0.1 定义

现将介子波函数重整化系数$Z_{\Sigma,k}$分开,定义

$$\bar{\sigma}_L = Z^{\frac{1}{2}}_{\phi_L,k}\sigma_L \quad \bar{\sigma}_S = Z^{\frac{1}{2}}_{\phi_S,k}\sigma_S \tag{1}$$

因为介子场的真空期待值$\Sigma_0 = \sigma_0 T^0 + \sigma_8 T^8$,并且考虑$\sigma_l, \sigma_s$与$\sigma_0, \sigma_8$的关系

$$\begin{pmatrix} \phi_L \\ \phi_S \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_0 \end{pmatrix} \tag{2}$$

易得

$$\Sigma_0 = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & \sigma_L & 0 \\ 0 & 0 & \sqrt{2}\sigma_S \end{pmatrix} \tag{3}$$

所以，我们定义的介子波函数重整化

$$Z_{\Sigma,k} = \begin{pmatrix} Z_{\phi_L,k} & 0 & 0 \\ 0 & Z_{\phi_L,k} & 0 \\ 0 & 0 & Z_{\phi_S,k} \end{pmatrix} \tag{4}$$

此外，反常量纲定义

$$\eta_{\phi_L} = -\frac{\partial_t Z_{\phi_L,k}}{Z_{\phi_L,k}} \quad \eta_{\phi_S} = -\frac{\partial_t Z_{\phi_S,k}}{Z_{\phi_S,k}} \tag{5}$$

考虑赝标介子场

$$\begin{aligned} \Sigma &= \sum_{a=0}^{8} \pi_a T^a \\ &= \frac{1}{2} \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta_8 + \sqrt{\frac{2}{3}}\eta_0 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 + \sqrt{\frac{2}{3}}\eta_0 & K^0 \\ \sqrt{2}K^- & \bar{K}^0 & -\frac{2}{\sqrt{3}}\eta_8 + \sqrt{\frac{2}{3}}\eta_0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \pi^0 + \eta_L & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta_L & K^0 \\ \sqrt{2}K^- & \bar{K}^0 & \sqrt{2}\eta_S \end{pmatrix} \end{aligned} \tag{6}$$

可以看到

$$Z_{\pi,k} = Z_{\phi_L,k} \tag{7}$$

对于其它介子，考虑到混合角等情况，建立严格的重整化关系比较复杂。我们做近似

$$Z_{K/\kappa,k} = Z_{\phi_S,k} \quad Z_{others,k} = Z_{\phi_L,k} \tag{8}$$

从另一个角度思考，相当于Z_Σ可能不仅仅是简单的对角阵，而是个3×3 的矩阵。

0.2 有效作用势

考虑有效作用势

$$\bar{U}_{total} = U_{total} = \bar{U}(\bar{\rho}_1, \bar{\rho}_2) - \bar{c}_A \bar{\xi} - \bar{j}_L \bar{\sigma}_L - \bar{j}_S \bar{\sigma}_S \tag{9}$$

硬破缺项：

$$\bar{j}_L = \frac{j_L}{Z^{\frac{1}{2}}_{\phi_L,k}} \quad \bar{j}_S = \frac{j_S}{Z^{\frac{1}{2}}_{\phi_S,k}} \tag{10}$$

所以

$$\begin{aligned} \partial_t \bar{j}_L &= -\frac{1}{2} Z^{-\frac{3}{2}}_{\phi_L,k} \partial_t Z_{\phi_L,k} j_L = \frac{1}{2}\eta_{\phi_L} \bar{j}_L \\ \partial_t \bar{j}_S &= -\frac{1}{2} Z^{-\frac{3}{2}}_{\phi_S,k} \partial_t Z_{\phi_S,k} j_S = \frac{1}{2}\eta_{\phi_S} \bar{j}_S \end{aligned} \tag{11}$$

