THERMODYNAMIC FORMALISM FOR SUBSYSTEMS OF EXPANDING THURSTON MAPS AND LARGE DEVIATIONS ASYMPTOTICS

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Abstract. Expanding Thurston maps were introduced by M. Bonk and D. Meyer with motivation from complex dynamics and Cannon's conjecture from geometric group theory via Sullivan's dictionary. In this paper, we introduce subsystems of expanding Thurston maps motivated via Sullivan's dictionary as analogs of certain subgroups. We develop thermodynamic formalism to prove the Variational Principle and the existence of equilibrium states for strongly irreducible subsystems and real-valued Hölder continuous potentials. Here, the sphere S^2 is equipped with a natural metric, called a visual metric, introduced by M. Bonk and D. Meyer. As an application, we establish large deviation asymptotics for expanding Thurston maps.

Contents

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1. Introduction

A Thurston map is a (non-homeomorphic) branched covering map on a topological 2-sphere S^2 such that each of its critical points has a finite orbit (postcritically-finite). The most important examples are given by postcritically-finite rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. While Thurston maps are purely topological objects, a deep theorem due to W. P. Thurston characterizes Thurston maps that are, in a suitable sense, described in the language of topology and combinatorics, equivalent to postcritically-finite rational maps (see [DH93]). This suggests that for the relevant rational maps, an explicit analytic expression is not so important, but rather a geometric-combinatorial description. This viewpoint is natural and fruitful for considering more general dynamics that are not necessarily conformal.

In the early 1980s, D. P. Sullivan introduced a "dictionary" that is now known as *Sullivan's dictionary*, which connects two branches of conformal dynamics, iterations of rational maps and actions of Kleinian groups. Under Sullivan's dictionary, the counterpart to Thurston's theorem in geometric group theory is Cannon's Conjecture [Can94]. An equivalent formulation of Cannon's Conjecture, viewed from a quasisymmetric uniformization perspective ([Bon06, Conjecture 5.2]), predicts that if the boundary at infinity $\partial_\infty G$ of a Gromov hyperbolic group G is homeomorphic to S^2, then $\partial_\infty G$ equipped with a visual metric is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$.

Inspired by Sullivan's dictionary and their interest in Cannon's Conjecture, M. Bonk and D. Meyer [BM10, BM17], as well as P. Haïssinsky and K. M. Pilgrim [HP09], studied a subclass of Thurston maps, call *expanding Thurston maps*, by imposing some additional condition of expansion. These maps are characterized by a contraction property for inverse images (see Subsection 3.2 for a precise definition). In particular, a postcritically-finite rational map on $\widehat{\mathbb{C}}$ is expanding if and only if its Julia set is equal to $\widehat{\mathbb{C}}$. For an expanding Thurston map on S^2, we can equip S^2 with a natural class of metrics d, called *visual metrics*, that are quasisymmetrically equivalent to each other and are constructed in a similar way as the visual metrics on the boundary $\partial_\infty G$ of a Gromov hyperbolic group G (see [BM17, Chapter 8] for details, and see [HP09] for a related construction). In the language above, the following theorem was obtained in [BM17, HP09], which can be seen as an analog of Cannon's conjecture for expanding Thurston maps as a positive result.

Theorem (Bonk–Meyer; Haïssinky–Pilgrim). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map with no periodic critical points and d be a visual metric for f. Then f is topologically conjugate to a rational map if and only (S^2, d) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$.*

The dynamical systems that we study in this paper are called *subsystems* of expanding Thurston maps (see Subsections 1.1 or 5.1 for a precise definition), inspired by a translation of the notion of subgroups from geometric group theory via Sullivan's dictionary. To reveal the connections between subsystems and subgroups, we first quickly review some backgrounds in Gromov hyperbolic groups and recall the notion of *tile graphs* for expanding Thurston maps (see [BM17, Chapters 4 and 10] for details).

Let G be a Gromov hyperbolic group and S a finite generating set of G. Then the *Cayley graph* $\mathcal{G}(G,S)$ of G is Gromov hyperbolic with respect to the word-metric. The boundary at infinity of G is defined as $\partial_\infty G := \partial_\infty \mathcal{G}(G,S)$, which is well-defined since a change of the generating set induces a quasi-isometry of the Cayley graphs.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ a Jordan curve with $\operatorname{post} f \subseteq S^2$. An associated *tile graph* $\mathcal{G}(f,\mathcal{C})$ is defined as follows. Its vertices are given by the tiles in the cell decompositions $\mathcal{D}^n(f,\mathcal{C})$ on all levels $n \in \mathbb{N}_0$. We consider $X^{-1} := S^2$ as a tile of level -1 and add it as a vertex. One joins two vertices by an edge if the corresponding tiles intersect and have levels differing by at most 1 (see [BM17, Chapter 10] for details). Then the graph $\mathcal{G}(f,\mathcal{C})$ is Gromov hyperbolic and its boundary at infinity $\partial_\infty \mathcal{G} := \partial_\infty \mathcal{G}(f,\mathcal{C})$ is well-defined and can be naturally identified with S^2. Moreover, under this identification, a metric on $\partial_\infty \mathcal{G} \cong S^2$ is visual in the sense of Gromov hyperbolic spaces if and only if it is visual in the sense of expanding Thurston maps.

From the point of view of tile graphs and Cayley graphs, roughly speaking, for expanding Thurston maps, 1-tiles together with the maps restricted to those tiles play the role of generators for Gromov hyperbolic groups. For example, one can construct the original expanding Thurston map f from all its 1-tiles and the maps restricted to those tiles. If we start with all n-tiles for some $n \in \mathbb{N}$, then we get an iteration f^n of f, which corresponds to a finite index subgroup of the original group in the group setting. Inspired by this similarity, it is natural to investigate more general cases, for example, a map generated by some 1-tiles, which leads to our study of subsystems. Moreover, the notion of tile graphs can be easily generalized to subsystems, and the similar identifications hold with S^2 generalized to the tile maximal invariant sets associated with subsystems.

Under Sullivan's dictionary, an expanding Thurston map corresponds to a Gromov hyperbolic group whose boundary at infinity is S^2. In this sense, a subsystem corresponds to a Gromov hyperbolic group whose boundary at infinity is a subset of S^2. In particular, for Gromov hyperbolic groups whose boundary at infinity is a Sierpiński carpet, there is an analog of Cannon's conjecture—the Kapovich–Kleiner conjecture. It predicts that these groups arise from some standard situation in hyperbolic geometry. Similar to Cannon's conjecture, one can reformulate the Kapovich–Kleiner conjecture in an equivalent way as a question related to quasisymmetric uniformization. For subsystems, it is easy to find examples where the tile maximal invariant set is homeomorphic to the standard Sierpiński carpet (see Subsection 5.1 for examples of subsystems). In this case, an analog of the Kapovich–Kleiner conjecture for subsystems is established in [BLL24].

In this paper, we study the dynamics of subsystems of expanding Thurston maps from the point of view of ergodic theory. Ergodic theory has been an essential tool in the study of dynamical systems. The investigation of the existence and uniqueness of invariant measures and their properties has been a central part of ergodic theory. However, a dynamical system may possess a large class of invariant measures, some of which may be more interesting than others. It is, therefore, crucial to examine the relevant invariant measures.

The *thermodynamic formalism* serves as a viable mechanism for generating invariant measures endowed with desirable properties. More precisely, for a continuous transformation on a compact metric space, we can consider the *topological pressure* as a weighted version of the *topological entropy*, with the weight induced by a real-valued continuous function, called *potential*. The Variational Principle identifies the topological pressure with the supremum of its measure-theoretic counterpart, the *measure-theoretic pressure*, over all invariant Borel probability measures [Bow75, Wal82]. Under additional regularity assumptions on the transformation and the potential, one gets the existence and uniqueness of an invariant Borel probability measure maximizing the measure-theoretic pressure, called the *equilibrium state* for the given transformation and the potential. The study of the existence and uniqueness of the equilibrium states and their various other properties, such as ergodic properties, equidistribution, fractal dimensions, etc., has been the primary motivation for much research in the area.

The ergodic theory for expanding Thurston maps has been investigated in [Li17] by the first-named author of the current paper. In [Li18], the first-named author of the current paper works out the thermodynamic formalism and investigates the existence, uniqueness, and other properties of equilibrium states for expanding Thurston maps. In particular, for each expanding Thurston map without periodic critical points, by using a general framework devised by Y. Kifer [Kif90], the first-named author of the current paper establishes level-2 large deviation principles for iterated preimages and periodic points in [Li15].

In the present paper, we develop the thermodynamic formalism for subsystems of expanding Thurston maps. We establish the Variational Principle and demonstrate the existence of equilibrium states for subsystems of expanding Thurston maps. As an application, we obtain large deviation asymptotics for (original) expanding Thurston maps.

In the next series of articles, we establish the uniqueness of equilibrium states and the equidistribution of iterated preimage and periodic points for subsystems. Moreover, for expanding Thurston maps, even in the presence of periodic critical points, we prove the entropy-approachability and establish a level-2 large deviation principle for the distribution of Birkhoff averages, periodic points, and iterated preimages. In particular, by constructing suitable subsystems, we prove that the entropy map of an expanding Thurston map f is upper semi-continuous if and only if f has no periodic critical points. This result gives a negative answer to the question posed in [Li15] by the first-named author of the current paper and shows that the method used there to prove large deviation principles does not apply to expanding Thurston maps with at least one periodic critical point.

1.1. **Main results.**

Our main results consist of two parts. We first establish the Variational Principle and the existence of equilibrium states for subsystems of expanding Thurston maps. As an application, we establish large deviation asymptotics for expanding Thurston maps. In particular, these results apply to postcritically-finite rational maps without periodic critical points.

Variational Principle and the existence of equilibrium states. In order to state our results more precisely, we quickly review some key concepts.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. We say that a map $F\colon \operatorname{dom}(F) \to S^2$ is a *subsystem of f with respect to $\mathcal{C}$* if $\operatorname{dom}(F) = \bigcup \mathfrak{X}$ for some non-empty subset $\mathfrak{X} \subseteq \mathbf{X}^1(f, \mathcal{C})$ and $F = f|_{\operatorname{dom}(F)}$. We denote by $\operatorname{Sub}(f, \mathcal{C})$ the set of all subsystems of f with respect to $\mathcal{C}$.

Consider a subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. For each $n \in \mathbb{N}_0$, we define the *set of n-tiles of F* to be

$$\mathfrak{X}^n(F, \mathcal{C}) \coloneqq \left\{ X^n \in \mathbf{X}^n(f, \mathcal{C}) : X^n \subseteq F^{-n}(F(\operatorname{dom}(F))) \right\},$$

where we set $F^0 \coloneqq \mathrm{id}_{S^2}$ when $n = 0$. We call each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ an *n-tile* of F. We define the *tile maximal invariant set* associated with F with respect to $\mathcal{C}$ to be

$$\Omega(F, \mathcal{C}) \coloneqq \bigcap_{n \in \mathbb{N}} \Big(\bigcup \mathfrak{X}^n(F, \mathcal{C}) \Big),$$

which is a compact subset of S^2.

One of the key properties of $\Omega(F, \mathcal{C})$ is $F(\Omega(F, \mathcal{C})) \subseteq \Omega(F, \mathcal{C})$ (see Proposition 5.4 (iii)). Therefore, we can restrict F to $\Omega(F, \mathcal{C})$ and consider the map $F|_{\Omega(F,\mathcal{C})}\colon \Omega(F, \mathcal{C}) \to \Omega(F, \mathcal{C})$ and its iterations.

We denote

$$Z_n(F, \varphi) \coloneqq \sum_{X^n \in \mathfrak{X}^n(F, \mathcal{C})} \exp\bigl(\sup\bigl\{ S_n^F \varphi(x) : x \in X^n \bigr\} \bigr)$$

for each $n \in \mathbb{N}$ and each $\varphi \in C(S^2)$. We define the *topological pressure* of F with respect to the *potential* φ by

$$P(F, \varphi) \coloneqq \liminf_{n \to +\infty} \frac{1}{n} \log(Z_n(F, \varphi)).$$

In the following theorem, we establish the Variational Principle for subsystems of expanding Thurston maps with respect to Hölder continuous potentials and demonstrate the existence of the equilibrium states. Here we require the Jordan curve to be f-invariant and the subsystem to be strongly irreducible (see Definition 5.15 in Subsection 5.5).

Theorem 1.1. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* $\operatorname{post} f$ *with the property that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f and ϕ be a real-valued Hölder continuous function on S^2 with respect to the metric d. Consider a strongly irreducible subsystem $F \in \operatorname{Sub}(f, \mathcal{C})$. Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$. Then*

$$P(F, \phi) = \sup\biggl\{ h_\mu(F|_\Omega) + \int_\Omega \phi \,\mathrm{d}\mu : \mu \in \mathcal{M}(\Omega, F|_\Omega) \biggr\}. \tag{1.1}$$

Moreover, there exists an equilibrium state $\mu_{F,\phi}$ for $F|_\Omega$ and ϕ attaining the supremum in (1.1).

Remark. The current one is the first in a series of articles on thermodynamic formalism for subsystems of expanding Thurston maps. We establish the uniqueness of the equilibrium states under some suitable conditions in the next article in the series.

Large deviation asymptotics. Large derivation theory was born in probability, and it usually studies asymptotic behavior as $n \to +\infty$ of the probability $\mathbb{P}\bigl\{ \frac{1}{n} \sum_{k=1}^n X_n \in I \bigr\}$ for a sequence $\{X\}_{n \in \mathbb{N}}$ of random valuables and an interval I. If a kind of law of large numbers $\frac{1}{n} \sum_{k=1}^n X_k \to \widehat{X}$ as $n \to +\infty$ holds and $\widehat{X} \notin I$ for a closed interval I, then the above probabilities tend to zero and sometimes it is possible to show that the convergence is exponentially fast allowing the large deviations theory to come into the picture whose prime goal is to describe the corresponding exponent.

Our second result is about the large deviations asymptotics for expanding Thurston maps. Applying Theorem 1.1, we provide exponential upper bounds for deviations. In order to state our result more precisely, we quickly review some key concepts.

For an expanding Thurston map $f\colon S^2 \to S^2$ and a continuous function $\varphi\colon S^2 \to \mathbb{R}$, each f-invariant Borel probability measure μ on S^2 corresponds to a quantity

$$P_\mu(f, \phi) \coloneqq h_\mu(f) + \int \varphi \,\mathrm{d}\mu$$

called the *measure-theoretic pressure* of f for μ and φ, where $h_\mu(f)$ is the measure-theoretic entropy of f for μ. The well-known Variational Principle (see for example, [PU10, Theorem 3.4.1]) asserts that

$$P(f,\varphi)=\sup\{P_\mu(f,\varphi):\mu\in\mathcal{M}(S^2,f)\},\tag{1.2}$$

where $\mathcal{M}(S^2,f)$ is the set of f-invariant Borel probability measures on S^2, and $P(f,\varphi)$ is the *topological pressure* of f with respect to φ. A measure μ that attains the supremum in (1.2) is called an *equilibrium state* for f and φ.

If $\phi\colon S^2\to\mathbb{R}$ is Hölder continuous (with respect to a given visual metric for f on S^2), then there exists a unique equilibrium state μ_ϕ for f and ϕ, and μ_ϕ is ergodic [Li18, Theorem 1.1]. Therefore, by the Birkhoff ergodic theorem, for μ_ϕ-a.e. $x\in S^2$, we have

$$\lim_{n\to+\infty}\frac{1}{n}S_n\phi(x)=\int\phi\,\mathrm{d}\mu_\phi,$$

where $S_n\phi(x)\coloneqq\sum_{i=0}^{n-1}\phi(f^i(x))$. For the convenience of notations, we denote the Birkhoff average $\int\phi\,\mathrm{d}\mu_\phi$ by γ_ϕ in the sequel.

Two real-valued functions $\varphi,\psi\in C(S^2)$ are called co-homologous in $C(S^2)$ if there exists a function $u\in C(S^2)$ such that $\varphi-\psi=u\circ f-u$. In the following, we assume that $\phi\colon S^2\to\mathbb{R}$ is Hölder continuous (with respect to some given visual metric for f on S^2) and is not co-homologous to a constant in $C(S^2)$.

By our notation, $\mathcal{M}(S^2,f)$ is the set of f-invariant Borel probability measures on S^2, which is convex and compact with respect to the weak* topology. We write $\mathcal{I}_\phi\coloneqq[\alpha_{\min},\alpha_{\max}]$ where

$$\alpha_{\max}\coloneqq\max_{\mu\in\mathcal{M}(S^2,f)}\int\phi\,\mathrm{d}\mu\quad\text{ and }\quad\alpha_{\min}\coloneqq\min_{\mu\in\mathcal{M}(S^2,f)}\int\phi\,\mathrm{d}\mu.\tag{1.3}$$

In particular, we have $\operatorname{int}(\mathcal{I}_\phi)\neq\emptyset$ and $\gamma_\phi\in(\alpha_{\min},\alpha_{\max})$ (see Proposition 7.1 (i)). We define a *rate function* $I=I_{f,\phi}\colon[\alpha_{\min},\alpha_{\max}]\to[0,+\infty)$ for f and ϕ by

$$I(\alpha)=I_{f,\phi}(\alpha)\coloneqq\inf\biggl\{P(f,\phi)-P_\mu(f,\phi):\int\phi\,\mathrm{d}\mu=\alpha,\ \mu\in\mathcal{M}(S^2,f)\biggr\}.\tag{1.4}$$

Now we are able to state our theorem on large derivations asymptotics as follows.

Theorem 1.2 (Large deviations asymptotics)**.** *Let $f\colon S^2\to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f. Let ϕ be a real-valued Hölder continuous function on S^2 with respect to the metric d that is not co-homologous to a constant in $C(S^2)$. Let μ_ϕ be the unique equilibrium state for the map f and the potential ϕ. Denote $\gamma_\phi\coloneqq\int\phi\,\mathrm{d}\mu_\phi$. Then for each $\alpha\in(\alpha_{\min},\alpha_{\max})$, there exists an integer $N\in\mathbb{N}$ and a real number $C_\alpha>0$ such that for each integer $n\geqslant N$,*

$$\mu_\phi\Bigl(\Bigl\{x:\operatorname{sgn}(\alpha-\gamma_\phi)\frac{1}{n}S_n\phi(x)\geqslant\operatorname{sgn}(\alpha-\gamma_\phi)\alpha\Bigr\}\Bigr)\leqslant C_\alpha e^{-I(\alpha)n},$$

where $\alpha_{\min}$ and $\alpha_{\max}$ are given in (1.3) *and $I(\alpha)$ in* (1.4)*.*

For the rate function, $I(\alpha)=0$ holds if and only if $\alpha=\gamma_\phi$ (see Proposition 7.1 (iii)). Hence, I gives control on exponential rates of the convergence. Moreover, it follows from Proposition 7.1 (ii) and [DPTUZ21, Theorems 1.1 (5) and 1.2] that the exponents $I(\alpha)$ in Theorem 1.2 are the best possible ones.

Misiurewicz–Thurston rational maps. Recall that a postcritically-finite rational map is expanding if and only if it has no periodic critical points (see [BM17, Proposition 2.3]). So when we restrict to postcritically-finite rational maps, we get the following immediate consequences of Theorems 1.1, 1.2, and Remark 3.12.

Theorem 1.3. *Let f be a Misiurewicz–Thurston rational map (i.e., a postcritically-finite rational map without periodic critical points) on the Riemann sphere $\widehat{\mathbb{C}}$ and $\mathcal{C}\subseteq\widehat{\mathbb{C}}$ be a Jordan curve containing* $\operatorname{post}f$ *with the property that $f(\mathcal{C})\subseteq\mathcal{C}$. Let ϕ be a real-valued Hölder continuous function on $\widehat{\mathbb{C}}$ with respect to the chordal metric. Consider a strongly irreducible subsystem $F\in\operatorname{Sub}(f,\mathcal{C})$. Denote $\Omega\coloneqq\Omega(F,\mathcal{C})$. Then*

$$P(F,\phi)=\sup\biggl\{h_\mu(F|_\Omega)+\int_\Omega\phi\,\mathrm{d}\mu:\mu\in\mathcal{M}(\Omega,F|_\Omega)\biggr\}.\tag{1.5}$$

Moreover, there exists an equilibrium state $\mu_{F,\phi}$ for $F|_\Omega$ and ϕ attaining the supremum in (1.5)*.*

Theorem 1.4 (Large deviations asymptotics). *Let f be a Misiurewicz–Thurston rational map (i.e., a postcritically-finite rational map without periodic critical points) on the Riemann sphere $\widehat{\mathbb{C}}$. Let ϕ be a real-valued Hölder continuous function on $\widehat{\mathbb{C}}$ with respect to the chordal metric that is not co-homologous to a constant in $C(\widehat{\mathbb{C}})$. Let μ_ϕ be the unique equilibrium state for the map f and the potential ϕ. Denote $\gamma_\phi := \int \phi \,\mathrm{d}\mu_\phi$. Then for each $\alpha \in (\alpha_{\min}, \alpha_{\max})$, there exists an integer $N \in \mathbb{N}$ and a real number $C_\alpha > 0$ such that for each integer $n \geqslant N$,*

$$\mu_\phi\Big(\Big\{x : \operatorname{sgn}(\alpha - \gamma_\phi)\frac{1}{n}S_n\phi(x) \geqslant \operatorname{sgn}(\alpha - \gamma_\phi)\alpha\Big\}\Big) \leqslant C_\alpha e^{-I(\alpha)n},$$

where $\alpha_{\min}$ and $\alpha_{\max}$ are given in (1.3) and $I(\alpha)$ in (1.4).

1.2. **Strategy and organization of the paper.** We now discuss the strategy of the proofs of our main results and describe the organization of the paper.

To prove Theorem 1.1, we define appropriate variants of the Ruelle operator called the *split Ruelle operator*. By definition, a subsystem F may not be a branched covering map on S^2. Then the local degree of F at $x \in \operatorname{dom}(F)$ may not make sense. This leads to inadequate combinatorial structures when one studies the dynamics. For example, the number of white 1-tiles of F may not equal the number of black 1-tiles of F. As a result, the Ruelle operator may not be well-defined and continuous. To address this, instead of a single number, we use 4 numbers written in the form of a 2×2 matrix, called the *local degree matrix*, to describe the local degree (see Subsection 5.3). Moreover, we show that such a local degree matrix is well-behaved under iterations. Then we can define the split Ruelle operator in our context (see Subsection 6.2). The idea is to "split" the Ruelle operator into two "pieces" so that the continuity is preserved under iterations in each piece, and to piece them together to get the split Ruelle operator on the product space. We next prove the existence of an eigenfunction of the split Ruelle operator and the existence of an eigenmeasure of the adjoint of the split Ruelle operator. Then by some extra work, this gives the existence of an equilibrium state and the Variational Principle.

To prove Theorem 1.2, we construct suitable subsystems and then apply them to the thermodynamic formalism in order to bound deviations by the topological pressures of the subsystems. Such strategy is built upon the approximation by subsystems in the work of Takahasi [Tak20] in continued fraction with the geometrical potential. However, due to the existence of critical points and the fact that our maps are branched covering maps on the topological 2-sphere, the construction on the subsystems for realizing Takahasi's method is not direct in this non-uniformly expanding setting. Indeed, we need to investigate the geometric properties of visual metrics and their interplay with the associated combinatorial structures to split and convert the difficulties arising from non-uniform expansion (see Subsection 7.2).

We now describe the structure of the paper in more detail.

In Section 2, we fix some notation that will be used throughout the paper. In Section 3, we review some notions from the ergodic theory and dynamical systems and go over some key concepts and results on Thurston maps. In Section 4, we state the assumptions on some of the objects in this paper, which we will repeatedly refer to later as the *Assumptions in Section 4*.

In Section 5, we set the stage for subsequent discussions. We use the framework set by M. Bonk and D. Meyer in [BM17]. We start with the definition of subsystems of expanding Thurston maps. In Subsection 5.2, we prove a few preliminary results for subsystems, which will be used frequently later. We categorize these results according to their assumptions. In Subsection 5.3, we introduce the notion of the local degree for a subsystem. In Subsection 5.4, we introduce the notion of tile matrices of subsystems, which are helpful tools to describe the combinatorial information of subsystems. In Subsection 5.5, we define irreducible (resp. strongly irreducible) subsystems, which have nice dynamical properties. In Subsection 5.6, we investigate some distortion estimates that serve as the cornerstones for the analysis of thermodynamic formalism for subsystems.

In Section 6, we focuses on thermodynamic formalism for subsystems, with the main results being Theorems 6.29 and 6.30. These theorems establish the Variational Principle and demonstrate the existence of equilibrium states for subsystems of expanding Thurston maps. In Subsection 6.1, we define the topological pressures of subsystems with respect to some potentials via the tile structures determined by subsystems. In Subsection 6.2, we define the split Ruelle operators (Definition 6.9).

In the rest of Section 6, we develop the thermodynamic formalism to establish the existence of equilibrium states for subsystems. We first introduce the split sphere $\widetilde{S}$ as the disjoint union of X_b^0 and X_w^0.

Then the product of function spaces and the product of measure spaces can be identified naturally with the space of functions and the space of measures on $\widetilde{S}$, respectively (Remark 6.13). Since the split Ruelle operator $\mathbb{L}_{F,\phi}$ acts on the product of function spaces, its adjoint operator $\mathbb{L}^*_{F,\phi}$ acts on the product of measure spaces. We have to deal with functions and measures on the split sphere $\widetilde{S}$ while the measures we want (for example, the equilibrium states) should be on S^2. Therefore, we make extra efforts to convert the measures on $\widetilde{S}$ provided by thermodynamic formalism into measures on S^2. Moreover, by the local degree defined in Subsection 5.3, we establish Theorem 6.16 and Proposition 6.26 so that we show that an eigenmeasure for $\mathbb{L}^*_{F,\phi}$ is a Gibbs measure on S^2 under our identifications (Proposition 6.28). Then in Theorem 6.24, we construct an f-invariant Gibbs measure $\mu_{F,\phi}$. Finally, we establish Theorem 1.1 by proving that $\mu_{F,\phi}$ is an equilibrium state in Theorem 6.30.

In Section 7, we establish the large deviation asymptotics (Theorem 1.2) for expanding Thurston maps. In Subsection 7.1, we investigate the rate function, with the main result being Proposition 7.1, which gives some properties of the rate function. In Subsection 7.2, we discuss pair structures associated with tile structures induced by an expanding Thurston map, which are used to build appropriate subsystems used in Subsection 7.3. Subsection 7.3 is devoted to the proof of key bounds in Proposition 7.16. The proof relies on the thermodynamic formalism for subsystems of expanding Thurston maps. We use results in Subsection 7.2 to construct strongly primitive subsystems for sufficiently high iterates of an expanding Thurston map. Then we apply the results in Section 6 and some distortion estimates to obtain the upper bounds in Proposition 7.16. Finally, in Subsection 7.4, we use the key bounds to establish Theorem 1.2.

2. Notation

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. Let S^2 denote an oriented topological 2-sphere. We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1 and write $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The symbol log denotes the logarithm to the base e. For $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the greatest integer $\leqslant x$, and $\lceil x \rceil$ the smallest integer $\geqslant x$. We denote by $\operatorname{sgn}(x)$ the sign function for each $x \in \mathbb{R}$. The cardinality of a set A is denoted by $\operatorname{card}(A)$.

Let $g\colon X \to Y$ be a map between two sets X and Y. We denote the restriction of g to a subset Z of X by $g|_Z$.

Consider a map $f\colon X \to X$ on a set X. The inverse map of f is denoted by f^{-1}. We write f^n for the n-th iterate of f, and $f^{-n} := (f^n)^{-1}$, for each $n \in \mathbb{N}$. We set $f^0 := \mathrm{id}_X$, the identity map on X. For a real-valued function $\varphi\colon X \to \mathbb{R}$, we write

$$S_n\varphi(x) = S_n^f\varphi(x) := \sum_{j=0}^{n-1} \varphi\big(f^j(x)\big) \tag{2.1}$$

for each $x \in X$ and each $n \in \mathbb{N}_0$. We omit the superscript f when the map f is clear from the context. Note that when $n = 0$, by definition we always have $S_0\varphi = 0$.

Let (X, d) be a metric space. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}_d(Y) := \sup\{d(x,y) : x, y \in Y\}$, the interior of Y by $\operatorname{int}(Y)$, and the characteristic function of Y by $\mathbb{1}_Y$, which maps each $x \in Y$ to $1 \in \mathbb{R}$ and vanishes otherwise. For each $r > 0$ and each $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x, r)$ (resp. $\overline{B_d}(x, r)$). We often omit the metric d in the subscript when it is clear from the context.

For a compact metric space (X, d), we denote by $C(X)$ (resp. $B(X)$) the space of continuous (resp. bounded Borel) functions from X to $\mathbb{R}$, by $\mathcal{M}(X)$ the set of finite signed Borel measures, and $\mathcal{P}(X)$ the set of Borel probability measures on X. By the Riesz representation theorem (see for example, [Fol13, Theorems 7.17 and 7.8]), we identify the dual of $C(X)$ with the space $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$, we use $\|\mu\|$ to denote the total variation norm of μ, $\operatorname{supp}\mu$ the support of μ (the smallest closed set $A \subseteq X$ such that $|\mu|(X \setminus A) = 0$), and

$$\langle \mu, u \rangle := \int u \,\mathrm{d}\mu$$

for each $u \in C(X)$. For a point $x \in X$, we define δ_x as the Dirac measure supported on $\{x\}$. For a continuous map $g\colon X \to X$, we set $\mathcal{M}(X, g)$ to be the set of g-invariant Borel probability measures on X. If we do not specify otherwise, we equip $C(X)$ with the uniform norm $\|\cdot\|_{C(X)} := \|\cdot\|_\infty$, and equip $\mathcal{M}(X)$, $\mathcal{P}(X)$, and $\mathcal{M}(X, g)$ with the weak* topology.

The space of real-valued Hölder continuous functions with an exponent $\beta \in (0,1]$ on a compact metric space (X,d) is denoted as $C^{0,\beta}(X,d)$. For each $\phi \in C^{0,\beta}(X,d)$,

$$|\phi|_{\beta,\,(X,d)} \coloneqq \sup\left\{ \frac{|\phi(x)-\phi(y)|}{d(x,y)^{\beta}} : x,\, y \in X,\, x \neq y \right\},$$

and the Hölder norm is defined as $\|\phi\|_{C^{0,\beta}(X,d)} \coloneqq |\phi|_{\beta,\,(X,d)} + \|\phi\|_{C(X)}$.

3. Preliminaries

3.1. **Thermodynamic formalism.** We first review some basic concepts from the ergodic theory and dynamical systems. We refer the reader to [PU10, Chapter 3], [Wal82, Chapter 9], or [KH95, Chapter 20] for more detailed studies of these concepts.

Let (X,d) be a compact metric space and $g\colon X \to X$ a continuous map. Given $n \in \mathbb{N}$,

$$d_g^n(x,y) \coloneqq \max\bigl\{ d\bigl(g^k(x), g^k(y)\bigr) : k \in \{0,\,1,\,\dots,\,n-1\} \bigr\}, \qquad \text{for } x,\, y \in X,$$

defines a metric on X. A set $F \subseteq X$ is *(n,ϵ)-separated* (with respect to g), for some $n \in \mathbb{N}$ and $\epsilon > 0$, if for each pair of distinct points $x,\, y \in F$, we have $d_g^n(x,y) \geqslant \epsilon$. Given $\epsilon > 0$ and $n \in \mathbb{N}$, let $F_n(\epsilon)$ be a maximal (in the sense of inclusion) (n,ϵ)-separated set in X.

For each real-valued continuous function $\psi \in C(X)$, the following limits exist and are equal, and we denote these limits by $P(g,\psi)$ (see for example, [PU10, Theorem 3.3.2]):

$$P(g,\psi) \coloneqq \lim_{\epsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{x \in F_n(\epsilon)} \exp(S_n\psi(x)), \tag{3.1}$$

where $S_n\psi(x) = \sum_{j=0}^{n-1} \psi(g^j(x))$ is defined in (2.1). We call $P(g,\psi)$ the *topological pressure* of g with respect to the *potential* ψ. Note that $P(g,\psi)$ is independent of d as long as the topology on X defined by d remains the same (see for example, [PU10, Section 3.2]). The quantity $h_{\mathrm{top}}(g) \coloneqq P(g,0)$ is called the *topological entropy* of g. The topological entropy is well-behaved under iterations. Indeed, if $n \in \mathbb{N}$, then $h_{\mathrm{top}}(g^n) = n h_{\mathrm{top}}(g)$ (see for example, [KH95, Proposition 3.1.7 (3)]).

We denote by $\mathcal{B}$ the σ-algebra of all Borel sets on X. We denote by $\mathcal{M}(X,g)$ the set of all g-invariant Borel probability measures on X.

A *measurable partition* ξ of X is a collection $\xi = \{A_i : i \in J\}$ consisting of countably many mutually disjoint sets in $\mathcal{B}$, where J is a countable (i.e., finite or countably infinite) index set. The measurable partition ξ is finite if the index set J is a finite set.

Let $\xi = \{A_j : j \in J\}$ and $\eta = \{B_k : k \in K\}$ be measurable partitions of X, where J and K are the corresponding index sets. We say ξ is a *refinement* of η if for each $A_j \in \xi$, there exists $B_k \in \eta$ such that $A_j \subseteq B_k$. The *common refinement* (or *join*) $\xi \vee \eta$ of ξ and η defined as

$$\xi \vee \eta \coloneqq \{A_j \cap B_k : j \in J,\, k \in K\}$$

is also a measurable partition. Put $g^{-1}(\xi) \coloneqq \bigl\{g^{-1}(A_j) : j \in J\bigr\}$, and for each $n \in \mathbb{N}$ define $\xi_g^n \coloneqq \bigvee\limits_{j=0}^{n-1} g^{-j}(\xi) = \xi \vee g^{-1}(\xi) \vee \cdots \vee g^{-(n-1)}(\xi)$.

Let $\xi = \{A_j : j \in J\}$ be a measurable partition of X and $\mu \in \mathcal{M}(X,g)$ be a g-invariant Borel probability measure on X. For $x \in X$, we denote by $\xi(x)$ the unique element of ξ that contains x. The *information function* I_μ maps a measurable partition ξ of X to a μ-a.e. defined real-valued function on X in the following way:

$$I_\mu(\xi)(x) \coloneqq -\log(\mu(\xi(x))), \qquad \text{for each} x \in X. \tag{3.2}$$

The *entropy* of ξ is $H_\mu(\xi) \coloneqq -\sum_{j \in J} \mu(A_j) \log(\mu(A_j)) \in [0,+\infty]$, where $0 \log 0$ is defined to be zero. One can show that (see for example, [Wal82, Chapter 4]) if $H_\mu(\xi) < +\infty$, then the following limit exists:

$$h_\mu(g,\xi) \coloneqq \lim_{n \to +\infty} \frac{1}{n} H_\mu(\xi_g^n) \in [0,+\infty). \tag{3.3}$$

The quantity $h_\mu(g,\xi)$ is called the *measure-theoretic entropy of g relative to ξ*. The *measure-theoretic entropy* of g for μ is defined as

$$h_\mu(g) \coloneqq \sup\{h_\mu(g,\xi) : \xi \text{ is a measurable partition of } X \text{ with } H_\mu(\xi) < +\infty\}. \tag{3.4}$$

If $\mu \in \mathcal{M}(X, g)$ and $n \in \mathbb{N}$, then (see for example, [KH95, Proposition 4.3.16 (4)])

$$h_{\mu}(g^n) = n h_{\mu}(g). \tag{3.5}$$

If $t \in [0,1]$ and $\nu \in \mathcal{M}(X, g)$ is another measure, then (see for example, [Wal82, Theorem 8.1])

$$h_{t\mu+(1-t)\nu}(g) = t h_{\mu}(g) + (1-t) h_{\nu}(g). \tag{3.6}$$

For each real-valued continuous function $\psi \in C(X)$, the *measure-theoretic pressure* $P_{\mu}(g, \psi)$ of g for the measure $\mu \in \mathcal{M}(X, g)$ and the potential ψ is

$$P_{\mu}(g, \psi) \coloneqq h_{\mu}(g) + \int \psi \, \mathrm{d}\mu. \tag{3.7}$$

The topological pressure is related to the measure-theoretic pressure by the so-called *Variational Principle*. It states that (see for example, [PU10, Theorem 3.4.1])

$$P(g, \psi) = \sup\{P_{\mu}(g, \psi) : \mu \in \mathcal{M}(X, g)\} \tag{3.8}$$

for each $\psi \in C(X)$. In particular, when ψ is the constant function 0,

$$h_{\mathrm{top}}(g) = \sup\{h_{\mu}(g) : \mu \in \mathcal{M}(X, g)\}. \tag{3.9}$$

A measure μ that attains the supremum in (3.8) is called an *equilibrium state* for the map g and the potential ψ. A measure μ that attains the supremum in (3.9) is called a *measure of maximal entropy* of g.

3.2. **Thurston maps.** In this subsection, we go over some key concepts and results on Thurston maps, and expanding Thurston maps in particular. For a more thorough treatment of the subject, we refer to [BM17].

Let S^2 denote an oriented topological 2-sphere. A continuous map $f \colon S^2 \to S^2$ is called a *branched covering map* on S^2 if for each point $x \in S^2$, there exists a positive integer $d \in \mathbb{N}$, open neighborhoods U of x and V of $y \coloneqq f(x)$, open neighborhoods U' and V' of 0 in $\widehat{\mathbb{C}}$, and orientation-preserving homeomorphisms $\varphi \colon U \to U'$ and $\eta \colon V \to V'$ such that $\varphi(x) = 0, \eta(y) = 0$, and $\big(\eta \circ f \circ \varphi^{-1}\big)(z) = z^d$ for each $z \in U'$. The positive integer d above is called the *local degree* of f at x and is denoted by $\deg_f(x)$ or $\deg(f, x)$.

The *degree* of f is

$$\deg f = \sum_{x \in f^{-1}(y)} \deg_f(x)$$

for $y \in S^2$ and is independent of y. If $f \colon S^2 \to S^2$ and $g \colon S^2 \to S^2$ are two branched covering maps on S^2, then so is $f \circ g$, and

$$\deg(f \circ g, x) = \deg(g, x) \deg(f, g(x)) \tag{3.10}$$

for each $x \in S^2$, and moreover, $\deg(f \circ g) = (\deg f)(\deg g)$.

A point $x \in S^2$ is a *critical point* of f if $\deg_f(x) \geqslant 2$. The set of critical points of f is denoted by $\operatorname{crit} f$. A point $y \in S^2$ is a *postcritical point* of f if $y = f^n(x)$ for some $x \in \operatorname{crit} f$ and $n \in \mathbb{N}$. The set of postcritical points of f is denoted by $\operatorname{post} f$. Note that $\operatorname{post} f = \operatorname{post} f^n$ for all $n \in \mathbb{N}$.

Definition 3.1 (Thurston maps)**.** A Thurston map is a branched covering map $f \colon S^2 \to S^2$ on S^2 with $\deg f \geqslant 2$ and $\operatorname{card}(\operatorname{post} f) < +\infty$.

We now recall the notation for cell decompositions of S^2 used in [BM17] and [Li17]. A *cell of dimension* n in S^2, $n \in \{1, 2\}$, is a subset $c \subseteq S^2$ that is homeomorphic to the closed unit ball $\overline{\mathbb{B}^n}$ in $\mathbb{R}^n$, where $\mathbb{B}^n$ is the open unit ball in $\mathbb{R}^n$. We define the *boundary of* c, denoted by ∂c, to be the set of points corresponding to $\partial \mathbb{B}^n$ under such a homeomorphism between c and $\overline{\mathbb{B}^n}$. The *interior of* c is defined to be $\operatorname{inte}(c) = c \setminus \partial c$. For each point $x \in S^2$, the set $\{x\}$ is considered as a *cell of dimension* 0 in S^2. For a cell c of dimension 0, we adopt the convention that $\partial c = \emptyset$ and $\operatorname{inte}(c) = c$.

We record the following definition of cell decompositions from [BM17, Definition 3.2].

Definition 3.2 (Cell decompositions)**.** Let $\mathbf{D}$ be a collection of cells in S^2. We say that $\mathbf{D}$ is a *cell decomposition of* S^2 if the following conditions are satisfied:

(i) the union of all cells in $\mathbf{D}$ is equal to S^2,

(ii) if $c \in \mathbf{D}$, then ∂c is a union of cells in $\mathbf{D}$,
(iii) for $c_1, c_2 \in \mathbf{D}$ with $c_1 \neq c_2$, we have $\mathrm{inte}(c_1) \cap \mathrm{inte}(c_2) = \emptyset$,
(iv) every point in S^2 has a neighborhood that meets only finitely many cells in $\mathbf{D}$.

We record [BM17, Lemma 5.3] here to review some facts about cell decompositions.

Lemma 3.3. *Let $\mathbf{D}$ be a cell decomposition of S^2.*

(i) *If σ and τ are two distinct cells in $\mathbf{D}$ with $\sigma \cap \tau \neq \emptyset$, then one of the following statements hold: $\sigma \subseteq \partial\tau$, $\tau \subseteq \partial\sigma$, or $\sigma \cap \tau = \partial\sigma \cap \partial\tau$ and this intersection consists of cells in $\mathbf{D}$ of dimension strictly less than $\min\{\dim\sigma, \dim\tau\}$.*
(ii) *If $\sigma, \tau_1, \dots, \tau_n$ are cells in $\mathbf{D}$ and $\mathrm{inte}(\sigma) \cap (\tau_1 \cup \cdots \cup \tau_n) \neq \emptyset$, then $\sigma \subseteq \tau_i$ for some $i \in \{1, \dots, n\}$.*

Definition 3.4 (Refinements)**.** Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of S^2. We say that $\mathbf{D}'$ is a *refinement* of $\mathbf{D}$ if the following conditions are satisfied:

(i) every cell $c \in \mathbf{D}$ is the union of all cells $c' \in \mathbf{D}'$ with $c' \subseteq c$.
(ii) for every cell $c' \in \mathbf{D}'$ there exists a cell $c \in \mathbf{D}$ with $c' \subseteq c$.

Definition 3.5 (Cellular maps and cellular Markov partitions)**.** Let $\mathbf{D}'$ and $\mathbf{D}$ be two cell decompositions of S^2. We say that a continuous map $f \colon S^2 \to S^2$ is *cellular* for $(\mathbf{D}', \mathbf{D})$ if for every cell $c \in \mathbf{D}'$, the restriction $f|_c$ of f to c is a homeomorphism of c onto a cell in $\mathbf{D}$. We say that $(\mathbf{D}', \mathbf{D})$ is a *cellular Markov partition* for f if f is cellular for $(\mathbf{D}', \mathbf{D})$ and $\mathbf{D}'$ is a refinement of $\mathbf{D}$.

