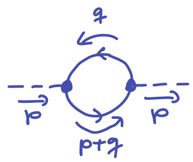
THE MOAT

can be understood from what condensed matter people refer to as Friedel oscillations in a more general sense.

We can understand what's going on from the quark contribution to the meson self-energy,

$$\Pi(p_0, \vec{p}^{\,2}) =$$



For a scalar field this is

$$\Pi(p_0,\vec{p}^{\,2}) \sim h^2 \left[I_1 + \tfrac{1}{2}(4m^2 + p_0^2 + \vec{p}^{\,2})\, I_2(p_0,\vec{p}^{\,2}) \right],$$

$$I_1 = \sum\!\!\!\!\!\!\int_q \frac{1}{(q_0 - i\mu)^2 + E_q^2} \longleftarrow \text{not interesting as it is indep. of } \vec{p}$$

$$I_2(p_0,\vec{p}^{\,2}) = \sum\!\!\!\!\!\!\int_q \frac{1}{(\tilde{q}_0^2 + E_q^2)\left[(\tilde{q}_0 + p_0)^2 + E_{p+q}^2\right]} \qquad \tilde{q}_0 = q_0 - i\mu \qquad \Big|\; E_p = \sqrt{p^2 + m^2}$$

$$= \int_{\vec{q}} \frac{1}{4E_q E_{p+q}} \left\{ \frac{1}{ip_0 + (E_q + E_{p+q})} - \frac{1}{ip_0 - (E_q + E_{p+q})} \right\} \qquad (*)$$

$$+ \int_{\vec{q}} \frac{1}{4E_q E_{p+q}} \left\{ \underbrace{\frac{n_F(E_q+\mu) + n_F(E_{p+q}-\mu)}{ip_0 - (E_q + E_{p+q})}}_{(a)} - \underbrace{\frac{n_F(E_q-\mu) + n_F(E_{p+q}+\mu)}{ip_0 + (E_q + E_{p+q})}}_{(b)} \right.$$

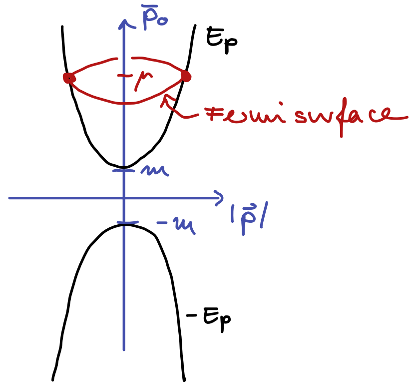
$$\left. + \underbrace{\frac{n_F(E_q-\mu) - n_F(E_{p+q}-\mu)}{ip_0 + (E_q - E_{p+q})}}_{(c)} - \underbrace{\frac{n_F(E_q+\mu) - n_F(E_{p+q}+\mu)}{ip_0 \pm (E_q - E_{p+q})}}_{(d)} \right\}$$

Landau damping $\longrightarrow$

The highlighted contribution contributes to the spacelike spectrum.
The first term of I_2 is of no interest, as it only contributes to the vacuum.
The second term is analytic for $\vec{p}^{\,2} \geq p_0^2$.

Now let's consider the limit $\mu \gg m$, i.e. a large Fermi surface:

In Minkowski space
$p_0 \to -i\bar{p}_0$
$\Rightarrow (p_0 - i\mu)^2 \to -(\bar{p}_0 + \mu)^2$
so the dispersion is
$(\bar{p}_0+\mu)^2 - E_p^2 = 0$
$\Leftrightarrow \bar{p}_0 = \pm E_p - \mu$



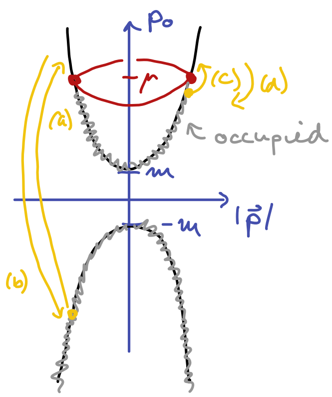
Fermi momentum:
$E_{p_F} - \mu = 0$
$\Leftrightarrow \vec{p}_F^{\,2} + m^2 = \mu^2$
$\Leftrightarrow |\vec{p}_F| = \sqrt{\mu^2 - m^2}$
$\approx \mu$ for $\mu \gg m$

Let's also consider $\mu \gg T$, so that

$$n_F(E \pm \mu) = \frac{1}{e^{(E\pm\mu)/T}+1} \sim \Theta(-E \mp \mu)$$

$\to$ drop the $n_F(E+\mu)$-terms (i.e. the $-E_p$-branch in the figure), as they are suppressed against $n_F(E-\mu)$

We can understand the processes (a)–(d) in (*), as follows



as long as interactions are not strong enough to allow processes (a) and (b)

So for $\mu \gg m$, where we can neglect the lower branch, also (a) and (b) are negligible, and we are left with only (c).

We can most directly see this as follows:

$$G_M \sim \frac{1}{(p_0 - i\mu)^2 + E^2} \overset{z = ip_0}{=} -\frac{1}{(z+\mu)^2 - E^2} = \frac{1}{\underbrace{[z-(E-\mu)]}_{\text{upper branch}}\underbrace{[z-(-E-\mu)]}_{\text{lower branch}}}$$

The residue from the lower branch in the Matsubara sum is

$$\mathrm{Res}_{low} \sim n_F(-E_p - \mu) = 1 - n_F(E_p+\mu)$$

$\rightarrow$ if the lower branch is neglected, all contributions involving the pole at $z = -E_p - \mu$ have to be neglected

$\rightarrow$ no vacuum terms, no $n_F(E+\mu)$-terms, and no $\frac{1}{E_p + E_{p+q}}$-terms

keep those, as it is not clear how they go away... $\rightarrow$ since they must stem from $(-E_p-\mu)-(E_{p+q}-\mu)$ denominators, which involve one pole from the lower branch

Note that this corresponds to the nonrelativistic limit, where

$$G_\psi \sim \frac{1}{z - \left(\frac{\vec{p}^2}{2m} - \mu\right)} \quad ,$$

so there is only an upper branch to begin with. Note:

$$E_p = \sqrt{\vec{p}^2 + m^2} = m\sqrt{1 + \vec{p}^2/m^2} = m + \frac{m}{2}\frac{\vec{p}^2}{m^2} + \mathcal{O}\left(\left(\frac{\vec{p}^2}{m^2}\right)^2\right)$$
$$\approx m + \frac{\vec{p}^2}{2m}$$

From this we conclude that the relevant contribution at large μ is

$$\Pi(p_0, \vec{p}^2) \sim \int_{\vec{q}} \frac{1}{4E_q E_{p+q}} \left\{ \frac{n_F(E_q - \mu) - n_F(E_{p+q}-\mu)}{ip_0 - (E_q - E_{p+q})} \right.$$
$$\left. + \frac{n_F(E_{p+q}-\mu)}{ip_0 - (E_q + E_{p+q})} - \frac{n_F(E_q - \mu)}{ip_0 + (E_q + E_{p+q})} \right\}$$

The first term is the only surviving term in the nonrel. case, i.e. process (c). The other terms survive from (a) and (b) for now. But note that they come from processes involving one pole from the lower branch.

Now consider $\mu \gg T$ / $T \rightarrow 0$, and the static limit $p_0 \rightarrow 0$

$$\Pi(0,\vec p^{\,2}) \sim \int_{\vec q} \frac{-h^2}{4E_q E_{p+q}} \left\{ \frac{\Theta(\mu - E_q) - \Theta(\mu - E_{p+q})}{E_q - E_{p+q}} \right. \quad \swarrow q \to q-p$$

$$\left. + \frac{\Theta(\mu - E_{p+q})}{E_q + E_{p+q}} + \frac{\Theta(\mu - E_q)}{E_q + E_{p+q}} \right\}$$

$$= -\frac{h^2}{4}\int_{\vec q} \Theta(\mu - E_q) \left\{ \frac{1}{E_q E_{q+p}} \left(\frac{1}{E_q - E_{q+p}} + \frac{1}{E_q + E_{q+p}} \right) \right.$$

$$\left. + \frac{1}{E_q E_{q-p}} \left(\frac{1}{E_q - E_{q-p}} + \frac{1}{E_q + E_{q-p}} \right) \right\}$$

Now we use that for $\mu \gg m$ we have for the Fermi energy

$$\mu = q_F \; .$$

The Θ-function shows that only $E_q \leq q_F$ contributes. The smallest energy is $E_0 = m$. Hence, E_q is constrained between m and q_F. Assuming that nothing special happens for $m=0$ as long as $p \neq 0$, we now also consider $m \to 0$. Then

$$E_q = |\vec q| \equiv q \; .$$

$$E_{q\pm p} = \sqrt{p^2 + q^2 \pm 2pqx} \, , \quad x = \cos\sphericalangle(\vec p, \vec q) = \cos\Theta$$

and we get in spherical coords.: $\quad \frac{d\cos\Theta}{d\Theta} = -\sin\Theta \Rightarrow d\Theta = dx \, \frac{1}{-\sin\Theta}$

$$\Pi(0,\vec p^{\,2}) \sim -\frac{h^2}{4}\frac{2\pi}{(2\pi)^3}\int_0^{q_F} dq\, q^2 \int_{-1}^{1} dx$$

$$\left\{ \frac{1}{q\sqrt{q^2+p^2+2pqx}} \left(\frac{1}{q - \sqrt{q^2+p^2+2pqx}} + \frac{1}{q + \sqrt{q^2+p^2+2pqx}} \right) \right.$$

$$\left. + \frac{1}{q\sqrt{q^2+p^2-2pqx}} \left(\frac{1}{q - \sqrt{q^2+p^2-2pqx}} + \frac{1}{q + \sqrt{q^2+p^2+2pqx}} \right) \right\}$$

$$\sqrt{p^2+q^2 \pm 2pqx} \overset{z=q/p}{=} p\sqrt{1+z^2 \pm 2zx} = p\left(1 + \frac{1}{2}\frac{2z \pm 2x}{\sqrt{1+z^2 \pm 2zx}}\bigg|_{z=0} \cdot z + \ldots \right)$$

$$= p\left(1 \pm xz\right) = p \pm qx$$

$\longleftarrow$ expand around $q = q_F$?

