(\widetilde{S})$, we have*

$$\big(\widetilde{\mathbb{L}}^*_{F,\phi}\big)^n (\nu_b, \nu_w) \xrightarrow{w^*} (\mu_b, \mu_w) \qquad \text{as } n \to +\infty.$$

Proof. Recall that for each $\widetilde{v} \in C(\widetilde{S})$, there exists some abstract modulus of continuity η such that $\widetilde{v} \in C^{\tau}_{\eta}(S^2, d)$, where $\tau \coloneqq \|\widetilde{v}\|_{C(\widetilde{S})}$. Recall from Theorem 3.35 that $(\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w)$.

Then by Lemma 5.7 and (5.16) in Theorem 5.11,

$$\begin{aligned}
&\lim_{n\to+\infty}\big\langle\big(\widetilde{\mathbb{L}}^*_{F,\phi}\big)^n(\nu_b,\nu_w),\widetilde{v}\big\rangle\\
&\quad=\lim_{n\to+\infty}\big(\big\langle(\nu_b,\nu_w),\widetilde{\mathbb{L}}^n_{F,\phi}\big(\widetilde{v}-\langle(\mu_b,\mu_w),\widetilde{v}\rangle\mathbb{1}_{\widetilde{S}}\big)\big\rangle+\big\langle(\nu_b,\nu_w),\widetilde{\mathbb{L}}^n_{F,\phi}\big(\langle(\mu_b,\mu_w),\widetilde{v}\rangle\mathbb{1}_{\widetilde{S}}\big)\big\rangle\big)\\
&\quad=0+\big\langle(\nu_b,\nu_w),\langle(\mu_b,\mu_w),\widetilde{v}\rangle\mathbb{1}_{\widetilde{S}}\big\rangle\\
&\quad=\langle(\mu_b,\mu_w),\widetilde{v}\rangle,
\end{aligned}$$

for each $\widetilde{v}\in C(\widetilde{S})$. This completes the proof. $\square$

5.3. **Uniqueness.** In this subsection, we finish the proof of Theorem 5.1.

Theorem 5.11 implies in particular the uniqueness of $(m_b,m_w)\in\mathcal{P}(\widetilde{S})$ from Theorem 3.34.

Proposition 5.13. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C})\subseteq\mathcal{C}$ and $F\in\operatorname{Sub}(f,\mathcal{C})$ is strongly primitive. Then the measure $m_{F,\phi}=(m_b,m_w)\in\mathcal{P}(\widetilde{S})$ from Theorem 3.34 is unique, i.e., (m_b,m_w) is the unique Borel probability measure on $\widetilde{S}$ that satisfies $\mathbb{L}^*_{F,\phi}(m_b,m_w)=\kappa(m_b,m_w)$ for some constant $\kappa\in\mathbb{R}$. Moreover, the measure $\mu_{F,\phi}=(\mu_b,\mu_w):=\widetilde{u}_{F,\phi}(m_b,m_w)$ from Theorem 3.35 is the unique Borel probability measure on $\widetilde{S}$ that satisfies $\widetilde{\mathbb{L}}^*_{F,\phi}(\mu_b,\mu_w)=(\mu_b,\mu_w)$.*

Note that by Theorem 3.36, $\mu_{F,\phi}$ is an equilibrium state for F_Ω and $\phi|_\Omega$.

Proof. Let (m_b,m_w), $(m'_b,m'_w)\in\mathcal{P}(\widetilde{S})$ be two measures, both of which arise from Theorem 3.34. Note that for each $\widetilde{v}=(v_b,v_w)\in C(\widetilde{S})$, there exists some abstract modulus of continuity η such that $\widetilde{v}=(v_b,v_w)\in C^\tau_\eta\big(X^0_b,d\big)\times C^\tau_\eta\big(X^0_w,d\big)$, where $\tau:=\|\widetilde{v}\|_{C(\widetilde{S})}$. Then by (5.15) in Theorem 5.11 and (3.32) in Theorem 3.35, we see that $\int\widetilde{v}\,\mathrm{d}(m_b,m_w)=\int\widetilde{v}\,\mathrm{d}(m'_b,m'_w)$ for each $\widetilde{v}\in C(\widetilde{S})$. Thus $(m_b,m_w)=(m'_b,m'_w)$.

Recall from (5.10) that $\widetilde{\mathbb{L}}^*_{F,\phi}(\mu_b,\mu_w)=(\mu_b,\mu_w)$. Suppose $(\nu_b,\nu_w)\in\mathcal{P}(\widetilde{S})$ is another measure with $\widetilde{\mathbb{L}}^*_{F,\phi}(\nu_b,\nu_w)=(\nu_b,\nu_w)$. It suffices to show that $(\nu_b,\nu_w)=(\mu_b,\mu_w)$. Note that by (3.32) in Theorem 3.35, there exists a constant $C>0$ such that $\widetilde{u}_{F,\phi}(\widetilde{x})\geqslant C$ for each $\widetilde{x}$. Then by (5.7) in Definition 5.5, for each $\widetilde{v}\in C(\widetilde{S})$, we have

$$\begin{aligned}
\Big\langle\widetilde{\mathbb{L}}^*_{F,\phi}(\nu_b,\nu_w),\widetilde{v}\Big\rangle&=\Big\langle(\nu_b,\nu_w),\widetilde{\mathbb{L}}_{F,\phi}(\widetilde{v})\Big\rangle=\Big\langle(\nu_b,\nu_w),\frac{1}{\widetilde{u}_{F,\phi}}\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}\widetilde{v})\Big\rangle\\
&=\bigg\langle\frac{(\nu_b,\nu_w)}{\widetilde{u}_{F,\phi}},\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}\widetilde{v})\bigg\rangle=\bigg\langle\widetilde{u}_{F,\phi}\mathbb{L}^*_{F,\overline{\phi}}\bigg(\frac{(\nu_b,\nu_w)}{\widetilde{u}_{F,\phi}}\bigg),\widetilde{v}\bigg\rangle.
\end{aligned}$$

This implies $\widetilde{u}_{F,\phi}\mathbb{L}^*_{F,\overline{\phi}}\Big(\frac{(\nu_b,\nu_w)}{\widetilde{u}_{F,\phi}}\Big)=\widetilde{\mathbb{L}}^*_{F,\phi}(\nu_b,\nu_w)=(\nu_b,\nu_w)$, i.e., $\mathbb{L}^*_{F,\overline{\phi}}\Big(\frac{(\nu_b,\nu_w)}{\widetilde{u}_{F,\phi}}\Big)=\frac{(\nu_b,\nu_w)}{\widetilde{u}_{F,\phi}}$. Denote $\lambda:=\Big\langle\frac{(\nu_b,\nu_w)}{\widetilde{u}_{F,\phi}},\mathbb{1}_{\widetilde{S}}\Big\rangle>0$. Then by (3.17) we have $\mathbb{L}^*_{F,\phi}\Big(\frac{(\nu_b,\nu_w)}{\lambda\widetilde{u}_{F,\phi}}\Big)=e^{P(F,\phi)}\frac{(\nu_b,\nu_w)}{\lambda\widetilde{u}_{F,\phi}}$. Noting that $\frac{(\nu_b,\nu_w)}{\lambda\widetilde{u}_{F,\phi}}$ is also a Borel probability measure on $\widetilde{S}$, by the uniqueness of (m_b,m_w) we have $\frac{(\nu_b,\nu_w)}{\lambda\widetilde{u}_{F,\phi}}=(m_b,m_w)$. Hence $(\nu_b,\nu_w)=\lambda\widetilde{u}_{F,\phi}(m_b,m_w)=\lambda(\mu_b,\mu_w)$. Since (ν_b,ν_w), $(\mu_b,\mu_w)\in\mathcal{P}(\widetilde{S})$, we get $\lambda=1$ and $(\nu_b,\nu_w)=(\mu_b,\mu_w)$. Thus (μ_b,μ_w) is the unique Borel probability measure on $\widetilde{S}$ that satisfies $\widetilde{\mathbb{L}}^*_{F,\phi}(\mu_b,\mu_w)=(\mu_b,\mu_w)$. $\square$

We follow the conventions discussed in Remarks 3.25 and 3.26.

Lemma 5.14. *Let f, $\mathcal{C}$, F, d satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C})\subseteq\mathcal{C}$ and $F\in\operatorname{Sub}(f,\mathcal{C})$ is strongly primitive. Let $\tau\geqslant0$ be a constant and η an abstract modulus of continuity. Let H be a bounded subset of $C^{0,\beta}(S^2,d)$ for some $\beta\in(0,1]$. Fix arbitrary* *sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in S^2 and sequence $\{c_n\}_{n\in\mathbb{N}}$ of colors in $\{b,w\}$ that satisfies $x_n \in X^0_{c_n}$ for each $n\in\mathbb{N}$. Then for each $v\in C^\tau_\eta(S^2,d)$ and each $\phi\in H$, we have*

$$\lim_{n\to+\infty} \frac{\frac{1}{n}\sum\limits_{X^n\in\mathfrak{X}^n_{c_n}(F,\mathcal{C})} \bigl(S^F_n v(x_{X^n})\bigr)\exp\bigl(S^F_n\phi(x_{X^n})\bigr)}{\sum\limits_{X^n\in\mathfrak{X}^n_{c_n}(F,\mathcal{C})} \exp\bigl(S^F_n\phi(x_{X^n})\bigr)} = \int_{S^2} v\,\mathrm{d}\mu_{F,\phi}, \tag{5.20}$$

where we denote $x_{X^n} \coloneqq (F^n|_{X^n})^{-1}(x_n)$ for each $X^n\in\mathfrak{X}^n_{c_n}(F,\mathcal{C})$, and $\mu_{F,\phi}\in\mathcal{P}(S^2)$ is defined in Theorem 3.35. Moreover, the convergence is uniform in $v\in C^\tau_\eta(S^2,d)$ and $\phi\in H$.

