1 effective action

The meson part of QCD effective action reads

$$\Gamma_{k,meson} = \int_x \left\{ tr\big(Z_{\Sigma,k}^{1/2} \cdot \partial_\mu \Sigma \cdot Z_{\Sigma,k}^{1/2} \cdot \partial_\mu \Sigma^\dagger\big) + \tilde{U}_k(\Sigma, \Sigma^\dagger) \right\}, \tag{1}$$

here, the meson field :

$$\Sigma = T^a(\sigma^a + i\pi^a) . \quad (a = 0, 1, ..., 8) \tag{2}$$

with $T^a = \lambda^a/2 (a = 1, ..., 8)$ and $T^0 = \frac{1}{\sqrt{2N_f}} \mathbb{I}_{N_f \times N_f}$ are generators of $SU(N_f = 3)$. σ^a and π^a mean the scalar and pseudoscalar fields, respectively. The physical meson can be written obviously:

$$\Sigma = \frac{1}{2} \begin{pmatrix} a_0^0 + \sigma_L + i\pi^0 + i\eta_L & \sqrt{2}(a_0^+ + i\pi^+) & \sqrt{2}(\kappa^+ + iK^+) \\ \sqrt{2}(a_0^- + i\pi^-) & -a_0^0 + \sigma_L - i\pi^0 + i\eta_L & \sqrt{2}(\kappa^0 + iK^0) \\ \sqrt{2}(\kappa^- + iK^-) & \sqrt{2}(\bar{\kappa}^0 + i\bar{K}^0) & \sqrt{2}(\sigma_S + i\eta_S) \end{pmatrix} \tag{3}$$

the meson effective potential can be devided into three parts

$$\tilde{U}_k(\Sigma) = U_k(\rho_1, \tilde{\rho}_2) - c_A\xi - j_L\sigma_L - j_S\sigma_S , \tag{4}$$

here $U_k(\rho_1, \tilde{\rho}_2)$ is an arbitrary function of chiral symmetry invariant variables $\rho_1, \tilde{\rho}_2$. $c_A\xi$ is Kobayashi-Maskawa-'t Hooft trem which breaks $U_A(1)$ symmetry. The last two terms of Eq.(4) are linear sigma terms, which break the chiral symmetry.

2 Result

Pressure:

