Series 12 - Renormalizing Yukawa theory

We consider Yukawa theory with bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_b)^2 - \frac{1}{2}m_b^2\varphi_b^2 + \bar{\psi}_b(i\partial\!\!\!/ - M_b)\psi_b - g_b\bar{\psi}_b\gamma^5\psi_b\varphi_b. \tag{1}$$

I. Divergence analysis

Write down the superficial degree of divergence for Yukawa theory in $d = 4$ and list the superficially divergent amplitudes.

Let us consider a connected Feynman diagram and let D, L and V be its superficial degree of divergence, number of loops and number of vertices, respectively. Moreover, let P_ϕ, P_ψ, N_ϕ and N_ψ be the number of scalar propagators, the number of fermion propagators, the number of external scalar legs and the number of external fermion legs, respectively. In d-dimensions, the measures in the loops bring $d \cdot L$ powers of k in the numerator, but the denominator has $2P_\phi + P_\psi$ powers of k, so the superficial degree of divergence is given by

$$D = d \cdot L - 2P_\phi - P_\psi. \tag{2}$$

Moreover, it is easy to see that $L = P_\phi + P_\psi - V + 1$ and $V = 2P_\phi + N_\phi = \frac{1}{2}(2P_\psi + N_\psi)$, which means that

$$\begin{aligned} D &= d + (d-2)P_\phi + (d-1)P_\psi - d\cdot V \\ &= d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\phi - \left(\frac{d-1}{2}\right)N_\psi \\ &= 4 - N_\phi - \frac{3}{2}N_\psi, \end{aligned} \tag{3}$$

where we used

$$(d-2)P_\phi + (d-1)P_\psi = \frac{d-2}{2}(V - N_\phi) + (d-1)\left(V - \frac{1}{2}N_\psi\right), \tag{4}$$

and finally set $d = 4$. Therefore, the counting is exactly the same as in section 19.1 of the notes, and the candidate diagrams are the same. In particular, the vacuum bubble (a) is just a constant shift that can be accounted for by a constant counterterm in the Lagrangian ; we thus ignore it. The tadpole (b) and the scalar three-point function (d) vanish due to parity. Indeed, you can easily check that $\mathcal{P}^\dagger\mathcal{L}(\vec{x},t)\mathcal{P} = \mathcal{L}(-\vec{x},t)$, where the parity operator $\mathcal{P}$ acts as

$$\mathcal{P}^\dagger\phi(\vec{x},t)\mathcal{P} = -\phi(-\vec{x},t) \qquad \text{and} \qquad \mathcal{P}^\dagger\psi(\vec{x},t)\mathcal{P} = \gamma^0\psi(-\vec{x},t). \tag{5}$$

We expect the scalar and the fermion two-point functions to be divergent, as well as the Yukawa interaction vertex (diagrams (c), (f) and (g)). However, the novelty is that we have no reasons to believe that the scalar four-point function (e) is finite, and actually it is not. The only way out is to add a new term in the Lagrangian:

$$\mathcal{L} \to \mathcal{L} - \frac{\lambda_b}{4!}\phi_b^4. \tag{6}$$

We therefore expect to deal with the diagrams consider in Series 8, as well as the box diagram (see below).

II. Counterterm structure

1. *Split the bare Lagrangian into physical Lagrangian and counterterm Lagrangian.*

   Introducing the renormalized fields ϕ and ψ through

$$\phi_b = \sqrt{Z_\phi}\phi \qquad \text{and} \qquad \psi_b = \sqrt{Z_\psi}\psi, \tag{7}$$

   as well as the renormalized masses and couplings m, M, g and λ via

$$\begin{aligned} Z_\phi m_b^2 &= m^2 + \delta_m, \\ Z_\psi M_b &= M + \delta_M, \\ \sqrt{Z_\phi} Z_\psi g_b &= g + \delta_g, \\ Z_\phi^2 \lambda_b &= \lambda + \delta_\lambda, \end{aligned} \tag{8}$$

   we can rewrite the Lagrangian as

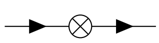
$$\begin{aligned} \mathcal{L} = &\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - g\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \\ &+ \frac{\delta_\phi}{2}(\partial_\mu\phi)^2 - \frac{\delta_m}{2} m^2\phi^2 + \bar{\psi}(i\delta_\psi \partial\!\!\!/ - \delta_M)\psi - \delta_g \bar{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4, \end{aligned} \tag{9}$$