Proof. By Lemma 3.30 and Definition 3.27, for each $n\in\mathbb{N}$, each $v\in C^\tau_\eta(S^2,d)$, and each $\phi\in H$,

$$\begin{aligned}\frac{\frac{1}{n}\sum\limits_{X^n\in\mathfrak{X}^n_{c_n}(F,\mathcal{C})} \bigl(S^F_n v(x_{X^n})\bigr)\exp\bigl(S^F_n\phi(x_{X^n})\bigr)}{\sum\limits_{X^n\in\mathfrak{X}^n_{c_n}(F,\mathcal{C})} \exp\bigl(S^F_n\phi(x_{X^n})\bigr)} &= \frac{\frac{1}{n}\sum\limits_{j=0}^{n-1}\sum\limits_{X^n\in\mathfrak{X}^n_{c_n}(F,\mathcal{C})} v(f^j(x_{X^n}))\exp\bigl(S^F_n\phi(x_{X^n})\bigr)}{\bigl(\mathbb{L}^n_{F,\phi}(\mathbb{1}_{\widetilde{S}})\bigr)(x_n,c_n)} \\ &= \frac{\frac{1}{n}\sum\limits_{j=0}^{n-1}\Bigl(\mathbb{L}^n_{F,\phi}\bigl(\widetilde{v\circ f^j}\bigr)\Bigr)(x_n,c_n)}{\bigl(\mathbb{L}^n_{F,\phi}(\mathbb{1}_{\widetilde{S}})\bigr)(x_n,c_n)},\end{aligned}$$

where, by abuse of notation, for each $u\in C(S^2)$ we denote by $\widetilde{u}$ the continuous function on $\widetilde{S}$ given by $\widetilde{u}(\widetilde{z})\coloneqq u(z)$ for each $\widetilde{z}=(z,c)\in\widetilde{S}$. Note that by Lemma 3.30 and Definition 3.27, for each $j\in\mathbb{N}_0$,

$$\mathbb{L}^j_{F,\phi}\bigl(\widetilde{v\circ f^j}\bigr) = \widetilde{v}\,\mathbb{L}^j_{F,\phi}(\mathbb{1}_{\widetilde{S}}).$$

Hence,

$$\begin{aligned}\frac{\frac{1}{n}\sum\limits_{j=0}^{n-1}\Bigl(\mathbb{L}^n_{F,\phi}\bigl(\widetilde{v\circ f^j}\bigr)\Bigr)(x_n,c_n)}{\bigl(\mathbb{L}^n_{F,\phi}(\mathbb{1}_{\widetilde{S}})\bigr)(x_n,c_n)} &= \frac{\frac{1}{n}\sum\limits_{j=0}^{n-1}\bigl(\mathbb{L}^{n-j}_{F,\phi}\bigl(\widetilde{v}\,\mathbb{L}^j_{F,\phi}(\mathbb{1}_{\widetilde{S}})\bigr)\bigr)(x_n,c_n)}{\bigl(\mathbb{L}^n_{F,\phi}(\mathbb{1}_{\widetilde{S}})\bigr)(x_n,c_n)} \\ &= \frac{\frac{1}{n}\sum\limits_{j=0}^{n-1}\bigl(\mathbb{L}^{n-j}_{F,\overline{\phi}}\bigl(\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\bigr)\bigr)(x_n,c_n)}{\bigl(\mathbb{L}^n_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\bigr)(x_n,c_n)}.\end{aligned}$$

By Proposition 5.9, $\bigl\{\mathbb{L}^n_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\colon n\in\mathbb{N}_0\bigr\}\subseteq C^{\widehat{\tau}}_{\widehat{\eta}}\bigl(X^0_b,d\bigr)\times C^{\widehat{\tau}}_{\widehat{\eta}}\bigl(X^0_w,d\bigr)$, for some constant $\widehat{\tau}\geqslant 0$ and some abstract modulus of continuity $\widehat{\eta}$, which are independent of the choice of $\phi\in H$. Thus by Lemma 5.8,

$$\bigl\{\widetilde{v}\,\mathbb{L}^n_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\colon n\in\mathbb{N}_0,\,v\in C^\tau_\eta(S^2,d)\bigr\}\subseteq C^{\tau_1}_{\eta_1}\bigl(X^0_b,d\bigr)\times C^{\tau_1}_{\eta_1}\bigl(X^0_w,d\bigr), \tag{5.21}$$

for some constant $\tau_1\geqslant 0$ and some abstract modulus of continuity η_1, which are independent of the choice of $\phi\in H$.

By Theorem 5.11 and Proposition 5.13, we have

$$\bigl\|\mathbb{L}^k_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})-\widetilde{u}_{F,\phi}\bigr\|_{C(\widetilde{S})}\longrightarrow 0, \tag{5.22}$$

as $k\to+\infty$, uniformly in $\phi\in H$. Moreover, by (5.21), the independence of τ_1 and η_1 on $\phi\in H$ in (5.21), Theorem 5.11, and Proposition 5.13, we have

$$\Bigl\|\mathbb{L}^k_{F,\overline{\phi}}\Bigl(\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\Bigr)-\widetilde{u}_{F,\phi}\int\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\,\mathrm{d}(m_b,m_w)\Bigr\|_{C(\widetilde{S})}\longrightarrow 0, \tag{5.23}$$

as $k\to+\infty$, uniformly in $j\in\mathbb{N}_0$, $\phi\in H$, and $v\in C^\tau_\eta(S^2,d)$.

Fix a constant $K \geqslant 0$ such that for each $\phi \in H$, $\|\phi\|_{C^{0,\beta}(S^2,d)} \leqslant K$. By (3.31) in Theorem 3.35, we have

$$\widetilde{C}^{-1} \leqslant \|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})} \leqslant \widetilde{C}, \tag{5.24}$$

where $\widetilde{C} \geqslant 1$ is the constant defined in (3.15) in Lemma 3.21 (ii) and depends only on F, $\mathcal{C}$, d, ϕ, and β. By (3.15) and (3.9), quantitatively,

$$\widetilde{C} = (\deg f)^{n_F} \exp\biggl(2n_F\|\phi\|_\infty + C_0\frac{|\phi|_{\beta,\,(S^2,d)}}{1-\Lambda^{-\beta}}(\operatorname{diam}_d(S^2))^\beta\biggr),$$

where $C_0 > 1$ is the constant depending only on f, $\mathcal{C}$, and d from Lemma 3.11 and $n_F \in \mathbb{N}$ is the constant depending only on F and $\mathcal{C}$ from Definition 3.18 since F is primitive. Define

$$\widetilde{C}' \coloneqq (\deg f)^{n_F} \exp\biggl(2n_F K + \frac{C_0 K}{1-\Lambda^{-\beta}}(\operatorname{diam}_d(S^2))^\beta\biggr).$$

Then we have $\widetilde{C} \leqslant \widetilde{C}'$ for each $\phi \in C^{0,\beta}(S^2,d)$ with $\|\phi\|_{C^{0,\beta}(S^2,d)} \leq K$, and the constant $\widetilde{C}'$ only depends on F, $\mathcal{C}$, d, K, and β.

Thus by (5.21), we get that for $j \in \mathbb{N}_0$, $v \in C^\tau_\eta(S^2,d)$, and $\phi \in H$,

$$\biggl\|\widetilde{u}_{F,\phi}\int \widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\,\mathrm{d}(m_b,m_w)\biggr\|_{C(\widetilde{S})} \leqslant \|\widetilde{u}_{F,\phi}\|_{C(\widetilde{S})}\Bigl\|\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\Bigr\|_{C(\widetilde{S})} \leqslant \tau_1\widetilde{C}'. \tag{5.25}$$

By (5.12) in Proposition 5.9 and (5.21), we get some constant $\tau_2 > 0$ such that for each j, $k \in \mathbb{N}_0$, each $v \in C^\tau_\eta(S^2,d)$, and each $\phi \in H$,

$$\Bigl\|\mathbb{L}^k_{F,\overline{\phi}}\Bigl(\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\Bigr)\Bigr\|_{C(\widetilde{S})} < \tau_2. \tag{5.26}$$

Hence we can conclude from (5.25), (5.26), and (5.23) that

$$\lim_{n\to+\infty}\frac{1}{n}\Biggl\|\sum_{j=0}^{n-1}\mathbb{L}^{n-j}_{F,\overline{\phi}}\Bigl(\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\Bigr) - \sum_{j=0}^{n-1}\widetilde{u}_{F,\phi}\int\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\,\mathrm{d}(m_b,m_w)\Biggr\|_{C(\widetilde{S})} = 0,$$

uniformly in $v \in C^\tau_\eta(S^2,d)$ and $\phi \in H$. Thus by (5.22) and (5.24), we have

$$\lim_{n\to+\infty}\left\|\frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathbb{L}^{n-j}_{F,\overline{\phi}}\Bigl(\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\Bigr)}{\mathbb{L}^n_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})} - \frac{\frac{1}{n}\sum_{j=0}^{n-1}\widetilde{u}_{F,\phi}\int\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\,\mathrm{d}(m_b,m_w)}{\widetilde{u}_{F,\phi}}\right\|_{C(\widetilde{S})} = 0,$$

uniformly in $v \in C^\tau_\eta(S^2,d)$ and $\phi \in H$. Combining the above with (5.21), (5.22), (5.24), and the calculation at the beginning of the proof, we can conclude, therefore, that the left-hand side of (5.20) is equal to

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int\widetilde{v}\,\mathbb{L}^j_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}})\,\mathrm{d}(m_b,m_w) &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int\widetilde{v}\,\widetilde{u}_{F,\phi}\,\mathrm{d}(m_b,m_w)\\ &= \int\widetilde{v}\,\mathrm{d}(\mu_b,\mu_w) = \int v\,\mathrm{d}\mu_{F,\phi},\end{aligned}$$

where $\mu_{F,\phi} \in \mathcal{P}(S^2)$ is defined in Theorem 3.35, and the convergence is uniform in $v \in C^\tau_\eta(S^2,d)$ and $\phi \in H$. $\square$

Theorem 5.15. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Let ϕ, $\gamma \in C^{0,\beta}(S^2, d)$ be real-valued Hölder continuous function with an exponent $\beta \in (0,1]$. Then for each $t \in \mathbb{R}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} P(F, \phi + t\gamma) = \int \gamma \,\mathrm{d}\mu_{F,\phi+t\gamma},$$

where $\mu_{F,\phi+t\gamma} \in \mathcal{P}(S^2)$ is defined in Theorem 3.35.