$$\frac{p}{T^4} = \frac{U(0,0) - U((T,\mu)}{T^4} \tag{5}$$

and n-th order cumulats

$$\chi_n^B = \frac{\partial^n}{\partial(\mu_B/T)^n} \frac{p}{T^4} \tag{6}$$

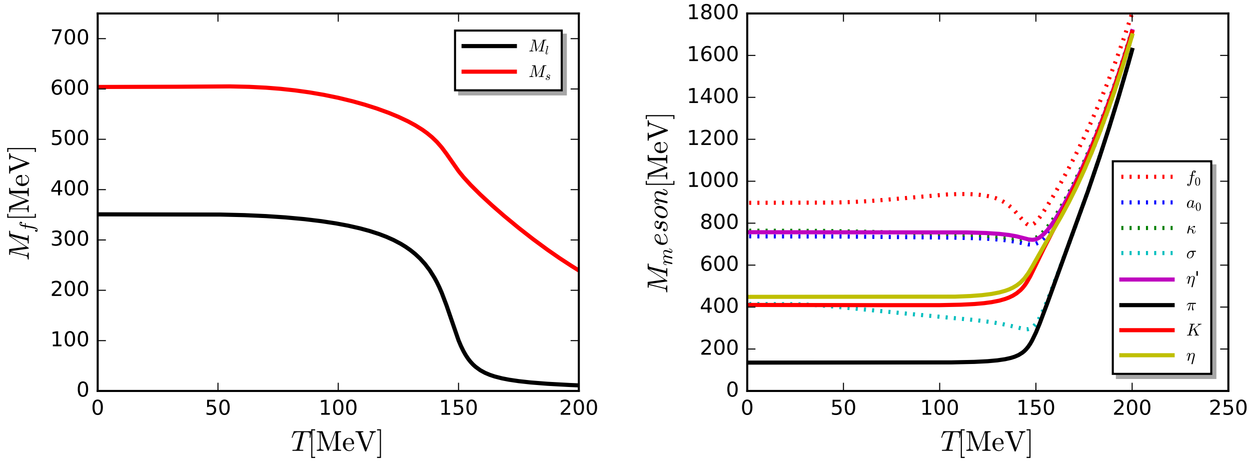

Figure 1: Quark and meson masses as functions of temperature with js/jl=26.5

3 Appendix.A

The meson masses can be obtained by Hessian matrix:

$$H_{p,LL} = \frac{c_A \sigma_S}{\sqrt{2}} + \frac{1}{6} U^{(0,1)} (\sigma_L^2 - 2\sigma_S^2) + U^{(1,0)} \tag{7}$$

$$H_{p,LS} = \frac{c_A \sigma_L}{\sqrt{2}} \tag{8}$$

$$H_{p,SS} = U^{(1,0)} - \frac{1}{3} U^{(0,1)} (\sigma_L^2 - 2\sigma_S^2) \tag{9}$$

$$H_{p,11} = -\frac{c_A \sigma_S}{\sqrt{2}} + \frac{1}{6} U^{(0,1)} (\sigma_L^2 - 2\sigma_S^2) + U^{(1,0)} \tag{10}$$

$$H_{p,44} = -\frac{c_A \sigma_L}{2} + U^{(1,0)} + \frac{1}{6} U^{(1,0)} (\sigma_L^2 - 3\sqrt{2}\sigma_L \sigma_S + 4\sigma_S^2) \tag{11}$$

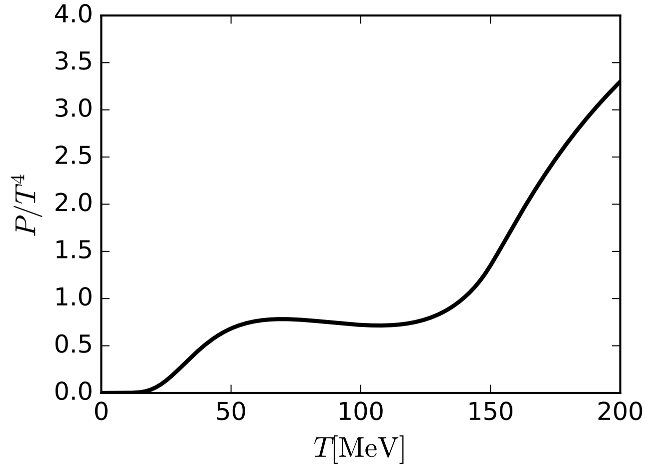

Figure 2: pressure

$$
\begin{aligned}
H_{s,LL} =& -\frac{c_A \sigma_S}{\sqrt{2}} + U^{(1,0)} + U^{(2,0)}\sigma_L^2 + \frac{1}{6}U^{(0,1)}(3\sigma_L^2 - 2\sigma_S^2) \\
&+ \frac{1}{36}\sigma_L^2(\sigma_L^2 - 2\sigma_S^2)(U^{(0,2)}(\sigma_L^2 - 2\sigma_S^2) + 12U^{(1,1)}) \qquad (12)\\
H_{s,LS} =& -\frac{c_A \sigma_L}{\sqrt{2}} + U^{(2,0)}\sigma_L\sigma_S - \frac{2}{3}U^{(0,1)}\sigma_L\sigma_S \\
&- \frac{1}{18}U^{(0,2)}\sigma_L\sigma_S(\sigma_L^2 - 2\sigma_S^2)^2 - \frac{1}{6}U^{(1,1)}\sigma_L\sigma_S(\sigma_L^2 - 2\sigma_S^2) \qquad (13)\\
H_{s,LS} =& U^{(1,0)} + U^{(2,0)}\sigma_S^2 - \frac{1}{3}U^{(0,1)}(\sigma_L^2 - 6\sigma_S^2) \\
&+ \frac{1}{9}\sigma_S^2(-6U^{(1,1)}(\sigma_L^2 - 2\sigma_S^2) + U^{(0,2)}(\sigma_L^2 - 2\sigma_S^2)^2) \qquad (14)\\
H_{s,11} =& \frac{c_A \sigma_S}{\sqrt{2}} + U^{(1,0)} + \frac{1}{6}U^{(0,1)}(7\sigma_L^2 - 2\sigma_S^2) \qquad (15)\\
H_{s,44} =& \frac{c_A \sigma_L}{2} + U^{(1,0)} + \frac{1}{6}U^{(0,1)}(\sigma_L^2 + 3\sqrt{2}\sigma_L\sigma_S + 4\sigma_S^2) \qquad (16)
\end{aligned}
$$