$q<p$

$$\approx -\frac{h^2}{4}\frac{2\pi}{(2\pi)^3}\int_0^{q_F} dq\, q^2 \int_{-1}^{1} dx$$

$$\left\{ \frac{1}{qp(1+qx)}\left(\frac{1}{q-(p+qx)} + \frac{1}{q+(p+qx)}\right)\right.$$

$$\left. + \frac{1}{q(p-qx)}\left(\frac{1}{q-(p-qx)} + \frac{1}{q+(p-qx)}\right)\right\}$$

show that only momenta around the Fermi-surface contribute and then go into the non-rel. limit (if that's possible) or expand E around Fermi momentum.

$\nwarrow$ can we neglect q_0^2 agains μq_0? Is this the usual limit? $\to$ check Degquitting EFT...

Kapusta found Friedel oscillations from the photon self-energy, so this should work. Maybe we should keep the vacuum contribution for now,

$$I_2(0,\vec{p}) \sim \frac{h^2}{4}\int_{\vec{q}} \frac{1}{E_q E_{p+q}}\left\{\frac{1}{(E_q+E_{p+q})} + \frac{1}{(E_q+E_{p+q})}\right\}$$

$$-\frac{h^2}{4}\int_{\vec{q}} \Theta(\mu-E_q)\left\{\frac{1}{E_q E_{q+p}}\left(\frac{1}{E_q-E_{q+p}} + \frac{1}{E_q+E_{q+p}}\right)\right.$$

$$\left. + \frac{1}{E_q E_{q-p}}\left(\frac{1}{E_q-E_{q-p}} + \frac{1}{E_q+E_{q-p}}\right)\right\}$$

$$= \frac{h^2}{4}\int_{\vec{q}}\left\{\frac{1}{E_q E_{q+p}}\left[\frac{\Theta(\mu-E_q)}{E_{q+p}-E_q} + \frac{\overbrace{1-\Theta(\mu-E_q)}^{\Theta(E_q-\mu)}}{E_{q+p}+E_q}\right]\right.$$

$$\left. + \frac{1}{E_q E_{q-p}}\left[\frac{\Theta(\mu-E_q)}{E_{q-p}-E_q} + \frac{1-\Theta(\mu-E_q)}{E_{q-p}+E_q}\right]\right\}$$

$$= \frac{h^2}{4(2\pi)^2}\int_0^\infty dq\, \frac{q^2}{E_q}\int_{-1}^{1} dx \left\{\frac{1}{E_{q+p}}\left[\frac{\Theta(\mu-E_q)}{E_{q+p}-E_q} + \frac{\Theta(E_q-\mu)}{E_{q+p}+E_q}\right]\right.$$

$$+\frac{1}{E_{q-p}}\left[\frac{\Theta(\mu-E_q)}{E_{q-p}-E_q}+\frac{\Theta(E_q-\mu)}{E_{q-p}+E_q}\right]\Bigg\}$$

$$\triangleright\ \frac{dE}{dq}=\frac{d\sqrt{q^2+m^2}}{dq}=\frac{1}{2}\frac{2q}{\sqrt{q^2+m^2}}=\frac{q}{E}\ \Rightarrow\ dq=dE\,\frac{E}{q}=dE\frac{E}{\sqrt{E^2-m^2}}$$

$$\triangleright\ E_q\lessgtr\mu\ \Leftrightarrow\ q^2+m^2\lessgtr\mu^2\ \Leftrightarrow\ q^2\lessgtr\mu^2-m^2\ \overset{q\geq 0}{\Longrightarrow}\ q\lessgtr\sqrt{\mu^2-m^2}=p_F$$

$$=\frac{h^2}{16\pi^2}\left\{\int_0^{p_F}dq\,\frac{q^2}{E_q}\int_{-1}^{1}dx\left[\frac{1}{E_{q+p}}\,\frac{1}{E_{q+p}-E_q}+\frac{1}{E_{q-p}}\,\frac{1}{E_{q-p}-E_q}\right]\right.$$

$$\left.+\int_{p_F}^{\infty}dq\,\frac{q^2}{E_q}\int_{-1}^{1}dx\left[\frac{1}{E_{q+p}}\,\frac{1}{E_{q+p}+E_q}+\frac{1}{E_{q-p}}\,\frac{1}{E_{q-p}+E_q}\right]\right\}$$

$$=I_{2a}+I_{2b}$$

$\uparrow$ needs regularization

$$I_{2a}=\frac{h^2}{16\pi^2}\int_0^{p_F}dq\,\frac{q^2}{E_q}\,\frac{2}{pq}\,\ln\left[\frac{\sqrt{q^2+m^2}-\sqrt{(q+p)^2+m^2}}{\sqrt{q^2+m^2}-\sqrt{(q-p)^2+m^2}}\right]$$

$$=\frac{h^2}{8\pi^2}\frac{1}{4p}\left\{2\sqrt{(q+p)^2+m^2}-2\sqrt{(q-p)^2+m^2}+4\sqrt{q^2+m^2}\,\ln\left(\frac{\sqrt{q^2+m^2}-\sqrt{(q+p)^2+m^2}}{\sqrt{q^2+m^2}-\sqrt{(q-p)^2+m^2}}\right)\right.$$

$$-p\ln\left(\frac{\left(q+p+\sqrt{(p+q)^2+m^2}\right)\cdot\left(q-p+\sqrt{(q-p)^2+m^2}\right)}{\left(q+\sqrt{q^2+m^2}\right)^2}\right)$$

$$\left.+\sqrt{p^2+4m^2}\,\ln\left(\frac{(p-2q)^2\left(2m^2-qp+\sqrt{(p^2+4m^2)(p^2+m^2)}\right)\left(2m^2+p(p+q)+\sqrt{(p^2+4m^2)((p+q)^2+m^2)}\right)}{(p+2q)^2\ \left(2m^2+qp+\sqrt{(p^2+4m^2)(p^2+m^2)}\right)\ \left(2m^2+p(p-q)+\sqrt{(p^2+4m^2)((p-q)^2+m^2)}\right)}\right)\right\}\Bigg|_0^{p_F}$$

the lower boundary vanishes, so only the upper boundary contributes. Thus, we have, noting that

$$\triangleright\ \sqrt{p_F^2+m^2}=\mu$$

$$\triangleright \quad (p_F \pm p)^2 + m^2 = p^2 \pm 2pp_F + p_F^2 + m^2 = p(p \pm 2p_F) + \mu^2$$

we get

$$I_{2a} = \frac{h^2}{32\pi^2}\frac{1}{p}\Bigg\{ 2\sqrt{p(p+2p_F)+\mu^2} - 2\sqrt{p(p-2p_F)+\mu^2} + 4\mu \ln\left[\frac{\sqrt{p(p+2p_F)^2+\mu^2}-\mu}{\sqrt{p(p-2p_F)^2+\mu^2}-\mu}\right]$$

$$- p\ln\left[\frac{\left(p_F+p+\sqrt{p(p+2p_F)+\mu^2}\right)\left(p_F-p+\sqrt{p(p-2p_F)+\mu^2}\right)}{(p_F+\mu)^2}\right]$$

$$+\sqrt{p^2+4m^2}\ln\left[\frac{(p-2p_F)^2\left(2m^2-pp_F+\mu\sqrt{(p^2+4m^2)}\right)\left(2m^2+p(p+p_F)+\sqrt{(p^2+4m^2)(p(p+2p_F)+\mu^2)}\right)}{(p+2p_F)^2\left(2m^2+pp_F+\mu\sqrt{(p^2+4m^2)}\right)\left(2m^2+p(p-p_F)+\sqrt{(p^2+4m^2)(p(p-2p_F)+\mu^2)}\right)}\right]\Bigg\}$$

The highlighted term has a branch point at $p = 2p_F$. This means that there is a dominant spacelike (because we are in Euclidean space) $\bar{\psi}(p)\psi(-p)$ - excitation around the Fermi surface.

The second log has vanishing argument for

$$p_F - p + \sqrt{p(p-2p_F)+\mu^2} = \sqrt{(p-p_F)^2+\mu^2-p_F^2} - (p-p_F) \overset{!}{=} 0$$

$m=0$: $\quad |p-\mu| - (p-\mu) = 0 \Rightarrow$ true $\forall\, p \geq \mu$

$\longrightarrow$ in the chiral limit the is a singularity for all $p \geq p_F$

This will presumably cancel with a corresponding singularity from I_2 ...

Aside from this, there is no spacelike singularity.

