

## Supplemental Materials

The supplemental materials provide some details of the 2+1 flavor low energy effective field theory within the functional renormalization group approach.

### S.1. 2+1 flavor low energy effective field theory within the fRG

In this appendix we recapitulate the setup of the 2+1 flavor low energy effective field theory (LEFT) within the functional renormalization group used in this work. More details about the 2+1 LEFT can be found in [1]. The effective action reads

$$\Gamma_k[\Phi] = \int_x \left\{ \bar{q} [\gamma_\mu \partial_\mu - \gamma_0 (\mu + ig A_0)] q + h_k \bar{q} \Sigma_5 q + \text{tr} (\partial_\mu \Sigma \cdot \partial_\mu \Sigma^\dagger) + V_{\text{matt},k}(\phi) + V_{\text{glue}}(L, \bar{L}) \right\}, \quad (1)$$

with the shorthand notation  $\int_x = \int_0^\beta dx_0 \int d^3x$  and  $\beta = 1/T$ , where  $T$  stands for the temperature. The subscript  $k$  in  $\Gamma_k$  indicates that an infrared (IR) cutoff is applied to the effective action, such that quantum and thermal fluctuations of momenta  $p \lesssim k$  are suppressed. The full effective action is resolved as  $k \rightarrow 0$ , and thus  $k$  plays a role as the renormalization group (RG) scale. The field  $\Phi = (q, \bar{q}, \phi)$  includes the three-flavor quark field  $q = (q_u, q_d, q_s)^\top$  and the scalar and pseudoscalar meson fields  $\phi = (\sigma, \pi)$ . The mesons are in the adjoint representation of the  $U(N_f = 3)$  group, which reads

$$\Sigma = T^a (\sigma^a + i\pi^a), \quad a = 0, 1, \dots, 8, \quad (2)$$

with

$$T^0 = \frac{1}{\sqrt{2N_f}} \mathbb{1}_{N_f \times N_f}, \quad (3)$$

and

$$T^a = \frac{\lambda^a}{2} \quad (a = 1, \dots, 8), \quad (4)$$

where  $\lambda^a$  are the Gell-Mann matrices. The quark and meson fields interact with each other through the Yukawa coupling  $h_k$  with

$$\Sigma_5 = T^a (\sigma^a + i\gamma_5 \pi^a). \quad (5)$$

The quark chemical potential  $\mu$  in (1) is related to the baryon chemical potential  $\mu_B$  with  $\mu = \mu_B/3$ , where other chemical potentials, e.g., the chemical potentials for the electric charge and strangeness, are assumed to be vanishing.

The mesonic potential in (1), i.e., the matter sector of the effective potential, reads

$$V_{\text{matt},k}(\phi) = \tilde{V}_k(\rho_1, \rho_2) - c_A \xi - c_l \sigma_l - c_s \sigma_s, \quad (6)$$

with

$$\rho_1 = \text{tr}(\Sigma \cdot \Sigma^\dagger), \quad (7)$$

$$\rho_2 = \text{tr} \left( \Sigma \cdot \Sigma^\dagger - \frac{1}{3} \rho_1 \mathbb{1}_{3 \times 3} \right)^2, \quad (8)$$

where  $\rho_1$  and  $\rho_2$  are invariant under the transformations  $SU_V(3) \times SU_A(3) \times U_V(1) \times U_A(1)$  in the flavor space. Here  $\sigma_l$  and  $\sigma_s$  indicate the scalar mesons of light and strange quarks, respectively. The relevant strength constants  $c_l$  and  $c_s$  result in explicit breaking of the chiral symmetry, as well as the breaking of the flavor symmetry from the three-flavor case to that of 2+1 flavors. The  $U_A(1)$  symmetry is broken by the Kobayashi-Maskawa-'t Hooft determinant, viz.,

$$\xi = \det(\Sigma) + \det(\Sigma^\dagger), \quad (9)$$

arising from quantum fluctuations, whose strength is controlled by the constant  $c_A$ .

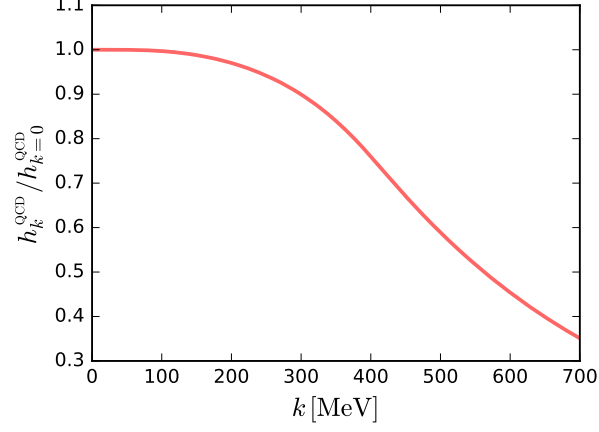


FIG. 1. Yukawa coupling, normalized to unity for  $k = 0$ , as a function of the RG scale  $k$ , computed from the first-principles QCD within the fRG in the vacuum [2].

