

1 Effective Action

$$\begin{aligned}
\Gamma_k = \int_x \Bigg\{ & \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + Z_c (\partial_\mu \bar{c}^a) D_\mu^{ab} c^b + \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 \\
& + \frac{1}{2} \int_p A_\mu^a(-p) (\Gamma_{AA\mu\nu}^{(2)ab} - Z_A \Pi_{\mu\nu}^\perp \delta^{ab} p^2) A_\nu^b(p) \\
& + \bar{q} [Z_q (\gamma_\mu D_\mu - \gamma_0 (\hat{\mu} + igA_0))] q - \lambda_q \sum_{a=0}^8 [(\bar{q} T_a q)^2 + (\bar{q} i \gamma_5 T_a q)^2] \\
& + \bar{q} h_k^{1/2} \cdot \Sigma_5 \cdot h_k^{1/2} q + tr(Z_{\Sigma,k}^{1/2} \cdot \partial_\mu \Sigma \cdot Z_{\Sigma,k}^{1/2} \cdot \partial_\mu \Sigma^\dagger) + \tilde{U}_k(\Sigma, \Sigma^\dagger) + V_{glue}(L, \bar{L}) \Bigg\}
\end{aligned} \tag{1}$$

here, the meson field :

$$\Sigma = T^a (\sigma^a + i\pi^a). \quad (a = 0, 1, \dots, 8) \tag{2}$$

and

$$\Sigma_5 = T^a (\sigma^a + i\gamma_5 \pi^a). \quad (a = 0, 1, \dots, 8) \tag{3}$$

with $T^a = \lambda^a/2 (a = 1, \dots, 8)$ and $T^0 = \frac{1}{\sqrt{2N_f}} \mathbb{I}_{N_f \times N_f}$ are generators of $SU(N_f = 3)$. σ^a and π^a mean the scalar and pseudoscalar fields, respectively. The physical meson can be written obviously:

$$\Sigma = \frac{1}{2} \begin{pmatrix} a_0^0 + \sigma_L + i\pi^0 + i\eta_L & \sqrt{2}(a_0^+ + i\pi^+) & \sqrt{2}(\kappa^+ + iK^+) \\ \sqrt{2}(a_0^- + i\pi^-) & -a_0^0 + \sigma_L - i\pi^0 + i\eta_L & \sqrt{2}(\kappa^0 + iK^0) \\ \sqrt{2}(\kappa^- + iK^-) & \sqrt{2}(\bar{\kappa}^0 + i\bar{K}^0) & \sqrt{2}(\sigma_S + i\eta_S) \end{pmatrix} \tag{4}$$

the meson effective potential can be divided into three parts

$$\tilde{U}_k(\Sigma) = U_k(\rho_1, \tilde{\rho}_2) - c_A \xi - j_L \sigma_L - j_S \sigma_S, \tag{5}$$

here $U_k(\rho_1, \tilde{\rho}_2)$ is an arbitrary function of chiral symmetry invariant variables $\rho_1, \tilde{\rho}_2$. $c_A \xi$ is Kobayashi-Maskawa-'t Hooft term which breaks $U_A(1)$ symmetry. The last two terms of Eq.(5) are linear sigma terms, which break the chiral symmetry.

The Yukawa coupling

$$h_k = \begin{pmatrix} h_{l,k} & 0 & 0 \\ 0 & h_{l,k} & 0 \\ 0 & 0 & h_{s,k} \end{pmatrix} \tag{6}$$

and meson and quark wave function renormalization

$$Z_{\sigma,k} = \begin{pmatrix} Z_{\phi_l,k} & 0 & 0 \\ 0 & Z_{\phi_l,k} & 0 \\ 0 & 0 & Z_{\phi_s,k} \end{pmatrix} \quad Z_{q,k} = \begin{pmatrix} Z_{l,k} & 0 & 0 \\ 0 & Z_{l,k} & 0 \\ 0 & 0 & Z_{s,k} \end{pmatrix} \quad (7)$$

At present, we assume $Z_{\sigma,k} = Z_{\pi,k}$ and $Z_{q,k} = Z_{l,k}$.

