Filter Design Using Transformed Variables

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Abstract—A concise description is given of some recently developed filter design techniques. The discussion includes equal-ripple and maximally flat passband filters with general stopbands, as well as equal-ripple stopband filters with general passbands. To solve the approximation problem and to improve numerical conditioning, the design is carried out exclusively in terms of one or two transformed frequency variables. A step-by-step description is given for the design of each filter type; the steps are so formulated that the erosion of significant digits is minimized. The design processes given are unique and are directly suitable for automatic computer programs. Experience with such programs, using the algorithms described in the paper, indicates that filters up to about degree 30 may be designed using only single precision in the calculations.

A discussion of some practical predistortion techniques, as well as a listing of available tabulated filter design information, is also included.

I. Introduction

A. Purpose

AMONG the attempts aimed at closing the technological gap between filter theorists and filter designers, few were as successful as the paper published by Saal and Ulbrich in the 1958 Special Issue of this Transactions [1]. It provided an eminently readable summary of those theoretical results, which had a direct bearing on design and gave explicit step-by-step descriptions of up-to-date synthesis procedures for most filters of practical importance. Finally, it included a useful set of tables (later supplemented by Saal[1]) of the element values in elliptic-function low-pass filters. Although some of the techniques described were novel, no effort was made to give theoretical derivations or proofs.

The intent of the present paper is to supplement this valuable work by providing filter design information based upon more recently developed techniques. The motive behind most of these new procedures was the wish to use digital computers. By removing the size limit of the largest designable filter, set by the sheer drudgery of hand calculation, the computer offered the exciting possibility of designing large, efficient high-quality filters. Initially, this hope was frustrated by the excessive loss of accuracy due to the ill-conditioning in the intermediate steps of the classical methods. This ill-conditioning rises rapidly with the size of filter; for hand-calculated filters, it was just tolerable, but with larger networks, one soon loses all accuracy in the element values, even with double-precision arithmetic.

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[1] See [34, Appendix I].

The urgency of this problem provoked the almost simultaneous appearance of several independently conceived and very similar solutions in different parts of the world. Basically, the solution is to handle the whole design with a different independent variable, with respect to which the various functions concerned are much better conditioned. The change of variable, which is most appropriate, simplifies some of the steps but complicates others; and even with this best choice of variable, there still remains some residual ill-conditioning, which demands special handling in certain steps. We have attempted to include all the important cases.

The improvement in the general numerical conditioning is such that, with normal specifications, filters up to the largest size one can construct (certainly up to degree 30) can be designed with single-precision arithmetic. Unfortunately, it is not possible, in framing the specification, to anticipate the occasional worsening of the conditioning that may arise due to near coincidence of infinite loss frequencies and the critical frequencies of the z or y matrices. To guard against such pathological cases, it is advisable to use double-precision arithmetic, or at least to have it available should the need arise.

It is now possible to write computer programs that will handle the complete design of most run-of-the-mill filters, from an input that specifies a required loss response, to an output that describes the structure and element values of the best filter. By maintaining and updating such a program, one can store a tremendous amount of design ability (unaffected by staff changes) and leave the design staff free to study the solution of more exciting problems.

B. Notations and Definitions

First, the terminology and notations[2] will be established. With reference to Fig. 1, define the following.

1) *Maximum Available Power:*

$$P_{\max} = \frac{E^2}{4R_1} \tag{1}$$

2) *Load Power:*

$$P_2 = \frac{V_2^2}{R_2} \tag{2}$$

3) *Driving-Point Impedances:*

$$Z_1(s) = \frac{V_1(s)}{I_1(s)}. \tag{3}$$

[2] Which differ only slightly from those of [1].

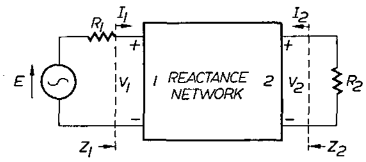


Fig. 1. Doubly terminated reactance two-port.

Z_2 is similarly obtained if the generator is in series with R_2.

4) *Reflection Coefficients:*

$$\rho_i(s) = \frac{R_i - Z_i(s)}{R_i + Z_i(s)} \qquad (i = 1, 2) \tag{4}$$

$$|\rho_i(j\omega)|^2 = \frac{P_{\max} - P_2}{P_{\max}} \leq 1 \tag{5}$$

$$\rho_1 = \frac{f(s)}{e(s)} \tag{6}$$

5) *Transducer Function:*[3]

$$H(s) = \frac{E}{V_2}\sqrt{\frac{R_2}{4R_1}} = \frac{e(s)}{p(s)} \tag{7}$$

$$|H(j\omega)|^2 = \frac{P_{\max}}{P_2} \geq 1 \tag{8}$$

6) *Characteristic Function:*[4]

$$K(s) = \rho_1(s)H(s) = \frac{f(s)}{p(s)} \tag{9}$$

$$|K(j\omega)|^2 = \frac{P_{\max} - P_2}{P_2} \tag{10}$$

7) *Open-Circuit and Short-Circuit Immittances:*

$$z_{11} = Z_1 \qquad \text{for} \qquad R_2 \to \infty \tag{11a}$$

$$z_{22} = Z_2 \qquad \text{for} \qquad R_1 \to \infty \tag{11b}$$

$$y_{11} = 1/Z_1 \qquad \text{for} \qquad R_2 = 0 \tag{11c}$$

$$y_{22} = 1/Z_2 \qquad \text{for} \qquad R_1 = 0 \tag{11d}$$

8) *Current Ratio:*

$$M = \frac{I_1}{-I_2} \tag{12}$$

9) *Voltage Ratio:*

$$N = \frac{V_1}{V_2} \tag{13}$$

10) *Loss:*

$$\alpha = 10 \log_{10} \frac{P_{\max}}{P_2} = 20 \log_{10} |H(j\omega)| \text{ in decibels} \tag{14}$$

11) *Phase:*

$$\beta = \text{angle of } H(j\omega) \text{ in radians} \tag{15}$$

12) *Critical Frequencies:*

The *natural modes* are the zeros of $H(s)$, and hence of $e(s)$. The *degree* of the network is the degree of $e(s)$, i.e., the number of natural modes.

The *primary* (*secondary*) *reflection zeros* are the zeros of $\rho_1(\rho_2)$. The finite zeros are the roots of $f(s)$. The zeros of ρ_2 are the negatives of those of ρ_1.

The *loss poles* are the poles of H (and of K); those in the finite part of the s plane are the roots of $p(s)$.

C. Relations Between Circuit Parameters

From the definitions given in Section I-B, the following relationships may be deduced.

The Feldtkeller Equation:

$$|H(j\omega)|^2 = 1 + |K(j\omega)|^2 \tag{16}$$

or

$$H(s)H(-s) = 1 + K(s)K(-s) \tag{17}$$

The Design Impedance Equations for Doubly Terminated Filters:[5]

$$\frac{z_{11}}{R_1} = \frac{H_e - K_e}{H_o + K_o} \tag{18a}$$

$$\frac{z_{22}}{R_2} = \frac{H_e + K_e}{H_o + K_o} \tag{18b}$$

$$y_{11}R_1 = \frac{H_e + K_e}{H_o - K_o} \tag{18c}$$

$$y_{22}R_2 = \frac{H_e - K_e}{H_o - K_o} \tag{18d}$$

The Design Impedance Equations for Singly Terminated Filters:[6]

$$\frac{z_{22}}{R_2} = \frac{M_e}{M_o} \qquad \text{for} \qquad R_1 \to \infty \tag{19a}$$

$$y_{22}R_2 = \frac{N_e}{N_o} \qquad \text{for} \qquad R_1 = 0. \tag{19b}$$

It can easily be seen that, by using the reciprocity theorem and, if necessary, interchanging the primary and secondary ports, all singly terminated filters may be treated using these two equations. Consider, e.g., the circuit of Fig. 2(a), where the specified transfer function is

$$F(s) = \frac{E}{-I_2}. \tag{20}$$

Going through the steps illustrated in Fig. 2, the problem is reduced to the realization of a prescribed $N(s)$ and (19) is applicable.

For unterminated networks, either M or N, or one of the open- (short-) circuit immittances $z_{12}(y_{12})$ is specified.

[3] "Effective transmission factor" in [1].
[4] This definition is somewhat arbitrary. $K = \rho_2 H$ is also possible.

[5] Here subscript $e(o)$ denotes the even (odd) part of the subscripted function.
[6] If either or both of the terminations are missing, $H \to K \to \infty$.

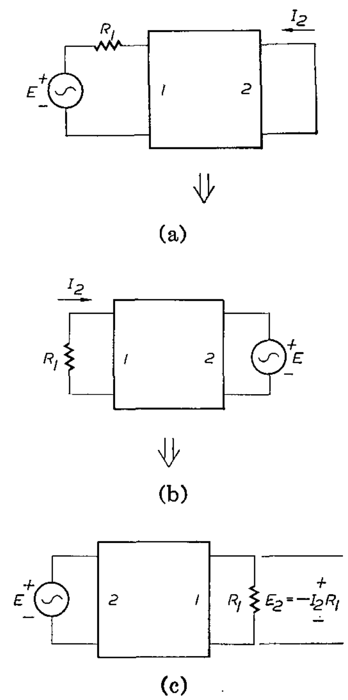


Fig. 2. Design steps for a singly terminated reactance two-port.

In the former case, (19) may be used; in the latter, the circuit may be chosen to be symmetric, and the open- or short-circuit driving-point immittances may be obtained via the residue conditions on the z or y parameters.

It may be verified from (18) that replacing $K_e(s)$ by $-K_e(s)$ is equivalent to interchanging the two ports, changing the sign of both K_e and K_o changes the network into its dual, and finally, changing the sign of K_o interchanges the ports *and* turns the network into its dual.

D. Properties of Circuit Parameters

Realizability in the form of the circuit shown in Fig. 1 requires the following conditions to hold for the circuit parameters:

1) H, K, ρ_1, ρ_2, M, and N must all be real and rational functions of s.

2) The numerators of H, M, and N, as well as the denominators of ρ_1 and ρ_2, must have their roots in the inside of the left-half plane (LHP).[7]

3) The denominators of H and K are purely even or odd polynomials[8] in s.

4) $|H| \geq 1$; $|\rho_1| = |\rho_2| \leq 1$ for $s = j\omega$.

5) The degree of $e(s)$ is equal to or greater than that of $f(s)$ and $p(s)$ as may be seen from

$$e(s)e(-s) = f(s)f(-s) + p(s)p(-s), \qquad (21)$$

which in turn follows from the Feldtkeller equation (17).

[7] That is, they are strictly Hurwitz. The only exceptions are the parameters of an unterminated network. This represents a trivial case and will not be considered herein.

[8] Although for important circuit configurations (such as symmetric lattices) cancellations may occur in $H(s)$ destroying this property.

E. The Filter Synthesis Process

Typically, the design of a doubly terminated filter proceeds in three steps:

1) *Construction of the Transfer Function:* This involves finding either K or H from the given specifications. Typically, if only the loss response is specified, it is preferable to search for K; if the phase or delay responses are also of interest, H should be found. For a singly terminated filter, of course, M or N must be calculated.

2) *Calculation of the Design Immittances:* H may be obtained from K (or vice versa) via the Feldtkeller equation (17) for a doubly terminated filter, after which one or more of the z_{ii}, y_{ii} are found from (18). For singly terminated filters y_{22} (or z_{22}) may be found directly from N (or M), without solving additional equations.

3) *The Calculation of Element Values:* For the usual ladder configuration, this is carried out using zero-shifting from one of the open- or short-circuit impedances.

The reader is referred to [1] for details of this procedure. In the remainder of the paper, they will be assumed to be known. Similarly, no effort will be made to recapitulate the well-known design of filters with Butterworth, Chebyshev, Bessel, elliptic, etc. responses. Design information on these networks may be obtained from the references tabulated in the Appendix.

II. Equal-Ripple Passband Filters

A. Conventional Filters

Only low-pass and frequency-asymmetric bandpass filters will be considered. To achieve a unified treatment, the frequency response will be normalized to the upper passband limit f_2 (Fig. 3); a low-pass filter will be treated as a bandpass with its lower passband limit f_1 equal to zero. Thus, all the formulas derived for the bandpass case will apply to low-pass filters if f_1 is set to zero.

To improve the numerical conditioning [2], [3], as well as to simplify the solution of the approximation problem [4], [5], the transformation of the complex frequency variable to

$$z^2 = \frac{f^2 - f_2^2}{f^2 - f_1^2} = \frac{\Omega^2 - 1}{\Omega^2 - a^2} \qquad (\text{Re}\ \ z \geq 0) \qquad (22)$$

is appropriate. Here,

$$\Omega = \frac{f}{f_2}; \qquad a = \frac{f_1}{f_2} < 1. \qquad (23)$$

References [3] and [6] give a detailed justification of this particular choice of transformation.

To take full advantage of the numerical properties of the transformation, the entire synthesis process must be performed with the z variable. The design steps are as follows.

1) *Finding* $|K|^2$*:* The transformed characteristic function is obtained from the loss poles $f_{i\infty}$ in the following steps.

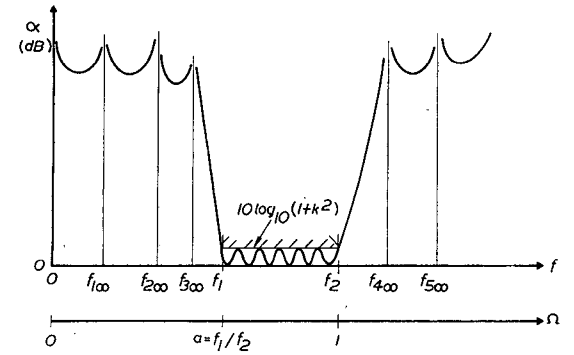


Fig. 3. Equal-ripple passband bandpass filter response.

a) All loss poles are transformed to the z domain via[9]

$$m_i = \sqrt{\frac{f_{i\infty}^2 - f_2^2}{f_{i\infty}^2 - f_1^2}} \qquad \text{(positive square root)} \tag{24}$$

$$i = 1, 2, \cdots, n.$$

The m_i must include the poles at the extreme frequencies: $f_{i\infty} \to \infty$ corresponds to $m_i = 1$; $f_{i\infty} = 0$ corresponds to $m_i = 1/a$. All other m_i must occur in identical pairs.

For conventional bandpass filters, n must be even; for low-pass filters it may be even or odd.

b) In frequent cases, the specified minimum stopband loss, rather than the $f_{i\infty}$, is given. The m_i may then be found by the following iterative procedure [5].

i) Transform the specified minimum stopband loss $\alpha_s(f)$ into $\alpha_s(\gamma)$ using the transformation

$$\gamma = \ln z = \frac{1}{2} \ln \frac{f^2 - f_2^2}{f^2 - f_1^2}. \tag{25}$$

This maps the lower stopband ($0 \leq f \leq f_{s_1}$) on the ($\ln(1/a), \gamma_u$) section of the positive γ-axis[10] and the upper stopband ($f_{s_2} \leq f \leq \infty$) on the ($\gamma_l$, 0) section of the negative γ-axis (Fig. 4).

ii) In the transformed domain, each finite nonzero loss pole $f_{i\infty}$ contributes

$$\alpha_i(\gamma) = 20 \log_{10} \coth \frac{|\gamma - \gamma_i|}{2} \tag{26}$$

to the total loss, where γ_i is the transform of $f_{i\infty}$. Each simple loss pole at zero and infinite frequency contributes one half of this. The total loss is given by

$$\alpha(\gamma, \gamma_i) = 20 \log_{10} \left(\frac{k}{2}\right) + 10 n_0 \log_{10} \coth \frac{|\gamma + \ln a|}{2}$$
$$+ 20 \sum_{i=1}^{n_1} \log_{10} \coth \frac{|\gamma - \gamma_i|}{2} + 10 n_\infty \log_{10} \coth \frac{|\gamma|}{2}. \tag{27}$$

[9] The reason for denoting the transforms of the $f_{i\infty}$ by m_i rather than $z_{i\infty}$ is historical. In the first treatment of Chebyshev passband filters [7], an image parameter filter was used as a reference filter and the m_i corresponded to the m parameters [8] of its m-derived sections.