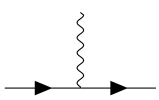
   where $\delta_\phi = Z_\phi - 1$ and $\delta_\psi = Z_\psi - 1$. We now have twice as many "constants" as we originally had, which might seems like an unnecessary large number of arbitrary definitions. However, by choosing the renormalization conditions appropriately, we can enforce the renormalized masses and couplings to actually be the *physical* ones, which are different from the bare quantities. Nevertheless, the renormalization conditions that we will choose here are not quite the physical (on-shell) ones, but the unphysical zero-momentum prescription. The point being that we merely want to extract the divergent part of the counterterms to practice a little bit, not to compute cross-sections.

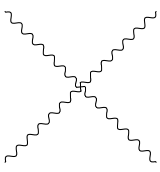
   Finally, note that one might argue that λ can be set to zero. Indeed, we started with a bare Lagrangian that had no such term, so the only thing we really need to include is the counterterm δ_λ. At the same time, the $\lambda\phi^4$-interaction term is marginal and there is no physical reason to ignore it. In what follows, we shall keep it non-zero.

2. *Give the Feynman rules for the counterterms.*

 $= i(\delta_\phi k^2 - \delta_m)$

 $= i(\delta_\psi p\!\!\!/ - \delta_M)$

 $= -i\delta_g\gamma^5$

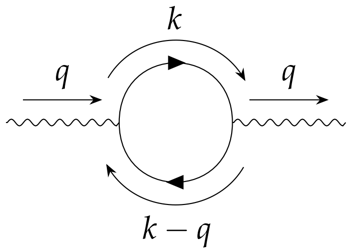
 $= -i\delta_\lambda$

   N.B.: the two vertices should have a circled cross in the center to indicate that they are counterterms, but for some reason, this is hard to do in LaTeX.