The third log also has a branch point at $p = 2p_F$. For I_1 to be finite, which it must be, there should be a cancellation. Indeed

$$\lim_{p\to 2p_F}\left\{ 4\mu\ln\left[\frac{\sqrt{p(p+2p_F)^2+\mu^2}-\mu}{\sqrt{p(p-2p_F)^2+\mu^2}-\mu}\right] + \sqrt{p^2+4m^2}\ln\left[\frac{(p-2p_F)^2}{(p+2p_F)^2}\right]\right\}$$

$$= \lim_{p\to 2p_F}\left\{ 4\mu\ln\left(\frac{p+2p_F}{p-2p_F}\right) + \underbrace{\sqrt{4p_F^2+4m^2}}_{=2\mu}\cdot 2\ln\left(\frac{p-2p_F}{p+2p_F}\right)\right\}$$

$$= 0$$

Furthermore, potential singularities can come from

$$(1)\quad 2m^2 - pp_F + \mu\sqrt{(p^2+4m^2)} = 0$$

$$(2)\quad 2m^2 + p(p-p_F) + \sqrt{(p^2+4m^2)(p(p-2p_F)+\mu^2)} = 0$$

(1) seems to only have complex singularities and (2) none at all for $m>0$. In the chiral limit we have with $p_F=\mu$

$$(1)\quad -p\mu + p\mu = 0$$

$$(2)\quad p(p-\mu) + p|p-\mu| = 0$$

(1) & (2) are both fulfilled for $p=0,\mu$. Since (1) is in the numerator and (2) in the denominator, these singularities cancel. This leaves us with a singularity for all p. from (1) Perhaps also this cancels with a corresponding term from I_{2b}, let's see...

regularized

$$I_{2b}^{\Lambda} = \frac{h^2}{16\pi^2}\int_{p_F}^{\Lambda} dq\,\frac{q^2}{E_q}\int_{-1}^{1}dx\left[\frac{1}{E_{q+p}}\,\frac{1}{E_{q+p}+E_q} + \frac{1}{E_{q-p}}\,\frac{1}{E_{q-p}+E_q}\right]$$

$$= \frac{h^2}{16\pi^2}\int_{p_F}^{\Lambda} dq\,\frac{q^2}{E_q}\,\frac{2}{pq}\ln\left[\frac{\sqrt{q^2+m^2}+\sqrt{(q+p)^2+m^2}}{\sqrt{q^2+m^2}+\sqrt{(q-p)^2+m^2}}\right]$$

$$= -\frac{h^2}{8\pi^2}\,\frac{1}{4p}\left\{2\sqrt{(p+q)^2+m^2} - 2\sqrt{(p-q)^2+m^2} + 4\sqrt{q^2+m^2}\ln\left[\frac{\sqrt{q^2+m^2}+\sqrt{(p+q)^2+m^2}}{\sqrt{q^2+m^2}+\sqrt{(p-q)^2+m^2}}\right]\right.$$

$$+ \ldots$$

In fact, there are many cancellations between I_1 and I_2^{Λ} in $I_1+I_2^{\Lambda}$ from the upper boundary of I_1 and the lower of I_2^{Λ}. So it is easier to show the sum from MATHEMATICA:

$$I_{2a} + I_{2b}^{\Lambda} = I_{2a}|_{p_F} - I_{2b}^{\Lambda}|_{p_F} + I_{2b}^{\Lambda}|_{\Lambda}$$

$$= \frac{h^2}{8\pi^2}\,\frac{1}{2p}\left\{2\sqrt{p(p+p_F)+\mu^2} - 2\sqrt{p(p-p_F)+\mu^2} + \sqrt{p^2+4m^2}\ln\left(\frac{p-2p_F}{p+2p_F}\right)\right.$$

$$+\sqrt{p^2+4m^2}\,\ln\left(\frac{2m^2+p(p+p_F)+\sqrt{p^2+4m^2}\sqrt{p(p+2p_F)+\mu^2}}{2m^2+p(p-p_F)+\sqrt{p^2+4m^2}\sqrt{p(p-2p_F)+\mu^2}}\right)\checkmark$$

$$-p\left[\ln\left(p_F+p+\sqrt{p(p+2p_F)+\mu^2}\right)+\ln\left(p_F-p+\sqrt{p(p-2p_F)+\mu^2}\right)\right]\checkmark$$

$$+2\mu\left[\ln\left(\frac{\mu+\sqrt{p(p-2p_F)+\mu^2}}{\mu+\sqrt{p(p+2p_F)+\mu^2}}\right)\checkmark+\ln\left(\frac{\mu-\sqrt{p(p+2p_F)+\mu^2}}{\mu-\sqrt{p(p-2p_F)+\mu^2}}\right)\checkmark\right]\Bigg\}$$

$$+\frac{h^2}{8\pi^2}\frac{1}{4p}\Bigg\{2\sqrt{m^2+(p-\Lambda)^2}\checkmark-2\sqrt{m^2+(p+\Lambda)^2}\checkmark$$

$$+p\left[\ln\left(\Lambda+p+\sqrt{m^2+(p+\Lambda)^2}\right)\checkmark+\ln\left(\Lambda-p+\sqrt{m^2+(p-\Lambda)^2}\right)+2\ln\left(\Lambda+\sqrt{m^2+\Lambda^2}\right)\checkmark\right]$$

$$+4\sqrt{m^2+\Lambda^2}\,\ln\left(\frac{\sqrt{m^2+\Lambda^2}+\sqrt{m^2+(p+\Lambda)^2}}{\sqrt{m^2+\Lambda^2}+\sqrt{m^2+(p-\Lambda)^2}}\right)\checkmark$$

$$+\sqrt{p^2+4m^2}\left[\ln\left(\frac{2m^2+p(p-\Lambda)+\sqrt{p^2+4m^2}\sqrt{m^2+(p-\Lambda)^2}}{2m^2+p(p+\Lambda)+\sqrt{p^2+4m^2}\sqrt{m^2+(p+\Lambda)^2}}\right)\checkmark\right.$$

$$\left.+\ln\left(\frac{2m^2-p\Lambda+\sqrt{p^2+4m^2}\sqrt{m^2+\Lambda^2}}{2m^2+p\Lambda+\sqrt{p^2+4m^2}\sqrt{m^2+\Lambda^2}}\right)\right]\Bigg\}\checkmark$$

$$\equiv I(p)+I_\Lambda(p)$$

For $\Lambda\to\infty$ we get

$$I_\Lambda\to\frac{h^2}{8\pi^2}\frac{1}{4p}\left\{4p\ln(2\Lambda)+2\sqrt{p^2+4m^2}\,\ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)\right\}$$

$$=\frac{h^2}{8\pi^2}\left[\ln\Lambda+\ln 2+\sqrt{\frac{p^2+4m^2}{4p^2}}\,\ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)\right],$$

reflecting the log-divergence of the vacuum contribution. We could minimally subtract the $\ln\Lambda$-term and send Λ to infinity. But let's do this properly now.

Since this divergence occurs in the meson two-point function, it is presumably related to mass renormalization. We hence add a counter-term Lagrangian,

$$\mathcal{L} = \mathcal{L}_{ren} + \mathcal{L}_{ct} \,,$$

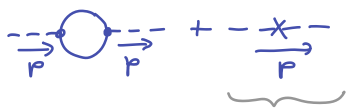
renormalized $\mathcal{L}$ counter terms

with

$$\mathcal{L}_{ct} = \frac{1}{2}\delta_Z (\partial_\mu \phi)^2 + \frac{1}{2}\delta_m \phi^2 + \ldots$$

Then the diagram we actually compute is

$\Pi(p) =$



$= \delta_Z p^2 + \delta_m$

For now, a "curvature mass renormalization" is sufficient. $\mathcal{L}$ contains a meson mass term

$$\frac{1}{2} m_\phi^2 \phi^2 \,.$$

If we want m_ϕ^2 to be the renormalized curvature mass, we get the following renormalization condition:

$$\Pi(0) = 0 \iff \delta_m = -I(0) - I_\Lambda(0) \,.$$

so

$$\delta_m = \frac{h^2}{16\pi^2}\left\{ \frac{2p_F}{\mu} + \underbrace{2m\,\overset{\text{from } \ln\left(\frac{p-2p_F}{p+2p_F}\right)}{\frac{i\pi}{(p)}}}_{?} + \frac{2p_F}{\mu}\frac{2m+\mu}{m+\mu} - 2\ln(p_F+\mu) - \frac{2p_F}{\mu} + \underbrace{2\mu\frac{i\pi}{(p)}}_{?}\right\}$$

$$\frac{h^2}{32\pi^2}\left\{ -\frac{4\Lambda}{\sqrt{m^2+\Lambda^2}} + 4\ln\left(\Lambda+\sqrt{m^2+\Lambda^2}\right) + \frac{4\Lambda}{\sqrt{m^2+\Lambda^2}} - \frac{2\Lambda\left(2m+\sqrt{m^2+\Lambda^2}\right)}{\Lambda^2+m^2\left(m+\sqrt{m^2+\Lambda^2}\right)} - \frac{2\Lambda}{m+\sqrt{m^2+\Lambda^2}}\right\}$$

I forgot I_1 and the prefactor of I_2, cf. p. 1 ...

Maybe it is better to compute the $p=0$ case separately, as it is a bit subtle. Note that we also ignored I_1, which we shouldn't.

$$I_1^\Lambda = \sum\!\!\!\!\!\!\int_q \frac{h^2}{(q_0-i\mu)^2+E_q^2} = \int_{\vec q} \frac{1}{2E_q}\Big[1-\underbrace{n_F(E_q-\mu)}_{\to\Theta(\mu-E_q)}-\underbrace{n_F(E_q+\mu)}_{\to 0}\Big]$$

$$\overset{T=0}{=} \int_{\vec q} \frac{h^2}{2E_q}\,\Theta(E_q-\mu) = \frac{4\pi}{(2\pi)^3}\frac{h^2}{2}\int dq\,\frac{q^2}{E_q}\,\Theta(E_q-\mu)$$

$$= \frac{h^2}{4\pi^2}\int_{p_F}^{\Lambda} dq\,\frac{q^2}{E_q} = \frac{h^2}{4\pi^2}\,\frac{1}{4}\left[2q\sqrt{q^2+m^2}+m^2\ln\left(\frac{\sqrt{q^2+m^2}-q}{\sqrt{q^2+m^2}+q}\right)\right]_{p_F}^{\Lambda}$$

$$= \frac{h^2}{16\pi^2}\left[-2\mu p_F+m^2\ln\left(\frac{\mu+p_F}{\mu-p_F}\right)\right]+\frac{h^2}{16\pi^2}\left[2\Lambda\sqrt{m^2+\Lambda^2}+m^2\ln\left(\frac{\sqrt{m^2+\Lambda^2}-\Lambda}{\sqrt{m^2+\Lambda^2}+\Lambda}\right)\right]$$