[10] For low-pass filters, of course, this part is absent. There, formally, $\gamma_u = \ln(a^{-1})$ should be used.

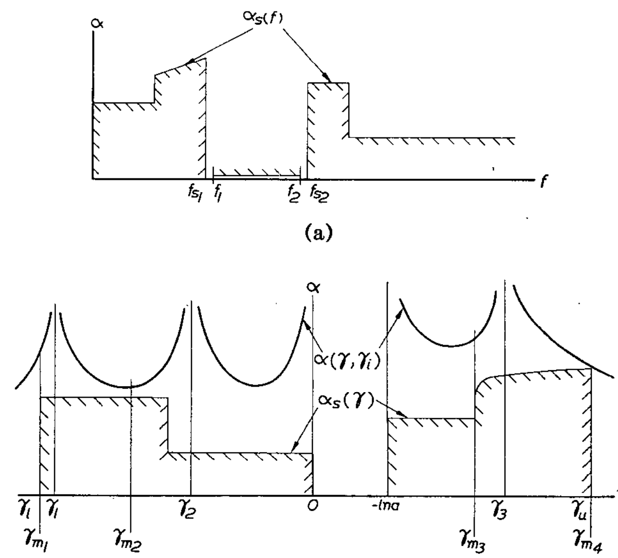


Fig. 4. Location of the loss poles along the γ-axis.

The parameter k is defined by

$$\alpha_p = 10 \log_{10} (1 + k^2) \tag{28}$$

where α_p is the prescribed passband ripple in dB, n_0 and n_∞ are the numbers of poles at zero and infinite frequency, respectively, and $n_1 = \frac{1}{2}(n - n_0 - n_\infty)$ is the number of finite nonzero poles. Find an initial approximation, e.g., by placing the γ_i equidistantly in the stopbands. See [5] and [6] for methods to obtain good initial estimates for n and the γ_i.

iii) Find the local minimum of $\alpha(\gamma) - \alpha_s(\gamma)$ between γ_l and γ_1; between γ_1 and γ_2; $\cdots$; between γ_k and γ_{k+1}; $\cdots$; between γ_{n_1} and γ_u. Call the corresponding $n_1 + 1$ abscissas γ_{m_j} (Fig. 4).

iv) Find a set of improved γ_i such that the margin between the new $\alpha(\gamma, \gamma_i)$ and the specified $\alpha_s(\gamma)$ takes on the same value Δ at all the previously determined minima γ_{m_j}. The new γ_i and Δ therefore have to satisfy

$$\alpha(\gamma_{m_j}, \gamma_i) - \alpha_s(\gamma_{m_j}) = \Delta \qquad j = 1, 2, \cdots, n_1 + 1. \tag{29}$$

The solution may readily be found by using Newton's method. None of the γ_i should lie in the range $0 \leq \gamma \leq \ln 1/a$ if a ladder realization is required.

v) Steps iii) and iv) above are then repeated, i.e., for the improved γ_i the new γ_{m_j} are found, and then, at all these γ_{m_j}, the "margins" are once more adjusted to have a common value by changing slightly the γ_i. Repetition of this cycle causes the γ_i to converge rapidly. When the γ_i are known accurately enough, the process is terminated and the desired m_i are given by

$$m_i = \exp \gamma_i \qquad i = 1, 2, \cdots, n_1. \tag{30}$$

This process can of course be used equally well for the design of elliptic filters. The computation time is somewhat longer than that needed for the power series computation of elliptic functions [9]. However, the structure of a general filter design program is much simplified by not having to include a special subroutine for this case.

c) From the transformed loss poles m_i, form the polynomial

$$E + zF = \prod_{i=1}^{n} (m_i + z). \tag{31}$$

Here, E and F are even polynomials.[11] Then, the squared modulus of the characteristic function is given by

$$|K|^2 = \frac{k^2E^2}{E^2 - z^2F^2}. \tag{32}$$

Polynomial kE is the z-plane equivalent of $f(s)$, except for an unimportant common factor $(s^2 + a^2)^{n/2}$.

2) *Finding H:* The transformed transducer function H is obtained next, in the following steps.

a) By the Feldtkeller equation (17)

$$|H|^2 = \frac{(1 + k^2)E^2 - z^2F^2}{E^2 - z^2F^2}. \tag{33}$$

To obtain the z-plane equivalent of $e(s)$, there is a choice of two methods.

i) The numerator of $|H|^2$ may be factored into $(E\sqrt{1 + k^2} + zF)\,(E\sqrt{1 + k^2} - zF)$. Then, one of the two factors, e.g., $E\sqrt{1 + k^2} + zF$ is separated into factors of the form $z^2 + mz + n$ and (for the odd-degree low-pass case) $z + l$. Next, the constants

$$M = 2n - m^2 \tag{34a}$$

$$N = n^2 \tag{34b}$$

$$L = l^2 \tag{34c}$$

are derived for each factor.

ii) Alternatively,

$$(1 + k^2)E^2 - z^2F^2 = \prod_{i=1}^{n} (m_i^2 - z^2) + k^2E^2 \tag{35}$$

may be separated[12] into factors

$$z^4 + Mz^2 + N \tag{36}$$

and (for low-pass filters) a linear factor

$$z^2 - L. \tag{37}$$

Method i) is normally somewhat better since the coefficients of $E\sqrt{1 + k^2} + zF$ can be found more accurately than those of $(1 + k^2)E^2 - z^2F^2$. Thus, by using method i), superior overall numerical accuracy may be achieved, despite the better conditioning[13] of $(1 + k^2)\cdot E^2 - z^2F^2$.

[11] E and F should not be confused with polynomials $E(s)$ and $F(s)$ used in [1]. The latter are denoted here by lower-case letters. The present notations preserve continuity with [3] and [4].

[12] The right-hand side of (35) is the more accurate expression.

[13] $(1+k^2)E^2-z^2F^2$ has, in z^2, the *squared* roots of $E\sqrt{1+k^2}\pm zF$.

Having obtained all M, N, and L, next calculate

$$R = \sqrt{1 + M + N}; \quad T = \sqrt{1 + a^2M + a^4N} \tag{38a, b}$$

$$p = \frac{1 + a^2(1 + M)}{T + a^2R}; \quad r = \frac{M + (1 + a^2)N}{T + R} \tag{38c, d}$$

$$q = \sqrt{\frac{r(r - Mp) + Np^2}{TR}} \tag{38e}$$

to form all quadratic factors in the form

$$pz^2 + q\sqrt{z^2 - 1}\,\sqrt{1 - a^2z^2} + r. \tag{39}$$

Linear factors (which occur only for odd-degree low-pass filters) transform (with $a = 0$) into

$$\sqrt{z^2 - 1} + \sqrt{L - 1}. \tag{40}$$

b) Next, all factors of the form (39) and (40) are multiplied together to obtain an expression

$$A + B\sqrt{z^2 - 1}\,\sqrt{1 - a^2z^2} \tag{41}$$

where A and B are polynomials in z^2. Their coefficients may be efficiently generated, using recursive relations, from the p, q, r, and $\sqrt{L - 1}$. Finally, one must include a constant factor that is the square root of the leading coefficient of $E\sqrt{1 + k^2} + zF$.

The expression (41) is the z-plane equivalent of $e(s)$. When n is even, A corresponds to $e_e(s)$, while $B\sqrt{z^2 - 1}\cdot\sqrt{1 - a^2z^2}$ corresponds to $e_o(s)$; when n is odd, the roles are reversed. The equivalent of $p(s)$ can also easily be derived in a similar way, but it is not needed in the synthesis.

3) *Finding the Open- and Short-Circuit Impedances:* The design impedances z_{11}, z_{22}, $1/y_{11}$, and $1/y_{22}$ may be found in two stages.

a) Find a constant l such that the polynomial $A - lE$ has a factor $z^2 - 1$[14]

$$l = \frac{A(1)}{E(1)}. \tag{42}$$

In addition, $A - lE$ must also have the factor $(1 - a^2z^2)$ if previous calculations, used to derive A, were accurate. Next calculate the even polynomials

$$A' = lE \tag{43}$$

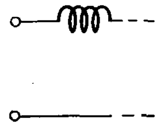
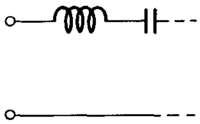
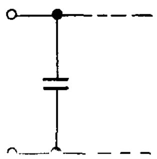
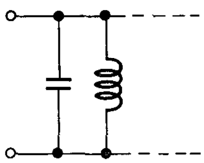
$$D = \frac{A - A'}{(z^2 - 1)(1 - a^2z^2)}. \tag{44}$$

b) From polynomials A, B, A', and D, the z-plane expressions for the design impedances may be obtained using Table I. Note that this table is universal in the sense that it is valid for both low-pass and bandpass filters, at both ports, and gives both open- and short-circuit impedances. The "lower order impedance" results of course if a branch on the far side of the network is either open- or short-circuited; the "higher order" one, if no such degeneracy is present.

[14] l should be very nearly equal to k.

TABLE I

DESIGN IMPEDANCES FOR CONVENTIONAL FILTERS

Impedance Behavior at Zero and Infinite Frequency Similar to:		Impedance of Singly Terminated Network	Doubly Terminated Network	
Low-Pass	Bandpass		Impedance of Lower Degree	Impedance of Higher Degree
		$\frac{A}{\sqrt{z^2-1}\sqrt{1-a^2z^2}B}$	$\frac{B}{\sqrt{z^2-1}\sqrt{1-a^2z^2}D}$	$\frac{A+A'}{\sqrt{z^2-1}\sqrt{1-a^2z^2}B}$
		$\frac{\sqrt{z^2-1}\sqrt{1-a^2z^2}B}{A}$	$\frac{\sqrt{z^2-1}\sqrt{1-a^2z^2}D}{B}$	$\frac{\sqrt{z^2-1}\sqrt{1-a^2z^2}B}{A+A'}$

Notes: 1) Multiply the impedance expression quoted by the value of the terminating resistance at the port concerned.
2) Put $a = 0$ for low-pass filter case.

4) *Low-Pass Filters with Prescribed Terminations:* The process described above yields the design impedances for low-pass filters, if a is set to zero in all formulas. It will result in a symmetric filter with $R_1 = R_2$ if n is odd; it will give an antimetric filter with

$$\frac{R_1}{R_2} = [\sqrt{k^2+1} + k]^2 \tag{45}$$

if n is even. [For even n, the smaller (greater) termination is at the port having the series inductor (shunt capacitor). For the choice of signs described, the series inductor and the smaller termination are at the secondary port. Naturally, the network may be turned around if necessary.]

In some cases, terminations different from the above are required.[15] For symmetric filters, this may, theoretically at least, be achieved by the following steps [1].

a) $A + B\sqrt{z^2-1}$ is obtained as before.

b) Let

$$\rho_0 = \frac{R_1 - R_2}{R_1 + R_2}. \tag{46}$$

Find the z-plane equivalent of a new $f(s)$ by factoring

$$E\sqrt{k^2+\rho_0^2} + \rho_0 zF \tag{47}$$

and rewriting, as well as multiplying, the factors to obtain an expression of the form $G + H\sqrt{z^2-1}$. This latter expression is now the equivalent of $f(s)$. Using (18), expressions similar to those in Table I may be obtained for the design impedances.

[15] This problem does not arise for bandpass filters, for which Norton transformations may be used to achieve the desired termination ratio.

A similar technique is applicable to the design of antimetric filters, provided R_1/R_2 is *not* in the range

$$(\sqrt{k^2+1} - k)^2 < \frac{R_1}{R_2} < (\sqrt{k^2+1} + k)^2. \tag{48}$$

In both the above cases, the desired termination ratio is achieved at the cost of adding a constant loss to the passband response. This, in turn, increases the sensitivity of the passband response to changes in the element values and also to at least one of the terminations. Thus, it is far preferable to accept the equal terminations (for n odd) or those given by (45) if n is even, and then to use a matching pad or transformer to obtain the desired termination ratio. It should be noted that the matching pad results in more constant loss than the steps a) and b) above. Nevertheless, it results in a sounder engineering design. For this reason, the details of steps a) and b) will not be given here.

Of more importance, however, is the case where n is even and terminations satisfying (48), for example $R_1 = R_2$, are required. The loss characteristic then takes the form shown in Fig. 5(b), where

$$\alpha(0) = 10 \log_{10} (1 + k_1^2) \tag{49}$$

with

$$k_1 = \frac{R_1 - R_2}{2\sqrt{R_1 R_2}}. \tag{50}$$

It is not possible to construct directly a transfer function having the required zero frequency behavior. Instead, one must first construct a standard antimetric filter function behaving in the manner of Fig. 5(a) and then apply to it a linear transformation of the f^2 variable (which becomes a bilinear transformation of z^2). The parameters of this transformation are chosen to keep both the passband edge frequency f_2 and the point at infinity unchanged,

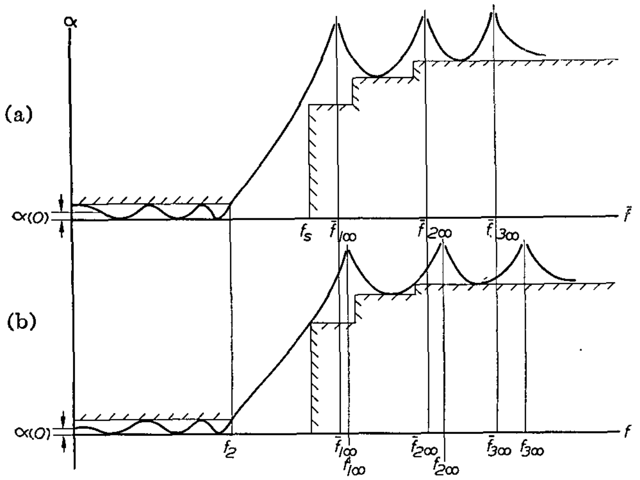


Fig. 5. The design of an antimetric low-pass filter with prescribed terminations.

while simultaneously adjusting the loss at zero frequency to $\alpha(0)$.

A side effect of this transformation is that the frequency of each loss pole is increased slightly, and so the poles used in the preliminary standard antimetric function must therefore be slightly lower than those required in the final filter. Unfortunately, the amount of this "predistortion" and, in fact, the specific linear transformation cannot be calculated explicitly beforehand but must be found via an iteration.

Let $\mathbf{z}$ and $\mathbf{m}_i$ be the variable and the m values, respectively, of the preliminary design, and z and m_i the corresponding quantities for the final filter. The m_i are thus defined by

$$m_i = \sqrt{1 - f_2^2/f_{i\infty}^2} \tag{51}$$

in terms of the final pole frequencies $f_{i\infty}$. The transformation relates these quantities by

$$\mathbf{z}^2 = \frac{z^2}{1 + \delta^2(1 - z^2)} \tag{52}$$

and

$$\mathbf{m}_i^2 = \frac{m_i^2}{1 + \delta^2(1 - m_i^2)}. \tag{53}$$

The points $z^2 = 1$ $(f \to \infty)$ and $z^2 = 0$ $(f = f_2)$ are left unchanged by (52) but the parameter δ remains to be chosen by the iteration. Specifically, $\mathbf{z} = j/\delta$ is the point in the $\mathbf{z}$ plane that must be transformed to the point at infinity.

The parameter δ must now be chosen so that the nonlinear equation

$$\frac{k}{2}\left\{\sqrt{\frac{\prod_{i=1}^{n}(\mathbf{m}_i + \mathbf{z})}{\prod_{i=1}^{n}(\mathbf{m}_i - \mathbf{z})}} + \sqrt{\frac{\prod_{i=1}^{n}(\mathbf{m}_i - \mathbf{z})}{\prod_{i=1}^{n}(\mathbf{m}_i + \mathbf{z})}}\right\} = k_1 \tag{54}$$

is satisfied for $\mathbf{z} = j/\delta$. Here k_1 is given by (50) and for each value of δ the $\mathbf{m}_i$ are assumed to be given by (53).

