Effective potential

$$\rho = \frac{1}{2}\sigma^2$$

$$E_q = \sqrt{q^2 + m_f^2}$$

$$V(\rho) = \frac{1}{2}\lambda\rho^2 + \nu\rho - c\sqrt{2\rho}$$
$$+ \frac{2}{3} N_f N_c \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{E_q}\left[1 - n_F(E_q,\mu) - n_F(E_q,-\mu)\right]$$

Vacuum part:

$$\int \frac{d^3q}{(2\pi)^3} \frac{q^2}{(q^2+m_f^2)^{\frac{1}{2}}} = \int \frac{d^3q}{(2\pi)^3}\left[\frac{q^2+m^2}{(q^2+m^2)^{\frac{1}{2}}} - \frac{m^2}{(q^2+m^2)^{\frac{1}{2}}}\right]$$

$$= \int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2+m^2)^{-\frac{1}{2}}} \quad (1) \quad - m^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2+m^2)^{\frac{1}{2}}} \quad (2)$$

For dimensional regularization:

$$\int \frac{d^Dp}{(2\pi)^D} \frac{1}{(p^2+L)^a} = \frac{\Gamma(a-\frac{D}{2})\, L^{\frac{D}{2}-a}}{(4\pi)^{\frac{D}{2}}\,\Gamma(a)}$$

So (1): $\displaystyle\int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2+m^2)^{-\frac{1}{2}}}$

set $\varepsilon = \frac{3-D}{2}$

then $D = 3 - 2\varepsilon$

and $a = -\frac{1}{2}$

$$\Rightarrow \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^{-\frac{1}{2}}}$$

$$= \mu^{2\varepsilon} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^{-\frac{1}{2}}}$$

$$= \mu^{2\varepsilon} \frac{\Gamma(-\frac{1}{2}-\frac{D}{2})\,(m^2)^{\frac{D}{2}+\frac{1}{2}}}{(4\pi)^{\frac{D}{2}}\,\Gamma(-\frac{1}{2})}$$

$$= \mu^{2\varepsilon} \frac{\Gamma(\varepsilon-2)\,(m^2)^{2-\varepsilon}}{(4\pi)^{\frac{3}{2}-\varepsilon}\,\Gamma(-\frac{1}{2})}$$ Expand around $\varepsilon = 0$ <2>

$$= -\frac{m^4}{32\pi^2}\left\{\frac{1}{\varepsilon} - \frac{1}{2}\left[-3+2\gamma_E + 4\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right]\right\}$$

② : $m^2 \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(q^2+m^2)^{\frac{1}{2}}}$ $a = \frac{1}{2}$

$$= m^2 \mu^{2\varepsilon} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)^{\frac{1}{2}}}$$

$$= m^2 \mu^{2\varepsilon} \frac{\Gamma(\frac{1}{2}-\frac{D}{2})\,(m^2)^{\frac{D}{2}-\frac{1}{2}}}{(4\pi)^{\frac{D}{2}}\,\Gamma(\frac{1}{2})}$$

$$= m^2 \mu^{2\varepsilon} \frac{\Gamma(\varepsilon-1)\,(m^2)^{1-\varepsilon}}{(4\pi)^{\frac{3}{2}-\varepsilon}\,\Gamma(\frac{1}{2})}$$ Expand around $\varepsilon = 0$

$$= \frac{m^4}{8\pi^2}\left\{-\frac{1}{\varepsilon} + \left[-1+\gamma_E + 2\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right]\right\}$$

So ① − ② :

$$= -\frac{m^4}{32\pi^2}\left\{\frac{1}{\varepsilon} - \frac{1}{2}\left[-3+2\gamma_E + 4\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right]\right\}$$

$$- \frac{m^4}{8\pi^2}\left\{-\frac{1}{\varepsilon} + \left[-1+\gamma_E + 2\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right]\right\}$$

$$= -\frac{m^4}{32\pi^2}\left\{\frac{1}{\varepsilon} + \frac{3}{2} - \gamma_E - 2\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right\}$$

$$- \frac{m^4}{32\pi^2}\left\{-\frac{4}{\varepsilon} - 4 + 4\gamma_E + 8\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right\}$$

$$= -\frac{m^4}{32\pi^2}\left\{-\frac{3}{\varepsilon}-\frac{5}{2}+3\gamma_E+6\,\mathrm{Log}\left(\frac{m}{2\sqrt{\pi}\mu}\right)\right\}$$

$$= -\frac{3m^4}{16\pi^2}\,\mathrm{Log}\left(\frac{m}{\mu}\right) + \text{divergent} + \text{constant}$$