Proof. We will use the well-known fact from real analysis that if a sequence $\{g_n\}_{n\in\mathbb{N}}$ of real-valued differentiable functions defined on a finite interval in $\mathbb{R}$ converges pointwise to some function g and the sequence of the corresponding derivatives $\bigl\{\frac{\mathrm{d}g_n}{\mathrm{d}t}\bigr\}_{n\in\mathbb{N}}$ converges uniformly to some function h, then g is differentiable and $\frac{dg}{\mathrm{d}t} = h$.

By (3.34) in Theorem 3.35 and (3.25) in Lemma 3.30, for each $c \in \{b, w\}$, each $x \in X_c^0$, and each $\psi \in C^{0,\beta}(S^2, d)$, we have

$$P(F, \psi) = \lim_{n\to+\infty} \frac{1}{n} \log\bigl(\mathbb{L}_{F,\psi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(x, c)\bigr) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{X^n \in \mathfrak{X}_c^n(F,\mathcal{C})} \exp\bigl(S_n^F \psi(x_{X^n})\bigr), \tag{5.27}$$

where $x_{X^n} \coloneqq (F^n|_{X^n})^{-1}(x)$ for each $X^n \in \mathfrak{X}_c^n(F, \mathcal{C})$.

Fix $c \in \{b, w\}$, $x \in X_c^0$, and $\ell \in (0, +\infty)$. For each $n \in \mathbb{N}$ and each $t \in \mathbb{R}$, define

$$P_n(t) \coloneqq \frac{1}{n} \log \sum_{X^n \in \mathfrak{X}_c^n(F,\mathcal{C})} \exp\bigl(S_n^F(\phi + t\gamma)(x_{X^n})\bigr),$$

where $x_{X^n} \coloneqq (F^n|_{X^n})^{-1}(x)$ for each $X^n \in \mathfrak{X}_c^n(F, \mathcal{C})$. Observe that there exists a bounded subset H of $C^{0,\beta}(S^2, d)$ such that $\phi + t\gamma \in H$ for each $t \in (-\ell, \ell)$. Then by Lemma 5.14,

$$\frac{\mathrm{d}P_n}{\mathrm{d}t}(t) = \frac{\frac{1}{n} \sum\limits_{X^n \in \mathfrak{X}_c^n(F,\mathcal{C})} \bigl(S_n^F \gamma(x_{X^n})\bigr) \exp\bigl(S_n^F(\phi + t\gamma)(x_{X^n})\bigr)}{\sum\limits_{X^n \in \mathfrak{X}_c^n(F,\mathcal{C})} \exp\bigl(S_n^F(\phi + t\gamma)(x_{X^n})\bigr)}$$

converges to $\int \gamma \,\mathrm{d}\mu_{F,\phi+t\gamma}$ as $n \to +\infty$, uniformly in $t \in (-\ell, \ell)$.

On the other hand, by (5.27), for each $t \in (-\ell, \ell)$, we have

$$\lim_{n\to+\infty} P_n(t) = P(F, \phi + t\gamma).$$

Hence $P(F, \phi + t\gamma)$ is differentiable with respect to t on $(-\ell, \ell)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} P(F, \phi + t\gamma) = \lim_{n\to+\infty} \frac{\mathrm{d}P_n}{\mathrm{d}t}(t) = \int \gamma \,\mathrm{d}\mu_{F,\phi+t\gamma}.$$

Since $\ell \in (0, +\infty)$ is arbitrary, the proof is complete. $\square$

We record the following well-known fact for the convenience of the reader.

Lemma 5.16. *Let (X, d) be a compact metric space. Then for each $\beta \in (0, 1]$, $C^{0,\beta}(X, d)$ is a dense subset of $C(X)$ with respect to the uniform norm.*

Proof. The lemma follows from the fact that the set of Lipschitz functions is dense in $C(X)$ with respect to the uniform norm (see for example, [Hei01, Theorem 6.8]). $\square$

Now we prove the uniqueness of the equilibrium states for subsystems.

Proof of Theorem 5.1. The existence is from Theorem 3.36.

We now prove the uniqueness.

Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$ and $F_\Omega \coloneqq F|_\Omega$. Recall that for each $\varphi \in C(S^2)$, $P(F_\Omega, \varphi|_\Omega)$ is the topological pressure of $F_\Omega \colon \Omega \to \Omega$ with respect to the potential $\varphi|_\Omega$.

Since $\phi \in C^{0,\beta}(S^2,d)$ for some $\beta \in (0,1]$, it follows from (3.35) in Theorem 3.36 and Theorem 5.15 that the function

$$t \mapsto P(F_\Omega, (\phi + t\gamma)|_\Omega)$$

is differentiable at 0 for each $\gamma \in C^{0,\beta}(S^2,d)$. Write

$$W := \bigl\{\psi|_\Omega \in C^{0,\beta}(\Omega,d) : \psi \in C^{0,\beta}(S^2,d)\bigr\}.$$

By Lemma 5.16, W is a dense subset of $C(\Omega)$ with respect to the uniform norm. In particular, W is a dense subset of $C(\Omega)$ in the weak topology. We note that the topological pressure function $P(F_\Omega,\cdot)\colon C(\Omega) \to \mathbb{R}$ is convex and continuous (see for example, [PU10, Theorem 3.6.1 and Theorem 3.6.2]). Thus by Theorem 5.3 with $V = C(\Omega)$, $x = \phi$, $U = W$, and $Q = P(F_\Omega,\cdot)$, we get $\operatorname{card}\bigl(C(\Omega)^*_{\phi,P(F_\Omega,\cdot)}\bigr) = 1$.

On the other hand, if $\mu \in \mathcal{M}(\Omega,F_\Omega)$ is an equilibrium state for F_Ω and $\phi|_\Omega$, then by (3.3) and (3.4),

$$h_\mu(F_\Omega) + \int \phi \,\mathrm{d}\mu = P(F_\Omega, \phi|_\Omega),$$

and for each $\gamma \in C(\Omega)$,

$$h_\mu(F_\Omega) + \int (\phi+\gamma)\,\mathrm{d}\mu \leqslant P(F_\Omega,(\phi+\gamma)|_\Omega).$$

Thus $\int \gamma\,\mathrm{d}\mu \leqslant P(F_\Omega,(\phi+\gamma)|_\Omega) - P(F_\Omega,\phi|_\Omega)$. Then by (5.1), the continuous functional $\gamma \mapsto \int \gamma\,\mathrm{d}\mu$ on $C(\Omega)$ is in $C(\Omega)^*_{\phi,P(F_\Omega,\cdot)}$. Since $\mu_{F,\phi}$ defined in Theorem 3.35 is an equilibrium state for F_Ω and $\phi|_\Omega$, and $\operatorname{card}\bigl(C(\Omega)^*_{\phi,P(F_\Omega,\cdot)}\bigr) = 1$, we get that each equilibrium state μ for F_Ω and $\phi|_\Omega$ must satisfy $\int\gamma\,\mathrm{d}\mu = \int \gamma\,\mathrm{d}\mu_{F,\phi}$ for $\gamma \in C(\Omega)$, i.e., $\mu = \mu_{F,\phi}$.

Finally, it follows from Theorem 3.36 that the map F_Ω is forward quasi-invariant and nonsingular with respect to $\mu_{F,\phi}$. □

Remark. Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly primitive. Since the entropy map $\mu \mapsto h_\mu(F_\Omega)$ for $F_\Omega\colon \Omega \to \Omega$ is *affine* (see for example, [Wal82, Theorem 8.1]), i.e., if μ, $\nu \in \mathcal{M}(\Omega,F_\Omega)$ and $p \in [0,1]$, then $h_{p\mu+(1-p)\nu}(F_\Omega) = ph_\mu(F_\Omega) + (1-p)h_\nu(F_\Omega)$, so is the pressure map $\mu \mapsto P_\mu(F_\Omega,\phi|_\Omega)$ for F_Ω and $\phi|_\Omega$. Thus, the uniqueness of the equilibrium state $\mu_{F,\phi}$ and the Variational Principle (3.4) imply that $\mu_{F,\phi}$ is an extreme point of the convex set $\mathcal{M}(\Omega,F_\Omega)$. It follows from the fact (see for example, [PU10, Theorem 2.2.8]) that the extreme points of $\mathcal{M}(\Omega,F_\Omega)$ are exactly the ergodic measures in $\mathcal{M}(\Omega,F_\Omega)$ that $\mu_{F,\phi}$ is ergodic. However, we are going to prove a much stronger ergodic property of $\mu_{F,\phi}$ in Section 6.

Theorem 5.1 implies in particular that there exists a unique equilibrium state μ_ϕ for each expanding Thurston map $f\colon S^2 \to S^2$ together with a real-valued Hölder continuous potential ϕ.

Corollary 5.17. *Let f, d, ϕ satisfy the Assumptions in Section 4. Then there exists a unique equilibrium state μ_ϕ for the map f and the potential ϕ.*

Proof. By Lemma 3.10 we can find a sufficiently high iterate $F := f^n$ of f for some $n \in \mathbb{N}$ that has an F-invariant Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} F = \operatorname{post} f \subseteq \mathcal{C}$. Then F is also an expanding Thurston map by Remark 3.7. In particular, by [Li18, Lemma 5.10] and Definition 3.18, F is a strongly primitive subsystem of F with respect to $\mathcal{C}$ and $\Omega(F,\mathcal{C}) = S^2$.