The two square roots in (54) are conjugate complex quantities and n here is even. The $\mathbf{m}_i$ may be found, and the design completed, in the following steps.

a) Solve iteratively the equation[16]

$$u\sqrt{\frac{k - k_1}{k + k_1}} - v = 0 \tag{55}$$

where

$$u = \mathrm{Re}\left[\prod_{i=1}^{n/2}(1 + j\,\delta\mathbf{m}_i)\right] \tag{56a}$$

$$v = \mathrm{Im}\left[\prod_{i=1}^{n/2}(1 + j\,\delta\mathbf{m}_i)\right]. \tag{56b}$$

For example, *regula falsi* may be used with

$$\delta = 0 \quad \text{and} \quad \delta = \pi\Big/\left[4\sum_{i=1}^{n/2} m_i\right] \tag{57}$$

as the two starting values. The smallest positive value of δ satisfying (55) is the one required.

b) Having found δ, calculate the $\mathbf{m}_i$ from (53). Then, find $|\mathbf{K}|^2$ as explained in Section II-A, 1), substituting $\mathbf{z}$ for z and $\mathbf{m}_i$ for m_i. Calculate $\mathbf{M}, \mathbf{N}$ and form all quadratic factors $\mathbf{z}^4 + \mathbf{M}\mathbf{z}^2 + \mathbf{N}$ as described in Section II-A, 2a), substituting bold-faced quantities for light-faced ones everywhere.

c) Carry out the change of variables (52) in all quadratic factors. Then all factors are transformed

$$\mathbf{z}^4 + \mathbf{M}\mathbf{z}^2 + \mathbf{N} \to z^4 + Mz^2 + N, \tag{58}$$

where

$$M = \frac{(1 + \delta^2)(\mathbf{M} - 2\,\delta^2\mathbf{N})}{1 - \delta^2\mathbf{M} + \delta^4\mathbf{N}} \tag{59a}$$

$$N = \frac{(1 + \delta^2)^2\mathbf{N}}{1 - \delta^2\mathbf{M} + \delta^4\mathbf{N}}. \tag{59b}$$

d) Carry out the same change of variables (52) in order to derive E from the $\mathbf{E}$, obtained in step b). Drop the unneeded $[1 + \delta^2(1 - z^2)]^n$ appearing as a common denominator in E.

e) Revert to Section II-A, 2a) to calculate the design impedances in the usual manner.

5) *Ladder Expansion:* The procedure used to develop the z-plane design impedances, obtained from Table I, into a ladder network follows closely the corresponding s-plane process (see Fig. 3 of [1], for example). Special care, however, must be taken with the detailed execution of these ladder development steps in order to preserve the maximum numerical accuracy. Using a poor numerical method can easily dissipate the initial higher accuracy obtained in the input impedance by use of the z variable. The z-plane operations needed to execute the circuit realization are listed in Tables III and IV. Examination of the tables reveals that in each stage of the development there is only one equation (in z^2) that needs to be solved. It has the general form

[16] If $R_1 = R_2$ is required, (55) simplifies to $u = v$.

TABLE II

AN ALGORITHM FOR LADDER EXPANSION

The general form of the equation to be solved is:

$$U(z^2) = a_1(pz^2 + q)\, V(z^2) + (z^2 - m^2)\, W(z^2)$$

where U, V, p, q, and m are given, and a_1 and W are required.

For example, let

$$U(z^2) = u_0 + u_1 z^2 + u_2 z^4 + u_3 z^6 + u_4 z^8 + u_5 z^{10}$$
$$V(z^2) = v_0 + v_1 z^2 + v_2 z^4 + v_3 z^6 + v_4 z^8 + v_5 z^{10}$$
$$W(z^2) = w_0 + w_1 z^2 + w_2 z^4 + w_3 z^6 + w_4 z^8.$$

Either:
1) $p = 0 \quad q = 1$

or

2) $p = 1 \quad q = -1 \quad v_5 = 0$

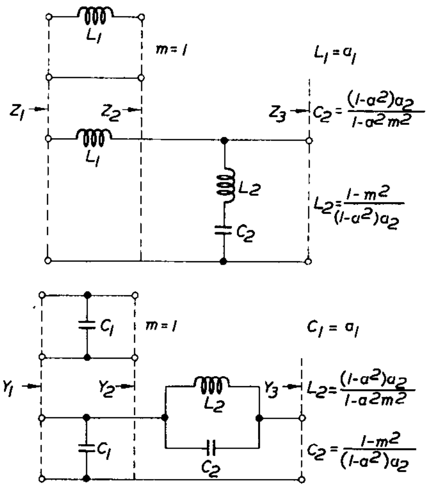
or

3) $p = -a^2 \quad q = 1 \quad v_5 = 0.$

$$\Delta = (pm^2 + q)\,(v_0 + v_1 m^2 + v_2 m^4 + v_3 m^6 + v_4 m^8 + v_5 m^{10})$$
$$a_1\Delta = (u_0 + u_1 m^2 + u_2 m^4 + u_3 m^6 + u_4 m^8 + u_5 m^{10})$$
$$w_0\Delta = (u_1 + u_2 m^2 + u_3 m^4 + u_4 m^6 + u_5 m^8)\, q v_0 - u_0[p v_0 + (pm^2 + q)\,(v_1 + v_2 m^2 + v_3 m^4 + v_4 m^6 + v_5 m^8)]$$
$$w_1\Delta = (u_2 + u_3 m^2 + u_4 m^4 + u_5 m^6)\,[(pm^2 + q)\, v_0 + q v_1 m^2] - (u_0 + u_1 m^2)\,[p v_1 + (pm^2 + q)\,(v_2 + v_3 m^2 + v_4 m^4 + v_5 m^6)]$$
$$w_2\Delta = (u_3 + u_4 m^2 + u_5 m^4)\,[(pm^2 + q)\,(v_0 + v_1 m^2) + q v_2 m^4] - (u_0 + u_1 m^2 + u_2 m^4)\,[p v_2 + (pm^2 + q)\,(v_3 + v_4 m^2 + v_5 m^4)]$$
$$w_3\Delta = (u_4 + u_5 m^2)\,[(pm^2 + q)\,(v_0 + v_1 m^2 + v_2 m^4) + q v_3 m^6] - (u_0 + u_1 m^2 + u_2 m^4 + u_3 m^6)\,[p v_3 + (pm^2 + q)\,(v_4 + v_5 m^2)]$$
$$w_4\Delta = u_5[(pm^2 + q)\,(v_0 + v_1 m^2 + v_2 m^4 + v_3 m^6) + q v_4 m^8] - (u_0 + u_1 m^2 + u_2 m^4 + u_3 m^6 + u_4 m^8)\,[p v_4 + (pm^2 + q)\, v_5].$$

TABLE III

LADDER EXPANSION FOR LOW-PASS AND BANDPASS FILTERS



	Either	Or
Z_1 or = Y_1	$\dfrac{N}{\sqrt{z^2 - 1}\,\sqrt{1 - a^2z^2}\,D}$	$\dfrac{\sqrt{1 - a^2z^2}\,N}{\sqrt{z^2 - 1}\,D}$
$N =$	$a_1(1 - a^2z^2)\,D + (z^2 - m^2)\,\mathbf{N}$	$a_1 D + (z^2 - m^2)\,\mathbf{N}$
Z_2 or = Y_2	$\dfrac{(z^2 - m^2)\,\mathbf{N}}{\sqrt{z^2 - 1}\,\sqrt{1 - a^2z^2}\,D}$	$\dfrac{(z^2 - m^2)\,\sqrt{1 - a^2z^2}\,\mathbf{N}}{\sqrt{z^2 - 1}\,D}$
$D =$	$a_2\mathbf{N} + (z^2 - m^2)\,\mathbf{D}$	$a_2(1 - a^2z^2)\,\mathbf{N} + (z^2 - m^2)\,\mathbf{D}$
Z_3 or = Y_3	$\dfrac{\mathbf{N}}{\sqrt{z^2 - 1}\,\sqrt{1 - a^2z^2}\,\mathbf{D}}$	$\dfrac{\sqrt{1 - a^2z^2}\,\mathbf{N}}{\sqrt{z^2 - 1}\,\mathbf{D}}$

Notes: Formulas apply in general bandpass case with

$$z^2 = \frac{f^2 - f_2^2}{f^2 - f_1^2}; \quad m^2 = \frac{f_\infty^2 - f_2^2}{f_\infty^2 - f_1^2}; \quad a = \frac{f_1}{f_2}.$$

For low-pass case, put $a = 0$.
For full removal, with $m = 1$, cycle terminates at $Z_2(Y_2)$.

$$U(z^2) = a_1(pz^2 + q)V(z^2) + (z^2 - m^2)W(z^2). \tag{60}$$

Here, U and V are given polynomials in z^2, while p, q and m are given constants. The left-hand side as well as both terms on the right-hand side of (60) are, for bandpass filters, of the same degree in z^2. The unknowns are the constant a_1, which determines directly or indirectly the next element value, and the polynomial W, which is the unknown numerator or denominator of the rational expression next needed in the development.

The solution of (60) is summarized in Table II. Rather than give the general expressions, which are quite obscure, the formulas are developed explicitly for the case where all the terms in (60) are of the fifth degree in z^2. The pattern for the general case should then be quite clear.

It should be noted that the expressions for all the coefficients of W are given in explicit form and are the best that can be obtained for numerical purposes. The various polynomials in m^2 may be built up using nested multiplication.

Table II may be used to create a subroutine, which can then be used, with various U, V, p, q, and m in all steps of the ladder development, as given in Tables III and IV.

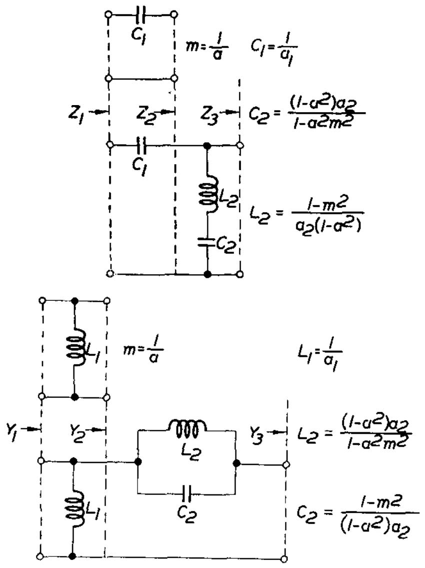
6) *Minimum Inductance Configurations* [64]: It is well known that the number of natural modes in an RLC network is

$$n = \text{total number of reactors} - (\text{total number of one-reactance-kind cutsets and tiesets}). \tag{61}$$

Assume that the filter initially contains only capacitors (no inductors, no resistors); then $n = 0$. Inserting the inductors one-by-one, each inductor increases the total number of reactors by one, and *may* also disrupt a purely capacitive cutset or tieset. Hence, each inductor may

TABLE IV

LADDER EXPANSION FOR BANDPASS FILTERS



	Either	Or
Z_1 or Y_1 =	$\dfrac{N}{\sqrt{z^2-1}\sqrt{1-a^2z^2}\,D}$	$\dfrac{\sqrt{z^2-1}\,N}{\sqrt{1-a^2z^2}\,D}$
N =	$a_1(z^2-1)\,D+(z^2-m^2)\,\mathbf{N}$	$a_1D+(z^2-m^2)\,\mathbf{N}$
Z_2 or Y_2 =	$\dfrac{(z^2-m^2)\,\mathbf{N}}{\sqrt{z^2-1}\sqrt{1-a^2z^2}\,D}$	$\dfrac{(z^2-m^2)\sqrt{z^2-1}\,\mathbf{N}}{\sqrt{1-a^2z^2}\,D}$
D =	$a_2\mathbf{N}+(z^2-m^2)\,\mathbf{D}$	$a_2(z^2-1)\,\mathbf{N}+(z^2-m^2)\,\mathbf{D}$
Z_3 or Y_3 =	$\dfrac{\mathbf{N}}{\sqrt{z^2-1}\sqrt{1-a^2z^2}\,\mathbf{D}}$	$\dfrac{\sqrt{z^2-1}\,\mathbf{N}}{\sqrt{1-a^2z^2}\,\mathbf{D}}$

Notes: Formulas apply in general bandpass case with

$$z^2=\frac{f^2-f_2^2}{f^2-f_1^2};\quad m^2=\frac{f_\infty^2-f_2^2}{f_\infty^2-f_1^2};\quad a=\frac{f_1}{f_2}.$$

For full removal, with $m=1/a$, cycle terminates Z_2 or Y_2.

increase the degree by two. Similarly, each termination *may* disrupt a purely capacitive cutset or tieset. Hence, the minimum total number of inductors n_L is related to the degree by

$$n\leq 2n_L+2 \tag{62}$$

for a doubly terminated LC two-port. For the conventional bandpass filter, all zeros of $f(s)$ are in the passband and hence, the sign of $\rho_1(s)=K(s)/H(s)$ is the same at $s=0$ and for $s\to\infty$. A similar conclusion holds for $\rho_2(s)=K(-s)/H(s)$. This, however, indicates that neither of the two terminations is connected to a purely capacitive node or loop. Thus, here,

$$n\leq 2n_L, \tag{63}$$

i.e., the minimum number of inductors is $n/2$.

B. Symmetric Parametric Filters

As mentioned in connection with (62), the truly minimum number of inductors $n_L=\frac{1}{2}n-1$ may be achieved only if both terminations are connected to a purely capacitive node or loop (Fig. 6). Then, both reflection coefficients tend to -1 for $s=0$ and to $+1$ for $s\to\infty$ and, hence, must have a single zero (or an odd number of zeros) on both the positive and the negative real s-axes.

A similar conclusion is easily reached for two-port circuits realized exclusively from capacitors and piezoelectric resonators. In both cases, therefore, $K(s)$ must contain a factor $(s-b_1)(s+b_2)$. If a symmetric circuit is desired, then the factor must be (s^2-b^2). The degree n must be even; hence, n_0+n_∞ is also even. The value of b can be varied over a considerable range, and it acts, in some respects, as a free parameter; hence, the name of this class of filter.

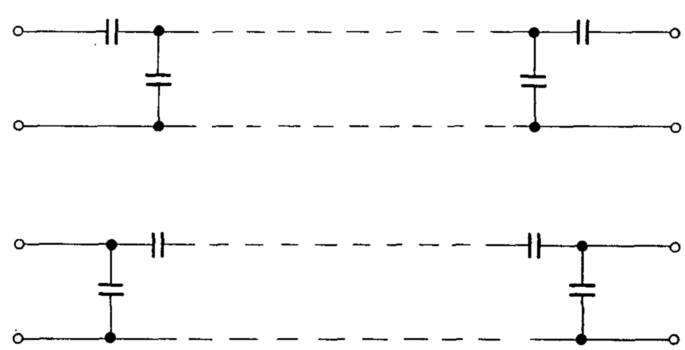


Fig. 6. Configurations for symmetric parametric filters.

An efficient technique for the z-plane design of symmetric parametric filters follows. The reader is referred to the tutorial material in [6] for a brief explanation of the theory and for the motivation behind the process. The design steps are as follows.

1) *Finding* $|K|^2$*:* This is accomplished in the following stages [2], [65].

a) If all finite nonzero loss poles $f_{i\infty}$ are known, find the m_i, in identical pairs, from (24). Assume one pole at $z=1$ and one at $z=1/a$ and include these among the m_i. Choose a value α in the range $1<\alpha<1/a$. Normally, $\alpha=a^{-1/2}$ is a good choice.[17]

b) If the $f_{i\infty}$ are not known, find them using the iterative method described in Section II-A, 1b). Use $n_0=n_\infty=1$; also, include in $\alpha(\gamma,\gamma_i)$ a fixed *negative* term

$$-20\log_{10}\coth\frac{|\gamma+\frac{1}{2}\ln a|}{2}<0. \tag{64}$$

[17] See [10] for a discussion of this choice.

c) Form

$$E + zF = \prod_{i=1}^{n} (m_i + z) \tag{65}$$

and calculate

$$(\alpha^2 + z^2)E - 2\alpha z^2 F$$
$$= \text{even part of } (-\alpha + z)^2 \prod_{i=1}^{n} (m_i + z). \tag{66}$$

It should be noted that this is equivalent to introducing temporarily a nonphysical double-loss pole at $z = -\alpha$.

