Supplemental Materials

The supplemental materials provide some details of the 2+1 flavor low energy effective field theory within the functional renormalization group approach.

S.1. 2+1 flavor low energy effective field theory within the fRG

In this appendix we recapitulate the setup of the 2+1 flavor low energy effective field theory (LEFT) within the functional renormalization group used in this work. More details about the 2+1 LEFT can be found in [1]. The effective action reads

$$\Gamma_k[\Phi] = \int_x \Big\{ \bar{q}\big[\gamma_\mu \partial_\mu - \gamma_0(\mu + igA_0)\big]q + h_k\,\bar{q}\,\Sigma_5 q + \mathrm{tr}\big(\partial_\mu \Sigma \cdot \partial_\mu \Sigma^\dagger\big) + V_{\mathrm{matt},k}(\phi) + V_{\mathrm{glue}}(L,\bar{L}) \Big\}\,, \tag{1}$$

with the shorthand notation $\int_x = \int_0^\beta dx_0 \int d^3x$ and $\beta = 1/T$, where T stands for the temperature. The subscript k in Γ_k indicates that an infrared (IR) cutoff is applied to the effective action, such that quantum and thermal fluctuations of momenta $p \lesssim k$ are suppressed. The full effective action is resolved as $k \to 0$, and thus k plays a role as the renormalization group (RG) scale. The field $\Phi = (q, \bar{q}, \phi)$ includes the three-flavor quark field $q = (q_u, q_d, q_s)^\intercal$ and the scalar and pseudoscalar meson fields $\phi = (\sigma, \pi)$. The mesons are in the adjoint representation of the $\mathrm{U}(N_f = 3)$ group, which reads

$$\Sigma = T^a(\sigma^a + i\pi^a)\,, \quad a = 0,\, 1, ..., 8\,, \tag{2}$$

with

$$T^0 = \frac{1}{\sqrt{2N_f}} \mathbb{1}_{N_f \times N_f}\,, \tag{3}$$

and

$$T^a = \frac{\lambda^a}{2} \quad (a = 1, ..., 8)\,, \tag{4}$$

where λ^a are the Gell-Mann matrices. The quark and meson fields interact with each other through the Yukawa coupling h_k with

$$\Sigma_5 = T^a(\sigma^a + i\gamma_5 \pi^a)\,. \tag{5}$$

The quark chemical potential μ in (1) is related to the baryon chemical potential μ_B with $\mu = \mu_B/3$, where other chemical potentials, e.g., the chemical potentials for the electric charge and strangeness, are assumed to be vanishing.

The mesonic potential in (1), i.e., the matter sector of the effective potential, reads

$$V_{\mathrm{matt},k}(\phi) = \tilde{V}_k(\rho_1, \rho_2) - c_A \xi - c_l \sigma_l - c_s \sigma_s\,, \tag{6}$$

with

$$\rho_1 = \mathrm{tr}(\Sigma \cdot \Sigma^\dagger)\,, \tag{7}$$

$$\rho_2 = \mathrm{tr}\Big(\Sigma \cdot \Sigma^\dagger - \frac{1}{3}\rho_1\, \mathbb{1}_{3\times 3}\Big)^2\,, \tag{8}$$

where ρ_1 and ρ_2 are invariant under the transformations $\mathrm{SU_V}(3) \times \mathrm{SU_A}(3) \times \mathrm{U_V}(1) \times \mathrm{U_A}(1)$ in the flavor space. Here σ_l and σ_s indicate the scalar mesons of light and strange quarks, respectively. The relevant strength constants c_l and c_s result in explicit breaking of the chiral symmetry, as well as the breaking of the flavor symmetry from the three-flavor case to that of 2+1 flavors. The $\mathrm{U_A}(1)$ symmetry is broken by the Kobayashi-Maskawa-'t Hooft determinant, viz.,

$$\xi = \det(\Sigma) + \det(\Sigma^\dagger)\,, \tag{9}$$

arising from quantum fluctuations, whose strength is controlled by the constant c_A.

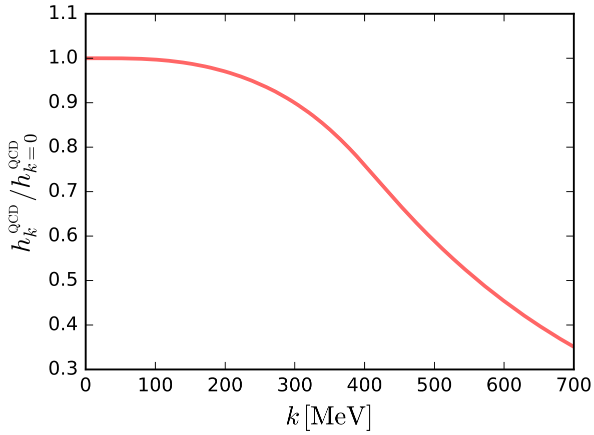

FIG. 1. Yukawa coupling, normalized to unity for $k = 0$, as a function of the RG scale k, computed from the first-principles QCD within the fRG in the vacuum [2].