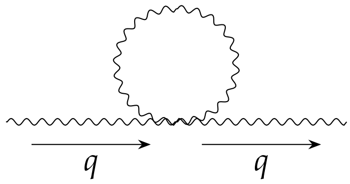
III. One-loop amplitudes

1. *Draw the one-loop diagrams for the divergent amplitudes.*

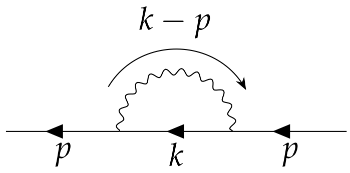
   We define



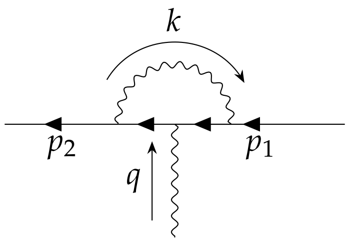
$$= -i\Pi_2(q^2)$$



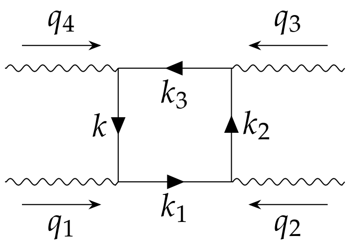
$$= -i\tilde{\Pi}_2(q^2)$$



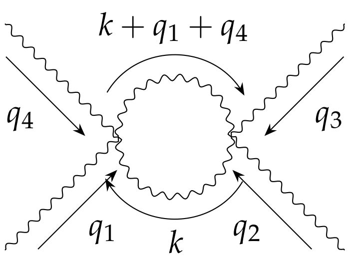
$$= -i\Sigma_2(\not{p})$$



$$= -ig\Gamma_2(p_1, p_2)$$



$$= -i\Lambda_2(\{q_i\}_i)$$



$$= -i\tilde{\Lambda}_2(\{q_i\}_i)$$

In the box diagram defining $\Lambda_2(\{p_i\}_i)$, we defined $k_1 \equiv k + q_1$ and $k_2 \equiv k + q_1 + q_2$ and $k_3 = k + q_1 + q_2 + q_3$ for convenience. There are six different diagrams of this form, but with the external momenta switched. Of course, if we use the zero-momentum scheme in which these momenta are 0, then the six diagrams give exactly the same amplitude and we thus pick up a factor of 6. The same is true for the last diagram, which was essentially worked out in series 8 and gives a factor of 3 in the zero-momentum scheme.

2. *Write down the amplitudes via the Feynman rules and apply the usual techniques for evaluating loop integrals via dimensional regularization.*

The one-loop correction to the scalar two-point function is

$$
\begin{aligned}
-i\Pi_2(q^2) &= (-1)\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{Tr}\left[(-ig\gamma^5)\frac{i(\not{k}+M)}{k^2-M^2}(-ig\gamma^5)\frac{i(\not{k}-\not{q}+M)}{(k-q)^2-M^2}\right]\\
&= -g^2\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{k_\alpha(k-q)_\beta\gamma^5\gamma^\alpha\gamma^5\gamma^\beta+M^2\gamma^5\gamma^5}{[(k-q)^2-M^2][k^2-M^2]}\right]\\
&= 4g^2\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{k(k-q)-M^2}{[(k-q)^2-M^2][k^2-M^2]}\\
&= 4g^2\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{k(k-q)-M^2}{[(q^2-2kq)x+k^2-M^2]^2}\\
&= -4g^2\mu^{4-d}\int\frac{\mathrm{d}^dl}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{-l^2+2M^2-\Delta(x)}{[l^2-\Delta(x)]^2}\\
&= -4ig^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\left[\frac{\mu^{4-d}l_E^2}{[l_E^2+\Delta(x)]^2}+(2M^2-\Delta(x))\frac{\mu^{4-d}}{[l_E^2+\Delta(x)]^2}\right]\\
&= \frac{-4ig^2}{16\pi^2}\int_0^1\mathrm{d}x\,\Gamma\left(\frac{\epsilon}{2}\right)\left[\frac{4\pi\mu^2}{\Delta(x)}\right]^{\frac{\epsilon}{2}}\left[\frac{4-\epsilon}{\epsilon-2}\Delta(x)+2M^2-\Delta(x)\right]\\
&= \frac{-ig^2}{4\pi^2}\int_0^1\mathrm{d}x\left[(2M^2-3\Delta(x))\left\{\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}-\ln\frac{\Delta(x)}{M^2}\right\}-\Delta(x)+\mathcal{O}(\epsilon)\right]\\
&= \frac{-ig^2}{24\pi^2}\left[(3q^2-6M^2)\left\{\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}\right\}+(q^2-6M^2)\right.\\
&\qquad\left.-6\int_0^1\mathrm{d}x\,[3x(1-x)q^2-M^2]\ln\left(1-x(1-x)\frac{q^2}{M^2}\right)+\mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{10}
$$

We have successively: written down the amplitude corresponding to the diagram (but dropping the $i\epsilon$-prescription in the denominators for simplicity); used the fact that the trace of an odd number of γ-matrices vanishes; used $\{\gamma^5,\gamma^\alpha\}=0$, $\gamma^5\gamma^5=1$ and $\mathrm{Tr}(1)=4$ (even in $d=4-\epsilon$); introduced the Feynman parameter x ; defined $l=k-qx$ and $\Delta(x)=M^2-x(1-x)q^2$ and dropped odd terms in l in the numerator; Wick rotated; integrated over l_E using the usual identities (see below); expanded for $\epsilon\ll 1$; performed part of the integration over x. The identities are