Denote $\Phi := S_n^f \phi$. By Theorem 5.1 there exists a unique equilibrium state $\mu_{F,\Phi} \in \mathcal{M}(S^2,F)$ for the map F and the potential Φ. Note that $\mu_{F,\Phi}$ is F-invariant. Set

$$\mu := \frac{1}{n}\sum_{i=0}^{n-1} f^i_* \mu_{F,\Phi} \in \mathcal{M}(S^2,f).$$

Then we get $nh_\mu(f) = h_{\mu_{F,\Phi}}(F)$ (see for example, [LS24, Lemma 5.2]) and

$$\int \phi \,\mathrm{d}\mu = \frac{1}{n} \int \sum_{i=0}^{n-1} \phi \,\mathrm{d} f^i_* \mu_{F,\Phi} = \frac{1}{n} \int \sum_{i=0}^{n-1} \phi \circ f^i \,\mathrm{d}\mu_{F,\Phi} = \frac{1}{n} \int \Phi \,\mathrm{d}\mu_{F,\Phi}.$$

Noting that $P(F,\Phi) = nP(f,\phi)$ (recall (3.1)), we obtain

$$h_\mu(f) + \int \phi \,\mathrm{d}\mu = \frac{1}{n}\biggl(h_{\mu_{F,\Phi}}(F) + \int \Phi \,\mathrm{d}\mu_{F,\Phi} \biggr) = \frac{1}{n} P(F,\Phi) = P(f,\phi).$$

Thus μ is an equilibrium state for the map f and the potential ϕ.

Now we prove the uniqueness. Suppose ν is an equilibrium state for the map f and the potential ϕ. By similar arguments as above one sees that ν is an equilibrium state for the map F and the potential Φ. Then it follows from the uniqueness part of Theorem 5.1 that $\nu = \mu_{F,\Phi}$. □

6. Ergodic Properties

In this section, we show in Theorem 6.3 that if f, $\mathcal{C}$, F, d, and ϕ satisfied the Assumptions in Section 4, $f(\mathcal{C}) \subseteq \mathcal{C}$, and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly primitive, then the measure-preserving transformation $F|_\Omega$ of the probability space $(\Omega, \mu_{F,\phi})$ is exact (see Definition 6.1), and as an immediate consequence, mixing (see Corollary 6.6). Another consequence of Theorem 6.3 is that $\mu_{F,\phi}$ is non-atomic (see Corollary 6.4).

For each Borel measure μ on a compact metric space (X,d), we denote by $\overline{\mu}$ the *completion* of μ, i.e., $\overline{\mu}$ is the unique measure defined on the smallest σ-algebra $\overline{\mathcal{B}}$ containing all Borel sets and all subsets of μ-null sets, satisfying $\overline{\mu}(E) = \mu(E)$ for each Borel set $E \subseteq X$.

Definition 6.1. Let $T\colon X \to X$ be a measure-preserving transformation of a probability space (X,μ). Then T is called *exact* if for every measurable set E with $\mu(E) > 0$ and measurable images $T(E)$, $T^2(E)$, $\dots$, the following holds:

$$\lim_{n \to +\infty} \mu(T^n(E)) = 1.$$

Remark 6.2. Note that in Definition 6.1, we do not require μ to be a Borel measure. In the case when $F \in \operatorname{Sub}(f,\mathcal{C})$ is a subsystem of some expanding Thurston map f with respect to some Jordan curve $\mathcal{C} \subseteq S^2$ containing $\operatorname{post} f$ and μ is a Borel measure on $\Omega \coloneqq \Omega(F,\mathcal{C})$, the set $(F|_\Omega)^n(E)$ is a Borel set for each $n \in \mathbb{N}$ and each Borel subset $E \subseteq \Omega$. Indeed, a Borel set $E \subseteq \Omega$ can be covered by n-tiles in the cell decompositions of S^2 induced by f and $\mathcal{C}$. For each n-tile $X \in \mathbf{X}^n(f,\mathcal{C})$, the restriction $f^n|_X$ of f^n to X is a homeomorphism from the closed set X onto $f^n(X)$ by [BM17, Proposition 5.16 (i)]. Thus the set $f^n(E)$ is Borel. Recall from Subsection 3.3 that $F|_\Omega = f|_\Omega$ and $F(\Omega) \subseteq \Omega$. It is then clear that the set $(F|_\Omega)^n(E)$ is also Borel.

We now prove that the measure-preserving transformation $F_\Omega \coloneqq F|_\Omega$ of the probability space $(\Omega, \mu_{F,\phi})$ is exact. We follow the conventions discussed in Remarks 3.25 and 3.26.

Theorem 6.3. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly primitive. Denote $F_\Omega \coloneqq F|_{F_\Omega}$. Let $\mu_{F,\phi}$ be the unique equilibrium state for F_Ω and $\phi|_\Omega$, and $\overline{\mu_{F,\phi}}$ its completion. Then the measure-preserving transformation F_Ω of the probability space $(\Omega, \mu_{F,\phi})$ (resp. $(\Omega, \overline{\mu_{F,\phi}})$) is exact.*

Proof. Recall from Theorems 3.36 and 3.35 that $\mu_{F,\phi} = (\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w)$, where $m_{F,\phi} = (m_b, m_w)$ is an eigenmeasure of $\mathbb{L}^*_{F,\phi}$ from Theorem 3.34 and $\widetilde{u}_{F,\phi}$ is an eigenfunction of $\mathbb{L}_{F,\phi}$ from Theorem 3.35. Then by (3.32) in Theorem 3.35, it suffices to prove that

$$\lim_{n \to +\infty} m_{F,\phi}(\Omega \setminus F^n_\Omega(A)) = 0$$

for each Borel set $A \subseteq \Omega$ with $m_{F,\phi}(A) > 0$. We follow the conventions discussed in Remark 5.2 so that $m_{F,\phi} \in \mathcal{P}(\Omega) \subseteq \mathcal{P}(S^2)$.

Let $A \subseteq \Omega$ be an arbitrary Borel subset of Ω with $m_{F,\phi}(A) > 0$. Fix an arbitrary $\varepsilon > 0$. By the regularity of $m_{F,\phi}$ there exists a compact set $K \subseteq A$ and an open set $U \subseteq S^2$ with $K \subseteq A \subseteq U$ and $m_{F,\phi}(U \setminus K) < \varepsilon$. Since the diameters of tiles approach 0 uniformly as their levels becomes larger, there exists $N \in \mathbb{N}$ such that for each integer $n \geqslant N$, every n-tile that meets K is contained in the open neighborhood U of K. For each $n \in \mathbb{N}$, we define

$$\mathbf{T}^n := \{X^n \in \mathfrak{X}^n(F, \mathcal{C}) : X^n \cap K \neq \emptyset\} \quad \text{and} \quad T^n := \bigcup \mathbf{T}^n.$$

Then for each integer $n \geqslant N$, we have $K \subseteq T^n \subseteq U$ and $m_{F,\phi}(T^n \setminus A) \leqslant m_{F,\phi}(U \setminus K) < \varepsilon$. Thus, it follows from Theorem 3.34 (i) that $\sum_{X^n \in \mathbf{T}^n} m_{F,\phi}(X^n \setminus K) < \varepsilon$. This implies

$$\frac{\sum_{X^n \in \mathbf{T}^n} m_{F,\phi}(X^n \setminus K)}{\sum_{X^n \in \mathbf{T}^n} m_{F,\phi}(X^n)} < \frac{\varepsilon}{m_{F,\phi}(K)}. \tag{6.1}$$

Hence for each integer $n \geqslant N$, there exists an n-tile $Y^n \in \mathbf{T}^n$ such that

$$\frac{m_{F,\phi}(Y^n \setminus K)}{m_{F,\phi}(Y^n)} < \frac{\varepsilon}{m_{F,\phi}(K)}.$$

By Proposition 3.15 (i), the map F^n is injective on Y^n. Then it follows from Theorem 3.34 (ii), Lemma 3.12, and (6.1) that

$$\begin{aligned}
\frac{m_{F,\phi}(F^n(Y^n) \setminus F^n(K))}{m_{F,\phi}(F^n(Y^n))} &\leqslant \frac{m_{F,\phi}(F^n(Y^n \setminus K))}{m_{F,\phi}(F^n(Y^n))} = \frac{\int_{Y^n \setminus K} \exp(-S_n\phi) \,\mathrm{d}m_{F,\phi}}{\int_{Y^n} \exp(-S_n\phi) \,\mathrm{d}m_{F,\phi}} \\
&\leqslant C \frac{m_{F,\phi}(Y^n \setminus K)}{m_{F,\phi}(Y^n)} \leqslant \frac{C\varepsilon}{m_{F,\phi}(K)},
\end{aligned}$$

where $C := \exp\bigl(C_1 (\operatorname{diam}_d(S^2))^\beta\bigr)$ and $C_1 \geqslant 0$ is the constant defined in (3.9) in Lemma 3.12 which depends only on f, $\mathcal{C}$, d, ϕ, and β. Let $n_F \in \mathbb{N}$ be the constant from Definition 3.18, which depends only on f and $\mathcal{C}$. Note that it follows from Proposition 3.15 (i) that $F^n(Y^n) = X^0_c$ for some $c \in \{b, w\}$. Since F is strongly primitive, by Lemma 3.20, there exist $X^{n_F}_b \in \mathfrak{X}^{n_F}_{bc}(F, \mathcal{C})$ and $X^{n_F}_w \in \mathfrak{X}^{n_F}_{wc}(F, \mathcal{C})$ such that $X^{n_F}_b \cup X^{n_F}_w \subseteq X^0_c = F^n(Y^n)$. We claim that

$$\Omega = F^{n+n_F}(Y^n \cap \Omega). \tag{6.2}$$

Indeed, since $F(\Omega) \subseteq \Omega$ (recall Proposition 3.15 (ii)), it suffices to show that $\Omega \subseteq F^{n+n_F}(Y^n \cap \Omega)$. For each $x \in \Omega$, by (3.11), there exists a sequence of tiles $\{X^k\}_{k \in \mathbb{N}}$ such that $\{x\} = \bigcap_{k \in \mathbb{N}} X^k$ and $X^k \in \mathfrak{X}^k(F, \mathcal{C})$ for each $k \in \mathbb{N}$. By Proposition 3.9, we may assume without loss of generality that $X^k \subseteq X^0_b$ for each $k \in \mathbb{N}$. Since F is strongly primitive, by Lemma 3.20, there exists $X^{n+n_F}_b \in \mathfrak{X}^{n+n_F}_b(F, \mathcal{C})$ such that $X^{n+n_F}_b \subseteq Y^n$ and $F^{n+n_F}\bigl(X^{n+n_F}_b\bigr) = X^0_b$. Then it follows from Proposition 3.15 (i) and Lemma 3.5 (i) that $Y^{k+n+n_F} := \bigl(F^{n+n_F}|_{X^{n+n_F}_b}\bigr)^{-1}\bigl(X^k\bigr) \in \mathfrak{X}^{k+n+n_F}(F, \mathcal{C})$ for each $k \in \mathbb{N}$. Set $y := \bigl(F^{n+n_F}|_{X^{n+n_F}_b}\bigr)^{-1}(x)$. Note that $y \in Y^{k+n+n_F} \subseteq Y^n$ for each $k \in \mathbb{N}$. Thus by (3.11) and Proposition 3.15 (iii), we conclude that $y \in Y^n \cap \Omega$ and (6.2) holds.