All but two of the roots (in z^2) of the even polynomial (66) lie on the negative real z^2-axis, i.e., in the passband. These are the transforms of the passband reflection zeros. The remaining two roots, however, lie on the positive real z^2-axis. One of them (z_a^2) in the range $1 < z_a^2 < \alpha^2$, the other (z_b^2) in the range $\alpha^2 < z_b^2 < a^{-2}$. These two roots must be located and removed from (66). In principle, this could be done with a general root-finding program, but it is better to use a special and rapidly convergent iteration to find them and then to divide them out.

Hence, starting with $z_0 = \alpha$, iterate by repeatedly using

$$z_{j+1} = \alpha \pm (\alpha + z_j)\sqrt{-\prod_{i=1}^{n} \frac{m_i - z_j}{m_i + z_j}}\,; \quad j = 0, 1, 2, \cdots . \tag{67}$$

Using the plus sign, the iteration converges to z_b; using the minus sign, to z_a. Terminate the iteration only after maximum possible accuracy has been obtained.

d) Find the even polynomial

$$U(z^2) = \frac{(\alpha^2 + z^2)E - 2\alpha z^2 F}{(z_a^2 - z^2)(z_b^2 - z^2)} \tag{68}$$

by removing the positive zeros from (66). To obtain good accuracy, the denominator in (68) should be divided out as a single quadratic factor in z^2, using high-accuracy deflation techniques [11].

e) Find

$$\beta \triangleq \frac{z_a z_b}{\alpha} \tag{69}$$

and form

$$|K|^2 = \frac{k^2(\beta^2 - z^2)^2 U^2}{E^2 - z^2 F^2}. \tag{70}$$

The meaning of this is that the effect of the two double real reflection zeros z_a^2 and z_b^2 and of the hypothetical double loss pole α^2 was approximated by $(\beta^2 - z^2)$. Thus

$$\frac{(z_a^2 - z^2)^2(z_b^2 - z^2)^2}{(\alpha^2 - z^2)^2} \cong (\beta^2 - z^2)^2. \tag{71}$$

The choice (69) of β assures that the passband loss will be the same at f_1 and f_2. A typical result is shown in Fig. 7. The overall scheme is illustrated in Fig. 8. Evidently, the desired real reflection zeros at $\pm b$ are realized (β being the z-transform of b), and the price paid is a microscopic deviation from the equal-ripple passband response, as well as some decrease in the selectivity as manifested by the negative term (64) in $\alpha(\gamma, \gamma_i)$.

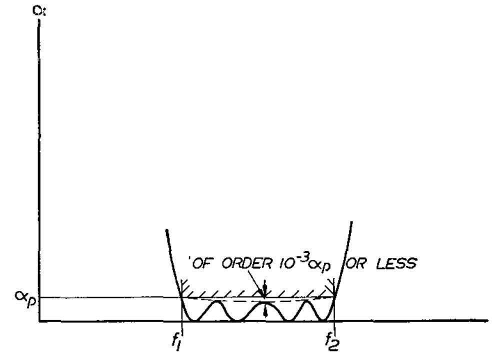
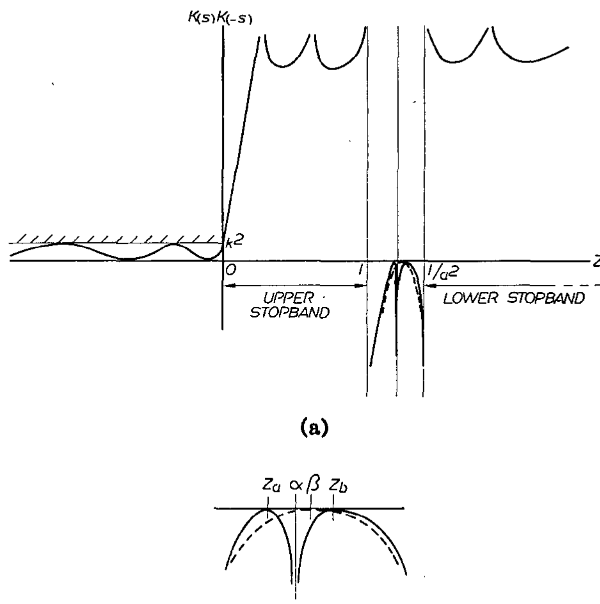


Fig. 7. Passband response of a symmetric parametric filter.



(b)

Fig. 8. The $|K|^2$ versus z^2 response of symmetric parametric filters.

2) *Finding H:*

a) By (16),

$$|H|^2 = \frac{E^2 - z^2F^2 + k^2(\beta^2 - z^2)^2 U^2}{E^2 - z^2F^2}. \tag{72}$$

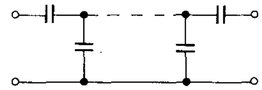
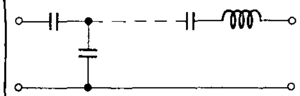
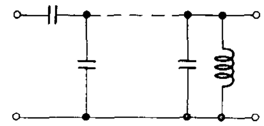
The numerator has two positive real roots z_c^2 and z_d^2 in z^2. They are located in $1 < z_c^2 < \beta^2$ and $\beta^2 < z_d^2 < a^{-2}$. As before, they are to be found by a special technique. Starting with $z_0^2 = \beta^2$, use repeatedly

$$z_{j+1}^2 = \beta^2 \pm \frac{\sqrt{-\prod_{i=1}^{n} (m_i^2 - z_j^2)}}{kU(z_j^2)} \quad j = 1, 2, \cdots . \tag{73}$$

For the plus sign, the iteration converges to z_d^2; for the minus sign, to z_c^2. Terminate the iteration after the maximum possible accuracy has been obtained. Divide the numerator of $|H|^2$ by $(z_c^2 - z^2)(z_d^2 - z^2)$, treating the latter as a single quadratic factor in z^2 and carrying out the division via the method of [11]. The rest of the quadratic factors can easily be found by standard methods, after the factor containing z_c^2 and z_d^2 has been removed.

TABLE V

DESIGN IMMITTANCES FOR PARAMETRIC FILTERS

$\frac{z_{11}}{R_1}$	$y_{11}R_1$	Structure	$\frac{z_{22}}{R_2}$	$y_{22}R_2$
$\frac{B\sqrt{z^2-1}}{C\sqrt{1-a^2z^2}}$	$\frac{B\sqrt{1-a^2z^2}}{D\sqrt{z^2-1}}$	 Symmetric	$\frac{B\sqrt{z^2-1}}{C\sqrt{1-a^2z^2}}$	$\frac{B\sqrt{1-a^2z^2}}{D\sqrt{z^2-1}}$
$\frac{D\sqrt{z^2-1}}{C\sqrt{1-a^2z^2}}$	$\frac{(B+B')\sqrt{1-a^2z^2}}{(A+A')\sqrt{z^2-1}}$	 Asymmetric	$\frac{(B+B')}{C\sqrt{z^2-1}\sqrt{1-a^2z^2}}$	$\frac{D\sqrt{z^2-1}\sqrt{1-a^2z^2}}{A+A'}$
$\frac{(A+A')\sqrt{z^2-1}}{(B+B')\sqrt{1-a^2z^2}}$	$\frac{C\sqrt{1-a^2z^2}}{D\sqrt{z^2-1}}$	 Asymmetric	$\frac{C\sqrt{z^2-1}\sqrt{1-a^2z^2}}{B+B'}$	$\frac{A+A'}{D\sqrt{z^2-1}\sqrt{1-a^2z^2}}$

b) Next, all quadratic factors, excepting the one with the real roots z_c^2 and z_d^2, are transformed into factors of the form

$$pz^2 + q\sqrt{z^2-1}\sqrt{1-a^2z^2} + r \tag{74}$$

using the same method and formulas as were used in Section II-A, 2) for conventional bandpass filters. To find the transformed factor for the two real roots, use

$$p = \frac{(1+a^2) - a^2(z_c^2 + z_d^2)}{\sqrt{1-a^2z_c^2}\sqrt{1-a^2z_d^2} + a^2\sqrt{z_c^2-1}\sqrt{z_d^2-1}} \tag{75a}$$

$$q = \frac{\sqrt{1-a^2z_c^2}\sqrt{z_d^2-1} + \sqrt{1-a^2z_d^2}\sqrt{z_c^2-1}}{1-a^2} \tag{75b}$$

$$r = \frac{(1+a^2)z_c^2z_d^2 - (z_c^2+z_d^2)}{\sqrt{z_c^2-1}\sqrt{z_d^2-1} + \sqrt{1-a^2z_c^2}\sqrt{1-a^2z_d^2}}. \tag{75c}$$

Even these expressions are sensitive to the accuracy of z_c^2 and z_d^2, which is the reason why they have to be found so accurately in the iteration of step a) above [see (73)].

c) Multiply all quadratic factors together to get the z equivalent of the natural mode polynomial

$$A + B\sqrt{z^2-1}\sqrt{1-a^2z^2}. \tag{76}$$

3) *Design Impedances and Element Values:* Find the constant l from

$$l = \frac{A(1)}{(\beta^2-1)U(1)}. \tag{77}$$

Let

$$A' = l(\beta^2 - z^2)U(z^2). \tag{78}$$

Then, $A - A'$ has a factor $(z^2 - 1)$. If the preceding calculations were accurate, then also $A + A'$ has a factor $(1 - a^2z^2)$. Hence, the even *polynomials*

$$C = \frac{A + A'}{1 - a^2z^2} \tag{79a}$$

$$D = \frac{A - A'}{z^2 - 1} \tag{79b}$$

may be defined. (The equations are analogous to (44) giving D for conventional bandpass filters; the same symbol is used here to denote a new polynomial, but no confusion should arise.)

From polynomials A, A', B, C, and D, the design impedance functions may be obtained via Table V. Having selected the desired impedance, Tables II, III, and IV may be used to obtain the circuit and the element values. It should be noted that it is meaningless to use a symmetric parametric filter with an extreme (zero or infinite) termination, since this destroys the symmetry and reduces the degree. For such application, the asymmetric parametric filter, discussed in Section II-C, should be used.

C. Asymmetric Parametric Filters

The parametric filter described in Section II-B uses the absolute minimum number of inductors possible for a given (even) degree. However, both its input impedances vary between infinity (at zero frequency) and zero (at infinite frequency). Hence, it cannot be safely connected either in parallel or in series with other filters at either port. If such a connection is required, or if an open- or short-circuit termination is desired at one end, one of the impedances may be suitably modified using the techniques next described. The price paid is a decrease in the degree by one for a given number of inductors. The

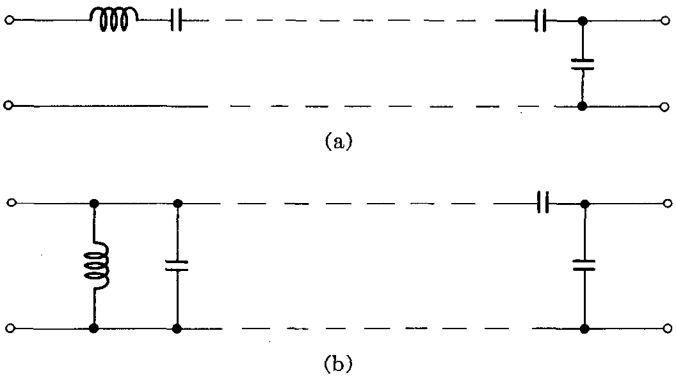


Fig. 9. Asymmetric parametric filter configurations.

required configurations (with the primary port modified) are shown in Fig. 9.

The design philosophy is similar to that followed for symmetric parametric filters. A single reflection zero is required on the real s-axis. It is achieved by temporarily placing a single hypothetical loss pole in the range $1 < z^2 < a^{-2}$. This attracts a double reflection zero near the pole. Performing an approximate cancellation of one of the zeros with the pole, the desired single real reflection zero remains. The departure from exact equal-ripple behavior in the passband proves to be very small.

1) *Finding* $|K|^2$*:* This is done in the following steps.

a) If all the finite nonzero $f_{i\infty}$ are given, find the corresponding m_i (which come in identical pairs). Assume either one pole at $z = 1$ and two at $1/a$, or vice versa, and include these among the m_i; n is thus *odd*. Select a value β in the range $1 < \beta < 1/a$; this will be the z-transform of the real reflection zero. Again, $\beta = a^{-1/2}$ is a good choice.[17]

b) If the $f_{i\infty}$ are not known, find them via the γ-variable iteration described in Section II-A, 1b). Here, $n_0 = 2$ and $n_\infty = 1$, or vice versa. A negative term, corresponding to a loss pole at $z = -\beta$,

$$-10 \log_{10} \coth \frac{|\gamma + \frac{1}{2} \ln a|}{2} < 0 \tag{80}$$

should be included.[18]

c) From the m_i and β, find the quantity

$$\lambda = \frac{\prod_{i=1}^{n} (m_i + \beta) - \prod_{i=1}^{n} (m_i - \beta)}{\prod_{i=1}^{n} (m_i + \beta) + \prod_{i=1}^{n} (m_i - \beta)} \tag{81}$$

and form the polynomial

$$E + zF = \prod_{i=1}^{n} (m_i + z). \tag{82}$$

Next, form $\lambda E + zF$. Due to the choice of λ, this polynomial has a factor $(\beta + z)$. It is therefore possible to form

$$U + zV = \frac{\lambda E + zF}{\beta + z} \tag{83}$$

where U and V are even polynomials in z. The removal of the factor $(\beta + z)$ must be carried out by high-accuracy deflation techniques [11].

The squared modulus of the characteristic function is then

$$|K|^2 = \frac{k^2(\beta^2 - z^2)U^2}{E^2 - z^2F^2}. \tag{84}$$

2) *Finding H:*

a) By the Feldtkeller equation (16)

$$|H|^2 = \frac{E^2 - z^2F^2 + k^2(\beta^2 - z^2)U^2}{E^2 - z^2F^2}. \tag{85}$$

The numerator has one positive real root z_e^2 in z^2, which lies in the range $1 < z_e^2 < a^{-2}$. Again, this should be found to a high accuracy using an iterative technique and then divided out. To this end, starting with $z_0^2 = \beta^2$, use repeatedly

$$z_{j+1}^2 = \beta^2 + \frac{\prod_{i=1}^{n} (m_i^2 - z_j^2)}{k^2 U^2(z_j^2)}; \qquad j = 0, 1, 2, \cdots. \tag{86}$$

Terminate the iteration after the maximum possible accuracy has been obtained. The limit is z_e^2.

Divide $E^2 - z^2F^2 + k^2(\beta^2 - z^2)U^2$ by $(z_e^2 - z^2)$ using the high-accuracy algorithm [11]. The remaining polynomial may be separated into quadratic factors by using a standard root-finding routine.

b) All quadratic factors should next be transformed using (38) of Section II-A, 2) into the form $pz^2 + q\sqrt{z^2 - 1}\ \sqrt{1 - a^2z^2} + r$. The transformed factors should then be multiplied together. This gives a product similar to the expression in (41) of Section II-A, 2). This product is in turn multiplied by the factor

$$\sqrt{1 - a^2z_e^2}\ \sqrt{z^2 - 1} + \sqrt{z_e^2 - 1}\ \sqrt{1 - a^2z^2}, \tag{87}$$

which is the transform of the linear factor $(z_e^2 - z^2)$. The final product is of the form

$$A\sqrt{z^2 - 1} + B\sqrt{1 - a^2z^2}. \tag{88}$$

Recursive formulas may easily be obtained to give the coefficients of A and B in terms of z_e, a, and the p, q, r values of the quadratic factors.