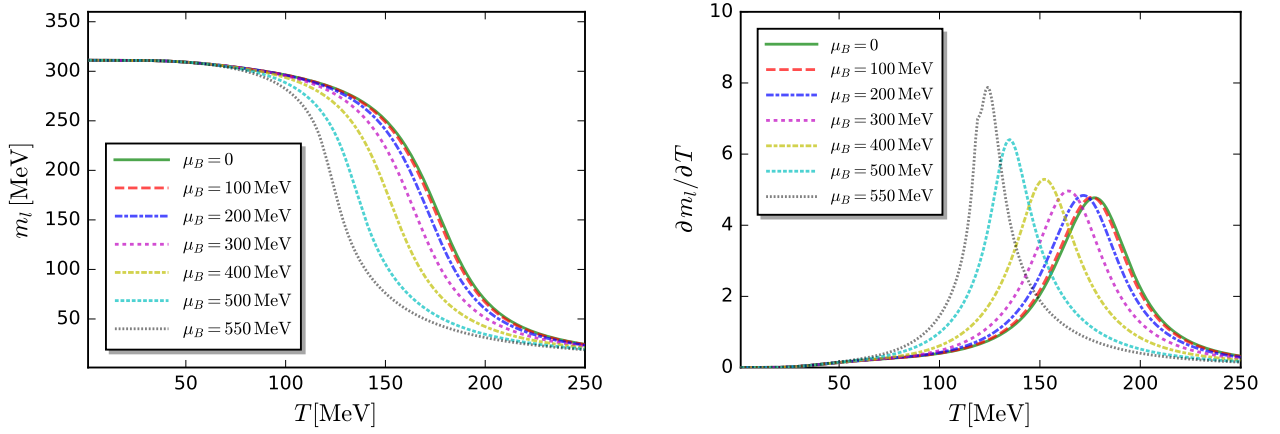


FIG. 2. Constituent light quark mass (left panel) and its derivative to the temperature (right panel) as functions of the temperature with several values of baryon chemical potential.

The glue dynamics is encoded in the glue potential  $V_{\text{glue}}$  in (1), also known as the Polyakov loop potential. The Polyakov loop is related to the temporal gluon background field  $A_0$ , that reads

$$L(\mathbf{x}) = \frac{1}{N_c} \langle \text{Tr } \mathcal{P}(\mathbf{x}) \rangle, \quad \bar{L}(\mathbf{x}) = \frac{1}{N_c} \langle \text{Tr } \mathcal{P}^\dagger(\mathbf{x}) \rangle, \quad (10)$$

with

$$\mathcal{P}(\mathbf{x}) = \mathcal{P} \exp \left( ig \int_0^\beta d\tau A_0(\mathbf{x}, \tau) \right), \quad (11)$$

where  $\mathcal{P}$  on the right side denotes the path ordering, and  $g$  is the strong coupling constant.

In this work we employ the Haar glue potential [1, 3],

$$V_{\text{glue}}(L, \bar{L}) = T^4 \bar{V}_{\text{glue-Haar}}, \quad (12)$$

with

$$\bar{V}_{\text{glue-Haar}} = -\frac{\bar{a}(T)}{2} \bar{L}L + \bar{b}(T) \ln M_H(L, \bar{L}) + \frac{\bar{c}(T)}{2} (L^3 + \bar{L}^3) + \bar{d}(T)(\bar{L}L)^2, \quad (13)$$

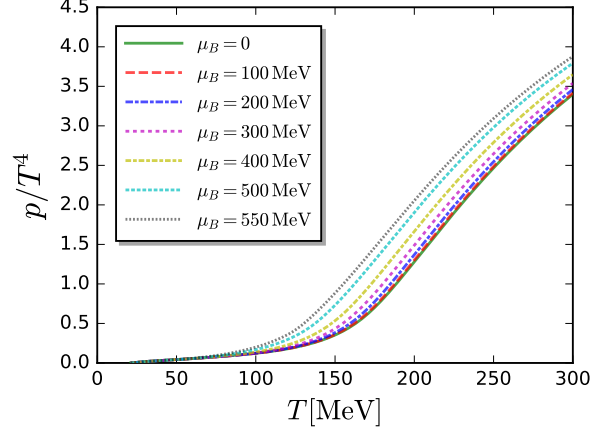


FIG. 3. Pressure normalized by  $T^4$  as a function of the temperature at baryon chemical potential  $\mu_B = 0, 100, 200, 300, 400, 500$  and  $550$  MeV. These results are obtained in the 2+1 flavor LEFT.