2 Flow Equations

The the Wetterich equation with dynamical hadronisation reads

$$\partial_t \Gamma_k[\Phi] + \int \langle \partial_t \hat{\phi}_{k,i} \rangle \left(\frac{\delta \Gamma_k[\Phi]}{\delta \phi_i} + j_\sigma \delta_{i\sigma} \right) = \frac{1}{2} \text{Tr}(G_k[\Phi] \partial_t R_k) + \text{Tr} \left(G_{\phi\Phi_j}[\Phi] \frac{\delta \langle \partial_t \hat{\phi}_{k,i} \rangle}{\delta \Phi_j} R_\phi \right) \quad (8)$$

we assume

$$\langle \partial_t \hat{\phi}_k \rangle = \dot{A}_{l,k} [(\bar{q} T_a q) + (\bar{q} i \gamma_5 T_a q)] + \dot{A}_{s,k} [(\bar{q} T_b q) + (\bar{q} i \gamma_5 T_b q)] + \dot{B}_k \Sigma, \quad (9)$$

for $a = L, 1, \cdot 3, b = 4, \cdot 7, S$

here T^L, T^S are given in Appendix ?? As pointed out in ref [], we choose $\dot{B}_k = 0$. By taking the derivative of of each side of Eq. (8)

$$\frac{\overrightarrow{\delta}}{\delta(\bar{q} T^a q)} (Eq.(8)) \frac{\overleftarrow{\delta}}{\delta(\bar{q} T^a q)}, \quad (10)$$

we get

$$-\partial_t \lambda_q + \dot{A} h_k = -\text{Flow}_{(\bar{q} T^a q)(\bar{q} T^a q)}^{(4)} \quad (11)$$

with the condication

$$\lambda_q \equiv 0, \quad \forall k \quad (12)$$

we get the renormalised hadronisation function

$$\dot{A} = -\frac{1}{\bar{h}_k} \overline{\text{Flow}}_{(\bar{q} T^a q)(\bar{q} T^a q)}^{(4)} \quad (13)$$

we split the expression

$$\dot{\bar{A}}_{l,k} = -\frac{1}{\bar{h}_{l,k}} \overline{\text{Flow}}_{(\bar{q}T^L q)(\bar{q}T^L q)}^{(4)} \quad (14)$$

$$\dot{\bar{A}}_{s,k} = -\frac{1}{\bar{h}_{s,k}} \overline{\text{Flow}}_{(\bar{q}T^S q)(\bar{q}T^S q)}^{(4)} \quad (15)$$

And to calculate the yukawa flow equation:

$$\frac{\delta}{\delta \sigma^a} \frac{\delta}{\delta (\bar{q}T^a q)} (Eq.(8)) \quad a = L/S \quad (16)$$

we get

$$\partial \bar{h}_{l,k} = \left(\eta_{l,k} + \frac{1}{2} \eta_{\phi,k} \right) - \frac{\delta^2 \tilde{U}(\Sigma)}{(\delta \sigma_L)^2} \dot{\bar{A}}_{l,k} + \overline{\text{Flow}}_{(\bar{q}T^L q) \sigma_L}^{(3)} \quad (17)$$

$$\partial \bar{h}_{s,k} = \left(\eta_{s,k} + \frac{1}{2} \eta_{\phi,k} \right) - \frac{\delta^2 \tilde{U}(\Sigma)}{(\delta \sigma_S)^2} \dot{\bar{A}}_{s,k} + \overline{\text{Flow}}_{(\bar{q}T^S q) \sigma_S}^{(3)} \quad (18)$$

the next step is to calculate the $\overline{\text{Flow}}$ terms.

3 Result

Pressure:

$$\frac{p}{T^4} = \frac{U(0,0) - U((T,\mu))}{T^4} \quad (19)$$

and n-th order cumulats

$$\chi_n^B = \frac{\partial^n}{\partial (\mu_B/T)^n} \frac{p}{T^4} \quad (20)$$

4 Appendix.A

The meson masses can be obtained by Hessian matrix:

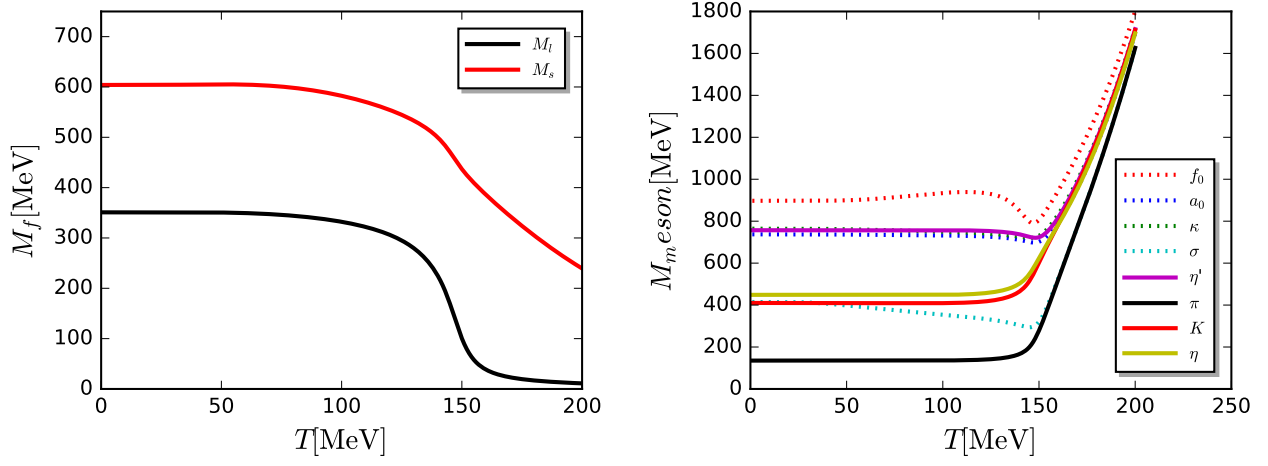


Figure 1: Quark and meson masses as functions of temperature with js/jl=17,ck=6000

$$H_{p,LL} = \frac{c_A \sigma_S}{\sqrt{2}} + \frac{1}{6} U^{(0,1)} (\sigma_L^2 - 2\sigma_S^2) + U^{(1,0)} \quad (21)$$

$$H_{p,LS} = \frac{c_A \sigma_L}{\sqrt{2}} \quad (22)$$

$$H_{p,SS} = U^{(1,0)} - \frac{1}{3} U^{(0,1)} (\sigma_L^2 - 2\sigma_S^2) \quad (23)$$

$$H_{p,11} = -\frac{c_A \sigma_S}{\sqrt{2}} + \frac{1}{6} U^{(0,1)} (\sigma_L^2 - 2\sigma_S^2) + U^{(1,0)} \quad (24)$$

$$H_{p,44} = -\frac{c_A \sigma_L}{2} + U^{(1,0)} + \frac{1}{6} U^{(1,0)} (\sigma_L^2 - 3\sqrt{2}\sigma_L\sigma_S + 4\sigma_S^2) \quad (25)$$

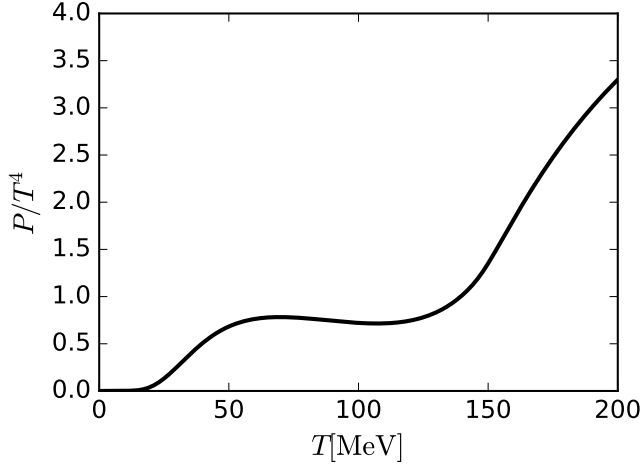


Figure 2: pressure

$$H_{s,LL} = -\frac{c_A \sigma_S}{\sqrt{2}} + U^{(1,0)} + U^{(2,0)} \sigma_L^2 + \frac{1}{6} U^{(0,1)} (3\sigma_L^2 - 2\sigma_S^2) \quad (26)$$

$$+ \frac{1}{36} \sigma_L^2 (\sigma_L^2 - 2\sigma_S^2) (U^{(0,2)} (\sigma_L^2 - 2\sigma_S^2) + 12U^{(1,1)})$$

$$H_{s,LS} = -\frac{c_A \sigma_L}{\sqrt{2}} + U^{(2,0)} \sigma_L \sigma_S - \frac{2}{3} U^{(0,1)} \sigma_L \sigma_S \quad (27)$$

$$- \frac{1}{18} U^{(0,2)} \sigma_L \sigma_S (\sigma_L^2 - 2\sigma_S^2)^2 - \frac{1}{6} U^{(1,1)} \sigma_L \sigma_S (\sigma_L^2 - 2\sigma_S^2)$$