Let $f \colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing $\operatorname{post} f$. Then the pair f and $\mathcal{C}$ induces natural cell decompositions $\mathbf{D}^n(f, \mathcal{C})$ of S^2, for each $n \in \mathbb{N}_0$, in the following way:

By the Jordan curve theorem, the set $S^2 \setminus \mathcal{C}$ has two connected components. We call the closure of one of them the *white* 0*-tile* for $(f, \mathcal{C})$, denoted by X^0_w, and the closure of the other one the *black* 0*-tile* for $(f, \mathcal{C})$, denoted be X^0_b. The set of 0*-tiles* is $\mathbf{X}^0(f, \mathcal{C}) \coloneqq \{X^0_b, X^0_w\}$. The set of 0*-vertices* is $\mathbf{V}^0(f, \mathcal{C}) \coloneqq \operatorname{post} f$. We set $\overline{\mathbf{V}}^0(f, \mathcal{C}) \coloneqq \{\{x\} : x \in \mathbf{V}^0(f, \mathcal{C})\}$. The set of 0*-edges* $\mathbf{E}^0(f, \mathcal{C})$ is the set of the closures of the connected components of $\mathcal{C} \setminus \operatorname{post} f$. Then we get a cell decomposition

$$\mathbf{D}^0(f, \mathcal{C}) \coloneqq \mathbf{X}^0(f, \mathcal{C}) \cup \mathbf{E}^0(f, \mathcal{C}) \cup \overline{\mathbf{V}}^0(f, \mathcal{C})$$

of S^2 consisting of *cells of level* 0, or 0*-cells*.

We can recursively define the unique cell decomposition $\mathbf{D}^n(f, \mathcal{C})$, $n \in \mathbb{N}$, consisting of n*-cells* such that f is cellular for $(\mathbf{D}^{n+1}(f, \mathcal{C}), \mathbf{D}^n(f, \mathcal{C}))$. We refer to [BM17, Lemma 5.12] for more details. We denote by $\mathbf{X}^n(f, \mathcal{C})$ the set of n-cells of dimension 2, called n*-tiles*; by $\mathbf{E}^n(f, \mathcal{C})$ the set of n-cells of dimension 1, called n*-edges*; by $\overline{\mathbf{V}}^n(f, \mathcal{C})$ the set of n-cells of dimension 0; and by $\mathbf{V}^n(f, \mathcal{C})$ the set $\{x : \{x\} \in \overline{\mathbf{V}}^n(f, \mathcal{C})\}$, called the set of n*-vertices*. The k*-skeleton*, for $k \in \{0, 1, 2\}$, of $\mathbf{D}^n(f, \mathcal{C})$ is the union of all n-cells of dimension k in this cell decomposition.

We record [BM17, Proposition 5.16] here in order to summarize properties of the cell decompositions $\mathbf{D}^n(f, \mathcal{C})$ defined above.

Proposition 3.6 (M. Bonk & D. Meyer [BM17])**.** *Let $k, n \in \mathbb{N}_0$, $f \colon S^2 \to S^2$ be a Thurston map, $\mathcal{C} \subseteq S^2$ be a Jordan curve with $\operatorname{post} f \subseteq \mathcal{C}$, and $m \coloneqq \operatorname{card}(\operatorname{post} f)$.*

(i) *The map f^k is cellular for $(\mathbf{D}^{n+k}(f, \mathcal{C}), \mathbf{D}^n(f, \mathcal{C}))$. In particular, if c is any $(n+k)$-cell, then $f^k(c)$ is an n-cell, and $f^k|_c$ is a homeomorphism of c onto $f^k(c)$.*
(ii) *Let c be an n-cell. Then $f^{-k}(c)$ is equal to the union of all $(n+k)$-cell c' with $f^k(c') = c$.*
(iii) *The* 1*-skeleton of $\mathbf{D}^n(f, \mathcal{C})$ is equal to $f^{-n}(\mathcal{C})$. The* 0*-skeleton of $\mathbf{D}^n(f, \mathcal{C})$ is the set $\mathbf{V}^n(f, \mathcal{C}) = f^{-n}(\operatorname{post} f)$, and we have $\mathbf{V}^n(f, \mathcal{C}) \subseteq \mathbf{V}^{n+k}(f, \mathcal{C})$.*
(iv) $\operatorname{card}(\mathbf{X}^n(f, \mathcal{C})) = 2(\deg f)^n$, $\operatorname{card}(\mathbf{E}^n(f, \mathcal{C})) = m(\deg f)^n$, *and* $\operatorname{card}(\mathbf{V}^n(f, \mathcal{C})) \leqslant m(\deg f)^n$.
(v) *The n-edges are precisely the closures of the connected components of $f^{-n}(\mathcal{C}) \setminus f^{-n}(\operatorname{post} f)$. The n-tiles are precisely the closures of the connected components of $S^2 \setminus f^{-n}(\mathcal{C})$.*
(vi) *Every n-tile is an m-gon, i.e., the number of n-edges and the number of n-vertices contained in its boundary are equal to m.*
(vii) *Let $F \coloneqq f^k$ be an iterate of f with $k \in \mathbb{N}$. Then $\mathbf{D}^n(F, \mathcal{C}) = \mathbf{D}^{nk}(f, \mathcal{C})$.*

We record [BM17, Lemma 5.17].

Lemma 3.7 (M. Bonk & D. Meyer [BM17])**.** *Let $k,\, n \in \mathbb{N}_0$, $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve with* $\operatorname{post} f \subseteq \mathcal{C}$*.*

(i) *If $c \subseteq S^2$ is a topological cell such that $f^k|_c$ is a homeomorphism onto its image and $f^k(c)$ is an n-cell, then c is an $(n+k)$-cell.*
(ii) *If X is an n-tile and $p \in S^2$ is a point with $f^k(p) \in \operatorname{inte}(X)$, then there exists a unique $(n+k)$-tile X' with $p \in X'$ and $f^k(X') = X$.*

Remark 3.8. Note that for each n-edge $e^n \in \mathbf{E}^n(f,\mathcal{C})$, $n \in \mathbb{N}_0$, there exist exactly two n-tiles in $\mathbf{X}^n(f,\mathcal{C})$ containing e^n.

For $n \in \mathbb{N}_0$, we define the *set of black n-tiles* as

$$\mathbf{X}^n_b(f,\mathcal{C}) \coloneqq \left\{ X \in \mathbf{X}^n(f,\mathcal{C}) : f^n(X) = X^0_b \right\},$$

and the *set of white n-tiles* as

$$\mathbf{X}^n_w(f,\mathcal{C}) \coloneqq \left\{ X \in \mathbf{X}^n(f,\mathcal{C}) : f^n(X) = X^0_w \right\}.$$

From now on, if the map f and the Jordan curve $\mathcal{C}$ are clear from the context, we will sometimes omit $(f,\mathcal{C})$ in the notation above.

We denote, for each $x \in S^2$ and each $n \in \mathbb{Z}$, the *n-bouquet of x*

$$U^n(x) \coloneqq \bigcup \left\{ Y^n \in \mathbf{X}^n : \text{there exists } X^n \in \mathbf{X}^n \text{ with } x \in X^n,\ X^n \cap Y^n \neq \emptyset \right\} \tag{3.11}$$

if $n \geqslant 0$, and set $U^n(x) \coloneqq S^2$ otherwise.

For each $n \in \mathbb{N}_0$, we define the *n-partition O_n* of S^2 induced by $(f,\mathcal{C})$ as

$$O_n \coloneqq \{\operatorname{inte}(X^n)\colon X^n \in \mathbf{X}^n\} \cup \{\operatorname{inte}(e^n)\colon e^n \in \mathbf{E}^n\} \cup \overline{\mathbf{V}}^n. \tag{3.12}$$

We can now give a definition of expanding Thurston maps.

Definition 3.9 (Expansion)**.** A Thurston map $f\colon S^2 \to S^2$ is called *expanding* if there exists a metric d on S^2 that induces the standard topology on S^2 and a Jordan curve $\mathcal{C} \subseteq S^2$ containing $\operatorname{post} f$ such that

$$\lim_{n\to+\infty} \max\{\operatorname{diam}_d(X)\colon X \in \mathbf{X}^n(f,\mathcal{C})\} = 0. \tag{3.13}$$

Remark 3.10. It is clear from Proposition 3.6 (vii) and Definition 3.9 that if f is an expanding Thurston map, so is f^n for each $n \in \mathbb{N}$. We observe that being expanding is a topological property of a Thurston map and independent of the choice of the metric d that generates the standard topology on S^2. By Lemma 6.2 in [BM17], it is also independent of the choice of the Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$. More precisely, if f is an expanding Thurston map, then

$$\lim_{n\to+\infty} \max\{\operatorname{diam}_{\widetilde{d}}(X)\colon X \in \mathbf{X}^n(f,\widetilde{\mathcal{C}})\} = 0,$$

for each metric $\widetilde{d}$ that generates the standard topology on S^2 and each Jordan curve $\widetilde{\mathcal{C}} \subseteq S^2$ that contains $\operatorname{post} f$.

For an expanding Thurston map f, we can fix a particular metric d on S^2 called a *visual metric for* f. For the existence and properties of such metrics, see [BM17, Chapter 8]. For a visual metric d for f, there exists a unique constant $\Lambda > 1$ called the *expansion factor* of d (see [BM17, Chapter 8] for more details). One major advantage of a visual metric d is that in (S^2,d) we have good quantitative control over the sizes of the cells in the cell decompositions discussed above. We summarize several results of this type ([BM17, Proposition 8.4, Lemmas 8.10, and 8.11]) in the lemma below.

Lemma 3.11 (M. Bonk & D. Meyer [BM17])**.** *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* $\operatorname{post} f$*. Let d be a visual metric on S^2 for f with expansion factor $\Lambda > 1$. Then there exist constants $C \geqslant 1$, $C' \geqslant 1$, $K \geqslant 1$, and $n_0 \in \mathbb{N}_0$ with the following properties:*

(i) *$d(\sigma,\tau) \geqslant C^{-1}\Lambda^{-n}$ whenever σ and τ are disjoint n-cells for some $n \in \mathbb{N}_0$.*
(ii) *$C^{-1}\Lambda^{-n} \leqslant \operatorname{diam}_d(\tau) \leqslant C\Lambda^{-n}$ for all n-edges and all n-tiles τ and for all $n \in \mathbb{N}_0$.*
(iii) *$B_d(x, K^{-1}\Lambda^{-n}) \subseteq U^n(x) \subseteq B_d(x, K\Lambda^{-n})$ for each $x \in S^2$ and each $n \in \mathbb{N}_0$.*

(iv) $U^{n+n_0}(x) \subseteq B_d(x,r) \subseteq U^{n-n_0}(x)$ *where* $n := \lceil -\log r/\log\Lambda \rceil$ *for all* $r > 0$ *and* $x \in S^2$.

(v) *For every* n*-tile* $X^n \in \mathbf{X}^n(f,\mathcal{C})$, $n \in \mathbb{N}_0$, *there exists a point* $p \in X^n$ *such that* $B_d(p, C^{-1}\Lambda^{-n}) \subseteq X^n \subseteq B_d(p, C\Lambda^{-n})$.

Conversely, if $\widetilde{d}$ *is a metric on* S^2 *satisfying conditions* (i) *and* (ii) *for some constant* $C \geqslant 1$, *then* $\widetilde{d}$ *is a visual metric with expansion factor* $\Lambda > 1$.

Recall $U^n(x)$ is defined in (3.11).

In addition, we will need the fact that a visual metric d induces the standard topology on S^2 ([BM17, Proposition 8.3]) and the fact that the metric space (S^2, d) is linearly locally connected ([BM17, Proposition 18.5]). A metric space (X,d) is *linearly locally connected* if there exists a constant $L \geqslant 1$ such that the following conditions are satisfied:

(1) For all $z \in X$, $r > 0$, and $x, y \in B_d(z,r)$ with $x \neq y$, there exists a continuum $E \subseteq X$ with $x, y \in E$ and $E \subseteq B_d(z, rL)$.

(2) For all $z \in X$, $r > 0$, and $x, y \in X \setminus B_d(z,r)$ with $x \neq y$, there exists a continuum $E \subseteq X$ with $x, y \in E$ and $E \subseteq X \setminus B_d(z, r/L)$.

We call such a constant $L \geqslant 1$ a *linear local connectivity constant of* d.

Remark 3.12. If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational expanding Thurston map, then a visual metric is quasisymmetrically equivalent to the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$ (see [BM17, Theorem 18.1 (ii)]). Here the chordal metric σ on $\widehat{\mathbb{C}}$ is given by $\sigma(z,w) := \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all $z, w \in \mathbb{C}$, and $\sigma(\infty, z) = \sigma(z, \infty) := \frac{2}{\sqrt{1+|z|^2}}$ for all $z \in \mathbb{C}$. We also note that quasisymmetric embeddings of bounded connected metric spaces are Hölder continuous (see [Hei01, Section 11.1 and Corollary 11.5]). Accordingly, the classes of Hölder continuous functions on $\widehat{\mathbb{C}}$ equipped with the chordal metric and on $S^2 = \widehat{\mathbb{C}}$ equipped with any visual metric for f are the same (up to a change of the Hölder exponent).

A Jordan curve $\mathcal{C} \subseteq S^2$ is f*-invariant* if $f(\mathcal{C}) \subseteq \mathcal{C}$. If $\mathcal{C}$ is f-invariant with $\operatorname{post} f \subseteq \mathcal{C}$, then the cell decompositions $\mathbf{D}^n(f,\mathcal{C})$ have nice compatibility properties. In particular, $\mathbf{D}^{n+k}(f,\mathcal{C})$ is a refinement of $\mathbf{D}^n(f,\mathcal{C})$, whenever $n, k \in \mathbb{N}_0$. Intuitively, this means that each cell $\mathbf{D}^n(f,\mathcal{C})$ is "subdivided" by the cells in $\mathbf{D}^{n+k}(f,\mathcal{C})$. A cell $c \in \mathbf{D}^n(f,\mathcal{C})$ is actually subdivided by the cells in $\mathbf{D}^{n+k}(f,\mathcal{C})$ "in the same way" as the cell $f^n(c) \in \mathbf{D}^0(f,\mathcal{C})$ by the cells in $\mathbf{D}^k(f,\mathcal{C})$.

For convenience we record Proposition 12.5 (ii) of [BM17] here, which is easy to check but useful.

Proposition 3.13 (M. Bonk & D. Meyer [BM17])**.** *Let* $k, n \in \mathbb{N}_0$, $f\colon S^2 \to S^2$ *be a Thurston map, and* $\mathcal{C} \subseteq S^2$ *be an* f*-invariant Jordan curve with* $\operatorname{post} f \subseteq \mathcal{C}$. *Then every* $(n+k)$*-tile* X^{n+k} *is contained in a unique* k*-tile* X^k.

We are interested in f-invariant Jordan curves that contain $\operatorname{post} f$, since for such a Jordan curve $\mathcal{C}$, we get a cellular Markov partition $\bigl(\mathbf{D}^1(f,\mathcal{C}), \mathbf{D}^0(f,\mathcal{C})\bigr)$ for f. According to Example 15.11 in [BM17], such f-invariant Jordan curves containing $\operatorname{post} f$ need not exist. However, M. Bonk and D. Meyer [BM17, Theorem 15.1] proved that there exists an f^n-invariant Jordan curve $\mathcal{C}$ containing $\operatorname{post} f$ for each sufficiently large n depending on f. We record it below for the convenience of the reader.

Lemma 3.14 (M. Bonk & D. Meyer [BM17])**.** *Let* $f\colon S^2 \to S^2$ *be an expanding Thurston map, and* $\widetilde{\mathcal{C}} \subseteq S^2$ *be a Jordan curve with* $\operatorname{post} f \subseteq \widetilde{\mathcal{C}}$. *Then there exists an integer* $N(f, \widetilde{\mathcal{C}}) \in \mathbb{N}$ *such that for each* $n \geqslant N(f, \widetilde{\mathcal{C}})$ *there exists an* f^n*-invariant Jordan curve* $\mathcal{C}$ *isotopic to* $\widetilde{\mathcal{C}}$ *rel.* $\operatorname{post} f$.

We summarize the existence, uniqueness, and some basic properties of equilibrium states for expanding Thurston maps in the following theorem.

Theorem 3.15 (Z. Li [Li18])**.** *Let* $f\colon S^2 \to S^2$ *be an expanding Thurston map and* d *a visual metric on* S^2 *for* f. *Let* $\phi, \eta \in C^{0,\beta}(S^2, d)$ *be real-valued Hölder continuous functions with an exponent* $\beta \in (0,1]$. *Then the following statements are satisfied:*

(i) *There exists a unique equilibrium state* μ_ϕ *for the map* f *and the potential* ϕ.

(ii) *For each* $t \in \mathbb{R}$, *we have* $\frac{\mathrm{d}}{\mathrm{d}t} P(f, \phi + t\gamma_\phi) = \int \gamma_\phi \,\mathrm{d}\mu_{\phi + t\gamma_\phi}$.

(iii) *If $\mathcal{C} \subseteq S^2$ is a Jordan curve containing* $\operatorname{post} f$ *with the property that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ for some $n_{\mathcal{C}} \in \mathbb{N}$, then $\mu_\phi\big(\bigcup_{i=0}^{+\infty} f^{-i}(\mathcal{C})\big) = 0$.*

Theorem 3.15 (i) is part of [Li18, Theorem 1.1]. Theorem 3.15 (ii) follows immediately from [Li18, Theorem 6.13] and the uniqueness of equilibrium states in Theorem 3.15 (i). Theorem 3.15 (iii) was established in [Li18, Proposition 7.1].

4. The Assumptions

We state below the hypotheses under which we will develop our theory in most parts of this paper. We will selectively use some of those assumptions in the later sections.

The Assumptions.

(1) $f \colon S^2 \to S^2$ is an expanding Thurston map.
(2) $\mathcal{C} \subseteq S^2$ is a Jordan curve containing $\operatorname{post} f$ with the property that there exists an integer $n_{\mathcal{C}} \in \mathbb{N}$ such that $f^{n_{\mathcal{C}}}(\mathcal{C}) \subseteq \mathcal{C}$ and $f^m(\mathcal{C}) \nsubseteq \mathcal{C}$ for each $m \in \{1, \dots, n_{\mathcal{C}} - 1\}$.
(3) $F \in \operatorname{Sub}(f, \mathcal{C})$ is a subsystem of f with respect to $\mathcal{C}$.
(4) d is a visual metric on S^2 for f with expansion factor $\Lambda > 1$ and a linear local connectivity constant $L \geqslant 1$.
(5) $\beta \in (0, 1]$.
(6) $\phi \in C^{0,\beta}(S^2, d)$ is a real-valued Hölder continuous function with exponent β. Denote $\alpha_{\min} \coloneqq \min\limits_{\mu \in \mathcal{M}(S^2, f)} \int \phi \,\mathrm{d}\mu$, $\alpha_{\max} \coloneqq \max\limits_{\mu \in \mathcal{M}(S^2, f)} \int \phi \,\mathrm{d}\mu$, and $\mathcal{I}_\phi \coloneqq [\alpha_{\min}, \alpha_{\max}]$.
(7) μ_ϕ is the unique equilibrium state for the map f and the potential ϕ. Denote $\gamma_\phi \coloneqq \int \phi \,\mathrm{d}\mu_\phi$.
(8) $e^0 \in \mathbf{E}^0(f, \mathcal{C})$ is a 0-edge.

Note that the notion of subsystems in (3) will be introduced in Definition 5.1.

Observe that by Lemma 3.14, for each f in (1), there exists at least one Jordan curve $\mathcal{C}$ that satisfies (2). Since for a fixed f, the number $n_{\mathcal{C}}$ is uniquely determined by $\mathcal{C}$ in (2), in the remaining part of the paper, we will say that a quantity depends on $\mathcal{C}$ even if it also depends on $n_{\mathcal{C}}$.

Recall that the expansion factor Λ of a visual metric d on S^2 for f is uniquely determined by d and f. We will say that a quantity depends on f and d if it depends on Λ.

Note that even though the value of L is not uniquely determined by the metric d, in the remainder of this paper, for each visual metric d on S^2 for f, we will fix a choice of linear local connectivity constant L. We will say that a quantity depends on the visual metric d without mentioning the dependence on L, even though if we had not fixed a choice of L, it would have depended on L as well.

In the discussion below, depending on the conditions we will need, we will sometimes say "Let f, $\mathcal{C}$, d, ϕ satisfy the Assumptions.", and sometimes say "Let f and $\mathcal{C}$ satisfy the Assumptions.", etc.

5. Subsystems

In this section, we set the stage for subsequent discussions. We start with the definition of subsystems of expanding Thurston maps and the associated cell decompositions. For readers unfamiliar with expanding Thurston maps, we recommend reviewing Subsection 3.2 first.

In Subsection 5.2, we prove a few preliminary results for subsystems, which will be used frequently later. We categorize these results according to their assumptions.

In Subsection 5.3, we introduce the notion of the local degree for a subsystem. Although a subsystem may not be a branched covering map or possess structures as good as those found in Thurston maps, such as the structures of flowers (see [BM17, Section 5.6]), we propose a solution to these challenges, namely, rather than relying on a single number, we use a 2×2 matrix to represent the local degree, consisting of 4 numbers. Our findings suggest that this approach is a natural way to capture the behavior of the local degrees under iterations similar to (3.10).

In Subsection 5.4, we introduce tile matrices of subsystems, which are useful tools to describe the combinatorial information of subsystems.

In Subsection 5.5, we define irreducible (resp. strongly irreducible) subsystems, which have nice dynamical properties. We will use these concepts frequently in Section 6. These requirements can be weakened, but for the brevity of the presentation, we will require the subsystem to be irreducible or strongly irreducible.

In Subsection 5.6, we investigate some distortion estimates that serve as the cornerstones for the analysis of thermodynamic formalism for subsystems.

5.1. **Definition of subsystem.** We first introduce the notion of subsystems along with relevant concepts and notations that will be used frequently throughout this section. Additionally, we will provide examples to illustrate these ideas.

Definition 5.1. Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. We say that a map $F\colon \operatorname{dom}(F) \to S^2$ is a *subsystem of f with respect to $\mathcal{C}$* if $\operatorname{dom}(F) = \bigcup \mathfrak{X}$ for some non-empty subset $\mathfrak{X} \subseteq \mathbf{X}^1(f,\mathcal{C})$ and $F = f|_{\operatorname{dom}(F)}$. We denote by $\operatorname{Sub}(f,\mathcal{C})$ the set of all subsystems of f with respect to $\mathcal{C}$. Define

$$\operatorname{Sub}_*(f,\mathcal{C}) \coloneqq \{F \in \operatorname{Sub}(f,\mathcal{C}) : \operatorname{dom}(F) \subseteq F(\operatorname{dom}(F))\}.$$

Consider a subsystem $F \in \operatorname{Sub}(f,\mathcal{C})$. For each $n \in \mathbb{N}_0$, we define the *set of n-tiles of F* to be

$$\mathfrak{X}^n(F,\mathcal{C}) \coloneqq \{X^n \in \mathbf{X}^n(f,\mathcal{C}) : X^n \subseteq F^{-n}(F(\operatorname{dom}(F)))\}, \tag{5.1}$$

where we set $F^0 \coloneqq \operatorname{id}_{S^2}$ when $n = 0$. We call each $X^n \in \mathfrak{X}^n(F,\mathcal{C})$ an *n-tile* of F. We define the *tile maximal invariant set* associated with F with respect to $\mathcal{C}$ to be

$$\Omega(F,\mathcal{C}) \coloneqq \bigcap_{n\in\mathbb{N}} \Bigl(\bigcup \mathfrak{X}^n(F,\mathcal{C})\Bigr), \tag{5.2}$$

which is a compact subset of S^2. Indeed, $\Omega(F,\mathcal{C})$ is forward invariant with respect to F, namely, $F(\Omega(F,\mathcal{C})) \subseteq \Omega(F,\mathcal{C})$ (see Proposition 5.4 (iii)). We denote by F_Ω the map $F|_{\Omega(F,\mathcal{C})}\colon \Omega(F,\mathcal{C}) \to \Omega(F,\mathcal{C})$.

Let X^0_b, $X^0_w \in \mathbf{X}^0(f,\mathcal{C})$ be the black 0-tile and the white 0-tile, respectively. We define the *color set of F* as

$$\mathfrak{C}(F,\mathcal{C}) \coloneqq \bigl\{c \in \{b,w\} : X^0_c \in \mathfrak{X}^0(F,\mathcal{C})\bigr\}.$$

For each $n \in \mathbb{N}_0$, we define the *set of black n-tiles of F* as

$$\mathfrak{X}^n_b(F,\mathcal{C}) \coloneqq \bigl\{X \in \mathfrak{X}^n(F,\mathcal{C}) : F^n(X) = X^0_b\bigr\},$$

and the *set of white n-tiles of F* as

$$\mathfrak{X}^n_w(F,\mathcal{C}) \coloneqq \bigl\{X \in \mathfrak{X}^n(F,\mathcal{C}) : F^n(X) = X^0_w\bigr\}.$$

Moreover, for each $n \in \mathbb{N}_0$ and each pair of c, $c' \in \{b,w\}$ we define

$$\mathfrak{X}^n_{cc'}(F,\mathcal{C}) \coloneqq \bigl\{X \in \mathfrak{X}^n_c(F,\mathcal{C}) : X \subseteq X^0_{c'}\bigr\}.$$

In other words, for example, a tile $X \in \mathfrak{X}^n_{bw}(F,\mathcal{C})$ is a *black n-tile of F contained in X^0_w*, i.e., an n-tile of F that is contained in the white 0-tile X^0_w as a set, and is mapped by F^n onto the black 0-tile X^0_b.

By abuse of notation, we often omit $(F,\mathcal{C})$ in the notations above when it is clear from the context.

Remark 5.2. Note that an expanding Thurston map $f\colon S^2 \to S^2$ itself is a subsystem of f with respect to every Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$, and we have $\mathfrak{X}^n(f,\mathcal{C}) = \mathbf{X}^n(f,\mathcal{C})$ for each $n \in \mathbb{N}_0$ and $\Omega(f,\mathcal{C}) = S^2$ in this case. In general, the map F defined in Definition 5.1 is not a *branched cover* (see for example, [DPTUZ21, Definition 2.1]). Therefore, the results in [DPTUZ21] cannot apply to our setting.

We discuss the following examples separately.

Example 5.3. Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4.

(i) The map F satisfies $\operatorname{dom}(F) = X^1_b \subseteq X^0_w$ for some $X^1_b \in \mathbf{X}^1_b(f,\mathcal{C})$. In this case, F is not surjective, and Ω is an empty set.

(ii) The map F satisfies $\operatorname{dom}(F) = X^1_c \subseteq X^0_c$ for some $c \in \{b,w\}$ and $X^1_c \in \mathbf{X}^1_c(f,\mathcal{C})$. In this case, F is not surjective, and one can check that Ω is non-empty and $\operatorname{card}(\Omega)= 1$.

(iii) The map F satisfies $\operatorname{dom}(F) = X^1_c \cup \widetilde{X}^1_c \subseteq X^0_c$ for some $c \in \{b,w\}$ and disjoint X^1_c, $\widetilde{X}^1_c \in \mathbf{X}^1_c(f,\mathcal{C})$. In this case, F is not surjective, and one can check that Ω is non-empty and uncountable.

(iv) The map F satisfies $\operatorname{dom}(F) = X_b^1 \cup \widetilde{X}_b^1$ for some X_b^1, $\widetilde{X}_b^1 \in \mathbf{X}_b^1(f, \mathcal{C})$ satisfying $X_b^1 \subseteq \operatorname{inte}(X_b^0)$ and $\widetilde{X}_b^1 \subseteq \operatorname{inte}(X_w^0)$. In this case, F is not surjective and $\Omega = \{p, q\}$ where $p \in X_b^1$ and $q \in \widetilde{X}_b^1$. One sees that $F(\Omega) = \{p\} \neq \Omega$ since $F(p) = p$ and $F(q) = p$.

(v) The map $F\colon \operatorname{dom}(F) \to S^2$ is represented by Figure 5.1. Here S^2 is identified with a pillow that is obtained by gluing two squares together along their boundaries. Moreover, each square is subdivided into 3×3 subsquares, and $\operatorname{dom}(F)$ is obtained from S^2 by removing the interior of the middle subsquare $X_w^1 \in \mathbf{X}_w^1(f, \mathcal{C})$ and $X_b^1 \in \mathbf{X}_b^1(f, \mathcal{C})$ of the respective squares. In this case, Ω is a Sierpiński carpet. It consists of two copies of the standard square Sierpiński carpet glued together along the boundary of the square.

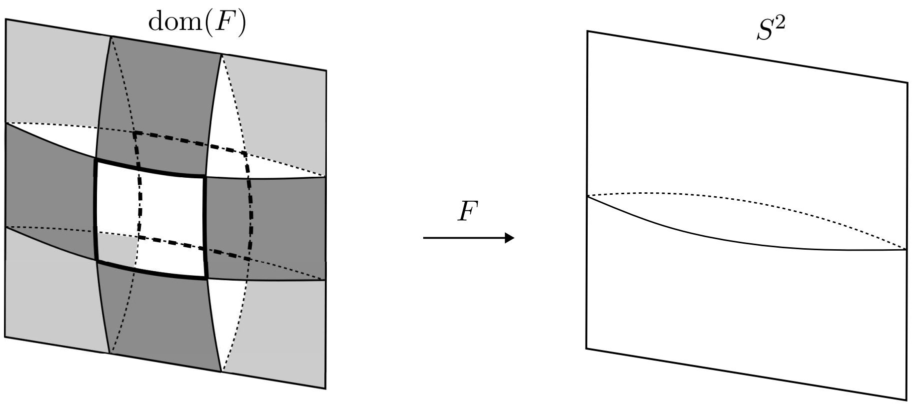

FIGURE 5.1. A Sierpiński carpet subsystem.

(vi) The map $F\colon \operatorname{dom}(F) \to S^2$ is represented by Figure 5.2. Here S^2 is identified with a pillow that is obtained by gluing two equilateral triangles together along their boundaries. Moreover, each triangle is subdivided into 4 small equilateral triangles, and $\operatorname{dom}(F)$ is obtained from S^2 by removing the interior of the middle small triangle $X_b^1 \in \mathbf{X}_b^1(f, \mathcal{C})$ and $X_w^1 \in \mathbf{X}_w^1(f, \mathcal{C})$ of the respective triangle. In this case, Ω is a Sierpiński gasket. It consists of two copies of the standard Sierpiński gasket glued together along the boundary of the triangle.

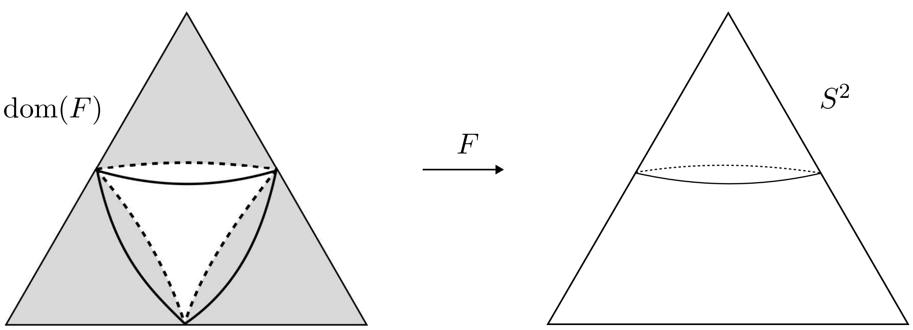

FIGURE 5.2. A Sierpiński gasket subsystem.

5.2. **Basic properties of subsystems.** In this subsection, we collect and prove a few preliminary results for subsystems.

Proposition 5.4. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider arbitrary n, $k \in \mathbb{N}_0$. Then the following statements hold:*

(i) *If $X \in \mathfrak{X}^{n+k}(F, \mathcal{C})$ is any $(n+k)$-tile of F, then $F^k(X)$ is an n-tile of F, and $F^k|_X$ is a homeomorphism of X onto $F^k(X)$. As a consequence we have $\{F^k(X) : X \in \mathfrak{X}^{n+k}(F, \mathcal{C})\} \subseteq \mathfrak{X}^n(F, \mathcal{C})$.*

(ii) *For each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$, the set $\bigcup\{Y \in \mathfrak{X}^{n+k}(F, \mathcal{C}) : Y \subseteq F^{-k}(X^n)\}$ is equal to the union of all $(n+k)$-tiles (of F) $X \in \mathfrak{X}^{n+k}(F, \mathcal{C})$ with $F^k(X) = X^n$.*

(iii) *The tile maximal invariant set Ω is forward invariant with respect to F, i.e., $F(\Omega) \subseteq \Omega$.*

Proof. (i) If $X \in \mathfrak{X}^{n+k}(F,\mathcal{C})$, then $X \in \mathbf{X}^{n+k}(f,\mathcal{C})$ and $X \subseteq F^{-(n+k)}(F(\mathrm{dom}(F)))$ (recall (5.1)). Thus $F^k(X) \subseteq F^k\bigl(F^{-(n+k)}(F(\mathrm{dom}(F)))\bigr) \subseteq F^{-n}(F(\mathrm{dom}(F)))$. Since $F^k|_X = f^k|_X$, it follows from Proposition 3.6 (i) that $F^k(X)$ is an n-tile of F and $F^k|_X$ is a homeomorphism of X onto $F^k(X)$.

(ii) It suffices to show that if $X^n \in \mathfrak{X}^n(F,\mathcal{C})$, then

$$\bigl\{X \in \mathfrak{X}^{n+k}(F,\mathcal{C}) : F^k(X) = X^n\bigr\} = \bigl\{X \in \mathfrak{X}^{n+k}(F,\mathcal{C}) : X \subseteq F^{-k}(X^n)\bigr\}.$$

If $X \in \mathfrak{X}^{n+k}(F,\mathcal{C})$ and $F^k(X) = X^n$, then $X \subseteq F^{-k}\bigl(F^k(X)\bigr) \subseteq F^{-k}(X^n)$. For the converse direction, suppose that $X \in \mathfrak{X}^{n+k}(F,\mathcal{C})$ and $X \subseteq F^{-k}(X^n)$. Then $F^k(X) \subseteq F^k(F^{-k}(X^n)) \subseteq X^n$. Thus $F^k(X) = X^n$ since both X^n and $F^k(X)$ are n-tiles by statement (i).

(iii) We write $\Omega^n := \bigcup \mathfrak{X}^n(F,\mathcal{C})$ for each $n \in \mathbb{N}_0$. Then it follows from statement (i) that $F(\Omega^{n+1}) \subseteq \Omega^n$ for each $n \in \mathbb{N}_0$. Thus $F(\Omega) = F\Bigl(\bigcap_{n=1}^{+\infty} \Omega^n\Bigr) \subseteq \bigcap_{n=1}^{+\infty} F(\Omega^n) \subseteq \bigcap_{n=0}^{+\infty} \Omega^n \subseteq \Omega$ by (5.2). $\square$

Proposition 5.5. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then the following statements hold:*

- (i) $\bigcup \mathfrak{X}^{n+k}(F,\mathcal{C}) \subseteq \bigcup \mathfrak{X}^n(F,\mathcal{C}) \subseteq \bigcup \mathfrak{X}^1(F,\mathcal{C}) = \mathrm{dom}(F)$ *for all n, $k \in \mathbb{N}$.*
- (ii) $\mathfrak{X}^m_c(F,\mathcal{C}) = \mathfrak{X}^m_{cb}(F,\mathcal{C}) \cup \mathfrak{X}^m_{cw}(F,\mathcal{C})$ *for each $m \in \mathbb{N}_0$ and each $c \in \{b,w\}$.*
- (iii) $F^{-1}(\Omega \setminus \mathcal{C}) \subseteq \Omega \setminus \mathcal{C}$. *Moreover, if $\mathcal{C} \subseteq \mathrm{int}(\mathrm{dom}(F))$, then $F^{-1}(\mathcal{C}) \subseteq \Omega$, and in particular, $F^{-1}(\Omega) = \Omega$.*
- (iv) *If Ω satisfies* $\mathrm{card}\bigl(\Omega \cap \mathrm{inte}\bigl(X^0_c\bigr)\bigr) \geqslant 2$ *for each $c \in \mathfrak{C}(F,\mathcal{C})$, then Ω has no isolated points, i.e., Ω is perfect.*

Proof. (i) By (5.1), it is clear that $\bigcup \mathfrak{X}^1(F,\mathcal{C}) = \mathrm{dom}(F)$ and $\bigcup \mathfrak{X}^n(F,\mathcal{C}) \subseteq \mathrm{dom}(F)$ for each $n \in \mathbb{N}$. Hence, it suffices to show that

$$\bigcup \mathfrak{X}^{n+1}(F,\mathcal{C}) \subseteq \bigcup \mathfrak{X}^n(F,\mathcal{C}) \tag{5.3}$$

for each $n \in \mathbb{N}$. We prove (5.3) by induction on $n \in \mathbb{N}$.

For $n = 1$, (5.3) holds trivially.

We now assume that (5.3) holds for $n = \ell$ for some $\ell \in \mathbb{N}$. Let $X^{\ell+2} \in \mathfrak{X}^{\ell+2}(F,\mathcal{C})$ be arbitrary. It suffices to show that $X^{\ell+2} \subseteq X^{\ell+1}$ for some $X^{\ell+1} \in \mathfrak{X}^{\ell+1}(F,\mathcal{C})$. Since $X^{\ell+2} \subseteq \bigcup \mathfrak{X}^1(F,\mathcal{C})$ and $f(\mathcal{C}) \subseteq \mathcal{C}$, by Proposition 3.13 and Lemma 3.3 (ii), there exists a unique $X^1 \in \mathfrak{X}^1(F,\mathcal{C})$ containing $X^{\ell+2}$. Denote $Y^{\ell+1} := F\bigl(X^{\ell+2}\bigr)$. Note that $Y^{\ell+1} \in \mathfrak{X}^{\ell+1}(F,\mathcal{C})$ by Proposition 5.4 (i). Then by the induction hypothesis, we have $Y^{\ell+1} \subseteq \bigcup \mathfrak{X}^{\ell}(F,\mathcal{C})$. Similarly, by Proposition 3.13 and Lemma 3.3 (ii), there exists a unique $Y^{\ell} \in \mathfrak{X}^{\ell}(F,\mathcal{C})$ containing $Y^{\ell+1}$. We set $X^{\ell+1} := (F|_{X^1})^{-1}\bigl(Y^{\ell}\bigr)$. It is clear that $X^{\ell+2} \subseteq X^{\ell+1}$ since $Y^{\ell+1} \subseteq Y^{\ell}$. By Proposition 5.4 (i) and Lemma 3.7 (i), we have $X^{\ell+1} \in \mathbf{X}^{\ell+1}(f,\mathcal{C})$. Then it follows from (5.1) and

$$X^{\ell+1} \subseteq F^{-1}\bigl(Y^{\ell}\bigr) \subseteq F^{-1}\bigl(F^{-\ell}(F(\mathrm{dom}(F)))\bigr) = F^{-(\ell+1)}(F(\mathrm{dom}(F)))$$

that $X^{\ell+1} \in \mathfrak{X}^{\ell+1}(F,\mathcal{C})$. The induction step is now complete.

(ii) Let $m \in \mathbb{N}_0$ and $c \in \{b,w\}$ be arbitrary. Since $f(\mathcal{C}) \subseteq \mathcal{C}$, it follows immediately from Proposition 3.13 that each m-tile $X^m_c \in \mathbf{X}^m_c(f,\mathcal{C})$ is contained in exactly one of X^0_b and X^0_w. Thus (ii) holds.

(iii) Let $y \in \Omega \setminus \mathcal{C}$ be arbitrary. Since $f(\mathcal{C}) \subseteq \mathcal{C}$ and $y \notin \mathcal{C}$, it suffices to show that $x \in \Omega$ for each $x \in F^{-1}(y)$. Let $x \in F^{-1}(y)$ be arbitrary. Without loss of generality we may assume that $y \in \mathrm{inte}\bigl(X^0_b\bigr)$. Then by Lemma 3.7 (ii) there exists a unique 1-tile $X^1 \in \mathfrak{X}^1(F,\mathcal{C})$ with $x \in X^1$ and $F(X^1) = X^0_b$. Since $y \in \Omega \cap \mathrm{inte}\bigl(X^0_b\bigr)$, by (5.2) and Lemma 3.11 (ii), for a sufficiently large integer $n_0 \in \mathbb{N}$ there exists a sequence of tiles $\{Y^{n_0+k}\}_{k\in\mathbb{N}}$ satisfying $Y^{n_0+k} \in \mathfrak{X}^{n_0+k}(F,\mathcal{C})$ and $y \in Y^{n_0+k+1} \subseteq Y^{n_0+k} \subseteq \mathrm{inte}\bigl(X^0_b\bigr)$. Then it follows from Proposition 5.4 (i) and Lemma 3.7 (i) that $x \in X^{n_0+k+1} := (F|_{X^1})^{-1}\bigl(Y^{n_0+k}\bigr) \in \mathfrak{X}^{n_0+k+1}(F,\mathcal{C})$ for each $k \in \mathbb{N}$. Hence $x \in \bigcap_{k\in\mathbb{N}} X^{n_0+k+1} \subseteq \Omega$ by (5.2) and statement (i).

We now assume that $\mathcal{C} \subseteq \mathrm{int}(\mathrm{dom}(F))$. We first show that $F^{-1}(\mathcal{C}) \subseteq \Omega$.

Let $y \in \mathcal{C}$ and $x \in F^{-1}(y)$ be arbitrary. It suffices to show that $x \in \Omega$. Since $x \in \mathrm{dom}(F) = \bigcup \mathfrak{X}^1(F,\mathcal{C})$, there exists a $X^1 \in \mathfrak{X}^1(F,\mathcal{C})$ such that $x \in X^1$. Then it follows from Proposition 5.4 (i) that $F(X^1) = X^0_c$ for some $c \in \{b,w\}$. Since $y \in \mathcal{C} \subseteq X^0_c$, by Proposition 3.13, there exists $Y^1 \in \mathbf{X}^1(f,\mathcal{C})$ such that $y \in Y^1 \subseteq X^0_c$. We claim that $Y^1 \in \mathfrak{X}^1(F,\mathcal{C})$. Indeed, since $y \in \mathcal{C} \subseteq \mathrm{int}(\mathrm{dom}(F))$, we have

$Y^1 \cap \operatorname{int}(\operatorname{dom}(F)) \neq \emptyset$. Thus $\operatorname{inte}(Y^1) \cap \operatorname{int}(\operatorname{dom}(F)) \neq \emptyset$. Then $\operatorname{inte}(Y^1) \cap \operatorname{dom}(F) \neq \emptyset$ and it follows from Lemma 3.3 (ii) and Definition 3.2 (iii) that $Y^1 \in \mathfrak{X}^1(F, \mathcal{C})$. Applying Proposition 5.4 (i) and Lemma 3.7 (i), we have $X^2 := (F|_{X^1})^{-1}(Y^1) \in \mathfrak{X}^2(F, \mathcal{C})$ and $x \in X^2 \subseteq X^1$ since $F(x) = y \in Y^1$. Then it follows from Proposition 5.4 (i) that $F^2(X^2) = X^0_{c'}$ for some $c' \in \{b, w\}$. Similarly, since $F(y) \in F(\mathcal{C}) \subseteq \mathcal{C}$, by Proposition 3.13, there exists a 1-tile $Z^1 \in \mathbf{X}^1(f, \mathcal{C})$ such that $F(y) \in Z^1 \subseteq X^0_{c'}$. By the same argument as before, with y and Y^1 replaced by $F(y)$ and Z^1, respectively, we deduce that $Z^1 \in \mathfrak{X}^1(F, \mathcal{C})$. Then we have $X^3 := (F^2|_{X^2})^{-1}(Z^1) \in \mathfrak{X}^3(F, \mathcal{C})$ and $x \in X^3 \subseteq X^2$ since $F^2(x) = F(y) \in Z^1$. Thus by induction, there exists a sequence of tiles $\{X^n\}_{n \in \mathbb{N}}$ such that $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ and $x \in X^{n+1} \subseteq X^n$ for each $n \in \mathbb{N}$. Hence $x \in \bigcap_{n \in \mathbb{N}} X^n \subseteq \Omega$ by (5.2).