The full self-energy thus is at $p^0=0$:

$$\Pi(0,\vec p) = \underbrace{h^2\left[I_1+\tfrac{1}{2}(4m^2+p^2)\,I_2(0,\vec p)\right]}_{=\Pi_\Lambda}+\delta_Z p^2+\delta_m$$

$$= \frac{h^2}{16\pi^2}\Bigg\{-2\mu p_F+m^2\ln\left(\frac{\mu+p_F}{\mu-p_F}\right)$$

$$+\frac{p^2+4m^2}{2p}\Bigg[2\sqrt{p(p+p_F)+\mu^2}-2\sqrt{p(p-p_F)+\mu^2}+\sqrt{p^2+4m^2}\,\ln\left(\frac{p-2p_F}{p+2p_F}\right)$$

$$+\sqrt{p^2+4m^2}\,\ln\left(\frac{2m^2+p(p+p_F)+\sqrt{p^2+4m^2}\,\sqrt{p(p+2p_F)+\mu^2}}{2m^2+p(p-p_F)+\sqrt{p^2+4m^2}\,\sqrt{p(p-2p_F)+\mu^2}}\right)$$

$$-p\left(\ln\left(p_F+p+\sqrt{p(p+2p_F)+\mu^2}\right)+\ln\left(p_F-p+\sqrt{p(p-2p_F)+\mu^2}\right)\right)$$

$$+2\mu\left(\ln\left(\frac{\mu+\sqrt{p(p-2p_F)+\mu^2}}{\mu+\sqrt{p(p+2p_F)+\mu^2}}\right)+\ln\left(\frac{\mu-\sqrt{p(p+2p_F)+\mu^2}}{\mu-\sqrt{p(p-2p_F)+\mu^2}}\right)\right)\Bigg]$$

$$+2\Lambda\sqrt{m^2+\Lambda^2}+m^2\ln\left(\frac{\sqrt{m^2+\Lambda^2}-\Lambda}{\sqrt{m^2+\Lambda^2}+\Lambda}\right)$$

$$+\frac{p^2+4m^2}{4p}\Big[2\underbrace{\sqrt{m^2+(p-\Lambda)^2}-2\sqrt{m^2+(p+\Lambda)^2}}_{\to 0}$$

$$+p\Big(\ln\big(\Lambda+p+\sqrt{m^2+(p+\Lambda)^2}\big)+\ln\big(\Lambda-p+\sqrt{m^2+(p-\Lambda)^2}\big)+2\ln\big(\Lambda+\sqrt{m^2+\Lambda^2}\big)\Big)$$

$$+\underbrace{4\sqrt{m^2+\Lambda^2}\ln\left(\frac{\sqrt{m^2+\Lambda^2}+\sqrt{m^2+(p+\Lambda)^2}}{\sqrt{m^2+\Lambda^2}+\sqrt{m^2+(p-\Lambda)^2}}\right)}_{\to p}$$

$$\frac{\Lambda+\Lambda}{\Lambda+\Lambda} \qquad \ln(1+x) = 0+x$$

$$+\sqrt{p^2+4m^2}\Bigg(\ln\left(\frac{2m^2+p(p-\Lambda)+\sqrt{p^2+4m^2}\sqrt{m^2+(p-\Lambda)^2}}{2m^2+p(p+\Lambda)+\sqrt{p^2+4m^2}\sqrt{m^2+(p+\Lambda)^2}}\right)$$

$$\underbrace{\ldots+\ln\left(\frac{2m^2-p\Lambda+\sqrt{p^2+4m^2}\sqrt{m^2+\Lambda^2}}{2m^2+p\Lambda+\sqrt{p^2+4m^2}\sqrt{m^2+\Lambda^2}}\right)\Bigg)\Big]}_{(\ldots)\to 2\ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)}\Bigg\}+\delta_Z p^2+\delta_m$$

Now we can properly fix the counter term...

$$\delta_m = -\Pi_\Lambda(0)$$

$$= -\frac{h^2}{16\pi^2}\left\{-2\mu p_F + m^2\ln\left(\frac{\mu-p_F}{\mu+p_F}\right) + 2\Lambda\sqrt{m^2+\Lambda^2} + m^2\ln\left(\frac{\sqrt{m^2+\Lambda^2}-\Lambda}{\sqrt{m^2+\Lambda^2}+\Lambda}\right)\right\}$$

$$-h^2\,2m^2\underbrace{\SumInt_q \frac{1}{\left[(q_0-i\mu)^2+E_q^2\right]^2}}_{=-\frac{1}{2E_q}\frac{\partial I_\Lambda}{\partial E_q}}$$

$$\triangleright\ \partial_E n_F(E\pm\mu) = n_F^2(E\pm\mu)\,\partial_E\, e^{(E\pm\mu)/T} = \frac{1}{T}\, n_F^2\big(\overbrace{n_F^{-1}}^{e^{E/T}}-1\big) = \mp\frac{1}{T}\, n_\mp(1-n_\mp)$$

$$\triangleright\ \frac{\partial_E I_\Lambda}{-2E} = \left[\partial_E \int_{\vec q}\frac{1}{2E}\big(1-n_F(E+\mu)-n_F(E-\mu)\big)\right]\frac{-1}{2E}$$

$$= \int_{\vec q}\Big\{+\frac{1}{4E^3}\left[1-n_F(E+\mu)-n_F(E-\mu)\right]$$

$$+\frac{1}{4TE^2}\Big[\underbrace{n_F(E-\mu)\big(1-n_F(E-\mu)\big)}_{\to\Theta(E-\mu)\Theta(\mu-E)}+\underbrace{n_F(E+\mu)\big(1-n_F(E+\mu)\big)}_{\to 0}\Big]\Big\}$$

$$\xrightarrow{T\to 0} \int_{\vec q} \frac{1}{4E^3}\,\Theta(E_q-\mu) = \frac{4\pi}{(2\pi)^3}\int_{p_F}^{\Lambda} dq\, q^2\, \frac{1}{4E_q^3}$$

$$= \frac{1}{8\pi^2}\left[\frac{-\Lambda}{\sqrt{m^2+\Lambda^2}} + \ln\left(\frac{\Lambda+\sqrt{m^2+\Lambda^2}}{m}\right) + \frac{p_F}{\mu} - \ln\left(\frac{p_F+\mu}{m}\right)\right]$$

$$= -\frac{h^2}{16\pi^2}\left\{-2\mu p_F + m^2\ln\left(\frac{\mu-p_F}{\mu+p_F}\right) + 2\Lambda\sqrt{m^2+\Lambda^2} + m^2\ln\left(\frac{\sqrt{m^2+\Lambda^2}-\Lambda}{\sqrt{m^2+\Lambda^2}+\Lambda}\right)\right\}$$

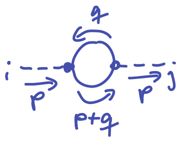
$$-\frac{h^2 m^2}{4\pi^2}\left\{\frac{p_F}{\mu} + \underbrace{\frac{-\Lambda}{\sqrt{m^2+\Lambda^2}}}_{\to -1\ (\Lambda\to\infty)} + \ln\left(\frac{\Lambda+\sqrt{m^2+\Lambda^2}}{p_F+\mu}\right)\right\}$$

Note that I_n completely cancels with δ_m.
There are still log-divergences $\propto p^2$, which require field renormalization δ_Z.
We impose $Z_\phi = 1$ at $p=0$,

$$\partial_p^2 \Pi(p)\Big|_{p=0} = 0 \iff \delta_Z = -\frac{1}{2}\,\partial_p^2 \Pi_\Lambda(p)\Big|_{p=0}$$

PIONS

Let's do this directly for pions. And with dim. reg., as it might be easier.



$$= \Pi_{ij}(p) = h^2 \sum\!\!\!\!\!\!\!\int_q \mathrm{tr}\left[i\gamma_5 T_i G_q(p+q)\, i\gamma_5 T_j G_q(q)\right]$$

$$= -\frac{N_c}{2} h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \bar{G}_q(p+q)\,\bar{G}_q(q)\, \mathrm{Tr}\left[\gamma_5(-i(\not{p}+\not{q})+m)\gamma_5(-i\not{q}+m)\right]$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \left[q(q+p)+m^2\right]\bar{G}_q(p+q)\,\bar{G}_q(q)$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \left[q\cdot p + \underbrace{q^2+m^2}_{\bar{G}_q^{-1}(q)}\right]\bar{G}_q(p+q)\,\bar{G}_q(q)$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \left[\bar{G}_q(q+p) + q\cdot p\, \bar{G}_q(q+p)\,\bar{G}_q(q)\right]$$

use: $\quad G_q^{-1}(p+q) = (p+q)^2+m^2 = p^2+2p\cdot q+\underbrace{q^2+m^2}_{G_q^{-1}(q)}$

$\Leftrightarrow \quad p\cdot q = \frac{1}{2}G_q^{-1}(p+q) - \frac{1}{2}G_q^{-1}(q) - \frac{1}{2}p^2$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \left\{ \bar{G}_q(q+p) + \frac{1}{2}\left[G_q^{-1}(q+p) - G_q^{-1}(q) - p^2\right]\bar{G}_q(q+p)\,\bar{G}_q(q)\right\}$$

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \left[\underbrace{\bar{G}_q(q+p)}_{q\to q+p} + \frac{1}{2}\bar{G}_q(q) - \frac{1}{2}\underbrace{\bar{G}_q(p+q)}_{q\to q+p} - \frac{1}{2}p^2\,\bar{G}_q(q+p)\,\bar{G}_q(q)\right]$$

integration by parts:

$$= -2N_c h^2 \delta_{ij} \sum\!\!\!\!\!\!\!\int_q \left[\bar{G}_q(q+p) - \frac{1}{2}p^2\,\bar{G}_q(q+p)\,\bar{G}_q(q)\right]$$

$$= -2N_c h^2 \delta_{ij}\left[I_1 - \frac{1}{2}p^2 I_2(p)\right].$$

From p. 11 we get

$$I_1 = \sum\!\!\!\!\!\!\int_q \frac{1}{(q_0-i\mu)^2+E_q^2} = \int_{\vec{q}} \frac{1}{2E_q}\Big[1-\underbrace{n_F(E_q-\mu)}_{\to\Theta(\mu-E_q)}-\underbrace{n_F(E_q+\mu)}_{\to 0}\Big] \quad \text{for } T\to 0$$