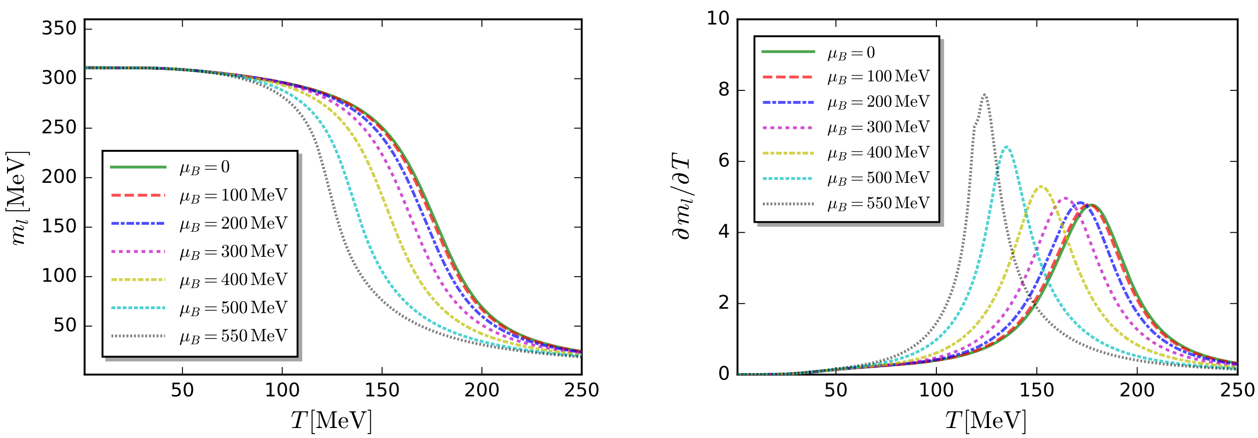

FIG. 2. Constituent light quark mass (left panel) and its derivative to the temperature (right panel) as functions of the temperature with several values of baryon chemical potential.

The glue dynamics is encoded in the glue potential $V_{\rm glue}$ in (1), also known as the Polyakov loop potential. The Polyakov loop is related to the temporal gluon background field A_0, that reads

$$L(\boldsymbol{x}) = \frac{1}{N_c}\left\langle \mathrm{Tr}\,\mathcal{P}(\boldsymbol{x})\right\rangle\,, \quad \bar{L}(\boldsymbol{x}) = \frac{1}{N_c}\langle \mathrm{Tr}\,\mathcal{P}^{\dagger}(\boldsymbol{x})\rangle\,, \tag{10}$$

with

$$\mathcal{P}(\boldsymbol{x}) = \mathcal{P}\exp\left(ig\int_0^{\beta} d\tau A_0(\boldsymbol{x},\tau)\right), \tag{11}$$

where $\mathcal{P}$ on the right side denotes the path ordering, and g is the strong coupling constant.

In this work we employ the Haar glue potential [1, 3],

$$V_{\rm glue}(L,\bar{L}) = T^4\,\bar{V}_{\text{glue-Haar}}\,, \tag{12}$$

with

$$\bar{V}_{\text{glue-Haar}} = -\frac{\bar{a}(T)}{2}\bar{L}L + \bar{b}(T)\ln M_H(L,\bar{L}) + \frac{\bar{c}(T)}{2}(L^3+\bar{L}^3) + \bar{d}(T)(\bar{L}L)^2\,, \tag{13}$$

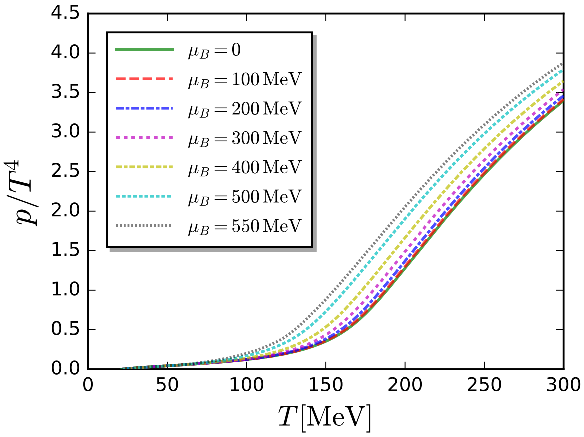

FIG. 3. Pressure normalized by T^4 as a function of the temperature at baryon chemical potential $\mu_B = 0$, 100, 200, 300, 400, 500 and 550 MeV. These results are obtained in the 2+1 flavor LEFT.