By (6.2), we get

$$\begin{aligned}
\Omega \setminus F^{n+n_F}_\Omega(K) &= F^{n+n_F}(Y^n \cap \Omega) \setminus F^{n+n_F}(K) \subseteq F^{n_F}(F^n(Y^n \cap \Omega) \setminus F^n(K)) \\
&\subseteq F^{n_F}((F^n(Y^n) \cap \Omega) \setminus F^n(K)) = F^{n_F}((F^n(Y^n) \setminus F^n(K)) \cap \Omega).
\end{aligned}$$

Hence, by Theorem 3.34 (ii) and (iii), for each integer $n \geqslant N$,

$$\begin{aligned}
m_{F,\phi}(\Omega \setminus F^{n+n_F}_\Omega(K)) &\leqslant m_{F,\phi}(F^{n_F}((F^n(Y^n) \setminus F^n(K)) \cap \Omega)) \\
&\leqslant \int_{F^n(Y^n) \setminus F^n(K)} \exp(n_F P(F, \phi) - S_{n_F}\phi) \,\mathrm{d}m_{F,\phi} \\
&\leqslant \exp(n_F P(F, \phi) + n_F \|\phi\|_\infty) C\varepsilon / m_{F,\phi}(K).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\lim_{n\to+\infty} m_{F,\phi}\bigl(\Omega \setminus F_\Omega^{n+n_F}(K)\bigr) = 0$. This implies

$$\lim_{n\to+\infty} m_{F,\phi}(F_\Omega^n(A)) \geqslant \lim_{n\to+\infty} m_{F,\phi}(F_\Omega^n(K)) = 1.$$

Hence the measure-preserving transformation F_Ω of the probability space $(\Omega, \mu_{F,\phi})$ is exact.

Next, we observe that since F_Ω is $\mu_{F,\phi}$-measurable, and is a non-singular measure-preserving transformation of the probability space $(\Omega, \mu_{F,\phi})$ by Theorem 5.1, it follows that F_Ω is also $\overline{\mu_{F,\phi}}$-measurable, and is a measure-preserving transformation of the probability space $(\Omega, \overline{\mu_{F,\phi}})$.

To prove that the measure-preserving transformation Ω of the probability space $(\Omega, \overline{\mu_{F,\phi}})$ is exact, we consider a $\overline{\mu_{F,\phi}}$-measurable set $B \subseteq \Omega$ with $\overline{\mu_{F,\phi}}(B) > 0$. Since $\overline{\mu_{F,\phi}} \in \mathcal{P}(\Omega)$ is the completion of the Borel probability measure $\mu_{F,\phi} \in \mathcal{P}(\Omega)$, we can choose Borel subsets A and C of Ω such that $A \subseteq B \subseteq C \subseteq \Omega$ and $\overline{\mu_{F,\phi}}(B) = \overline{\mu_{F,\phi}}(A) = \overline{\mu_{F,\phi}}(C) = \mu_{F,\phi}(A) = \mu_{F,\phi}(C)$. For each $n \in \mathbb{N}$, we have $F_\Omega^n(A) \subseteq F_\Omega^n(B) \subseteq F_\Omega^n(C)$, and both $F_\Omega^n(A)$ and $F_\Omega^n(C)$ are Borel sets by Remark 6.2. Since F_Ω is forward quasi-invariant with respect to $\mu_{F,\phi}$ by Theorem 5.1, it follows that $\mu_{F,\phi}(F_\Omega^n(A)) = \mu_{F,\phi}(F_\Omega^n(C))$. Thus

$$\mu_{F,\phi}(F_\Omega^n(A)) = \overline{\mu_{F,\phi}}(F_\Omega^n(A)) = \overline{\mu_{F,\phi}}(F_\Omega^n(B)) = \overline{\mu_{F,\phi}}(F_\Omega^n(C)) = \mu_{F,\phi}(F_\Omega^n(C)).$$

Therefore, $\lim_{n\to+\infty} \overline{\mu_{F,\phi}}(F_\Omega^n(B)) = \lim_{n\to+\infty} \mu_{F,\phi}(F_\Omega^n(A)) = 1$. $\square$

The following corollary strengthens [LSZ24, Theorem 6.16 (ii)] for strongly primitive subsystems.

Corollary 6.4. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Let $\mu_{F,\phi}$ be the unique equilibrium state for F_Ω and $\phi|_\Omega$, and $m_{F,\phi}$ be as in Proposition 5.13. Then both $\mu_{F,\phi}$ and $m_{F,\phi}$ as well as their corresponding completions are non-atomic.*

Recall that a measure μ on a topological space X is called *non-atomic* if $\mu(\{x\}) = 0$ for each $x \in X$.

Proof. Recall from Theorems 3.36 and 3.35 that $\mu_{F,\phi} = (\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w)$, where $m_{F,\phi} = (m_b, m_w)$ is an eigenmeasure of $\mathbb{L}_{F,\phi}^*$ and $\widetilde{u}_{F,\phi}$ is an eigenfunction of $\mathbb{L}_{F,\phi}$. Then by (3.32) in Theorem 3.35, it suffices to prove that $\mu_{F,\phi}$ is non-atomic.

Suppose that there exists a point $x \in \Omega$ with $\mu_{F,\phi}(\{x\}) > 0$, then for each $y \in \Omega$, we have

$$\mu_{F,\phi}(\{y\}) \leqslant \max\{\mu_{F,\phi}(\{x\}), 1 - \mu_{F,\phi}(\{x\})\}.$$

Since the transformation F_Ω of $(\Omega, \mu_{F,\phi})$ is exact by Theorem 6.3, it follows that $\mu_{F,\phi}(\{x\}) = 1$ and $F_\Omega(x) = x$. By Lemma 3.20, there exist $n \in \mathbb{N}$ and $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ such that $x \notin X^n$. This implies $\mu_{F,\phi}(X^n) = 0$, which contradicts with the fact that $\mu_{F,\phi}$ is a Gibbs measure for F, $\mathcal{C}$, and ϕ (see Theorem 3.35 and Definition 3.33).

The fact that the completions are non-atomic now follows immediately. $\square$

Remark 6.5. Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Let $\mu_{F,\phi}$ be the unique equilibrium state for F_Ω and $\phi|_\Omega$ from Theorem 5.1, and $\overline{\mu_{F,\phi}}$ its completion. Then by Theorem 2.7 in [Roh49], the complete separable metric space (Ω, d) equipped the complete non-atomic measure $\overline{\mu_{F,\phi}}$ is a Lebesgue space in the sense of V. Rokhlin. We omit V. Rokhlin's definition of a *Lebesgue space* here and refer the reader to [Roh49, Section 2], since the only results we will use about Lebesgue spaces are V. Rokhlin's definition of exactness of a measure-preserving transformation on a Lebesgue space and its implication to the mixing properties. More precisely, in [Roh64], V. Rokhlin gave a definition of exactness for a measure-preserving transformation on a Lebesgue space equipped with a complete non-atomic measure, and showed [Roh64, Section 2.2] that in such a context, it is equivalent to our definition of exactness in Definition 6.1. Moreover, he proved [Roh64, Section 2.6] that if a measure-preserving transformation on a Lebesgue space equipped with a complete non-atomic measure is exact, then it is *mixing* (he actually proved that it is *mixing of all degrees*, which we will not discuss here).

It is well-known and easy to see that if g is mixing (recall (3.2)), then it is ergodic (see for example, [Wal82, Theorem 1.17]).

Corollary 6.6. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \mathrm{Sub}(f, \mathcal{C})$ is strongly primitive. Let $\mu_{F,\phi}$ be the unique equilibrium state for F_Ω and $\phi|_\Omega$, and $\overline{\mu_{F,\phi}}$ its completion. Then the measure-preserving transformation F_Ω of the probability space $(\Omega, \mu_{F,\phi})$ (resp. $(\Omega, \overline{\mu_{F,\phi}})$) is mixing and ergodic.*

Proof. By Remark 6.5, the measure-preserving transformation F_Ω of $(\Omega, \overline{\mu_{F,\phi}})$ is mixing and thus ergodic. Since any $\mu_{F,\phi}$-measurable sets A, $B \subseteq S^2$ are also $\overline{\mu_{F,\phi}}$-measurable, the measure-preserving transformation F_Ω of $(\Omega, \mu_{F,\phi})$ is also mixing and ergodic. □

Definition 6.7. Let $T\colon X \to X$ be a continuous map on a topological space X. We say $T\colon X \to X$ is *topologically transitive* if for any non-empty open subsets U, V of X, there exists $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$. We say $T\colon X \to X$ is *topologically mixing* if for any non-empty open subsets U, V of X, there exists $N \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$ for any integer $n \geqslant N$.