$$H_{s,LS} = U^{(1,0)} + U^{(2,0)} \sigma_S^2 - \frac{1}{3} U^{(0,1)} (\sigma_L^2 - 6\sigma_S^2) \quad (28)$$

$$+ \frac{1}{9} \sigma_S^2 (-6U^{(1,1)} (\sigma_L^2 - 2\sigma_S^2) + U^{(0,2)} (\sigma_L^2 - 2\sigma_S^2)^2)$$

$$H_{s,11} = \frac{c_A \sigma_S}{\sqrt{2}} + U^{(1,0)} + \frac{1}{6} U^{(0,1)} (7\sigma_L^2 - 2\sigma_S^2) \quad (29)$$

$$H_{s,44} = \frac{c_A \sigma_L}{2} + U^{(1,0)} + \frac{1}{6} U^{(0,1)} (\sigma_L^2 + 3\sqrt{2} \sigma_L \sigma_S + 4\sigma_S^2) \quad (30)$$

5 Appendix.B

As we defined

$$\rho_1 = \frac{1}{2} (\sigma_l^2 + \sigma_s^2)$$

$$\rho_2 = \frac{1}{24} (\sigma_l^2 - 2\sigma_s^2)^2 \quad (31)$$

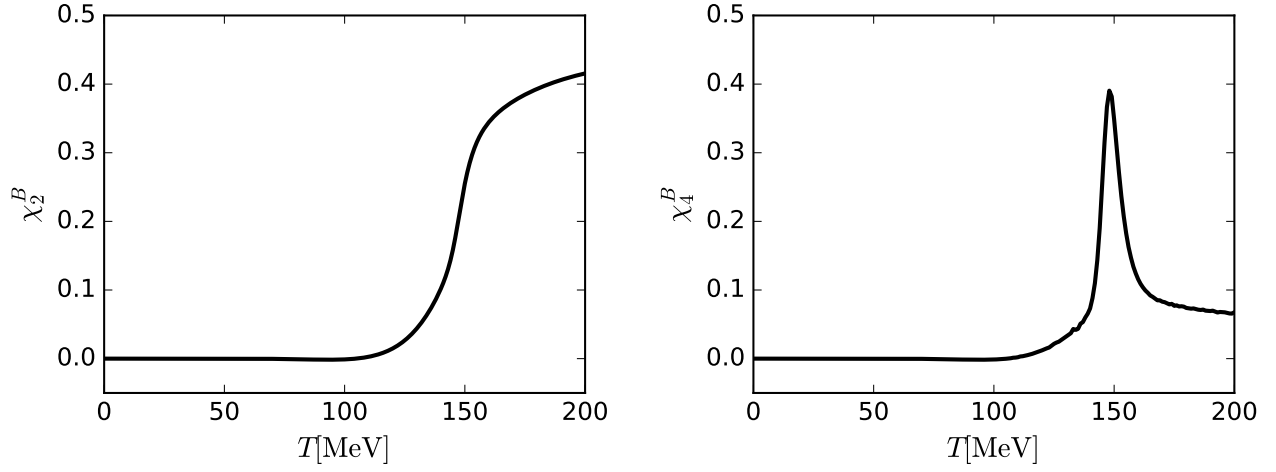


Figure 3: cumulats

Note that

$$\sigma_l < \sqrt{2}\sigma_s \quad (32)$$

then

$$\begin{aligned} 2\rho_1 &= \sigma_l^2 + \sigma_s^2 \\ -2\sqrt{6\rho_2} &= \sigma_l^2 - 2\sigma_s^2 \end{aligned} \quad (33)$$

then

$$\begin{aligned} \sigma_l^2 &= \frac{2}{3}(2\rho_1 - \sqrt{6\rho_2}) \\ \sigma_s^2 &= \frac{2}{3}(\rho_1 + \sqrt{6\rho_2}) \end{aligned} \quad (34)$$

so

$$\begin{aligned} \frac{\partial \sigma_l^2}{\partial \rho_1} &= \frac{4}{3} & \frac{\partial \sigma_l^2}{\partial \rho_2} &= -\frac{\sqrt{6}}{3}\rho_2^{-1/2} \\ \frac{\partial \sigma_s^2}{\partial \rho_1} &= \frac{2}{3} & \frac{\partial \sigma_s^2}{\partial \rho_2} &= \frac{\sqrt{6}}{3}\rho_2^{-1/2} \end{aligned} \quad (35)$$

so we get

$$\frac{\partial \sigma_l^2}{\partial \rho_2} = -\frac{\partial \sigma_s^2}{\partial \rho_2} \quad (36)$$

and

$$\frac{\partial^2 \sigma_l^2}{\partial \rho_2^2} = -\frac{\partial^2 \sigma_s^2}{\partial \rho_2^2} = \frac{\sqrt{6}}{6}\rho_2^{-3/2} \quad (37)$$