On p. 11 we considered the whole $T=0$ contribution for the regularization,

$$I_1(T=0) = \frac{4\pi}{(2\pi)^3}\frac{1}{2}\int dq\,\frac{q^2}{E_q}\,\Theta(E_q-\mu) = \frac{1}{4\pi^2}\int_{p_F}^{\infty} dq\,\frac{q^2}{E_q}$$

$$E_q \geq \mu \Leftrightarrow q^2+m^2 \geq \mu^2 \Leftrightarrow q^2 \geq \mu^2-m^2 \Leftrightarrow q^2 \geq p_F^2$$

This is the combination of a finite integral from 0 to p_F from $n_F(E_q-\mu)$ and the divergent integral from 0 to ∞ from the 1. For dim. reg. we want to split these contributions,

$$I_1(T=0) = \int\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q} - \frac{1}{4\pi^2}\int_0^{p_F} dq\,\frac{q^2}{E_q}$$

$$= -\frac{1}{16\pi^2}\Big[2\mu p_F + m^2\ln\Big(\frac{\overbrace{\mu-p_F}^{p_F\leq\mu\text{, so no problem here}}}{\mu+p_F}\Big)\Big] + \Delta I_1^{\varepsilon}$$

$(\star\star)_{96}$ in my QFT script

$$\Delta I_1^{\varepsilon} = \frac{1}{2}\Lambda^{3-d}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2+m^2)^{1/2}} = \frac{1}{2}\frac{(m^2)^{3/2}}{(4\pi)^2}\Lambda^{3-d}\left(\frac{4\pi}{m^2}\right)^{2-\frac{d}{2}}\frac{\Gamma(\frac{1}{2}-\frac{d}{2})}{\underbrace{\Gamma(1/2)}_{=\sqrt{\pi}}}$$

$d=3-2\varepsilon$ (so that the integral converges for $\varepsilon\geq 1$)

$$= \frac{1}{2}\frac{m^3}{(4\pi)^2}\Lambda^{2\varepsilon}\left(\frac{4\pi}{m^2}\right)^{2-\frac{3}{2}+\varepsilon}\frac{\Gamma(\frac{1}{2}-\frac{3}{2}+\varepsilon)}{\sqrt{\pi}}$$

$$= \frac{1}{2\sqrt{\pi}}\frac{m^3}{(4\pi)^2}\Lambda^{2\varepsilon}\left(\frac{4\pi}{m^2}\right)^{1/2}\left(\frac{4\pi}{m^2}\right)^{\varepsilon}\Gamma(-1+\varepsilon)$$

$$= \frac{m^2}{(4\pi)^2}\left(\frac{4\pi\Lambda^2}{m^2}\right)^{\varepsilon}\Gamma(-1+\varepsilon)$$

p. 97 in my QFT script

$$= \frac{m^2}{(4\pi)^2}\left[-\frac{1}{\varepsilon}+\gamma_E-1+\ln\left(\frac{m^2}{4\pi\Lambda^2}\right)\right]$$

$$\Leftrightarrow\ \boxed{I_1^{\varepsilon}(T=0) = \frac{m^2}{16\pi^2}\left[-\frac{1}{\varepsilon}+\gamma_E-1+\ln\left(\frac{m^2}{4\pi\Lambda^2}\right)\right] - \frac{1}{16\pi^2}\left[2\mu p_F + m^2\ln\left(\frac{\mu-p_F}{\mu+p_F}\right)\right]}$$

Now onto I_2 :

$$I_2 = \sum\!\!\!\!\!\!\int_q \frac{1}{[(q_0-i\mu)^2+E_q^2][(q_0+p_0-i\mu)^2+E_{q+p}^2]}$$

$T\to 0,\ p.1$

$$= \int_{\vec q} \frac{1}{4E_qE_{q+p}}\left\{\frac{1}{ip_0+(E_q+E_{q+p})} - \frac{1}{ip_0-(E_q+E_{q+p})}\right\}$$

$$+ \int_{\vec q} \frac{1}{4E_qE_{q+p}}\left\{\frac{\Theta(\mu-E_{q+p})}{ip_0-(E_q+E_{q+p})} - \frac{\Theta(\mu-E_q)}{ip_0+(E_q+E_{q+p})}\right.$$

$$\left. + \frac{\Theta(\mu-E_q)-\Theta(\mu-E_{q+p})}{ip_0+(E_q-E_{q+p})}\right\}$$

$q\to q-p$

$$+\frac{\Theta(\mu-E_q)}{-ip_0-E_{q-p}+E_q}$$

$$= \frac{1}{4}\int_{\vec q} \frac{1}{E_qE_{q+p}}\left(\frac{1}{E_{q+p}+E_q+ip_0} + \frac{1}{E_{q+p}+E_q-ip_0}\right)$$

$$+\frac{1}{4}\int_{\vec q}\frac{1}{E_q}\Theta(\mu-E_q)\left\{\frac{1}{E_{q+p}}\left[-\frac{1}{E_{q+p}+E_q+ip_0}+\frac{1}{-E_{q+p}+E_q+ip_0}\right]\right.$$

$$\left.+\frac{1}{E_{q-p}}\left[-\frac{1}{E_{q-p}+E_q-ip_0}+\frac{1}{-E_{q-p}+E_q-ip_0}\right]\right\}$$

$$= \Delta I_2^{\varepsilon}(p_0,\vec p) \leftarrow \text{needs regularization}$$

$$+\frac{\pi}{2(2\pi)^3}\int_0^{p_F}dq\,\frac{q^2}{E_q}\int_{-1}^{1}dx\left\{\frac{1}{E_{q+p}}\left[-\frac{1}{E_{q+p}+E_q+ip_0}+\frac{1}{-E_{q+p}+E_q+ip_0}\right]\right.$$

$$\left.+\frac{1}{E_{q-p}}\left[-\frac{1}{E_{q-p}+E_q-ip_0}+\frac{1}{-E_{q-p}+E_q-ip_0}\right]\right\}$$

$p_0=0\ ,\ p,q\equiv|\vec p|,|\vec q|$

$$= \Delta I_2^{\varepsilon}(0,\vec p)$$

$$+\frac{1}{16\pi^2}\int_0^{p_F}dq\,\frac{q^2}{E_q}\left\{\overbrace{\frac{2}{pq}\ln\left|\frac{\sqrt{(q-p)^2+m^2}-\sqrt{q^2+m^2}}{\sqrt{(q+p)^2+m^2}-\sqrt{q^2+m^2}}\right|}^{\text{see next page}}\right.$$

$$\left.-\frac{2}{pq}\ln\left|\frac{\sqrt{(q+p)^2+m^2}+\sqrt{q^2+m^2}}{\sqrt{(q-p)^2+m^2}+\sqrt{q^2+m^2}}\right|\right\}$$

$$= \Delta I_2^{\varepsilon}(0,\vec p)$$

$$+\frac{1}{16\pi^2}\int_0^{p_F}dq\,\frac{q^2}{E_q}\frac{2}{pq}\ln\left|\frac{p-2q}{p+2q}\right|$$

$$\frac{B-E}{A-E}\,\frac{B+E}{A+E} = \frac{B^2-E^2}{A^2-E^2} = \frac{(q-p)^2-q^2}{(q+p)^2-q^2} = \frac{p(p-2q)}{p(p+2q)}$$

x-integral : $\qquad A \equiv p^2+q^2+m^2, \; B \equiv 2pq$

① $\displaystyle\int dx \frac{1}{\sqrt{p^2+q^2+2pqx+m^2}} \frac{1}{E_q - \sqrt{p^2+q^2+2pqx+m^2}} = \int dx \frac{1}{\sqrt{A+Bx}} \frac{1}{E-\sqrt{A+Bx}}$

$\triangleright \; z = \sqrt{A+Bx} \Rightarrow \frac{dz}{dx} = \frac{B}{2z} \Leftrightarrow dx = dz \frac{2z}{B}$,

- $\displaystyle\int dz \frac{2z}{B} \cdot \frac{1}{z} = \frac{2}{B} z$

- $\displaystyle\frac{2}{B}\int dz\, z \frac{1}{z} \frac{1}{E-z} = -\frac{2}{B}\int dz \frac{1}{z-E} \overset{w \equiv z-E \,\Rightarrow\, dw = dz}{=} -\frac{2}{B}\int dw \frac{1}{w} = -\frac{2}{B}\ln|w|$

$\displaystyle = -\frac{2}{B}\ln|z-E| = -\frac{2}{B}\ln\left|\sqrt{A+Bx}-E\right|$

$\displaystyle = -\frac{1}{pq}\ln\left|\sqrt{p^2+q^2+2pqx+m^2} - \sqrt{q^2+m^2}\right|$

$\displaystyle \xrightarrow{\int_{-1}^{1} dx} = -\frac{1}{pq}\left(\ln\left|\sqrt{(q+p)^2+m^2} - \sqrt{q^2+m^2}\right|\right) - \ln\left|\sqrt{(q-p)^2+m^2} - \sqrt{q^2+m^2}\right|\Big)$

$\displaystyle = \frac{1}{pq}\ln\left|\frac{\sqrt{(q-p)^2+m^2} - \sqrt{q^2+m^2}}{\sqrt{(q+p)^2+m^2} - \sqrt{q^2+m^2}}\right| \qquad (*)$

② $\displaystyle\int_{-1}^{1} dx \frac{1}{\sqrt{p^2+q^2-2pqx+m^2}} \frac{1}{E_q - \sqrt{p^2+q^2-2pqx+m^2}} \overset{B \to -B}{=} \frac{2}{B}\ln\left|\sqrt{A-Bx}-E\right|\Big|_{-1}^{1}$

$= (*)$

③ $\displaystyle\int_{-1}^{1} dx \frac{1}{\sqrt{p^2+q^2+2pqx+m^2}} \frac{1}{E_q + \sqrt{p^2+q^2+2pqx+m^2}} = \frac{2}{B}\int dz\, z \frac{1}{z}\frac{1}{z+E} \overset{\bar{w} = z+E}{=} \frac{2}{B}\int d\bar{w} \frac{1}{\bar{w}}$