The following proposition is not used in this paper but should be of independent interest.

Proposition 6.8. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then the following statements hold:*

(i) *If F is irreducible, then $F|_\Omega \colon \Omega \to \Omega$ is topological transitive and has a dense forward orbit.*
(ii) *If F is primitive, then $F|_\Omega \colon \Omega \to \Omega$ is topological mixing.*

Proof. (i) Assume that F is irreducible. Note that $F(\Omega) = \Omega$ by Proposition 3.15 (iv).

We first show that $F|_\Omega \colon \Omega \to \Omega$ is topological transitive. Consider arbitrary non-empty open subsets $U \cap \Omega$ and $V \cap \Omega$ of Ω, where U and V are open subsets of S^2. Since $U \cap \Omega \neq \emptyset$ and U is open, by (3.11), Definition 3.6, and Remark 3.7, there exist $N \in \mathbb{N}$ and $X^N \in \mathfrak{X}^N(F, \mathcal{C})$ such that $X^N \subseteq U$.

Fix arbitrary $y \in V \cap \Omega$. Then by (3.11), there exists a sequence $\{Y^k\}_{k \in \mathbb{N}}$ of tiles such that $\{y\} = \bigcap_{k \in \mathbb{N}} Y^k$ and $Y^k \in \mathfrak{X}^k(F, \mathcal{C})$ for each $k \in \mathbb{N}$. By Proposition 3.9, we may assume without loss of generality that $Y^k \subseteq X^0_b$ for each $k \in \mathbb{N}$. Since F is irreducible, by Lemma 3.19, there exist $n_0 \in \mathbb{N}$ and $X^{N+n_0}_b \in \mathfrak{X}^{N+n_0}_b(F, \mathcal{C})$ such that $X^{N+n_0}_b \subseteq X^N$. Then it follows from Lemma 3.5 (i) and Proposition 3.15 (i) that $X^{k+N+n_0} \coloneqq \bigl(F^{N+n_0}|_{X^{N+n_0}_b}\bigr)^{-1}\bigl(Y^k\bigr) \in \mathfrak{X}^{k+N+n_0}(F, \mathcal{C})$ for each $k \in \mathbb{N}$. Set $x \coloneqq \bigl(F^{N+n_0}|_{X^{N+n_0}_b}\bigr)^{-1}(y)$. Note that for each $k \in \mathbb{N}$ we have $x \in X^{k+N+n_0} \subseteq X^N$ since $y \in Y^k$. Thus by (3.11) and Proposition 3.15 (iii), we conclude that $x \in X^N \cap \Omega$. This implies $y = F^{N+n_0}(x) \in F^{N+n_0}\bigl(X^N \cap \Omega\bigr) \subseteq F^{N+n_0}(U \cap \Omega)$. Since $y \in V \cap \Omega$, it follows immediately from Definition 6.7 that $F|_\Omega \colon \Omega \to \Omega$ is topologically transitive.

We now prove that there exists $x \in \Omega$ such that $\{F^n(x)\}_{n \in \mathbb{N}}$ is dense in Ω. By (3.11), Definition 3.6, and Remark 3.7, it suffices to show that there exists $x \in \Omega$ such that for each $Y \in \bigcup_{n \in \mathbb{N}} \mathfrak{X}^n(F, \mathcal{C})$, there exists $k \in \mathbb{N}$ satisfying $F^k(x) \in Y$.

Since for each $n \in \mathbb{N}$ the set $\mathfrak{X}^n(F, \mathcal{C})$ is finite, we can write the countable set $\bigcup_{n \in \mathbb{N}} \mathfrak{X}^n(F, \mathcal{C})$ as $\{Y_n\}_{n \in \mathbb{N}}$. By Proposition 3.9, for each $n \in \mathbb{N}$ there exists $c_n \in \{b, w\}$ such that $Y_n \subseteq X^0_{c_n}$. Since F is irreducible, by Lemma 3.19, there exist $n_1 \in \mathbb{N}$ and $X^{n_1}_{c_1} \in \mathfrak{X}^{n_1}_{c_1}(F, \mathcal{C})$ such that $X^{n_1}_{c_1} \subseteq Z_0 \coloneqq Y_1$. Then it follows from Lemma 3.5 (i) and Proposition 3.15 (i) that $Z_1 \coloneqq \bigl(F^{n_1}|_{X^{n_1}_{c_1}}\bigr)^{-1}\bigl(Y_1\bigr) \in \mathfrak{X}^{k_1}(F, \mathcal{C})$ for some $k_1 \in \mathbb{N}$. Note that $k_1 > n_1$ and $Z_1 \subseteq X^{n_1}_{c_1}$. Similarly, by Lemma 3.19, there exist $n_2 \in \mathbb{N}$ and $X^{n_2}_{c_2} \in \mathfrak{X}^{n_2}_{c_2}(F, \mathcal{C})$ such that $X^{n_2}_{c_2} \subseteq Z_1$ and $n_2 > k_1$. In particular, we have $F^{n_1}(X^{n_2}_{c_2}) \subseteq F^{n_1}(Z_1) = Y_1$. It follows from Lemma 3.5 (i) and Proposition 3.15 (i) that $Z_2 := \bigl(F^{n_2}|_{X^{n_2}_{c_2}}\bigr)^{-1}\bigl(Y_2\bigr) \in \mathfrak{X}^{k_2}(F, \mathcal{C})$ for some $k_2 \in \mathbb{N}$. Note that $k_2 > n_2$ and $Z_2 \subseteq X^{n_2}_{c_2}$. Then we can inductively construct a strictly increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ of integers and a sequence $\bigl\{X^{n_j}_{c_j}\bigr\}_{j\in\mathbb{N}}$ of tiles such that $X^{n_j}_{c_j} \in \mathfrak{X}^{n_j}_{c_j}(F, \mathcal{C})$, $X^{n_{j+1}}_{c_{j+1}} \subseteq X^{n_j}_{c_j}$, and $F^{n_j}\bigl(X^{n_{j+1}}_{c_{j+1}}\bigr) \subseteq Y_j$ for each $j \in \mathbb{N}$. Since $\lim_{j\to+\infty} n_j = +\infty$, it follows from Definition 3.6 and Remark 3.7 that the set $\bigcap_{j\in\mathbb{N}} X^{n_j}_{c_j}$ is a singleton set. We write $\{x\} = \bigcap_{j\in\mathbb{N}} X^{n_j}_{c_j}$. Then by (3.11) and Proposition 3.15 (iii), we have $x \in \Omega$. This finishes the proof since $F^{n_j}(x) \in F^{n_j}\bigl(X^{n_{j+1}}_{c_{j+1}}\bigr) \subseteq Y_j$ for each $j \in \mathbb{N}$.

(ii) Let F be primitive and n_F be the constant from Definition 3.18, which depends only on F and $\mathcal{C}$. Note that $F(\Omega) = \Omega$ by Proposition 3.15 (iv). Consider arbitrary non-empty open subsets $U \cap \Omega$ and $V \cap \Omega$ of Ω, where U and V are open subsets of S^2. Since $U \cap \Omega \neq \emptyset$ and U is open, by (3.11), Definition 3.6, and Remark 3.7, there exist $N \in \mathbb{N}$ and $X^N \in \mathfrak{X}^N(F, \mathcal{C})$ such that $X^N \subseteq U$. We claim that $\Omega \subseteq F^{N+n_F}\bigl(X^N \cap \Omega\bigr)$.

Indeed, for each $y \in \Omega$, by (3.11), there exists a sequence $\bigl\{Y^k\bigr\}_{k\in\mathbb{N}}$ of tiles such that $\{y\} = \bigcap_{k\in\mathbb{N}} Y^k$ and $Y^k \in \mathfrak{X}^k(F, \mathcal{C})$ for each $k \in \mathbb{N}$. By Proposition 3.9, we may assume without loss of generality that $Y^k \subseteq X^0_b$ for each $k \in \mathbb{N}$. Since F is primitive, by Lemma 3.20, there exists $X^{N+n_F}_b \in \mathfrak{X}^{N+n_F}_b(F, \mathcal{C})$ such that $X^{N+n_F}_b \subseteq X^N$. Then it follows from Lemma 3.5 (i) and Proposition 3.15 (i) that $X^{k+N+n_F} := \bigl(F^{N+n_F}|_{X^{N+n_F}_b}\bigr)^{-1}\bigl(Y^k\bigr) \in \mathfrak{X}^{k+N+n_F}(F, \mathcal{C})$ for each $k \in \mathbb{N}$. Set $x := \bigl(F^{N+n_F}|_{X^{N+n_F}_b}\bigr)^{-1}(y)$. Note that for each $k \in \mathbb{N}$ we have $x \in X^{k+N+n_F} \subseteq X^N$ since $y \in Y^k$. Thus by (3.11) and Proposition 3.15 (iii), we conclude that $x \in X^N \cap \Omega$. This implies $\Omega \subseteq F^{N+n_F}(X^N \cap \Omega)$ since $y \in \Omega$ is arbitrary and $y = F^{N+n_F}(x)$.

Hence, for each integer $n \geqslant N + n_F$, we have $(F|_\Omega)^n(U \cap \Omega) = F^n(U \cap \Omega) \supseteq F^n\bigl(X^N \cap \Omega\bigr) \supseteq F^{n-N-n_F}(\Omega) = \Omega \supseteq V \cap \Omega \neq \emptyset$. This implies that $F|_\Omega \colon \Omega \to \Omega$ is topologically mixing by Definition 6.7. $\square$

7. Equidistribution

In this section, we establish equidistribution results for preimages for subsystems of expanding Thurston maps.