To verify that $F^{-1}(\Omega) = \Omega$, by Proposition 5.4 (iii), it suffices to show that $F^{-1}(\Omega) \subseteq \Omega$. Since $F^{-1}(\Omega \setminus \mathcal{C}) \subseteq \Omega \setminus \mathcal{C} \subseteq \Omega$ and $F^{-1}(\mathcal{C}) \subseteq \Omega$, the proof is complete.

(iv) We fix a visual metric d on S^2 for f. It suffices to show that for each $p \in \Omega$ and each $r > 0$, the set $(B_d(p, r) \setminus \{p\}) \cap \Omega$ is non-empty. Since $p \in \Omega$, for each $n \in \mathbb{N}_0$ there exists $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ which contains p. Thus by Lemma 3.11 (ii), for each sufficiently large integer n there exists $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ such that $p \in X^n$ and $X^n \subseteq B_d(p, r)$. We fix such an integer $n \in \mathbb{N}_0$ and an n-tile $X^n \in \mathfrak{X}^n(F, \mathcal{C})$. By Proposition 5.4 (i), we have $F^n(X^n) = X^0_c$ for some $c \in \mathfrak{C}(F, \mathcal{C})$, and $F^n|_{X^n}$ is a homeomorphism of X^n onto X^0_c. Then it follows from statement (iii) that $(F^n|_{X^n})^{-1}(\Omega \setminus \mathcal{C}) \subseteq F^{-n}(\Omega \setminus \mathcal{C}) \subseteq \Omega \setminus \mathcal{C} \subseteq \Omega$. Thus, by the assumption in statement (iv), we have $\operatorname{card}(X^n \cap \Omega) \geqslant \operatorname{card}\bigl(\Omega \cap \operatorname{inte}\bigl(X^0_c\bigr)\bigr) \geqslant 2$, which completes the proof. $\square$

Proposition 5.6. *Let f and $\mathcal{C}$ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $F \in \operatorname{Sub}_*(f, \mathcal{C})$. Then the following statements hold:*

(i) *$\bigl\{F(X) : X \in \mathfrak{X}^{n+1}(F, \mathcal{C})\bigr\} = \mathfrak{X}^n(F, \mathcal{C})$ for each $n \in \mathbb{N}_0$. In particular, if $F(\operatorname{dom}(F)) = S^2$, then $\mathfrak{X}^n_c(F, \mathcal{C}) \neq \emptyset$ for each $c \in \{b, w\}$ and each $n \in \mathbb{N}_0$.*

(ii) *$F(\Omega) = \Omega \neq \emptyset$.*

Proof. (i) Let $n \in \mathbb{N}_0$ be arbitrary. We first establish

$$\bigl\{F(X) : X \in \mathfrak{X}^{n+1}(F, \mathcal{C})\bigr\} = \mathfrak{X}^n(F, \mathcal{C}). \tag{5.4}$$

It follows from Proposition 5.4 (i) that $\bigl\{F(X) : X \in \mathfrak{X}^{n+1}(F, \mathcal{C})\bigr\} \subseteq \mathfrak{X}^n(F, \mathcal{C})$. Thus, it suffices to show that for each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$, there exists $X \in \mathfrak{X}^{n+1}(F, \mathcal{C})$ such that $F(X) = X^n$.

Fix arbitrary $X^n \in \mathfrak{X}^n(F, \mathcal{C})$. Since $\operatorname{dom}(F) \subseteq F(\operatorname{dom}(F))$, we have $X^n \subseteq F(\operatorname{dom}(F))$ by Proposition 5.5 (i). Then by Proposition 3.13, there exists a unique $X^0 \in \mathbf{X}^0(f, \mathcal{C})$ containing X^n. Thus $\operatorname{inte}(X^0) \cap F(\operatorname{dom}(F))$ is non-empty as it contains $\operatorname{inte}(X^n)$. Noting that $F(\operatorname{dom}(F))$ is a union of 0-tiles in $\mathbf{X}^0(f, \mathcal{C})$, by Lemma 3.3, we conclude that there exists $X^1 \in \mathfrak{X}^1(F, \mathcal{C})$ such that $F(X^1) = X^0$.

We denote by X the set $(F|_{X^1})^{-1}(X^n)$. Since $F(X) = X^n$, it suffices to show that $X \in \mathfrak{X}^{n+1}(F, \mathcal{C})$. By Proposition 5.4 (i) and Lemma 3.7 (i), $X \in \mathbf{X}^{n+1}(f, \mathcal{C})$. Since

$$X = (F|_{X^1})^{-1}(X^n) \subseteq F^{-1}(X^n) \subseteq F^{-(n+1)}(F(\operatorname{dom}(F))),$$

it follows from (5.1) that $X \in \mathfrak{X}^{n+1}(F, \mathcal{C})$, which establishes (5.4).

If $F(\operatorname{dom}(F)) = S^2$, then $\mathfrak{X}^0(F, \mathcal{C}) = \mathbf{X}^0(f, \mathcal{C})$. Then it follows from (5.4), Proposition 5.4 (i), and induction that $\mathfrak{X}^n_c(F, \mathcal{C}) \neq \emptyset$ for each $c \in \{b, w\}$.

(ii) We write $\Omega^n := \bigcup \mathfrak{X}^n(F, \mathcal{C})$ for each $n \in \mathbb{N}_0$. Then it follows from statement (i) and induction that $\Omega^n \neq \emptyset$ and $F(\Omega^{n+1}) = \Omega^n$ for each $n \in \mathbb{N}_0$. Thus, by Proposition 5.5 (i) and (5.2) we have $\Omega = \bigcap_{n=1}^{+\infty} \Omega^n \neq \emptyset$.

By Proposition 5.4 (iii) we have $F(\Omega) \subseteq \Omega$. Next, we prove that $\Omega \subseteq F(\Omega)$. It suffices to show that for each $x \in \Omega$, there exists a point $y \in \Omega$ such that $x = F(y) \in F(\Omega)$.

Let $x \in \Omega$ be arbitrary. For each $n \in \mathbb{N}$, we have $x \in \Omega^{n-1} = F(\Omega^n)$ so that there exists a point $y_n \in \Omega^n$ satisfying $F(y_n) = x$. This gives a sequence $\{y_n\}_{n \in \mathbb{N}}$ in S^2. Then there exists a point $y \in S^2$ and a subsequence $\{y_{n_i}\}_{i \in \mathbb{N}}$ of $\{y_n\}_{n \in \mathbb{N}}$ such that $\{y_{n_i}\}_{i \in \mathbb{N}}$ converges to y. Since S^2 is first countable, by [Fol13, Proposition 4.6], we have $y \in \overline{\{y_n,\, y_{n+1},\, \dots\}}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, noting that Ω^n is closed and $y_{n+k} \in \Omega^{n+k} \subseteq \Omega^n$ for each $k \in \mathbb{N}_0$, we have $\overline{\{y_n,\, y_{n+1},\, \dots\}} \subseteq \Omega^n$. Thus $y \in \Omega^n$ for each $n \in \mathbb{N}$, i.e., $y \in \Omega$. Finally, since the sequence $\{y_{n_i}\}_{i \in \mathbb{N}}$ converges to y, it follows from the the continuity of F that the sequence $\{F(y_{n_i})\}_{i \in \mathbb{N}}$ converges to $F(y)$. Then we deduce that $x = F(y)$ and finish the proof. $\square$

5.3. **Local degree.** In this subsection, we define the degree and local degrees for a subsystem. We show that local degrees are well-behaved under iterations.

Definition 5.7. Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. The *degree* of F is defined as

$$\deg(F) \coloneqq \sup\bigl\{\operatorname{card}\bigl(F^{-1}(\{y\})\bigr) : y \in S^2\bigr\}.$$

Fix arbitrary $x \in S^2$ and $n \in \mathbb{N}$. We define the *black degree* of F^n at x as

$$\deg_b(F^n, x) \coloneqq \operatorname{card}(\mathfrak{X}_b^n(F, \mathcal{C}, x)),$$

where $\mathfrak{X}_b^n(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}_b^n(F, \mathcal{C}) : x \in X\}$ is the *set of black n-tiles of F at x*. Similarly, we define the *white degree* of F^n at x as

$$\deg_w(F^n, x) \coloneqq \operatorname{card}(\mathfrak{X}_w^n(F, \mathcal{C}, x)),$$

where $\mathfrak{X}_w^n(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}_w^n(F, \mathcal{C}) : x \in X\}$ is the *set of white n-tiles of F at x*. Moreover, the *local degree* of F^n at x is defined as

$$\deg(F^n, x) \coloneqq \max\{\deg_b(F^n, x), \deg_w(F^n, x)\}, \tag{5.5}$$

and the *set of n-tiles of F at x* is $\mathfrak{X}^n(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}^n(F, \mathcal{C}) : x \in X\}$. Furthermore, for each pair of c, $c' \in \{b, w\}$ we define

$$\mathfrak{X}_{cc'}^n(F, \mathcal{C}, x) \coloneqq \{X \in \mathfrak{X}_{cc'}^n(F, \mathcal{C}) : x \in X\},$$
$$\deg_{cc'}(F^n, x) \coloneqq \operatorname{card}(\mathfrak{X}_{cc'}^n(F, \mathcal{C}, x)),$$

and the *local degree matrix of F^n at x* is

$$\operatorname{Deg}(F^n, x) \coloneqq \begin{bmatrix} \deg_{bb}(F^n, x) & \deg_{wb}(F^n, x) \\ \deg_{bw}(F^n, x) & \deg_{ww}(F^n, x) \end{bmatrix}.$$

One sees that $\deg(F, x) \geqslant 1$ if and only if $x \in \operatorname{dom}(F)$, and that if $\deg(F, x) > 1$ then $x \in \operatorname{crit} f$.

Remark 5.8. If we assume that $f(\mathcal{C}) \subseteq \mathcal{C}$, then for each $n \in \mathbb{N}$, by Proposition 3.13, each n-tile $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ is contained in exactly one of X_b^0 and X_w^0 so that $\mathfrak{X}^n(F, \mathcal{C}) = \bigcup_{c,\, c' \in \{b,w\}} \mathfrak{X}_{cc'}^n(F, \mathcal{C})$. Thus it follows immediately from Definition 5.7 that $\mathfrak{X}^n(F, \mathcal{C}, x) = \bigcup_{c,\, c' \in \{b,w\}} \mathfrak{X}_{cc'}^n(F, \mathcal{C}, x)$ and $\|\operatorname{Deg}(F^n, x)\|_{\mathrm{sum}} = \operatorname{card}(\mathfrak{X}^n(F, \mathcal{C}, x))$ for each $n \in \mathbb{N}$ and each $x \in S^2$, where

$$\|\operatorname{Deg}(F^n, x)\|_{\mathrm{sum}} \coloneqq \sum_{c,\, c' \in \{b,w\}} \deg_{cc'}(F^n, x).$$

Moreover, for all $n \in \mathbb{N}$, $x \in S^2$, and $c \in \{b, w\}$, by Proposition 5.5 (ii), we have

$$\deg_c(F^n, x) = \sum_{c' \in \{b,w\}} \deg_{cc'}(F^n, x). \tag{5.6}$$

To describe the local degree for a subsystem, instead of a single number, we use 4 numbers written in the form of a 2×2 matrix. Indeed, we prove that the local degree matrix is well-behaved under iterations.

Lemma 5.9. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then for all $x \in S^2$ and n, $m \in \mathbb{N}$, we have*

$$\operatorname{Deg}(F^{n+m}, x) = \operatorname{Deg}(F^n, x) \operatorname{Deg}(F^m, f^n(x)). \tag{5.7}$$

Proof. Let $x \in S^2$, n, $m \in \mathbb{N}$, and c, $c' \in \{b, w\}$ be arbitrary. It suffices to show that

$$\deg_{cc'}(F^{n+m}, x) = \sum_{c'' \in \{b,w\}} \deg_{c''c'}(F^n, x) \deg_{cc''}(F^m, f^n(x)).$$

By Propositions 3.13 and 5.5 (i), every $X^{n+m} \in \mathfrak{X}_{cc'}^{n+m}(F, \mathcal{C}, x)$ is contained in a unique $X^n \in \mathfrak{X}^n(F, \mathcal{C})$. Moreover, X^n is contained in $X_{c'}^0$ since X^{n+m} is contained in $X_{c'}^0$ by definition. Thus X^n is contained in $\bigcup_{c'' \in \{b,w\}} \mathfrak{X}_{c''c'}^n(F, \mathcal{C}, x)$.

Now let $c'' \in \{b, w\}$ and $X^n \in \mathfrak{X}_{c''c'}^n(F, \mathcal{C}, x)$ be arbitrary, and define

$$\mathcal{X} \coloneqq \{X^{n+m} \in \mathfrak{X}_{cc'}^{n+m}(F, \mathcal{C}, x) : X^{n+m} \subseteq X^n\}.$$

By Proposition 5.4 (i), each $X^{n+m} \in \mathcal{X}$ is mapped by F^n homeomorphically to $Y^m \coloneqq F^n(X^{n+m})$, and $Y^m \in \mathfrak{X}_{cc''}^m(F, \mathcal{C}, f^n(x))$ since $Y^m \subseteq F^n(X^n) = X_{c''}^0$ and Y^m has the same color as X^{n+m}. Moreover, it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that the map $F^n|_{X^n}$ induces a bijection $X^{n+m} \mapsto F^n(X^{n+m})$ between $\mathcal{X}$ and $\mathfrak{X}^m_{cc''}(F,\mathcal{C},f^n(x))$. Thus $\operatorname{card}(\mathcal{X})= \operatorname{card}(\mathfrak{X}^m_{cc''}(F,\mathcal{C},f^n(x)))= \deg_{cc''}(F^m,f^n(x))$, which is independent of X^n.

Combining the two arguments above, we get

$$\operatorname{card}(\mathfrak{X}^{n+m}_{cc'}(F,\mathcal{C},x))= \sum_{c''\in\{b,w\}} \operatorname{card}(\mathfrak{X}^n_{c''c'}(F,\mathcal{C},x))\deg_{cc''}(F^m,f^n(x)).$$

This completes the proof. □

5.4. **Tile matrix.** In this subsection, we introduce a 2×2 matrix called the tile matrix to describe tiles of a subsystem according to their colors and locations. We show that the tile matrix is well-behaved under iterations.

Let $f\colon S^2\to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C}\subseteq S^2$ satisfying $\operatorname{post} f\subseteq\mathcal{C}$ and $f(\mathcal{C})\subseteq\mathcal{C}$. Recall that X^0_b, $X^0_w\in\mathbf{X}^0(f,\mathcal{C})$ is the black 0-tile and the white 0-tile, respectively.

Definition 5.10 (Tile matrices)**.** Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We define the *tile matrix* of F with respect to $\mathcal{C}$ as

$$A=A(F,\mathcal{C})\coloneqq\begin{bmatrix}N_{ww} & N_{bw}\\ N_{wb} & N_{bb}\end{bmatrix} \tag{5.8}$$

where

$$N_{cc'}=N_{cc'}(A)\coloneqq\operatorname{card}\bigl\{X\in\mathfrak{X}^1_c(F,\mathcal{C}):X\subseteq X^0_{c'}\bigr\}=\operatorname{card}\bigl(\mathfrak{X}^1_{cc'}(F,\mathcal{C})\bigr)$$

for each pair of colors c, $c'\in\{b,w\}$. For example, N_{bw} is the number of black tiles in $\mathfrak{X}^1(F,\mathcal{C})$ which are contained in the white 0-tile X^0_w.

Remark 5.11. Note that the tile matrix $A(F,\mathcal{C})$ of F with respect to $\mathcal{C}$ is completely determined by the set $\mathfrak{X}^1(F,\mathcal{C})$. Thus for each integer $n\in\mathbb{N}_0$ and each set of n-tiles $\mathbf{E}\subseteq\mathbf{X}^n(f,\mathcal{C})$, similarly, we can define the tile matrix of $\mathbf{E}$ and denote it by $A(\mathbf{E})$. For example, when $\mathbf{E}=\mathfrak{X}^n(F,\mathcal{C})$ for some $n\in\mathbb{N}_0$, we define

$$A(\mathfrak{X}^n(F,\mathcal{C}))\coloneqq\begin{bmatrix}N_{ww}(\mathfrak{X}^n(F,\mathcal{C})) & N_{bw}(\mathfrak{X}^n(F,\mathcal{C}))\\ N_{wb}(\mathfrak{X}^n(F,\mathcal{C})) & N_{bb}(\mathfrak{X}^n(F,\mathcal{C}))\end{bmatrix}$$

where $N_{cc'}(\mathfrak{X}^n(F,\mathcal{C}))\coloneqq\operatorname{card}\bigl\{X\in\mathfrak{X}^n(F,\mathcal{C}):X\in\mathbf{X}^n_c(f,\mathcal{C}),\,X\subseteq X^0_{c'}\bigr\}=\operatorname{card}(\mathfrak{X}^n_{cc'}(F,\mathcal{C}))$ for each pair of c, $c'\in\{b,w\}$.

Proposition 5.12. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C})\subseteq\mathcal{C}$. Then for each integer $n\in\mathbb{N}$, we have*

$$A(\mathfrak{X}^n(F,\mathcal{C}))=\bigl(A\bigl(\mathfrak{X}^1(F,\mathcal{C})\bigr)\bigr)^n, \tag{5.9}$$

i.e., the tile matrix of $\mathfrak{X}^n(F,\mathcal{C})$ equals the n-th power of the tile matrix of $\mathfrak{X}^1(F,\mathcal{C})$.

Remark 5.13. Note that if the map $F(\operatorname{dom}(F))=S^2$, then $A(\mathfrak{X}^0)$ is a 2×2 identity matrix and (5.9) holds for $n=0$.

Proof. For convenience, we write for each $n\in\mathbb{N}_0$,

$$\begin{bmatrix}w_n & b_n\\ w'_n & b'_n\end{bmatrix}\coloneqq A(\mathfrak{X}^n(F,\mathcal{C}))=\begin{bmatrix}N_{ww}(A(\mathfrak{X}^n(F,\mathcal{C}))) & N_{bw}(A(\mathfrak{X}^n(F,\mathcal{C})))\\ N_{wb}(A(\mathfrak{X}^n(F,\mathcal{C}))) & N_{bb}(A(\mathfrak{X}^n(F,\mathcal{C})))\end{bmatrix}.$$

Let k, ℓ, $m\in\mathbb{N}_0$ with $m\geqslant\ell\geqslant k$ be arbitrary. By Proposition 5.4 (i), the map F^k preserves colors of tiles of F, i.e., if X^m is an m-tile of F, then $F^k(X^m)$ is an $(m-k)$-tile of F with the same color as X^m. Moreover, if Y^ℓ is an ℓ-tile of F, then it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that the map $F^k|_{Y^\ell}$ induces a bijection $X^m\mapsto F^k(X^m)$ between the m-tiles of F contained in Y^ℓ and the $(m-k)$-tiles of F contained in the $(\ell-k)$-tile $Y^{\ell-k}\coloneqq F^k(Y^\ell)$.

If we use this for $m=k+1$ and $\ell=k$, then we see that a white k-tile of F contains w_1 white and b_1 black $(k+1)$-tiles of F, and similarly each black k-tile of F contains w'_1 white and b'_1 black $(k+1)$-tiles of F. This leads to the identity

$$\begin{bmatrix}w_{k+1} & b_{k+1}\\ w'_{k+1} & b'_{k+1}\end{bmatrix}=\begin{bmatrix}w_k & b_k\\ w'_k & b'_k\end{bmatrix}\begin{bmatrix}w_1 & b_1\\ w'_1 & b'_1\end{bmatrix}$$

for each $k\in\mathbb{N}_0$. This implies (5.9). □

The following proposition is not used in this paper but should be of independent interest.

Proposition 5.14. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. If the tile matrix A of F with respect to $\mathcal{C}$ is not degenerate, then for each $n \in \mathbb{N}_0$ and each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ we have $X^n \cap \Omega \neq \emptyset$.*

We say that the tile matrix A of F with respect to $\mathcal{C}$ is *degenerate* if A has one of the following forms:

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

where a, $b \in \mathbb{N}_0$.

Proof. Fix arbitrary integer $n \in \mathbb{N}_0$ and tile $X^n \in \mathfrak{X}^n(F, \mathcal{C})$. Recall that we set $F^0 = \mathrm{id}_{S^2}$. By Proposition 5.4 (i), we have $F^n(X^n) = X^0_c \in \mathbf{X}^0(f, \mathcal{C})$ for some $c \in \{b, w\}$ and $F^n|_{X^n}$ is a homeomorphism of X^n onto X^0_c. Since the tile matrix A is not degenerate, there exists a tile $Y^1_c \in \mathfrak{X}^1(F, \mathcal{C})$ such that $Y^1_c \subseteq X^0_c$. Hence by Lemma 3.7 (i) and Proposition 5.4 (i), there exists a tile $X^{n+1} \in \mathfrak{X}^{n+1}(F, \mathcal{C})$ such that $X^{n+1} \subseteq X^n$ and $F^n(X^{n+1}) = Y^1_c$. Since $F^{n+1}(X^{n+1}) \in \mathfrak{X}^0(F, \mathcal{C})$, similarly, there exists a tile $X^{n+2} \in \mathfrak{X}^{n+2}(F, \mathcal{C})$ such that $X^{n+2} \subseteq X^{n+1}$. Thus by induction, there exists a sequence of tiles $\{X^{n+k}\}_{k \in \mathbb{N}_0}$ that satisfies $X^{n+k} \in \mathfrak{X}^{n+k}(F, \mathcal{C})$ and $X^{n+k+1} \subseteq X^{n+k}$ for each $k \in \mathbb{N}_0$. By Lemma 3.11 (ii), the set $\bigcap_{k \in \mathbb{N}_0} X^{n+k}$ is the intersection of a nested sequence of closed sets with radii convergent to zero. Thus, it contains exactly one point in S^2. Since $\bigcap_{k \in \mathbb{N}_0} X^{n+k} \subseteq X^n \cap \Omega$ by (5.2) and Proposition 5.5 (i), the proof is complete. $\square$

5.5. **Irreducible subsystems.** In this subsection, we specialize in irreducible (resp. strongly irreducible) subsystems and primitive (resp. strongly primitive) subsystems, which have additional properties.

Definition 5.15 (Irreducibility)**.** Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We say F is an *irreducible* (resp. a *strongly irreducible*) subsystem (of f with respect to $\mathcal{C}$) if for each pair of c, $c' \in \{b, w\}$, there exists an integer $n_{cc'} \in \mathbb{N}$ and $X^{n_{cc'}} \in \mathfrak{X}^{n_{cc'}}_c(F, \mathcal{C})$ satisfying $X^{n_{cc'}} \subseteq X^0_{c'}$ (resp. $X^{n_{cc'}} \subseteq \mathrm{inte}\big(X^0_{c'}\big)$). We denote by n_F the constant $\max_{c,\, c' \in \{b, w\}} n_{cc'}$, which depends only F and $\mathcal{C}$.

Obviously, if F is irreducible then $\mathfrak{C}(F, \mathcal{C}) = \{b, w\}$ and $F(\mathrm{dom}(F)) = S^2$.
Note that the subsystem F defined in Example 5.3 (vi) is strongly irreducible if the front side of the left pillow shown in Picture 5.2 is black.

Definition 5.16 (Primitivity)**.** Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We say that F is a *primitive* (resp. *strongly primitive*) subsystem (of f with respect to $\mathcal{C}$) if there exists an integer $n_F \in \mathbb{N}$ such that for each pair of c, $c' \in \{b, w\}$ and each integer $n \geqslant n_F$, there exists $X^n \in \mathfrak{X}^n_c(F, \mathcal{C})$ satisfying $X^n \subseteq X^0_{c'}$ (resp. $X^n \subseteq \mathrm{inte}\big(X^0_{c'}\big)$).

Obviously, if F is primitive (resp. strongly irreducible), then F is irreducible (resp. strongly irreducible).
Note that the subsystem F defined in Example 5.3 (v) is strongly primitive.

Remark 5.17. By [Li18, Lemma 5.10], an expanding Thurston map f is a strongly primitive subsystem of itself with respect to every Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\mathrm{post}\, f \subseteq \mathcal{C}$.

Definition 5.18. A matrix all of whose entries are positive (resp. non-negative) is called *positive* (resp. *non-negative*). Let A be a square non-negative matrix. If for any i, j there is $n \in \mathbb{N}$ such that $(A^n)_{ij} > 0$, then A is called *irreducible*; otherwise A is called *reducible*. If some power of A is positive, A is called *primitive*.

Remark 5.19. Consider $F \in \mathrm{Sub}(f, \mathcal{C})$. If we assume that $f(\mathcal{C}) \subseteq \mathcal{C}$, then it follows immediately from Definitions 5.10 and Proposition 5.12 that F is irreducible (resp. primitive) if and only if the tile matrix of F is irreducible (resp. primitive).

Proposition 5.20. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then the following statements hold:*

(i) *If F is irreducible, then $\bigcup_{i \in \mathbb{N}} F^{-i}(x)$ is dense in Ω for each $x \in S^2$.*
(ii) *If F is strongly irreducible, then $\Omega \cap \mathrm{inte}\big(X^0_c\big) \neq \emptyset$ for each $c \in \{b, w\}$.*

Proof. (i) Fix an arbitrary point $x \in S^2$. It suffices to show that the closure of $\bigcup_{i\in\mathbb{N}} F^{-i}(x)$ in S^2 contains Ω. By (5.1), Proposition 5.5 (i), (5.2), and (3.13), it suffices to show that for each $n \in \mathbb{N}$ and each $X^n \in \mathfrak{X}^n(F, \mathcal{C})$, $X^n \cap \bigcup_{i\in\mathbb{N}} F^{-i}(x) \neq \emptyset$.

Fix arbitrary $n \in \mathbb{N}$ and $X^n \in \mathfrak{X}^n(F, \mathcal{C})$. Since $x \in S^2 = X^0_b \cup X^0_w$, there exists $c \in \{b, w\}$ such that $x \in X^0_c$. By Proposition 5.4 (i), X^n is mapped by F^n homeomorphically to a 0-tile $X^0_{c'}$ for some $c' \in \{b, w\}$. Since F is irreducible, by Definition 5.15, there exist $k \in \mathbb{N}$ and $Y^k \in \mathfrak{X}^k_c(F, \mathcal{C})$ such that $Y^k \subseteq X^0_{c'}$ and $F^k(Y^k) = X^0_c$. Then it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that $X^{k+n} \coloneqq (F^n|_{X^n})^{-1}(Y^k) \in \mathfrak{X}^{k+n}(F, \mathcal{C})$. Since $x \in X^0_c = F^{k+n}(X^{k+n})$ and $X^{k+n} \subseteq X^n$, we conclude that $X^n \cap \bigcup_{i\in\mathbb{N}} F^{-i}(x) \neq \emptyset$.

(ii) Fix arbitrary $c \in \{b, w\}$. Assume that $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Then by Definition 5.15, there exist $n \in \mathbb{N}$ and $X^n \in \mathfrak{X}^n(F, \mathcal{C})$ such that $X^n \subseteq \operatorname{inte}(X^0_c)$ and $F^n(X^n) = X^0_c$. Thus, by Lemma 3.7 (i) and Proposition 5.4 (i), there exists $X^{2n} \in \mathfrak{X}^{2n}(F, \mathcal{C})$ such that $X^{2n} \subseteq X^n$ and $F^n(X^{2n}) = X^n$. Since $X^n \subseteq \operatorname{inte}(X^0_c)$ and $F^{2n}(X^{2n}) = X^0_c$, similarly, there exists $X^{3n} \in \mathfrak{X}^{3n}(F, \mathcal{C})$ such that $X^{3n} \subseteq X^{2n}$ and $F^{2n}(X^{3n}) = X^n$. Thus by induction, there exists a sequence of tiles $\{X^{kn}\}_{k\in\mathbb{N}}$ such that $X^{kn} \in \mathfrak{X}^{kn}(F, \mathcal{C})$ and $X^{(k+1)n} \subseteq X^{kn} \subseteq \operatorname{inte}(X^0_c)$ for each $k \in \mathbb{N}$. Hence the set $\bigcap_{k\in\mathbb{N}} X^{kn} \subseteq \operatorname{inte}(X^0_c)$ is the intersection of a nested sequence of closed sets so that it is non-empty. Since $\bigcap_{k\in\mathbb{N}} X^{kn} \subseteq \Omega$ by (5.2) and Proposition 5.5 (i), we deduce that $\Omega \cap \operatorname{inte}(X^0_c) \neq \emptyset$. □

Lemma 5.21. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Suppose that $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible and let $n_F \in \mathbb{N}$ be the constant in Definition 5.15, which depends only on F and $\mathcal{C}$. Then for each $k \in \mathbb{N}_0$, each $c \in \{b, w\}$, and each k-tile $X^k \in \mathfrak{X}^k(F, \mathcal{C})$, there exists an integer $n \in \mathbb{N}$ with $n \leqslant n_F$ and $X^{k+n}_c \in \mathfrak{X}^{k+n}_c(F, \mathcal{C})$ satisfying $X^{k+n}_c \subseteq \operatorname{inte}(X^k)$.*

Proof. Fix arbitrary $k \in \mathbb{N}_0$, $c \in \{b, w\}$, and $X^k \in \mathfrak{X}^k(F, \mathcal{C})$. By Proposition 5.4 (i), X^k is mapped by F^k homeomorphically to $X^0 \coloneqq F^k(X^k) \in \mathbf{X}^0(f, \mathcal{C})$. Since F is strongly irreducible, by Definition 5.15, there exists $n \in \mathbb{N}$ with $n \leqslant n_F$ such that there exists $X^n_c \in \mathfrak{X}^n_c(F, \mathcal{C})$ satisfying $X^n_c \subseteq \operatorname{inte}(X^0)$. Then it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that $X^{k+n}_c \coloneqq (F^k|_{X^k})^{-1}(X^n_c)$ is an $(k + n)$-tile satisfying $X^{k+n}_c \subseteq \operatorname{inte}(X^k)$. Since $F^{k+n}(X^{k+n}_c) = F^n(X^n_c) = X^0_c$ and $X^{k+n}_c \subseteq F^{-k}(X^n_c) \subseteq F^{-(k+n)}(F(\operatorname{dom}(F)))$, we conclude that $X^{k+n}_c \in \mathfrak{X}^{k+n}_c(F, \mathcal{C})$. □

Lemma 5.22. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly primitive. Let $n_F \in \mathbb{N}$ be the constant from Definition 5.16, which depends only on F and $\mathcal{C}$. Then for each $n \in \mathbb{N}$ with $n \geqslant n_F$, each $m \in \mathbb{N}_0$, each $c \in \{b, w\}$, and each m-tile $X^m \in \mathfrak{X}^m(F, \mathcal{C})$, there exists an $(n + m)$-tile $X^{n+m}_c \in \mathfrak{X}^{n+m}_c(F, \mathcal{C})$ such that $X^{n+m}_c \subseteq \operatorname{inte}(X^m)$.*

Proof. Let integer $n \geqslant n_F$, $m \in \mathbb{N}_0$, $c \in \{b, w\}$, and $X^m \in \mathfrak{X}^m(F, \mathcal{C})$ be arbitrary. By Proposition 5.4 (i), X^m is mapped by F^m homeomorphically to $X^0 \coloneqq F^m(X^m)$. Since F is strongly primitive, there exists $X^n_c \in \mathfrak{X}^n_c(F, \mathcal{C})$ satisfying $X^n_c \subseteq \operatorname{inte}(X^0)$. Then it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that $X^{n+m}_c \coloneqq (F^m|_{X^m})^{-1}(X^n_c)$ is an $(n + m)$-tile satisfying $X^{n+m}_c \subseteq \operatorname{inte}(X^m)$. Since $F^{n+m}(X^{n+m}_c) = F^n(X^n_c) = X^0_c$ and $X^{n+m}_c \subseteq F^{-m}(X^n_c) \subseteq F^{-(n+m)}(F(\operatorname{dom}(F)))$, we conclude that $X^{n+m}_c \in \mathfrak{X}^{n+m}_c(F, \mathcal{C})$. □

5.6. **Distortion lemmas.** For the convenience of the reader, we first record the following lemma from [Li18, Lemma 3.13], which generalizes [BM17, Lemma 15.25].

Lemma 5.23 (M. Bonk & D. Meyer [BM17], Z. Li [Li18])**.** *Let f, $\mathcal{C}$, d, Λ satisfy the Assumptions in Section 4. Then there exists a constant $C_0 > 1$, depending only on f, $\mathcal{C}$, and d, with the following property: If $n \in \mathbb{N}_0$, $X^{n+k} \in \mathbf{X}^{n+k}(f, \mathcal{C})$, and x, $y \in X^{n+k}$, then*

$$C_0^{-1} d(x, y) \leqslant d(f^n(x), f^n(y))/\Lambda^n \leqslant C_0 d(x, y). \tag{5.10}$$

The next distortion lemma for expanding Thurston maps follows immediately from [Li18, Lemma 5.1].

Lemma 5.24. *Let f, $\mathcal{C}$, d, Λ, ϕ, β satisfy the Assumptions in Section 4. Then there exists a constant $C_1 \geqslant 0$ depending only on f, $\mathcal{C}$, d, ϕ and β such that for all $n \in \mathbb{N}_0$, $X^n \in \mathbf{X}^n(f, \mathcal{C})$, and $x, y \in X^n$,*

$$|S_n\phi(x) - S_n\phi(y)| \leqslant C_1 d(f^n(x), f^n(y))^\beta \leqslant C_1 (\operatorname{diam}_d(S^2))^\beta. \tag{5.11}$$

Quantitatively, we choose

$$C_1 := C_0 |\phi|_{\beta,\,(S^2,d)} \big/ \big(1 - \Lambda^{-\beta}\big), \tag{5.12}$$

where $C_0 > 1$ is the constant depending only on f, $\mathcal{C}$, and d from Lemma 5.23.

We establish the following distortion lemma for subsystems of expanding Thurston maps.

Lemma 5.25. *Let f, $\mathcal{C}$, F, d, Λ, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then the following statements hold:*

(i) *For each $n \in \mathbb{N}_0$, each $c \in \mathfrak{C}(F,\mathcal{C})$, and each pair of $x, y \in X^0_c$, we have*

$$\frac{\sum_{X^n \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\big(S^F_n \phi\big((F^n|_{X^n})^{-1}(x)\big)\big)}{\sum_{X^n \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\big(S^F_n \phi\big((F^n|_{X^n})^{-1}(x)\big)\big)} \leqslant \exp\big(C_1 d(x,y)^\beta\big) \leqslant \exp\big(C_1 (\operatorname{diam}_d(S^2))^\beta\big), \tag{5.13}$$

where $C_1 \geqslant 0$ is the constant defined in (5.12) *in Lemma 5.24 and depends only on f, $\mathcal{C}$, d, ϕ, and β.*

(ii) *If F is irreducible, then there exists a constant $\widetilde{C} \geqslant 1$ depending only on F, $\mathcal{C}$, d, ϕ, and β such that for each $n \in \mathbb{N}_0$, each pair of $c, c' \in \mathfrak{C}(F,\mathcal{C})$, each $x \in X^0_c$, and each $y \in X^0_{c'}$, we have*

$$\frac{\sum_{X^n_c \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\big(S^F_n \phi\big((F^n|_{X^n_c})^{-1}(x)\big)\big)}{\sum_{X^n_{c'} \in \mathfrak{X}^n_{c'}(F,\mathcal{C})} \exp\big(S^F_n \phi\big((F^n|_{X^n_{c'}})^{-1}(y)\big)\big)} \leqslant \widetilde{C}. \tag{5.14}$$

Quantitatively, we choose

$$\widetilde{C} := (\deg f)^{n_F} \exp\big(2 n_F \|\phi\|_\infty + C_1 (\operatorname{diam}_d(S^2))^\beta\big), \tag{5.15}$$

where $n_F \in \mathbb{N}$ is the constant in Definition 5.15 and depends only on F and $\mathcal{C}$, and $C_1 \geqslant 0$ is the constant defined in (5.12) *in Lemma 5.24 and depends only on f, $\mathcal{C}$, d, ϕ, and β.*

Proof. (i) We fix arbitrary $n \in \mathbb{N}_0$, $c \in \mathfrak{C}(F,\mathcal{C})$, and $x, y \in X^0_c$. For each $X^n \in \mathfrak{X}^n_c(F,\mathcal{C})$, it follows from Proposition 5.4 (i) that $F^n|_{X^n}$ is a homeomorphism from X^n onto X^0_c. Then by Lemma 5.24, we have

$$\exp\big(S^F_n \phi\big((F^n|_{X^n})^{-1}(x)\big) - S^F_n \phi\big((F^n|_{X^n})^{-1}(y)\big)\big) \leqslant \exp\big(C_1 d(x,y)^\beta\big).$$

Thus

$$\exp\big(S^F_n \phi\big((F^n|_{X^n})^{-1}(x)\big)\big) \leqslant \exp\big(C_1 d(x,y)^\beta\big) \exp\big(S^F_n \phi\big((F^n|_{X^n})^{-1}(y)\big)\big).$$

By summing the last inequality over all $X^n \in \mathfrak{X}^n_c(F,\mathcal{C})$, we can conclude that (5.13) holds.

(ii) We assume that F is irreducible and let $n_F \in \mathbb{N}$ be the constant from Definition 5.15, which depends only on F and $\mathcal{C}$.

For convenience, we write $I^n_c(z) := \sum_{X^n \in \mathfrak{X}^n_c(F,\mathcal{C})} \exp\big(S^F_n \phi\big((F^n|_{X^n})^{-1}(z)\big)\big)$ for $n \in \mathbb{N}_0$, $c \in \mathfrak{C}(F,\mathcal{C})$, and $z \in X^0_c$. Since F is irreducible, by Definition 5.15 and Proposition 5.6 (i), we have $\mathfrak{C}(F,\mathcal{C}) = \{b, w\}$ and $\mathfrak{X}^n_c(F,\mathcal{C}) \neq \emptyset$ for each $n \in \mathbb{N}_0$ and each $c \in \{b, w\}$. Thus $I^n_c(z) > 0$ for each $n \in \mathbb{N}_0$, each $c \in \{b, w\}$, and each $z \in X^0_c$.

In the rest of the proof we fix arbitrary $c, c' \in \{b, w\}$, $x \in X^0_c$, and $y \in X^0_{c'}$.

We first consider an arbitrary integer $n \in \{0, 1, 2, \dots, n_F\}$. Since $\operatorname{card}(\mathfrak{X}^n_c(F,\mathcal{C})) \leqslant \operatorname{card}(\mathbf{X}^n_c(f,\mathcal{C})) = (\deg f)^n$ and $\mathfrak{X}^n_{c'}(F,\mathcal{C}) \neq \emptyset$, we have

$$\frac{I^n_c(x)}{I^n_{c'}(y)} \leqslant \frac{\operatorname{card}(\mathfrak{X}^n_c(F,\mathcal{C})) \exp(n\|\phi\|_\infty)}{\operatorname{card}(\mathfrak{X}^n_{c'}(F,\mathcal{C})) \exp(-n\|\phi\|_\infty)} \leqslant (\deg f)^{n_F} \exp(2 n_F \|\phi\|_\infty) \tag{5.16}$$

Thus (5.14) holds.

We now consider an arbitrary integer $n > n_F$. Since F is irreducible, by Definition 5.15, there exists $n_{c'c} \in \mathbb{N}$ with $n_{c'c} \leqslant n_F$ such that there exists $Y^{n_{c'c}}_{c'} \in \mathfrak{X}^{n_{c'c}}_{c'}(F,\mathcal{C})$ satisfying $Y^{n_{c'c}}_{c'} \subseteq X^0_c$. Then for each $X \in \mathfrak{X}^{n-n_{c'c}}_c(F,\mathcal{C})$, it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that $Y := (F^{n-n_{c'c}}|_X)^{-1}\big(Y^{n_{c'c}}_{c'}\big) \in \mathfrak{X}^n_{c'}(F,\mathcal{C})$. This defines an injective map from $\mathfrak{X}^{n-n_{c'c}}_c(F,\mathcal{C})$ to $\mathfrak{X}^n_{c'}(F,\mathcal{C})$ that maps each $X \in \mathfrak{X}^{n-n_{c'c}}_c(F,\mathcal{C})$ to $Y = (F^{n-n_{c'c}}|_X)^{-1}\big(Y^{n_{c'c}}_{c'}\big) \in \mathfrak{X}^n_{c'}(F,\mathcal{C})$. Moreover, applying Lemma 5.24, we have

$$\begin{aligned} S^F_n \phi\big((F^n|_Y)^{-1}(y)\big) &\geqslant S^F_{n-n_{c'c}} \phi\big((F^n|_Y)^{-1}(y)\big) - n_{c'c}\|\phi\|_\infty \\ &\geqslant S^F_{n-n_{c'c}} \phi\big((F^{n-n_{c'c}}|_X)^{-1}(x)\big) - C_1 (\operatorname{diam}_d(S^2))^\beta - n_F \|\phi\|_\infty \end{aligned}$$

since $(F^n|_Y)^{-1}(y) \in Y \subseteq X \in \mathbf{X}^{n-n_{c'c}}(f,\mathcal{C})$ and $n_{c'c} \leqslant n_F$. Hence,

$$I_c^{n-n_{c'c}}(x) \leqslant I_{c'}^n(y) \exp\bigl(n_F\|\phi\|_\infty + C_1(\operatorname{diam}_d(S^2))^\beta\bigr). \tag{5.17}$$

In order to establish (5.14), it suffices to show that

$$I_c^n(x) \leqslant (\deg f)^{n_F} \exp(n_F\|\phi\|_\infty) I_c^{n-n_{c'c}}(x).$$

For each $X \in \mathfrak{X}_c^{n-n_{c'c}}(F,\mathcal{C})$, we set $E(X) := \{X^n \in \mathfrak{X}_c^n(F,\mathcal{C}) : F^{n_{c'c}}(X^n) = X\}$. It follows immediately from Proposition 5.4 (i) that $\mathfrak{X}_c^n(F,\mathcal{C}) = \bigcup_{X \in \mathfrak{X}_c^{n-n_{c'c}}(F,\mathcal{C})} E(X)$. By Proposition 3.6 (ii), for each $X \in \mathfrak{X}_c^{n-n_{c'c}}(F,\mathcal{C})$, we have $\operatorname{card}(E(X)) \leqslant (\deg f)^{n_{c'c}} \leqslant (\deg f)^{n_F}$. Moreover, for each $X^n \in E(X)$,

$$\begin{aligned} S_n^F\phi\bigl((F^n|_{X^n})^{-1}(x)\bigr) &= S_{n_{c'c}}^F\phi\bigl((F^n|_{X^n})^{-1}(x)\bigr) + S_{n-n_{c'c}}^F\phi\bigl((F^{n-n_{c'c}}|_X)^{-1}(x)\bigr) \\ &\leqslant n_F\|\phi\|_\infty + S_{n-n_{c'c}}^F\phi\bigl((F^{n-n_{c'c}}|_X)^{-1}(x)\bigr). \end{aligned}$$

Therefore, we get

$$\begin{aligned} I_c^n(x) &= \sum_{X \in \mathfrak{X}_c^{n-n_{c'c}}(F,\mathcal{C})} \sum_{X^n \in E(X)} \exp\bigl(S_n^F\phi\bigl((F^n|_{X^n})^{-1}(z)\bigr)\bigr) \\ &\leqslant \sum_{X \in \mathfrak{X}_c^{n-n_{c'c}}(F,\mathcal{C})} (\deg f)^{n_F} \exp(n_F\|\phi\|_\infty)\exp\bigl(S_{n-n_{c'c}}^F\phi\bigl((F^{n-n_{c'c}}|_X)^{-1}(x)\bigr)\bigr) \\ &= (\deg f)^{n_F} \exp(n_F\|\phi\|_\infty) I_c^{n-n_{c'c}}(x). \end{aligned}$$

Combining this with (5.17), we establish (5.14) by choosing $\widetilde{C}$ as in (5.15), which depends only on F, $\mathcal{C}$, d, ϕ, and β. $\square$

6. Thermodynamic formalism for subsystems

This section focuses on thermodynamic formalism for subsystems, with the main results being Theorems 6.29 and 6.30. These theorems establish the Variational Principle and demonstrate the existence of equilibrium states for subsystems of expanding Thurston maps.