4 Appendix.B

As we defined

$$
\begin{aligned}
\rho_1 &= \frac{1}{2}(\sigma_l^2 + \sigma_s^2) \\
\rho_2 &= \frac{1}{24}(\sigma_l^2 - 2\sigma_s^2)^2
\end{aligned}
\qquad (17)
$$

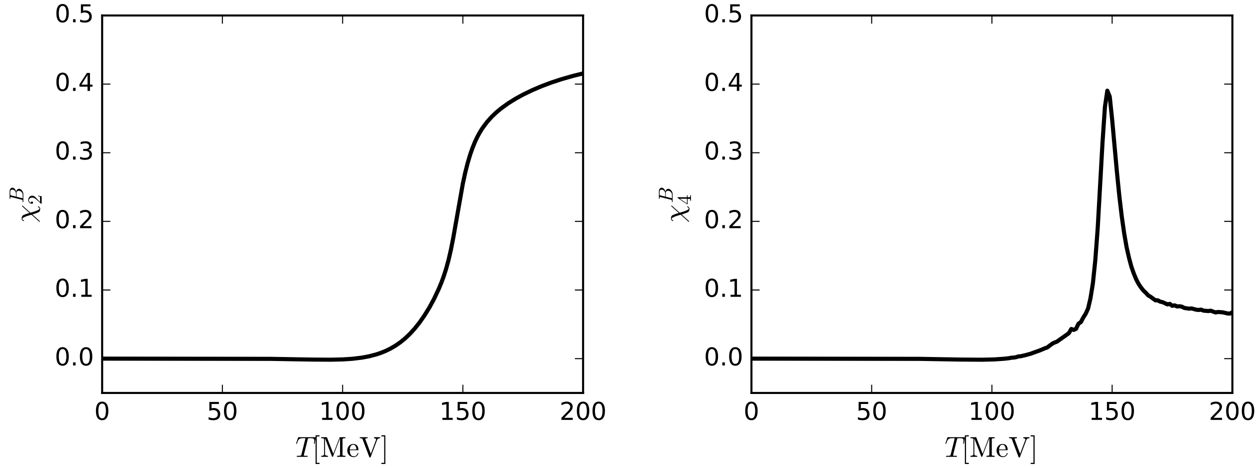

Figure 3: pressure

Note that

$$\sigma_l < \sqrt{2}\sigma_s \tag{18}$$

then

$$\begin{aligned} 2\rho_1 &= \sigma_l^2 + \sigma_s^2 \\ -2\sqrt{6\rho_2} &= \sigma_l^2 - 2\sigma_s^2 \end{aligned} \tag{19}$$

then

$$\begin{aligned} \sigma_l^2 &= \frac{2}{3}(2\rho_1 - \sqrt{6\rho_2}) \\ \sigma_s^2 &= \frac{2}{3}(\rho_1 + \sqrt{6\rho_2}) \end{aligned} \tag{20}$$