Theorem 7.1. *Let f, $\mathcal{C}$, F, d, and ϕ satisfied the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Let $\mu_{F,\phi}$ be the unique equilibrium state for $F|_\Omega$ and $\phi|_\Omega$, and let $m_{F,\phi}$ be as in Proposition 5.13. Fix arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in S^2 and sequence $\{c_n\}_{n\in\mathbb{N}}$ of colors in $\{b, w\}$ that satisfies $x_n \in X^0_{c_n}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define the Borel probability measures*

$$\nu_n := \frac{1}{Z_n(\phi)} \sum_{y\in F^{-n}(x_n)} \deg_{c_n}(F^n, y) \exp\bigl(S^F_n \phi(y)\bigr)\delta_y,$$

$$\widehat{\nu}_n := \frac{1}{Z_n(\phi)} \sum_{y\in F^{-n}(x_n)} \deg_{c_n}(F^n, y) \exp\bigl(S^F_n \phi(y)\bigr)\frac{1}{n}\sum_{i=0}^{n-1} \delta_{F^i(y)},$$

where $Z_n(\phi) := \sum_{y\in F^{-n}(x_n)} \deg_{c_n}(F^n, y) \exp\bigl(S^F_n \phi(y)\bigr)$. Then we have

$$\nu_n \xrightarrow{w^*} m_{F,\phi} \quad \text{as } n \to +\infty, \tag{7.1}$$

$$\widehat{\nu}_n \xrightarrow{w^*} \mu_{F,\phi} \quad \text{as } n \to +\infty. \tag{7.2}$$

We follow the conventions discussed in Remarks 3.25 and 3.26 in this subsection. Recall that $\widetilde{S}$ is the split sphere defined in Definition 3.23.

Proof. Fix an arbitrary $u \in C(S^2)$. We denote by $\widetilde{u}$ the continuous function on $\widetilde{S}$ given by $\widetilde{u}(\widetilde{z}) \coloneqq u(z)$ for each $\widetilde{z} = (z, c) \in \widetilde{S}$.

We prove (7.1) by showing that $\lim_{n \to +\infty} \langle \nu_n, u \rangle = \langle m_{F,\phi}, u \rangle$. Indeed, by Lemma 3.30, Definition 3.27, and (3.17), we have

$$\langle \nu_n, u \rangle = \frac{\mathbb{L}^n_{F,\phi}(\widetilde{u})(x_n, c_n)}{\mathbb{L}^n_{F,\phi}\big(\mathbb{1}_{\widetilde{S}}\big)(x_n, c_n)} = \frac{\mathbb{L}^n_{F,\overline{\phi}}(\widetilde{u})(x_n, c_n)}{\mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(x_n, c_n)}.$$

Note that by (5.15) in Theorem 5.11,

$$\big\| \mathbb{L}^n_{F,\overline{\phi}}(\mathbb{1}_{\widetilde{S}}) - \widetilde{u}_{F,\phi} \big\|_{C(\widetilde{S})} \longrightarrow 0 \quad \text{and} \quad \bigg\| \mathbb{L}^n_{F,\overline{\phi}}(\widetilde{u}) - \widetilde{u}_{F,\phi} \int \widetilde{u} \,\mathrm{d}(m_b, m_w) \bigg\|_{C(\widetilde{S})} \longrightarrow 0$$

as $n \to +\infty$, where $\widetilde{u}_{F,\phi} \in C(\widetilde{S})$ is an eigenfunction of $\mathbb{L}_{F,\phi}$ from Theorem 3.35, and $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ is an eigenmeasure of $\mathbb{L}^*_{F,\phi}$ from Theorem 3.34. Then it follows from (3.32) in Theorem 3.35 that

$$\lim_{n \to +\infty} \frac{\mathbb{L}^n_{F,\overline{\phi}}(\widetilde{u})(x_n, c_n)}{\mathbb{L}^n_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)(x_n, c_n)} = \int \widetilde{u} \,\mathrm{d}(m_b, m_w) = \int u \,\mathrm{d}m_{F,\phi}.$$

Hence, (7.1) holds.

Finally, (7.2) follows directly from Lemma 5.14. □

8. Large deviation principles for subsystems

In this section, we establish level-2 large deviation principles for iterated preimages for subsystems of expanding Thurston maps without periodic critical point.

8.1. **Level-2 large deviation principles.**

We first review some basic concepts and results from large deviation theory. We refer the reader to [DZ09, Ell12, RAS15] for more details.

Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence of Borel probability measures on a topological space $\mathcal{X}$. We say that $\{\xi_n\}_{n \in \mathbb{N}}$ satisfies a *large deviation principle* if there exists a lower semi-continuous function $I \colon \mathcal{X} \to [0, +\infty]$ such that

$$\liminf_{n \to +\infty} \frac{1}{n} \log \xi_n(\mathcal{G}) \geqslant - \inf_{\mathcal{G}} I \quad \text{for all open } \mathcal{G} \subseteq \mathcal{X}, \tag{8.1}$$

and

$$\limsup_{n \to +\infty} \frac{1}{n} \log \xi_n(\mathcal{K}) \leqslant - \inf_{\mathcal{K}} I \quad \text{for all closed } \mathcal{K} \subseteq \mathcal{X}, \tag{8.2}$$

where $\log 0 = -\infty$ and $\inf \emptyset = +\infty$ by convention. Such a function I is then unique when $\mathcal{X}$ is metrizable; it is called the *rate function*.

Definition 8.1. Let $T \colon X \to X$ be a continuous map on a compact metric space X. The *entropy map* of T is the map $\mu \mapsto h_\mu(T)$ which is defined on $\mathcal{M}(X, T)$ and has values in $[0, +\infty]$. Here $\mathcal{M}(X, T)$ is equipped with the weak* topology. We say that the entropy map of T is *upper semi-continuous* if $\limsup_{n \to +\infty} h_{\mu_n}(T) \leqslant h_\mu(T)$ holds for every sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of Borel probability measures on X which converges to $\mu \in \mathcal{M}(X, T)$ in the weak* topology.

We recall the following theorem due to Y. Kifer [Kif90, Theorem 4.3], reformulated by H. Comman and J. Rivera-Letelier [CRL11, Theorem C].

Theorem 8.2 (Y. Kifer [Kif90]; H. Comman & J. Rivera-Letelier [CRL11])**.** *Let X be a compact metrizable topological space, and let $g \colon X \to X$ be a continuous map. Fix $\varphi \in C(X)$, and let H*

be a dense vector subspace of $C(X)$ with respect to the uniform norm. Let $I_\phi \colon \mathcal{P}(X) \to [0,+\infty]$ be the function defined by

$$I_\varphi(\mu) \coloneqq \begin{cases} P(g,\varphi) - h_\mu(g) - \int \varphi \,\mathrm{d}\mu & \text{if } \mu \in \mathcal{M}(X,g); \\ +\infty & \text{if } \mu \in \mathcal{P}(X) \setminus \mathcal{M}(X,g). \end{cases}$$

We assume the following conditions are satisfied:

(i) *The measure-theoretic entropy $h_\mu(g)$ of g, as a function of μ defined on $\mathcal{M}(X,g)$ (equipped with the weak* topology), is finite and upper semi-continuous.*

(ii) *For each $\psi \in H$, there exists a unique equilibrium state for the map g and the potential $\varphi + \psi$.*

Then every sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of Borel probability measures on $\mathcal{P}(X)$ that satisfies the property that for each $\psi \in H$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \int_{\mathcal{P}(X)} \exp\biggl(n \int \psi \,\mathrm{d}\mu\biggr) \,\mathrm{d}\xi_n(\mu) = P(g,\varphi+\psi) - P(g,\varphi) \tag{8.3}$$

satisfies a large deviation principle with rate function I_φ, and it converges in the weak* topology to the Dirac measure supported on the unique equilibrium state for the map g and the potential φ. Furthermore, for each convex open subset $\mathcal{G}$ of $\mathcal{P}(X)$ containing some invariant measure, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log \xi_n(\mathcal{G}) = \lim_{n\to+\infty} \frac{1}{n} \log \xi_n(\overline{\mathcal{G}}) = -\inf_{\mathcal{G}} I_\varphi = -\inf_{\overline{\mathcal{G}}} I_\varphi.$$

Recall that $P(g,\varphi)$ is the topological pressure of the map g with respect to the potential φ.

8.2. **Characterizations of pressures.** Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. Suppose that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. In this subsection, we characterize the pressure function $P(F,\phi)$ in terms of Birkhoff averages (Proposition 8.3) and iterated preimages (Proposition 8.4).

Proposition 8.3. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. Denote $\Omega \coloneqq \Omega(F,\mathcal{C})$. Then for each $\psi \in C^{0,\beta}(S^2,d)$, we have*

$$P(F,\phi+\psi) - P(F,\phi) = \lim_{n\to+\infty} \frac{1}{n} \log \int \exp\bigl(S_n^F \psi\bigr) \,\mathrm{d}\mu_{F,\phi}, \tag{8.4}$$

where $P(F,\phi)$ and $P(F,\phi+\psi)$ are defined in (3.16).

Proof. Recall from Theorems 3.36 and 3.35 that $\mu_{F,\phi} = (\mu_b,\mu_w) = \widetilde{u}_{F,\phi}(m_b,m_w)$, where $m_{F,\phi} = (m_b,m_w)$ is an eigenmeasure of $\mathbb{L}_{F,\phi}^*$ from Theorem 3.34 and $\widetilde{u}_{F,\phi}$ is an eigenfunction of $\mathbb{L}_{F,\phi}$ from Theorem 3.35. Recall from Definition 3.23 and Remark 3.25 that $\widetilde{S} = X_b^0 \sqcup X_w^0$ is the disjoint union of X_b^0 and X_w^0. Note that by Theorem 3.34 (iii), we have $\mathbb{L}_{F,\phi}^*(m_b,m_w) = e^{P(F,\phi)}(m_b,m_w)$. Since $\inf_{\widetilde{S}} \widetilde{u}_{F,\phi} > 0$ and $\sup_{\widetilde{S}} \widetilde{u}_{F,\phi} < +\infty$, it is enough to prove the limit with $\mu_{F,\phi}$ replaced by $m_{F,\phi}$.