$$
\begin{aligned}
\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\frac{\mu^{4-d}}{[l_E^2+\Delta(x)]^2} &= \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{16\pi^2}\left[\frac{4\pi\mu^2}{\Delta(x)}\right]^{\frac{\epsilon}{2}}\\
\int\frac{\mathrm{d}^dl_E}{(2\pi)^d}\frac{\mu^{4-d}l_E^2}{[l_E^2+\Delta(x)]^2} &= \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{16\pi^2}\left[\frac{4\pi\mu^2}{\Delta(x)}\right]^{\frac{\epsilon}{2}}\cdot\frac{4-\epsilon}{\epsilon-2}\Delta(x).
\end{aligned}
\tag{11}
$$

Moreover, the scalar two-point function receives another correction at one-loop due to the extra $\lambda\phi^4$-term. We have

$$\begin{aligned}
-i\tilde{\Pi}_2(q^2) &= \frac{-i\lambda}{2}\mu^{4-d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m^2} \\
&= \frac{-i\lambda}{2}\int\frac{\mathrm{d}^d k_E}{(2\pi)^d}\frac{\mu^{4-d}}{k_E^2+m^2} \\
&= \frac{-i\lambda}{16\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)\left[\frac{4\pi\mu^2}{\Delta(x)}\right]^{\frac{\epsilon}{2}}\cdot\frac{m^2}{\epsilon-2} \\
&= \frac{-\lambda m^2}{32\pi^2}\left[\frac{2}{\epsilon}+1-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+\mathcal{O}(\epsilon)\right].
\end{aligned} \tag{12}$$

Note that for this diagram, we have to account for a symmetry factor of $\frac{4\cdot 3}{4!}=\frac{1}{2}$. Then, we simply Wick rotated and performed the integration over k_E, which we expanded for $\epsilon \ll 1$.

For the fermion two-point function, we have

$$\begin{aligned}
-i\Sigma_2(\not{p}) &= \mu^{4-d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{(k-p)^2-m^2}(-ig\gamma^5)\frac{i(\not{k}+M)}{k^2-M^2}(-ig\gamma^5) \\
&= g^2\mu^{4-d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{M-\not{k}}{[(p^2-2kp-m^2+M^2)x+k^2-M^2]^2} \\
&= ig^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^d l_E}{(2\pi)^d}\frac{\mu^{4-d}(M-\not{p}x)}{[l_E^2+\Delta(x)]^2} \\
&= \frac{ig^2}{16\pi^2}\int_0^1\mathrm{d}x\,(M-\not{p}x)\Gamma\left(\frac{\epsilon}{2}\right)\left[\frac{4\pi\mu^2}{\Delta(x)}\right]^{\frac{\epsilon}{2}} \\
&= \frac{ig^2}{16\pi^2}\int_0^1\mathrm{d}x\,(M-\not{p}x)\left\{\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{\Delta(x)}+\mathcal{O}(\epsilon)\right\} \\
&= \frac{-ig^2}{16\pi^2}\left[(\not{p}-2M)\left\{\frac{2}{\epsilon}-\gamma_E+\ln 4\pi\right\}\right. \\
&\quad \left. -2\int_0^1\mathrm{d}x\,(\not{p}x-M)\ln\left(\frac{M^2+(m^2-M^2)x-x(1-x)p^2}{\mu^2}\right)+\mathcal{O}(\epsilon)\right],
\end{aligned} \tag{13}$$

where we used again $\{\gamma^5,\gamma^\alpha\}=0$, then we introduced the Feynman parameter x together with $l=k-px$ and $\Delta(x)=M^2+(m^2-M^2)x-x(1-x)p^2$ and we Wick rotated. We then performed the integration over l_E, expanded the result for $\epsilon \ll 1$ and partly integrated over x.