In Subsection 6.1, we define the topological pressures of subsystems with respect to some potentials via the tile structures determined by subsystems.

In Subsection 6.2 we define appropriate variants of the Ruelle operator called split Ruelle operators (Definition 6.9). One cannot use the Ruelle operator introduced in [Li18] for a subsystem in the proofs of Theorems 6.29 and 6.30 directly. One fundamental problem is that the direct adaptation of the Ruelle operator $\mathcal{L}_\phi \colon C(S^2) \to C(S^2)$ may not be well-defined for subsystems. More precisely, it is possible that $\mathcal{L}_\phi(u) \notin C(S^2)$ for some $u \in C(S^2)$ due to the subtle combinatorial structures of subsystems. Our strategy here is to "split" the Ruelle operator into two pieces so that in each piece, the continuity is preserved under iterations. We note that the concept of splitting the Ruelle operators was first introduced in [LZ18]. We extend this definition to split Ruelle operators for subsystems.

We next develop the thermodynamic formalism to establish the existence of equilibrium states for subsystems. We follow [Li18] in several places, but with some notable changes due to the difficulties from subsystems. We first introduce the split sphere $\widetilde{S}$ as the disjoint union of X_b^0 and X_w^0. Then the product of function spaces and the product of measure spaces can be identified naturally with the space of functions and the space of measures on $\widetilde{S}$, respectively (Remark 6.13). Since the split Ruelle operator $\mathbb{L}_{F,\phi}$ acts on the product of function spaces, its adjoint operator $\mathbb{L}_{F,\phi}^*$ acts on the product of measure spaces. We have to deal with functions and measures on the split sphere $\widetilde{S}$ while the measures we want (for example, the equilibrium states) should be on S^2. Therefore, we make extra efforts to convert the measures on $\widetilde{S}$ provided by thermodynamic formalism into measures on S^2. Moreover, by the local degree defined in Subsection 5.3, we establish Theorem 6.16 and Proposition 6.26 so that we show that an eigenmeasure for $\mathbb{L}_{F,\phi}^*$ is a Gibbs measure on S^2 under our identifications (Proposition 6.28). Then we construct an f-invariant Gibbs measure $\mu_{F,\phi}$ with desired properties in Theorem 6.24. Finally, in Theorem 6.30, we prove that such $\mu_{F,\phi}$ is an equilibrium state.

6.1. **Pressures for subsystems.** In this subsection, we define the topological pressures of subsystems.

Definition 6.1. Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. For a real-valued function $\varphi\colon S^2 \to \mathbb{R}$, we denote

$$Z_n(F,\varphi) \coloneqq \sum_{X^n \in \mathfrak{X}^n(F,\mathcal{C})} \exp\bigl(\sup\bigl\{S_n^F\varphi(x) : x \in X^n\bigr\}\bigr)$$

for each $n \in \mathbb{N}$. We define the *topological pressure* of F with respect to the *potential* φ by

$$P(F,\varphi) \coloneqq \liminf_{n\to+\infty} \frac{1}{n}\log(Z_n(F,\varphi)). \tag{6.1}$$

In particular, when φ is the constant function 0, the quantity $h_{\mathrm{top}}(F) \coloneqq P(F,0)$ is called the *topological entropy* of F.

Lemma 6.2. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider a real-valued function $\varphi\colon S^2 \to \mathbb{R}$. Then for all k, $\ell \in \mathbb{N}$, we have*

$$Z_{k+\ell}(F,\varphi) \leqslant Z_k(F,\varphi)Z_\ell(F,\varphi). \tag{6.2}$$

Proof. For each k, $\ell \in \mathbb{N}$,

$$\begin{aligned}
Z_{k+\ell}(F,\varphi) &= \sum_{X^{k+\ell} \in \mathfrak{X}^{k+\ell}(F,\mathcal{C})} \exp\bigl(\sup\bigl\{S_{k+\ell}^F\varphi(x) : x \in X^{k+\ell}\bigr\}\bigr) \\
&\leqslant \sum_{X^{k+\ell} \in \mathfrak{X}^{k+\ell}(F,\mathcal{C})} e^{\sup\{S_k^F\varphi(x) : x \in X^{k+\ell}\}} e^{\sup\{S_\ell^F\varphi(F^k(x)) : x \in X^{k+\ell}\}} \\
&\leqslant \sum_{X^\ell \in \mathfrak{X}^\ell(F,\mathcal{C})} e^{\sup\{S_\ell^F\varphi(y) : y \in X^\ell\}} \sum_{\substack{X^{k+\ell} \in \mathfrak{X}^{k+\ell}(F,\mathcal{C}) \\ F^k(X^{k+\ell}) = X^\ell}} e^{\sup\{S_k^F\varphi(x) : x \in X^{k+\ell}\}} \\
&\leqslant \sum_{X^\ell \in \mathfrak{X}^\ell(F,\mathcal{C})} e^{\sup\{S_\ell^F\varphi(y) : y \in X^\ell\}} \sum_{X^k \in \mathfrak{X}^k(F,\mathcal{C})} e^{\sup\{S_k^F\varphi(y) : y \in X^k\}} \\
&= Z_k(F,\varphi)Z_\ell(F,\varphi).
\end{aligned}$$

Here, the first inequality uses the fact $S_{k+\ell}^F\varphi(x) = S_k^F\varphi(x) + S_\ell^F\varphi\bigl(F^k(x)\bigr)$, the second inequality follows from Proposition 5.4 (i), and the last inequality follows from Proposition 5.4 (i), which shows that for each ℓ-tile $X^\ell \in \mathfrak{X}^\ell(F,\mathcal{C})$ the map from $\bigl\{X \in \mathfrak{X}^{k+\ell}(F,\mathcal{C}) : F^k(X) = X^\ell\bigr\}$ to $\mathfrak{X}^k(F,\mathcal{C})$ induced by Proposition 3.13 is injective. $\square$

We record the following well-known lemma and refer the reader to [MU03, Lemma 2.11] for a proof.

Lemma 6.3. *If a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real number is subadditive (i.e., $a_{i+j} \leqslant a_i + a_j$ for all i, $j \in \mathbb{N}$), then $\lim_{n\to+\infty} a_n/n$ exists in $\mathbb{R} \cup \{-\infty\}$ and is equal to $\inf\{a_n/n : n \in \mathbb{N}\}$.*

Lemma 6.4. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Consider $\varphi \in C(S^2)$. Then*

$$P(F,\varphi) = \lim_{n\to+\infty} \frac{1}{n}\log(Z_n(F,\varphi)) \in [-\|\varphi\|_\infty, +\infty).$$

Proof. By Proposition 5.6 (ii), $\Omega(F,\mathcal{C}) \neq \emptyset$. Then for each $n \in \mathbb{N}$ we have $\bigcup \mathfrak{X}^n(F,\mathcal{C}) \neq \emptyset$ and $Z_n(F,\varphi) > 0$. Thus by Lemmas 6.2 and 6.3, the sequence $\bigl\{\frac{1}{n}\log(Z_n(F,\varphi))\bigr\}_{n\in\mathbb{N}}$ is subadditive and

$$P(F,\varphi) = \lim_{n\to+\infty} \frac{1}{n}\log(Z_n(F,\varphi)) = \inf_{n\in\mathbb{N}}\biggl\{\frac{1}{n}\log(Z_n(F,\varphi))\biggr\} \in \mathbb{R} \cup \{-\infty\}.$$

It suffices now to prove that $\frac{1}{n}\log(Z_n(F,\varphi))$ is bounded from below by $-\|\varphi\|_\infty$. Let $x_0 \in \Omega$ be arbitrary. Then for each $n \in \mathbb{N}$, we have $x_0 \in \bigcup \mathfrak{X}^n(F,\mathcal{C})$ and

$$Z_n(F,\varphi) = \sum_{X^n \in \mathfrak{X}^n(F,\mathcal{C})} \exp\bigl(\sup\bigl\{S_n^F\varphi(x) : x \in X^n\bigr\}\bigr) \geqslant e^{S_n^F\varphi(x_0)} \geqslant e^{-n\|\varphi\|_\infty}.$$

Therefore, we get $\frac{1}{n}\log(Z_n(F,\varphi)) \geqslant -\|\varphi\|_\infty$ for each $n \in \mathbb{N}$ and $P(F,\varphi) \geqslant -\|\varphi\|_\infty$. $\square$

The topological entropy of F can in fact be computed explicitly via tile matrices (defined in Subsection 5.4).

Proposition 6.5. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let A be the tile matrix of F with respect to $\mathcal{C}$. Then we have*

$$h_{\mathrm{top}}(F) = \log(\rho(A)), \tag{6.3}$$

where $\rho(A)$ is the spectral radius of A.

Remark. The spectral radius $\rho(A)$ can easily be computed from any matrix norm. If for an $(m \times m)$-matrix $B = (b_{ij})$ we set $\|B\|_{\mathrm{sum}} := \sum_{i,j=1}^{m} |b_{ij}|$ for example, then $\rho(A) = \lim_{n\to+\infty} (\|A^n\|_{\mathrm{sum}})^{1/n}$.

Proof. By (6.1) and Proposition 5.5 (ii), we have

$$h_{\mathrm{top}}(F) = \liminf_{n\to+\infty} \frac{1}{n} \log(\mathrm{card}(\mathfrak{X}^n(F,\mathcal{C}))) = \liminf_{n\to+\infty} \frac{1}{n} \log \sum_{c,\, c' \in \{b,w\}} \mathrm{card}(\mathfrak{X}^n_{cc'}(F,\mathcal{C})).$$

Then it follows from Remark 5.11 and Proposition 5.12 that

$$h_{\mathrm{top}}(F) = \liminf_{n\to+\infty} \frac{1}{n} \log\|A^n\|_{\mathrm{sum}}.$$

Since $\rho(A) = \lim_{n\to+\infty} (\|A^n\|_{\mathrm{sum}})^{1/n}$, we deduce that $h_{\mathrm{top}}(F) = \log(\rho(A))$. $\square$

6.2. **Split Ruelle operators.** In this subsection, we define appropriate variants of the Ruelle operator (see Definition 6.9) on the suitable function spaces in our context.

Let $f \colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Let X^0_b, $X^0_w \in \mathbf{X}^0(f,\mathcal{C})$ be the black 0-tile and the white 0-tile, respectively.

Definition 6.6 (Partial split Ruelle operators)**.** Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider $\varphi \in C(S^2)$. We define a map $\mathcal{L}^{(n)}_{F,\varphi,c,c'} \colon B\bigl(X^0_{c'}\bigr) \to B\bigl(X^0_c\bigr)$, for c, $c' \in \{b,w\}$, and $n \in \mathbb{N}_0$, by

$$\begin{aligned} \mathcal{L}^{(n)}_{F,\varphi,c,c'}(u)(y) &:= \sum_{x \in F^{-n}(y)} \deg_{cc'}(F^n, x) u(x) \exp\bigl(S^F_n \varphi(x)\bigr) \\ &= \sum_{X^n \in \mathfrak{X}^n_{cc'}(F,\mathcal{C})} u\bigl((F^n|_{X^n})^{-1}(y)\bigr) \exp\bigl(S^F_n \varphi\bigl((F^n|_{X^n})^{-1}(y)\bigr)\bigr) \end{aligned} \tag{6.4}$$

for each real-valued bounded Borel function $u \in B\bigl(X^0_{c'}\bigr)$ and each point $y \in X^0_c$.

Note that by default, a summation over an empty set is equal to 0. We will always use this convention in this paper.

If $X^0_c \subseteq F(\mathrm{dom}(F))$ for some $c \in \{b,w\}$, then $\mathcal{L}^{(0)}_{F,\varphi,c,c'}(u) = \begin{cases} u & \text{if } c' = c \\ 0 & \text{if } c' \neq c \end{cases}$, for each $c' \in \{b,w\}$, which means that $\mathcal{L}^{(0)}_{F,\varphi,c,c}$ is the identity map on $B\bigl(X^0_c\bigr)$ for each $c \in \{b,w\}$ satisfying $X^0_c \subseteq F(\mathrm{dom}(F))$.

Remark 6.7. If we assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$, then by Propositions 3.13 and 5.4 (i), for all $n \in \mathbb{N}$ and $y \in \mathrm{inte}\bigl(X^0_c\bigr)$, the summation (with respect to tiles) on the right-hand side of (6.4) becomes a summation with respect to preimages, i.e.,

$$\mathcal{L}^{(n)}_{F,\varphi,c,c'}(u)(y) = \sum_{x \in F^{-k}(y) \cap \mathrm{inte}(X^0_{c'})} u(x) \exp\bigl(S^F_n \varphi(x)\bigr).$$

Lemma 6.8. *Let f, $\mathcal{C}$, F, d satisfy the Assumptions in Section 4. Consider $\varphi \in C(S^2)$. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \mathrm{Sub}_*(f,\mathcal{C})$. Then for all n, $k \in \mathbb{N}_0$, c, $c' \in \{b,w\}$, and $u \in C\bigl(X^0_{c'}\bigr)$, we have*

$$\mathcal{L}^{(n)}_{F,\varphi,c,c'}(u) \in C\bigl(X^0_c\bigr) \qquad \textit{and} \tag{6.5}$$

$$\mathcal{L}^{(n+k)}_{F,\varphi,c,c'}(u) = \sum_{c'' \in \{b,w\}} \mathcal{L}^{(n)}_{F,\varphi,c,c''}\bigl(\mathcal{L}^{(k)}_{F,\varphi,c'',c'}(u)\bigr). \tag{6.6}$$

Proof. The case of Lemma 6.8 where either $n = 0$ or $k = 0$ follows immediately from Definition 6.6. Thus, we may assume without loss of generality that $n,\, k \in \mathbb{N}$.

The continuity of $\mathcal{L}^{(n)}_{F,\varphi,c,c'}(u)$ follows trivially from (6.4) and Proposition 5.4 (i). Note that the number of terms in the summation in (6.4) is fixed as y in X^0_c varies, and each term is continuous with respect to y when $y \in X^0_c$ by Proposition 5.4 (i).

We next establish (6.6) by proving that the two sides of (6.6) are equal at each point $y \in X^0_c$. Indeed, by continuity, we only need to prove for $y \in \operatorname{inte}(X^0_c)$. Since $f(\mathcal{C}) \subseteq \mathcal{C}$, each preimage $x \in F^{-k}(y)$ belongs to the set $S^2 \setminus \mathcal{C} = \operatorname{inte}(X^0_b) \cup \operatorname{inte}(X^0_w)$ when $y \in \operatorname{inte}(X^0_c)$. Then by Remark 6.7, we get the first two equalities and the second equality of the following:

$$
\begin{aligned}
\sum_{c'' \in \{b,w\}} \mathcal{L}^{(n)}_{F,\varphi,c,c''}\bigl(\mathcal{L}^{(k)}_{F,\varphi,c'',c'}(u)\bigr)(y) &= \sum_{c'' \in \{b,w\}} \sum_{x \in F^{-n}(y) \cap \operatorname{inte}(X^0_{c''})} e^{S^F_n \varphi(x)} \sum_{z \in F^{-k}(x) \cap \operatorname{inte}(X^0_{c'})} u(z) e^{S^F_k \varphi(z)} \\
&= \sum_{x \in F^{-n}(y)} \sum_{z \in F^{-k}(x) \cap \operatorname{inte}(X^0_{c'})} u(z) e^{S^F_n \varphi(x) + S^F_k \varphi(z)} \\
&= \sum_{z \in F^{-(n+k)}(y) \cap \operatorname{inte}(X^0_{c'})} u(z) e^{S^F_{n+k} \varphi(z)} = \mathcal{L}^{(n+k)}_{F,\varphi,c,c'}(u)(y),
\end{aligned}
$$

where the second-to-last equality holds since $x = F^k(z)$ and the last equality follows from Remark 6.7. $\square$

Definition 6.9 (Split Ruelle operators)**.** Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider $\varphi \in C(S^2)$. The *split Ruelle operator* for the subsystem F and potential φ

$$\mathbb{L}_{F,\varphi} \colon C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr) \to C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$$

on the product space $C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$ is defined by

$$\mathbb{L}_{F,\varphi}(u_b, u_w) := \bigl(\mathcal{L}^{(1)}_{F,\varphi,b,b}(u_b) + \mathcal{L}^{(1)}_{F,\varphi,b,w}(u_w), \mathcal{L}^{(1)}_{F,\varphi,w,b}(u_b) + \mathcal{L}^{(1)}_{F,\varphi,w,w}(u_w)\bigr) \tag{6.7}$$

for each $u_b \in C\bigl(X^0_b\bigr)$ and each $u_w \in C\bigl(X^0_w\bigr)$.

Note that by (6.5) in Lemma 6.8, the operator $\mathbb{L}_{F,\varphi}$ is well-defined, and it follows immediately from Definition 6.6 that $\mathbb{L}^0_{F,\varphi}$ is the identity map on $C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$ if $F(\operatorname{dom}(F)) = S^2$. Moreover, one sees that $\mathbb{L}_{F,\varphi} \colon C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr) \to C\bigl(X^0_b\bigr) \times C\bigl(X^0_w\bigr)$ has a natural extension to the space $B\bigl(X^0_b\bigr) \times B\bigl(X^0_w\bigr)$ given by (6.7) for each $u_b \in B\bigl(X^0_b\bigr)$ and each $u_w \in B\bigl(X^0_w\bigr)$.

For each color $c \in \{b, w\}$, we define the projection $\pi_c \colon B\bigl(X^0_b\bigr) \times B\bigl(X^0_w\bigr) \to B\bigl(X^0_c\bigr)$ by

$$\pi_c(u_b, u_w) := u_c, \qquad \text{for } (u_b, u_w) \in B\bigl(X^0_b\bigr) \times B\bigl(X^0_w\bigr). \tag{6.8}$$

We show that the split Ruelle operator $\mathbb{L}_{F,\varphi}$ is well-behaved under iterations.

Lemma 6.10. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider $\varphi \in C(S^2)$. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F(\operatorname{dom}(F)) = S^2$. Then for all $n \in \mathbb{N}_0$, $u_b \in C\bigl(X^0_b\bigr)$, and $u_w \in C\bigl(X^0_w\bigr)$,*

$$\mathbb{L}^n_{F,\varphi}(u_b, u_w) = \bigl(\mathcal{L}^{(n)}_{F,\varphi,b,b}(u_b) + \mathcal{L}^{(n)}_{F,\varphi,b,w}(u_w),\ \mathcal{L}^{(n)}_{F,\varphi,w,b}(u_b) + \mathcal{L}^{(n)}_{F,\varphi,w,w}(u_w)\bigr). \tag{6.9}$$

Proof. We prove (6.9) by induction. The case where $n = 0$ and the case where $n = 1$ both hold by definition. Assume now (6.9) holds for $n = k$ for some $k \in \mathbb{N}$. Then by Definition 6.9 and (6.6) in Lemma 6.8, for each $c \in \{b, w\}$, we have

$$
\begin{aligned}
\pi_c\bigl(\mathbb{L}^{k+1}_{F,\varphi}(u_b, u_w)\bigr) &= \pi_c\bigl(\mathbb{L}_{F,\varphi}\bigl(\mathcal{L}^{(k)}_{F,\varphi,b,b}(u_b) + \mathcal{L}^{(k)}_{F,\varphi,b,w}(u_w),\ \mathcal{L}^{(k)}_{F,\varphi,w,b}(u_b) + \mathcal{L}^{(k)}_{F,\varphi,w,w}(u_w)\bigr)\bigr) \\
&= \sum_{c' \in \{b,w\}} \mathcal{L}^{(1)}_{F,\varphi,c,c'}\bigl(\mathcal{L}^{(k)}_{F,\varphi,c',b}(u_b) + \mathcal{L}^{(k)}_{F,\varphi,c',w}(u_w)\bigr) \\
&= \sum_{c'' \in \{b,w\}} \sum_{c' \in \{b,w\}} \mathcal{L}^{(1)}_{F,\varphi,c,c'}\bigl(\mathcal{L}^{(k)}_{F,\varphi,c',c''}(u_{c''})\bigr) = \sum_{c'' \in \{b,w\}} \mathcal{L}^{(k+1)}_{F,\varphi,c,c''}(u_{c''}),
\end{aligned}
$$

for $u_b \in C\bigl(X^0_b\bigr)$ and $u_w \in C\bigl(X^0_w\bigr)$. This completes the inductive step, establishing (6.9). $\square$

Remark. Similarly, one can show that (6.9) holds for $(u_b, u_w) \in B\bigl(X^0_b\bigr) \times B\bigl(X^0_w\bigr)$.

6.3. **Split sphere.** In this subsection, we introduce the notion of a split sphere, and set up some identifications and conventions (see Remarks 6.13 and 6.14), which will be used frequently in this section.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map with a Jordan curve $\mathcal{C} \subseteq S^2$ satisfying $\operatorname{post} f \subseteq \mathcal{C}$. Let $X_b^0,\, X_w^0 \in \mathbf{X}^0(f,\mathcal{C})$ be the black 0-tile and the white 0-tile, respectively.

Definition 6.11. We define the *split sphere* $\widetilde{S}$ to be the disjoint union of X_b^0 and X_w^0, i.e.,

$$\widetilde{S} \coloneqq X_b^0 \sqcup X_w^0 = \bigl\{(x,c) : c \in \{b,w\},\, x \in X_c^0\bigr\}.$$

For each $c \in \{b,w\}$, let

$$i_c\colon X_c^0 \to \widetilde{S} \tag{6.10}$$

be the natural injection (defined by $i_c(x) \coloneqq (x,c)$). Recall that the topology on $\widetilde{S}$ is defined as the finest topology on $\widetilde{S}$ for which both the natural injections i_b and i_w are continuous. In particular, $\widetilde{S}$ is compact and metrizable. Obviously, a subset U of $\widetilde{S}$ is open in $\widetilde{S}$ if and only if its preimage $i_c^{-1}(U)$ is open in X_c^0 for each $c \in \{b,w\}$.

Let X and Y be normed vector spaces. Recall that a bounded linear map T from X to Y is said to be an *isomorphism* if T is bijective and T^{-1} is bounded (in other words, $\|T(x)\| \geqslant C\|x\|$ for some $C > 0$), and T is called an *isometry* if $\|T(x)\| = \|x\|$ for all $x \in X$.

Proposition 6.12 (Dual of the product space is isometric to the product of the dual spaces)**.** *Let X and Y be normed vector spaces and define $T\colon X^* \times Y^* \to (X\times Y)^*$ by $T(u,v)(x,y) = u(x) + v(y)$. Then T is an isomorphism which is an isometry with respect to the norm $\|(x,y)\| = \max\{\|x\|,\|y\|\}$ on $X \times Y$, the corresponding operator norm on $(X\times Y)^*$, and the norm $\|(u,v)\| = \|u\| + \|v\|$ on $X^*\times Y^*$.*

Proof. It is easy to see that the map T is bijective. Then it suffices to show that $\|T(u,v)\| = \|(u,v)\|$ for all $(u,v) \in X^*\times Y^*$. For one direction, we have

$$\begin{aligned}\|T(u,v)\| &= \sup_{\|(x,y)\|\leqslant 1} |T(u,v)(x,y)| = \sup_{\|x\|\leqslant 1,\, \|y\|\leqslant 1} |u(x)+v(y)| \\ &\leqslant \sup_{\|x\|\leqslant 1}|u(x)| + \sup_{\|y\|\leqslant 1}|v(y)| = \|u\| + \|v\| = \|(u,v)\|.\end{aligned}$$

For an inequality in the opposite direction, it is enough to show that $(1-\varepsilon)\|(u,v)\| \leqslant \|T(u,v)\|$ for each $\varepsilon > 0$. Indeed, for each $\varepsilon > 0$, there exist $x' \in X$ and $y' \in Y$ satisfying $u(x') \geqslant (1-\varepsilon)\|u\|\|x'\|$ and $v(y') \geqslant (1-\varepsilon)\|v\|\|y'\|$, respectively. Without loss of generality, we may assume that $\|x'\| = \|y'\| = 1$. Since $\|(x',y')\| = \max\{\|x'\|,\|y'\|\} = 1$, we have

$$(1-\varepsilon)\|(u,v)\| \leqslant u(x') + v(y') = T(u,v)(x',y') \leqslant \|T(u,v)\|$$

for each $\varepsilon > 0$, which completes the proof. $\square$

By Proposition 6.12 and the Riesz representation theorem (see [Fol13, Theorems 7.17 and 7.8]), we can identify $\bigl(C\bigl(X_b^0\bigr)\times C\bigl(X_w^0\bigr)\bigr)^*$ with the product of spaces of finite signed Borel measures $\mathcal{M}\bigl(X_b^0\bigr)\times\mathcal{M}\bigl(X_w^0\bigr)$, where we use the norm $\|(u_b,u_w)\| = \max\{\|u_b\|,\|u_w\|\}$ on $C\bigl(X_b^0\bigr)\times C\bigl(X_w^0\bigr)$, the corresponding operator norm on $\bigl(C\bigl(X_b^0\bigr)\times C\bigl(X_w^0\bigr)\bigr)^*$, and the norm $\|(\mu_b,\mu_w)\| = \|\mu_b\| + \|\mu_w\|$ on $\mathcal{M}\bigl(X_b^0\bigr)\times\mathcal{M}\bigl(X_w^0\bigr)$.

From now on, we write

$$(\mu_b,\mu_w)(A_b,A_w) \coloneqq \mu_b(A_b) + \mu_w(A_w), \tag{6.11}$$

$$\langle(\mu_b,\mu_w),(u_b,u_w)\rangle \coloneqq \langle\mu_b,u_b\rangle + \langle\mu_w,u_w\rangle = \int_{X_b^0} u_b\,\mathrm{d}\mu_b + \int_{X_w^0} u_w\,\mathrm{d}\mu_w, \tag{6.12}$$

whenever $(\mu_b,\mu_w) \in \mathcal{M}\bigl(X_b^0\bigr)\times\mathcal{M}\bigl(X_w^0\bigr)$, $(u_b,u_w) \in B\bigl(X_b^0\bigr)\times B\bigl(X_w^0\bigr)$, and A_b and A_w are Borel subset of X_b^0 and X_w^0, respectively. In particular, for each Borel set $A \subseteq S^2$, we define

$$(\mu_b,\mu_w)(A) \coloneqq (\mu_b,\mu_w)\bigl(A\cap X_b^0, A\cap X_w^0\bigr) = \mu_b\bigl(A\cap X_b^0\bigr) + \mu_w\bigl(A\cap X_w^0\bigr). \tag{6.13}$$

Remark 6.13. In the natural way, the product space $C\big(X_b^0\big) \times C\big(X_w^0\big)$ (resp. $B\big(X_b^0\big) \times B\big(X_w^0\big)$) can be identified with $C(\widetilde{S})$ (resp. $B(\widetilde{S})$). Similarly, the product space $\mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$ can be identified with $\mathcal{M}(\widetilde{S})$. Under such identifications, we write

$$\int (u_b, u_w) \,\mathrm{d}(\mu_b, \mu_w) := \langle (\mu_b, \mu_w), (u_b, u_w) \rangle \qquad \text{and} \qquad (u_b, u_w)(\mu_b, \mu_w) := (u_b \mu_b, u_w \mu_w)$$

whenever $(\mu_b, \mu_w) \in \mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$ and $(u_b, u_w) \in B\big(X_b^0\big) \times B\big(X_w^0\big)$.

Moreover, we have the following natural identification of $\mathcal{P}(\widetilde{S})$:

$$\mathcal{P}(\widetilde{S}) = \big\{ (\mu_b, \mu_w) \in \mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big) : \mu_b \text{ and } \mu_w \text{ are positive measures, } \mu_b\big(X_b^0\big) + \mu_w\big(X_w^0\big) = 1 \big\}.$$

Here we follow the terminology in [Fol13, Section 3.1] that a *positive measure* is a signed measure that takes values in $[0, +\infty]$.

Remark 6.14. It is easy to see that (6.13) defines a finite signed Borel measure $\mu := (\mu_b, \mu_w)$ on S^2. Here we use the notation μ (resp. (μ_b, μ_w)) when we view the measure as a measure on S^2 (resp. $\widetilde{S}$), and we will always use these conventions in this paper. In this sense, for each $u \in B(S^2)$ we have

$$\langle \mu, u \rangle = \int u \,\mathrm{d}\mu = \int (u_b, u_w) \,\mathrm{d}(\mu_b, \mu_w) = \int_{X_b^0} u \,\mathrm{d}\mu_b + \int_{X_w^0} u \,\mathrm{d}\mu_w, \tag{6.14}$$

where $u_b := u|_{X_b^0}$ and $u_w := u|_{X_w^0}$. Moreover, if both μ_b and μ_w are positive measures and $\mu_b\big(X_b^0\big) + \mu_w\big(X_w^0\big) = 1$, then $\mu = (\mu_b, \mu_w)$ defined in (6.13) is a Borel probability measure on S^2. In view of the identifications in Remark 6.13, this means that if $(\mu_b, \mu_w) \in \mathcal{P}(\widetilde{S})$, then $\mu \in \mathcal{P}(S^2)$.

For each color $c \in \{b, w\}$, we define the projection $\pi_c \colon \mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big) \to \mathcal{M}\big(X_c^0\big)$ by

$$\pi_c(\mu_b, \mu_w) := \mu_c, \qquad \text{for } (\mu_b, \mu_w) \in \mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big). \tag{6.15}$$

6.4. **Adjoint operators of split Ruelle operators.** In this subsection, we investigate the adjoint operators of split Ruelle operators.

Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider $\varphi \in C(S^2)$. Note that the split Ruelle operator $\mathbb{L}_{F,\varphi}$ (see Definition 6.9) is a positive, continuous operator on $C\big(X_b^0\big) \times C\big(X_w^0\big)$. Thus, the adjoint operator

$$\mathbb{L}_{F,\varphi}^* \colon \big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^* \to \big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$$

of $\mathbb{L}_{F,\varphi}$ acts on the dual space $\big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$ of the Banach space $C\big(X_b^0\big) \times C\big(X_w^0\big)$. Recall in Subsection 6.3 we identify $\big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$ with the product of spaces of finite signed Borel measures $\mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$, where we use the norm $\|(u_b, u_w)\| = \max\{\|u_b\|, \|u_w\|\}$ on $C\big(X_b^0\big) \times C\big(X_w^0\big)$, the corresponding operator norm on $\big(C\big(X_b^0\big) \times C\big(X_w^0\big)\big)^*$, and the norm $\|(\mu_b, \mu_w)\| = \|\mu_b\| + \|\mu_w\|$ on $\mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$. Then by Remark 6.13, we can also view $\mathbb{L}_{F,\varphi}$ (resp. $\mathbb{L}_{F,\varphi}^*$) as an operator on $C(\widetilde{S})$ (resp. $\mathcal{M}(\widetilde{S})$).

In the following proposition, we summarize properties of the adjoint operator $\mathbb{L}_{F,\varphi}^*$.

Proposition 6.15. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider $\varphi \in C(S^2)$. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F(\operatorname{dom}(F)) = S^2$. Consider arbitrary $n \in \mathbb{N}$ and $(\mu_b, \mu_w) \in \mathcal{M}\big(X_b^0\big) \times \mathcal{M}\big(X_w^0\big)$. Then the following statements hold:*

(i) $\big\langle \mathbb{L}_{F,\varphi}^*(\mu_b, \mu_w), (u_b, u_w) \big\rangle = \langle (\mu_b, \mu_w), \mathbb{L}_{F,\varphi}(u_b, u_w) \rangle$ *for* $(u_b, u_w) \in B\big(X_b^0\big) \times B\big(X_w^0\big)$.

(ii) *For each Borel set $A \subseteq \bigcup \mathfrak{X}^n(F, \mathcal{C})$ on which F^n is injective, we have that $F^n(A)$ is a Borel set, and*

$$\big(\mathbb{L}_{F,\varphi}^*\big)^n(\mu_b, \mu_w)(A) = \sum_{c \in \{b, w\}} \int_{F^n(A) \cap X_c^0} \big(\deg_c(F^n, \cdot) \exp\big(S_n^F \varphi\big)\big) \circ (F^n|_A)^{-1} \,\mathrm{d}\mu_c. \tag{6.16}$$

Here $\big(\mathbb{L}_{F,\varphi}^*\big)^n(\mu_b, \mu_w)(A)$ is defined in (6.13).

(iii) *For each color $c \in \{b, w\}$ and each Borel set $A_c \subseteq \operatorname{dom}(F) \cap X_c^0$ on which F is injective, we have that $F(A_c)$ is a Borel set, and*

$$\pi_c\bigl(\mathbb{L}_{F,\varphi}^*(\mu_b,\mu_w)\bigr)(A_c) = \sum_{c' \in \{b,w\}} \int_{F(A_c) \cap X_{c'}^0} (\deg_{c'c}(F,\cdot)\exp(\varphi)) \circ (F|_{A_c})^{-1} \,\mathrm{d}\mu_{c'}. \tag{6.17}$$

(iv) $\bigl(\mathbb{L}_{F,\varphi}^*\bigr)^n(\mu_b,\mu_w)\bigl(\bigcup \mathfrak{X}^{n-1}(F,\mathcal{C})\bigr) = \bigl(\mathbb{L}_{F,\varphi}^*\bigr)^n(\mu_b,\mu_w)\bigl(\bigcup \mathfrak{X}^n(F,\mathcal{C})\bigr)$.

Recall that a collection $\mathfrak{P}$ of subsets of a set X is a *π-system* if it is closed under the intersection, i.e., if A, $B \in \mathfrak{P}$ then $A \cap B \in \mathfrak{P}$. A collection $\mathfrak{L}$ of subsets of X is a *λ-system* if the following are satisfied: (1) $X \in \mathfrak{L}$. (2) If B, $C \in \mathfrak{L}$ and $B \subseteq C$, then $C \setminus B \in \mathfrak{L}$. (3) If $A_n \in \mathfrak{L}$, $n \in \mathbb{N}$, with $A_n \subseteq A_{n+1}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{L}$.

Proof. We follow the identifications discussed in Remark 6.13. Recall from Definition 6.11 that $\widetilde{S} = X_b^0 \sqcup X_w^0$ is the disjoint union of X_b^0 and X_w^0.

(i) It suffices to show that for each $\mu \in \mathcal{M}(\widetilde{S})$ and each Borel set $A \subseteq \widetilde{S}$,

$$\bigl\langle \mathbb{L}_{F,\varphi}^*(\mu), \mathbb{1}_A \bigr\rangle = \langle \mu, \mathbb{L}_{F,\varphi}(\mathbb{1}_A)\rangle. \tag{6.18}$$

Let $\mathfrak{L}$ be the collection of Borel sets $A \subseteq \widetilde{S}$ for which (6.18) holds. Denote the collection of open subsets of $\widetilde{S}$ by $\mathfrak{G}$. Then $\mathfrak{G}$ is a π-system.

We observe from (6.7) that if $\{u_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of real-valued continuous functions on $\widetilde{S}$, then so is $\bigl\{\mathbb{L}_{F,\varphi}(u_n)\bigr\}_{n\in\mathbb{N}}$.

By the definition of $\mathbb{L}_{F,\varphi}^*$, we have

$$\langle \mathbb{L}_{F,\varphi}^*(\mu), u\rangle = \langle \mu, \mathbb{L}_{F,\varphi}(u)\rangle \tag{6.19}$$

for $u \in C(\widetilde{S})$. Fix an open set $U \subseteq \widetilde{S}$, then there exists a non-decreasing sequence $\{g_n\}_{n\in\mathbb{N}}$ of real-valued continuous functions on $\widetilde{S}$ supported in U such that g_n converges to $\mathbb{1}_U$ pointwise as $n \to +\infty$. Then $\bigl\{\mathbb{L}_{F,\varphi}(g_n)\bigr\}_{n\in\mathbb{N}}$ is also a non-decreasing sequence of continuous functions, whose pointwise limit is $\mathbb{L}_{F,\varphi}(\mathbb{1}_U)$. By the Lebesgue Monotone Convergence Theorem and (6.19), we can conclude that (6.18) holds for $A = U$. Thus $\mathfrak{G} \subseteq \mathfrak{L}$.

We now prove that $\mathfrak{L}$ is a λ-system. Indeed, since (6.19) holds for $u = \mathbb{1}_{\widetilde{S}}$, we get $\widetilde{S} \in \mathfrak{L}$. For each pair of B, $C \in \mathfrak{L}$ with $B \subseteq C$, it follows from (6.7) that $\mathbb{1}_C - \mathbb{1}_B = \mathbb{1}_{C\setminus B}$ and $\mathbb{L}_{F,\varphi}(\mathbb{1}_C) - \mathbb{L}_{F,\varphi}(\mathbb{1}_B) = \mathbb{L}_{F,\varphi}(\mathbb{1}_C - \mathbb{1}_B) = \mathbb{L}_{F,\varphi}(\mathbb{1}_{C\setminus B})$. Thus $C \setminus B \in \mathfrak{L}$. Finally, given $A_n \in \mathfrak{L}$, $n \in \mathbb{N}$, with $A_n \subseteq A_{n+1}$, and denote $A := \bigcup_{n\in\mathbb{N}} A_n$. Then $\{\mathbb{1}_{A_n}\}_{n\in\mathbb{N}}$ and $\bigl\{\mathbb{L}_{F,\varphi}(\mathbb{1}_{A_n})\bigr\}_{n\in\mathbb{N}}$ are non-decreasing sequences of real-valued Borel functions on $\widetilde{S}$ that converge pointwise to $\mathbb{1}_A$ and $\mathbb{L}_{F,\varphi}(\mathbb{1}_A)$, respectively, as $n \to +\infty$. Then by the the Lebesgue Monotone Convergence Theorem, we get $A \in \mathfrak{L}$. Hence $\mathfrak{L}$ is a λ-system.

Recall that Dynkin's π-λ theorem (see for example, [Bil08, Theorem 3.2]) states that if $\mathfrak{P}$ is a π-system and $\mathfrak{L}$ is a λ-system that contains $\mathfrak{P}$, then the σ-algebra $\sigma(\mathfrak{P})$ generated by $\mathfrak{P}$ is a subset of $\mathfrak{L}$. Thus by Dynkin's π-λ theorem, the Borel σ-algebra $\sigma(\mathfrak{G})$ is a subset of $\mathfrak{L}$, i.e., equality (6.18) holds for each Borel set $A \subseteq \widetilde{S}$. This finishes the proof of statement (i).

(ii) We fix an arbitrary Borel set $A \subseteq \bigcup \mathfrak{X}^n(F,\mathcal{C})$ on which F^n is injective. Denote $A_b := A \cap X_b^0$ and $A_w := A \cap X_w^0$.

For each $X^n \in \mathfrak{X}^n(F,\mathcal{C})$, it follows from Proposition 5.4 (i) that $F^n|_{X^n}$ is a homeomorphism from X^n onto $F^n(X^n)$, which maps Borel sets to Borel sets. Thus $F^n(A)$ is a Borel set since

$$F^n(A) = F^n\Biggl(\bigcup_{X^n\in\mathfrak{X}^n(F,\mathcal{C})} A \cap X^n\Biggr) = \bigcup_{X^n\in\mathfrak{X}^n(F,\mathcal{C})} F^n\left(A \cap X^n\right) = \bigcup_{X^n\in\mathfrak{X}^n(F,\mathcal{C})} F^n|_{X^n}(A).$$

We now prove (6.16). By (6.13), statement (i), and (6.12), we get

$$\bigl(\mathbb{L}_{F,\varphi}^*\bigr)^n(\mu_b,\mu_w)(A) = \langle(\mu_b,\mu_w), \mathbb{L}_{F,\varphi}^n(\mathbb{1}_{A_b},\mathbb{1}_{A_w})\rangle = \sum_{c\in\{b,w\}} \bigl\langle \mu_c, \pi_c\bigl(\mathbb{L}_{F,\varphi}^n(\mathbb{1}_{A_c},\mathbb{1}_{A_w})\bigr)\bigr\rangle.$$

Then it suffices to show that for each $c \in \{b,w\}$ and each $x \in X_c^0$,

$$\pi_c\bigl(\mathbb{L}_{F,\varphi}^n(\mathbb{1}_{A_b},\mathbb{1}_{A_w})\bigr)(x) = \mathbb{1}_{F^n(A)}(x) \cdot \bigl(\deg_c(F^n,\cdot)\exp\bigl(S_n^F\varphi\bigr)\bigr) \circ (F^n|_A)^{-1}(x).$$

Indeed, by (6.9), (6.8), and (6.4),

$$\begin{aligned}
\pi_c\big(\mathbb{L}^n_{F,\varphi}(\mathbb{1}_{A_b},\mathbb{1}_{A_w})\big)(x) &= \mathcal{L}^{(n)}_{F,\phi,c,b}(\mathbb{1}_{A_b})(x)+\mathcal{L}^{(n)}_{F,\phi,c,w}(\mathbb{1}_{A_w})(x)\\
&= \sum_{c'\in\{b,w\}}\sum_{X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})}\big(\mathbb{1}_{A_{c'}}\cdot\exp\big(S^F_n\varphi\big)\big)\circ(F^n|_{X^n})^{-1}(x)\\
&= \sum_{c'\in\{b,w\}}\sum_{X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})}\big(\mathbb{1}_{A}\cdot\exp\big(S^F_n\varphi\big)\big)\circ(F^n|_{X^n})^{-1}(x)\\
&= \sum_{X^n\in\mathfrak{X}^n_{c}(F,\mathcal{C})}\big(\mathbb{1}_{A}\cdot\exp\big(S^F_n\varphi\big)\big)\circ(F^n|_{X^n})^{-1}(x)\\
&= \sum_{y\in F^{-n}(x)}\deg_c(F^n,y)\mathbb{1}_A(y)\exp\big(S^F_n\varphi(y)\big)\\
&= \mathbb{1}_{F^n(A)}(x)\big(\deg_c(F^n,\cdot)\exp\big(S^F_n\varphi\big)\big)\circ(F^n|_A)^{-1}(x),
\end{aligned}$$

where the third equality follows from the fact that for each $c'\in\{b,w\}$ and each $X^n\in\mathfrak{X}^n_{cc'}(F,\mathcal{C})$, the point $z=(F^n|_{X^n})^{-1}(x)\in A$ if and only if $z\in A_{c'}$ since $z\in X^n\subseteq X^0_{c'}$, and the last equality holds since F is injective on A. Thus, we finish the proof of statement (ii).