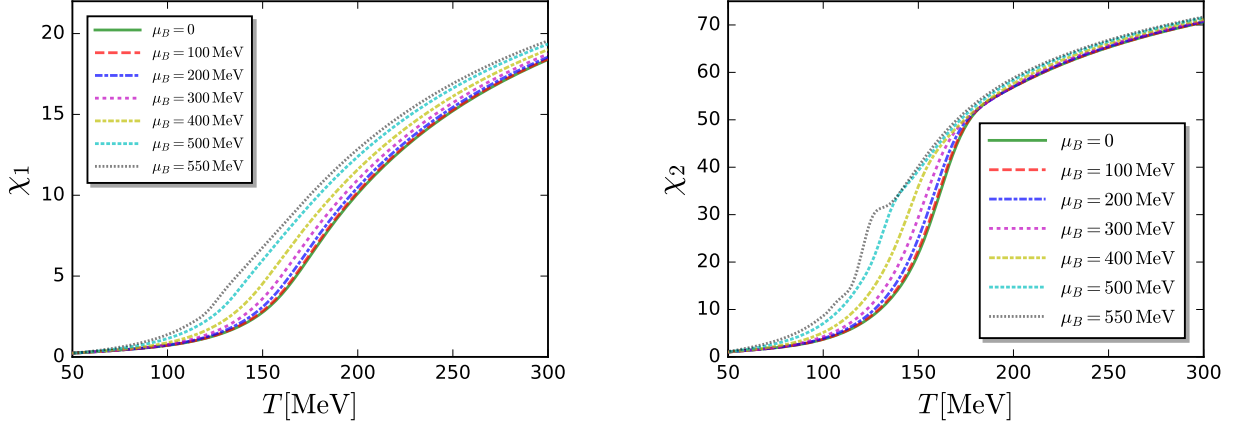


FIG. 4. Dimensionless entropy (left panel) and heat capacity (right panel) normalized by appropriate powers of  $T$ , i.e.,  $\chi_1$  and  $\chi_2$ , as functions of the temperature with several values of baryon chemical potential.

where the Haar measure reads

$$M_H(L, \bar{L}) = 1 - 6\bar{L}L + 4(L^3 + \bar{L}^3) - 3(\bar{L}L)^2. \quad (14)$$

The temperature dependence of the coefficients  $\bar{a}$ ,  $\bar{c}$ ,  $\bar{d}$  in (13) is parameterized as

$$x(T) = \frac{x_1 + x_2/(t+1) + x_3/(t+1)^2}{1 + x_4/(t+1) + x_5/(t+1)^2}, \quad x \in (\bar{a}, \bar{c}, \bar{d}), \quad (15)$$

and that of  $\bar{b}$  as

$$\bar{b}(T) = \bar{b}_1(t+1)^{-\bar{b}_4} \left( 1 - e^{\bar{b}_2/(t+1)^{\bar{b}_3}} \right). \quad (16)$$

Here  $t$  in (15) and (16) is the reduced temperature, that is

$$t = \alpha(T - T_c^{\text{glue}})/T_c^{\text{glue}}, \quad (17)$$

where the parameters  $T_c^{\text{glue}} = 250$  MeV and  $\alpha = 0.6$  are used throughout this work. The values of other parameters in (15) and (16) can be found in [1, 3].