so

$$\begin{aligned} \frac{\partial \sigma_l^2}{\partial \rho_1} &= \frac{4}{3} \quad \frac{\partial \sigma_l^2}{\partial \rho_2} = -\frac{\sqrt{6}}{3}\rho_2^{-1/2} \\ \frac{\partial \sigma_s^2}{\partial \rho_1} &= \frac{2}{3} \quad \frac{\partial \sigma_s^2}{\partial \rho_2} = \frac{\sqrt{6}}{3}\rho_2^{-1/2} \end{aligned} \tag{21}$$

so we get

$$\frac{\partial \sigma_l^2}{\partial \rho_2} = -\frac{\partial \sigma_s^2}{\partial \rho_2} \tag{22}$$

and

$$\frac{\partial^2 \sigma_l^2}{\partial \rho_2^2} = -\frac{\partial^2 \sigma_s^2}{\partial \rho_2^2} = \frac{\sqrt{6}}{6}\rho_2^{-3/2} \tag{23}$$

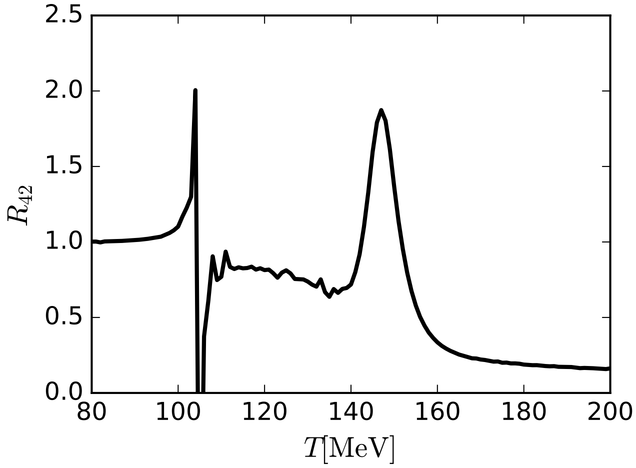

Figure 4: pressure

The quark masses are given as

$$m_l^2 = h^2 \frac{\sigma_l^2}{4} \quad m_s^2 = h^2 \frac{\sigma_s^2}{2} \tag{24}$$

therefore

$$\begin{aligned} \frac{\partial m_l^2}{\partial \rho_2} &= -\frac{1}{2}\frac{\partial m_s^2}{\partial \rho_2} = -\frac{\sqrt{6}}{12} h^2 \rho_2^{-1/2} \\ \frac{\partial^2 m_l^2}{\partial \rho_2^2} &= -\frac{1}{2}\frac{\partial^2 m_s^2}{\partial \rho_2^2} = \frac{\sqrt{6}}{24} h^2 \rho_2^{-3/2} \end{aligned} \tag{25}$$

The quark loop function

$$l^{(f)} = \frac{1}{3}\left(1 - \frac{\eta_q}{4}\right) \frac{1}{\sqrt{1+\bar{m}_f^2}} (1 - n_f(E+\mu) - n_f(E-\mu)) \tag{26}$$

here $\bar{m}_f$ is dimensionless mass.

Then the light and strange quarks part of the the potential flow:

$$\partial_t U = -4N_c \frac{k^4}{4\pi^2}\left[2l^{(f)}(\bar{m}_l^2) + l^{(f)}(\bar{m}_s^2)\right] \tag{27}$$

For simplify,we only consider the square brackets above:

$$A_{qk} = 2l^{(f)}(\bar{m}_l^2) + l^{(f)}(\bar{m}_s^2) \tag{28}$$

and

$$\begin{aligned}\frac{\partial A_{qk}}{\partial \rho_2} &= 2\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2}\frac{\partial \bar{m}_l^2}{\partial \rho_2} + \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2}\frac{\partial \bar{m}_s^2}{\partial \rho_2} \\ &= 2\frac{\partial \bar{m}_l^2}{\partial \rho_2}\Big(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} - \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2}\Big)\end{aligned} \tag{29}$$