Here we use $\overline{MS}$

$$= -\frac{3m^4}{16\pi^2}\,\mathrm{Log}\left(\frac{m}{\mu}\right)$$

So vacuum part of $V(\rho)$ is

$$V_{vac}(\rho) = \frac{2}{3}N_c N_f\,(-1)\,\frac{3m^4}{16\pi^2}\,\mathrm{Log}\left(\frac{m}{\mu}\right)$$

$$= -\frac{N_c N_f}{8\pi^2}\,m^4\,\mathrm{Log}\left(\frac{m}{\mu}\right)$$

this result is same to arXiv: 1005.3166

Pion wave function renormalization at $P_0=0$ and $\vec{p}=0$.

$$\Sigma = T\sum_n \int \frac{d^3q}{(2\pi)^3} T_r\left[i\gamma_5 h T_i\, G_{\bar{q}q}(q)\, i\gamma_5 h T_j\, G_{\bar{q}q}(q-p)\right]$$

$$= -\frac{1}{2}\delta_{ij} h^2 \sum\!\!\!\!\!\!\int \bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\, T_r\left[\gamma_5(-i\not{q}+m)\gamma_5(-i(\not{q}-\not{p})+m)\right]$$

$$= -\frac{1}{2}\delta_{ij} h^2 \cdot 4N_c \sum\!\!\!\!\!\!\int \bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\left[q(q-p)+m^2\right]$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\int \bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\left[q_0(q_0-p_0)+\vec{q}(\vec{q}-\vec{p})+m^2\right]$$

Set $P_0=0$ then,

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\int \bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\left[q_0^2+\vec{q}(\vec{q}-\vec{p})+m^2\right]$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\int \bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\left[q_0^2+\vec{q}^{\,2}+m^2-\vec{q}\cdot\vec{p}\right]$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\int \bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\left[\bar{G}^{-1}_{\bar{q}q}(q)-\vec{q}\cdot\vec{p}\right]$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\int \left[\bar{G}_{\bar{q}q}(q-p)-\bar{G}_{\bar{q}q}(q)\bar{G}_{\bar{q}q}(q-p)\,\vec{q}\cdot\vec{p}\right] \quad ①$$

Here we can expand $\bar{G}_{\bar{q}q}(q-p)$ around $\vec{p}=0$.

$$\bar{G}_{\bar{q}q}(q-p) = \frac{1}{q_0^2+x'+m^2}$$

Here $x=\vec{q}^{\,2}$

$x'=(\vec{q}-\vec{p})^2$

$$= \bar{G}_{\bar{q}q}(q) - \bar{G}^2_{\bar{q}q}(q)(x'-x) + \bar{G}^3_{\bar{q}q}(q)(x'-x)^2$$

$$= \bar{G}_{\bar{q}q}(q) - \bar{G}^2_{\bar{q}q}(q)(\vec{q}^{\,2}+\vec{p}^{\,2}-2\vec{p}\cdot\vec{q}-\vec{q}^{\,2})$$

$$+\bar{G}_q^{3}(q)\left(\vec{q}^{\,2}+\vec{p}^{\,2}-2\vec{p}\cdot\vec{q}-\vec{q}^{\,2}\right)^2$$

$$=\bar{G}_q(q)-\bar{G}_q^{2}(q)\left(-2\vec{p}\cdot\vec{q}+\vec{p}^{\,2}\right)+\bar{G}_q^{3}(q)\left(4(\vec{p}\cdot\vec{q})^2-4\vec{p}^{\,2}(\vec{p}\cdot\vec{q})+\vec{p}^{\,4}\right)$$

then

$$① = -2N_c h^2\delta_{ij}\sum\!\!\!\!\!\!\!\int\Big\{\bar{G}_q(q)-\bar{G}_q^{2}(q)\left(-2\vec{p}\cdot\vec{q}+\vec{p}^{\,2}\right)+\bar{G}_q^{3}(q)\left(4(\vec{p}\cdot\vec{q})^2-4\vec{p}^{\,2}(\vec{p}\cdot\vec{q})+\vec{p}^{\,4}\right)$$