3) *Design Impedances and Element Values:* The formulas for the design impedances use the z-plane equivalent of the reflection zero polynomial $f(s)$. This may be obtained from $|K|^2$ in (84). Transforming the $(\beta^2 - z^2)$ factor as the $(z_e^2 - z^2)$ factor above, we obtain for the z-plane equivalent of $f(s)$

$$A'\sqrt{z^2 - 1} + B'\sqrt{1 - a^2z^2} = lU(z^2)[\sqrt{1 - a^2\beta^2}\ \sqrt{z^2 - 1} + \sqrt{\beta^2 - 1}\ \sqrt{1 - a^2z^2}] \tag{89}$$

[18] This represents an approximation, since the hypothetical loss pole is in fact at $\lambda\beta$. However, λ is exceedingly close to unity and, hence, the error due to this approximation is negligible.

in close analogy to (88), the transformed natural mode polynomial. In (89), l is a constant

$$l = \frac{B(1)}{\sqrt{\beta^2 - 1}\, U(1)}, \tag{90}$$

which is chosen[19] such that $B - B'$ has a factor $(z^2 - 1)$. Furthermore, if the computation of A and A' was correct, $A - A'$ should then have a factor $(1 - a^2z^2)$. Hence the polynomials

$$D = \frac{B - B'}{z^2 - 1} \tag{91a}$$

$$C = \frac{A - A'}{1 - a^2z^2} \tag{91b}$$

may be formed.

From polynomials A, A', B, B', C, and D, all design impedances can be found using Table V. After that, Tables II, III, and IV may be used to obtain the element values.

4) *Open- or Short-Circuit Termination:* It is evident that the circuit of Fig. 9(a) may be fed from a voltage source and that of Fig. 9(b) from a current source without any degeneracy or reduction in degree. For this case, the only design immittance may be obtained from Table V, if H is replaced by M or N and if $A' = B' = 0$ is substituted into the formula giving the design immittance. The ladder expansion is then carried out in the usual way.

III. Maximally Flat Passband Filters

A. Conventional Filters

For a maximally flat passband bandpass filter there are n reflection zeros at some bandcenter frequency f_0. If the passband limits, defined as the frequencies at which the loss reaches α_p on either side of f_0, are f_1 and f_2 (Fig. 10), then the normalization and frequency variable mapping of (22) and (23) of Section II-A may again be used. Since f_0 lies between f_1 and f_2, it will transform to two points on the imaginary axis in the z plane. Denote these points by $\pm jy_0$, $y_0 > 0$. The characteristic function is then

$$|K|^2 = \frac{k^2(y_0^2 + z^2)^n}{\prod_{i=1}^{n} (m_i^2 - z^2)} \tag{92}$$

where, as before, k is defined via

$$\alpha_p = 10 \log_{10} (1 + k^2) \tag{93}$$

and the m_i are the transformed loss poles, including those at zero and infinite frequency. In order that the loss at f_1 may be the same as at f_2, y_0 must satisfy

$$y_0^{2n} = \prod_{i=1}^{n} m_i^2 . \tag{94}$$

Examination of (92) shows that n should be even.

[19] Of course, l is close to k, if reasonable accuracy was maintained in the preceding calculations.

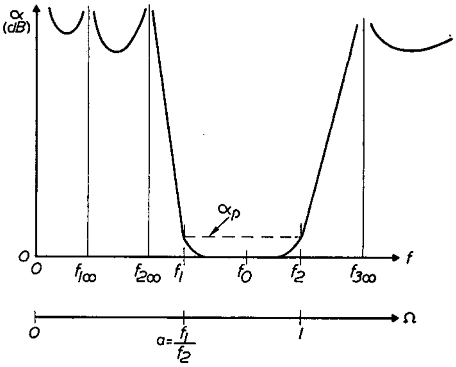


Fig. 10. Maximally flat passband filter.

For a maximally flat passband low-pass filter both f_1 and f_0 must be set to zero; f_2 is still the frequency at which the loss reaches α_p. In terms of the z variable, the characteristic function then becomes

$$|K|^2 = \frac{k^2 \prod_{i=1}^{n} m_i^2}{\prod_{i=1}^{n} (m_i^2 - z^2)}. \tag{95}$$

Here n may be either even or odd. The design steps are as follows.

1) *Finding* $|K|^2$*:* From given loss poles $f_{i\infty}$, the m_i are obtained exactly as in Section II-A, 1a). If the minimum stopband loss is given and the $f_{i\infty}$ are not, then an iterative method similar to that of Section II-A, 1b) may be used. For low-pass filters, the transformation remains the same, namely,

$$\gamma = \ln z = \frac{1}{2} \ln \frac{f^2 - f_2^2}{f^2}, \tag{96}$$

but for bandpass filters it must be modified slightly to

$$\gamma = \frac{1}{2} \ln \frac{(y_0^2 + 1)z^2}{y_0^2 + z^2} = \frac{1}{2} \ln \frac{f^2 - f_2^2}{f^2 - f_0^2}. \tag{97}$$

In both cases, the stopband loss is then given by

$$\alpha(\gamma, \gamma_i) = 20 \log_{10} k - 10 \sum_{i=1}^{n} \log_{10} |1 - e^{2(\gamma - \gamma_i)}|. \tag{98}$$

(Note that each finite, nonzero loss pole corresponds to a pair of identical γ_i. In (27), for clarity, they were shown explicitly.)

Except for the difference in the definition of γ and $\alpha(\gamma, \gamma_i)$, the iteration may be performed in the same way as for equal-ripple passband filters. From the resulting γ_i the required m_i^2 are obtained by

$$m_i^2 = \frac{y_0^2 e^{2\gamma_i}}{y_0^2 + 1 - e^{2\gamma_i}} \quad \text{for bandpass filters} \tag{99a}$$

$$= e^{2\gamma_i} \quad \text{for low-pass filters.} \tag{99b}$$

Although in the bandpass case it is theoretically possible to specify all three of f_1, f_0, and f_2 independently, it is not very practicable because it implies a constraint on the m_i that they satisfy (94). If, as is most likely, f_1 and f_2

are given, but f_0 is arbitrary, the following iteration may be used.

i) An estimate is made of f_0 (and hence of y_0).

ii) The γ_i and m_i are found by the iteration described above.

iii) Condition (94) is used to find an improved estimate of y_0, and hence of f_0, and the cycle is repeated until the sequence of f_0 has converged.

On the other hand, if f_0 and f_2 are given, but f_1 is arbitrary, the γ-transformation of (97) can be used and the γ_i found. One can then obtain y_0^2 by solving iteratively the equation

$$\prod_{i=1}^{n} [(y_0^2 + 1)e^{-2\gamma_i} - 1] = 1, \tag{100}$$

which results from substituting (99) into (94). Finally, the value of f_1^2 is calculable from

$$\frac{f_0^2 - f_2^2}{f_0^2 - f_1^2} = -y_0^2 . \tag{101}$$

A similar technique is applicable if f_1 and f_0 are the prescribed quantities. Then f_1, rather than f_2, is used in formula (96) for γ. Having found the m_i, $|K|^2$ can be found from (92).

2) *Finding H:* The transformed natural mode polynomial is now obtained by factoring, for bandpass filters, the polynomial

$$k^2(y_0^2 + z^2)^n + \prod_{i=1}^{n} (m_i^2 - z^2) \tag{102}$$

and for low-pass filters, the polynomial

$$k^2 \prod_{i=1}^{n} m_i^2 + \prod_{i=1}^{n} (m_i^2 - z^2). \tag{103}$$

The resulting quadratic factors (and, for odd-degree low-pass filters, also the one linear factor) must next be transformed via (36) to (40). After that, the transformed factors are multiplied together, and by the square root of the leading coefficient of (102) or (103), to get the expression

$$A + B\sqrt{z^2 - 1}\ \sqrt{1 - a^2z^2}. \tag{104}$$

3) *Finding the Open- and Short-Circuit Impedances:* This is done in two steps.

a) For bandpass filters, a constant l

$$l = \frac{A(1)}{(y_0^2 + 1)^{n/2}} \tag{105}$$

is found so that the polynomial

$$A(z^2) - l(y_0^2 + z^2)^{n/2} \tag{106}$$

has a factor $z^2 - 1$. For adequate accuracy, $l \simeq k$. Next, one calculates the even polynomials

$$A' = l(y_0^2 + z^2)^{n/2} \tag{107a}$$

$$D = \frac{A - A'}{(z^2 - 1)(1 - a^2z^2)}. \tag{107b}$$

In the low-pass case, $A' = A(1)$, i.e., merely a constant.

b) From polynomials A, B, A', and D, the design impedances may again be obtained via Table I. Note that for this class of low-pass filter one always has $R_1 = R_2$, since $K(0) = 0$. Hence, any termination ratio may be accommodated by associating a flat loss with the filter, as described in Section II-A, 4), or preferably by using a matching pad.

4) *Ladder Expansion:* This is carried out in the same way as for equal-ripple passband filters, using Tables II, III, and IV.

B. Symmetric Parametric Filters

For minimum-inductor or capacitor-crystal filters with maximally flat passbands, symmetric parametric filters may be used. These networks are characterized by the property that two of the reflection zeros are located symmetrically on the positive and negative real s-axis at $s = \pm b$, while the remainder are coincident in the passband at f_0. The difficulty that exists with equal-ripple passband parametric filters regarding the construction of the characteristic function is absent here since the function can be described immediately. If, as before, β is the z-variable equivalent of b, then the characteristic function is

$$|K|^2 = \frac{k^2(\beta^2 - z^2)^2(y_0^2 + z^2)^{n-2}}{\prod_{i=1}^{n} (m_i^2 - z^2)} \tag{108}$$

where the m_i and k are defined as before and β lies in the range $1 < \beta < a^{-1}$. In order to give equal loss at f_1 and f_2, y_0 and β must now jointly satisfy

$$y_0^{2n-4} = \beta^{-4} \prod_{i=1}^{n} m_i^2 . \tag{109}$$

If a special choice of β is not required for some other purpose, it may, to a very limited extent, be set so as to achieve a prescribed value for y_0 (or f_0). However, the permissible range for β, namely $1 < \beta < a^{-1}$, is fairly restricted, especially in narrow-band filters, and not much can be accomplished with this artifice. In practice, it is better to set β at $a^{-1/2}$.

The design steps are as follows.

1) *Finding $|K|^2$:* From given loss poles $f_{i\infty}$, the m_i are obtained. Assume one loss pole at $z = 1$ and one at $z = a^{-1}$. (Note that n must be even.) If the minimum stopband loss is given and the $f_{i\infty}$ are not, the same iterative methods as described for conventional maximally flat passband bandpass filters may be used. The γ transformation and stopband loss formulas of (97) and (98) apply, with the exception that the single loss poles at $\gamma = 0$ and $\gamma = \ln (f_2/f_0)$ should *not* be included.[20] From the m_i, one can get $|K|^2$ by using (108) and (109).

It may be seen that (108) is of the same form as (70) in Section II-B, 1e) if the identification

$$U(z^2) = (y_0^2 + z^2)^{n/2-1} \tag{110}$$

is made.

[20] This is equivalent to neglecting the effect of the factor $(\beta^2 - z^2)^2/[(1 - z^2)(a^{-2} - z^2)]$ of $|K|^2$ in the stopband. This effect is indeed very small for most filters [10].

2) *Finding H:* Using the new definition of U, the z-plane equivalent of the natural mode polynomial $e(s)$ is obtained in precisely the same way as for equal-ripple passband symmetric parametric filters described in Section II-B, 2). The process results in the polynomial

$$A + B\sqrt{z^2 - 1}\,\sqrt{1 - a^2z^2}, \qquad (111)$$

which is the equivalent of $e(s)$.

3) *Design Impedances and Element Values:* The process is exactly the same as that described for equal-ripple filters in Section I-B, 3).

C. Asymmetric Parametric Filters

Maximally flat passband bandpass filters, which are to be parallel- or series-connected at one end, or which have to have an open- or short-circuit termination, may be realized as asymmetric parametric filters. These networks have one single reflection zero on the negative real s-axis at $s = -b$ and the remainder coincident in the passband at f_0. The characteristic function takes the form

$$|K|^2 = \frac{k^2(\beta^2 - z^2)(y_0^2 + z^2)^{n-1}}{\prod_{i=1}^{n} (m_i^2 - z^2)}. \qquad (112)$$

The m_i and k have the usual definition, β lies in the range $1 < \beta < a^{-1}$, and, to give equal loss at f_1 and f_2, y_0 and β must jointly satisfy

$$y_0^{2n-2} = \beta^{-2} \prod_{i=1}^{n} m_i^2 . \qquad (113)$$

The design steps are as follows.

1) *Finding* $|K|^2$*:* From the given loss poles $f_{i\infty}$, the m_i are obtained. Assume either one loss pole at $z = a^{-1}$ and two at $z = 1$, or vice versa. (Note that n is now odd.) If the $f_{i\infty}$ are not given, they can be obtained using the same iterative methods as described for conventional maximally flat passband bandpass filters. The same transformation and loss formulas apply, with the exception that *one* of the loss poles assumed at $\gamma = 0$ or $\gamma = \ln (f_2/f_0)$ should be ignored. This assumes that either the factor

$$\frac{\beta^2 - z^2}{1 - z^2} \quad \text{or} \quad \frac{\beta^2 - z^2}{a^{-2} - z^2} \qquad (114)$$

is nearly unity in the stopbands. Since $1 < \beta^2 < a^{-2}$, this will indeed be true if the passband is not too wide. From the m_i, one can get $|K|^2$ by using (112) and (113).

2) *Finding H:* A comparison of the expression (112) for $|K|^2$ with (84) in Section II-C, 1) shows that they become identical if the formal definitions

$$U(z^2) = (y_0^2 + z^2)^{(n-1)/2}, \quad E + zF = \prod_{i=1}^{n} (m_i + z) \qquad (115)$$

are introduced. With these notations, all the formulas and procedures of Section II-C, 2) remain applicable for finding $A\sqrt{z^2 - 1} + B\sqrt{1 - a^2z^2}$, the z-plane equivalent of the natural mode polynomial $e(s)$.

3) *Design Impedances and Element Values:* All equations and operations of Section II-C, 3) are applicable using the notations (115) above. This also remains true for open- or short-circuit terminations.

IV. Equal-Ripple Stopband Filters

A. Low-Pass Filters [12], [13]

Filters that have to suppress a constant spectrum noise in their stopbands and satisfy specifications on their passband loss, phase or transient responses are conveniently realized as equal-ripple stopband filters. In this section, design techniques will be given for such low-pass filters. It should be noted that the transformed variable w, to be used here, is analogous in many ways to the z variable used in Sections II and III and that it simplifies greatly the solution of the approximation problem, as does the z variable. However, it does not improve the numerical conditioning of the realization process. In fact, the numerical accuracy is liable to become worse rather than better. Hence, it should only be applied in the approximation process, after which the z variable should be introduced at the earliest possible stage and used for the remainder of the calculation.

1) *Mapping:* A typical equal-ripple stopband low-pass characteristic is shown in Fig. 11. With the notations used there, the transformed frequency variable applicable is

$$w^2 = 1 - \frac{f^2}{f_1^2} = 1 - \Omega^2 = 1 + s^2 \quad (\mathrm{Re}\, w \geq 0) \qquad (116)$$

where $\Omega \triangleq f/f_1$. This transformation maps the passband $0 \leq f \leq f_p$ on the $\sqrt{1 - f_p^2/f_1^2} \leq w \leq 1$ portion of the real w-axis. It maps the stopband over the whole imaginary w-axis.