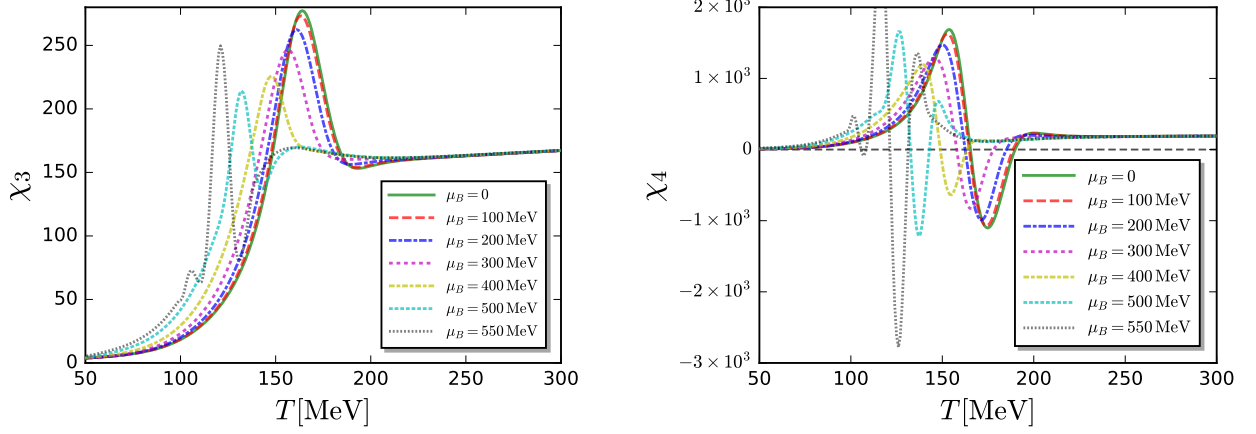


FIG. 5. Skewness (left panel) and kurtosis (right panel) of the entropy fluctuations, i.e.,  $\chi_3$  and  $\chi_4$ , as functions of the temperature with several values of baryon chemical potential.

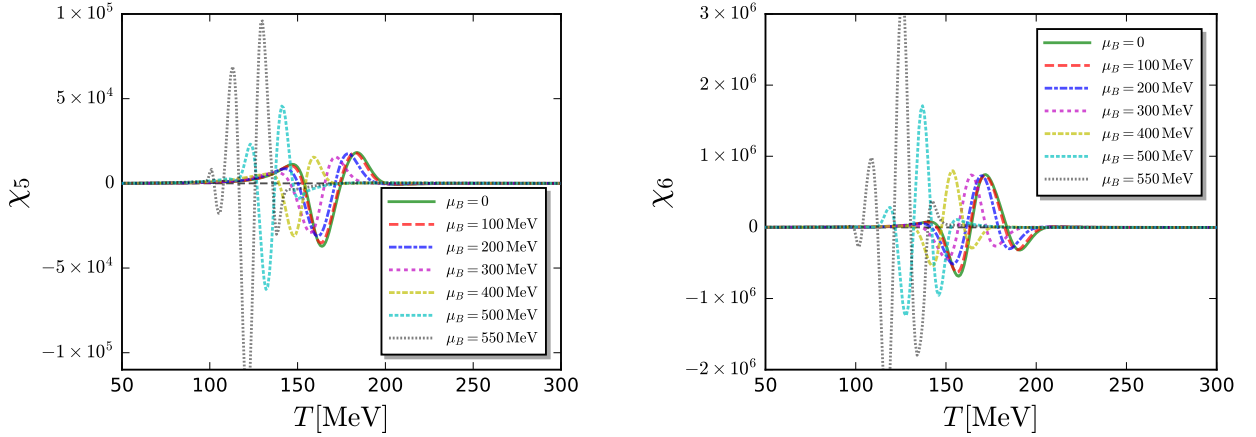


FIG. 6. Fifth (left panel) and sixth (right panel) order fluctuations of the entropy, i.e.,  $\chi_5$  and  $\chi_6$ , as functions of the temperature with several values of baryon chemical potential.

Moreover, using the same method in [4], we employ the dependence of the Yukawa coupling on the RG scale  $k$  calculated from the first-principles QCD [2], as an input for the LEFT. Then the Yukawa coupling in the LEFT now reads

$$h_k = h_0 \frac{h_k^{\text{QCD}}}{h_{k=0}^{\text{QCD}}}, \quad (18)$$

where  $h_k^{\text{QCD}}$  is computed by using the fRG approach to the first-principles QCD in the vacuum [2], as shown in Figure 1. Here the parameter in the LEFT  $h_0 = 12$  is determined by fitting the constituent light  $u$  and  $d$  quark mass  $m_l = 311$  MeV. Furthermore, we use the same values of parameters in the matter sector in (6) as those in [1].

In the left panel of Figure 2 we show the constituent masses for the  $u$ ,  $d$  light quarks calculated in the 2+1 flavor LEFT, depicted as functions of the temperature at several values of  $\mu_B$ . Their respective derivatives with respect to the temperature are shown in the right panel of Figure 2, from which one can determine the pseudo-critical temperature for the chiral crossover through the location of the peak. Figure 3 displays the temperature dependence of pressure calculated in our 2+1 flavor LEFT-fRG framework for baryon chemical potential  $\mu_B$  ranging from  $\mu_B = 0$  to 550 MeV, from which one can compute the temperature derivatives of pressure. The relevant results, from the first to sixth order derivatives, are presented in Figures 4 to 6, which stand for the entropy and its fluctuations of different

orders.

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  - [4] W.-j. Fu, X. Luo, J. M. Pawłowski, F. Rennecke, and S. Yin, Ripples of the QCD critical point, *Phys. Rev. D* **111**, L031502 (2025), [arXiv:2308.15508 \[hep-ph\]](#).