$$-\bar{G}_q(q)\left[\bar{G}_q(q)-\bar{G}_q^{2}(q)\left(-2\vec{p}\cdot\vec{q}+\vec{p}^{\,2}\right)+\bar{G}_q^{3}(q)\left(4(\vec{p}\cdot\vec{q})^2-4\vec{p}^{\,2}(\vec{p}\cdot\vec{q})+\vec{p}^{\,4}\right)\right]\vec{q}\cdot\vec{p}\Big\}$$

$$=-2N_c h^2\delta_{ij}\sum\!\!\!\!\!\!\!\int\Big\{\bar{G}_q(q)-\bar{G}_q^{2}(q)\left(-2\vec{p}\cdot\vec{q}+\underline{\vec{p}^{\,2}}\right)+\bar{G}_q^{3}(q)\left(\underline{4(\vec{p}\cdot\vec{q})^2}-4\vec{p}^{\,2}(\vec{p}\cdot\vec{q})+\vec{p}^{\,4}\right)$$

$$-\bar{G}_q^{2}(q)\,\vec{q}\cdot\vec{p}+\bar{G}_q^{3}(q)\left(\underline{-2(\vec{p}\cdot\vec{q})^2}+\vec{p}^{\,2}\vec{q}\cdot\vec{p}\right)-\bar{G}_q^{4}(q)\left(4(\vec{p}\cdot\vec{q})^3-4\vec{p}^{\,2}(\vec{p}\cdot\vec{q})^2+\vec{p}^{\,4}\vec{q}\cdot\vec{p}\right)\Big\}$$

Keep $\vec{p}^{\,2}$ term

$$=-2N_c h^2\delta_{ij}\sum\!\!\!\!\!\!\!\int\Big\{-\vec{p}^{\,2}\bar{G}_q^{2}(q)+4(\vec{p}\cdot\vec{q})^2\bar{G}_q^{3}(q)-2(\vec{p}\cdot\vec{q})^2\bar{G}_q^{3}(q)\Big\}$$

$$=-2N_c h^2\delta_{ij}\sum\!\!\!\!\!\!\!\int\Big\{-\vec{p}^{\,2}\bar{G}_q^{2}(q)+2(\vec{p}\cdot\vec{q})^2\bar{G}_q^{3}(q)\Big\}$$

$$=-2N_c h^2\delta_{ij}|\vec{p}|^2\sum\!\!\!\!\!\!\!\int\left[-\bar{G}_q^{2}(q)+2|\vec{q}|^2\cos^2\theta\,\bar{G}_q^{3}(q)\right]$$

$$=-2N_c h^2\delta_{ij}|\vec{p}|^2\,T\sum_n\int\frac{d^3q}{(2\pi)^3}\left[-\bar{G}_q^{2}(q)+2|\vec{q}|^2\cos^2\theta\,\bar{G}_q^{3}(q)\right]$$

extract $\vec{q}^{\,2}$ from propagator

$$=-2N_c h^2\delta_{ij}\frac{\vec{p}^{\,2}}{q^3}\,\frac{T}{q}\sum_n\int\frac{d^3q}{(2\pi)^3}\left[-\tilde{G}_q^{2}(q)+2\cos^2\theta\,\tilde{G}_q^{3}(q)\right]$$

Here $\tilde{G}_q(\tilde{q})$ is the dimensionless propagator

$$\tilde{G}_q(\tilde{q}) = \frac{1}{\frac{q_0^2}{\vec{q}^2} + 1 + \frac{m^2}{\vec{q}^2}} = \frac{1}{\tilde{q}_0^2 + 1 + \tilde{m}^2}$$

Then perform the Matsubara summation :

$$= -2 N_c h^2 \delta_{ij} \frac{\vec{p}^2}{\vec{q}^3} \int \frac{d^3q}{(2\pi)^3} \left[- F_2(q, m^2) + 2\cos^2\theta\, F_3(q, m^2) \right]$$

Then

$$Z_\pi^s(0) = -2 N_c h^2 \frac{1}{q^3} \int \frac{d^3q}{(2\pi)^3} \left(- F_2 + 2\cos^2\theta\, F_3 \right)$$