For each $n \in \mathbb{N}$ we have

$$\begin{aligned} \int_{S^2} \exp\bigl(S_n^F\psi\bigr) \,\mathrm{d}m_{F,\phi} &= \int_{\widetilde{S}} \bigl(e^{S_n^F\psi}|_{X_b^0}, e^{S_n^F\psi}|_{X_w^0}\bigr) \,\mathrm{d}(m_b,m_w) \\ &= \int_{\widetilde{S}} \bigl(e^{S_n^F\psi}|_{X_b^0}, e^{S_n^F\psi}|_{X_w^0}\bigr) \,\mathrm{d}\bigl(e^{-nP(F,\phi)}(\mathbb{L}_{F,\phi}^*)^n(m_b,m_w)\bigr) \\ &= e^{-nP(F,\phi)} \int_{\widetilde{S}} \mathbb{L}_{F,\phi}^n\bigl(e^{S_n^F\psi}|_{X_b^0}, e^{S_n^F\psi}|_{X_w^0}\bigr) \,\mathrm{d}(m_b,m_w). \end{aligned}$$

Using $\mathbb{L}^n_{F,\phi}\big(e^{S_n^F\psi}|_{X_b^0}, e^{S_n^F\psi}|_{X_w^0}\big) = \mathbb{L}^n_{F,\phi+\psi}\mathbb{1}_{\widetilde{S}}$, the assertion of the proposition is then a direct consequence of (3.34) in Theorem 3.35. □

The following proposition characterizes topological pressures in terms of iterated preimages.

Proposition 8.4. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$. Then for each $y_0 \in \Omega \setminus \mathcal{C}$, we have*

$$P(F,\phi) = P(F|_\Omega, \phi|_\Omega) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{x\in(F|_\Omega)^{-n}(y_0)} \exp\big(S_n^F\phi(x)\big), \tag{8.5}$$

where $P(F,\phi)$ is defined in (3.16) *and $P(F|_\Omega, \phi|_\Omega)$ is defined in* (3.1)*.*

Proposition 8.4 follows immediately from [LSZ24, Proposition 6.21 and Theorem 6.30]. Note that $\Omega \setminus \mathcal{C} \neq \emptyset$ by [LSZ24, Proposition 5.20 (ii)].

8.3. **Proof of large deviation principles.** In this subsection, we establish Theorem 1.2 by applying Theorem 8.2.

By the following two lemmas, we can show that conditions (i) and (ii) in Theorem 8.2 are satisfied in our context.

Lemma 8.5. *Let $T\colon X \to X$ be a continuous map on a compact metric space X. Suppose that the entropy map of T is upper semi-continuous. Let Y be a compact subset of X with $T(Y) \subseteq Y$. Then the entropy map of $T|_Y$ is upper semi-continuous.*

Proof. Since $\mathcal{M}(Y, T|_Y) \subseteq \mathcal{M}(X,T)$ and $h_\mu(T|_Y) = h_\mu(T)$ for each $\mu \in \mathcal{M}(Y, T|_Y)$, the statement follows. □

Lemma 8.6. *Let (X,d) be a metric space and Y be a subset of X. Then for each $\beta \in (0,1]$, we have*

$$C^{0,\beta}(Y,d) = \big\{\psi|_Y : \psi \in C^{0,\beta}(X,d)\big\}.$$

Proof. For each $\psi \in C^{0,\beta}(X,d)$, it follows immediately from the definition of Hölder continuity that $\psi|_Y \in C^{0,\beta}(Y,d)$. The converse direction also holds since every function in $C^{0,\beta}(Y,d)$ can extend to a function in $C^{0,\beta}(X,d)$ (see for example, [Hei01, Theorem 6.2 and p. 44]). □

Now we are ready to prove the level-2 large deviation principles.

Proof of Theorem 1.2. First note that by Remark 3.8, if $f\colon X \to X$ is a postcritically-finite rational map with no periodic critical points on the Riemann sphere $X = \widehat{\mathbb{C}}$, then the classes of Hölder continuous functions on $\widehat{\mathbb{C}}$ equipped with the chordal metric and on $S^2 = \widehat{\mathbb{C}}$ equipped with any visual metric for f are the same. Thus we only need to prove for the case where $f\colon S^2 \to S^2$ is an expanding Thurston map with no periodic critical points on a topological 2-sphere S^2 equipped with a visual metric d for f.

Let $\phi \in C^{0,\beta}(S^2,d)$ for some $\beta \in (0,1]$.

We apply Theorem 8.2 with $X = \Omega$, $g = F|_\Omega$, $\varphi = \phi|_\Omega$, and $H = C^{0,\beta}(\Omega,d)$. Note that $P(F,\phi) = P(F|_\Omega, \phi|_\Omega)$ by Proposition 8.4, and $C^{0,\beta}(\Omega,d)$ is dense in $C(\Omega)$ with respect to the uniform norm by Lemma 5.16. By [LSZ24, Lemma 6.4] and (3.35) in Theorem 3.36, the measure-theoretic entropy $h_\mu(F|_\Omega)$ is finite for each $\mu \in \mathcal{M}(\Omega, F|_\Omega)$. Since f has no periodic critical points, it follows from [LS24, Theorem 1.1] that the entropy of f is upper semi-continuous. Then by Lemma 8.5, the entropy map of $F|_\Omega = f|_\Omega$ is upper semi-continuous. Thus, condition (i) in Theorem 8.2 is satisfied. Condition (ii) in Theorem 8.2 follows from Theorem 5.1 and Lemma 8.6.

It now suffices to verify (8.3) for each of the sequences $\{\Sigma_n\}_{n\in\mathbb{N}}$ and $\{\Omega_n(x_n)\}_{n\in\mathbb{N}}$ of Borel probability measures on $\mathcal{P}(\Omega)$.

Fix an arbitrary $\psi \in C^{0,\beta}(\Omega,d)$. By Lemma 8.6, there exists $\widetilde{\psi} \in C^{0,\beta}(S^2,d)$ such that $\widetilde{\psi}|_\Omega = \psi$.

For the sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$, by (8.4) in Proposition 8.3 and (3.35) in Theorem 3.36, we have

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{n}\log\int_{\mathcal{P}(\Omega)}\exp\Bigl(n\int\psi\,\mathrm{d}\mu\Bigr)\,\mathrm{d}\Sigma_n(\mu) &= \lim_{n\to+\infty}\frac{1}{n}\log\int_{\Omega}\exp(S_n^F\psi)\,\mathrm{d}\mu_{F,\phi}\\ &= P(F|_\Omega,\phi|_\Omega+\psi)-P(F|_\Omega,\phi|_\Omega).\end{aligned}$$

Similarly, for the sequence $\{\Omega_n(x_n)\}_{n\in\mathbb{N}}$, by (8.5) in Proposition 8.4, we have

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{n}\log\int_{\mathcal{P}(\Omega)}\exp\Bigl(n\int\psi\,\mathrm{d}\mu\Bigr)\,\mathrm{d}\Omega_n(x_n)(\mu)\\ &\quad=\lim_{n\to+\infty}\frac{1}{n}\log\sum_{y\in(F|_\Omega)^{-n}(x_n)}\frac{\exp(S_n^F\phi(y))}{\sum_{y'\in(F|_\Omega)^{-n}(x_n)}\exp(S_n^F\phi(y'))}e^{\sum_{i=1}^{n-1}\psi(F^i(y))}\\ &\quad=\lim_{n\to+\infty}\frac{1}{n}\Biggl(\log\sum_{y\in(F|_\Omega)^{-n}(x_n)}e^{S_n^F(\phi+\widetilde{\psi})(y)}-\log\sum_{y'\in(F|_\Omega)^{-n}(x_n)}e^{S_n^F\phi(y')}\Biggr)\\ &\quad=P(F|_\Omega,\phi|_\Omega+\psi)-P(F|_\Omega,\phi|_\Omega).\end{aligned}$$

Therefore, all the assertions of Theorem 1.2 follow from Theorem 8.2. □

We finally prove Corollary 1.3, which gives a characterization of measure-theoretic pressure.

Proof of Corollary 1.3. Fix $\mu\in\mathcal{M}(\Omega,F|_\Omega)$ and a convex local basis G_μ at μ. We show that (1.5) in Corollary 1.3 holds. By (1.3) and the upper semi-continuity of $h_\mu(F|_\Omega)$ ([LS24, Theorem 1.1] and Lemma 8.5), we get

$$-I_\phi(\mu)=\inf_{\mathcal{G}\in G_\mu}\sup_{\mathcal{G}}(-I_\phi)=\inf_{\mathcal{G}\in G_\mu}\bigl(-\inf_{\mathcal{G}}I_\phi\bigr).$$

Then it follows from (1.3) and (1.4) in Theorem 1.2 that

$$\begin{aligned}-P(F,\phi)+h_\mu(F|_\Omega)+\int\phi\,\mathrm{d}\mu&=-I_\phi(\mu)=\inf_{\mathcal{G}\in G_\mu}\bigl(-\inf_{\mathcal{G}}I_\phi\bigr)\\ &=\inf_{\mathcal{G}\in G_\mu}\Bigl\{\lim_{n\to+\infty}\frac{1}{n}\log\mu_{F,\phi}(\{x\in\Omega:V_n(x)\in\mathcal{G}\})\Bigr\}\\ &=\inf_{\mathcal{G}\in G_\mu}\Biggl\{\lim_{n\to+\infty}\frac{1}{n}\log\sum_{y\in(F|_\Omega)^{-n}(x_n),V_n(y)\in\mathcal{G}}\frac{\exp\bigl(S_n^F\phi(y)\bigr)}{Z_n(\phi)}\Biggr\},\end{aligned}$$

where we write $Z_n(\phi):=\sum_{y\in(F|_\Omega)^{-n}(x_n)}\exp\bigl(S_n^F\phi(y)\bigr)$. Note that by Propositions 8.4 we have $P(F,\phi)=\lim_{n\to+\infty}\frac{1}{n}\log Z_n(\phi)$. Thus (1.5) holds. □

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