The correction to the Yukawa interaction vertex at one-loop is (we define $k_1 \equiv k+p_1$ and $k_2 \equiv k+p_2$)

$$-ig\Gamma_2(p,p') = \mu^{4-d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m^2}(-ig\gamma^5)\frac{i(\not{k}_2+M)}{k_2^2-M^2}(-ig\gamma^5)\frac{i(\not{k}_1+M)}{k_1^2-M^2}(-ig\gamma^5). \tag{14}$$

We will now evaluate it in the zero-momentum scheme: $p_1=p_2=q=0$. We have to impose this constraint now because it breaks the equation of motion, so we cannot use the Gordon identity or other physical properties of this sort for what follows. However, this prescription allows us to extract almost straightforwardly the divergent part of the amplitude, which is pedagogically nice. Indeed,

$$
\begin{aligned}
\Gamma_2(0,0) &= ig^2\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma^5(\not{k}+M)\gamma^5(\not{k}+M)}{[k^2-m^2][k^2-M^2]^2} \\
&= -ig^2\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(\not{k}-M)(\not{k}+M)}{[k^2-m^2][k^2-M^2]^2} \\
&= -ig^2\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{[k^2-m^2][k^2-M^2]} \\
&= -ig^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{\mu^{4-d}}{[k^2-\Delta(x)]^2} \\
&= g^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{\mu^{4-d}}{[k_E^2+\Delta(x)]^2} \\
&= \frac{g^2}{16\pi^2}\int_0^1\mathrm{d}x\,\Gamma\left(\frac{\epsilon}{2}\right)\left[\frac{4\pi\mu^2}{\Delta(x)}\right]^{\frac{\epsilon}{2}} \\
&= \frac{g^2}{16\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}-\int_0^1\mathrm{d}x\,\ln\left(1-\frac{M^2-m^2}{M^2}x\right)+\mathcal{O}(\epsilon)\right] \\
&= \frac{g^2}{16\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}+1+\frac{m^2}{M^2-m^2}\ln\frac{m^2}{M^2}+\mathcal{O}(\epsilon)\right],
\end{aligned}
\tag{15}
$$

where we used $(\not{k}-M)(\not{k}+M)=k^2-M^2$ in the second line, and defined $\Delta(x)=M^2-(M^2-m^2)x$ in the fourth.

The correction to the $\lambda\phi^4$-interaction vertex at one-loop is (we define $k_1\equiv k+p_1$ and $k_2\equiv k+p_1+p_2$ and $k_3=k+p_1+p_2+p_3$)

$$
\begin{aligned}
-i\Lambda_2(\{p_i\}_i) &= (-1)\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{i(\not{k}_3+M)}{k_3^2-M^2}(-ig\gamma^5)\frac{i(\not{k}_2+M)}{k_2^2-M^2}\right. \\
&\qquad\left.\times(-ig\gamma^5)\frac{i(\not{k}_1+M)}{k_1^2-M^2}(-ig\gamma^5)\frac{i(\not{k}+M)}{k^2-M^2}(-ig\gamma^5)\right] \\
&= -g^4\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{(M-\not{k}_3)(M+\not{k}_2)(M-\not{k}_1)(M+\not{k})}{[k_3^2-M^2][k_2^2-M^2][k_1^2-M^2][k^2-M^2]}\right].
\end{aligned}
\tag{16}
$$

Remember that there are a total of 6 such contributions, coming from the exchange of p_i on external legs. In the zero-momentum prescription $p_1=p_2=p_3=p_4=0$, the six diagrams give exactly the same contribution, and we therefore include this factor in what follows:

$$
\begin{aligned}
\Lambda_2(\{0\}_i) &= -6ig^4\mu^{4-d}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{[k^2-M^2]^2}{[k^2-M^2]^4}\right] \\
&= -24ig^4\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mu^{4-d}}{[k^2-M^2]^2} \\
&= 24g^4\int\frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{\mu^{4-d}}{[k_E^2+M^2]^2} \\
&= \frac{3g^2}{2\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}+\mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{17}
$$