t'Hooft项

$$\xi = \frac{\sigma_L^2 \sigma_S}{2\sqrt{2}} \tag{12}$$

所以

$$\bar{c}_A = \frac{c_A}{Z_{\phi_L,k} Z^{\frac{1}{2}}_{\phi_S,k}} \tag{13}$$

然而，我们依然令

$$\partial_t \bar{c}_A = 0 \tag{14}$$

我们定义手征对称的重整化变量

$$\begin{aligned} \bar{\rho}_1 &= \frac{1}{2} Z_{\phi_L,k}(\sigma_L^2 + \sigma_S^2) = \frac{1}{2}(\bar{\sigma}_L^2 + \frac{Z_{\phi_L,k}}{Z_{\phi_S,k}} \bar{\sigma}_S^2) \\ \bar{\rho}_2 &= \frac{1}{24} Z^2_{\phi_L,k}(\sigma_L^2 - 2\sigma_S^2)^2 = \frac{1}{24}(\bar{\sigma}_L^2 - 2\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}} \bar{\sigma}_S^2)^2 \end{aligned} \tag{15}$$

和有效作用势的手征对称部分

$$\bar{U}(\bar{\rho}_1,\bar{\rho}_2)=U(\rho_1,\rho_2) \tag{16}$$

为计算有效作用势，我们做泰勒展开

$$\bar{U}(\bar{\rho}_1,\bar{\rho}_2)=\sum_{m,n=0}^{N_v}\frac{\bar{\lambda}_{mn,k}}{m!n!}(\bar{\rho}_1-\bar{\kappa}_{1,k})^m(\bar{\rho}_2-\bar{\kappa}_{2,k})^n \tag{17}$$

展开系数和展开点

$$\bar{\lambda}_{mn,k}=\frac{\lambda_{mn,k}}{Z^{m+2n}_{\phi_L,k}}\quad \bar{\kappa}_{1,k}=Z_{\phi_L,k}\kappa_{1,k}\quad \bar{\kappa}_{2,k}=Z^2_{\phi_L,k}\kappa_{2,k} \tag{18}$$

我们计算展开点$\kappa_{1,k}$的流$\partial_t\bar{\kappa}_{1,k}$

$$\begin{aligned}\partial_t\bar{\kappa}_{1,k}&=\partial_t Z_{\phi_L,k}\cdot\frac{1}{2}(\sigma_{L0}^2+\sigma_{S0}^2)+Z_{\phi_L,k}\cdot\partial_t\big(\frac{1}{2}(\sigma_{L0}^2+\sigma_{S0}^2)\big)\\&=-\eta_{\phi_L}\bar{\kappa}_{1,k}+Z_{\phi_L,k}\partial_t\kappa_{1,k}\\&=-\eta_{\phi_L}\bar{\kappa}_{1,k}+\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{\kappa}_{1,k}\end{aligned} \tag{19}$$

同理：

$$\begin{aligned}\partial_t\bar{\kappa}_{2,k}&=-2\eta_{\phi_L}\bar{\kappa}_{2,k}+\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{\kappa}_{2,k}\\\partial_t\bar{\lambda}_{mn,k}&=(m+2n)\eta_{\phi_L}\bar{\lambda}_{mn,k}+\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{\lambda}_{mn,k}\end{aligned} \tag{20}$$

因此

$$\begin{aligned}\partial^m_{\bar{\rho}_1}\partial^n_{\bar{\rho}_2}\big(\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{U}(\bar{\rho}_1,\bar{\rho}_2)\big)\big|_{\substack{\bar{\rho}_1=\bar{\kappa}_{1,k}\\\bar{\rho}_2=\bar{\kappa}_{2,k}}}&=\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{\lambda}_{mn,k}-\bar{\lambda}_{(m+1)n,k}\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{\kappa}_{1,k}-\bar{\lambda}_{m(n+1),k}\partial_t|_{\bar{\rho}_1,\bar{\rho}_2}\bar{\kappa}_{2,k}\\&=\partial_t\bar{\lambda}_{mn,k}-(m+2n)\eta_{\phi_L}\bar{\lambda}_{mn,k}-\bar{\lambda}_{(m+1)n,k}(\partial_t\bar{\kappa}_{1,k}+\eta_{\phi_L}\bar{\kappa}_{1,k})\\&-\bar{\lambda}_{m(n+1),k}(\partial_t\bar{\kappa}_{2,k}+2\eta_{\phi_L}\bar{\kappa}_{2,k})\\&\equiv\mathrm{drmndtV}\end{aligned} \tag{21}$$