(iii) We arbitrarily fix a color $c\in\{b,w\}$ and a Borel set $A_c\subseteq\operatorname{dom}(F)\cap X^0_c$ on which F is injective. Then it follows immediately from statement (ii) that $F(A_c)$ is a Borel set.

We now prove (6.17). Without loss of generality, we can assume that $c=b$. Then, by (6.15), statement (i), and (6.12), we get

$$\begin{aligned}
\pi_b\big(\mathbb{L}^*_{F,\varphi}(\mu_b,\mu_w)\big)(A_b) &= \mathbb{L}^*_{F,\varphi}(\mu_b,\mu_w)(A_b,\emptyset)=\langle(\mu_b,\mu_w),\mathbb{L}_{F,\varphi}(\mathbb{1}_{A_b},0)\rangle\\
&= \big\langle\mu_b,\pi_b\big(\mathbb{L}_{F,\varphi}(\mathbb{1}_{A_b},0)\big)\big\rangle+\big\langle\mu_w,\pi_w\big(\mathbb{L}_{F,\varphi}(\mathbb{1}_{A_b},0)\big)\big\rangle.
\end{aligned}$$

It suffices to show that for each $c'\in\{b,w\}$ and each $x\in X^0_{c'}$,

$$\pi_{c'}\big(\mathbb{L}_{F,\varphi}(\mathbb{1}_{A_b},0)\big)(x)=\mathbb{1}_{F(A_b)}(x)\cdot(\deg_{c'b}(F,\cdot)\exp(\varphi))\circ(F|_{A_b})^{-1}(x).$$

Indeed, by (6.7), (6.8), and (6.4),

$$\begin{aligned}
\pi_{c'}\big(\mathbb{L}_{F,\varphi}(\mathbb{1}_{A_b},0)\big)(x) &= \mathcal{L}^{(1)}_{F,\phi,c',b}(\mathbb{1}_{A_b})(x)\\
&= \sum_{X^1\in\mathfrak{X}^1_{c'b}(F,\mathcal{C})}\big(\mathbb{1}_{A_b}\cdot\exp(\varphi)\big)\circ(F|_{X^1})^{-1}(x)\\
&= \sum_{y\in F^{-1}(x)}\deg_{c'b}(F,y)\mathbb{1}_{A_b}(y)\exp(\varphi(y))\\
&= \mathbb{1}_{F(A_b)}(x)(\deg_{c'b}(F,\cdot)\exp(\varphi))\circ(F|_{A_b})^{-1}(x),
\end{aligned}$$

where the last equality holds since F is injective on A_b. Thus we finish the proof of statement (iii).

(iv) For convenience we set $\Omega^k:=\bigcup\mathfrak{X}^k(F,\mathcal{C})$ and $\Omega^k_c:=\Omega^k\cap X^0_c$ for each $k\in\mathbb{N}_0$ and each $c\in\{b,w\}$. Note that $\Omega^1=\operatorname{dom}(F)\subseteq\Omega^0$ since $\Omega^0=F(\operatorname{dom}(F))=S^2$. By (6.13), statement (i), and (6.12), we have

$$\begin{aligned}
\big(\mathbb{L}^*_{F,\varphi}\big)^n(\mu_b,\mu_w)\big(\Omega^{n-1}\big) &= \big(\mathbb{L}^*_{F,\varphi}\big)^n(\mu_b,\mu_w)\big(\Omega^{n-1}_b,\Omega^{n-1}_w\big)=\big\langle(\mu_b,\mu_w),\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^{n-1}_b},\mathbb{1}_{\Omega^{n-1}_w}\big)\big\rangle\\
&= \big\langle\mu_b,\pi_b\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^{n-1}_b},\mathbb{1}_{\Omega^{n-1}_w}\big)\big)\big\rangle+\big\langle\mu_w,\pi_w\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^{n-1}_b},\mathbb{1}_{\Omega^{n-1}_w}\big)\big)\big\rangle,\\
\big(\mathbb{L}^*_{F,\varphi}\big)^n(\mu_b,\mu_w)(\Omega^n) &= \big(\mathbb{L}^*_{F,\varphi}\big)^n(\mu_b,\mu_w)\big(\Omega^n_b,\Omega^n_w\big)=\big\langle(\mu_b,\mu_w),\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^n_b},\mathbb{1}_{\Omega^n_w}\big)\big\rangle\\
&= \big\langle\mu_b,\pi_b\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^n_b},\mathbb{1}_{\Omega^n_w}\big)\big)\big\rangle+\big\langle\mu_w,\pi_w\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^n_b},\mathbb{1}_{\Omega^n_w}\big)\big)\big\rangle.
\end{aligned}$$

It suffices to show that $\pi_c\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^{n-1}_b}, \mathbb{1}_{\Omega^{n-1}_w}\big)\big) = \pi_c\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^n_b}, \mathbb{1}_{\Omega^n_w}\big)\big)$ for each $c \in \{b, w\}$. Indeed, by (6.8), (6.9), and (6.4),

$$\begin{aligned}\pi_c\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^{n-1}_b}, \mathbb{1}_{\Omega^{n-1}_w}\big)\big) &= \mathcal{L}^{(n)}_{F,\phi,c,b}\big(\mathbb{1}_{\Omega^{n-1}_b}\big) + \mathcal{L}^{(n)}_{F,\phi,c,w}\big(\mathbb{1}_{\Omega^{n-1}_w}\big) \\ &= \sum_{c'\in\{b,w\}} \sum_{X^n \in \mathfrak{X}^n_{cc'}(F,\mathcal{C})} \big(\mathbb{1}_{\Omega^{n-1}_{c'}} \cdot \exp\big(S^F_n \varphi\big)\big) \circ (F|_{X^n})^{-1} \\ &= \sum_{c'\in\{b,w\}} \sum_{X^n \in \mathfrak{X}^n_{cc'}(F,\mathcal{C})} \big(\mathbb{1}_{\Omega^{n}_{c'}} \cdot \exp\big(S^F_n \varphi\big)\big) \circ (F|_{X^n})^{-1} \\ &= \pi_c\big(\mathbb{L}^n_{F,\varphi}\big(\mathbb{1}_{\Omega^n_b}, \mathbb{1}_{\Omega^n_w}\big)\big),\end{aligned}$$

where the third equality holds since $X^n \subseteq \Omega^n_{c'} \subseteq \Omega^{n-1}_{c'}$ for each $c' \in \{b, w\}$ and each $X^n \in \mathfrak{X}^n_{cc'}(F,\mathcal{C})$ by Proposition 5.5 (i). $\square$

6.5. **The eigenmeasure.** By applying the Schauder–Tikhonov Fixed Point Theorem, we establish in Theorem 6.16 the existence of an eigenmeasure of the adjoint $\mathbb{L}^*_{F,\phi}$ of the split Ruelle operator $\mathbb{L}_{F,\phi}$, where f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We also show in Theorem 6.16 (ii) that if F is strongly irreducible (see Definition 5.15), then the set of vertices is of measure zero with respect to such an eigenmeasure.

We follow the conventions discussed in Remarks 6.13 and 6.14 in this subsection.

Theorem 6.16. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F(\operatorname{dom}(F)) = S^2$. Then there exists a Borel probability measure $m_{F,\phi} = (m_b, m_w) \in \mathcal{P}(\widetilde{S})$ such that*

$$\mathbb{L}^*_{F,\phi}(m_b, m_w) = \kappa(m_b, m_w), \tag{6.20}$$

where $\kappa = \big\langle \mathbb{L}^*_{F,\phi}(m_b, m_w), \mathbb{1}_{\widetilde{S}}\big\rangle$. Moreover, any $m_{F,\phi} = (m_b, m_w) \in \mathcal{P}(\widetilde{S})$ that satisfies (6.20) *for some $\kappa > 0$ has the following properties:*

(i) $m_{F,\phi}(\Omega(F,\mathcal{C})) = 1$.

(ii) *If F is strongly irreducible, then $m_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)\big) = 0$.*

Note that we use the notation (m_b, m_w) (resp. $m_{F,\phi}$) to emphasize that we treat the eigenmeasure as a Borel probability measure on $\widetilde{S}$ (resp. S^2).

The proof of Theorem 6.16 will be given at the end of this subsection.

Under the same assumption as in Theorem 6.16 (ii), we will show $m_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = 0$ in Proposition 6.26 by using property (ii).

By some elementary calculations in linear algebra, we have the following results for a 2×2 matrix with non-negative entries.

Lemma 6.17. *Let A be a 2×2 matrix with non-negative entries. We denote by $\|A\|_{\mathrm{sum}}$ the sum of all the absolute values of entries in A. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda > 0$ and $0 \neq \mathbf{x} \in \mathbb{R}^2$ with non-negative entries, then $\|A^n\|_{\mathrm{sum}} \geqslant \lambda^n$ for each $n \in \mathbb{N}$.*

Proof. If A is an upper-triangular matrix, then the conclusion follows immediately. For general cases, one can establish the conclusion by calculating the iterates of A via Schur decomposition and summarizing the entries. $\square$

Proof of Theorem 6.16. We first show the existence. Define $\tau\colon \mathcal{P}(\widetilde{S}) \to \mathcal{P}(\widetilde{S})$ by $\tau(\mu_b, \mu_w) := \frac{\mathbb{L}^*_{F,\phi}(\mu_b,\mu_w)}{\langle \mathbb{L}^*_{F,\phi}(\mu_b,\mu_w), \mathbb{1}_{\widetilde{S}}\rangle}$. Then τ is a continuous transformation on the non-empty, convex, compact (in the weak* topology, by Alaoglu's theorem) space $\mathcal{P}(\widetilde{S})$ of Borel probability measures on $\widetilde{S}$. By the Schauder–Tikhonov Fixed Point Theorem (see for example, [PU10, Theorem 3.1.7]), there exists a measure $m_{F,\phi} = (m_b, m_w) \in \mathcal{P}(\widetilde{S})$ such that $\tau(m_b, m_w) = (m_b, m_w)$. Thus $\mathbb{L}^*_{F,\phi}(m_b, m_w) = \kappa(m_b, m_w)$ with $\kappa := \big\langle \mathbb{L}^*_{F,\phi}(m_b, m_w), \mathbb{1}_{\widetilde{S}}\big\rangle$. Note that $\kappa > 0$ since $F(\operatorname{dom}(F)) = S^2$.

We now show that any $m_{F,\phi} = (m_b, m_w) \in \mathcal{P}(\widetilde{S})$ that satisfies (6.20) for some $\kappa > 0$ has properties (i) and (ii). By Proposition 6.15 (iv), for each $n \in \mathbb{N}$, we have $m_{F,\phi}\big(\bigcup \mathfrak{X}^n(F,\mathcal{C})\big) = m_{F,\phi}\big(\bigcup \mathfrak{X}^{n-1}(F,\mathcal{C})\big)$. Then by Proposition 5.5 (i) and (5.2) we have

$$m_{F,\phi}(\Omega) = \lim_{n\to+\infty} m_{F,\phi}\Big(\bigcup \mathfrak{X}^n(F,\mathcal{C})\Big) = m_{F,\phi}(S^2) = 1,$$

which proves property (i).

Next, we verify property (ii). Assume that F is strongly irreducible.

We will prove that $m_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)\big) = 0$. Since $\bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)$ is a countable set, by property (i), the conclusion follows if we can prove that $m_{F,\phi}(\{y\}) = 0$ for each $y \in \Omega \cap \bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)$.

We claim that it suffices to show that $m_{F,\phi}(\{x\}) = 0$ for each periodic $x \in \Omega \cap \operatorname{post} f$. To see this, let $y \in \Omega \cap \bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)$ be arbitrary. We follow the convention that if $p \notin X_c^0$ for some color $c \in \{b, w\}$, then $m_c(\{p\}) = 0$. For each $c \in \{b, w\}$, since $\pi_c\big(\mathbb{L}_{F,\phi}^*(m_b, m_w)\big) = \pi_c(\kappa(m_b, m_w)) = \kappa m_c$, by Proposition 6.15 (iii), we have

$$\begin{aligned}
\kappa m_c(\{y\}) &= \sum_{c'\in\{b,w\}} \int_{F(\{y\})\cap X_{c'}^0} (\deg_{c'c}(F,\cdot)\exp(\phi)) \circ \big(F|_{\{y\}}\big)^{-1} \,\mathrm{d}m_{c'} \\
&= \sum_{c'\in\{b,w\}} \deg_{c'c}(F,y)\exp(\phi(y)) m_{c'}(\{F(y)\}).
\end{aligned}$$

By using the notion of local degree matrix (see Definition 5.7), we can write the equation above as

$$\begin{bmatrix} m_b(\{y\}) \\ m_w(\{y\}) \end{bmatrix} = \frac{\exp(\phi(y))}{\kappa} \operatorname{Deg}(F,y) \begin{bmatrix} m_b(\{F(y)\}) \\ m_w(\{F(y)\}) \end{bmatrix}. \tag{6.21}$$

Since $F^n(y) \in \Omega \subseteq \operatorname{dom}(F)$ for each $n \in \mathbb{N}$ by Proposition 5.5 (i), we can iterate (6.21) under F. Then it follows from Lemma 5.9 and induction that

$$\begin{bmatrix} m_b(\{y\}) \\ m_w(\{y\}) \end{bmatrix} = \frac{\exp(S_n\phi(y))}{\kappa^n} \operatorname{Deg}(F^n,y) \begin{bmatrix} m_b(\{F^n(y)\}) \\ m_w(\{F^n(y)\}) \end{bmatrix} \tag{6.22}$$

for each $n \in \mathbb{N}$. Hence, since $y \in \bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)$ and $m_{F,\phi}(\{y\}) = 0$ if and only if $m_b(\{y\}) = m_w(\{y\}) = 0$, by (6.22), it suffices to show that $m_{F,\phi}(\{x\}) = 0$ for each periodic $x \in \Omega \cap \operatorname{post} f$.

It remains to show that $m_{F,\phi}(\{x\}) = 0$ for each periodic $x \in \Omega \cap \operatorname{post} f$. We argue by contradiction and assume that there exists $x \in \Omega \cap \operatorname{post} f$ such that $F^\ell(x) = x$ for some $\ell \in \mathbb{N}$ and $m_{F,\phi}(\{x\}) \neq 0$. Since $m_{F,\phi}(\{x\}) = m_b(\{x\}) + m_w(\{x\})$, we may assume without loss of generality that $m_b(\{x\}) > 0$.

Since $x \in \Omega \cap \operatorname{post} f$ and $F^\ell(x) = x$, it follows immediately from (6.22) that

$$\begin{bmatrix} m_b(\{x\}) \\ m_w(\{x\}) \end{bmatrix} = \frac{\exp(S_\ell\phi(x))}{\kappa^\ell} \operatorname{Deg}\big(F^\ell,x\big) \begin{bmatrix} m_b(\{x\}) \\ m_w(\{x\}) \end{bmatrix}. \tag{6.23}$$

Similarly, by using Proposition 6.15 (iii) repeatedly, for each $k \in \mathbb{N}$ and each $y \in F^{-k\ell}(x)$, we have

$$\begin{bmatrix} m_b(\{y\}) \\ m_w(\{y\}) \end{bmatrix} = \frac{\exp(S_{k\ell}\phi(y))}{\kappa^{k\ell}} \operatorname{Deg}\big(F^{k\ell},y\big) \begin{bmatrix} m_b(\{x\}) \\ m_w(\{x\}) \end{bmatrix}. \tag{6.24}$$

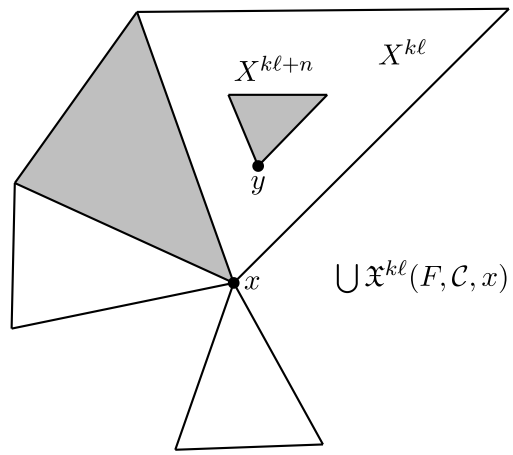

FIGURE 6.1. $\bigcup \mathfrak{X}^{k\ell}(F,\mathcal{C},x)$, with card(post f)$=3$.

Recall from Definition 5.7 and Remark 5.8 that for each $k\in\mathbb{N}$, $\mathfrak{X}^{k\ell}(F,\mathcal{C},x)$ is the collection of $(k\ell)$-tiles of F that intersect $\{x\}$, and contains exactly $\left\|\operatorname{Deg}\bigl(F^{k\ell},x\bigr)\right\|_{\operatorname{sum}}$ distinct $(k\ell)$-tiles of F. Since F is strongly irreducible, it follows from Lemma 5.21 that for each $k\in\mathbb{N}$ and each $X^{k\ell}\in\mathfrak{X}^{k\ell}(F,\mathcal{C},x)$, there exists an integer $n\in\mathbb{N}$ with $n\leqslant n_F$ and a black $(k\ell+n)$-tile $X_b^{k\ell+n}\in\mathfrak{X}_b^{k\ell+n}(F,\mathcal{C})$ satisfying $X_b^{k\ell+n}\subseteq\operatorname{inte}\bigl(X^{k\ell}\bigr)$. Here $n_F\in\mathbb{N}$ is the constant in Definition 5.15, which depends only on F and $\mathcal{C}$. Then by Proposition 5.4 (i), there exists a unique $y\in X_b^{k\ell+n}\subseteq\operatorname{inte}\bigl(X^{k\ell}\bigr)$ such that

$$F^{k\ell+n}(y)=x$$

(see Figure 6.1). For each $k\in\mathbb{N}$, we denote by T_k the set consisting of one such y from each $X^{k\ell}\in\mathfrak{X}^{k\ell}(F,\mathcal{C},x)$, and we have

$$\operatorname{card}(T_k)=\left\|\operatorname{Deg}\bigl(F^{k\ell},x\bigr)\right\|_{\operatorname{sum}}. \tag{6.25}$$

Then $\{T_k\}_{k\in\mathbb{N}}$ is a sequence of subsets of $\bigcup_{j=0}^{+\infty}f^{-j}(\operatorname{post}f)$. Since f is expanding, we can choose an increasing sequence $\{k_i\}_{i\in\mathbb{N}}$ of integers recursively in such a way that $\bigcup\mathfrak{X}^{k_{i+1}\ell}(F,\mathcal{C},x)\cap\bigcup_{j=1}^{i}T_{k_j}=\emptyset$ for each $i\in\mathbb{N}$. Then $\{T_{k_i}\}_{i\in\mathbb{N}}$ is a sequence of mutually disjoint sets. Thus, by (6.24) and Lemma 5.24,

$$\begin{aligned}
m_{F,\phi}\Biggl(\bigcup_{j=0}^{+\infty}f^{-j}(\operatorname{post}f)\Biggr)&\geqslant\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}m_{F,\phi}(\{y\})=\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}\begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}m_b(\{y\})\\m_w(\{y\})\end{bmatrix}\\
&=\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}\frac{\exp(S_{k_i\ell+n}\phi(y))}{\kappa^{k_i\ell+n}}\begin{bmatrix}1&1\end{bmatrix}\operatorname{Deg}\bigl(F^{k_i\ell+n},y\bigr)\begin{bmatrix}m_b(\{x\})\\m_w(\{x\})\end{bmatrix}\\
&=\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}\frac{e^{S_{k_i\ell+n}\phi(y)}}{e^{S_{k_i\ell}\phi(x)}}\frac{e^{S_{k_i\ell}\phi(x)}}{\kappa^{k_i\ell+n}}\begin{bmatrix}1&1\end{bmatrix}\operatorname{Deg}\bigl(F^{k_i\ell+n},y\bigr)\begin{bmatrix}m_b(\{x\})\\m_w(\{x\})\end{bmatrix}\\
&=\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}\frac{e^{S_n\phi(f^{k_i\ell}(y))}}{e^{S_{k_i\ell}\phi(x)-S_{k_i\ell}\phi(y)}}\biggl(\frac{e^{S_\ell\phi(x)}}{\kappa^\ell}\biggr)^{k_i}\kappa^{-n}\begin{bmatrix}1&1\end{bmatrix}\operatorname{Deg}\bigl(F^{k_i\ell+n},y\bigr)\begin{bmatrix}m_b(\{x\})\\m_w(\{x\})\end{bmatrix}\\
&\geqslant\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}\frac{e^{-n_F\|\phi\|_\infty}}{e^{C_1(\operatorname{diam}_d(S^2))^\beta}}\biggl(\frac{e^{S_\ell\phi(x)}}{\kappa^\ell}\biggr)^{k_i}\min\{1,\kappa^{-n_F}\}\begin{bmatrix}1&1\end{bmatrix}\operatorname{Deg}\bigl(F^{k_i\ell+n},y\bigr)\begin{bmatrix}m_b(\{x\})\\m_w(\{x\})\end{bmatrix},
\end{aligned}\tag{6.26}$$

where $C_1\geqslant 0$ is the constant defined in (5.12) in Lemma 5.24 and depends only on f, $\mathcal{C}$, d, ϕ, and β.

To reach a contradiction, it suffices to show that $m_{F,\phi}\bigl(\bigcup_{j=0}^{+\infty}f^{-j}(\operatorname{post}f)\bigr)=+\infty$ since $m_{F,\phi}$ is a Borel probability measure. For each $i\in\mathbb{N}$ and each $y\in T_{k_i}$, we have

$$\begin{aligned}
\begin{bmatrix}1&1\end{bmatrix}\operatorname{Deg}\bigl(F^{k_i\ell+n},y\bigr)\begin{bmatrix}m_b(\{x\})\\m_w(\{x\})\end{bmatrix}&\geqslant\begin{bmatrix}1&1\end{bmatrix}\operatorname{Deg}\bigl(F^{k_i\ell+n},y\bigr)\begin{bmatrix}1\\0\end{bmatrix}m_b(\{x\})\\
&=\deg_b(F^{k_i\ell+n},y)m_b(\{x\})\geqslant m_b(\{x\}),
\end{aligned}\tag{6.27}$$

where the equality follows from (5.6) in Remark 5.8, and the last inequality holds since $y \in X_b^{k_i\ell+n}$ for some $X_b^{k_i\ell+n} \in \mathfrak{X}_b^{k_i\ell+n}(F,\mathcal{C})$. Thus, it follows from (6.26), (6.27), and (6.25) that

$$\begin{aligned} m_{F,\phi}\biggl(\bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)\biggr) &\geqslant C\sum_{i=1}^{+\infty}\sum_{y\in T_{k_i}}\biggl(\frac{\exp(S_\ell\phi(x))}{\kappa^\ell}\biggr)^{k_i} \\ &= C\sum_{i=1}^{+\infty}\operatorname{card}(T_{k_i})\biggl(\frac{\exp(S_\ell\phi(x))}{\kappa^\ell}\biggr)^{k_i} \\ &= C\sum_{i=1}^{+\infty}\bigl\|\operatorname{Deg}\bigl(F^{k_i\ell},x\bigr)\bigr\|_{\mathrm{sum}}\biggl(\frac{\exp(S_\ell\phi(x))}{\kappa^\ell}\biggr)^{k_i}, \end{aligned} \tag{6.28}$$

where $C \coloneqq m_b(\{x\})\min\{1,\kappa^{-n_F}\}\exp\bigl(-n_F\ell\|\phi\|_\infty - C_1(\operatorname{diam}_d(S^2))^\beta\bigr) > 0$. It suffices to show that

$$\bigl\|\operatorname{Deg}\bigl(F^{k_i\ell},x\bigr)\bigr\|_{\mathrm{sum}} \geqslant \bigl(\kappa^\ell/\exp(S_\ell\phi(x))\bigr)^{k_i} \tag{6.29}$$

for each $i\in\mathbb{N}$. We denote by M the matrix $\operatorname{Deg}\bigl(F^\ell,x\bigr)$ and λ the number $\kappa^\ell/\exp(S_\ell\phi(x))$. For each $i\in\mathbb{N}$, by Lemma 5.9, we have $\operatorname{Deg}\bigl(F^{k_i\ell},x\bigr) = M^{k_i}$. Then (6.29) follows from (6.23) and Lemma 6.17. Combining (6.29) and (6.28), we get

$$m_{F,\phi}\biggl(\bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)\biggr) \geqslant C\sum_{i=1}^{+\infty} 1 = +\infty.$$

This contradicts the fact that $m_{F,\phi}$ is a finite Borel measure. The proof of property (ii) is complete. □

6.6. **The eigenfunction.** In this subsection, we establish some useful estimates for the split Ruelle operator and construct its eigenfunction.

We follow the conventions discussed in Remarks 6.13 and 6.14 in this subsection.

Proposition 6.18. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C})\subseteq\mathcal{C}$ and $F\in\operatorname{Sub}(f,\mathcal{C})$ is irreducible. Let $(m_b,m_w)\in\mathcal{P}(\widetilde{S})$ be a Borel probability measure defined in Theorem 6.16 which satisfies $\mathbb{L}_{F,\phi}^*(m_b,m_w)=\kappa(m_b,m_w)$ where $\kappa=\bigl\langle\mathbb{L}_{F,\phi}^*(m_b,m_w),\mathbb{1}_{\widetilde{S}}\bigr\rangle$. Then for each $\widetilde{x}\in\widetilde{S}$, $\frac{1}{n}\log\bigl(\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x})\bigr)$ converges to $\log\kappa$ as n tends to $+\infty$.*

Proof. Note that by Lemmas 5.25 (ii) and 6.10, for all $n\in\mathbb{N}_0$ and $\widetilde{x},\widetilde{y}\in\widetilde{S}$, we have

$$\widetilde{C}^{-1} \leqslant \frac{\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x})}{\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y})} \leqslant \widetilde{C}, \tag{6.30}$$

where $\widetilde{C}\geqslant 1$ is the constant depending only on F, $\mathcal{C}$, d, ϕ, and β from Lemma 5.25 (ii). Since $\bigl\langle(m_b,m_w),\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)\bigr\rangle = \bigl\langle\bigl(\mathbb{L}_{F,\phi}^*\bigr)^n(m_b,m_w),\mathbb{1}_{\widetilde{S}}\bigr\rangle = \bigl\langle\kappa^n(m_b,m_w),\mathbb{1}_{\widetilde{S}}\bigr\rangle = \kappa^n$, it follows from (6.9) and (6.30) that

$$\begin{aligned} \log\kappa &= \lim_{n\to+\infty}\frac{1}{n}\log\int\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y})\,\mathrm{d}(m_b,m_w)(\widetilde{y}) \\ &= \lim_{n\to+\infty}\frac{1}{n}\log\int\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x})\,\mathrm{d}(m_b,m_w)(\widetilde{y}) \\ &= \lim_{n\to+\infty}\frac{1}{n}\log\bigl(\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x})\bigr) \end{aligned}$$

for each arbitrarily chosen $\widetilde{x}\in\widetilde{S}$. □

Corollary 6.19. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C})\subseteq\mathcal{C}$ and $F\in\operatorname{Sub}(f,\mathcal{C})$ is irreducible. Then the limit $\lim_{n\to+\infty}\frac{1}{n}\log\bigl(\mathbb{L}_{F,\phi}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x})\bigr)$ exists for each $\widetilde{x}\in\widetilde{S}$ and is independent of $\widetilde{x}\in\widetilde{S}$.*

We denote the limit as $D_{F,\phi}\in\mathbb{R}$.

Proof. By Theorem 6.16, there exists a measure $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ such as the one in Proposition 6.18. The limit then clearly depends only on F, $\mathcal{C}$, d, ϕ, and β, and in particular, does not depend on the choice of (m_b, m_w). □

Proposition 6.20. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Then for each $y_0 \in \Omega \setminus \mathcal{C}$, we have*

$$D_{F,\phi} = \lim_{n \to +\infty} \frac{1}{n} \log\Biggl(\sum_{x \in (F|_\Omega)^{-n}(y_0)} \exp\bigl(S_n^F \phi(x)\bigr) \Biggr).$$

Note that it follows from Propositions 5.6 (ii) and 5.20 (ii) that $F(\Omega) = \Omega$ and $\Omega \setminus \mathcal{C} \neq \emptyset$.

Proof. Let $y_0 \in \Omega \setminus \mathcal{C}$ be arbitrary. Without loss of generality we may assume that $y_0 \in \operatorname{inte}\bigl(X_b^0\bigr)$. By Corollary 6.19 and (6.9), it suffices to show that

$$(F|_\Omega)^{-n}(y_0) = \bigl\{ (F^n|_{X^n})^{-1}(y_0) : X^n \in \mathfrak{X}_b^n(F, \mathcal{C}) \bigr\}$$

for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary in the following two paragraphs.

For each $x \in (F|_\Omega)^{-n}(y_0)$, we have $x \in \Omega \subseteq \bigcup \mathfrak{X}^n(F, \mathcal{C})$ and $(F|_\Omega)^n(x) = F^n(x) = y_0 \in \operatorname{inte}\bigl(X_b^0\bigr)$. Thus by Lemma 3.7 (ii) and Proposition 5.4 (i), there exists a unique n-tile $X^n \in \mathfrak{X}_b^n(F, \mathcal{C})$ with $x \in \operatorname{inte}(X^n)$ and $F^n(X^n) = X_b^0$. Thus we deduce that $x = (F^n|_{X^n})^{-1}(y_0)$ for some $X^n \in \mathfrak{X}_b^n(F, \mathcal{C})$.

To deduce the reverse inclusion, it suffices to show that $(F^n|_{X^n})^{-1}(y_0) \in \Omega$ for each $X^n \in \mathfrak{X}_b^n(F, \mathcal{C})$. Let $X^n \in \mathfrak{X}_b^n(F, \mathcal{C})$ be arbitrary and denote $x \coloneqq (F^n|_{X^n})^{-1}(y_0)$. Then it follows from Proposition 5.5 (iii) and induction that $x \in \Omega$ since $y_0 \in \Omega \cap \operatorname{inte}\bigl(X_b^0\bigr)$ and $x \in F^{-n}(y_0)$. □

Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $F \in \operatorname{Sub}(f, \mathcal{C})$ is irreducible. We define the function

$$\overline{\phi} \coloneqq \phi - D_{F,\phi} \in C^{0,\beta}(S^2, d). \tag{6.31}$$

Then

$$\mathbb{L}_{F,\overline{\phi}} = e^{-D_{F,\phi}} \mathbb{L}_{F,\phi}. \tag{6.32}$$

If (m_b, m_w) is an eigenmeasure from Theorem 6.16, then by Proposition 6.18 we have

$$\mathbb{L}_{F,\phi}^*(m_b, m_w) = e^{D_{F,\phi}}(m_b, m_w) \qquad \text{and} \qquad \mathbb{L}_{F,\overline{\phi}}^*(m_b, m_w) = (m_b, m_w). \tag{6.33}$$

We summarize in the following lemma the properties of $\mathbb{L}_{F,\overline{\phi}}$ that we will need.

Lemma 6.21. *Let f, $\mathcal{C}$, F, d, Λ, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is irreducible. Then there exists a constant $\widetilde{C}_1 \geqslant 0$ depending only on F, $\mathcal{C}$, d, ϕ, and β such that for each $n \in \mathbb{N}$, each $c \in \{b, w\}$, and each pair of x, $y \in X_c^0$, the following inequalities holds:*

$$\mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x}) / \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y}) \leqslant \exp\bigl(C_1 d(x,y)^\beta\bigr) \leqslant \widetilde{C}, \tag{6.34}$$

$$\widetilde{C}^{-1} \leqslant \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x}) \leqslant \widetilde{C}, \tag{6.35}$$

$$\bigl| \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x}) - \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y}) \bigr| \leqslant \widetilde{C}\bigl(\exp\bigl(C_1 d(x,y)^\beta\bigr) - 1\bigr) \leqslant \widetilde{C}_1 d(x,y)^\beta, \tag{6.36}$$

where $\widetilde{x} \coloneqq i_c(x) = (x, c) \in \widetilde{S}$, $\widetilde{y} \coloneqq i_c(y) = (y, c) \in \widetilde{S}$ (recall Remark 6.13), $C_1 \geqslant 0$ is the constant in Lemma 5.24 depending only on f, $\mathcal{C}$, d, ϕ, and β, and $\widetilde{C} \geqslant 1$ is the constant in Lemma 5.25 (ii) *depending only on F, $\mathcal{C}$, d, ϕ, and β.*

Proof. Inequality (6.34) follows immediately from Lemmas 5.25, 6.10, and (6.32).

Fix arbitrary $n \in \mathbb{N}$, $c \in \{b, w\}$, and x, $y \in X_c^0$.

By Lemma 5.25 (ii), (6.32), and (6.9), we have

$$0 < \inf_{\widetilde{z} \in \widetilde{S}} \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{z}) \leqslant \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x}) \leqslant \sup_{\widetilde{z} \in \widetilde{S}} \mathbb{L}_{F,\overline{\phi}}^n\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{z}) < +\infty$$

and $\sup_{\widetilde{z}\in\widetilde{S}} \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{z}) \leqslant \widetilde{C} \inf_{\widetilde{z}\in\widetilde{S}} \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{z})$, where $\widetilde{C}$ is the constant defined in (5.15) in Lemma 5.25 (ii) and depends only on F, $\mathcal{C}$, d, ϕ, β. By Theorem 6.16, Proposition 6.18, and Corollary 6.19, we can choose a Borel probability measure $(\mu_b, \mu_w) \in \mathcal{P}(\widetilde{S})$ such that $\mathbb{L}^*_{F,\overline{\phi}}(\mu_b, \mu_w) = (\mu_b, \mu_w)$. Then we have

$$\bigl\langle(\mu_b, \mu_w), \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)\bigr\rangle = \bigl\langle\bigl(\mathbb{L}^*_{F,\overline{\phi}}\bigr)^n(\mu_b, \mu_w), \mathbb{1}_{\widetilde{S}}\bigr\rangle = \bigl\langle(\mu_b, \mu_w), \mathbb{1}_{\widetilde{S}}\bigr\rangle = 1.$$

Thus $1 \leqslant \sup_{\widetilde{z}\in\widetilde{S}} \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{z}) \leqslant \widetilde{C} \inf_{\widetilde{z}\in\widetilde{S}} \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{z}) \leqslant \widetilde{C}$ and (6.35) holds.

Applying (6.34) and (6.35), we get

$$\bigl|\mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x}) - \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y})\bigr| = \mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y}) \left|\frac{\mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{x})}{\mathbb{L}^n_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y})} - 1\right| \leqslant \widetilde{C}\bigl(e^{C_1 d(x,y)^\beta} - 1\bigr) \leqslant \widetilde{C}_1 d(x, y)^\beta$$

for some constant $\widetilde{C}_1$ depending only on C_1, $\widetilde{C}$, and $\operatorname{diam}_d(S^2)$. Therefore, we establish (6.36) as the constant $\widetilde{C}_1 > 0$ depends only on F, $\mathcal{C}$, d, ϕ, and β. □

By Lemma 5.25, we can construct eigenfunctions of $\mathbb{L}_{F,\overline{\phi}}$.

Proposition 6.22. *Let f, $\mathcal{C}$, F, d, Λ, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is irreducible. Then the sequence $\bigl\{\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)\bigr\}_{n\in\mathbb{N}}$ has a subsequential limit (with respect to the uniform norm). Moreover, if $\widetilde{u}_{F,\phi} \in C(\widetilde{S})$ is such a subsequential limit, then*

$$\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}) = \widetilde{u}_{F,\phi} \qquad \textit{and} \tag{6.37}$$

$$\widetilde{C}^{-1} \leqslant \widetilde{u}_{F,\phi}(\widetilde{x}) \leqslant \widetilde{C}, \qquad \textit{for each } \widetilde{x} \in \widetilde{S}, \tag{6.38}$$

where $\widetilde{C} \geqslant 1$ is the constant from Lemma 5.25 (ii) *depending only on f, $\mathcal{C}$, d, ϕ, and β. Furthermore, if we let $(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ be an eigenmeasure from Theorem 6.16, then*

$$\int_{\widetilde{S}} \widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = 1 \tag{6.39}$$

and $(\mu_b, \mu_w) \coloneqq \widetilde{u}_{F,\phi}(m_b, m_w) \in \mathcal{P}(\widetilde{S})$ is well-defined as a Borel probability measure on $\widetilde{S}$.

We will show in Theorem 6.24 that a subsequential limit $\widetilde{u}_{F,\phi}$ as defined above is unique and the sequence $\bigl\{\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{L}^j_{F,\phi}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)\bigr\}_{n\in\mathbb{N}}$ converges uniformly to $\widetilde{u}_{F,\phi} \in C(\widetilde{S})$.

Proof. Define $\widetilde{u}_n \coloneqq \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)$ for each $n \in \mathbb{N}$. Then $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of equicontinuous functions on $\widetilde{S}$ by (6.35) and (6.36) in Lemma 6.21. By the Arzelà–Ascoli Theorem, there exists a continuous function $\widetilde{u}_{F,\phi} \in C(\widetilde{S})$ and an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $\widetilde{u}_{n_i} \to \widetilde{u}_{F,\phi}$ uniformly on $\widetilde{S}$ as $i \to +\infty$.

To prove (6.37), we note that by the definition of $\widetilde{u}_n$ and (6.35) in Lemma 6.21, we have that for each $i \in \mathbb{N}$,

$$\bigl\|\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{n_i}) - \widetilde{u}_{n_i}\bigr\|_\infty = \frac{1}{n_i}\bigl\|\mathbb{L}^{n_i}_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr) - \mathbb{1}_{\widetilde{S}}\bigr\|_\infty \leqslant \frac{1 + \widetilde{C}}{n_i},$$

where $\widetilde{C} \geqslant 1$ is the constant from Lemma 5.25 (ii) depending only on f, $\mathcal{C}$, d, ϕ, and β. By letting $i \to +\infty$, we can conclude that $\bigl\|\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}) - \widetilde{u}_{F,\phi}\bigr\|_\infty = 0$. Thus (6.37) holds.

By (6.35), we have that $\widetilde{C}^{-1} \leqslant \widetilde{u}_n(\widetilde{x}) \leqslant \widetilde{C}$ for each $n \in \mathbb{N}$ and each $\widetilde{x} \in \widetilde{S}$. Thus (6.38) follows.

By (6.33) and definition of $\widetilde{u}_n$, we have $\int \widetilde{u}_n \,\mathrm{d}(m_b, m_w) = \int \mathbb{1}_{\widetilde{S}} \,\mathrm{d}(m_b, m_w) = 1$ for each $n \in \mathbb{N}$. Then by the Lebesgue Dominated Theorem, we can conclude that

$$\int \widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = \lim_{i\to+\infty} \int \widetilde{u}_{n_i} \,\mathrm{d}(m_b, m_w) = 1,$$

establishing (6.39). Therefore, $\widetilde{u}_{F,\phi}(m_b, m_w)$ is a Borel probability measure on $\widetilde{S}$. □

6.7. **Invariant Gibbs measures.** The goal of this subsection is to establish Theorem 6.24, namely, the existence of invariant Gibbs measures for subsystems of expanding Thurston maps. We investigate the properties of the eigenmeasures (of the adjoint split Ruelle operators) from Theorem 6.16, with the main results being Propositions 6.25, 6.26, and 6.28.

In this subsection, we follow the conventions discussed in Remarks 6.13 and 6.14. In particular, we use the notation $m_{F,\phi}$ (resp. (m_b, m_w)) for emphasis when we view the eigenmeasure as a Borel probability measure on S^2 (resp. $\widetilde{S}$), and we follow the same conventions for $\mu_{F,\phi}$ (resp. (μ_b, μ_w)), where $\mu_{F,\phi} = (\mu_b, \mu_w) \coloneqq \widetilde{u}_{F,\phi}(m_b, m_w)$ comes from Proposition 6.22.

Definition 6.23 (Gibbs measures for subsystems)**.** Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. A Borel probability measure $\mu \in \mathcal{P}(S^2)$ is called a *Gibbs measure* with respect to F, $\mathcal{C}$, and ϕ if there exist constants $P_\mu \in \mathbb{R}$ and $C_\mu \geqslant 1$ such that for each $n \in \mathbb{N}_0$, each n-tile $X^n \in \mathfrak{X}^n(F,\mathcal{C})$, and each $x \in X^n$, we have

$$\frac{1}{C_\mu} \leqslant \frac{\mu(X^n)}{\exp(S_n^F \phi(x) - nP_\mu)} \leqslant C_\mu. \tag{6.40}$$

One observes that for each Gibbs measure μ with respect to F, $\mathcal{C}$, and ϕ, the constant P_μ is unique.

Theorem 6.24. *Let f, $\mathcal{C}$, F, d, Λ, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. Then the sequence $\bigl\{\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{L}^j_{F,\overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)\bigr\}_{n\in\mathbb{N}}$ converges uniformly to a function $\widetilde{u}_{F,\phi} = (u_b, u_w) \in C^{0,\beta}\bigl(X^0_b, d\bigr) \times C^{0,\beta}\bigl(X^0_w, d\bigr)$, which satisfies*

$$\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}) = \widetilde{u}_{F,\phi} \qquad \text{and} \tag{6.41}$$

$$\widetilde{C}^{-1} \leqslant \widetilde{u}_{F,\phi}(\widetilde{x}) \leqslant \widetilde{C}, \qquad \text{for each } \widetilde{x} \in \widetilde{S}, \tag{6.42}$$

where $\widetilde{C} \geqslant 1$ is the constant from Lemma 5.25 (ii) *depending only on f, $\mathcal{C}$, d, ϕ, and β. Moreover, if we let $m_{F,\phi} = (m_b, m_w)$ be an eigenmeasure from Theorem 6.16, then*

$$\int_{\widetilde{S}} \widetilde{u}_{F,\phi} \,\mathrm{d}(m_b, m_w) = 1, \tag{6.43}$$

and $\mu_{F,\phi} = (\mu_b, \mu_w) \coloneqq \widetilde{u}_{F,\phi}(m_b, m_w)$ is an f-invariant Gibbs measure with respect to F, $\mathcal{C}$, and ϕ, with $\mu_{F,\phi}(\Omega(F,\mathcal{C})) = 1$ and

$$P_{\mu_{F,\phi}} = P_{m_{F,\phi}} = P(F,\phi) = D_{F,\phi} = \lim_{n\to+\infty} \frac{1}{n} \log\bigl(\mathbb{L}^n_{F,\phi}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)(\widetilde{y})\bigr), \tag{6.44}$$

for each $\widetilde{y} \in \widetilde{S}$. In particular, $\mu_{F,\phi}(U) \neq 0$ for each open set $U \subseteq S^2$ with $U \cap \Omega(F,\mathcal{C}) \neq \emptyset$.

See (6.1) for the definition of $P(F,\phi)$. The proof of Theorem 6.24 will be presented at the end of this subsection.

We will show in Theorem 6.30 that a measure $\mu_{F,\phi}$ as defined above is, in fact, an equilibrium state for the map $F_\Omega = F|_\Omega$ and the potential ϕ. We will also show in Theorem 6.29 that $P(F,\phi) = P(F_\Omega,\phi)$.