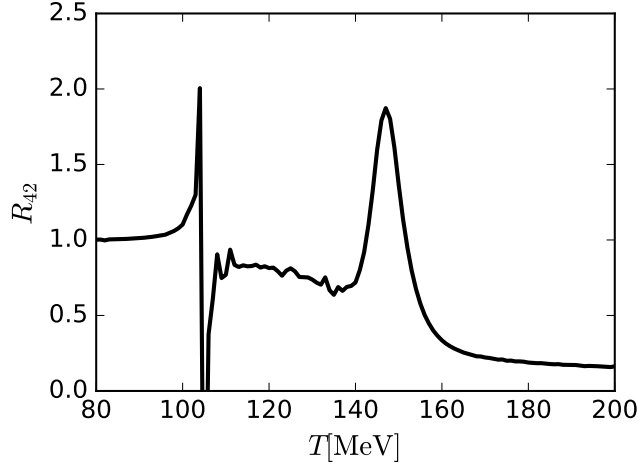


Figure 4: cumulates

The quark masses are given as

$$m_l^2 = h^2 \frac{\sigma_l^2}{4} \quad m_s^2 = h^2 \frac{\sigma_s^2}{2} \quad (38)$$

therefore

$$\begin{aligned} \frac{\partial m_l^2}{\partial \rho_2} &= -\frac{1}{2} \frac{\partial m_s^2}{\partial \rho_2} = -\frac{\sqrt{6}}{12} h^2 \rho_2^{-1/2} \\ \frac{\partial^2 m_l^2}{\partial \rho_2^2} &= -\frac{1}{2} \frac{\partial^2 m_s^2}{\partial \rho_2^2} = \frac{\sqrt{6}}{24} h^2 \rho_2^{-3/2} \end{aligned} \quad (39)$$

The quark loop function

$$l^{(f)} = \frac{1}{3} \left(1 - \frac{\eta_q}{4} \right) \frac{1}{\sqrt{1 + \bar{m}_f^2}} (1 - n_f(E + \mu) - n_f(E - \mu)) \quad (40)$$

here \bar{m}_f is dimensionless mass.

Then the light and strange quarks part of the the potential flow:

$$\partial_t U = -4N_c \frac{k^4}{4\pi^2} \left[2l^{(f)}(\bar{m}_l^2) + l^{(f)}(\bar{m}_s^2) \right] \quad (41)$$

For simplify, we only consider the square brackets above:

$$A_{qk} = 2l^{(f)}(\bar{m}_l^2) + l^{(f)}(\bar{m}_s^2) \quad (42)$$

and

$$\begin{aligned}\frac{\partial A_{qk}}{\partial \rho_2} &= 2 \frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} \frac{\partial \bar{m}_l^2}{\partial \rho_2} + \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2} \frac{\partial \bar{m}_s^2}{\partial \rho_2} \\ &= 2 \frac{\partial \bar{m}_l^2}{\partial \rho_2} \left(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} - \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2} \right)\end{aligned}\quad (43)$$

We consider $T = 0$ case:

$$l^{(f)} = \frac{1}{3} \left(1 - \frac{\eta_q}{4} \right) \frac{1}{\sqrt{1 + \bar{m}_f^2}} \quad (44)$$

then

$$\begin{aligned}\frac{\partial A_{qk}}{\partial \rho_2} &= 2 \frac{\partial \bar{m}_l^2}{\partial \rho_2} \left(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} - \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2} \right) \\ &= -\frac{\partial \bar{m}_l^2}{\partial \rho_2} \frac{1}{3} \left(1 - \frac{\eta_q}{4} \right) \left((1 + \bar{m}_l^2)^{-3/2} - (1 + \bar{m}_s^2)^{-3/2} \right)\end{aligned}\quad (45)$$