Vacuum part of threshold functions:

$$F_2 = \frac{q^3}{4(q^2+m^2)^{\frac{3}{2}}} \qquad F_3 = \frac{3\, q^5}{16\,(q^2+m^2)^{\frac{5}{2}}}$$

Regularization of F_2 :

$$\int \frac{d^3q}{(2\pi)^3} \frac{1}{q^3} \frac{q^3}{4(q^2+m^2)^{\frac{3}{2}}} = \frac{1}{4} \int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2+m^2)^{\frac{3}{2}}}$$

$$= \frac{1}{4} \mu^{2\varepsilon} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^{\frac{3}{2}}}$$

$$= \frac{1}{4} \mu^{2\varepsilon} \frac{\Gamma(\frac{3}{2}-\frac{D}{2})\,(m^2)^{\frac{D}{2}-\frac{3}{2}}}{(4\pi)^{\frac{D}{2}}\,\Gamma(\frac{3}{2})} \quad / \quad D \to 3-2\varepsilon$$

$$= \frac{1}{4} \mu^{2\varepsilon} \frac{\Gamma(\varepsilon)\, m^{-2\varepsilon}}{(4\pi)^{\frac{3}{2}-\varepsilon}\,\Gamma(\frac{3}{2})}$$ Expand around $\varepsilon = 0$

$$= \frac{1}{16\pi^2}\left(\frac{1}{\varepsilon} - \gamma_E + \mathrm{Log}(4\pi) - 2\,\mathrm{Log}\left(\frac{m}{\mu}\right)\right)$$

Regularization of F_3:

$$\frac{2}{q^3}\int \frac{d^3q}{(2\pi)^3}\cos^2\theta\, F_3$$

$$= \int \frac{d^3q}{(2\pi)^3}\frac{2}{q^3}\cos^2\theta \frac{3q^5}{16(q^2+m^2)^{\frac{5}{2}}}$$

$$= \frac{3}{8}\int \frac{d^Dq}{(2\pi)^D}\frac{q^2\cos^2\theta}{(q^2+m^2)^{\frac{5}{2}}}$$

$$= \frac{3}{8}\int \frac{q^{D-1}dq}{(2\pi)^D}\frac{q^2}{(q^2+m^2)^{\frac{5}{2}}}\int d\Omega_D \cos^2\theta$$

$$= \frac{3}{8}\cdot\frac{4\pi}{3}\int \frac{q^{D-1}dq}{(2\pi)^D}\frac{q^2}{(q^2+m^2)^{\frac{5}{2}}}$$

$$= \frac{\pi}{2}\int \frac{q^{D-1}dq}{(2\pi)^D}\left[\frac{q^2+m^2}{(q^2+m^2)^{\frac{5}{2}}} - \frac{m^2}{(q^2+m^2)^{\frac{5}{2}}}\right]$$

$$= \frac{\pi}{2}\left[\mu^{2\varepsilon}\int \frac{q^{D-1}dq}{(2\pi)^D}\frac{1}{(q^2+m^2)^{\frac{3}{2}}} - m^2\mu^{2\varepsilon}\int \frac{q^{D-1}dq}{(2\pi)^D}\frac{1}{(q^2+m^2)^{\frac{5}{2}}}\right]$$

$$= \frac{\pi}{2}\left[\mu^{2\varepsilon}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{3}{2}-\frac{D}{2})(m^2)^{\frac{D}{2}-\frac{3}{2}}}{2(2\pi)^D\Gamma(\frac{3}{2})} - m^2\mu^{2\varepsilon}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{5}{2}-\frac{D}{2})(m^2)^{\frac{D}{2}-\frac{5}{2}}}{2(2\pi)^D\Gamma(\frac{5}{2})}\right]$$

$D \to 3-2\varepsilon$

finite.

$$= \frac{\pi}{2}\left[\mu^{2\varepsilon}\frac{\Gamma(\frac{3}{2}-\varepsilon)\Gamma(\varepsilon)\,m^{-2\varepsilon}}{2(2\pi)^{3-2\varepsilon}\Gamma(\frac{3}{2})} - m^2\mu^{2\varepsilon}\frac{\Gamma(\frac{3}{2}-\varepsilon)\Gamma(1+\varepsilon)\,m^{-2-2\varepsilon}}{2(2\pi)^{3-2\varepsilon}\Gamma(\frac{5}{2})}\right]$$