Proposition 6.25. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is irreducible. Let $m_{F,\phi} = (m_b, m_w)$ be an eigenmeasure from Theorem 6.16 and $\widetilde{u}_{F,\phi} \in C(\widetilde{S})$ be an eigenfunction of $\mathbb{L}_{F,\overline{\phi}}$ from Proposition 6.22. Let $\mu_{F,\phi} = (\mu_b, \mu_w) \coloneqq \widetilde{u}_{F,\phi}(m_b, m_w)$. Then $\mu_{F,\phi}$ is an f-invariant Borel probability measure on S^2.*

Proof. By Proposition 6.22 and Remark 6.14, we get $\mu_{F,\phi} \in \mathcal{P}(S^2)$. It suffices to prove that $\langle \mu_{F,\phi}, g\circ f\rangle = \langle \mu_{F,\phi}, g\rangle$ for each $g \in C(S^2)$. Indeed, by (6.33), (6.37), (6.4), and (6.7), we get

$$\begin{aligned}
\langle \mu_{F,\phi}, g\circ f\rangle &= \bigl\langle (\mu_b,\mu_w), \widetilde{g\circ f}\bigr\rangle = \langle (m_b,m_w), \widetilde{u}_{F,\phi}(\widetilde{g\circ f})\rangle \\
&= \bigl\langle \mathbb{L}^*_{F,\overline{\phi}}(m_b,m_w), \widetilde{u}_{F,\phi}(\widetilde{g\circ f})\bigr\rangle = \bigl\langle (m_b,m_w), \mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}(\widetilde{g\circ f}))\bigr\rangle \\
&= \bigl\langle (m_b,m_w), \widetilde{g}\,\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi})\bigr\rangle = \langle (m_b,m_w), \widetilde{g}\,\widetilde{u}_{F,\phi}\rangle \\
&= \langle \widetilde{u}_{F,\phi}(m_b,m_w), \widetilde{g}\rangle = \langle \mu_{F,\phi}, g\rangle,
\end{aligned}$$

where $\widetilde{g}$ is a continuous function on $\widetilde{S}$ defined by $\widetilde{g}(\widetilde{x}) \coloneqq g(x)$ for each $\widetilde{x} = (x,c) \in \widetilde{S}$, and $\widetilde{g\circ f} \in C(\widetilde{S})$ is defined similarly. $\square$

Proposition 6.26. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Let $m_{F,\phi} = (m_b, m_w)$ be an eigenmeasure from Theorem 6.16 and $\widetilde{u}_{F,\phi}$ be an eigenfunction of $\mathbb{L}_{F,\overline{\phi}}$ from Proposition 6.22. Denote $\mu_{F,\phi} = (\mu_b, \mu_w) \coloneqq \widetilde{u}_{F,\phi}(m_b, m_w)$. Then $m_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = \mu_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = 0$.*

Proof. By Proposition 6.25, the measure $\mu_{F,\phi} \in \mathcal{P}(S^2)$ is f-invariant. Note that $\mathcal{C} \subseteq f^{-j}(\mathcal{C})$ for each $j \in \mathbb{N}$. Thus we have $\mu_{F,\phi}(f^{-j}(\mathcal{C}) \setminus \mathcal{C}) = 0$ for each $j \in \mathbb{N}$. Since F is strongly irreducible, by Definition 5.15, for each $c \in \{b, w\}$, there exists an integer $n_c \in \mathbb{N}$ and $X_c \in \mathfrak{X}_c^{n_c}(F, \mathcal{C})$ such that $X_c \cap \mathcal{C} = \emptyset$. Then $\partial X_c \subseteq f^{-n_c}(\mathcal{C}) \setminus \mathcal{C}$ for each $c \in \{b, w\}$. So $\mu_{F,\phi}(\partial X_c) = 0$ for each $c \in \{b, w\}$. Since $\mu_{F,\phi} = (\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w)$, where $\widetilde{u}_{F,\phi} \in C(\widetilde{S})$ is bounded away from 0 by Theorem 6.24, we have $m_{F,\phi}(\partial X_c) = (m_b, m_w)(\partial X_c) = 0$ for each $c \in \{b, w\}$.

We next show that $m_{F,\phi}(\mathcal{C}) = 0$. For each $c \in \{b, w\}$, it follows from Proposition 5.4 (i) that $F^{n_c}|_{\partial X_c}$ is a homeomorphism from ∂X_c to $\mathcal{C}$. Then for each $c \in \{b, w\}$, by Proposition 6.15 (ii), we have

$$
\begin{aligned}
0 = m_{F,\phi}(\partial X_c) &= \sum_{c' \in \{b,w\}} \int_{\mathcal{C}} \big(\deg_{c'}(F^{n_c}, \cdot\,) \exp\big(S_{n_c}^F \phi\big)\big) \circ (F^{n_c}|_{\partial X_c})^{-1} \,\mathrm{d}m_{c'} \\
&\geqslant \int_{\mathcal{C}} \big(\deg_{c}(F^{n_c}, \cdot\,) \exp\big(S_{n_c}^F \phi\big)\big) \circ (F^{n_c}|_{\partial X_c})^{-1} \,\mathrm{d}m_{c} \\
&\geqslant \int_{\mathcal{C}} \big(\exp\big(S_{n_c}^F \phi\big)\big) \circ (F^{n_c}|_{\partial X_c})^{-1} \,\mathrm{d}m_{c} \geqslant \exp(-n_c \|\phi\|_\infty) m_c(\mathcal{C}) \geqslant 0,
\end{aligned}
$$

where the second inequality holds since $\deg_c(F^{n_c}, y) \geqslant 1$ for each $y \in \partial X_c \subseteq X_c \in \mathfrak{X}_c^{n_c}(F, \mathcal{C})$. Thus $m_{F,\phi}(\mathcal{C}) = (m_b, m_w)(\mathcal{C}) = m_b(\mathcal{C}) + m_w(\mathcal{C}) = 0$.

Now suppose that there exist $k \in \mathbb{N}$ and $e^k \in \mathbf{E}^k(f, \mathcal{C})$ such that $m_{F,\phi}(e^k) > 0$. Then it follows Theorem 6.16 (ii) that $m_{F,\phi}\big(\operatorname{inte}(e^k)\big) = m_{F,\phi}(e^k) > 0$. For convenience we set $\Omega^n \coloneqq \bigcup \mathfrak{X}^n(F, \mathcal{C})$ for each $n \in \mathbb{N}$. Note that $\Omega = \bigcap_{n \in \mathbb{N}} \Omega^n$ by (5.2). By Theorem 6.16 (i), we have $m_{F,\phi}(\Omega^n) = 1$ for each $n \in \mathbb{N}$. If $\operatorname{inte}(e^k) \cap \Omega^k = \emptyset$, then it follows from $m_{F,\phi}(\Omega^k) = 1$ that $m_{F,\phi}\big(\operatorname{inte}(e^k)\big) = 0$, a contradiction. Thus $\operatorname{inte}(e^k) \cap \Omega^k \neq \emptyset$. Then by Lemma 3.3 (ii), we conclude that $e^k \subseteq \Omega^k$.

Note that $F^k|_{e^k}$ is a homeomorphism from e^k to $e^0 \coloneqq F^k(e^k) \subseteq \mathcal{C}$. Since $e^k \subseteq \Omega^k$, it follows from Proposition 6.15 (ii), we

$$
\begin{aligned}
m_{F,\phi}(e^k) &= \sum_{c \in \{b,w\}} \int_{e^0} \big(\deg_c(F^k, \cdot\,) \exp\big(S_k^F \phi\big)\big) \circ \big(F^k|_{e^k}\big)^{-1} \,\mathrm{d}m_c \\
&\leqslant \sum_{c \in \{b,w\}} \int_{e^0} \big(\deg\big(F^k, \cdot\,\big) \exp\big(S_k^F \phi\big)\big) \circ \big(F^k|_{e^k}\big)^{-1} \,\mathrm{d}m_c \\
&= \int_{e^0} \big(\deg\big(F^k, \cdot\,\big) \exp\big(S_k^F \phi\big)\big) \circ \big(F^k|_{e^k}\big)^{-1} \,\mathrm{d}m_{F,\phi} \\
&\leqslant (\deg f)^k \int_{e^0} \big(\exp\big(S_k^F \phi\big)\big) \circ \big(F^k|_{e^k}\big)^{-1} \,\mathrm{d}m_{F,\phi} \\
&\leqslant (\deg f)^k \exp(k \|\phi\|_\infty) m_{F,\phi}(e^0),
\end{aligned}
$$

where the first and second inequalities follow from (5.5), and the second equality follows from (6.14) in Remark 6.14. Then we get $m_{F,\phi}(e^0) > 0$, a contradiction. Hence $(m_b, m_w)\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = m_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = 0$. Since $\mu_{F,\phi} = (\mu_b, \mu_w) = \widetilde{u}_{F,\phi}(m_b, m_w)$, we get $\mu_{F,\phi}\big(\bigcup_{j=0}^{+\infty} f^{-j}(\mathcal{C})\big) = 0$. □

Definition 6.27 (Jacobian)**.** Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4. Consider a Borel probability measure $\mu \in \mathcal{P}(S^2)$ on S^2. A Borel function $J \colon \operatorname{dom}(F) \to [0, +\infty)$ is a *Jacobian (function)* for F with respect to μ if for every Borel $A \subseteq \operatorname{dom}(F)$ on which F is injective, the following equation holds:

$$
\mu(F(A)) = \int_A J \,\mathrm{d}\mu. \tag{6.45}
$$

Proposition 6.28. *Let f, $\mathcal{C}$, F, d, Λ, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f, \mathcal{C})$ is strongly irreducible. Consider an arbitrary $\mu = (\mu_b, \mu_w) \in \mathcal{P}(\widetilde{S})$. If $\mathbb{L}_{F,\phi}^*(\mu_b, \mu_w) = \kappa(\mu_b, \mu_w)$ for some constant $\kappa > 0$, then the following statements hold:*

(i) *The function* $J\colon \operatorname{dom}(F) \to [0, +\infty)$ *given by* $J := \kappa \exp(-\phi)$ *is a Jacobian for* F *with respect to* μ.

(ii) *The measure* μ *is a Gibbs measure with respect to* F, $\mathcal{C}$, *and* ϕ *with* $P_\mu = P(F, \phi) = \log \kappa$. *Here* $P(F, \phi)$ *is defined in* (6.1) *from Subsection 6.1. In particular,* $\sum_{X^n \in \mathfrak{X}^n(F,\mathcal{C})} \mu(X^n) \leqslant 2$.

(iii) *The map* $F_\Omega := F|_{\Omega(F,\mathcal{C})}$ *with respect to* μ *is forward quasi-invariant (i.e., for each Borel set* $A \subseteq \Omega(F, \mathcal{C})$, *if* $\mu(A) = 0$, *then* $\mu(F_\Omega(A)) = 0$*) and non-singular (i.e., for each Borel set* $A \subseteq \Omega(F, \mathcal{C})$, $\mu(A) = 0$ *if and only if* $\mu(f^{-1}(A)) = 0$*).*

Proof. Assume that $\mathbb{L}^*_{F,\phi}(\mu_b, \mu_w) = \kappa(\mu_b, \mu_w)$ for some constant $\kappa > 0$.

(i) By Proposition 6.15 (ii), for each Borel $A \subseteq \operatorname{dom}(F)$ on which F is injective, we have that $F(A)$ is Borel, and

$$\mu(A) = \kappa^{-1}\mathbb{L}^*_{F,\phi}(\mu_b, \mu_w)(A) = \kappa^{-1} \sum_{c\in\{b,w\}} \int_{F(A)\cap X^0_c} (\deg_c(F, \cdot\,) \exp(\phi)) \circ (F|_A)^{-1} \,\mathrm{d}\mu_c.$$

Since F is strongly irreducible, it follows from Proposition 6.26 that $\mu(\mathcal{C}) = 0$. Thus $\mu_b(\mathcal{C}) = \mu_w(\mathcal{C}) = 0$. Note that $\deg_c(F, y) = 1$ for each $c \in \{b, w\}$, each $x \in \operatorname{inte}(X^0_c)$, and each $y \in F^{-1}(x)$. Thus by (6.14) we have

$$\begin{aligned}\mu(A) &= \kappa^{-1} \sum_{c\in\{b,w\}} \int_{F(A)\cap X^0_c} \exp(\phi)\circ(F|_A)^{-1} \,\mathrm{d}\mu_c \\ &= \kappa^{-1} \int_{F(A)} \exp(\phi)\circ(F|_A)^{-1} \,\mathrm{d}\mu \\ &= \int_{F(A)} \frac{1}{J \circ (F|_A)^{-1}} \,\mathrm{d}\mu,\end{aligned} \tag{6.46}$$

where the function J is given in (i).

Since F is injective on each 1-tile $X^1 \in \mathfrak{X}^1(F, \mathcal{C})$, and both X^1 and $F(X^1)$ are closed subsets of S^2 by Proposition 5.4 (i), in order to verify (6.45), it suffices to assume that $A \subseteq X$ for some 1-tile $X \in \mathfrak{X}^1(F, \mathcal{C})$. Denote the restriction of μ on X by μ_X, i.e., μ_X assigns $\mu(B)$ to each Borel subset B of X.

Let $\widetilde{\mu}$ be a function defined on the set of Borel subsets of X in such a way that $\widetilde{\mu}(B) = \mu(F(B))$ for each Borel $B \subseteq X$. It is clear that $\widetilde{\mu}$ is a Borel measure on X. In this notation, we can write (6.46) as

$$\mu_X(A) = \int_A \frac{1}{J|_X} \,\mathrm{d}\widetilde{\mu}, \tag{6.47}$$

for each Borel $A \subseteq X$.

By (6.47), we know that μ_X is absolutely continuous with respect to $\widetilde{\mu}$. On the other hand, since J is positive and uniformly bounded away from $+\infty$ on X, we can conclude that $\widetilde{\mu}$ is absolutely continuous with respect to μ_X. Therefore, by the Radon–Nikodym theorem, for each Borel $A \subseteq X$, we get $\mu(F(A)) = \widetilde{\mu}(A) = \int_A J|_X \,\mathrm{d}\mu_X = \int_A J \,\mathrm{d}\mu$. Thus we finish the proof of statement (i).

(ii) We observe that

$$\mu(F^m(B)) = \int_B \frac{\kappa^m}{\exp(S_m\phi)} \,\mathrm{d}\mu \tag{6.48}$$

for all $n \in \mathbb{N}$, $m \in \{0, 1, \dots, n\}$, and Borel set $B \subseteq \operatorname{dom}(F)$ on which F^n is defined and injective. Recall from Definition 5.1 that F^n is defined at $x \in \operatorname{dom}(F)$ if and only if $F^i(x) \in \operatorname{dom}(F)$ for each $i \in \{0, 1, \dots, n-1\}$. Indeed, by statement (i), for a given Borel set $A \subseteq \operatorname{dom}(F)$ on which F is injective, we have

$$\int_{F(A)} g \,\mathrm{d}\mu = \int_A \frac{\kappa g \circ F}{\exp(\phi)} \,\mathrm{d}\mu$$

for each simple function g on $\operatorname{dom}(F)$, thus also for each integrable function g.

Fix arbitrary $n \in \mathbb{N}$ and Borel set $B \subseteq \operatorname{dom}(F)$ on which F^n is defined and injective. We establish (6.48) for each $m \in \{0, \dots, n\}$ by induction. For $m = 0$, (6.48) holds trivially. Assume that (6.48) is established for some $m \in \{0, \dots, n-1\}$, then since F^m is injective on $F(B)$, we get

$$\mu(F^{m+1}(B)) = \int_{F(B)} \frac{\kappa^m}{\exp(S_m \phi)} \,\mathrm{d}\mu = \int_B \frac{\kappa^{m+1}}{\exp(S_{m+1}\phi)} \,\mathrm{d}\mu.$$

The induction is now complete. In particular, by Proposition 5.4 (i),

$$\mu(F^n(X^n)) = \int_{X^n} \frac{\kappa^n}{\exp(S_n \phi)} \,\mathrm{d}\mu \tag{6.49}$$

for all $n \in \mathbb{N}$ and $X^n \in \mathfrak{X}^n(F, \mathcal{C})$.

By (6.49) and Lemma 5.24, for all $n \in \mathbb{N}_0$, $X^n \in \mathfrak{X}^n(F, \mathcal{C})$, and $x \in X^n$, we have

$$e^{-C} \mu(F^n(X^n)) \leqslant \kappa^n \mu(X^n) / \exp(S_n \phi(x)) \leqslant e^C \mu(F^n(X^n)), \tag{6.50}$$

where $C \coloneqq C_1 (\operatorname{diam}_d(S^2))^\beta \geqslant 0$ and C_1 is the constant defined in (5.12) in Lemma 5.24 and depends only on f, $\mathcal{C}$, d, ϕ, and β. Note that $F^n(X^n) \in \mathfrak{X}^0(F, \mathcal{C})$ is either the black 0-tile X^0_b or the white 0-tile X^0_w. We claim that for each $c \in \{b, w\}$,

$$\mu\bigl(X^0_c\bigr) \geqslant \bigl(1 + e^C \kappa^{n_{c'c}} \exp(n_{c'c} \|\phi\|_\infty)\bigr)^{-1}. \tag{6.51}$$

Indeed, for $c' \in \{b, w\} \setminus \{c\}$, since F is irreducible, by Definition 5.15, there exist $n_{c'c} \in \mathbb{N}$ and $X^{n_{c'c}}_{c'} \in \mathfrak{X}^{n_{c'c}}_{c'c}(F, \mathcal{C})$ such that $X^{n_{c'c}}_{c'} \subseteq X^0_c$ and $F^{n_{c'c}}\bigl(X^{n_{c'c}}_{c'}\bigr) = X^0_{c'}$. Then it follows from (6.50) that

$$e^{-C} \mu\bigl(X^0_{c'}\bigr) \leqslant \kappa^{n_{c'c}} \exp(n_{c'c} \|\phi\|_\infty) \mu\bigl(X^{n_{c'c}}_{c'}\bigr) \leqslant \kappa^{n_{c'c}} \exp(n_{c'c} \|\phi\|_\infty) \mu\bigl(X^0_c\bigr).$$

This implies (6.51) since $\mu\bigl(X^0_c\bigr) + \mu\bigl(X^0_{c'}\bigr) \geqslant \mu(S^2) = 1$. Hence (6.40) holds, and μ is a Gibbs measure with respect to F, $\mathcal{C}$, and ϕ with $P_\mu = \log \kappa$.

Finally, we show that $P_\mu = P(F, \phi)$. By Theorem 6.16 (i), we get

$$\sum_{X^n \in \mathfrak{X}^n(F, \mathcal{C})} \mu(X^n) \geqslant \mu(\Omega(F, \mathcal{C})) = 1. \tag{6.52}$$

Since μ is a Gibbs measure with respect to F, $\mathcal{C}$, and ϕ with $P_\mu = \log \kappa$, by (6.40), for each $n \in \mathbb{N}_0$, we have

$$\sum_{X^n \in \mathfrak{X}^n(F, \mathcal{C})} \mu(X^n) \leqslant C_\mu e^{-n P_\mu} \sum_{X^n \in \mathfrak{X}^n(F, \mathcal{C})} \exp(\sup\{S_n \phi(x) : x \in X^n\})$$

for some constant $C_\mu \geqslant 1$. Combining this with (6.52), we obtain the inequality $P_\mu \leqslant P(F, \phi)$. For the other direction, since F is strongly irreducible, by Theorem 6.16 (ii), we get $\mu\bigl(\bigcup_{j=0}^{+\infty} f^{-j}(\operatorname{post} f)\bigr) = 0$. Then it follows from Remark 3.8 and Proposition 6.26 that

$$\sum_{X^n \in \mathfrak{X}^n(F, \mathcal{C})} \mu(X^n) \leqslant \mu\bigl(\bigcup \mathfrak{X}^n(F, \mathcal{C})\bigr) + \mu\Biggl(\bigcup_{j=0}^{+\infty} F^{-j}(\mathcal{C})\Biggr) \leqslant 1.$$

Then $P_\mu \geqslant P(F, \phi)$ holds similarly by the Gibbs property of μ. Thus we get $P_\mu = P(F, \phi)$ and finish the proof of statement (ii).

(iii) Denote $\Omega \coloneqq \Omega(F, \mathcal{C})$. Fix a Borel set $A \subseteq \Omega$ with $\mu(A) = 0$. For each 1-tile $X^1 \in \mathbf{X}^1(f, \mathcal{C})$, the map F_Ω is injective both on $A \cap X^1$ and $F_\Omega^{-1}(A) \cap X^1$ by Proposition 5.4 (i). Then it follows from the definition of the Jacobian (recall (6.27)) and statement (i) that $\mu(F_\Omega(A \cap X^1)) = 0$ and $\mu(F_\Omega^{-1}(A) \cap X^1) = 0$. Thus $\mu(F_\Omega(A)) = 0$ and $\mu(F_\Omega^{-1}(A)) = 0$. This implies F_Ω is forward quasi-invariant and non-singular with respect to μ since $F_\Omega \colon \Omega \to \Omega$ is surjective by Proposition 5.6 (ii). $\square$

Now we can prove the existence of an f-invariant Gibbs measure with respect to F, $\mathcal{C}$, and ϕ.

Proof of Theorem 6.24. Define, for each $n \in \mathbb{N}$, $\widetilde{u}_n \coloneqq \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{L}^j_{F, \overline{\phi}}\bigl(\mathbb{1}_{\widetilde{S}}\bigr)$. Then $\{\widetilde{u}_n\}_{n \in \mathbb{N}}$ is a uniformly bounded sequence of equicontinuous functions on $\widetilde{S}$ by (6.35) and (6.36) in Lemma 6.21. By the Arzelà–Ascoli Theorem, every subsequence $\{\widetilde{u}_{n_k}\}_{k \in \mathbb{N}}$, which is a uniformly bounded sequence of equicontinuous functions on $\widetilde{S}$, has a further subsequential limit with respect to the uniform norm. Recall in Proposition 6.22 we show that the sequence $\{\widetilde{u}_n\}_{n \in \mathbb{N}}$ has a subsequential limit $\widetilde{u}_{F, \phi} \in C(\widetilde{S})$ satisfying (6.41), (6.42), and (6.43). In order to prove this theorem, it suffices to show that $\{\widetilde{u}_n\}_{n \in \mathbb{N}}$ has a unique subsequential limit with respect to the uniform norm, and finally justify (6.44).

Suppose that $\widetilde{v}_{F,\phi}$ is another subsequential limit of $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ with respect to the uniform norm. Then $\widetilde{v}_{F,\phi}$ is also a continuous function on $\widetilde{S}$ satisfying (6.41), (6.42), and (6.43) by Proposition 6.22. To prove the uniqueness, it suffices to show that $\widetilde{u}_{F,\phi} = \widetilde{v}_{F,\phi}$ on $\widetilde{S} = X_b^0 \sqcup X_w^0$. The proof proceeds in two steps. We first show that $\widetilde{u}_{F,\phi} = \widetilde{v}_{F,\phi}$ on a subset $\widetilde{\Omega}$ (defined in (6.54) below) of $\widetilde{S}$, which satisfies $(m_b, m_w)\big(\widetilde{\Omega}\big) = 1$. Then we extend this identity $\widetilde{u}_{F,\phi} = \widetilde{v}_{F,\phi}$ to $\widetilde{S}$.

For each $n \in \mathbb{N}_0$, we set

$$\widetilde{\mathfrak{X}}^n(F,\mathcal{C}) \coloneqq \bigcup_{c\in\{b,w\}} \big\{ i_c(X^n) : X^n \in \mathfrak{X}^n(F,\mathcal{C}),\, X^n \subseteq X_c^0 \big\}, \tag{6.53}$$

where i_c is defined by (6.10). Let $\widetilde{\Omega}$ be the subset of $\widetilde{S}$ defined by

$$\widetilde{\Omega} \coloneqq \bigcap_{n\in\mathbb{N}} \bigcup \widetilde{\mathfrak{X}}^n(F,\mathcal{C}). \tag{6.54}$$

Mimicking the proofs of Proposition 6.15 (iv) and Theorem 6.16 (i), we can show that $(m_b, m_w)\big(\widetilde{\Omega}\big) = 1$. Indeed, by Proposition 5.5 (i) and (6.53), $\big\{\bigcup \widetilde{\mathfrak{X}}^n(F,\mathcal{C})\big\}_{n\in\mathbb{N}}$ is a decreasing sequence of sets. Then $(m_b, m_w)\big(\widetilde{\Omega}\big) = \lim_{n\to+\infty}(m_b, m_w)\big(\bigcup \widetilde{\mathfrak{X}}^n(F,\mathcal{C})\big)$. Noting that F is surjective since F is irreducible, we have $\bigcup \widetilde{\mathfrak{X}}^0(F,\mathcal{C}) = \widetilde{S}$. Thus, it suffices to show that

$$(m_b, m_w)\Big(\bigcup \widetilde{\mathfrak{X}}^n(F,\mathcal{C})\Big) = (m_b, m_w)\Big(\bigcup \widetilde{\mathfrak{X}}^{n-1}(F,\mathcal{C})\Big) \tag{6.55}$$

for each $n \in \mathbb{N}$.

For convenience we set $\widetilde{\Omega}^n \coloneqq \bigcup \widetilde{\mathfrak{X}}^n(F,\mathcal{C}) \subseteq \widetilde{S}$ and $\Omega_c^n \coloneqq i_c^{-1}\big(\bigcup \widetilde{\mathfrak{X}}^n(F,\mathcal{C})\big) \subseteq X_c^0$ for each $n \in \mathbb{N}_0$ and each $c \in \{b,w\}$. Note that $\widetilde{\Omega}^n = \Omega_b^n \sqcup \Omega_w^n$ and $\Omega_c^n = \bigcup\big\{X^n \in \mathfrak{X}^n(F,\mathcal{C}) : X^n \subseteq X_c^0\big\}$ for each $n \in \mathbb{N}_0$ and each $c \in \{b,w\}$. Let $n \in \mathbb{N}$ be arbitrary. By Proposition 6.15 (i) and (6.12), we have

$$\begin{aligned}
\big(\mathbb{L}_{F,\phi}^*\big)^n(m_b, m_w)\big(\widetilde{\Omega}^{n-1}\big) &= \big(\mathbb{L}_{F,\phi}^*\big)^n(m_b, m_w)\big(\Omega_b^{n-1}, \Omega_w^{n-1}\big) = \big\langle (m_b, m_w), \mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n-1}}, \mathbb{1}_{\Omega_w^{n-1}}\big)\big\rangle \\
&= \big\langle m_b, \pi_b\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n-1}}, \mathbb{1}_{\Omega_w^{n-1}}\big)\big)\big\rangle + \big\langle m_w, \pi_w\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n-1}}, \mathbb{1}_{\Omega_w^{n-1}}\big)\big)\big\rangle, \\
\big(\mathbb{L}_{F,\phi}^*\big)^n(m_b, m_w)\big(\widetilde{\Omega}^{n}\big) &= \big(\mathbb{L}_{F,\phi}^*\big)^n(m_b, m_w)\big(\Omega_b^{n}, \Omega_w^{n}\big) = \big\langle (m_b, m_w), \mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n}}, \mathbb{1}_{\Omega_w^{n}}\big)\big\rangle \\
&= \big\langle m_b, \pi_b\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n}}, \mathbb{1}_{\Omega_w^{n}}\big)\big)\big\rangle + \big\langle m_w, \pi_w\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n}}, \mathbb{1}_{\Omega_w^{n}}\big)\big)\big\rangle.
\end{aligned}$$

Since $\mathbb{L}_{F,\phi}^*(m_b, m_w) = \kappa(m_b, m_w)$, it suffices to show that

$$\pi_c\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n-1}}, \mathbb{1}_{\Omega_w^{n-1}}\big)\big) = \pi_c\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n}}, \mathbb{1}_{\Omega_w^{n}}\big)\big)$$

for each $c \in \{b,w\}$. Indeed, by (6.8), (6.9), and (6.4),

$$\begin{aligned}
\pi_c\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n-1}}, \mathbb{1}_{\Omega_w^{n-1}}\big)\big) &= \mathcal{L}_{F,\phi,c,b}^{(n)}\big(\mathbb{1}_{\Omega_b^{n-1}}\big) + \mathcal{L}_{F,\phi,c,w}^{(n)}\big(\mathbb{1}_{\Omega_w^{n-1}}\big) \\
&= \sum_{c'\in\{b,w\}} \sum_{X^n\in\mathfrak{X}_{cc'}^n(F,\mathcal{C})} \big(\mathbb{1}_{\Omega_{c'}^{n-1}} \cdot \exp\big(S_n^F\phi\big)\big) \circ (F|_{X^n})^{-1} \\
&= \sum_{c'\in\{b,w\}} \sum_{X^n\in\mathfrak{X}_{cc'}^n(F,\mathcal{C})} \big(\mathbb{1}_{\Omega_{c'}^{n}} \cdot \exp\big(S_n^F\phi\big)\big) \circ (F|_{X^n})^{-1} \\
&= \pi_c\big(\mathbb{L}_{F,\phi}^n\big(\mathbb{1}_{\Omega_b^{n}}, \mathbb{1}_{\Omega_w^{n}}\big)\big),
\end{aligned}$$

where the third equality holds since $X^n \subseteq \Omega_{c'}^n \subseteq \Omega_{c'}^{n-1}$ for each $c' \in \{b,w\}$ and each $X^n \in \mathfrak{X}_{cc'}^n(F,\mathcal{C})$. Thus (6.55) holds, and we deduce that $(m_b, m_w)\big(\widetilde{\Omega}\big) = 1$.

Now we can show that $\widetilde{v}_{F,\phi}(\widetilde{x}) = \widetilde{u}_{F,\phi}(\widetilde{x})$ for each $\widetilde{x} \in \widetilde{\Omega}$. Let

$$t \coloneqq \sup\big\{ s \in \mathbb{R} : \widetilde{u}_{F,\phi}(\widetilde{x}) - s\widetilde{v}_{F,\phi}(\widetilde{x}) > 0 \text{ for all } \widetilde{x} \in \widetilde{S} \big\}.$$

It follows from (6.42) that $t \in (0,+\infty)$. Then there is a point $\widetilde{y} \in \widetilde{S}$ such that $\widetilde{u}_{F,\phi}(\widetilde{y}) - t\widetilde{v}_{F,\phi}(\widetilde{y}) = 0$. Without loss of generality, we may assume that $\widetilde{y} = (y, b)$ for some point $y \in X_b^0$. By (6.4), (6.7), and the equality

$$\mathbb{L}_{F,\overline{\phi}}(\widetilde{u}_{F,\phi} - t\widetilde{v}_{F,\phi}) = \widetilde{u}_{F,\phi} - t\widetilde{v}_{F,\phi},$$

which comes from (6.41), we get that $\widetilde{u}_{F,\phi}(\widetilde{z})-t\widetilde{v}_{F,\phi}(\widetilde{z})=0$ for each $\widetilde{z}\in\widetilde{F}^{-1}(\widetilde{y})$, where we define $\widetilde{F}^{-n}(\widetilde{y})$ to be the set

$$\big\{(z,c): z=\big(F^n|_{X^n_{bc}}\big)^{-1}(y),\, X^n_{bc}\in\mathfrak{X}^n_{bc}(F,\mathcal{C}),\, c\in\{b,w\}\big\}. \tag{6.56}$$

for each $n\in\mathbb{N}$. Inductively, we can conclude that $\widetilde{u}_{F,\phi}(\widetilde{z})-t\widetilde{v}_{F,\phi}(\widetilde{z})=0$ for all $\widetilde{z}\in\bigcup_{i\in\mathbb{N}}\widetilde{F}^{-i}(\widetilde{y})$. Noting that F is irreducible and mimicking the proof of Proposition 5.20 (i), we can show that the closure of $\bigcup_{i\in\mathbb{N}}\widetilde{F}^{-i}(\widetilde{y})$ in $\widetilde{S}$ contains $\widetilde{\Omega}$. Indeed, by (6.53), (6.54), and (3.13), it suffices to show that for each $n\in\mathbb{N}$ and each $\widetilde{X}^n\in\widetilde{\mathfrak{X}}^n(F,\mathcal{C})$, $\widetilde{X}^n\cap\bigcup_{i\in\mathbb{N}}\widetilde{F}^{-i}(\widetilde{y})\neq\emptyset$. Fix arbitrary $n\in\mathbb{N}$ and $\widetilde{X}^n\in\widetilde{\mathfrak{X}}^n(F,\mathcal{C})$. By (6.53), there exist $c\in\{b,w\}$ and $X^n\in\mathfrak{X}^n(F,\mathcal{C})$ such that $X^n\subseteq X^0_c$ and $\widetilde{X}^n=i_c(X^n)$. By Proposition 5.4 (i), X^n is mapped by F^n homeomorphically to a 0-tile $X^0_{c'}$ for some $c'\in\{b,w\}$. Since F is irreducible, by Definition 5.15, there exist $k\in\mathbb{N}$ and $Y^k\in\mathfrak{X}^k(F,\mathcal{C})$ such that $Y^k\subseteq X^0_{c'}$ and $F^k\big(Y^k\big)=X^0_b$. Then it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that $X^{k+n}:=(F^n|_{X^n})^{-1}\big(Y^k\big)\in\mathfrak{X}^{k+n}_{bc}(F,\mathcal{C})$. Denote $z:=(F^{k+n}|_{X^{k+n}})^{-1}(y)\in X^{k+n}$. Then by (6.56) we have $(z,c)\in\bigcup_{i\in\mathbb{N}}\widetilde{F}^{-i}(\widetilde{y})$. Since $z\in X^{k+n}\subseteq X^n\subseteq X^0_c$, we have $(z,c)\in\widetilde{X}^n\cap\bigcup_{i\in\mathbb{N}}\widetilde{F}^{-i}(\widetilde{y})\neq\emptyset$. Thus the closure of $\bigcup_{i\in\mathbb{N}}\widetilde{F}^{-i}(\widetilde{y})$ in $\widetilde{S}$ contains $\widetilde{\Omega}$. Hence $\widetilde{u}_{F,\phi}=t\widetilde{v}_{F,\phi}$ on $\widetilde{\Omega}$. Since both $\widetilde{u}_{F,\phi}$ and $\widetilde{v}_{F,\phi}$ satisfy (6.43) and $(m_b,m_w)(\widetilde{\Omega})=1$, we get $t=1$. Thus $\widetilde{v}_{F,\phi}(\widetilde{x})=\widetilde{u}_{F,\phi}(\widetilde{x})$ for each $\widetilde{x}\in\widetilde{\Omega}$.

Mimicking the proof of Proposition 5.20 (ii), we can show that $\widetilde{\Omega}\cap\widetilde{X}^n\neq\emptyset$ for each $n\in\mathbb{N}$ and each $\widetilde{X}^n\in\widetilde{\mathfrak{X}}^n(F,\mathcal{C})$. Fix arbitrary $n\in\mathbb{N}$ and $\widetilde{X}^n\in\widetilde{\mathfrak{X}}^n(F,\mathcal{C})$. By (6.53), there exist $c\in\{b,w\}$ and $X^n\in\mathfrak{X}^n(F,\mathcal{C})$ such that $X^n\subseteq X^0_c$ and $\widetilde{X}^n=i_c(X^n)$. Since F is irreducible, by Definition 5.15, there exist $m\in\mathbb{N}$ and $Y^m_c\in\mathfrak{X}^m_c(F,\mathcal{C})$ such that $Y^m_c\subseteq X^0_c$ and $F^m(Y^m_c)=X^0_c$. Then it follows from Lemma 3.7 (i) and Proposition 5.4 (i) that $X^{n+m}:=(F^n|_{X^n})^{-1}(Y^m_c)\in\mathfrak{X}^{n+m}(F,\mathcal{C})$. Note that $X^{n+m}\subseteq X^n\subseteq X^0_c$. Since $F^{n+m}(X^{n+m})=X^0_c$, similarly, we have $X^{n+2m}:=(F^{n+m}|_{X^{n+m}})^{-1}(Y^m_c)\in\mathfrak{X}^{n+2m}(F,\mathcal{C})$ and $X^{n+2m}\subseteq X^{n+m}\subseteq X^0_c$. Thus by induction, there exists a sequence of tiles $\big\{X^{n+km}\big\}_{k\in\mathbb{N}_0}$ such that $X^{n+km}\in\mathfrak{X}^{n+km}(F,\mathcal{C})$ and $X^{n+(k+1)m}\subseteq X^{n+km}\subseteq X^0_c$ for each $k\in\mathbb{N}_0$. For each $k\in\mathbb{N}_0$, we denote by $\widetilde{X}^{n+km}$ the set $i_c\big(X^{n+km}\big)$. Note that $\widetilde{X}^{n+km}\in\widetilde{\mathfrak{X}}^{n+km}(F,\mathcal{C})$ by (6.53). Then it follows from (6.54) that $\bigcap_{k\in\mathbb{N}_0}\widetilde{X}^{n+km}\subseteq\widetilde{\Omega}$ is the intersection of a nested sequence of closed sets so that it is non-empty. Thus we deduce that $\widetilde{\Omega}\cap\widetilde{X}^n\neq\emptyset$.

We next show that $\widetilde{u}_{F,\phi}=\widetilde{v}_{F,\phi}$ on $\widetilde{S}$. Recall for each $c\in\{b,w\}$, $\widetilde{u}_{F,\phi}$ and $\widetilde{v}_{F,\phi}$ are continuous on X^0_c and X^0_c is compact. Fix an arbitrary number $\varepsilon>0$, then we can choose $\delta>0$ such that for each $c\in\{b,w\}$ and each pair of $x,\,y\in X^0_c$ with $d(x,y)<\delta$, we have $|\widetilde{u}_{F,\phi}(\widetilde{x})-\widetilde{u}_{F,\phi}(\widetilde{y})|<\varepsilon$ and $|\widetilde{v}_{F,\phi}(\widetilde{x})-\widetilde{v}_{F,\phi}(\widetilde{y})|<\varepsilon$, where $\widetilde{x}:=(x,c)$ and $\widetilde{y}:=(y,c)$. By Lemma 3.11 (ii), there exists $n\in\mathbb{N}$ such that for each $X^n\in\mathfrak{X}^n(F,\mathcal{C})$, $\operatorname{diam}_d(X^n)<\delta$. We fix such an integer n in the rest of this paragraph. We also fix an arbitrary point $\widetilde{y}_{\widetilde{X}^n}\in\widetilde{\Omega}\cap\widetilde{X}^n$ for each $\widetilde{X}^n\in\widetilde{\mathfrak{X}}^n(F,\mathcal{C})$. Since there is a natural bijection $X^n\mapsto\widetilde{X}^n$ between $\mathfrak{X}^n(F,\mathcal{C})$ and $\widetilde{\mathfrak{X}}^n(F,\mathcal{C})$, we can define $\widetilde{y}_{X^n}:=\widetilde{y}_{\widetilde{X}^n}$ for each $X^n\in\mathfrak{X}^n(F,\mathcal{C})$. Hence by (6.4), (6.9), Proposition 6.28 (ii), and Definition 6.23, for each $c\in\{b,w\}$ and each $\widetilde{x}=(x,c)\in i_c\big(X^0_c\big)$,

$$\begin{aligned}
|\widetilde{u}_{F,\phi}(\widetilde{x})-\widetilde{v}_{F,\phi}(\widetilde{x})| &= \big|\mathbb{L}^n_{F,\overline{\phi}}(\widetilde{u}_{F,\phi}-\widetilde{v}_{F,\phi})(\widetilde{x})\big| \\
&\leqslant \sum_{X^n\in\mathfrak{X}^n_c(F,\mathcal{C})}|\widetilde{u}_{F,\phi}(\widetilde{x}_{X^n})-\widetilde{v}_{F,\phi}(\widetilde{x}_{X^n})|\exp\big(S_n\phi(x_{X^n})-nP_{m_{F,\phi}}\big) \\
&\leqslant C_{m_{F,\phi}}\sum_{X^n\in\mathfrak{X}^n_c(F,\mathcal{C})}\big(|\widetilde{u}_{F,\phi}(\widetilde{x}_{X^n})-\widetilde{u}_{F,\phi}(\widetilde{y}_{X^n})|+|\widetilde{u}_{F,\phi}(\widetilde{y}_{X^n})-\widetilde{v}_{F,\phi}(\widetilde{y}_{X^n})| \\
&\qquad\qquad +|\widetilde{v}_{F,\phi}(\widetilde{y}_{X^n})-\widetilde{v}_{F,\phi}(\widetilde{x}_{X^n})|\big)m_{F,\phi}(X^n) \\
&\leqslant C_{m_{F,\phi}}\sum_{X^n\in\mathfrak{X}^n_c(F,\mathcal{C})}2\varepsilon m_{F,\phi}(X^n)\leqslant 4\varepsilon C_{m_{F,\phi}},
\end{aligned}$$

where we write $x_{X^n}:=(F^n|_{X^n})^{-1}(x)$ and $\widetilde{x}_{X^n}:=(x_{X^n},c')$ with $c'\in\{b,w\}$ uniquely determined by the relation $X^n\subseteq X^0_{c'}$. Since $\varepsilon>0$, $c\in\{b,w\}$, and $\widetilde{x}\in i_c\big(X^0_c\big)$ are arbitrary, we conclude that $\widetilde{u}_{F,\phi}=\widetilde{v}_{F,\phi}$ on $\widetilde{S}$.

We have proved that $\widetilde{u}_n$ converges to $\widetilde{u}_{F,\phi}$ uniformly as $n\to+\infty$.

It now follows immediately from (6.35), (6.36), and the uniform convergence of $\widetilde{u}_n$ that $\widetilde{u}_{F,\phi} \in C^{0,\beta}\big(X_b^0, d\big) \times C^{0,\beta}\big(X_w^0, d\big)$.

By Proposition 6.25, $\mu_{F,\phi}$ is f-invariant. Since $m_{F,\phi}$ is a Gibbs measure (with respect to F, $\mathcal{C}$ and ϕ) supported on Ω with $P_{m_{F,\phi}} = P(F,\phi)$ by Proposition 6.28 (ii), then by (6.42) and Proposition 6.16 (i), $\mu_{F,\phi}$ is also a Gibbs measure (with respect to F, $\mathcal{C}$ and ϕ) supported on Ω with $P_{\mu_{F,\phi}} = P_{m_{F,\phi}} = P(F,\phi) = D_{F,\phi} = \lim_{n\to+\infty} \frac{1}{n} \log\big(\mathbb{L}^n_{F,\phi}\big(\mathbb{1}_{\widetilde{S}}\big)(\widetilde{y})\big)$ for each $\widetilde{y} \in \widetilde{S}$, establishing (6.44).

Finally, let $U \subseteq S^2$ be an open set with $U \cap \Omega \neq \emptyset$. Then by (5.2) and (3.13), there exists $X^k \in \mathfrak{X}^k(F,\mathcal{C})$ with $X^k \subseteq U$ for some $k \in \mathbb{N}$. Since $\mu_{F,\phi}$ is a Gibbs measure, we get $\mu_{F,\phi}(X^k) > 0$. Thus $\mu_{F,\phi}(U) \geqslant \mu_{F,\phi}(X^k) > 0$. $\square$

Remark. By a similar argument to that in the proof of the uniqueness of the subsequential limit of $\Big\{\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{L}^j_{F,\overline{\phi}}\big(\mathbb{1}_{\widetilde{S}}\big)\Big\}_{n\in\mathbb{N}}$, one can show that $\widetilde{u}_{F,\phi}$ is the unique eigenfunction, up to scalar multiplication, of $\mathbb{L}_{F,\overline{\phi}}$ corresponding to the eigenvalue 1.