$\displaystyle = \frac{2}{B}\ln|\bar{w}| = \frac{2}{B}\ln\left|\sqrt{A+Bx}+E\right|$

$\displaystyle = \frac{1}{pq}\ln\left|\frac{\sqrt{(q+p)^2+m^2} + \sqrt{q^2+m^2}}{\sqrt{(q-p)^2+m^2} + \sqrt{q^2+m^2}}\right| \qquad (**)$

④ $\displaystyle\int_{-1}^{1} dx \frac{1}{\sqrt{p^2+q^2-2pqx+m^2}} \frac{1}{E_q + \sqrt{p^2+q^2-2pqx+m^2}} = (**)$

Evaluate

$$\int_0^{p_F} dq\, \frac{2q}{pE_q} \ln\left|\frac{p-2q}{p+2q}\right| = \begin{cases} p_F > \frac{p}{2}: & \int_0^{p/2} dq\, \frac{q}{E_q} \ln\left(\frac{p-2q}{p+2q}\right) + \int_{p/2}^{p_F} dq\, \frac{q}{E_q} \ln\left(\frac{2q-p}{2q+p}\right) \\ p_F < \frac{p}{2}: & \int_0^{p_F} dq\, \frac{q}{E_q} \ln\left(\frac{p-2q}{p+2q}\right) \end{cases}$$

$$\triangleright \int dq\, \frac{2q}{pE_q} \ln \frac{p-2q}{p+2q} = \frac{1}{p}\Big\{ 2E_q \ln\left(\frac{p-2q}{p+2q}\right) - 2p\ln(q+E_q) + \sqrt{p^2+4m^2}\left[\ln\left(\frac{p+2q}{p-2q}\right) + \ln\left(\frac{2m^2+pq+\sqrt{p^2+4m^2}\,E_q}{2m^2-pq+\sqrt{p^2+4m^2}\,E_q}\right)\right]\Big\}$$

$E_q \to \sqrt{\frac{p^2}{4}+m^2} = \frac{1}{2}\sqrt{p^2+4m^2}$

$$\xrightarrow{q\to p/2} \frac{1}{p}\Big\{ \cancel{\sqrt{p^2+4m^2}\ln\left(\frac{p-p}{p+p}\right) + \sqrt{p^2+4m^2}\ln\left(\frac{p+p}{p-p}\right)} - 2p \underbrace{\ln\left(\frac{p}{2}+\frac{1}{2}\sqrt{p^2+4m^2}\right)}_{\ln\left(p+\sqrt{p^2+4m^2}\right) - \ln 2}$$

$$+ \sqrt{p^2+4m^2}\, \underbrace{\ln\left[\frac{\frac{1}{2}(p^2+4m^2)+\frac{1}{2}(p^2+4m^2)}{\frac{1}{2}(-p^2+4m^2)+\frac{1}{2}(p^2+4m^2)}\right]}_{}$$

$$= \ln\left(\frac{2(p^2+4m^2)}{8m^2}\right) = \ln\frac{p^2+4m^2}{4m^2}$$

cancels!

$$\xrightarrow{q\to 0} \frac{1}{p}\left\{-2p\ln m\right\} = -2\ln m$$

need to verify numerically; Mathematica screws this up analytically!

$$\triangleright \int dq\, \frac{2q}{pE_q} \ln\left(\frac{2q-p}{2q+p}\right) = \frac{1}{p}\Big\{ 2E_q \ln\frac{2q-p}{2q+p} - 2p\ln(q+E_q) + \sqrt{p^2+4m^2}\left[\ln\frac{2q+p}{2q-p} + \ln\frac{2m^2+pq+\sqrt{p^2+4m^2}\,E_q}{2m^2-pq+\sqrt{p^2+4m^2}\cdot E_q}\right]\Big\}$$

$$\xrightarrow{q\to \frac{p}{2}} \frac{1}{p}\Big\{ \cancel{\sqrt{p^2+4m^2}\left[\ln\frac{p-p}{p+p} + \ln\frac{p+p}{p-p}\right]} - 2p\ln\left(\frac{p}{2}+\frac{\sqrt{p^2+4m^2}}{2}\right) + \ln\frac{(p^2+4m^2)+(p^2+4m^2)}{(-p^2+4m^2)+(p^2+4m^2)}\Big\}$$

$$\xrightarrow{q\to p_F} \frac{1}{p}\Big\{ 2\mu \ln\frac{2p_F-p}{2p_F+p} - 2p\ln(p_F+\mu) + \sqrt{p^2+4m^2}\left[\ln\frac{p+2p_F}{p-2p_F} + \ln\frac{2m^2+pp_F+\sqrt{p^2+4m^2}\cdot\mu}{2m^2-pp_F+\sqrt{p^2+4m^2}\cdot\mu}\right]\Big\}$$

$E_q \to \sqrt{p_F^2+m^2} = \mu$

$$\Rightarrow \int_0^{p_F} dq\, \frac{2q}{pE_q} \ln\left|\frac{p-2q}{p+2q}\right|$$

$$= \begin{cases} p_F > \frac{p}{2}: & \frac{1}{p}\Big\{ 2\mu\ln\frac{2p_F-p}{2p_F+p} - 2p\ln\frac{p_F+\mu}{m} + \sqrt{p^2+4m^2}\left[\ln\frac{2p_F+p}{2p_F-p} + \ln\frac{2m^2+pp_F+\sqrt{p^2+4m^2}\cdot\mu}{2m^2-pp_F+\sqrt{p^2+4m^2}\cdot\mu}\right]\Big\} \\ p_F < \frac{p}{2}: & \frac{1}{p}\Big\{ 2\mu\ln\frac{p-2p_F}{p+2p_F} - 2p\ln\frac{p_F+\mu}{m} + \sqrt{p^2+4m^2}\left[\ln\frac{p+2p_F}{p-2p_F} + \ln\frac{2m^2+pp_F+\sqrt{p^2+4m^2}\cdot\mu}{2m^2-pp_F+\sqrt{p^2+4m^2}\cdot\mu}\right]\Big\} \end{cases}$$

$$\Rightarrow I_2 = \Delta I_2^{\varepsilon}(0,\vec{p})$$

$$+ \frac{1}{16\pi^2}\frac{1}{p}\left[\left(\sqrt{p^2+4m^2}-2\mu\right)\ln\left|\frac{p+2p_F}{p-2p_F}\right| - 2p\ln\left(\frac{p_F+\mu}{m}\right) + \sqrt{p^2+4m^2}\,\ln\left(\frac{2m^2+pp_F+\mu\sqrt{p^2+4m^2}}{2m^2-pp_F+\mu\sqrt{p^2+4m^2}}\right)\right]$$

checked numerically that this is correct!

old, incorrect result:

$$+\frac{1}{16\pi^2}\frac{1}{p}\Bigg\{2\sqrt{p(p+2p_F)+\mu^2} - 2\sqrt{p(p-2p_F)+\mu^2} + \sqrt{p^2+4m^2}\,\ln\left(\frac{p-2p_F}{p+2p_F}\right)$$

$$+\sqrt{p^2+4m^2}\,\ln\left(\frac{2m^2+p(p+p_F)+\sqrt{p^2+4m^2}\sqrt{p(p+2p_F)+\mu^2}}{2m^2+p(p-p_F)+\sqrt{p^2+4m^2}\sqrt{p(p-2p_F)+\mu^2}}\right)$$

$$-p\left[\ln\left(p_F+p+\sqrt{p(p+2p_F)+\mu^2}\right)+\ln\left(p_F-p+\sqrt{p(p-2p_F)+\mu^2}\right)\right]$$

$$-2\mu\left[\ln\left(\frac{\mu+\sqrt{p(p+2p_F)+\mu^2}}{\mu+\sqrt{p(p-2p_F)+\mu^2}}\right)-\ln\left(\frac{\mu-\sqrt{p(p+2p_F)+\mu^2}}{\mu-\sqrt{p(p-2p_F)+\mu^2}}\right)\right]\Bigg\}$$

$$= \Delta I_2^{\varepsilon}(0,\vec{p})$$

$$+ \frac{1}{16\pi^2}\frac{1}{p}\Bigg[\left(\sqrt{p^2+4m^2}-2\mu\right)\ln\left|\frac{p+2p_F}{p-2p_F}\right| - 2p\ln\left(\frac{p_F+\mu}{m}\right)$$

$$+\sqrt{p^2+4m^2}\,\ln\left(\frac{2m^2+pp_F+\mu\sqrt{p^2+4m^2}}{2m^2-pp_F+\mu\sqrt{p^2+4m^2}}\right)\Bigg]$$

The regular part disagrees with the regular piece on p. 9. For the divergent piece we get

$$\Delta I_2^{\varepsilon}(0,\vec{p}) = \frac{1}{2}\int_q \frac{1}{E_q E_{q+p}}\,\frac{1}{E_{q+p}+E_q}$$

$$= \int\frac{d^4q}{(2\pi)^4}\frac{1}{\left(q_0^2+E_q^2\right)\left(q_0^2+E_{p+q}^2\right)}$$

It might be easier to work with $p_0 \neq 0$ and to set it to zero afterwards as then the integral is standard, (p, q are 4-momenta now!)

$$\Delta I_2^{\varepsilon}(p) = \int\frac{d^4q}{(2\pi)^4}\frac{1}{[q^2+m^2][(p+q)^2+m^2]} = \int_q \frac{1}{AB} = \int_0^1 dx\int_q\frac{1}{[xA+(1-x)B]^2}$$