Because $\bar{m}_q^2 \ll 1, \bar{m}_l^2 \sim 5 \times 10^{-16}$ at $k = \Lambda$, we use Taylor expansion

$$\begin{aligned}\frac{\partial A_{qk}}{\partial \rho_2} &= -\frac{1}{3} \left(1 - \frac{\eta_q}{4} \right) \frac{\partial \bar{m}_l^2}{\partial \rho_2} \left(\left(1 - \frac{3}{2} \bar{m}_l^2 + \frac{15}{8} \bar{m}_l^4 \dots \right) - \left(1 - \frac{3}{2} \bar{m}_s^2 + \frac{15}{8} \bar{m}_s^4 + \dots \right) \right) \\ &= -\frac{1}{3} \left(1 - \frac{\eta_q}{4} \right) \left(-\frac{\sqrt{6}}{12} \frac{h^2}{k^2} \rho_2^{-1/2} \right) \left(-\frac{3}{2} (\bar{m}_l^2 - \bar{m}_s^2) + \frac{15}{8} (\bar{m}_l^4 - \bar{m}_s^4) + \dots \right) \\ &= -\frac{\sqrt{6}}{24} \left(1 - \frac{\eta_q}{4} \right) \left(\frac{h^2}{k^2} \rho_2^{-1/2} \right) \left((\bar{m}_l^2 - \bar{m}_s^2) - \frac{5}{4} (\bar{m}_l^4 - \bar{m}_s^4) + \frac{35}{24} (\bar{m}_l^6 - \bar{m}_s^6) \dots \right)\end{aligned}\quad (46)$$

If we consider second order derivative, it will be worse:

$$\begin{aligned}
\frac{\partial}{\partial \rho_2} \left(\frac{\partial A_{qk}}{\partial \rho_2} \right) &= 2 \left[\frac{\partial^2 \bar{m}_l^2}{\partial \rho_2^2} \left(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} - \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2} \right) + \frac{\partial \bar{m}_l^2}{\partial \rho_2} \left(\frac{\partial \bar{m}_l^2}{\partial \rho_2} \frac{\partial^2 l^{(f)}(\bar{m}_l^2)}{\partial (\bar{m}_l^2)^2} - \frac{\partial \bar{m}_s^2}{\partial \rho_2} \frac{\partial^2 l^{(f)}(\bar{m}_s^2)}{\partial (\bar{m}_s^2)^2} \right) \right] \\
&= 2 \left[\frac{\partial^2 \bar{m}_l^2}{\partial \rho_2^2} \left(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} - \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2} \right) + \left(\frac{\partial \bar{m}_l^2}{\partial \rho_2} \right)^2 \left(\frac{\partial^2 l^{(f)}(\bar{m}_l^2)}{\partial (\bar{m}_l^2)^2} + 2 \frac{\partial^2 l^{(f)}(\bar{m}_s^2)}{\partial (\bar{m}_s^2)^2} \right) \right] \\
&= \frac{2}{3} \left(1 - \frac{\eta_q}{4} \right) \left[-\frac{\sqrt{6} h^2}{48 k^2} \rho_2^{-3/2} \left((1 + \bar{m}_l^2)^{-3/2} - (1 + \bar{m}_s^2)^{-3/2} \right) \right. \\
&\quad \left. + \frac{1}{32} \frac{h^4}{k^4} \rho_2^{-1} \left((1 + \bar{m}_l^2)^{-5/2} + 2(1 + \bar{m}_s^2)^{-5/2} \right) \right] \\
&= \frac{1}{24} \left(1 - \frac{\eta_q}{4} \right) \frac{h^2}{k^2} \rho_2^{-3/2} \left[-\frac{\sqrt{6}}{3} \left(\left(1 - \frac{3}{2} \bar{m}_l^2 + \frac{15}{8} \bar{m}_l^4 + \dots \right) - \left(1 - \frac{3}{2} \bar{m}_s^2 + \frac{15}{8} \bar{m}_s^4 + \dots \right) \right) \right. \\
&\quad \left. + \frac{1}{2} \frac{h^2}{k^2} \rho_2^{1/2} \left(\left(1 - \frac{5}{2} \bar{m}_l^2 + \dots \right) + 2 \left(1 - \frac{5}{2} \bar{m}_s^2 + \dots \right) \right) \right] \\
&= \frac{1}{24} \left(1 - \frac{\eta_q}{4} \right) \frac{h^2}{k^2} \rho_2^{-3/2} \left[-\frac{\sqrt{6}}{3} \left(\left(1 - \frac{3}{2} \bar{m}_l^2 + \frac{15}{8} \bar{m}_l^4 - \frac{35}{16} \bar{m}_l^6 + \dots \right) \right. \right. \\
&\quad \left. \left. - \left(1 - \frac{3}{2} \bar{m}_s^2 + \frac{15}{8} \bar{m}_s^4 - \frac{35}{16} \bar{m}_s^6 + \dots \right) \right) \right. \\
&\quad \left. + \frac{1}{2} \frac{2}{\sqrt{6}} (\bar{m}_s^2 - \bar{m}_l^2) \left(\left(1 - \frac{5}{2} \bar{m}_l^2 + \frac{35}{8} \bar{m}_l^4 + \dots \right) + 2 \left(1 - \frac{5}{2} \bar{m}_s^2 + \frac{35}{8} \bar{m}_s^4 + \dots \right) \right) \right]
\end{aligned} \tag{47}$$