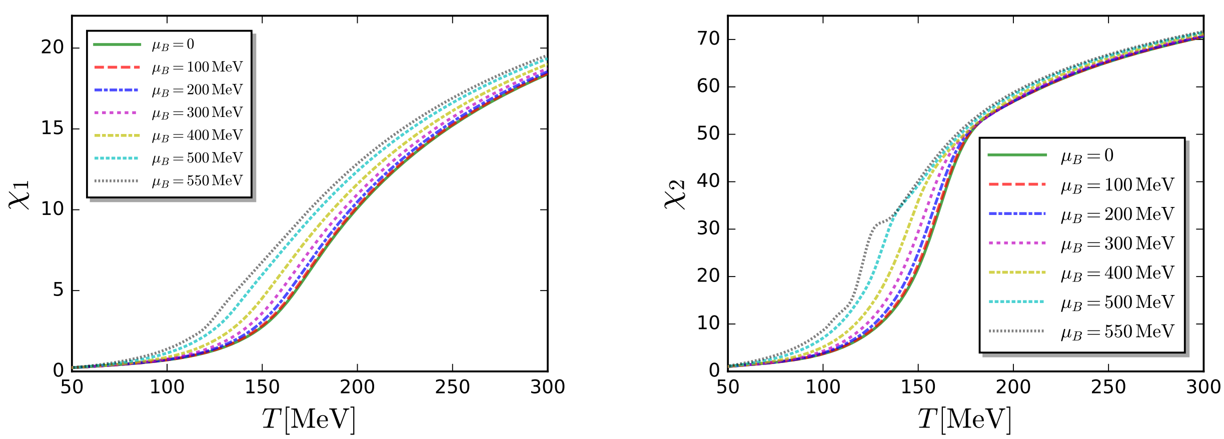

FIG. 4. Dimensionless entropy (left panel) and heat capacity (right panel) normalized by appropriate powers of T, i.e., χ_1 and χ_2, as functions of the temperature with several values of baryon chemical potential.

where the Haar measure reads

$$M_H(L, \bar{L}) = 1 - 6\bar{L}L + 4(L^3 + \bar{L}^3) - 3(\bar{L}L)^2 \,. \tag{14}$$

The temperature dependence of the coefficients $\bar{a}$, $\bar{c}$, $\bar{d}$ in (13) is parameterized as

$$x(T) = \frac{x_1 + x_2/(t+1) + x_3/(t+1)^2}{1 + x_4/(t+1) + x_5/(t+1)^2} \,, \quad x \in (\bar{a}, \bar{c}, \bar{d}) \,, \tag{15}$$

and that of $\bar{b}$ as

$$\bar{b}(T) = \bar{b}_1 (t+1)^{-\bar{b}_4} \left(1 - e^{\bar{b}_2/(t+1)^{\bar{b}_3}}\right) \,. \tag{16}$$

Here t in (15) and (16) is the reduced temperature, that is

$$t = \alpha(T - T_c^{\mathrm{glue}})/T_c^{\mathrm{glue}} \,, \tag{17}$$

where the parameters $T_c^{\mathrm{glue}} = 250$ MeV and $\alpha = 0.6$ are used throughout this work. The values of other parameters in (15) and (16) can be found in [1, 3].

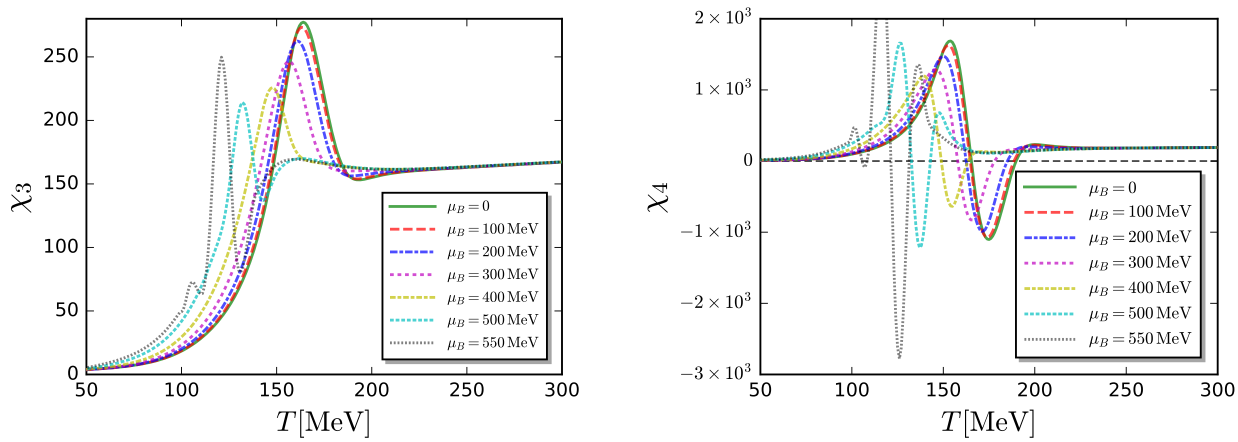

FIG. 5. Skewness (left panel) and kurtosis (right panel) of the entropy fluctuations, i.e., χ_3 and χ_4, as functions of the temperature with several values of baryon chemical potential.