Introduce Feynman parameter x, $A \equiv (p+q)^2+m^2$, $B \equiv q^2+m^2$

$$xA+(1-x)B = xp^2+2xp\cdot q+\cancel{xq^2}+\cancel{xm^2}+q^2-\cancel{xq^2}+m^2-\cancel{xm^2}$$

$$= q^2+2xp\cdot q+\left(x^2p^2-x^2p^2\right)+xp^2+m^2$$

$$= \underbrace{(q+xp)^2}_{\equiv\, l^2}+\underbrace{x(1-x)p^2+m^2}_{\equiv\, M^2}$$

$$d^4l = d^4q$$

$$= \int_0^1 dx \int_\ell \frac{1}{(\ell^2+M^2)^2} = \int_0^1 dx\, \Lambda^{4-d} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2+M^2)^2}$$

$$= \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi\Lambda^2}{M^2}\right)^{2-\frac{d}{2}} \frac{\Gamma(2-\frac{d}{2})}{\underbrace{\Gamma(2)}_{=1}}$$

$d=4-2\varepsilon$

$$= \int_0^1 dx \frac{1}{(4\pi)^2} \left(\frac{4\pi\Lambda^2}{M^2}\right)^{\varepsilon} \Gamma(\varepsilon)$$

$$= \frac{1}{16\pi^2} \int_0^1 dx \left[1 + \varepsilon \ln\left(\frac{4\pi\Lambda^2}{M^2}\right) + \mathcal{O}(\varepsilon^2)\right] \left(\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)\right)$$

$\varepsilon \to 0$

$$= \frac{1}{16\pi^2} \int_0^1 dx \left[\frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\Lambda^2}{M^2}\right)\right]$$

$$= \frac{1}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + 2 + \ln\left(\frac{4\pi\Lambda^2}{m^2}\right) + \frac{\sqrt{p^2+4m^2}}{\sqrt{p^2}} \ln\left(\frac{\sqrt{p^2+4m^2}-\sqrt{p^2}}{\sqrt{p^2+4m^2}+\sqrt{p^2}}\right)\right]$$

Hence for $p_0 = 0$ we have (with $p = \sqrt{\vec{p}^2}$ now)

$$\Delta I_2^\varepsilon(0,\vec{p}) = \frac{1}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + 2 + \ln\left(\frac{4\pi\Lambda^2}{m^2}\right) + \frac{1}{p}\sqrt{p^2+4m^2}\, \ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)\right]$$

Note that the last term is identical to the last term of I_1 at the bottom of page 9. Hence, the regularization scheme independent parts are identical, as they should.

Putting all this together, we get $\left(\Pi = -2N_c h^2 \delta_{ij}\left[I_1 - \frac{1}{2}p^2 I_2(p)\right]\right)$

$$\Pi_{ij}(0,\vec{p}) = \frac{N_c h^2}{8\pi^2}\delta_{ij} \Big\{ \tfrac{1}{2}(p^2+2m^2)\left[\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{m^2}{4\pi\Lambda^2}\right)\right] + \tfrac{1}{2}p^2$$

$2N_c h^2 \delta_{ij}\left(-I_1 + \frac{1}{2}p^2 I_2\right)$

$$+ \tfrac{1}{2}p\sqrt{p^2+4m^2}\, \ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)$$

vacuum

$$+ 2\mu p_F + m^2 \ln\left(\frac{\mu - p_F}{\mu + p_F}\right)$$

$$+ \tfrac{1}{2}p\Big[\left(\sqrt{p^2+4m^2} - 2\mu\right)\ln\left|\frac{p+2p_F}{p-2p_F}\right| - 2p\ln\left(\frac{p_F+\mu}{m}\right)$$

$$+ \sqrt{p^2+4m^2}\, \ln\left(\frac{2m^2 + pp_F + \mu\sqrt{p^2+4m^2}}{2m^2 - pp_F + \mu\sqrt{p^2+4m^2}}\right)\Big]\Big\}$$

$$+ \delta_{ij}\left(\delta_Z p^2 + \delta_m\right)$$

counter terms

We use some sort of minimal subtraction. To this end, choose

$$\delta_Z p^2 + \delta_m \stackrel{!}{=} -\frac{Nch^2}{8\pi^2}\left[\frac{1}{2}(p^2+2m^2)\left(\frac{1}{\varepsilon}-\gamma_E+1+\ln(4\pi)\right)+\frac{1}{2}p^2\right]$$

$$= \underbrace{-\frac{Nch^2}{16\pi^2}\left[\frac{1}{\varepsilon}-\gamma_E+2+\ln\left(\frac{4\pi\Lambda}{\bar{\Lambda}}\right)\right]}_{=\delta_Z} p^2 \underbrace{- \frac{h^2}{8\pi^2} m^2\left[\frac{1}{\varepsilon}-\gamma_E+1+\ln\left(\frac{4\pi\Lambda}{\bar{\Lambda}}\right)\right]}_{\delta_m}$$

Note that this formally replaces the scale-parameter Λ with the renormalization scale $\bar{\Lambda}$. We will henceforth use $\Lambda = \bar{\Lambda}$.

The renormalized self-energy is then:

$$\boxed{\begin{aligned}\Pi_{ij}(0,\vec{p}) = \frac{Nch^2}{8\pi^2}\delta_{ij}\Bigg\{&-\frac{1}{2}(p^2+2m^2)\ln\left(\frac{m^2}{\Lambda^2}\right)+\frac{1}{2}p\sqrt{p^2+4m^2}\,\ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)\\ &+2\mu p_F + m^2\ln\left(\frac{\mu-p_F}{\mu+p_F}\right)\\ &+\frac{1}{2}p\Bigg[\left(\sqrt{p^2+4m^2}-2\mu\right)\ln\left|\frac{p+2p_F}{p-2p_F}\right| - 2p\ln\left(\frac{p_F+\mu}{m}\right)\\ &+\sqrt{p^2+4m^2}\,\ln\left(\frac{2m^2+pp_F+\mu\sqrt{p^2+4m^2}}{2m^2-pp_F+\mu\sqrt{p^2+4m^2}}\right)\Bigg]\Bigg\}\end{aligned}}$$

$$\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}f(p)$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dp\,p^2\int_{-1}^{1}dz\,e^{iprz}f(p)$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dp\,p^2\frac{1}{ipr}2i\sin(pr)f(p)$$

$$= \frac{1}{2\pi^2}\frac{1}{r}\int_0^\infty dp\,p\sin(pr)f(p)$$

the following discussion is based on an old, wrong result!

To simplify things, consider the ultrarelativistic limit $m \to 0$. This limit is somewhat subtle due to the $\ln m^2$-term. There is a cancellation:

$$\lim_{m\to 0}\left[-\frac{1}{2}p^2\ln\left(\frac{m^2}{\Lambda^2}\right)+\frac{1}{2}p^2\sqrt{1+\frac{4m^2}{p^2}}\ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right)\right]=-\frac{1}{2}p^2\ln\left(\frac{p^2}{\Lambda^2}\right)$$

Also

$\triangleright$ $\lim_{m\to 0} p_F = \mu \implies p(p\pm 2p_F)+\mu^2 \to p^2 \pm 2p_F p + p_F^2 = (p\pm p_F)^2$

$\triangleright$ $\ln\left(\frac{2m^2+p(p+p_F)+\sqrt{p^2+4m^2}\sqrt{p(p+2p_F)+\mu^2}}{2m^2+p(p-p_F)+\sqrt{p^2+4m^2}\sqrt{p(p-2p_F)+\mu^2}}\right) \to \ln\left(\frac{2(p+p_F)}{|p-p_F|+(p-p_F)}\right)$

$\triangleright$ $-\ln\left(p_F+p+\sqrt{p(p+2p_F)+\mu^2}\right)-\ln\left(p_F-p+\sqrt{p(p-2p_F)+\mu^2}\right)$

$$\to -\ln\left[2(p+p_F)\left(|p-p_F|-(p-p_F)\right)\right]$$

Ok, there might be an IR divergence here, so $m\to$ isn't such a great idea...

It might make things easier if we fix $\Pi(0,0)=0$, so the curvature mass is not changed by Π. Since

$$\lim_{p\to 0} p^2 I_2(p) = 0,$$

this cancels I_1 entirely. However, there are μ-dependent contributions that we might want to keep? This contribution gives a μ-dependent correction to the curvature mass of the pion

$$\Delta m_\pi^c = \frac{N_c h^2}{8\pi^2}\left[2\mu p_F + m^2\ln\left(\frac{\mu-p_F}{\mu+p_F}\right)\right]$$

THE STATIC PROPAGATOR

The pion propagator in the static limit is

$$G_\pi(0,\vec{p}) = \frac{1}{\vec{p}^2 + m_\pi^2 + \Pi(0,\vec{p})}$$

$\equiv \Pi_{11}(0,\vec{p})$

doesn't matter due to isospin symmetry.

We want to look at this in position space,

$$G_\pi(0,r) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} \frac{1}{\vec{p}^2 + m_\pi^2 + \Pi(0,\vec{p})} = \frac{1}{4\pi^2} \int dp \int_{-1}^{1} dx \frac{p^2 e^{iprx}}{p^2 + m_\pi^2 + \Pi(0,p)}$$

$$= -\frac{i}{(2\pi)^2 r} \int_0^\infty dp \frac{p\left(e^{ipr} - e^{-ipr}\right)}{p^2 + m_\pi^2 + \Pi(0,p)}$$

$\Pi(0,p) = \Pi(0,-p)$

$$= -\frac{i}{4\pi^2 r} \left[\int_0^\infty dp \frac{p e^{ipr}}{p^2 + m_\pi^2 + \Pi(0,p)} + \int_{-\infty}^{0} dp \frac{p e^{ipr}}{p^2 + m_\pi^2 + \Pi(0,p)} \right.$$

$$= -\frac{i}{4\pi^2 r} \int_{-\infty}^{\infty} dp \frac{p e^{ipr}}{p^2 + m_\pi^2 + \Pi(0,p)} \quad = \frac{1}{2\pi^2 r} \int_0^\infty dp \frac{p \sin(pr)}{p^2 + m_\pi^2 + \Pi(0,p)}$$

To carry out this integral, we need to know the analytic structure of the propagator

▷ there are screening poles at $p = \pm i m_s^2$,

$$-m_s^2 + m_\pi^2 + \Pi(0, i m_s^2) \stackrel{!}{=} 0$$

These seem to be impossible to find analytically. So let's just assume they are there. They give rise to the well-known contribution

$$\int_{-\infty}^{\infty} dp \frac{p e^{ipr}}{p^2 + m_\pi^2 + \Pi(0,p)} = i\pi e^{-m_s r} + (\text{cut contributions})$$

▷ there are also cuts. Two of those correspond to the decay into two quarks for

$$p^2 + (2m)^2 \leq 0 ,$$

so they are along the imaginary axis. This contribution is not relevant for us now.