There is yet another correction to this vertex, which boils down to the computation performed in series 8:

$$
\tilde{\Lambda}_2(\{0\}_i) = \frac{-3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+\mathcal{O}(\epsilon)\right].
\tag{18}
$$

IV. Determining the counterterms

Determine the counterterms at one loop via the zero-momentum *subtraction scheme (i.e. corrections to the amplitudes vanish at zero momentum, for the 1PI two-point functions also their derivatives vanish at zero momentum.)*

We define the renormalized scalar two-point function at one-loop as

$$
\Pi_R(q^2) = \Pi_2(q^2)+\tilde{\Pi}_2(q^2)+\delta_m-q^2\delta_\phi,
\tag{19}
$$

on which we impose $\Pi_R(q=0)=0$ and $\left.\frac{\mathrm{d}\Pi_R}{\mathrm{d}(q^2)}\right|_{q=0}=0$, which imply

$$
\begin{aligned}
\delta_m &= -\Pi_2(q=0)-\tilde{\Pi}_2(q=0) \\
&= \frac{g^2M^2}{4\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}+1\right] \\
&\quad+\frac{\lambda^2m^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+1\right],
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
\delta_\phi &= \left.\frac{\mathrm{d}\Pi_2}{\mathrm{d}(q^2)}\right|_{q=0}+\left.\frac{\mathrm{d}\tilde{\Pi}_2}{\mathrm{d}(q^2)}\right|_{q=0} \\
&= \frac{g^2M^2}{8\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+\frac{1}{3}-2\int_0^1\mathrm{d}x\,x(1-x)\right] \\
&= \frac{g^2M^2}{8\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}\right].
\end{aligned}
\tag{21}
$$

Next, we define the renormalized fermion two-point function at one-loop as

$$
\Sigma_R(\not{p}) = \Sigma_2(\not{p})+\delta_M-\not{p}\delta_\psi,
\tag{22}
$$

on which we impose $\Sigma_R(\not{p}=0)=0$ and $\left.\frac{\mathrm{d}\Sigma_R}{\mathrm{d}\not{p}}\right|_{\not{p}=0}=0$, which imply

$$\begin{aligned}\delta_m &= -\Sigma_2(\not{p}=0)\\ &= \frac{g^2M}{16\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}-\int_0^1\mathrm{d}x\,\ln\left(1-\frac{M^2-m^2}{M^2}x\right)\right]\\ &= \frac{g^2M}{16\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}+1+\frac{m^2}{M^2-m^2}\ln\frac{m^2}{M^2}\right],\end{aligned} \tag{23}$$

and

$$\begin{aligned}\delta_\psi &= \left.\frac{\mathrm{d}\Sigma_2}{\mathrm{d}\not{p}}\right|_{\not{p}=0}\\ &= \frac{g^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}-2\int_0^1\mathrm{d}x\,x\ln\left(1-\frac{M^2-m^2}{M^2}x\right)\right]\\ &= \frac{g^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}+\frac{r-3}{2(r-1)}-\frac{(r-2)r\ln r}{(r-1)^2}\right],\end{aligned} \tag{24}$$

with $r\equiv\frac{m^2}{M^2}$.

Finally, we impose that renormalized corrections to the vertices vanish, which imposes that

$$\begin{aligned}\delta_g &= -g\Gamma_2(0,0)\\ &= \frac{-g^3}{16\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}+1+\frac{m^2}{M^2-m^2}\ln\frac{m^2}{M^2}\right],\end{aligned} \tag{25}$$

and

$$\begin{aligned}\delta_\lambda &= -\Lambda_2(\{0\}_i)-\tilde{\Lambda}_2(\{0\}_i)\\ &= \frac{-3g^2}{2\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{M^2}\right]+\frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\ln\frac{4\pi\mu^2}{m^2}\right].\end{aligned} \tag{26}$$