所以

$$\partial_t\bar{\lambda}_{mn,k}=(m+2n)\eta_{\phi_L}\bar{\lambda}_{mn,k}+\bar{\lambda}_{(m+1)n,k}(\partial_t\bar{\kappa}_{1,k}+\eta_{\phi_L}\bar{\kappa}_{1,k})+\bar{\lambda}_{m(n+1),k}(\partial_t\bar{\kappa}_{2,k}+2\eta_{\phi_L}\bar{\kappa}_{2,k})+\mathrm{drmndtV} \tag{22}$$

特别的，我们写出$\bar{\lambda}_{10}$和$\bar{\lambda}_{01}$阶的流方程

$$\begin{aligned}\partial_t\bar{\lambda}_{10,k}&=\eta_{\phi_L}\bar{\lambda}_{10,k}+\lambda_{20,k}(\partial_t\bar{\kappa}_{1,k}+\eta_{\phi_L}\bar{\kappa}_{1,k})+\bar{\lambda}_{11,k}(\partial_t\bar{\kappa}_{2,k}+2\eta_{\phi_L}\bar{\kappa}_{2,k})+\mathrm{dr10dtV}\\\partial_t\bar{\lambda}_{01,k}&=2\eta_{\phi_L}\bar{\lambda}_{01,k}+\lambda_{11,k}(\partial_t\bar{\kappa}_{1,k}+\eta_{\phi_L}\bar{\kappa}_{1,k})+\bar{\lambda}_{02,k}(\partial_t\bar{\kappa}_{2,k}+2\eta_{\phi_L}\bar{\kappa}_{2,k})+\mathrm{dr01dtV}\end{aligned} \tag{23}$$

考虑到运动学方程：

$$\frac{\partial \bar{U}_{total}}{\partial \bar{\sigma}_L}\bigg|_{\substack{\bar{\sigma}_L=\bar{\sigma}_{L0}\\ \bar{\sigma}_S=\bar{\sigma}_{S0}}} = \frac{\partial \bar{U}_{total}}{\partial \bar{\sigma}_S}\bigg|_{\substack{\bar{\sigma}_L=\bar{\sigma}_{L0}\\ \bar{\sigma}_S=\bar{\sigma}_{S0}}} = 0 \tag{24}$$

我们得到

$$\begin{aligned}
0 &= \bar{\lambda}_{10,k}\bar{\sigma}_{L0} + \frac{1}{6}\bar{\lambda}_{10,k}\bar{\sigma}_{L0}(\bar{\sigma}_{L0}^2 - 2\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0}^2) - \bar{c}_A\frac{\bar{\sigma}_{L0}\bar{\sigma}_{S0}}{\sqrt{2}} - \bar{j}_L \\
0 &= \bar{\lambda}_{10,k}\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0} - \frac{1}{3}\bar{\lambda}_{10,k}\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0}(\bar{\sigma}_{L0}^2 - 2\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0}^2) - \bar{c}_A\frac{\bar{\sigma}_{L0}^2}{2\sqrt{2}} - \bar{j}_S
\end{aligned} \tag{25}$$

上式对t求偏导，再加上之前算得的$\bar{\lambda}_{10}$和$\bar{\lambda}_{01}$阶的流方程(23)，以及$\partial_t\bar{\kappa}_{1,k}$，$\partial_t\bar{\kappa}_{2,k}$：

$$\begin{aligned}
\partial_t\bar{\kappa}_{1,k} &= \bar{\sigma}_{L0}\bar{\partial}_t\bar{\sigma}_{L0} + \partial_t\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\cdot\frac{1}{2}\bar{\sigma}_{S0}^2 + \frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0}\bar{\partial}_t\bar{\sigma}_{S0} \\
\partial_t\bar{\kappa}_{2,k} &= \frac{1}{6}\left(\bar{\sigma}_{L0}^2 - 2\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0}^2\right)\left(\bar{\sigma}_{L0}\partial_t\bar{\sigma}_{L0} - \partial_t\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\cdot\bar{\sigma}_{S0}^2 - 2\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\bar{\sigma}_{S0}\partial_t\bar{\sigma}_{S0}\right)
\end{aligned} \tag{26}$$