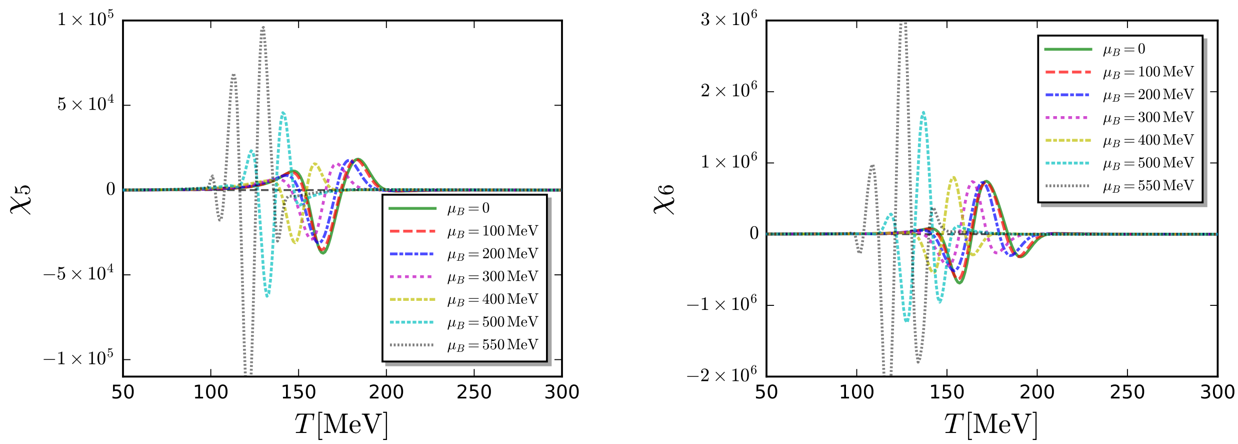

FIG. 6. Fifth (left panel) and sixth (right panel) order fluctuations of the entropy, i.e., χ_5 and χ_6, as functions of the temperature with several values of baryon chemical potential.

Moreover, using the same method in [4], we employ the dependence of the Yukawa coupling on the RG scale k calculated from the first-principles QCD [2], as an input for the LEFT. Then the Yukawa coupling in the LEFT now reads

$$h_k = h_0 \frac{h_k^{\rm QCD}}{h_{k=0}^{\rm QCD}}\,, \tag{18}$$

where $h_k^{\rm QCD}$ is computed by using the fRG approach to the first-principles QCD in the vacuum [2], as shown in Figure 1. Here the parameter in the LEFT $h_0 = 12$ is determined by fitting the constituent light u and d quark mass $m_l = 311$ MeV. Furthermore, we use the same values of parameters in the matter sector in (6) as those in [1].

In the left panel of Figure 2 we show the constituent masses for the u, d light quarks calculated in the 2+1 flavor LEFT, depicted as functions of the temperature at several values of μ_B. Their respective derivatives with respect to the temperature are shown in the right panel of Figure 2, from which one can determine the pseudo-critical temperature for the chiral crossover through the location of the peak. Figure 3 displays the temperature dependence of pressure calculated in our 2+1 flavor LEFT-fRG framework for baryon chemical potential μ_B ranging from $\mu_B = 0$ to 550 MeV, from which one can compute the temperature derivatives of pressure. The relevant results, from the first to sixth order derivatives, are presented in Figures 4 to 6, which stand for the entropy and its fluctuations of different orders.

[1] R. Wen, C. Huang, and W.-J. Fu, Baryon number fluctuations in the 2+1 flavor low energy effective model, Phys. Rev. D **99**, 094019 (2019), arXiv:1809.04233 [hep-ph].

[2] W.-j. Fu, J. M. Pawlowski, and F. Rennecke, QCD phase structure at finite temperature and density, Phys. Rev. D **101**, 054032 (2020), arXiv:1909.02991 [hep-ph].

[3] P. M. Lo, B. Friman, O. Kaczmarek, K. Redlich, and C. Sasaki, Polyakov loop fluctuations in SU(3) lattice gauge theory and an effective gluon potential, Phys. Rev. **D88**, 074502 (2013), arXiv:1307.5958 [hep-lat].

[4] W.-j. Fu, X. Luo, J. M. Pawlowski, F. Rennecke, and S. Yin, Ripples of the QCD critical point, Phys. Rev. D **111**, L031502 (2025), arXiv:2308.15508 [hep-ph].