For dimensional regularization:

$$\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+L)^a} = \int\frac{q^{D-1}dq\,d\Omega_D}{(2\pi)^D}\frac{1}{(q^2+L)^a}$$

$$= \frac{\Gamma(a-\frac{D}{2})L^{\frac{D}{2}-a}}{(4\pi)^{\frac{D}{2}}\Gamma(a)}$$

and $\int d\Omega_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$

So

$$\int\frac{q^{D-1}dq}{(2\pi)^D} = \frac{\Gamma(a-\frac{D}{2})L^{\frac{D}{2}-a}}{(4\pi)^{\frac{D}{2}}\Gamma(a)}\frac{\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}}$$

$$= \frac{\Gamma(\frac{D}{2})\Gamma(a-\frac{D}{2})L^{\frac{D}{2}-a}}{2(2\pi)^D\Gamma(a)}$$

Expand around $\varepsilon = 0$

$$= \frac{\pi}{2}\left[\frac{1}{16\pi^3}\left(\frac{1}{\varepsilon} - \gamma_E + 2\,\mathrm{Log}(2\pi) - \mathrm{PolyГ}(0,\tfrac{3}{2}) - 2\,\mathrm{Log}\left(\frac{m}{\mu}\right)\right) - \frac{1}{24\pi^3}\right] \quad ②$$

So Z_π^S :

$$Z_\pi^S(0) = -2N_c h^2 \frac{1}{q^3}\int \frac{d^3q}{(2\pi)^3}\left(-F_2 + 2\cos^2\theta\, F_3\right)$$

$$= -2N_c h^2 \left(-① + ②\right)$$

$$= -2N_c h^2 \left\{ -\frac{1}{16\pi^2}\left(\frac{1}{\varepsilon} - \gamma_E + \mathrm{Log}(4\pi) - 2\,\mathrm{Log}\left(\frac{m}{\mu}\right)\right) + \frac{\pi}{2}\left[\frac{1}{16\pi^3}\left(\frac{1}{\varepsilon} - \gamma_E + 2\,\mathrm{Log}(2\pi) - \mathrm{PolyГ}(0,\tfrac{3}{2}) - 2\,\mathrm{Log}\left(\frac{m}{\mu}\right)\right) - \frac{1}{24\pi^3}\right]\right\}$$

$$= -2N_c h^2 \left\{ -\frac{1}{16\pi^2}\left(\frac{1}{\varepsilon} - \gamma_E + \mathrm{Log}(4\pi) - 2\,\mathrm{Log}\left(\frac{m}{\mu}\right)\right) + \frac{1}{16\pi^2}\left(\frac{1}{2}\frac{1}{\varepsilon} - \frac{1}{2}\gamma_E + \mathrm{Log}(2\pi) - \frac{1}{2}\mathrm{PolyГ}(0,\tfrac{3}{2}) - \mathrm{Log}\left(\frac{m}{\mu}\right)\right) - \frac{1}{48\pi^2}\right\}$$

$$= -\frac{2N_c h^2}{16\pi^2}\left\{ -\frac{1}{\varepsilon} + \gamma_E - \mathrm{Log}(4\pi) + 2\,\mathrm{Log}\left(\frac{m}{\mu}\right) + \frac{1}{2}\frac{1}{\varepsilon} - \frac{1}{2}\gamma_E + \mathrm{Log}(2\pi) - \frac{1}{2}\mathrm{PolyГ}(0,\tfrac{3}{2}) - \mathrm{Log}\left(\frac{m}{\mu}\right) - \frac{1}{3\pi^2}\right\}$$

$$= -\frac{N_c h^2}{8\pi^2}\left\{ -\frac{1}{2}\frac{1}{\epsilon} + \frac{1}{2}\gamma_E - \mathrm{Log}(4\pi) + \mathrm{Log}(2\pi) - \frac{1}{2}\mathrm{PolyGamma}\left(0,\frac{3}{2}\right) - \frac{1}{3\pi^2} + \mathrm{Log}\left(\frac{m}{\mu}\right)\right\}$$

Here we use $\overline{MS}$ then

$$Z^s_\pi(0) = -\frac{N_c h^2}{8\pi^2}\,\mathrm{Log}\left(\frac{m}{\mu}\right)$$