2) *Finding* $|K|^2$ *and* $|H|^2$ *from Given Natural Modes or Reflection Zeros:* If the natural modes s_{ni} are known, $|H|^2$ may be found in the following steps.

a) All n natural modes are transformed via (116) into the w plane:

$$w_{ni} = \sqrt{1 + s_{ni}^2} \qquad (\mathrm{Re}\, w_{ni} \geq 0). \qquad (117)$$

b) The w-plane Hurwitz polynomial

$$E + wF = \prod_{i=1}^{n} (w_{ni} + w) \qquad (118)$$

is formed. Here, E and F are even polynomials in w. Then

$$|H|^2 = 10^{\alpha_s/10} \frac{E^2 - w^2F^2}{E^2} \qquad (119)$$

and hence

$$|K|^2 = 10^{\alpha_s/10} \frac{(1 - 10^{-\alpha_s/10})E^2 - w^2F^2}{E^2}. \qquad (120)$$

In a similar way, if the n reflection zeros s_{0i} are given, then $|K|^2$ may be found by the following steps.

a) Find the w transforms w_{0i} of the s_{0i}.

b) Form the polynomial

$$C + wD = \prod_{i=1}^{n} (w_{0i} + w) \qquad (121)$$

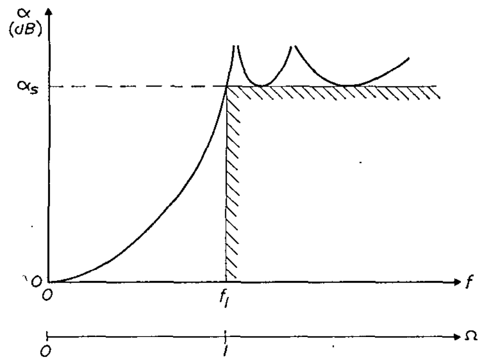


Fig. 11. Equal-ripple stopband low-pass filter.

where C, D are even polynomials in w. Then,

$$|K|^2 = k^2 \frac{C^2 - w^2D^2}{C^2} \tag{122}$$

where now

$$k = (10^{\alpha_s/10} - 1)^{1/2}. \tag{123}$$

Comparison with (120) shows that

$$C = E \tag{124a}$$

$$D = \sqrt{1 + k^{-2}}\, F. \tag{124b}$$

Then,

$$|H|^2 = k^2 \frac{(1 + k^{-2})C^2 - w^2D^2}{C^2}. \tag{125}$$

3) *Ladder Realizability:* For equal-terminated ladder realization, the necessary conditions

$$|H|_{w=1} = 1 \tag{126}$$

$$|H|_{w\to\infty} \to \infty \tag{127}$$

must hold. The first of these conditions, (126), requires that the natural modes satisfy

$$\frac{E(1) + F(1)}{E(1) - F(1)} = \prod_{i=1}^{n} \frac{w_{ni} + 1}{w_{ni} - 1} = (\sqrt{k^2 + 1} + k)^2. \tag{128}$$

For given reflection zeros, one of the w_{0i} should be at $w = 1$.

The second condition, (127), requires that the numerator of $H(w)$ be of higher degree than its denominator. This is automatically satisfied if n is odd. If n is even, however, then (125) gives $|H|_{w\to\infty} \to 10^{\alpha_s/20}$. Therefore, a transformation similar to that discussed in Section II-A, 4) for even-degree equal-ripple passband low-pass filters should be used. It is illustrated in Fig. 12. Again, a preliminary design is implicitly utilized with natural modes

$$\mathbf{w}_{ni}^2 = \frac{w_{ni}^2}{1 + \delta^2(1 - w_{ni}^2)} \tag{129}$$

where the w_{ni} are the transforms of the *desired* natural modes and $\mathbf{w} = j/\delta$ the largest loss pole of the preliminary design. The unknown parameters $\mathbf{w}_{ni}$ and δ satisfy the nonlinear equation obtained from (119):

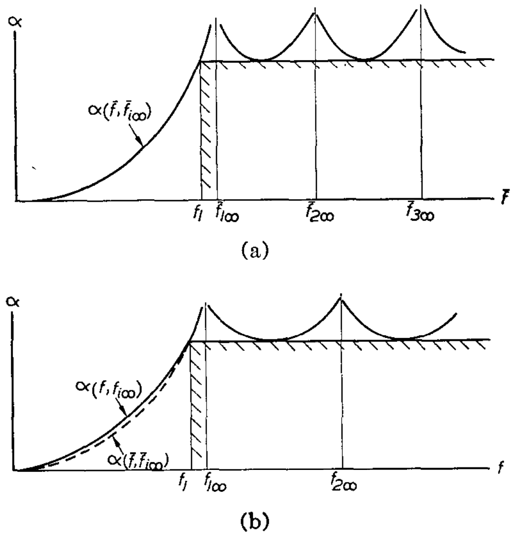


Fig. 12. The design of an antimetric equal-ripple stopband low-pass filter.

$$\text{Even part of} \left\{\prod_{i=1}^{n} (\mathbf{w}_{ni} + \mathbf{w})\right\}_{\mathbf{w}=j/\delta} = 0 \tag{130}$$

where the $\mathbf{w}_{ni}$, through (129), are also functions of δ. The solution may again be obtained via *regula falsi*, using

$$\delta = 0 \quad \text{and} \quad \delta = \frac{\pi}{2\sum_{i=1}^{n} w_{ni}} \tag{131}$$

as starting values.

Having obtained δ, the $\mathbf{w}_{ni}$ are obtained from (129). Then the preliminary functions $|\mathbf{H}|^2$ and $|\mathbf{K}|^2$ can be found as in (118) to (120), substituting bold-faced quantities for light-faced ones. Finally, using the

$$\mathbf{w}^2 = \frac{w^2}{1 + \delta^2(1 - w^2)} \tag{132}$$

substitution, the final functions $|H|^2$, $|K|^2$ may be obtained.

Obviously, for most active filter realizations, (127) is not a necessary condition.

An exactly analogous procedure may be used if the reflection zeros, rather than the natural modes, are to be preserved. Then replace w_{ni} by w_{0i} in (129) to (131) and calculate $\mathbf{C} + \mathbf{w}\mathbf{D}$, $|\mathbf{K}|^2$, $|\mathbf{H}|^2$, etc. using (121) to (125).

4) *Finding the Natural Mode Polynomial for Prescribed Passband Loss Response*: If a desired passband loss response $\alpha_p(f)$ is given (with $\alpha_p(0) = 0$), it should be transformed to the $(\sqrt{1 - f_p^2/f_1^2}, 1)$ part of the positive real w-axis. Then the approximation may be carried out using one of the following methods.

a) *Point Matching:* (119) may be rewritten, using

$$E(w) = 1 + \sum_{k=1}^{[n/2]} e_k w^{2k} \tag{133a}$$

$$F(w) = \sum_{l=0}^{[(n-1)/2]} f_l w^{2l} \tag{133b}$$

(where $[N]$ indicates the integer part of N), in the form

$$\sum_{l=0}^{[(n-1)/2]} f_l w^{2l+1} - \sqrt{1 - a(w)}\left[1 + \sum_{k=1}^{[n/2]} e_k w^{2k}\right] = 0. \tag{134}$$

Here, the notation

$$10 \log_{10} a(w) = \alpha_p(w) - \alpha_s \tag{135}$$

has been used. Equation (134) contains n unknown coefficients f_l, e_k. Hence, if n points w_i (including the $w = 1$ point) are matched on the $\alpha_p(w)$ curve, a system of n linear equations in n unknowns results, which may be solved to determine $E + wF$.

An alternative (and preferable) approach is to select a larger number ($>5n$) points and solve the resulting overdetermined system of linear equations via least-squares or least-pth approximation [14].

b) *Equal-Ripple Approximation:* Equal-ripple error may be obtained using linearized Remez techniques [14]. The design proceeds in the following steps [12].

i) Using point matching, an initial approximation is found, which satisfies (134) at $w = 1$ and also at $n - 1$ internal points of the passband. Let the resulting coefficients be e_k, f_l.

ii) The n alternating local extrema (largest in absolute value) of the error function $\alpha_p(w) - \alpha_s - 10 \log_{10}(1 - w^2F^2/E^2)$ are found in the passband. Let the corresponding abscissas be w_i, $i = 1, 2, \cdots, n$ (Fig. 13).

iii) Solve the equations

$$\begin{aligned} &\sqrt{1 - 10^{-\alpha_s/10}}\, E_\Delta(1) + F_\Delta(1) = 0 \\ &\sqrt{1 - \epsilon^{(-1)^i} a(w_i)}\, E_\Delta(w_i) + w_i F_\Delta(w_i) \\ &\quad - \frac{(-1)^i a(w_i) E(w_i)\, \Delta\epsilon}{2\epsilon^{1-(-1)^i}\sqrt{1 - \epsilon^{(-1)^i} a(w_i)}} = 0, \\ &\qquad i = 1, 2, \cdots, n \end{aligned} \tag{136}$$

where

$$E_\Delta(w) = 1 + \sum_{k=1}^{[n/2]} (e_k + \Delta e_k) w^{2k} \tag{137a}$$

$$F_\Delta(w) = \sum_{l=0}^{[(n-1)/2]} (f_l + \Delta f_l) w^{2l} \tag{137b}$$

and $E(w)$ is as in (133), for Δe_k, Δf_l, and $\Delta\epsilon$. Here, $\epsilon = 10^{\Delta\alpha/10}$ and $\Delta\alpha$ is the expected error ripple in dB. Initially, $\epsilon = 1$ can be used.

iv) Next, all parameters are incremented

$$e_k \rightarrow e_k + \Delta e_k \tag{138a}$$

$$f_l \rightarrow f_l + \Delta f_l \tag{138b}$$

$$\epsilon \rightarrow \epsilon + \Delta\epsilon \tag{138c}$$

and steps ii) to iv) repeated until the relative increments become small enough.

c) *Maximally Flat Approximation* [12]: If the power series of $\alpha_p(\Omega)$ around $\Omega = 0$ is known

$$\alpha_p(\Omega) \cong a_0 + a_1\Omega^2 + a_2\Omega^4 + \cdots + a_{n-1}\Omega^{2n-2} \tag{139}$$

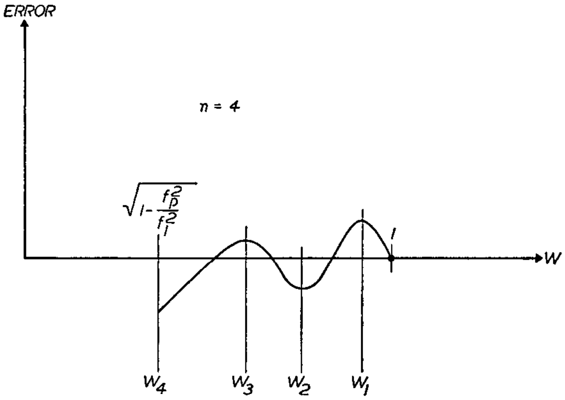


Fig. 13. Equal-ripple passband loss approximation.

then a multistage procedure [12], [14] involving only simple linear operations may be used to obtain an approximation with a power series matching all a_i up to $i = n - 1$. It consists of the following steps.

i) Let $b_0 = 10^{\alpha_s/10}$. Find

$$b_k = \frac{\ln 10}{10k} \sum_{j=1}^{k} j\, a_j\, b_{k-j}, \quad k = 1, 2, \cdots, n-1. \tag{140}$$

ii) Let $c_0 = 1 - 10^{(a_0-\alpha_s)/10}$. Find

$$c_k = c_{k-1} - 10^{-\alpha_s/10} b_k, \qquad k = 1, 2, \cdots, n-1. \tag{141}$$

iii) Let $d_0 = \sqrt{c_0}$. Find

$$d_j = \frac{1}{d_0}\left[\frac{c_j}{2} - \frac{1}{j}\sum_{k=1}^{j-1} k\, d_k\, d_{j-k}\right], \quad j = 1, 2, \cdots, n-1. \tag{142}$$

iv) Match the coefficients of $\Omega^0, \Omega^2, \Omega^4, \cdots, \Omega^{2n-2}$ in the equation

$$\sum_{l=0}^{[(n-1)/2]} f_l(1 - \Omega^2)^l \overset{n-1}{=} \left[\sum_{k=0}^{[n/2]} e_k(1 - \Omega^2)^k\right]\left[\sum_{i=0}^{n-1} d_i\Omega^{2i}\right] \tag{143}$$

to find n linear equations for e_k, f_l. One of these may again be chosen as 1. Solution of these equations gives the natural mode polynomial $E + wE$ via (133).

5) *Finding the Natural Mode Polynomial for a Prescribed Phase Response:* For a ladder network, the passband phase is exclusively determined by the natural modes. Hence, published polynomial approximations to maximally flat delay, equal-ripple delay, etc. [55], [51] may be transformed, using (116), to find the $E + wF$ polynomial. For equal terminations, the relationship (128) must hold between the w_{ni}, α_s, and f_1. It may simply be solved for

$$\alpha_s = -10 \log_{10}\left[1 - \frac{F^2(1)}{E^2(1)}\right]. \tag{144}$$

Alternatively, an iterative technique [13] may be used to find f_1 for n odd, or f_1 and δ simultaneously for n even.

6) *Finding the Natural Mode Polynomial for Prescribed Time Response:* It can be shown [12], [14] that the time response of an equal-ripple stopband low-pass filter, satisfying (126) and (127), to a 2-volt step voltage is

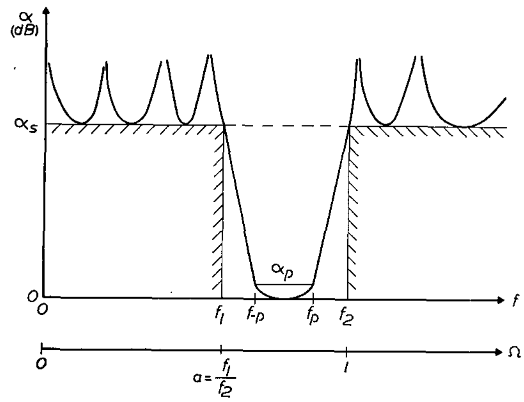


Fig. 14. Equal-ripple stopband bandpass filter response.

$$g(t) = 1 + \sum_{i=1}^{n} r_i e^{s_{ni} t} \qquad (t \geq 0). \tag{145}$$

Here, the residues can be expressed by the s_i via

$$r_i = 10^{-\alpha_s/20} \frac{w_{ni}}{s_{ni}} \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{w_{ni} + w_{nk}}{s_{ni} - s_{nk}} \tag{146}$$

and satisfy

$$\sum_{i=1}^{n} r_i = -1. \tag{147}$$

The corresponding impulse-response is

$$h(t) = \sum_{i=1}^{n} r_i s_{ni} e^{s_{ni} t}. \tag{148}$$

Using these expressions, linearized Remez and least-pth techniques may be used to find the s_{ni} for optimum time response [12]. Care must be taken to keep the realization conditions (126) and (127) satisfied in the course of the iteration. The significant advantage of using these equations for $g(t)$ and $h(t)$ is, of course, that they use only n parameters to achieve simultaneous time- and frequency-domain optimization.

B. Bandpass Filters [12], [14]

The design of frequency-asymmetric bandpass filters with equal-ripple stopbands (Fig. 14) may be accomplished using Moebius transformations [8], [14] on the squared frequency variable, if a suitable equal-ripple stopband low-pass prototype is available. Otherwise, methods analogous to those followed for low-pass filters should be used. The mapping needed is

$$w^2 = \frac{f_2^2 - f^2}{f^2 - f_1^2} = \frac{1 - \Omega^2}{\Omega^2 - a^2} = -\frac{1 + s^2}{a^2 + s^2} \quad (\operatorname{Re} w \geq 0). \tag{149}$$

This transforms the passband $f_{-p} \leq f \leq f_p$ to the

$$\sqrt{\frac{f_2^2 - f_p^2}{f_p^2 - f_1^2}} \leq w \leq \sqrt{\frac{f_2^2 - f_{-p}^2}{f_{-p}^2 - f_1^2}} \tag{150}$$

segment of the real positive w-axis. It maps the upper stopband onto the $[0, j]$ part, the lower stopband onto the $[j/a, \infty]$ part of the imaginary w-axis.

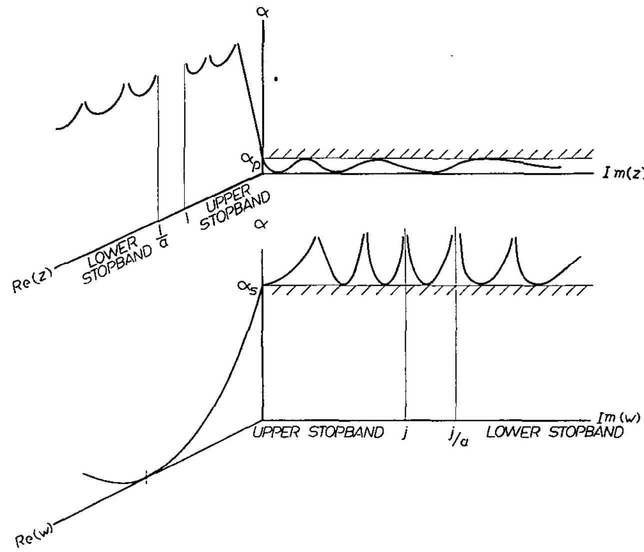


Fig. 15. The z transformation and the w transformation for bandpass filters.