联立求解，即可求得展开点的流方程。具体计算过程在mathematica里。最小值点的流方程可以用红外时是否为势U_{total}来检验。注意到做替换

$$\sigma_{S0}^* = \sqrt{\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}}\bar{\sigma}_{S0} \quad j_S^* = \sqrt{\frac{Z_{\phi_S,k}}{Z_{\phi_L,k}}}\bar{j}_S \tag{27}$$

也就是全部回到$Z_{\phi_L,k}$的情景，然后通过

$$\begin{aligned}
\partial_t\bar{\sigma}_{S0} &= \partial_t\left(\frac{Z_{\phi_S,k}}{Z_{\phi_L,k}}\right)^{1/2}\sigma_{S0}^* + \left(\frac{Z_{\phi_S,k}}{Z_{\phi_L,k}}\right)^{1/2}\partial_t\sigma_{S0}^* \\
&= \left(\frac{Z_{\phi_S,k}}{Z_{\phi_L,k}}\right)^{1/2}\left(\frac{1}{2}(\eta_{\phi_L} - \eta_{\phi_S})\sigma_{S0}^* + \partial_t\sigma_{S0}^*\right)
\end{aligned} \tag{28}$$

计算出$\bar{\sigma}_{S0}$的流。

0.3 质量

夸克质量

$$\bar{M}_{q,k} = \frac{m_{q,k}}{Z_{q,k}} \tag{29}$$

Yukawa项我们用$Z_{\phi_L,k}$来重整化

$$\bar{h}_k = \frac{h_k}{Z_{q,k}Z_{\phi_L,k}^{\frac{1}{2}}} \tag{30}$$

所以轻和奇异夸克质量

$$\begin{aligned} \bar{M}_{l,k} &= \frac{m_{l,k}}{Z_{q,k}} = \frac{1}{Z_{q,k}}\frac{h}{2}\sigma_L = \frac{\bar{h}}{2}\bar{\sigma}_L \\ \bar{M}_{s,k} &= \frac{m_{s,k}}{Z_{q,k}} = \frac{1}{Z_{q,k}}\frac{h}{\sqrt{2}}\sigma_S = \frac{\bar{h}}{\sqrt{2}}\left(\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\right)^{\frac{1}{2}}\bar{\sigma}_S \end{aligned} \tag{31}$$

质量的平方以

$$\begin{aligned} \bar{M}^2_{l,k} &= \frac{\bar{h}^2}{4}\bar{\sigma}^2_L \\ \bar{M}^2_{s,k} &= \frac{\bar{h}^2}{2}\left(\frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}\right)\bar{\sigma}^2_S \end{aligned} \tag{32}$$

计算$\bar{\rho}_1$,$\bar{\rho}_2$的偏导前，为方便我们引入：

$$z_{LS} = \frac{Z_{\phi_L,k}}{Z_{\phi_S,k}} \tag{33}$$

由(15)式得

$$\begin{aligned} \bar{\sigma}^2_L &= \frac{2}{3}(2\bar{\rho}_1 - \sqrt{6\bar{\rho}_2}) \\ \bar{\sigma}^2_S &= \frac{2}{3}\frac{1}{z_{LS}}(\bar{\rho}_1 + \sqrt{6\bar{\rho}_2}) \end{aligned} \tag{34}$$