6.8. **Variational Principle and existence of the equilibrium states for subsystems.** In this subsection, we prove the main results Theorems 6.29 and 6.30 of this subsection.

In the following theorem we establish the Variational Principle for strongly irreducible subsystems with respect to Hölder continuous potentials. The constant $D_{F,\phi}$ is defined in Corollary 6.19 and $P(F,\phi)$ is the topological pressure of the subsystem F with respect to the Hölder continuous potential ϕ (see (6.1) for the definition of $P(F,\phi)$). Recall that $F(\Omega) \subseteq \Omega$ by Proposition 5.4 (iii) and $F_\Omega = F|_\Omega \colon \Omega \to \Omega$.

Theorem 6.29. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. Then we have*

$$D_{F,\phi} = P(F,\phi) = \sup\Big\{h_\mu(F_\Omega) + \int_\Omega \phi \,\mathrm{d}\mu : \mu \in \mathcal{M}(\Omega, F_\Omega)\Big\}. \tag{6.57}$$

The following theorem gives the existence of the equilibrium states for strongly irreducible subsystems and Hölder continuous potentials.

Theorem 6.30. *Let f, $\mathcal{C}$, F, d, ϕ, β satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \operatorname{Sub}(f,\mathcal{C})$ is strongly irreducible. Then there exists an equilibrium state for F_Ω and ϕ. Moreover, any measure $\mu_{F,\phi} \in \mathcal{M}(S^2, f)$ defined in Theorem 6.24 is an equilibrium state for F_Ω and ϕ.*

The proofs of Theorems 6.29 and 6.30 will be given at the end of this subsection. Note that Theorem 1.1 follows immediately from Theorems 6.29 and 6.30.

We use the following convention in this subsection.

Remark 6.31. Given a non-empty Borel subset $X \subseteq S^2$ and a Borel probability measure $\mu \in \mathcal{P}(X)$, by abuse of notation, we can view μ as a Borel probability measure on S^2 by setting $\mu(A) := \mu(A \cap X)$ for all Borel subsets $A \subseteq S^2$. Conversely, for each measure $\nu \in \mathcal{P}(S^2)$ supported on X, we can view ν as a Borel probability measure on X.

Proposition 6.32. *Let f, $\mathcal{C}$, F, d, ϕ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then for each F_Ω-invariant Gibbs measure $\mu \in \mathcal{M}(\Omega, F_\Omega) \subseteq \mathcal{P}(S^2)$ with respect to F, $\mathcal{C}$, and ϕ, we have*

$$P_\mu \leqslant h_\mu(F_\Omega) + \int_\Omega \phi \,\mathrm{d}\mu \leqslant P(F_\Omega, \phi), \tag{6.58}$$

where $P(F_\Omega,\phi)$ is the topological pressure of the map $F_\Omega = F|_\Omega \colon \Omega \to \Omega$ with respect to the potential ϕ.

Recall that $F(\Omega) \subseteq \Omega$ by Proposition 5.4 (iii) and $F_\Omega = F|_\Omega$.

Proof. Note that the second inequality in (6.58) follows from the Variational Principle (3.8) in Subsection 3.1.

Recall measurable partitions O_n, $n \in \mathbb{N}$, of S^2 defined in (3.12). Observe that by Proposition 3.6 (i) and induction, we can conclude that for each $n \in \mathbb{N}$,

$$O_1 \vee f^{-1}(O_1) \vee \cdots \vee f^{-n}(O_n) = O_{n+1}. \tag{6.59}$$

For each $n \in \mathbb{N}$, we define $\widehat{O}_n := \{U \cap \Omega : U \in O_n\}$, which is a measurable partition of Ω. Noting that $F_\Omega^{-1}(A) = \Omega \cap f^{-1}(A)$ for each subset $A \subseteq \Omega$, it follows from (6.59) that

$$\widehat{O}_1 \vee F_\Omega^{-1}(\widehat{O}_1) \vee \cdots \vee F_\Omega^{-n}(\widehat{O}_n) = \widehat{O}_{n+1} \tag{6.60}$$

for each $n \in \mathbb{N}$.

Let $\mu \in \mathcal{M}(\Omega, F_\Omega) \subseteq \mathcal{P}(S^2)$ be an F_Ω-invariant Gibbs measure with respect to F, $\mathcal{C}$, and ϕ. Then by Shannon–McMillan–Breiman Theorem (see for example, [PU10, Theorem 2.5.4]), we have $h_\mu(F_\Omega, \widehat{O}_1) = \int F_{\mathcal{I}} \, \mathrm{d}\mu$, where

$$F_{\mathcal{I}} := \lim_{n \to +\infty} \frac{1}{n+1} I_\mu \biggl(\bigvee_{j=0}^{n} F_\Omega^{-j}(\widehat{O}_1) \biggr) \qquad \mu\text{-a.e. and in } L^1(\mu),$$

$h_\mu\bigl(F_\Omega, \widehat{O}_1\bigr)$ is defined in (3.3), and the information function I_μ is defined in (3.2).

Note that for all $n \in \mathbb{N}$, $\widehat{U} \in \widehat{O}_n$, and $X^n \in \mathfrak{X}^n(F, \mathcal{C})$, either $\widehat{U} \cap X^n = \emptyset$ or $\widehat{U} \subseteq X^n$.

For $n \in \mathbb{N}_0$ and $x \in \Omega$, we denote by $X^n(x) \in \mathfrak{X}^n(F, \mathcal{C})$ any one of the n-tiles of F containing x. Recall that $\widehat{O}_n(x)$ denotes the unique set in the measurable partition $\widehat{O}_n$ that contains x. Note that $\widehat{O}_n(x) \subseteq X^n(x)$. By (6.60) and (6.40) we get

$$\begin{aligned}
\int F_{\mathcal{I}} \, \mathrm{d}\mu &= \int \lim_{n \to +\infty} \frac{1}{n+1} I_\mu \biggl(\bigvee_{j=0}^{n} F_\Omega^{-j}(\widehat{O}_1) \biggr)(x) \, \mathrm{d}\mu(x) \\
&\geqslant \limsup_{n \to +\infty} \int \frac{I_\mu\bigl(\widehat{O}_{n+1}\bigr)(x)}{n+1} \, \mathrm{d}\mu(x) \geqslant \limsup_{n \to +\infty} \int \frac{-\log\bigl(\mu\bigl(X^{n+1}(x)\bigr)\bigr)}{n+1} \, \mathrm{d}\mu(x) \\
&\geqslant \limsup_{n \to +\infty} \int \frac{(n+1)P_\mu - S_{n+1}^F \phi(x) - \log C_\mu}{n+1} \, \mathrm{d}\mu(x) \\
&= P_\mu - \liminf_{n \to +\infty} \frac{1}{n+1} \int S_{n+1}^F \phi(x) \, \mathrm{d}\mu(x) = P_\mu - \int \phi \, \mathrm{d}\mu,
\end{aligned}$$

where the last equality follows from (2.1) and the identity $\int \psi \circ F_\Omega \, \mathrm{d}\mu = \int \psi \, \mathrm{d}\mu$ for each $\psi \in C(\Omega)$, which is equivalent to the fact that μ is F_Ω-invariant. Since $\widehat{O}_1$ is a finite measurable partition, the condition than $H_\mu(\widehat{O}_1) < +\infty$ in (3.4) is fulfilled. By (3.4), we get that

$$h_\mu(F_\Omega) \geqslant h_\mu(F_\Omega, \widehat{O}_1) \geqslant P_\mu - \int \phi \, \mathrm{d}\mu.$$

Therefore, $P_\mu \leqslant h_\mu(F_\Omega) + \int \phi \, \mathrm{d}\mu$. ◻

Proposition 6.33. *Let f, $\mathcal{C}$, F satisfy the Assumptions in Section 4.*

(i) *If $\mu \in \mathcal{P}(S^2)$ is f-invariant and is supported on $\Omega(F, \mathcal{C})$, then μ is F_Ω-invariant, i.e., $\mu(A) = \mu\bigl(F_\Omega^{-1}(A)\bigr)$ for each Borel subset A of Ω.*

(ii) *If measure $\mu \in \mathcal{P}(\Omega)$ is F_Ω-invariant, then $\mu \in \mathcal{P}(S^2)$ is f-invariant. Moreover, $\mathcal{M}(\Omega, F_\Omega) \subseteq \mathcal{M}(S^2, f)$.*

Recall that $F(\Omega) \subseteq \Omega$ by Proposition 5.4 (iii). We use the convention in Remark 6.31.

Proof. (i) Assume that $\mu \in \mathcal{P}(S^2)$ is f-invariant and is supported on $\Omega(F, \mathcal{C})$. Let $A \subseteq \Omega$ be an arbitrary Borel subset. Noting that $(F|_\Omega)^{-1}(A) = \Omega \cap f^{-1}(A)$, we have

$$\mu\bigl(F_\Omega^{-1}(A)\bigr) = \mu\bigl(\Omega \cap f^{-1}(A)\bigr) = \mu\bigl(f^{-1}(A)\bigr) = \mu(A).$$

Thus μ is F_Ω-invariant.

(ii) Assume that $\mu \in \mathcal{P}(\Omega)$ is F_Ω-invariant. Let $A \subseteq S^2$ be an arbitrary Borel subset. Noting that $(F|_\Omega)^{-1}(A \cap \Omega) = \Omega \cap f^{-1}(A)$, we have

$$\mu\bigl(f^{-1}(A)\bigr) = \mu\bigl(\Omega \cap f^{-1}(A)\bigr) = \mu\bigl(F_\Omega^{-1}(A \cap \Omega)\bigr) = \mu(A \cap \Omega) = \mu(A).$$

Thus μ is f-invariant. ◻

Proposition 6.34. *Let f, $\mathcal{C}$, F, d, Λ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and $F \in \mathrm{Sub}_*(f, \mathcal{C})$. Consider $\varphi \in C(S^2)$. Then*

$$P(F, \varphi) \geqslant P(F_\Omega, \varphi), \tag{6.61}$$

where $P(F_\Omega, \varphi)$ is the topological pressure of the map $F_\Omega \colon \Omega \to \Omega$ with respect to the potential ϕ.

Recall that $F(\Omega) \subseteq \Omega$ by Proposition 5.4 (iii) and $F_\Omega = F|_\Omega$.

Proof. By (3.1), it suffices to show that

$$P(F, \varphi) \geqslant \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log(N_d(F_\Omega, \varphi, \varepsilon, n)), \tag{6.62}$$

where $N_d(F_\Omega, \varphi, \varepsilon, m) \coloneqq \sup\bigl\{\sum_{x \in E} \exp\bigl(S_m^F \varphi(x)\bigr) \colon E \subseteq \Omega \text{ is } (m, \varepsilon)\text{-separated with respect to } F_\Omega\bigr\}$. Recall the definition of separated sets in the beginning of Subsection 3.1.

For fixed $\varepsilon > 0$ and $n \in \mathbb{N}$, by Lemma 3.11 (ii), we have

$$\operatorname{diam}_d(X^{n+i}) \leqslant C\Lambda^{-(n+i)} \qquad \text{for all } i \in \mathbb{N}_0 \text{ and } X^{n+i} \in \mathfrak{X}^{n+i}(F, \mathcal{C}), \tag{6.63}$$

where $C \geqslant 1$ is the constant from Lemma 3.11. Let $k = k(\varepsilon)$ be the smallest non-negative integer satisfying $C\Lambda^{-(k+1)} < \varepsilon$. Let $E \subseteq \Omega$ be an arbitrary (n, ε)-separated set. For each $x \in E$, let $X^{n+k}(x)$ be an element of $\mathfrak{X}^{n+k}(F, \mathcal{C})$ containing x. Next, we prove that the map $x \mapsto X^{n+k}(x)$ is injective by contradiction. Suppose this map is not injective, i.e., there exist two distinct points $x, y \in E$ such that $X^{n+k}(x) = X^{n+k}(y)$. Then by (6.63) and Proposition 5.4 (i), we have

$$d(F^i(x), F^i(y)) \leqslant \operatorname{diam}_d\bigl(F^i\bigl(X^{n+k}\bigr)\bigr) \leqslant C\Lambda^{-(n+k-i)} \leqslant C\Lambda^{-(k+1)} < \varepsilon$$

for each $i \in \{0, 1, \dots, n-1\}$, i.e., $d_n(x, y) < \varepsilon$. This contradicts the fact that $E \subseteq \Omega$ is (n, ε)-separated. Hence the map $x \mapsto X^{n+k}(x)$ is injective. Thus,

$$\begin{aligned}
\sum_{x \in E} \exp\bigl(S_n^F \varphi(x)\bigr) &\leqslant \sum_{X^{n+k} \in \mathfrak{X}^{n+k}(F, \mathcal{C})} \exp\bigl(\sup\bigl\{S_n^F \varphi(x) : x \in X^{n+k}\bigr\}\bigr) \\
&\leqslant \sum_{X^{n+k} \in \mathfrak{X}^{n+k}(F, \mathcal{C})} \exp\bigl(k\|\varphi\|_\infty + \sup\bigl\{S_{n+k}^F \varphi(x) : x \in X^{n+k}\bigr\}\bigr) \\
&= e^{k\|\varphi\|_\infty} Z_{n+k}(F, \varphi).
\end{aligned}$$

Noting that the above inequality holds for every (n, ε)-separated set $E \subseteq \Omega$ since $k = k(\varepsilon)$ is independent of E, we can take the supremum of the left-hand side of the above inequality over all (n, ε)-separated set in Ω. Then it follows from Lemma 6.4 that

$$\limsup_{n \to +\infty} \frac{1}{n} \log(N_d(F_\Omega, \varphi, \varepsilon, n)) \leqslant \lim_{n \to +\infty} \biggl(\frac{k\|\varphi\|_\infty}{n} + \frac{n+k}{n} \frac{\log(Z_{n+k}(F, \varphi))}{n+k}\biggr) = P(F, \varphi).$$

Note that $N_d(F_\Omega, \varphi, \varepsilon, n)$ increases as ε decreases. By letting ε decrease to 0, we establish (6.62) and finish the proof. □

Proof of Theorem 6.29. Let $P(F_\Omega, \phi)$ be the topological pressure of the map $F_\Omega \colon \Omega \to \Omega$ with respect to the potential ϕ. Then it follows from the Variational Principle (see (3.8)) that $P(F_\Omega, \phi) = \sup\bigl\{h_\mu(F_\Omega) + \int_\Omega \phi \,\mathrm{d}\mu : \mu \in \mathcal{M}(\Omega, F_\Omega)\bigr\}$. By Proposition 6.34, we have $P(F, \phi) \geqslant P(F_\Omega, \phi)$. Thus, it suffices to show that $D_{F,\phi} = P(F, \phi) \leqslant P(F_\Omega, \phi)$.

Let $\mu_{F,\phi} \in \mathcal{P}(S^2)$ be an f-invariant Gibbs measure with respect to F, $\mathcal{C}$, and ϕ with $\mu_{F,\phi}(\Omega) = 1$ and $P_{\mu_{F,\phi}} = P(F, \phi) = D_{F,\phi}$ from Theorem 6.24. By Proposition 6.33, $\mu_{F,\phi}$ is F_Ω-invariant. Thus, by abuse of notation, we have $\mu_{F,\phi} \in \mathcal{M}(\Omega, F_\Omega)$. Then it follows from Proposition 6.32 that $P(F, \phi) = P_{\mu_{F,\phi}} \leqslant P(F_\Omega, \phi)$. □

Proof of Theorem 6.30. Consider an f-invariant Gibbs measure $\mu_{F,\phi} \in \mathcal{M}(S^2, f)$ with respect to F, $\mathcal{C}$, and ϕ with $\mu_{F,\phi}(\Omega) = 1$ and $P_{\mu_{F,\phi}} = P(F, \phi)$ from Theorem 6.24. Then by Proposition 6.29, we have $P_{\mu_{F,\phi}} = P(F_\Omega, \phi)$. By Proposition 6.33, $\mu_{F,\phi}$ is F_Ω-invariant. Thus by abuse of notation we have $\mu_{F,\phi} \in \mathcal{M}(\Omega, F_\Omega)$. Then it follows from Proposition 6.32 that $P_{\mu_{F,\phi}} = h_{\mu_{F,\phi}}(F_\Omega) + \int_\Omega \phi \,\mathrm{d}\mu_{F,\phi} = P(F_\Omega, \phi)$. Therefore, $\mu_{F,\phi}$ is an equilibrium state for F_Ω and ϕ. □

Proof of Theorem 1.1. The statement follows from Theorems 6.29 and 6.30. □

7. Large deviation asymptotics for expanding Thurston maps

In this section, we establish the large deviation asymptotics (Theorem 1.2) for expanding Thurston maps. Our main strategy for establishing Theorem 1.2 is built upon the approximation by subsystems in the work of Takahasi [Tak20] in continued fraction with the geometrical potential. However, due to the existence of critical points and the fact that our maps are branched covering maps on the topological 2-sphere, the construction on the subsystems for realizing Takahasi's method is not direct in this non-uniformly expanding setting. Indeed, we need to investigate the geometric properties of visual metrics and their interplay with the associated combinatorial structures to split and convert the difficulties arising from non-uniform expansion.

This section is organized as follows. In Subsection 7.1, we investigate the rate function, with the main result being Proposition 7.1, which gives some properties of the rate function.

In Subsection 7.2, we discuss pair structures associated with tile structures induced by an expanding Thurston map, which are used to build appropriate subsystems used in Subsection 7.3.

Subsection 7.3 is devoted to the proof of key bounds in Proposition 7.16. The proof relies on the thermodynamic formalism for subsystems of expanding Thurston maps. We use results in Subsection 7.2 (especially Lemma 7.9) to construct strongly primitive subsystems for sufficiently high iterates of an expanding Thurston map. Then we apply the results in Section 6 and some distortion estimates to obtain the upper bounds in Proposition 7.16.

Finally, in Subsection 7.4, we use the key bounds in Proposition 7.16 to establish Theorem 1.2.

7.1. The rate function.

In this subsection, we are going to prove the following results about the rate function I as defined in (1.4). The proof will be given at the end of this subsection.

Proposition 7.1. *Let $f : S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f. Let $\phi \in C^{0,\beta}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0,1]$ and not co-homologous to a constant in $C(S^2)$. Let μ_ϕ be the unique equilibrium state for the map f and the potential ϕ. Denote $\gamma_\phi := \int \phi \, \mathrm{d}\mu_\phi$, $\alpha_{\min} := \min\limits_{\mu \in \mathcal{M}(S^2,f)} \int \phi \, \mathrm{d}\mu$, and $\alpha_{\max} := \max\limits_{\mu \in \mathcal{M}(S^2,f)} \int \phi \, \mathrm{d}\mu$. Then the following statements hold:*

(i) *$\gamma_\phi \in \operatorname{int}(\mathcal{I}_\phi) = (\alpha_{\min}, \alpha_{\max})$. In particular, the closed interval $\mathcal{I}_\phi$ cannot degenerate into a singleton set consisting of γ_ϕ.*

(ii) *For $\alpha \in (\alpha_{\min}, \alpha_{\max})$,*
$$I(\alpha) = P(f,\phi) - P(f,(p')^{-1}(\alpha)\phi) + ((p')^{-1}(\alpha) - 1)\alpha,$$
where the function $p\colon \mathbb{R} \to \mathbb{R}$ is defined by $p(t) := P(f, t\phi)$.

(iii) *The rate function $I\colon [\alpha_{\min}, \alpha_{\max}] \to [0, +\infty)$ is twice differentiable and strictly convex on $(\alpha_{\min}, \alpha_{\max})$. Moreover, $I(\alpha) = 0$ if and only if $\alpha = \gamma_\phi$. Furthermore, $\lim\limits_{\alpha \to {\alpha_{\min}}^+} I'(\alpha) = -\infty$ and $\lim\limits_{\alpha \to {\alpha_{\max}}^-} I'(\alpha) = +\infty$.*

The graph of the rate function I is shown in Figure 7.1.

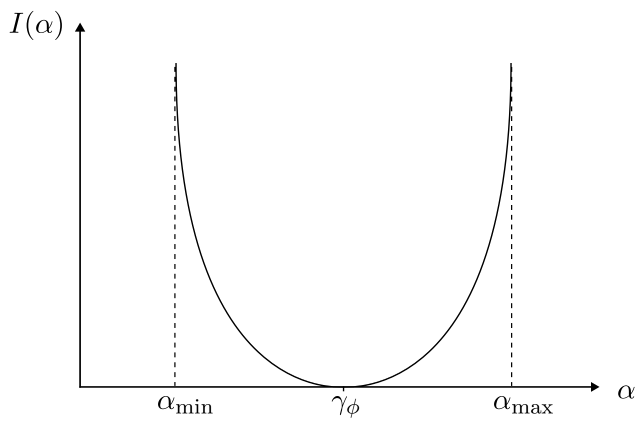

Figure 7.1. The graph of the rate function I.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map and d be a visual metric on S^2 for f. If ϕ is a Hölder continuous function on S^2 with respect to the metric d, then there is a unique equilibrium state for f and ϕ (recall Theorem 3.15 (i)); we denote it by μ_ϕ.

Two continuous functions φ and ψ are called *co-homologous* in $C(S^2)$ if there exists a continuous function $u \in C(S^2)$ such that $\varphi - \psi = u \circ f - u$. If φ and ψ are both Hölder continuous functions (with respect to the metric d) with the same exponent, then $\mu_\varphi = \mu_\psi$ if and only if there exists a constant $K \in \mathbb{R}$ such that $\varphi - \psi$ and $K \mathbb{1}_{S^2}$ are co-homologous in the $C(S^2)$ (see [Li18, Theorem 8.2]).

Let ϕ and $\mathcal{I}_\phi$ satisfy the Assumptions in Section 4. For each $\alpha \in \mathcal{I}_\phi$, we define

$$H(\alpha) \coloneqq \sup\biggl\{ h_\mu(f) : \mu \in \mathcal{M}(S^2, f) \text{ with } \int \phi \,\mathrm{d}\mu = \alpha \biggr\}. \tag{7.1}$$

The following lemma is analog to [SS22, Lemma 3.3]. The proof is essentially the same, and we retain this proof for the convenience of the reader.

Lemma 7.2. *Let f, d, ϕ, γ_ϕ, $\mathcal{I}_\phi$, $\alpha_{\min}$, $\alpha_{\max}$ satisfy the Assumptions in Section 4. We assume in addition that ϕ is not co-homologous to a constant in $C(S^2)$. Then $\gamma_\phi \in \operatorname{int}(\mathcal{I}_\phi) = (\alpha_{\min}, \alpha_{\max})$ and for each $\alpha \in (\alpha_{\min}, \alpha_{\max})$, there exists a unique number $\xi = \xi(\alpha) \in \mathbb{R}$ such that $H(\alpha) = h_{\mu_{\xi\phi}}(f)$ and*

$$\frac{\mathrm{d}P(f, t\phi)}{\mathrm{d}t}\bigg|_{t=\xi} = \int \phi \,\mathrm{d}\mu_{\xi\phi} = \alpha. \tag{7.2}$$

Here $\mu_{\xi\phi}$ is the equilibrium state for f and $\xi\phi$. Moreover, the supremum in (7.1) *is uniquely attained by $\mu_{\xi\phi}$.*

Proof. We write $p(t) \coloneqq P(f, t\phi)$ for $t \in \mathbb{R}$ for convenience. It is a standard fact that the function $p\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable (whose proof is verbatim the same as that of [PU10, Theorem 5.7.4]). Moreover, since ϕ is not co-homologous to a constant, the function p is strictly convex (see [PU10, Theorem 2.11.3] or [DPTUZ21, Theorem 1.1]).

Now we consider the set

$$\mathcal{D} \coloneqq \{ p'(t) : t \in \mathbb{R} \} = \biggl\{ \int \phi \,\mathrm{d}\mu_{t\phi} : t \in \mathbb{R} \biggr\} \subseteq \mathcal{I}_\phi.$$

Here $\mathcal{D}$ is an open interval since the function p is twice differentiable and strictly convex. Then we have $\gamma_\phi \in \operatorname{int}(\mathcal{I}_\phi) = (\alpha_{\min}, \alpha_{\max})$ since $\gamma_\phi = p'(1) \in \mathcal{D}$.

By the definition of pressure, for all $t \in \mathbb{R}$ and $\mu \in \mathcal{M}(S^2, f)$, $p(t) \geqslant h_\mu(f) + t \int \phi \,\mathrm{d}\mu$. In particular, the graph of the strictly convex function p lies above a line with slope $\int \phi \,\mathrm{d}\mu$ (possibly touching it tangentially) so that $\int \phi \,\mathrm{d}\mu \in \overline{\mathcal{D}}$. Since $\mu \in \mathcal{M}(S^2, f)$ is arbitrary, we have $\mathcal{I}_\phi \subseteq \overline{\mathcal{D}}$. Then it follows that $\mathcal{D} = \operatorname{int}(\mathcal{I}_\phi)$. Note that the function $p'\colon \mathbb{R} \to \mathbb{R}$ is differentiable and strictly increasing. Thus, for each $\alpha \in \operatorname{int}(\mathcal{I}_\phi) = p'(\mathbb{R})$, there exists a unique number $\xi = \xi(\alpha) \in \mathbb{R}$ with $\alpha = p'(\xi) = \int \phi \,\mathrm{d}\mu_{\xi\phi}$. Since $\mu_{\xi\phi}$ is the unique equilibrium state for f and $\xi\phi$ (recall Theorem 3.15 (i)), we have, for any $\mu \in \mathcal{M}(S^2, f)$ with $\mu \neq \mu_{\xi\phi}$, $h_{\mu_{\xi\phi}}(f) + \xi \int \phi \,\mathrm{d}\mu_{\xi\phi} > h_\mu(f) + \xi \int \phi \,\mathrm{d}\mu$. In particular, if $\int \phi \,\mathrm{d}\mu = \alpha$ then $h_{\mu_{\xi\phi}}(f) > h_\mu(f)$. This means that the supremum in (7.1) is uniquely attained by $h_{\mu_{\xi\phi}}$. Therefore, $\mu_{\xi\phi}$ is the unique measure with the desired properties. $\square$

Corollary 7.3. *Let f, d, ϕ, γ_ϕ, $\alpha_{\min}$, $\alpha_{\max}$ satisfy the Assumptions in Section 4. We assume in addition that ϕ is not co-homologous to a constant in $C(S^2)$. Let $\xi\colon (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$ be the function defined via Lemma 7.2. Then this function $\xi\colon (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$ is differentiable and strictly increasing. Moreover, we have $\xi(\gamma_\phi) = 1$ and $\lim_{\alpha \to {\alpha_{\min}}^+} \xi(\alpha) = -\infty$, $\lim_{\alpha \to {\alpha_{\max}}^-} \xi(\alpha) = +\infty$.*

Proof. We write $p(t) \coloneqq P(f, t\phi)$ for $t \in \mathbb{R}$. Then the function $p\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and strictly convex. Note that it follows from Lemma 7.2 that $\xi(\alpha) = (p')^{-1}(\alpha)$ for each $\alpha \in (\alpha_{\min}, \alpha_{\max})$. Therefore, the function $\xi\colon (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$ is differentiable and strictly increasing, and $\lim_{\alpha \to {\alpha_{\min}}^+} \xi(\alpha) = -\infty$, $\lim_{\alpha \to {\alpha_{\max}}^-} \xi(\alpha) = +\infty$.

For $\alpha = \gamma_\phi$, by (7.1), we have

$$\begin{aligned} H(\gamma_\phi) &= \sup\biggl\{ h_\mu(f) + \int \phi \,\mathrm{d}\mu : \mu \in \mathcal{M}(S^2, f),\ \int \phi \,\mathrm{d}\mu = \gamma_\phi \biggr\} - \gamma_\phi \\ &\leqslant P(f,\phi) - \gamma_\phi = h_{\mu_\phi}(f) \leqslant H(\gamma_\phi) = h_{\mu_{\xi(\gamma_\phi)\phi}}(f). \end{aligned}$$

Thus $h_{\mu_\phi}(f) = h_{\mu_{\xi(\gamma_\phi)\phi}}(f) = H(\gamma_\phi)$. Therefore, it follows from Lemma 7.2 that $\xi(\gamma_\phi) = 1$. $\square$

Now we can prove Proposition 7.1.

Proof of Proposition 7.1. (i) Statement (i) follows immediately from Lemma 7.2.

(ii) Fix arbitrary $\alpha \in (\alpha_{\min}, \alpha_{\max})$. By Lemma 7.2 and (7.1), we have $H(\alpha) = h_{\mu_{\xi(\alpha)\phi}}(f) = P(f, \xi(\alpha)\phi) - \xi(\alpha)\alpha$. Thus we get

$$I(\alpha) = P(f,\phi) - \alpha - H(\alpha) = P(f,\phi) - P(f,\xi(\alpha)\phi) + (\xi(\alpha) - 1)\alpha$$

by the definition of the rate function I (see (1.4)). By Lemma 7.2, we have $\xi(\alpha) = (p')^{-1}(\alpha)$, where the function $p\colon \mathbb{R} \to \mathbb{R}$ is defined by $p(t) := P(f, t\phi)$. Hence, statement (ii) holds.

(iii) Let $\alpha \in (\alpha_{\min}, \alpha_{\max})$. We write $p(t) := P(f, t\phi)$ for $t \in \mathbb{R}$. Then the function $p\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and strictly convex. By statement (ii), we can write $I(\alpha)$ as

$$I(\alpha) = p(1) - \alpha + \xi(\alpha)\alpha - p(\xi(\alpha)). \tag{7.3}$$

Then it follows from Corollary 7.3 that the rate function I is differentiable on $(\alpha_{\min}, \alpha_{\max})$.

Differentiating the rate function $I(\alpha)$ with respect to α and noting that $p'(\xi(\alpha)) = \alpha$ (Lemma 7.2), we obtain $I'(\alpha) = -1 + \xi'(\alpha)\alpha + \xi(\alpha) - p'(\xi(\alpha))\xi'(\alpha) = \xi(\alpha) - 1$. Thus by Corollary 7.3, I is twice differentiable and strictly convex since $I''(\alpha) = \xi'(\alpha) > 0$, and it follows from Corollary 7.3 that $\lim_{\alpha \to \alpha_{\min}^+} I'(\alpha) = -\infty$ and $\lim_{\alpha \to \alpha_{\max}^-} I'(\alpha) = +\infty$. Moreover, since $\xi(\gamma_\phi) = 1$ and $I'(\gamma_\phi) = 0$, by the strict convexity of I and (7.3), $I(\alpha) = 0$ if and only if $\alpha = \gamma_\phi$. $\square$

7.2. **Pair structures.** In this subsection, we discuss pair structures associated with the tile structures induced by an expanding Thurston map, which will be used to build appropriate subsystems in Subsection 7.3.

Definition 7.4 (Pair structures)**.** Let f, $\mathcal{C}$, e^0 satisfy the Assumptions in Section 4. For each $n \in \mathbb{N}$, we can pair a white n-tile $X_w^n \in \mathbf{X}_w^n$ and a black n-tile $X_b^n \in \mathbf{X}_b^n$ whose intersection $X_w^n \cap X_b^n$ contains an n-edge contained in $f^{-n}(e^0)$. We define the *set of n-pairs* (with respect to f, $\mathcal{C}$, and e^0), denoted by $\mathbf{P}^n(f, \mathcal{C}, e^0)$, to be the collection of the union $X_w^n \cup X_b^n$ of such pairs (called *n-pairs*), i.e.,

$$\mathbf{P}^n(f,\mathcal{C},e^0) := \bigl\{ X_w^n \cup X_b^n : X_w^n \in \mathbf{X}_w^n,\ X_b^n \in \mathbf{X}_b^n,\ X_w^n \cap X_b^n \cap f^{-n}\bigl(e^0\bigr) \in \mathbf{E}^n(f,\mathcal{C}) \bigr\}. \tag{7.4}$$

Figure 7.2 illustrates the structure of n-pairs. There are a total of $(\deg f)^n$ such pairs, and each n-tile is in precisely one such pair (see Lemma 7.6).

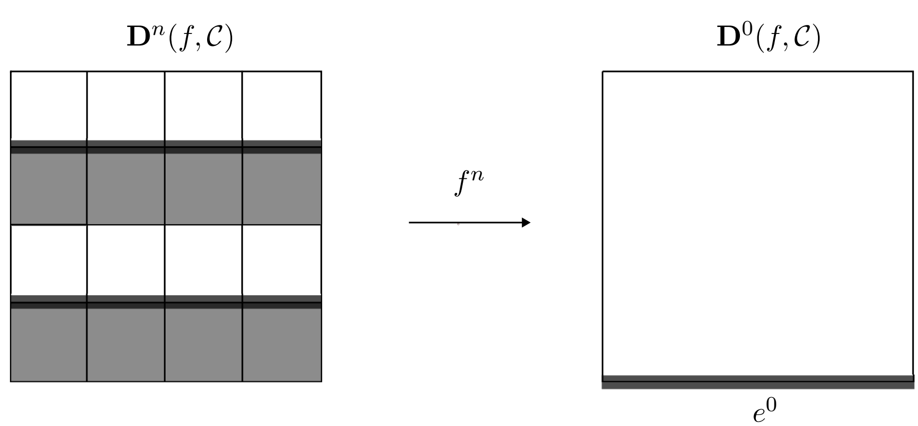

FIGURE 7.2. The graph of n-pairs.

Remark 7.5. For each integer $n \in \mathbb{N}$, one sees that $f^n(P^n) = S^2$ for each n-pair $P^n \in \mathbf{P}^n(f, \mathcal{C}, e^0)$. This basic property is crucial for our constructions and proofs in this Section (for example, see Proposition 7.16). Indeed, this is the main reason why we define and use the pair structures.

From now on, if the map f, the Jordan curve $\mathcal{C}$ and the 0-edge e^0 are clear from the context, we will sometimes omit $(f, \mathcal{C}, e^0)$ in the notation above.

Lemma 7.6. *Let f, $\mathcal{C}$, e^0 satisfy the Assumptions in Section 4. Then for each $n \in \mathbb{N}$ and any two distinct n-pairs P^n, $\widetilde{P}^n \in \mathbf{P}^n$, their interiors are disjoint.*

Proof. We argue by contradiction and assume that there exists two distinct n-pairs P^n, $\widetilde{P}^n \in \mathbf{P}^n$ for some $n \in \mathbb{N}$ such that their interiors intersect. By the definition of n-pairs, there exist white n-tiles X^n_w, $\widetilde{X}^n_w \in \mathbf{X}^n_w$ and black n-tiles X^n_b, $\widetilde{X}^n_b \in \mathbf{X}^n_b$ satisfying $P^n = X^n_w \cup X^n_b$ and $\widetilde{P}^n = \widetilde{X}^n_w \cup \widetilde{X}^n_b$, and there exist n-edges e^n, $\widetilde{e}^n \in \mathbf{E}^n(f, \mathcal{C})$ with $f^n(e^n) = f^n(\widetilde{e}^n) = e^0$ satisfying $e^n \subseteq X^n_w \cap X^n_b$ and $\widetilde{e}^n \subseteq \widetilde{X}^n_w \cap \widetilde{X}^n_b$. Since the interiors of P^n and $\widetilde{P}^n$ intersect, we conclude that $\{X^n_w, X^n_b\} \cap \{\widetilde{X}^n_w, \widetilde{X}^n_b\} \neq \emptyset$. Noting that two tiles with distinct colors cannot be the same, without loss of generality, we can assume that $X^n_b = \widetilde{X}^n_b$. Thus X^n_b contains both e^n and $\widetilde{e}^n$. Moreover, since P^n and $\widetilde{P}^n$ are distinct and $f^n(e^n) = f^n(\widetilde{e}^n) = e^0 \subseteq X^0_b$, it follows that e^n and $\widetilde{e}^n$ are distinct. This contradicts Proposition 3.6 (i), which says that $f^n|_{X^n_b} \colon X^n_b \to f^n(X^n_b) = X^0_b$ is a homeomorphism. ☐

Corollary 7.7. *Let f, $\mathcal{C}$, e^0 satisfy the Assumptions in Section 4. Then for each $n \in \mathbb{N}$, we have $\bigcup \mathbf{P}^n = S^2$ and* card$(\mathbf{P}^n)$= $(\deg f)^n$.

Proof. For each n-tile $X^n \in \mathbf{X}^n$, by Proposition 3.6 (i) and the definition of $\mathbf{P}^n$, there exists an n-pair $P^n \in \mathbf{P}^n$ such that $X^n \subseteq P^n$. Thus $S^2 = \bigcup \mathbf{X}^n \subseteq \bigcup \mathbf{P}^n$ and we get $\bigcup \mathbf{P}^n = S^2$. By Lemma 7.6 and Proposition 3.6 (iv), we have card$(\mathbf{P}^n)$= card$(\mathbf{X}^n)/2 = (\deg f)^n$. ☐

Lemma 7.8. *Let f, $\mathcal{C}$, e^0 satisfy the Assumptions in Section 4. Then $\mathbf{P}^k(f^n, \mathcal{C}, e^0) = \mathbf{P}^{kn}(f, \mathcal{C}, e^0)$ for each n, $k \in \mathbb{N}$.*

Proof. It follows immediately from Proposition 3.6 (vii) that $\mathbf{P}^k(f^n, \mathcal{C}, e^0) = \mathbf{P}^{kn}(f, \mathcal{C}, e^0)$ for all n, $k \in \mathbb{N}$. ☐

We formulate the next lemma to prove Lemma 7.12. Recall $U^n(x)$ is the *n-bouquet of x* (see (3.11)).

Lemma 7.9. *Let f, $\mathcal{C}$, d, e^0 satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$. Then there exists an integer $M \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, and e^0 such that for each color $c \in \{b, w\}$, there exists an M-pair $P^M_c \in \mathbf{P}^M$ such that for each integer $n \geqslant M$ and each $x \in P^M_c$, we have $U^n(x) \subseteq$* inte(X^0_c).

Proof. We first show that there exists an integer $m \in \mathbb{N}$ satisfying the requirement, then let the integer M be the smallest one among all such integers so that M depends only on f, $\mathcal{C}$, d, and e^0.

For two fixed point $x_b \in \operatorname{inte}(X^0_b)$ and $x_w \in \operatorname{inte}(X^0_w)$, we can find a sufficiently small number $r > 0$ such that $B_d(x_c, r) \subseteq \operatorname{inte}(X^0_c)$ for each $c \in \{b, w\}$. By Lemma 3.11 (iii) and (iv), there exists an integer $m \in \mathbb{N}$ such that $2K\Lambda^{-m} \leqslant r$ and $U^m(x_c) \subseteq B_d(x_c, r)$ for each $c \in \{b, w\}$, where K is the constant from Lemma 3.11. By Corollary 7.7, for each color $c \in \{b, w\}$ there exists an m-pair $P^m_c \in \mathbf{P}^m$ containing x_c. Then for each color $c \in \{b, w\}$, by the definition of $U^m(x_c)$ and P^m_c, we have $P^m_c \subseteq U^m(x_c)$.

Let $c \in \{b, w\}$ be arbitrary. For each $y \in P^m_c$ and each $z \in U^m(y)$, by Lemma 3.11 (iii), we have

$$d(z, x_c) \leqslant d(z, y) + d(y, x_c) < K\Lambda^{-m} + K\Lambda^{-m} = 2K\Lambda^{-m} \leqslant r.$$

Thus $z \in B_d(x_c, r)$ for each $z \in U^m(y)$ and each $y \in P^m_c$. Since $B_d(x_c, r) \subseteq \operatorname{inte}(X^0_c)$, we have $U^m(y) \subseteq \operatorname{inte}(X^0_c)$ for each $y \in P^m_c$. Then for each integer $n \geqslant m$ and each $y \in P^m_c$, it follows from Proposition 3.13 and (3.11) that $U^n(y) \subseteq U^m(y) \subseteq \operatorname{inte}(X^0_c)$. ☐

Remark. The visual metric plays an important role in the proof of Lemma 7.9 since it gives good quantitative control over the sizes of the cells in the cell decompositions.

7.3. **Key bounds.** The main goal in this subsection is to establish Proposition 7.16, namely, an exponential upper bound for the measure of the set $P^n(\alpha)$ with respect to the equilibrium state μ_ϕ, where $P^n(\alpha)$ is defined in Definition 7.10. Roughly speaking, we use pairs defined in Subsection 7.2 to cover the set in (1.2) and establish an upper bound for the measure of those tiles by the Gibbs property of μ_ϕ.

Definition 7.10. Let f, $\mathcal{C}$, d, ϕ, e^0, γ_ϕ, $\alpha_{\min}$, $\alpha_{\max}$ satisfy the Assumptions in Section 4. We assume in addition that ϕ is not co-homologous to a constant in $C(S^2)$. For each $\alpha \in (\alpha_{\min}, \alpha_{\max}) \setminus \{\gamma_\phi\}$ and each $n \in \mathbb{N}$, we define

$$\mathbf{P}^n(\alpha) := \begin{cases} \{P^n \in \mathbf{P}^n(f, \mathcal{C}, e^0) : \exists x \in P^n \text{ s.t. } \frac{1}{n} S_n\phi(x) \geqslant \alpha\} & \text{if } \gamma_\phi < \alpha < \alpha_{\max}; \\ \{P^n \in \mathbf{P}^n(f, \mathcal{C}, e^0) : \exists x \in P^n \text{ s.t. } \frac{1}{n} S_n\phi(x) \leqslant \alpha\} & \text{if } \alpha_{\min} < \alpha < \gamma_\phi, \end{cases} \tag{7.5}$$

and $P^n(\alpha) := \bigcup \mathbf{P}^n(\alpha)$.

Note that the sets $\mathbf{P}^n(\alpha)$ and $P^n(\alpha)$ defined above depend on the choice of e^0.

Remark 7.11. For each $\alpha \in (\alpha_{\min}, \alpha_{\max}) \setminus \{\gamma_\phi\}$ and each integer $n \in \mathbb{N}$, it is easy to see that $P^n(\alpha)$ is non-empty and is a union of some n-tiles in $\mathbf{X}^n(f, \mathcal{C})$. Thus by Definition 5.1 and Proposition 3.6 (vii), the map $f^n|_{P^n(\alpha)} : P^n(\alpha) \to S^2$ is a subsystem of f^n with respect to $\mathcal{C}$. Recall that $\Omega(f^n|_{P^n(\alpha)}, \mathcal{C})$ is the tile maximal invariant set associated with $f^n|_{P^n(\alpha)}$ with respect to $\mathcal{C}$ (see Definition 5.1). Denote $\Omega := \Omega\big(f^n|_{P^n(\alpha)}, \mathcal{C}\big)$. Then by Proposition 5.4 (iii), we have $f^n(\Omega) \subseteq \Omega$.

The next lemma shows that the subsystem $f^n|_{P^n(\alpha)}$ (with respect to f^n and $\mathcal{C}$) is strongly primitive (see Definition 5.16) for each $\alpha \in (\alpha_{\min}, \alpha_{\max}) \setminus \{\gamma_\phi\}$ and each sufficiently large integer $n \in \mathbb{N}$.