▷ finally, there are branch points at

$$p = \pm 2p_F .$$

This is true for almost all logs:

- $\ln\left(\frac{p-2p_F}{p+2p_F}\right) \longrightarrow p = \pm 2p_F$

- $\ln\left(\frac{\sqrt{p^2+4m^2}-p}{\sqrt{p^2+4m^2}+p}\right) \longrightarrow$ no branch point ($\equiv$ BP)

- $\ln\left(\frac{2m^2+p(p+p_F)+\sqrt{p^2+4m^2}\sqrt{p(p+2p_F)+\mu^2}}{2m^2+p(p-p_F)+\sqrt{p^2+4m^2}\sqrt{p(p-2p_F)+\mu^2}}\right) \longrightarrow$ no BP

  $$\text{numerator} \xrightarrow[\text{at } p=-2p_F]{} 2m^2 - 2p_F(-2p_F+p_F) + 2\underbrace{\sqrt{p_F^2+m^2}}_{\mu^2}\cdot\mu = 2m^2+2\mu^2+2p_F^2 = 4\mu^2$$

  $$\text{denominator} \xrightarrow[\text{at } p=-2p_F]{} -4m^2+6\mu^2+\sqrt{9\mu^2-8m^2}$$

  and the other way around for $p=+2p_F$. There are no other potential zeros, so this log is finite.

- $\ln\left(p_F+p+\sqrt{p(p+2p_F)+\mu^2}\right) \longrightarrow$ no BP

- $\ln\left(p_F-p+\sqrt{p(p-2p_F)+\mu^2}\right) \longrightarrow$ no BP

- $\ln\left(\frac{\mu+\sqrt{p(p+2p_F)+\mu^2}}{\mu+\sqrt{p(p-2p_F)+\mu^2}}\right) \longrightarrow$ no BP

- $\ln\left(\frac{\mu-\sqrt{p(p+2p_F)+\mu^2}}{\mu-\sqrt{p(p-2p_F)+\mu^2}}\right) \longrightarrow p = \pm 2p_F$

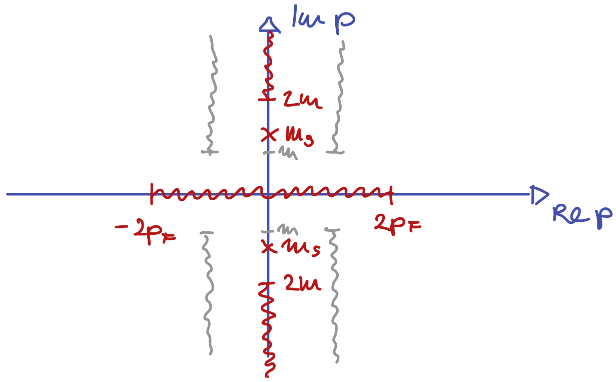
Note that the last BP cancels with the first, so Π is finite.

Now usually the cuts of the log are set to be along the negative real axis. Then we get cuts at

$$\left.\begin{array}{l} \bullet\ \ln(p-2p_F) : [-\infty, 2p_F] \\ \bullet\ \ln(p+2p_F) : [-\infty, -2p_F] \end{array}\right\} \ln\left(\frac{p-2p_F}{p+2p_F}\right) : [-2p_F, 2p_F]$$

$$\left.\begin{array}{l} \bullet\ \ln\left(\mu-\sqrt{p(p-2p_F)+\mu^2}\right) : [2p_F,\infty] \cup [-\infty, 0] \\ \bullet\ \ln\left(\mu-\sqrt{p(p+2p_F)+\mu^2}\right) : [-\infty,-2p_F] \cup [0,\infty] \end{array}\right\} \ln\left(\frac{\mu-\sqrt{p(p-2p_F)+\mu^2}}{\mu-\sqrt{p(p+2p_F)+\mu^2}}\right) : [-2p_F, 2p_F]$$

So the following analytic structure emerges for $G_n(0,p)$:



Note that there are additional potential cuts from

$$\sqrt{p(p(\pm)2p_F)+\mu^2},$$

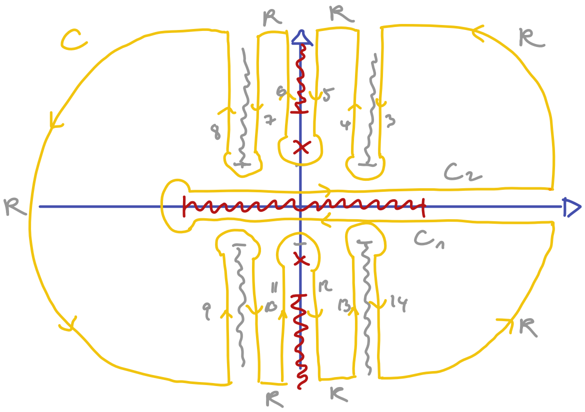
with BPs at

$$p = (\pm) p_F \pm \sqrt{p_F^2-\mu^2} \in \mathbb{C} \quad (\text{since } p_F \leq \mu)$$

$$\begin{aligned} &\left(p_F + i\alpha\sqrt{\mu^2-p_F^2}\right)\left(p_F + i\alpha\sqrt{\mu^2-p_F^2} - 2p_F\right) + \mu^2 \\ &= \left(p_F + i\alpha\sqrt{\mu^2-p_F^2}\right)\left(-p_F + i\alpha\sqrt{\mu^2-p_F^2}\right) + \mu^2 \\ &= -p_F^2 - \alpha^2(\mu^2-p_F^2) + \mu^2 = -(1-\alpha^2)p_F^2 + (1-\alpha^2)\mu^2 \\ &\underline{\leq 0 \ \forall \alpha^2 \geq 1} \end{aligned}$$

$\rightarrow$ cuts go parallel to the imaginary axis (gray in Fig. above)

If we choose the following contour,



there are no singularities inside, so

$$0 = \oint_C dp\, p\, e^{ipr} G_n(p) = \Big(\int_R + \int_{C_1} + \int_{C_2} + \underbrace{\sum_{n=3}^{14} \int_{C_n}}_{\text{ignore these for now}} \Big) dp\, p\, e^{ipr} G_n(p)$$

The "R"-sections of the integral should vanish for $\operatorname{Im} p \to \infty$, but it doesn't in the lower half plane?

Maybe we should redefine the branch cut locations so this makes more sense. Let's move them parallel to the imaginary axis (the other cuts come from square roots, so they won't change) and give them a little kick into the complex plane. How does this work?

Consider $\ln(p-2p_F)$, which has a BP at $p=2p_F$. In general, we have

$$\ln z = \ln|z| + i\arg(z),$$

so $\ln z$ is multivalued since $z_1 = |z|e^{i\varphi} = z_2 = |z|e^{i(\varphi+n2\pi)}$, $n\in\mathbb{Z}$, but

$$\ln(z_1) = \ln|z| + i\varphi \neq \ln(z_2) = \ln|z| + i\varphi + in2\pi \text{ for } n\neq 0$$

putting the cut on the negative real axis

The principal branch is where $\arg(z) \in (-\pi,\pi)$. But we might as well choose something else, e.g.,

$$\arg(z) \in \left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) \longrightarrow \text{cut along the positive imaginary axis}$$

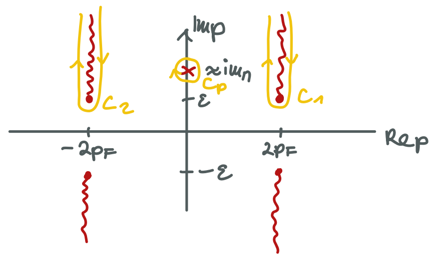
we can freely make this choice for each log, as long as we keep track of the individual definitions.

Try again with the Fourier trafo, using the result on p.19, and following [Kapusta, Toimela (1988)], so we choose the branch cuts of the log to lie at $p = \pm 2p_F + iy$

Expand at $p = \pm 2p_F + iy \pm \varepsilon$ around $y = 0$; $\varepsilon \to 0$ to pick-up contribution from the cuts.

$$G_\eta(0,r) = -\frac{i}{4\pi^2 r}\int_{-\infty}^{\infty} dp\, \frac{p\, e^{ipr}}{p^2 + m_\eta^2 + \Pi(0,p)}$$

$$= -\frac{i}{4\pi r}\left(\oint_{C_p} + \oint_{C_1} + \oint_{C_2}\right)\frac{p\, e^{ipr}}{p^2 + m_\eta^2 + \Pi(0,p)}$$



$$y = a e^{\alpha x}$$

$$\ln y = \ln a + \alpha x$$

▷ $+2p_F + iy \pm \varepsilon$:

$$\Pi_{med} = 2\mu p_F + m^2 \ln\left(\frac{\mu - p_F}{\mu + p_F}\right) + \frac{1}{2} p \left[\left(\sqrt{p^2 + 4m^2} - 2\mu\right) \ln\left|\frac{p + 2p_F}{p - 2p_F}\right| - 2p \ln\left(\frac{p_F + \mu}{m}\right) + \sqrt{p^2 + 4m^2}\, \ln\left(\frac{2m^2 + p p_F + \mu\sqrt{p^2 + 4m^2}}{2m^2 - p p_F + \mu\sqrt{p^2 + 4m^2}}\right)\right]$$

$$= \Pi_{med}(y)\Big|_{y=0} + \Pi'_{med}(y)\Big|_{y=0} \cdot y$$