Obviously, the leading-order equal zero, and the next-leading-order is also vanished, and

$$\begin{aligned}
\frac{\partial}{\partial \rho_2} \left(\frac{\partial A_{qk}}{\partial \rho_2} \right) &= \frac{1}{24} \left(1 - \frac{\eta_q}{4} \right) \frac{h^2}{k^2} \rho_2^{-3/2} \left[-\frac{5\sqrt{6}}{8} (\bar{m}_l^4 - \bar{m}_s^4) + \frac{5\sqrt{6}}{12} (\bar{m}_l^2 - \bar{m}_s^2) (\bar{m}_l^2 + 2\bar{m}_s^2) + \mathcal{O}(\bar{m}_f^6) \right] \\
&= \frac{5\sqrt{6}}{96} \left(1 - \frac{\eta_q}{4} \right) \frac{h^2}{k^2} \rho_2^{-3/2} \left[-\frac{1}{2} (\bar{m}_l^4 - \bar{m}_s^4) + \frac{1}{3} (\bar{m}_l^2 - \bar{m}_s^2) (\bar{m}_l^2 + 2\bar{m}_s^2) \right. \\
&\quad \left. + \frac{7}{12} (\bar{m}_l^6 - \bar{m}_s^6) - \frac{7}{12} (\bar{m}_l^2 - \bar{m}_s^2) (\bar{m}_l^4 + 2\bar{m}_s^4) + \dots \right]
\end{aligned} \tag{48}$$

As calculated above, the leading-order of $\partial A_{qk}/\partial \rho_2$ and the leading-order and next leading-order of $\partial^2 A_{qk}/(\partial \rho_2)^2$ are canceled out. This will cause numerical problems and we introduce a scalar $\Lambda_2 \sim 2\text{GeV}$. Above the scalar Λ_2 , Taylor expansion of quark loop function at zero temperature are employed.

6 Appendix.C

The Kobayashi-Maskawa-'t Hooft coupling \bar{c}_A , should decrease at high scalar and high temperatures. However, if we keep c_A a constant, and

$$\bar{c}_A = \frac{c_A}{Z_\phi^{3/2}} \quad (49)$$

\bar{c}_A will increase with scalar and temperature. One scheme is given in ref [?], which assume that \bar{c}_A is a constant. However, this scheme cause a very sharp phase transitions. We assume c_A is a infrared enhancement function

$$c_A = c_{A,IR} \frac{1}{e^{\frac{k-k_{cut}}{\Delta_k}} + 1} \quad (50)$$

here $k_{cut} = 1GeV$ and $\Delta_k = 20MeV$. And \bar{c}_A still dressed as $\bar{c}_A = c_A/Z_\phi^{3/2}$.

7 Appendix.D

As we known that

$$T_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (51)$$

We have

$$\Sigma_0 = T_0 \sigma_0 + T_8 \sigma_8 = \begin{pmatrix} \frac{1}{\sqrt{6}} \sigma_0 + \frac{1}{2\sqrt{3}} \sigma_8 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} \sigma_0 + \frac{1}{2\sqrt{3}} \sigma_8 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} \sigma_0 - \frac{1}{\sqrt{3}} \sigma_8 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & \sigma_L & 0 \\ 0 & 0 & \sqrt{2} \sigma_S \end{pmatrix} = T_L \sigma_L + T_S \sigma_S \quad (52)$$

so

$$T_L = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (53)$$