Lemma 7.12. *Let f, $\mathcal{C}$, d, ϕ, e^0, γ_ϕ, $\alpha_{\min}$, $\alpha_{\max}$ satisfy the Assumptions in Section 4. We assume in addition that $f(\mathcal{C}) \subseteq \mathcal{C}$ and ϕ is not co-homologous to a constant in $C(S^2)$. Then for each $\alpha \in (\alpha_{\min}, \alpha_{\max}) \setminus \{\gamma_\phi\}$, there exists an integer $N \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, ϕ, e^0, and α such that for each integer $n \geqslant N$ and each color $c \in \{b, w\}$, there exists an n-pair $P^n_c \in \mathbf{P}^n(\alpha)$ such that $P^n_c \subseteq \operatorname{inte}\big(X^0_c\big)$. In particular, the subsystem $f^n|_{P^n(\alpha)}$ (with respect to f^n and $\mathcal{C}$) is strongly primitive.*

Proof. Let $\alpha \in (\alpha_{\min}, \alpha_{\max}) \setminus \{\gamma_\phi\}$ be arbitrary. Without loss of generality, we may assume that $\gamma_\phi < \alpha < \alpha_{\max}$.

We first prove the following claim, which follows from the definition of $\alpha_{\max}$ and the compactness of $\mathcal{M}(S^2, f)$ (equipped with the weak* topology).

Claim 1. For each integer $m \in \mathbb{N}$, there exists $y \in S^2$ such that $\frac{1}{m} S_m\phi(y) \geqslant \alpha_{\max}$.

To establish Claim 1, we argue by contradiction and assume that there exists an integer $m \in \mathbb{N}$ such that $\frac{1}{m} S_m\phi(y) < \alpha_{\max}$ for all $y \in S^2$. Since ϕ is continuous and S^2 is compact, there exists a number $\delta > 0$ such that $\frac{1}{m} S_m\phi(y) < \alpha_{\max} - \delta$ for all $y \in S^2$. Then for each $\mu \in \mathcal{M}(S^2, f)$ we have

$$\int \frac{1}{m} S_m\phi \,\mathrm{d}\mu \leqslant \int (\alpha_{\max} - \delta) \,\mathrm{d}\mu = \alpha_{\max} - \delta.$$

Note that $\mathcal{M}(S^2, f)$ is compact in the weak* topology and the continuous map $\mu \mapsto \int \phi \,\mathrm{d}\mu$ maps $\mathcal{M}(S^2, f)$ onto the closed interval $[\alpha_{\min}, \alpha_{\max}]$. Hence there exists an f-invariant measure $\nu \in \mathcal{M}(S^2, f)$ such that $\int \phi \,\mathrm{d}\nu = \alpha_{\max}$. Since

$$\int \frac{1}{m} S_m\phi \,\mathrm{d}\nu = \frac{1}{m} \int \sum_{j=0}^{m-1} \phi \circ f^j \,\mathrm{d}\nu = \frac{1}{m} \sum_{j=0}^{m-1} \int \phi \,\mathrm{d}f^j_*\nu = \frac{1}{m} \cdot m \int \phi \,\mathrm{d}\nu = \int \phi \,\mathrm{d}\nu,$$

we get $\alpha_{\max} = \int \phi \,\mathrm{d}\nu = \int \frac{1}{m} S_m\phi \,\mathrm{d}\nu \leqslant \alpha_{\max} - \delta$, which is a contradiction, and so Claim 1 follows.

By Lemma 7.9, there exists an integer $M \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, and e^0 such that for each color $c \in \{b, w\}$, there exists an M-pair $P^M_c \in \mathbf{P}^M$ such that for each integer $n \geqslant M$ and each $x \in P^M_c$, we have $U^n(x) \subseteq \operatorname{inte}\big(X^0_c\big)$.

We fix such an integer M and the corresponding M-pairs P^M_b and P^M_w in the following.

Set

$$N := \left\lfloor M \cdot \frac{\alpha_{\max} - \inf_{x \in S^2} \phi(x)}{\alpha_{\max} - \alpha} \right\rfloor + 1. \tag{7.6}$$

Note that the integer N depends only on f, $\mathcal{C}$, d, ϕ, e^0, and α. Moreover, we have $N \geqslant M+1$ since $\inf_{x\in S^2}\phi(x) \leqslant \alpha_{\min} < \alpha$.

Claim 2. For each integer $n \geqslant N$ and each color $c \in \{b, w\}$, there exists an n-pair $P_c^n \in \mathbf{P}^n(\alpha)$ such that $P_c^n \subseteq \operatorname{inte}\bigl(X_c^0\bigr)$.

To establish Claim 2, let integer $n \geqslant N$ and color $c \in \{b, w\}$ be arbitrary. Since $N \geqslant M+1$, we have $n - M \in \mathbb{N}$. By Claim 1 there exists $y \in S^2$ such that $\frac{1}{n-M} S_{n-M}\phi(y) \geqslant \alpha_{\max}$. Then there exists $x_c \in P_c^M$ such that $f^M(x_c) = y$ since $f^M(P_c^M) = S^2$. Thus we have

$$\begin{aligned}\frac{1}{n}S_n\phi(x_c) &= \frac{1}{n}S_M\phi(x_c) + \frac{1}{n}S_{n-M}\phi(y) \geqslant \frac{M}{n}\inf_{z\in S^2}\phi(z) + \frac{n-M}{n}\cdot\alpha_{\max} = \alpha_{\max} - \frac{M}{n}\bigl(\alpha_{\max} - \inf_{z\in S^2}\phi(z)\bigr)\\ &\geqslant \alpha_{\max} - \frac{M}{N}\bigl(\alpha_{\max} - \inf_{z\in S^2}\phi(z)\bigr) \geqslant \alpha_{\max} - (\alpha_{\max} - \alpha) = \alpha,\end{aligned}$$

where the last inequality follows from the definition of N (see (7.6)) and the fact that $\lfloor t \rfloor + 1 \geqslant t$ for all $t \in \mathbb{R}$. By Corollary 7.7, there exists an n-pair $P_c^n \in \mathbf{P}^n$ containing x_c. Thus we have $x_c \in P_c^n \in \mathbf{P}^n(\alpha)$ since $\frac{1}{n}S_n\phi(x_c) \geqslant \alpha$. Noting that $x_c \in P_c^M$ and $n \geqslant M$, by Lemma 7.9, we get $U^n(x_c) \subseteq \operatorname{inte}\bigl(X_c^0\bigr)$. Then it follows from the definition of $U^n(x_c)$ and P_c^n that $x_c \in P_c^n \subseteq U^n(x_c) \subseteq \operatorname{inte}\bigl(X_c^0\bigr)$, and so Claim 2 follows.

By Claim 2, it follows immediately from Definition 5.16 and Definition 7.4 that the subsystem $f^n|_{P^n(\alpha)}$ (with respect to f^n and $\mathcal{C}$) is strongly primitive for each integer $n \geqslant N$. $\square$

Definition 7.13. Let f, $\mathcal{C}$, d, ϕ satisfy the Assumptions in Section 4. A Borel probability measure $\mu \in \mathcal{P}(S^2)$ is a *Gibbs measure* with respect to f, $\mathcal{C}$, and ϕ if there exist constants $P_\mu \in \mathbb{R}$ and $C_\mu \geqslant 1$ such that for each $n \in \mathbb{N}_0$, each n-tile $X^n \in \mathbf{X}^n(f, \mathcal{C})$, and each $x \in X^n$, we have

$$\frac{1}{C_\mu} \leqslant \frac{\mu(X^n)}{\exp(S_n\phi(x) - nP_\mu)} \leqslant C_\mu.$$

One observes that for each Gibbs measure μ with respect to f, $\mathcal{C}$, and ϕ, the constant P_μ is unique. Actually, the equilibrium state μ_ϕ is an f-invariant Gibbs measure with respect to f, $\mathcal{C}$, and ϕ, with $P_{\mu_\phi} = P(f, \phi)$ (see [Li18, Theorem 5.16, Proposition 5.17]). We record this result below for the convenience of the reader.

Proposition 7.14 (Z. Li [Li18])**.** *Let f, $\mathcal{C}$, d, ϕ, μ_ϕ satisfy the Assumptions in Section 4. Then μ_ϕ is a Gibbs measure with respect to f, $\mathcal{C}$, and ϕ, with the constant $P_{\mu_\phi} = P(f, \phi)$, i.e., there exists a constant $C_{\mu_\phi} \geqslant 1$ such that for each $n \in \mathbb{N}_0$, each n-tile $X^n \in \mathbf{X}^n(f, \mathcal{C})$, and each $x \in X^n$, we have*

$$\frac{1}{C_{\mu_\phi}} \leqslant \frac{\mu_\phi(X^n)}{\exp(S_n\phi(x) - nP(f, \phi))} \leqslant C_{\mu_\phi}. \tag{7.7}$$

We need the following estimates to prove Proposition 7.16.

Lemma 7.15. *Let f, $\mathcal{C}$, d, ϕ, β, μ_ϕ, e^0 satisfy the Assumptions in Section 4. Let $C_1 \geqslant 0$ be the constant defined in* (5.12) *in Lemma 5.24. Then the following statements are satisfied:*

(i) *For each integer $n \in \mathbb{N}$ and each n-pair $P^n \in \mathbf{P}^n$, we have*

$$\mu_\phi(P^n) \leqslant \frac{2C_{\mu_\phi}e^{C_1(\operatorname{diam}_d(S^2))^\beta}}{e^{P(f,\phi)n}} \inf_{x\in P^n} e^{S_n\phi(x)},$$

where C_{μ_ϕ} is the constant from Proposition 7.14.

(ii) *We assume in addition that ϕ is not co-homologous to a constant in $C(S^2)$. Then for each $n \in \mathbb{N}$ and each $\alpha \in (\gamma_\phi, \alpha_{\max})$, we have*

$$\inf_{x\in P^n(\alpha)} S_n\phi(x) \geqslant n\alpha - 2C_1(\operatorname{diam}_d(S^2))^\beta.$$

Similarly, for each $n \in \mathbb{N}$ and each $\alpha \in (\alpha_{\min}, \gamma_\phi)$, we have

$$\sup_{x\in P^n(\alpha)} S_n\phi(x) \leqslant n\alpha + 2C_1(\operatorname{diam}_d(S^2))^\beta.$$

Proof. We first prove statement (i). For each integer $n \in \mathbb{N}$ and each n-pair $P^n = X_b^n \cup X_w^n \in \mathbf{P}^n$, denote $e^n := X_b^n \cap X_w^n$ and choose an arbitrary point $x_e \in e^n$. By Lemma 5.24, we have

$$\inf_{x \in X_b^n} S_n\phi(x) \geqslant S_n\phi(x_e) - C_1(\mathrm{diam}_d(S^2))^\beta \quad \text{and} \quad \inf_{x \in X_w^n} S_n\phi(x) \geqslant S_n\phi(x_e) - C_1(\mathrm{diam}_d(S^2))^\beta.$$

Thus, we deduce that

$$\inf_{x \in P^n} S_n\phi(x) = \min\Big\{ \inf_{x \in X_b^n} S_n\phi(x), \inf_{x \in X_w^n} S_n\phi(x) \Big\} \geqslant S_n\phi(x_e) - C_1(\mathrm{diam}_d(S^2))^\beta. \tag{7.8}$$

Since the equilibrium state μ_ϕ for f and ϕ is a Gibbs measure with respect to f, $\mathcal{C}$, and ϕ, with constants $P_{\mu_\phi} = P(f,\phi)$ and C_{μ_ϕ}, by Proposition 7.14, we have

$$\mu_\phi(X_b^n) \leqslant C_{\mu_\phi} e^{-P(f,\phi)n} e^{S_n\phi(x_e)} \quad \text{and} \quad \mu_\phi(X_w^n) \leqslant C_{\mu_\phi} e^{-P(f,\phi)n} e^{S_n\phi(x_e)}.$$

Thus we get

$$\mu_\phi(P^n) \leqslant \mu_\phi(X_b^n) + \mu_\phi(X_w^n) \leqslant 2C_{\mu_\phi} e^{-P(f,\phi)n} e^{S_n\phi(x_e)} \leqslant \frac{2C_{\mu_\phi} e^{C_1(\mathrm{diam}_d(S^2))^\beta}}{e^{P(f,\phi)n}} \inf_{x \in P^n} e^{S_n\phi(x)}.$$

We next prove statement (ii). For each $n \in \mathbb{N}$, each $\alpha \in (\gamma_\phi, \alpha_{\max})$, and each $P^n = X_b^n \cup X_w^n \in \mathbf{P}^n(\alpha)$, denote $e^n := X_b^n \cap X_w^n$ and choose an arbitrary point $x_e \in e^n$. By the definition of $\mathbf{P}^n(\alpha)$ in (7.5), there exists $x \in P^n = X_b^n \cup X_w^n$ such that $\frac{1}{n} S_n\phi(x) \geqslant \alpha$. Since $x_e \in X_b^n \cap X_w^n$, by Lemma 5.24, we have

$$S_n\phi(x_e) \geqslant S_n\phi(x) - C_1(\mathrm{diam}_d(S^2))^\beta.$$

Thus, for each $P^n \in \mathbf{P}^n(\alpha)$, using the inequality (7.8) deduced in the proof of statement (i), we have

$$\inf_{x \in P^n} S_n\phi(x) \geqslant S_n\phi(x_e) - C_1(\mathrm{diam}_d(S^2))^\beta \geqslant S_n\phi(x) - 2C_1(\mathrm{diam}_d(S^2))^\beta \geqslant n\alpha - 2C_1(\mathrm{diam}_d(S^2))^\beta.$$

Then by taking infimum of P^n over $\mathbf{P}^n(\alpha)$, we get

$$\inf_{x \in P^n(\alpha)} S_n\phi(x) = \min_{P^n \in \mathbf{P}^n(\alpha)} \big\{ \inf_{x \in P^n} S_n\phi(x) \big\} \geqslant n\alpha - 2C_1(\mathrm{diam}_d(S^2))^\beta.$$

This completes the proof of statement (ii) when $\alpha \in (\gamma_\phi, \alpha_{\max})$. For $\alpha \in (\alpha_{\min}, \gamma_\phi)$, the proof is similar, and we omit it here. $\square$

Now we are ready to prove the main results of this section.

Proposition 7.16 (Key bounds)**.** *Let $f \colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* $\operatorname{post} f$ *with the property that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f. Let $\phi \in C^{0,\beta}(S^2, d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0,1]$ and not co-homologous to a constant in $C(S^2)$. Let μ_ϕ be the unique equilibrium state for the map f and the potential ϕ. Put $C := 2C_{\mu_\phi} \exp\big(C_1(\mathrm{diam}_d(S^2))^\beta\big)$, where $C_{\mu_\phi} \geqslant 1$ and $C_1 \geqslant 0$ are the constants from Proposition 7.14 and (5.12) in Lemma 5.24, respectively. Fix a 0-edge $e^0 \in \mathbf{E}^0(f, \mathcal{C})$ and denote $\gamma_\phi := \int \phi \,\mathrm{d}\mu_\phi$, $\alpha_{\min} := \min_{\mu \in \mathcal{M}(S^2,f)} \int \phi \,\mathrm{d}\mu$, and $\alpha_{\max} := \max_{\mu \in \mathcal{M}(S^2,f)} \int \phi \,\mathrm{d}\mu$. Then the following statements holds:*

(i) *For each $\alpha \in (\gamma_\phi, \alpha_{\max})$, there exists an integer $N \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, ϕ, e^0, and α such that for each integer $n \geqslant N$, there exists a measure $\mu \in \mathcal{M}(S^2, f)$ such that*

$$\mu_\phi(P^n(\alpha)) \leqslant C e^{n(P_\mu(f,\phi) - P(f,\phi))} \quad \text{and} \quad \int \phi \,\mathrm{d}\mu \in \bigg[\alpha - \frac{2C_1(\mathrm{diam}_d(S^2))^\beta}{n}, \alpha_{\max}\bigg].$$

(ii) *For each $\alpha \in (\alpha_{\min}, \gamma_\phi)$, there exists an integer $N \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, ϕ, e^0, and α such that for each integer $n \geqslant N$, there exists a measure $\mu \in \mathcal{M}(S^2, f)$ such that*

$$\mu_\phi(P^n(\alpha)) \leqslant C e^{n(P_\mu(f,\phi) - P(f,\phi))} \quad \text{and} \quad \int \phi \,\mathrm{d}\mu \in \bigg[\alpha_{\min}, \alpha + \frac{2C_1(\mathrm{diam}_d(S^2))^\beta}{n}\bigg].$$

Proof. We first consider the case where $\alpha \in (\gamma_\phi, \alpha_{\max})$. Let $\alpha \in (\gamma_\phi, \alpha_{\max})$ be arbitrary.

By Lemma 7.12, there exists an integer $N \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, ϕ, e^0, and α such that for each integer $n \geqslant N$, the subsystem $f^n|_{P^n(\alpha)}$ (with respect to f^n and $\mathcal{C}$) is strongly primitive. Let integer $n \geqslant N$ be arbitrary. Denote $\Omega := \Omega(f^n|_{P^n(\alpha)}, \mathcal{C})$. Then it follows from Propositions 5.6 (ii) and 5.20 (ii) that $f^n(\Omega) = \Omega$ and $\Omega \setminus \mathcal{C} \neq \emptyset$.

Let $y_0 \in \Omega \setminus \mathcal{C}$ be arbitrary. By Proposition 6.20 and Theorem 6.29, we have

$$\sup_{\nu \in \mathcal{M}(\Omega, f^n|_\Omega)} \Big\{ h_\nu(f^n|_\Omega) + \int S_n\phi \,\mathrm{d}\nu \Big\} = \lim_{m\to+\infty} \frac{1}{m} \log \sum_{x \in (f^n|_\Omega)^{-m}(y_0)} e^{\sum_{k=0}^{m-1} S_n\phi(f^{nk}(x))}. \tag{7.9}$$

Here $\mathcal{M}(\Omega, f^n|_\Omega)$ denotes the set of $f^n|_\Omega$-invariant Borel probability measures on Ω endowed with the weak* topology, and $h_\nu(f^n|_\Omega)$ denotes the measure-theoretic entropy of $f^n|_\Omega$ for ν. For the summand inside the logarithm in (7.9), we have

$$\sum_{x \in (f^n|_\Omega)^{-m}(y_0)} e^{\sum_{k=0}^{m-1} S_n\phi(f^{nk}(x))} = \prod_{i=0}^{m-1} \sum_{y_{i+1} \in (f^n|_\Omega)^{-1}(y_i)} e^{S_n\phi(y_{i+1})}. \tag{7.10}$$

Claim. For each point $y \in \Omega \setminus \mathcal{C}$, we have $\operatorname{card}((f^n|_\Omega)^{-1}(y)) = \operatorname{card}(\mathbf{P}^n(\alpha))$, and each n-pair $P^n \in \mathbf{P}^n(\alpha)$ contains exactly one preimage $x \in (f^n|_\Omega)^{-1}(y)$, which satisfies $x \in \Omega \setminus \mathcal{C}$.

To establish this Claim, we consider an arbitrary point $y \in \Omega \setminus \mathcal{C}$. Without loss of generality we may assume that $y \in \operatorname{inte}(X_b^0)$. Then by Proposition 3.6, we have $\operatorname{card}(f^{-n}(y)) = (\deg f)^n = \operatorname{card}(\mathbf{X}_b^n)$, and each black n-tile $X_b^n \in \mathbf{X}_b^n$ contains exactly one preimage $x \in f^{-n}(y)$, which satisfies $x \in \operatorname{inte}(X_b^n)$. Thus each n-pair $P^n \in \mathbf{P}^n(\alpha)$ contains exactly one preimage $x \in f^{-n}(y) \cap P^n(\alpha)$, which satisfies $x \in \operatorname{inte}(P^n)$, and we have

$$\operatorname{card}(f^{-n}(y) \cap P^n(\alpha)) = \operatorname{card}(\mathbf{P}^n(\alpha)).$$

Let preimage $x \in f^{-n}(y) \cap P^n(\alpha)$ be arbitrary. Noting that $f^{-n}(y) \cap P^n(\alpha) = (f^n|_{P^n(\alpha)})^{-1}(y)$ and $y \in \Omega \setminus \mathcal{C}$, by Proposition 5.5 (iii), we have $x \in \Omega \setminus \mathcal{C}$. Since $(f^n|_\Omega)^{-1}(y) = f^{-n}(y) \cap \Omega = f^{-n}(y) \cap P^n(\alpha)$, the claim follows.

By the claim, we know that all the preimages y_i in the summation in (7.10) belong to $\Omega \setminus \mathcal{C}$. Moreover, for each point $y \in \Omega \setminus \mathcal{C}$, every n-pair $P^n \in \mathbf{P}^n(\alpha)$ contains exactly one preimage $x \in (f^n|_\Omega)^{-1}(y)$, and every preimage $x \in (f^n|_\Omega)^{-1}(y)$ is contained in a unique n-pair $P^n \in \mathbf{P}^n(\alpha)$. Thus, we get the first two inequalities of the following:

$$\begin{aligned} \sum_{x \in (f^n|_\Omega)^{-m}(y_0)} e^{\sum_{k=0}^{m-1} S_n\phi(f^{nk}(x))} &= \prod_{i=0}^{m-1} \sum_{y_{i+1} \in (f^n|_\Omega)^{-1}(y_i)} e^{S_n\phi(y_{i+1})} \geqslant \Big(\inf_{y \in \Omega\setminus\mathcal{C}} \sum_{x \in (f^n|_\Omega)^{-1}(y)} e^{S_n\phi(x)} \Big)^m \\ &\geqslant \Big(\sum_{P^n \in \mathbf{P}^n(\alpha)} \inf_{x \in P^n} e^{S_n\phi(x)} \Big)^m \geqslant \bigg(\frac{e^{nP(f,\phi)} \cdot \sum_{P^n \in \mathbf{P}^n(\alpha)} \mu_\phi(P^n)}{2C_{\mu_\phi} e^{C_1 (\operatorname{diam}_d(S^2))^\beta}} \bigg)^m = \bigg(\frac{e^{nP(f,\phi)} \cdot \mu_\phi\left(P^n(\alpha)\right)}{2C_{\mu_\phi} e^{C_1 (\operatorname{diam}_d(S^2))^\beta}} \bigg)^m. \end{aligned}$$

The last inequality follows from Lemma 7.15 (i) and the last equality follows from Lemma 7.6 and Theorem 3.15 (iii). Taking logarithms of both sides, dividing by m, and plugging the result into the previous inequality, we get

$$\begin{aligned} &\lim_{m\to+\infty} \frac{1}{m} \log\Big(\sum_{x \in (f^n|_\Omega)^{-m}(y_0)} \exp\Big(\sum_{k=0}^{m-1} S_n\phi\big(f^{nk}(x)\big) \Big) \Big) \\ &\qquad \geqslant \log\big(\mu_\phi(P^n(\alpha))\big) + nP(f,\phi) - \big(C_1 (\operatorname{diam}_d(S^2))^\beta + \log(2C_{\mu_\phi}) \big). \end{aligned}$$

Plugging this inequality into (7.9) yields

$$\sup_{\nu \in \mathcal{M}(\Omega, f^n|_\Omega)} \Big\{ h_\nu(f^n|_\Omega) + \int S_n\phi \,\mathrm{d}\nu \Big\} \geqslant \log\big(\mu_\phi(P^n(\alpha))\big) + nP(f,\phi) - \big(C_1 (\operatorname{diam}_d(S^2))^\beta + \log(2C_{\mu_\phi}) \big). \tag{7.11}$$

By Theorem 6.30 and Proposition 6.33 (ii), there exists an equilibrium state $\widehat{\mu} \in \mathcal{M}(\Omega, f^n|_\Omega) \subseteq \mathcal{M}(S^2, f^n)$ which attains the supremum in (7.11). Denote $\mu := \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \widehat{\mu}$. Then $\mu \in \mathcal{M}(S^2, f)$ and we have

$$\int \phi \,\mathrm{d}\mu = \frac{1}{n}\int \sum_{i=0}^{n-1} \phi \,\mathrm{d} f_*^i \widehat{\mu} = \frac{1}{n}\int \sum_{i=0}^{n-1} \phi \circ f^i \,\mathrm{d}\widehat{\mu} = \frac{1}{n}\int S_n\phi \,\mathrm{d}\widehat{\mu}.$$

By Lemma 7.15 (ii), we have $\inf_{x\in P^n(\alpha)} S_n\phi(x) \geqslant n\alpha - 2C_1(\operatorname{diam}_d(S^2))^\beta$. Noting that $\operatorname{supp}\widehat{\mu} \subseteq \Omega \subseteq P^n(\alpha)$, we have

$$\int \phi \,\mathrm{d}\mu = \frac{1}{n}\int S_n\phi \,\mathrm{d}\widehat{\mu} \geqslant \alpha - \frac{2C_1(\operatorname{diam}_d(S^2))^\beta}{n}.$$

Thus the measure μ satisfies $\int \phi \,\mathrm{d}\mu \in \bigl[\alpha - 2n^{-1}C_1(\operatorname{diam}_d(S^2))^\beta, \alpha_{\max}\bigr]$.

By (3.5) and (3.6), we have

$$nh_\mu(f) = h_\mu(f^n) = \frac{1}{n}\sum_{i=0}^{n-1} h_{f_*^i\widehat{\mu}}(f^n). \tag{7.12}$$

We now show that $h_{f_*^i\widehat{\mu}}(f^n) = h_{\widehat{\mu}}(f^n)$ for each $i \in \{0, 1 \dots, n-1\}$. Indeed, the measure $f_*\widehat{\mu}$ is f^n-invariant and the triple $(S^2, f^n, f_*\widehat{\mu})$ is a factor of $(S^2, f^n, \widehat{\mu})$ by the map f. It follows that $h_{f_*\widehat{\mu}}(f^n) \leqslant h_{\widehat{\mu}}(f^n)$ (see for example, [KH95, Proposition 4.3.16]). Iterating this and noting that $f_*^n\widehat{\mu} = (f^n)_*\widehat{\mu} = \widehat{\mu}$ by f^n-invariance of $\widehat{\mu}$, we obtain

$$h_{\widehat{\mu}}(f^n) = h_{f_*^n\widehat{\mu}}(f^n) \leqslant h_{f_*^{n-1}\widehat{\mu}}(f^n) \leqslant \cdots \leqslant h_{f_*\widehat{\mu}}(f^n) \leqslant h_{\widehat{\mu}}(f^n).$$

Hence $h_{f_*^i\widehat{\mu}}(f^n) = h_{\widehat{\mu}}(f^n)$ for each $i \in \{0, 1 \dots, n-1\}$. Combining this with (7.12), we obtain

$$nh_\mu(f) = h_{\widehat{\mu}}(f^n) = h_{\widehat{\mu}}(f^n|_\Omega).$$

Thus

$$\begin{aligned} n\left(h_\mu(f) + \int \phi \,\mathrm{d}\mu\right) &= h_{\widehat{\mu}}(f^n|_\Omega) + \int S_n\phi \,\mathrm{d}\widehat{\mu} \\ &\geqslant \log\bigl(\mu_\phi(P^n(\alpha))\bigr) + nP(f,\phi) - \bigl(C_1(\operatorname{diam}_d(S^2))^\beta + \log(2C_{\mu_\phi})\bigr), \end{aligned}$$

i.e., $\log(\mu_\phi(P^n(\alpha))) \leqslant n(P_\mu(f,\phi) - P(f,\phi)) + C_1(\operatorname{diam}_d(S^2))^\beta + \log(2C_{\mu_\phi})$. This completes the proof of Proposition 7.16 (a).

For $\alpha \in (\alpha_{\min}, \gamma_\phi)$, the proof is similar. □

7.4. **Proof of the large deviation asymptotics.** In this subsection, we establish large deviation asymptotics for expanding Thurston maps. More precisely, we first prove the results under the assumption that there exists an f-invariant Jordan curve $\mathcal{C}$ with $\operatorname{post} f \subseteq \mathcal{C}$. Then by Lemma 3.14, we remove this assumption and prove Theorem 1.2.

Proposition 7.17. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and $\mathcal{C} \subseteq S^2$ be a Jordan curve containing* $\operatorname{post} f$ *with the property that $f(\mathcal{C}) \subseteq \mathcal{C}$. Let d be a visual metric on S^2 for f. Let $\phi \in C^{0,\beta}(S^2,d)$ be a real-valued Hölder continuous function with an exponent $\beta \in (0,1]$ and not co-homologous to a constant in $C(S^2)$. Let μ_ϕ be the unique equilibrium state for the map f and the potential ϕ. Denote $\gamma_\phi := \int \phi \,\mathrm{d}\mu_\phi$, $\alpha_{\min} := \min\limits_{\mu\in\mathcal{M}(S^2,f)} \int \phi \,\mathrm{d}\mu$, and $\alpha_{\max} := \max\limits_{\mu\in\mathcal{M}(S^2,f)} \int \phi \,\mathrm{d}\mu$. Then for each $\alpha \in (\alpha_{\min}, \alpha_{\max})$, there exists an integer $N \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, ϕ, and α such that for each integer $n \geqslant N$,*

$$\mu_\phi\Bigl(\Bigl\{x : \operatorname{sgn}(\alpha-\gamma_\phi)\frac{1}{n}S_n\phi(x) \geqslant \operatorname{sgn}(\alpha-\gamma_\phi)\alpha\Bigr\}\Bigr) \leqslant C_\alpha e^{-I(\alpha)n},$$

where $C_\alpha > 0$ is a constant depending only on f, $\mathcal{C}$, d, ϕ, β, and α. Quantitatively, we choose

$$C_\alpha = 2C_{\mu_\phi} e^{C_1(\operatorname{diam}_d(S^2))^\beta(|2I'(\alpha)|+1)},$$

where $C_{\mu_\phi} \geqslant 1$ is the constant from Proposition 7.14 and $C_1 \geqslant 0$ is the constant defined in (5.12) *in Lemma 5.24.*

Proof. We denote $D(\phi) := C_1(\operatorname{diam}_d(S^2))^\beta$ in this proof. We first consider the case where $\alpha \in (\gamma_\phi, \alpha_{\max})$. Let $\alpha \in (\gamma_\phi, \alpha_{\max})$ be arbitrary.

For each 0-edge $e^0 \in \mathbf{E}^0(f,\mathcal{C})$, by Proposition 7.16, there exists an integer $N_{e^0} \in \mathbb{N}$ depending only on f, $\mathcal{C}$, d, ϕ, α, and e^0 such that for each integer $n \geqslant N_{e^0}$, there exists a measure $\mu \in \mathcal{M}(S^2,f)$ satisfying

$$\mu_\phi(P^n(\alpha)) \leqslant Ce^{(P_\mu(f,\phi)-P(f,\phi))n} \quad \text{and} \quad \int \phi \,\mathrm{d}\mu \in \bigl[\alpha - 2D(\phi)n^{-1}, \alpha_{\max}\bigr],$$

where $C = 2C_{\mu_\phi}e^{D(\phi)}$. Since $\mathbf{E}^0(f,\mathcal{C})$ is a finite set, there exists an 0-edge $\widetilde{e}^0 \in \mathbf{E}^0(f,\mathcal{C})$ such that

$$N_{\widetilde{e}^0} = \min\bigl\{N_{e^0} : e^0 \in \mathbf{E}^0(f,\mathcal{C})\bigr\}.$$

We fix such 0-edge $\widetilde{e}^0$ and let

$$N \coloneqq \max\{N_{\widetilde{e}^0}, \lceil 2D(\phi)/(\alpha-\gamma_\phi)\rceil\},$$

where $\lceil x \rceil$ denotes the smallest integer no less than x. Note that this integer N satisfies $\gamma_\phi + \frac{2D(\phi)}{N} \leqslant \alpha$ and depends only on f, $\mathcal{C}$, d, ϕ, and α.

For each integer $n \geqslant N$, by the definition and properties of rate function I (see (1.4) and Proposition 7.1), we have

$$\begin{aligned}
\mu_\phi\bigl(\{x : n^{-1}S_n\phi(x) \geqslant \alpha\}\bigr) &\leqslant \mu_\phi(P^n(\alpha)) \leqslant Ce^{(P_\mu(f,\phi)-P(f,\phi))n} \\
&\leqslant C\exp\Bigl(\sup\Bigl\{P_\nu(f,\phi) - P(f,\phi) : \nu \in \mathcal{M}(S^2,f),\, \int \phi\,\mathrm{d}\nu \in \bigl[\alpha - 2D(\phi)n^{-1}, \alpha_{\max}\bigr]\Bigr\}n\Bigr) \\
&= C\exp\bigl(-\inf\bigl\{I(\eta) : \eta \in [\alpha - 2D(\phi)n^{-1}, \alpha_{\max}]\bigr\}n\bigr) \\
&= Ce^{-I(\alpha - 2D(\phi)n^{-1})n} \leqslant Ce^{2D(\phi)I'(\alpha)}e^{-I(\alpha)n}.
\end{aligned}$$

The last inequality is shown as follows. By Taylor's formula, there exists $\theta \in \bigl[-2D(\phi)n^{-1}, 0\bigr]$ such that

$$I\bigl(\alpha - 2D(\phi)n^{-1}\bigr) = I(\alpha) - 2D(\phi)n^{-1}I'(\alpha+\theta).$$

Since I is convex and C^1, I' is increasing on $(\gamma_\phi, \alpha_{\max})$ and $I'(\alpha+\theta) \leqslant I'(\alpha)$. Hence,

$$I\bigl(\alpha - 2D(\phi)n^{-1}\bigr) \geqslant I(\alpha) - 2D(\phi)n^{-1}I'(\alpha).$$

Thus we obtain

$$\mu_\phi\bigl(\{x : n^{-1}S_n\phi(x) \geqslant \alpha\}\bigr) \leqslant Ce^{2D(\phi)I'(\alpha)}e^{-I(\alpha)n} = 2C_{\mu_\phi}e^{D(\phi)(2I'(\alpha)+1)}e^{-I(\alpha)n}$$

as required.

For $\alpha \in (\alpha_{\min}, \gamma_\phi)$, the proof is similar. In this case, we choose $C_\alpha = 2C_{\mu_\phi}e^{D(\phi)(-2I'(\alpha)+1)}$.

For $\alpha = \gamma_\phi$, the conclusion holds trivially since $C_{\gamma_\phi} \geqslant 1$ and $I(\gamma_\phi) = 0$. This completes the proof. $\square$

Now we can prove the Theorem 1.2. The key point of the proof is to iterate the map and apply Proposition 7.17.

Proof of Theorem 1.2. We first consider the case where $\alpha \in (\gamma_\phi, \alpha_{\max})$. By Lemma 3.14 we can find a sufficiently high iterate $F \coloneqq f^K$ of f that has an F-invariant Jordan curve $\mathcal{C} \subseteq S^2$ with $\operatorname{post} F = \operatorname{post} f \subseteq \mathcal{C}$. Then F is also an expanding Thurston map (recall Remark 3.10).

Denote $\Phi \coloneqq S_K^f\phi$. Let $\mu_{F,\Phi}$ and $I_{F,\Phi}$ be the unique equilibrium state and the rate function, respectively, for the map F and the potential Φ. Note that $P(F,\Phi) = KP(f,\phi)$ (recall (3.1)). Then it follows from $P_{\mu_\phi}(F,\Phi) = KP_{\mu_\phi}(f,\phi)$ and the uniqueness of the equilibrium state that $\mu_{F,\phi} = \mu_\phi$.

We next show that $I_{F,\Phi}(K\widetilde{\alpha}) = KI(\widetilde{\alpha})$ for each $\widetilde{\alpha} \in (\alpha_{\min}, \alpha_{\max})$. By Proposition 7.1 (ii) and Lemma 7.2, we have

$$I(\widetilde{\alpha}) = P(f,\phi) - P(f,\xi(\widetilde{\alpha})\phi) + (\xi(\widetilde{\alpha}) - 1)\widetilde{\alpha}$$

and

$$I_{F,\Phi}(K\widetilde{\alpha}) = P(F,\Phi) - P(F,\xi_F(K\widetilde{\alpha})\Phi) + (\xi_F(K\widetilde{\alpha}) - 1)K\widetilde{\alpha},$$

where $\xi(\widetilde{\alpha})$ and $\xi_F(K\widetilde{\alpha})$ are defined by

$$\widetilde{\alpha} = \frac{\mathrm{d}P(f,t\phi)}{\mathrm{d}t}\bigg|_{t=\xi(\widetilde{\alpha})} \quad \text{and} \quad K\widetilde{\alpha} = \frac{\mathrm{d}P(F,t\Phi)}{\mathrm{d}t}\bigg|_{t=\xi_F(K\widetilde{\alpha})},$$

respectively. Since $P(F, t\Phi) = KP(f, t\phi)$ for each $t \in \mathbb{R}$, we get

$$\widetilde{\alpha} = \frac{1}{K} \frac{\mathrm{d}P(F, t\Phi)}{\mathrm{d}t}\bigg|_{t=\xi_F(K\widetilde{\alpha})} = \frac{\mathrm{d}P(f, t\phi)}{\mathrm{d}t}\bigg|_{t=\xi_F(K\widetilde{\alpha})}.$$

Thus $\xi_F(K\widetilde{\alpha}) = \xi(\widetilde{\alpha})$ by the uniqueness of $\xi(\widetilde{\alpha})$ (see Lemma 7.2). Then it follows immediately from the expressions of $I(\widetilde{\alpha})$ and $I_{F,\Phi}(K\widetilde{\alpha})$ that $I_{F,\Phi}(K\widetilde{\alpha}) = KI(\widetilde{\alpha})$.

Applying Proposition 7.17, we obtain the large deviation asymptotics for the map F and the potential Φ. Thus for each $\widetilde{\alpha} \in [\gamma_\phi, \alpha_{\max})$ there exists an integer $M \in \mathbb{N}$ such that for each integer $m \geqslant M$,

$$\mu_\phi\big(\big\{x : m^{-1} S_m^F \Phi(x) \geqslant K\widetilde{\alpha}\big\}\big) \leqslant C_{K\widetilde{\alpha}} e^{-I_{F,\Phi}(K\widetilde{\alpha})m}, \tag{7.13}$$

where $C_{K\widetilde{\alpha}} = 2C_{\mu_{F,\Phi}} e^{D(\Phi)(2I'_{F,\Phi}(K\alpha)+1)} \leqslant 2C_{\mu_\phi} e^{C_1(\operatorname{diam}_d(S^2))^\beta(2I'(\alpha)+1)}$. We will derive the large deviation asymptotics for f and ϕ from (7.13). Indeed, for each integer $m \geqslant M$ and each $k \in \{0, 1\, \dots, K-1\}$, we have

$$\begin{aligned} \big\{x : S_{mK+k}^f \phi(x) \geqslant (mK+k)\alpha\big\} &\subseteq \big\{x : S_{mK}^f \phi(x) \geqslant (mK+k)\alpha - k\|\phi\|_\infty\big\} \\ &\subseteq \big\{x : S_{mK}^f \phi(x) \geqslant mK\alpha - 2K\|\phi\|_\infty\big\} \\ &= \big\{x : m^{-1} S_m^F \Phi(x) \geqslant K\big(\alpha - 2m^{-1}\|\phi\|_\infty\big)\big\}. \end{aligned} \tag{7.14}$$

Put

$$N := K \max\{M, \lceil 2\|\phi\|_\infty / (\alpha - \gamma_\phi)\rceil\}, \tag{7.15}$$

where $\lceil x \rceil$ denotes the smallest integer no less than x. Note that this integer N satisfies $\alpha - \frac{2\|\phi\|_\infty}{N/K} \geqslant \gamma_\phi$.

Recall that $\alpha \in (\gamma_\phi, \alpha_{\max})$. For each integer $n \geqslant N$, we can write $n = mK + k$ for some integers $k \in \{0, \dots, K-1\}$ and $m \geqslant M$ satisfying $\alpha - \frac{2\|\phi\|_\infty}{m} \geqslant \gamma_\phi$. Then by (7.14) and (7.13), we get the first two inequalities of the following:

$$\begin{aligned} \mu_\phi\big(\big\{x : n^{-1} S_n^f \phi(x) \geqslant \alpha\big\}\big) &= \mu_\phi\big(\big\{x : S_{mK+k}^f \phi(x) \geqslant (mK+k)\alpha\big\}\big) \\ &\leqslant \mu_\phi\big(\big\{x : m^{-1} S_m^F \Phi(x) \geqslant K\big(\alpha - 2m^{-1}\|\phi\|_\infty\big)\big\}\big) \\ &\leqslant 2C_{\mu_\phi} e^{C_1(\operatorname{diam}_d(S^2))^\beta(2I'(\alpha - 2m^{-1}\|\phi\|_\infty)+1)}\, e^{-mKI\left(\alpha - 2m^{-1}\|\phi\|_\infty\right)} \\ &\leqslant 2C_{\mu_\phi} e^{C_1(\operatorname{diam}_d(S^2))^\beta(2I'(\alpha)+1) + 2K\|\phi\|_\infty I'(\alpha) + KI(\alpha)} e^{-nI(\alpha)}. \end{aligned} \tag{7.16}$$

The last inequality in (7.16) is shown as follows. By Taylor's formula, there exists $\theta \in [-\frac{2\|\phi\|_\infty}{m}, 0]$ such that

$$I\big(\alpha - 2m^{-1}\|\phi\|_\infty\big) = I(\alpha) - 2m^{-1}\|\phi\|_\infty I'(\alpha + \theta).$$

Since I is convex and C^1, I' is increasing on $(\gamma_\phi, \alpha_{\max})$ and $I'(\alpha + \theta) \leqslant I'(\alpha)$. Hence,

$$I\big(\alpha - 2m^{-1}\|\phi\|_\infty\big) \geqslant I(\alpha) - 2m^{-1}\|\phi\|_\infty I'(\alpha).$$

Then

$$e^{-mKI\left(\alpha - 2m^{-1}\|\phi\|_\infty\right)} \leqslant e^{-mKI(\alpha) + 2K\|\phi\|_\infty I'(\alpha)} = e^{2K\|\phi\|_\infty I'(\alpha) + kI(\alpha)} e^{-nI(\alpha)} \leqslant e^{2K\|\phi\|_\infty I'(\alpha) + KI(\alpha)} e^{-nI(\alpha)}.$$

Since $I'\big(\alpha - 2m^{-1}\|\phi\|_\infty\big) \leqslant I'(\alpha)$, we obtain the last inequality in (7.16). Thus, for each $\alpha \in (\gamma_\phi, \alpha_{\max})$, there exists an integer $N \in \mathbb{N}$ (given by (7.15)) such that for each integer $n \geqslant N$,

$$\mu_\phi\big(\big\{x : n^{-1} S_n^f \phi(x) \geqslant \alpha\big\}\big) \leqslant C_\alpha e^{-I(\alpha)n},$$

where $C_\alpha = 2C_{\mu_\phi} e^{C_1(\operatorname{diam}_d(S^2))^\beta(2I'(\alpha)+1) + 2K\|\phi\|_\infty I'(\alpha) + KI(\alpha)}$.

For $\alpha \in (\alpha_{\min}, \gamma_\phi)$, the proof is similar. In this case, we choose the constant

$$C_\alpha = 2C_{\mu_\phi} e^{C_1(\operatorname{diam}_d(S^2))^\beta(-2I'(\alpha)+1) - 2K\|\phi\|_\infty I'(\alpha) + KI(\alpha)}.$$

For $\alpha = \gamma_\phi$, the conclusion holds trivially since $C_{\gamma_\phi} \geqslant 1$ and $I(\gamma_\phi) = 0$. This completes the proof. $\square$

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