We consider $T = 0$ case:

$$l^{(f)} = \frac{1}{3}\Big(1 - \frac{\eta_q}{4}\Big)\frac{1}{\sqrt{1+\bar{m}_f^2}} \tag{30}$$

then

$$\begin{aligned}\frac{\partial A_{qk}}{\partial \rho_2} &= 2\frac{\partial \bar{m}_l^2}{\partial \rho_2}\Big(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial \bar{m}_l^2} - \frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial \bar{m}_s^2}\Big) \\ &= -\frac{\partial \bar{m}_l^2}{\partial \rho_2}\frac{1}{3}\Big(1 - \frac{\eta_q}{4}\Big)\Big((1+\bar{m}_l^2)^{-3/2} - (1+\bar{m}_s^2)^{-3/2}\Big)\end{aligned} \tag{31}$$

Because $\bar{m}_q^2 \ll 1, \bar{m}_l^2 \sim 5 \times 10^{-16}$ at $k = \Lambda$, we use Taylor expansion

$$\begin{aligned}\frac{\partial A_{qk}}{\partial \rho_2} &= -\frac{1}{3}\Big(1 - \frac{\eta_q}{4}\Big)\frac{\partial \bar{m}_l^2}{\partial \rho_2}\Big((1 - \frac{3}{2}\bar{m}_l^2 + \frac{15}{8}\bar{m}_l^4 \cdots) - (1 - \frac{3}{2}\bar{m}_s^2 + \frac{15}{8}\bar{m}_s^4 + \cdots)\Big) \\ &= -\frac{1}{3}\Big(1 - \frac{\eta_q}{4}\Big)\Big(-\frac{\sqrt{6}}{12}\frac{h^2}{k^2}\rho_2^{-1/2}\Big)\Big(-\frac{3}{2}(\bar{m}_l^2 - \bar{m}_s^2) + \frac{15}{8}(\bar{m}_l^4 - \bar{m}_s^4) + \cdots\Big) \\ &= -\frac{\sqrt{6}}{24}\Big(1 - \frac{\eta_q}{4}\Big)\Big(\frac{h^2}{k^2}\rho_2^{-1/2}\Big)\Big((\bar{m}_l^2 - \bar{m}_s^2) - \frac{5}{4}(\bar{m}_l^4 - \bar{m}_s^4) + \frac{35}{24}(\bar{m}_l^6 - \bar{m}_s^6)\cdots\Big)\end{aligned} \tag{32}$$

If we consider second order derivative, it will be worse:

$$\begin{aligned}
\frac{\partial}{\partial\rho_2}\Big(\frac{\partial A_{qk}}{\partial\rho_2}\Big) &= 2\Big[\frac{\partial^2\bar{m}_l^2}{\partial\rho_2^2}\Big(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial\bar{m}_l^2}-\frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial\bar{m}_s^2}\Big)+\frac{\partial\bar{m}_l^2}{\partial\rho_2}\Big(\frac{\partial\bar{m}_l^2}{\partial\rho_2}\frac{\partial^2 l^{(f)}(\bar{m}_l^2)}{\partial(\bar{m}_l^2)^2}-\frac{\partial\bar{m}_s^2}{\partial\rho_2}\frac{\partial^2 l^{(f)}(\bar{m}_s^2)}{\partial(\bar{m}_s^2)^2}\Big)\Big]\\
&= 2\Big[\frac{\partial^2\bar{m}_l^2}{\partial\rho_2^2}\Big(\frac{\partial l^{(f)}(\bar{m}_l^2)}{\partial\bar{m}_l^2}-\frac{\partial l^{(f)}(\bar{m}_s^2)}{\partial\bar{m}_s^2}\Big)+\Big(\frac{\partial\bar{m}_l^2}{\partial\rho_2}\Big)^2\Big(\frac{\partial^2 l^{(f)}(\bar{m}_l^2)}{\partial(\bar{m}_l^2)^2}+2\frac{\partial^2 l^{(f)}(\bar{m}_s^2)}{\partial(\bar{m}_s^2)^2}\Big)\Big]\\
&= \frac{2}{3}\Big(1-\frac{\eta_q}{4}\Big)\Big[-\frac{\sqrt{6}}{48}\frac{h^2}{k^2}\rho_2^{-3/2}\big((1+\bar{m}_l^2)^{-3/2}-(1+\bar{m}_s^2)^{-3/2}\big)\\
&+\frac{1}{32}\frac{h^4}{k^4}\rho_2^{-1}\big((1+\bar{m}_l^2)^{-5/2}+2(1+\bar{m}_s^2)^{-5/2}\big)\Big]\\
&= \frac{1}{24}\Big(1-\frac{\eta_q}{4}\Big)\frac{h^2}{k^2}\rho_2^{-3/2}\Big[-\frac{\sqrt{6}}{3}\big((1-\frac{3}{2}\bar{m}_l^2+\frac{15}{8}\bar{m}_l^4+\cdots)-(1-\frac{3}{2}\bar{m}_s^2+\frac{15}{8}\bar{m}_s^4\bar{m}_l^4+\cdots)\big)\\
&+\frac{1}{2}\frac{h^2}{k^2}\rho_2^{1/2}\big((1-\frac{5}{2}\bar{m}_l^2+\cdots)+2(1-\frac{5}{2}\bar{m}_s^2+\cdots)\big)\Big]\\
&= \frac{1}{24}\Big(1-\frac{\eta_q}{4}\Big)\frac{h^2}{k^2}\rho_2^{-3/2}\Big[-\frac{\sqrt{6}}{3}\big((1-\frac{3}{2}\bar{m}_l^2+\frac{15}{8}\bar{m}_l^4-\frac{35}{16}\bar{m}_l^6+\cdots)\\
&-(1-\frac{3}{2}\bar{m}_s^2+\frac{15}{8}\bar{m}_s^4-\frac{35}{16}\bar{m}_s^6+\cdots)\big)\\
&+\frac{1}{2}\frac{2}{\sqrt{6}}(\bar{m}_s^2-\bar{m}_l^2)\big((1-\frac{5}{2}\bar{m}_l^2+\frac{35}{8}\bar{m}_l^4+\cdots)+2(1-\frac{5}{2}\bar{m}_s^2+\frac{35}{8}\bar{m}_s^4+\cdots)\big)\Big]
\end{aligned} \tag{33}$$

Obviously, the leading-order equal zero, and the next-leading-order is also vanished, and

$$\begin{aligned}
\frac{\partial}{\partial\rho_2}\Big(\frac{\partial A_{qk}}{\partial\rho_2}\Big) &= \frac{1}{24}\Big(1-\frac{\eta_q}{4}\Big)\frac{h^2}{k^2}\rho_2^{-3/2}\Big[-\frac{5\sqrt{6}}{8}(\bar{m}_l^4-\bar{m}_s^4)+\frac{5\sqrt{6}}{12}(\bar{m}_l^2-\bar{m}_s^2)(\bar{m}_l^2+2\bar{m}_s^2)+\mathscr{O}(\bar{m}_f^6)\Big]\\
&= \frac{5\sqrt{6}}{96}\Big(1-\frac{\eta_q}{4}\Big)\frac{h^2}{k^2}\rho_2^{-3/2}\Big[-\frac{1}{2}(\bar{m}_l^4-\bar{m}_s^4)+\frac{1}{3}(\bar{m}_l^2-\bar{m}_s^2)(\bar{m}_l^2+2\bar{m}_s^2)\\
&+\frac{7}{12}(\bar{m}_l^6-\bar{m}_s^6)-\frac{7}{12}(\bar{m}_l^2-\bar{m}_s^2)(\bar{m}_l^4+2\bar{m}_s^4)+\cdots\Big]
\end{aligned} \tag{34}$$

As calculated above, the leading-order of $\partial A_{qk}/\partial\rho_2$ and the leading-order and next leading-order of $\partial^2 A_{qk}/(\partial\rho_2)^2$ are canceled out. This will cause numerical problems and we introduce a scalar $\Lambda_2 \sim 2GeV$. Above the scalar Λ_2, Taylor expansion of quark loop function at zero temperature are employed.