可以得到

$$\begin{aligned} \frac{\partial\bar{\sigma}^2_L}{\partial\bar{\rho}_1} &= \frac{4}{3} \quad \frac{\partial\bar{\sigma}^2_L}{\partial\bar{\rho}_2} = -\frac{\sqrt{6}}{3}\bar{\rho}_2^{-1/2} \\ \frac{\partial\bar{\sigma}^2_S}{\partial\bar{\rho}_1} &= \frac{2}{3}\frac{1}{z_{LS}} \quad \frac{\partial\bar{\sigma}^2_S}{\partial\bar{\rho}_2} = \frac{\sqrt{6}}{3}\frac{1}{z_{LS}}\bar{\rho}_2^{-1/2} \end{aligned} \tag{35}$$

所以

$$\begin{aligned} \frac{\partial\bar{M}^2_{l,k}}{\partial\bar{\rho}_1} &= \frac{\bar{h}^2}{3} \\ \frac{\partial\bar{M}^2_{l,k}}{\partial\bar{\rho}_2} &= -\frac{\sqrt{6}\bar{h}^2}{12}\bar{\rho}_2^{-1/2} \\ \frac{\partial\bar{M}^2_{s,k}}{\partial\bar{\rho}_1} &= \frac{\bar{h}^2}{3} \\ \frac{\partial\bar{M}^2_{s,k}}{\partial\bar{\rho}_2} &= \frac{\sqrt{6}\bar{h}^2}{6}\bar{\rho}_2^{-1/2} \end{aligned} \tag{36}$$

所以夸克质量及其导数相对于不区分时没有改变。

介子质量

$$\bar{M}_{\phi,k} = \frac{m_{\phi,k}}{Z_{\phi,k}^{1/2}} \tag{37}$$

$m_{\phi,k}$代表裸的介子质量，这里我们以π介子和K介子为例：

$$\begin{aligned} m_{\pi,k}^2 &= U_{total}^{(1,0)} + \frac{1}{6}U_{total}^{(0,1)}(\sigma_L^2 - 2\sigma_S^2) - \frac{c_A}{\sqrt{2}}\sigma_S \\ m_{K,k}^2 &= U_{total}^{(1,0)} + \frac{1}{6}U_{total}^{(0,1)}(-3\sqrt{2}\sigma_L\sigma_S + \sigma_L^2 + 4\sigma_S^2) - \frac{c_A}{2}\sigma_L \end{aligned} \tag{38}$$

所以重整化的介子质量：

$$\begin{aligned} \bar{M}_{\pi,k}^2 &= \bar{U}_{total}^{(1,0)} + \frac{1}{6}\bar{U}_{total}^{(0,1)}(\bar{\sigma}_L^2 - 2z_{LS}\bar{\sigma}_S^2) - \frac{\bar{c}_A}{\sqrt{2}}\bar{\sigma}_S \\ \bar{M}_{K,k}^2 &= z_{LS}\big(\bar{U}_{total}^{(1,0)} + \frac{1}{6}\bar{U}_{total}^{(0,1)}(-3\sqrt{2}z_{LS}^{1/2}\bar{\sigma}_L\bar{\sigma}_S + \bar{\sigma}_L^2 + 4z_{LS}\bar{\sigma}_S^2) - z_{LS}^{-1/2}\frac{\bar{c}_A}{2}\bar{\sigma}_L\big) \end{aligned} \tag{39}$$

也就是重整化的$\bar{M}_{\pi,k}$与不区分时没变化，$\bar{M}_{K,k}$与单一重整化波函数的情况差一个系数$z_{LS} = \frac{Z_{\phi_L,k}}{Z_{\phi_S,k}}$。

0.4 阈值函数

介子的规制函数需要做相应的调整：

$$R_k^{\pi/K}(\vec{p}^2) = Z_{\pi/K,k}\vec{p}^2 r_b(\vec{p}^2/k^2) \tag{40}$$

这最终导致介子圈函数的分开：

$$l_{(0)}^{(B)}(\pi/K) = \frac{2}{3}\frac{k}{E_{\pi/K}}\left(1 - \frac{\eta_{\pi/K}}{5}\right)\left(\frac{1}{2} + n_B(\bar{M}_{\pi/K};T,\mu)\right) \tag{41}$$

这可能是我觉得的实质上唯一分离的地方。但K介子和κ介子对整体的影响很小。