Fig. 15 compares the w mapping of (149) with the z transformation (22).

For bandpass filters, (126) is no longer necessary for equal terminations. However, there must be an even number of poles (at least two) both at $f = 0$ and $f \to \infty$ for ladder realization. In addition, *no* loss pole should exist between j and j/a, since this segment corresponds to the real s-axis. These conditions are not necessary for many active filter realizations. For ladder filters, however, they are valid and make the design of these networks a very complicated problem, beyond the scope of this paper.

V. Predistortion

Parasitic losses in the reactances can distort both the loss and the phase response of the filter. Preferably, this distortion should be compensated either by cascading an equalizer with the network or possibly by associating negative resistances with the inductors. This will not result in unduly high sensitivities for the combination. In some cases, however, the additional elements cannot be afforded and the increase in sensitivity caused by predistorting the network is tolerable. Then, one of several available predistortion methods may be used.

The most elegant and accurate technique, and also the earliest, which compensates for uniform and semiuniform losses, is due to Darlington [7]. This has been extended and interpreted by many authors (see, e.g., [66], which also contains an extensive bibliography on the subject). Both uniform and semiuniform predistortion can also be carried out in terms of the z variable. However, even semiuniform losses approximate many physical situations poorly. In general, all elements in the circuit will have different Q-factors. No exact synthesis procedure is available for this situation, and one must normally resort to an iterative scheme. One such technique, which has proved to be very effective, is described next. It is based upon the representation of losses shown in Fig. 16 and is carried out in the following steps.

a) The lossless transducer function

$$H = \frac{e}{p} \tag{151}$$

is found in the usual way.

b) The anticipated constant passband loss α_0 is estimated from the known quality of the available components. Alternatively, the lossless circuit may be synthesized, the element Q-factors assigned, and the computed loss at one or both of the passband limits used as α_0.

c) Next, H is multiplied by $10^{\alpha_0/20}$ to get

$$H^{(0)} = 10^{\alpha_0/20}H = \frac{e^{(0)}}{p}. \tag{152}$$

d) From the modified $H^{(0)}$, the reactance filter is designed. Knowing the element values, realistic Q-factors may be assigned to each element.

e) A *negative* resistance is connected in series with each inductor and across each capacitor so that the resulting Q of each element is the negative of the value assigned in step d).

f) The natural mode polynomial $e^{(1)} = e_{-L}^{(0)}$ of this hypothetical network is found, either from a driving-point impedance or from a transfer function. It is multiplied by a constant so that the new and old natural mode polynomials $e^{(1)}$ and $e^{(0)}$, respectively, take on the same value E for $z = 1$. This last step is equivalent to arranging for identical coefficients of the highest power of s in the corresponding s-plane polynomials.[21]

g) A new lossless network is now synthesized from the transducer function

$$H^{(1)} = \frac{e^{(1)}}{p} \tag{153}$$

and *positive* resistors are connected to each reactor to achieve the Q-factors assigned in step d). The resulting lossy network is analyzed. If the response is acceptable, the process is terminated; otherwise, one proceeds to step h).

h) The natural mode polynomial $e_L^{(1)}$ of the lossy network found in step g) is calculated and multiplied by a constant so that it takes on the value E at $z = 1$ [see step f)]. If the previous predistortion had been accurate, we would have obtained $e_L^{(1)} = e^{(0)}$. But if not, the error in the predistortion can be reduced by subtracting from $e^{(1)}$ the difference between $e_L^{(1)}$ and $e^{(0)}$, i.e., by replacing $e^{(1)}$ by

$$e^{(2)} = e^{(1)} + e^{(0)} - e_L^{(1)}.$$

Steps g) and h) are repeated with successive updating of the superscripts until, for some i, $e_L^{(i)} = e^{(0)}$ to within the required accuracy [67].

[21] This is necessary because the assumed dissipation leaves the response unchanged for $s \to \infty$.

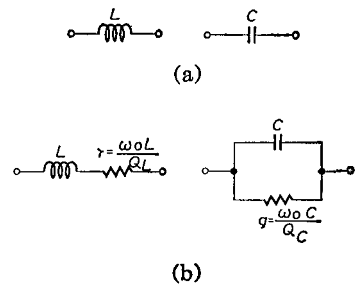


Fig. 16. Loss representations.

In summary, the sequence of natural mode polynomials generated is

$$\begin{aligned} e^{(0)} &= 10^{\alpha_0/20}e \\ e^{(1)} &= e_{-L}^{(0)} \\ e^{(2)} &= e^{(1)} + e^{(0)} - e_L^{(1)} \\ e^{(3)} &= e^{(2)} + e^{(0)} - e_L^{(2)} \\ &\cdots\cdots\cdots\cdots\cdots \\ e^{(i)} &= e^{(i-1)} + e^{(0)} - e_L^{(i-1)} \end{aligned} \tag{154}$$

The denominator p remains unchanged throughout the precedure. Hence, because of the dissipation, the loss poles of the final network will not be the zeros of p. This uncompensated shift has only second-order effects on the loss response [67]. However, for low-Q and exacting design specifications, it may be desirable to eliminate even this small distortion. This may be accomplished in the following steps [68].

i) Estimate the quality factor Q_i of each tuned circuit, which creates a loss pole at its resonant frequency ω_i. The loss pole will shift to about

$$s_i = -\frac{\omega_i}{2Q_i} \pm j\omega_i \tag{155}$$

from its original position at $\pm j\omega_i$, due to the losses.

ii) Obtain the natural modes by assuming that the loss poles are the (shifted) s_i of (155). Note that the procedures of Sections II and III are equally applicable for prescribed complex loss poles. Let the corresponding natural mode polynomial be e.

iii) Choose as the lossless transducer factor

$$H = \frac{e}{p} \tag{156}$$

where the denominator p is obtained from the *unshifted* loss poles $j\omega_i$. Enter step a) of the iteration used for predistorting the natural modes.

Appendix

An Index to Instant Design Information

Thanks to the unselfish efforts of many workers in the field of computer-aided filter design, a large amount of design information is now available in the form of tables and charts. It is the purpose of this Appendix to give a catalog of the books and papers containing this information. The catalog is organized by filter classes.

It is hoped that this catalog contains reference to most tabulated filter design aids. However, no claim is made to completeness and absence of reference to any work indicates simply that the authors did not have the material in their files; it implies no value judgement (nor does the order of tabulation).

Notes:

1) The notation $a(c)b$ indicates that the parameter in question has been tabulated from value a to value b, in steps equal to c. The symbol a to b means that the parameter may be read in a continuous way between a and b.

2) Unless otherwise noted, the information is presented in the form of tables.

A. *Butterworth Filters*

1) *Necessary Degree:*

a) Christian *et al.* [15]: $n = 2(1)19$, $\alpha_p = 0.0001$ to 3 dB, $\alpha_s = 20$ to 90 dB (nomographs).

b) Geffe [16]: $n = 2(1)10$, $\alpha_s = 3$ to 60 dB, $\alpha_p = 3$ dB (chart).

c) Kawakami [17] and Zverev [18]: $n = 1(1)23$, $\alpha_s = 0.5$ to 140 dB, $\alpha_p = 0.005$ to 40 dB (nomograph).

d) Maclean [19]: $n = 2(1)7$, $\alpha_s = 20$ to 90 dB, $\alpha_p = 0.01$ to 4 dB (charts).

e) White Electromagnetics [20]: $n = 1(1)10$, $\alpha_s = 20(10)60$ dB (charts).

2) *Transfer Function, Responses:*

a) Christian *et al.* [15]: natural modes for $n = 2(1)9$, $\rho = 5, 10, 15, 25, 50$ percent.

b) Zverev [18]: loss, group delay, and time responses for $n = 1(1)10$ (charts).

c) Weinberg [21]: natural modes for $n = 1(1)10$.

d) Henderson *et al.* [22]: impulse and step responses for $n = 1(1)10$, low-pass and high-pass filters.

e) Fritzsche [24]: natural modes, coefficients for $n = 11(1)20$.

f) Craig [26]: natural modes, coefficients for $n = 2(1)5$; also, loss, phase, impulse, and step responses (charts).

g) ITT [27]: cutoff curves, minimum Q for $n = 2(1)7$ (charts).

h) White Electromagnetics [20]: loss response for $n = 1(1)10$, 12, 15, 20; midband and band-edge delay for $n = 1$ to 30; time responses for $n = 1(1)10$ (charts). Also, rise time, overshoot for $n = 1(1)10$.

i) Matthaei *et al.* [28]: loss response for $n = 1(1)15$; also phase and delay for $n = 5$.

3) *Element Values:*

a) Geffe [16], [29]: $n = 2(1)10$, $R_1/R_2 = 1$. For lossless elements, for uniform losses, and for lossy inductor filters.

b) Zverev [18]: $n = 2(1)10$ lossless; $n = 2(1)8$ predistorted.

c) Weinberg [21]: $n = 1(1)10$, $R_2/R_1 = 1, 2, 3, 4, 8, \infty$ lossless; $n = 1(1)10$, $Q = 4, 10, 20, 30$ predistorted (singly loaded).

d) Fritzsche [24]: $n = 11(1)20$, $R_2/R_1 = 1, 1.5, 2, 5, \infty$.

e) Glowatzky [30], [31]: $n = 2(1)9$, $\rho = 5(5)20$ percent, and $\alpha_p = 3.01$ dB. Also open- and short-circuit at one port.

f) Maclean [19]: $n = 2(1)7$, $\alpha_p = 0$ to 4 dB (charts). Also, charts for minimum Q.

g) Craig [26]: elements of doubly terminated, uniformly lossy low-pass and bandpass filters, for $n = 2(1)5$ (charts).

h) ITT [27]: elements for $n = 2(1)7$ lossless doubly and singly terminated networks; also, for finite Q low-pass and bandpass, $n = 3(1)5$ (charts).

i) White Electronics [20]: elements for lossless ladder $n = 1(1)20$, $R_2/R_1 = 1$; $n = 1(1)10$ for $R_2/R_1 = 2, 3, 4, 8$. Also for lossy singly terminated filters, $n = 1(1)10$, $Q = 5, 10, 20, 30, 50, \infty$.

j) Matthaei *et al.* [28]: elements for lossless ladder, $n = 1(1)15$, $R_2/R_1 = 1$; also, $n = 1(1)10$ for $R_2 = \infty$.

B. *Chebyshev Passband Polynomial Filters*

1) *Necessary Degree:*

a) Christian *et al.* [15]: $n = 2(1)19$, $\alpha_p = 0.0001$ to 3 dB, $\alpha_s = 20$ to 90 dB (nomograph).

b) Geffe [16]: $n = 3(1)7$, $\alpha_p = 0.9$ dB, $\alpha_s = 0.9$ to 70 dB (nomograph).

c) Kawakami [17] and Zverev [18]: $n = 1(1)15$, $\alpha_p = 0.00005$ to 40 dB, $\alpha_s = 0.5$ to 140 dB (nomograph).

d) Matthaei [32]: Chebyshev passband impedance matching low-pass filters, $n = 2(2)10$, $R_1/R_2 = 1.5, 2, 2.5, 3(1)6, 8, 10(5)25, 30, 40, 50$.

e) Humpherys [33]: even-order filters with $R_1 = R_2$, $\alpha_p = 0.1, 0.5, 1$ dB, $n = 4(2)10$.

f) Maclean [19]: $n = 2(1)7$, $\alpha_p = 0.01$ to 4 dB, $\alpha_s = 20$ to 90 dB (charts).

2) *Transfer Function, Responses:*

a) Christian *et al.* [15]: natural modes for $n = 2(1)9$, $\rho = 5, 10, 15, 25, 50$ percent.

b) Zverev [18]: loss, group delay, and time response for $n = 1(1)10$ and $\alpha_p = 0.01, 0.1, 0.5$ dB (charts). Natural modes for $n = 3(1)7$, $\rho = 1(1)5, 8, 10(5)25, 50$ percent.

c) Weinberg [21]: natural modes for $n = 1(1)10$, $\alpha_p = 0.5, 1, 2, 3$ dB.

d) Henderson *et al.* [22]: impulse and step responses for $n = 1(1)10$, $\alpha_p = 0.5, 1, 2$ dB; low-pass and high-pass filters.

e) Fritzsche [24]: natural modes, coefficients for $n = 1(1)16, 20$, $\rho = 5, 10, 20, 50$ percent.

f) Humpherys [33]: natural modes, coefficients for $n = 4(2)10$, $\alpha_p = 0.1, 0.5$, 1-dB filters with $R_1 = R_2$.

g) Craig [26]: natural modes, coefficients for $n = 2(1)5$, $\alpha_p = 10^{-3}, 10^{-2}, 0.03, 0.1, 0.3$ and 1 dB; also, loss, phase, impulse, and step responses (charts).

h) ITT [27]: cutoff curves, minimum Q for $n = 2(1)7$, $\alpha_p = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 0.1, 0.3, 1, 3$ dB (charts).

i) White Electromagnetics [20]: loss responses for $n = 1(1)10, 12, 15, 20$; $\alpha_p = 0.1, 0.25, 0.5, 1, 2, 3$ dB, midband delay for $n = 1$ to 30, $\alpha_p = 0.5, 1, 2$ dB; time responses for $n = 1(1)10$, $\alpha_p = 0.5, 1, 2$ dB (charts). Also, rise time, overshoot for $n = 1(1)10$.

j) Matthaei *et al.* [28]: loss responses for $n = 1(1)15$, $\alpha_p = 0.01, 0.1, 0.2, 0.5, 1, 2, 3$ dB. Also, phase and delay for $n = 5$.

3) *Element Values:*

a) Saal [34]: for $n = 4(1)9$, $\rho = 1(1)5, 8, 10, 15, 20, 25, 50$ percent.

b) Geffe [16]: $n = 6(1)11$, $\alpha_p = 0.177$ dB lossless; $n = 3(1)7$, $\alpha_p = 0.9$ dB uniformly lossy.

c) Saal *et al.* [1]: $n = 6(1)11$, $\alpha_p = 0.177$ dB.

d) Zverev [18]: $n = 3(1)7$, $\rho = 1(1)5, 8, 10(5)25, 50$ percent. Also, $n = 2(1)10$, $\alpha_p = 0.01, 0.1$, and 0.5 dB lossless; $n = 2(1)8$, $\alpha_p = 0.01, 0.1$, and 0.5 dB predistorted.

e) Weinberg [21]: $n = 1(1)10$, $R_2/R_1 = 1(1)4, 8, \infty$, $\alpha_p = 0.1, 0.25, 0.5, 1, 2, 3$ dB lossless; $n = 1(1)10$, $Q = 4, 10, 20, 30$, $\alpha_p = 0.1, 0.25, 0.5, 1, 2, 3$ dB predistorted (singly loaded).

f) Matthaei [32]: Chebyshev passband impedance matching low-pass filters, $n = 2(2)10$, $R_1/R_2 = 1.5(0.5)3$ (1)6, 8, 10(5)25, 30, 40, 50.

g) Fritzsche [24]: element values for $n = 1(1)16, 20$, $\rho = 5, 10, 20, 50$ percent.

h) Humpherys [33]: elements for $n = 4(2)10$, $\alpha_p = 0.1, 0.5, 1$ dB; $R_1 = R_2$.

i) Glowatzky [30], [31]: elements for $n = 2(1)9$, $\rho = 5(5)20$. Also for open- or short-circuit at one end.

j) Maclean [19]: elements for $n = 2(1)7$, $\alpha_p = 0$ to 4 dB (charts). Also, charts for minimum Q.

k) Craig [26]: elements of doubly terminated, uniformly lossy low-pass and bandpass filters for $n = 2(1)5$, $\alpha_p = 0.001, 0.01, 0.03, 0.1, 0.3$, and 1 dB (charts).

l) ITT [27]: lossless ladder elements, doubly and singly terminated for $n = 2(1)7$, $\alpha_p = 10^{-5}, 10^{-3}, 10^{-2}, 0.1, 0.3, 1, 3$ dB.

m) White Electromagnetics [20]: lossless ladder elements for $n = 1(1)10$, $\alpha_p = 0.1, 0.25, 0.5, 1, 2, 3$; $R_2/R_1 = 1, 2, 3, 4, 8$; also, lossy singly terminated filters for $n = 1(1)10$, $Q = 10, 20, 30, \infty$; $\alpha_p = 0.1, 0.25, 0.5, 1, 2, 3$ dB.

n) Matthaei *et al.* [28]: lossless ladder elements for $n = 1(1)15$, $\alpha_p = 0.01, 0.1, 0.2, 0.5, 1, 2, 3$ dB, $R_2/R_1 = 1$. Also, $n = 1(1)10$, $R_2 = \infty$, same α_p.

C. Chebyshev Stopband Filters

1) *Necessary Degree:*

a) Christian *et al.* [15]: $n = 2(1)19$, $\alpha_p = 0.0001$ to 3 dB, $\alpha_s = 20$ to 90 dB (nomographs).

b) Kawakami [17] and Zverev [18]: $n = 1(1)15$, $\alpha_p = 0.00005$ to 40 dB, $\alpha_s = 0.5$ to 140 dB (nomograph).

c) Feistel *et al.* [35]: maximally flat passband delay, $n = 3(1)10$.

2) *Transfer Function, Responses:*

a) Christian *et al.* [15]: natural modes and loss poles for $n = 2(1)9$, $\alpha_s = 20(10)60$ dB.

b) Fritzsche [24]: natural modes, loss poles for $n = 3, 5, 7$, $\alpha_p = 3, 6$ dB, $R_2/R_1 = 1, \infty$.

c) Fritzsche [25]: natural modes, loss poles for RC–Chebyshev stopband filters, $n = 3, 5, 7$; $\alpha_p = 3, 6$ dB; $\alpha_s = 15(5)60$ dB.

d) Feistel *et al.* [35]: maximally flat passband delay $n = 3(1)10$, $\alpha_s = 10(4)98$ dB; 3-dB point, stopband limit frequency, loss poles.

e) Jess *et al.* [36]–[38]: natural modes, loss poles for equal-ripple time response.

3) *Element Values:*

a) Fritzsche [24]: elements for $n = 3, 5, 7$, $\alpha_p = 3, 6$ dB, $R_2/R_1 = 1$.

b) Feistel *et al.* [35]: maximally flat passband delay $n = 3(1)10$, $\alpha_s = 10(4)98$.

D. Elliptic Filters

1) *Necessary Degree:*

a) Christian *et al.* [15]: $n = 2(1)19$, $\alpha_p = 0.0001$ to 3 dB, $\alpha_s = 20$ to 90 dB (nomographs).

b) Henderson [39]: $n = 1(1)20$, $\alpha_p = 0.05$ to 3 dB, $\alpha_s = 3$ to ∞ dB, $f_s/f_p = 1.001$ to 2 (nomographs).

c) Skwirzynski [40]: $n = 3(1)7$, $\alpha_s = 0$ to 125 dB, $f_p/f_s = 0$ to 1 (chart).

d) Saal [34]: $n = 3(1)12$, $\alpha_s = 3$ to 17 N, $f_p/f_s = 0.05$ to 1, $\rho = 1(1)5, 8, 10, 15, 20, 25, 50$ percent (chart).

e) Saal *et al.* [1]: $n = 3(1)11$, $\alpha_p = 0.177$ dB, $\alpha_s = 0$ to 120 dB (chart).

f) Kawakami [17] and Zverev [18]: $n = 3(1)11$, $\alpha_p = 0.005$ to 40 dB, $\alpha_s = 1$ to 140 dB (nomograph).

g) Szentirmai [41]: $n = 2(1)30$, $\alpha_p = 0.025$ to 3 dB, $\alpha_s = 0$ to 70 dB (nomographs).

2) *Transfer Function, Responses:*

a) Christian *et al.* [15]: natural modes, loss poles, and zeros for $n = 2(1)9$, $\rho = 5, 10, 15, 25, 50$ percent, $\theta = \sin^{-1} f_p/f_s = 1°(1°)85°$.

b) Skwirzynski [40]: loss poles, loss zeros, minima and maxima of loss; natural modes (implicitly) for $n = 3(1)7$, $\alpha_p = 0.01, 0.02, 0.05, 0.1, 0.15, 0.2, 0.3, 0.5, 0.7, 1$ dB; $f_p/f_s = 0.10(0.01)\ 0.99$.

c) Saal [34]: loss poles for $n = 4(1)9$, $\theta = 6°(1°)87°$, $\rho = 1(1)5, 8, 10, 15, 20, 25, 50$ percent.

d) Saal *et al.* [1]: loss poles for $n = 6(1)11$, $\theta = 16°(1°)85°$, $\rho = 20$ percent.

e) Zverev [18]: natural modes, loss poles for $n = 3(1)7$, $\rho = 1(1)5, 8, 10(5)25, 50$ percent, $\theta = 1°(1°)60°$.

f) Sinozaki *et al.* [42]: loss poles for $n = 11(1)20$ (for $n = 1(1)10$ given implicitly). $f_p/f_s = 0(0.002)0.9\ (0.001)\ 0.999$.

g) Fritzsche [23]: natural modes, loss poles for $n = 3$, $\rho = 1(1)5, 8, 10(5)25, 50$ percent, $\theta = 1°(1°)75°$.

h) Fritzsche [24]: natural modes, loss poles for $n = 5, 7$; $\rho = 5, 10, 20, 50$ percent, $\theta = 6°(1°)89°$; $R_2 = \infty$.

i) Glowatzki [43]: tables of $\varphi = am(mK/n, \sin\theta)$ for $m = 1(1)n - 1$, $n = 2(1)12$, $\theta = 0°(1°)90°$.

j) Glowatzki [44]: tables of loss poles for $n = 1(1)12$, $\theta = 0°(1°)90°$, products of odd-order loss poles.

k) Glowatzki [45]: tables of loss poles for even n, transformed as in (53) of this paper.

l) Bedrosian *et al.* [46]: chain matrix coefficients for $n = 3(1)6$, $\alpha_p = 0.1, 0.5, 1$ dB, $\alpha_s = 20(5)80$ dB.

m) ITT [27]: cutoff curves for $n = 2(1)7$, $\alpha_p = 10^{-4}$, 10^{-2}, 0.3, 1, 3 dB.

3) *Element Values:*

a) Skwirzynski [40]: elements for $n = 3(1)7$, $R_1/R_2 = 0, 1, \infty$ (other for even n), $\alpha_p = 0.01, 0.02, 0.05, 0.1, 0.15, 0.2, 0.3, 0.5, 0.7, 1$ dB. Also, curves giving minimum necessary Q, distortion due to this Q, predistorted element values.

b) Saal [34]: elements for $n = 4(1)9$, $\theta = 6°(1°)87°$, $\rho = 1(1)5$, 8, 10, 15, 20, 25, 50 percent.

c) Geffe [16]: $n = 3, 5$; $\alpha_p = 0.1, 0.5, 1.0$ dB; $n = 6(1)11$, $\alpha_p = 0.177$ dB, $\theta = 16°(1°)85°$. Also, frequency-symmetric elliptic "zigzag" bandpass

$$\frac{f_s - f_{-s}}{f_p - f_{-p}} = 1.278,$$

$$\frac{f_{\text{center}}}{f_p - f_{-p}} = 1.6(0.1)2(0.2)13(0.5)20(1)35.$$

d) Saal *et al.* [1]: $n = 6(1)11$, $\rho = 20$ percent, $\theta = 16°(1°)85°$.

e) VEB [47]: predistorted filters for $n = 5(1)9$, $\alpha_p = 0.005, 0.01(0.01)0.04N$, $Q_{\text{average}} \times$ (smallest real part of any natural mode) $= 1.5, 2, 3, 4, 8$.

f) Zverev [18]: $n = 3(1)7$; $\rho = 1(1)5$, 8, 10(5)25, 50 percent, $\theta = 1°(1°)60°$.

g) Fritzsche [23]: $n = 3$, $\rho = 1(1)5$, 8, 10(5)25, 50 percent, $\theta = 1°(1°)75°$.

h) Bedrosian *et al.* [46]: $n = 3, 5$, $\alpha_p = 0.1, 0.5, 1$ dB, $\alpha_s = 20(5)70$ dB.

i) Skwirzynski *et al.* [48]: $n = 3, 5$, $\alpha_p = 0.01, 0.1, 0.2, 0.5, 1, 3$ dB, $f_p/f_s = 0.35$ to 0.9 (curves). Also, curves of required minimum Q.

j) Fritzsche [24]: $n = 5, 7$, $\rho = 5, 10, 20, 50$ percent, $\theta = 6°(1°)89°$, $R_2 = \infty$.

E. Legendre Filters

1) *Necessary Degree:*

a) Zverev [18]: $n = 1(1)10$, $\alpha_p = 0$ to 3 dB, $\alpha_s = 0$ to 140 dB (chart).

2) *Transfer Function, Responses:*

a) Zverev [18]: loss, group delay, and time responses for $n = 1(1)10$ (charts).

3) *Element Values:*

a) Zverev [18]: $n = 3(1)10$ lossless; $n = 3(1)8$ predistorted.

F. Linear-Phase Filters

1) *Necessary Degree:*

a) Zverev [18]: Bessel equal-ripple phase, Gaussian filters, $n = 1(1)10$.

b) Kulmann [49]: equal-ripple group delay all-pass, $n = 1(1)12$.

c) Hausner *et al.* [50]: equal-ripple phase delay error and equal-ripple phase error, $n = 1(1)12$, delay = 1 second, delay error 0.01 to 10 percent, phase error 0.0005 to 0.5 rad, bandwidth = 0 to 40 rad/s.

d) Ulbrich *et al.* [51]: equal-ripple group delay, $\omega = 0$ to 1, $n = 1(1)10$, delay = 0.1 to 50, delay ripple = ± 0.01 to 0.5.

e) Feistel *et al.* [35]: maximally flat passband delay, Chebyshev stopband filters, $n = 3(1)10$.

2) *Transfer Function, Responses:*

a) Zverev [18]: loss, group delay, and time responses for Bessel, Gaussian, equal-ripple phase filters, $n = 1(1)10$.

b) Weinberg [21]: natural modes for Bessel, $n = 1(1)11$.

c) Humpherys [52]: natural modes, impulse responses for equal-ripple phase filters, $n = 2(1)10$.

d) Kulmann [49]: natural modes for all-passes with equal-ripple group delay, $n = 3(1)12$, error ripple = 0.01 to 10 percent, $0 \leq \omega \leq 1$.

e) Henderson *et al.* [22]: impulse and step responses, overshoot for Bessel filters, $n = 1(1)10$.

f) Hausner *et al.* [50]: ratios of polynomial coefficients for all-passes with equal-ripple phase error and equal-ripple phase delay error; $n = 2, 4, 6, 12$, delay = 1 second, delay error = 10^{-4}, 2×10^{-4}, 5×10^{-4}, 10^{-3}, 2×10^{-3}, 5×10^{-3}, 10^{-2}, 2×10^{-2}, 5×10^{-2} seconds, phase error = 5×10^{-4}, 10^{-3}, 2×10^{-3}, 5×10^{-3}, 10^{-2}, 2×10^{-2}, 5×10^{-2}, 0.1, 0.2, 0.5 rad.

g) Ulbrich *et al.* [51]: equal-ripple group delay, $n = 1(1)10$, delay ripple = $\pm 0.01(0.01)0.1(0.1)0.5$, $\omega = 0$ to 1; natural modes for all-passes.

h) Abele [53]: equal-ripple group delay, $n = 1(1)10$, delay ripple = 0(0.1)1(1)10, 15, 20, 30 percent, delay = 1; coefficients of natural mode polynomial for all-passes.

i) Macnee [54]: equal-ripple group delay, $n = 2(1)6$, natural modes, rise time, overshoot for polynomial low-pass filters.

j) Feistel *et al.* [35]: maximally flat passband group delay, equal-ripple stopband filters, $n = 3(1)10$, $\alpha_s = 10(4)98$ dB; 3-dB point, stopband limit frequency, loss poles.

k) Orchard [55]: natural modes for Bessel filters, $n = 1(1)31$.

l) Craig [26]: natural modes, coefficients for $n = 2(1)5$ Bessel filters; also loss, phase, impulse and step responses (charts).

m) Peless *et al.* [56]: natural modes, coefficients, $n = 2(1)5$ transitional Butterworth–Bessel filters; also amplitude and group delay response, step response, bandwidth, rise time and overshoot.

n) ITT [27]: passband and stopband loss response, minimum Q, group delay distortion for $n = 2(1)8$ Bessel filters.

o) Matthaei *et al.* [28]: passband loss and delay for $n = 1(1)11$ Bessel filters.

3) *Element Values:*

a) Zverev [18]: Bessel, equal-ripple phase, Gaussian for $n = 2(1)10$ lossless; $n = 2(1)8$ for predistorted filters.

b) Weinberg [21]: Bessel, $n = 1(1)11$, $R_2/R_1 = 1(1)4$, 8, ∞, lossless ladder; also, $n = 1(1)11$, $Q = 4, 10, 20, 30$ predistorted singly loaded ladder. All-pass lattice, $n = 1(1)11$.

c) Feistel *et al.* [35]: maximally flat passband delay, Chebyshev stopband filters; $n = 3(1)10$, $\alpha_s = 10(4)98$.

d) Craig [26]: Bessel, $n = 2(1)5$, doubly terminated, uniformly lossy low-pass and bandpass filters (charts).

e) Peless *et al.* [56]: elements of singly terminated transitional Butterworth–Bessel filters, $n = 2(1)5$.

f) ITT [27]: elements, minimum Qs for singly and doubly terminated lossless ladders, for $n = 2(1)7$ Bessel filters. Also, for finite-Q low-pass and bandpass, $n = 3(1)5$ (charts).

g) Matthaei *et al.* [28]: elements for lossless Bessel, $R_2/R_1 = 1, n = 1(1)11$.

G. Pulse Filters

1) *Necessary Degree:*

a) Dishal [57]: $n = 4(1)10$ for Gaussian filter, $f_s/f_{3dB} = 1 - 8$, $\alpha_s = 3$ to 100 dB.

2) *Transfer Function, Responses:*

a) Dishal [57]: natural modes for Gaussian, $n = 1(1)9$.

b) Zverev [18]: loss and group delay responses for Gaussian and transitional Gaussian filters, $n = 1(1)10$.

c) Jess *et al.* [36]–[38]: natural modes and loss poles for polynomial and rational approximations to simultaneous equal-ripple time and stopband loss response.

d) Meyer [58]: natural modes, loss poles for least-squares approximation to a square impulse response, $n = 1(1)11$; for complex and for $j\omega$-axis loss poles.

e) Mullick [59]: natural modes, overshoot, rise time for filters with natural modes lying on a parabola, $n = 2(1)5$.

f) Scanlan [60]: natural modes, amplitude, phase and group delay responses (charts) for filters with natural modes on an ellipse at equal frequency spacings, $n = 2(1)6$.

g) Ghausi *et al.*[61]: natural modes, coefficients, loss, group delay, step response, rise time, overshoot for filters whose natural modes lie on a catenary.

3) *Element Values:*

a) Dishal [57]: for Gaussian, $n = 1(1)9$, elements for singly and doubly loaded lossless ladders, for $n = 5, 8$ uniformly lossy ladders.

b) Zverev [18]: elements for Gaussian and transitional Gaussian filters, $n = 2(1)8$.

c) Jess *et al.* [36]: elements for polynomial filters, $n = 2(1)9$, for equal-ripple time responses.

d) Meyer [58]: ladder filters approximating a square impulse response in a least-square sense, $n = 1(1)11$.

e) Mullick [59]: elements for filters with natural modes lying on a parabola, $n = 2(1)5$.

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