Sharp Cutoff Filters with Monotonic Passband Response

Dragana Živaljević
Faculty of Electronic Engineering
University of Niš
Niš, Serbia
dragana.zivaljevic@elfak.ni.ac.rs

Negovan Stamenković
Faculty of Natural Science and Mathematics
University of Priština
K. Mitrovica, Serbia
negovanstamenkovic@gmail.com

Jeroslav Živanić
Faculty of Technical Sciences
University of Kragujevac
Čačak, Serbia
jeroslav.zivanic@ftn.kg.ac.rs

***Abstract*—Rational functions approximately monotonic in the passband, with one pair of zeros on the imaginary axis, have been compared with allpole functions, monotonic in the passband, for the lowpass filter applications. It is shown that usage of the zeros as a parameter can accomplish a trade between minimum stopband attenuation and sharpness of the cutoff characteristic. The paper concludes with a detailed example showing the efficiency of the proposed technique.**

***Keywords*—Legendre polynomials, Chirstoffel-Darboux identity, orthogonal function, all-pole filter, LC ladder network.**

I. Introduction

Many types of classical orthogonal polynomials for designing analog low-pass filters have been presented in the literature [1]–[3]. All-pole low-pass filters is an important filter category where transfer functions have all theirs zeros at the infinity. Those functions are easier to implement in comparison to low-pass transfer functions with finite zeros on the imaginary axis, as for example, inverse Chebyshev and Elliptic filters [1]. Thus, all-pole approximations are always considered as a first option for the filter design.

There are approximations that have very good attenuation characteristic at the expense of their phase characteristic, as for example, Chebyshev [1] and Legendre [4], [5]. Opposite case occurs with some other approximations, as for example, Bessel filters [6] which are optimized for maximally-flat constant group delay.

The Legendre orthogonal polynomials have been often used to approximate filter function. The Legendre-Papoulis (known as an "Optimum L" or just "Optimum") filter with monotonic magnitude response in the passband, proposed by A. Papoulis, [5], [7] has the maximum rolloff rate for a given filter degree. It provides a compromise between maximally flat Butterworth filter and Legendre filter [4] with ripples in the pass-band. Legendre-Papoulis filters can be useful in applications that need a steep cutoff at the passband edge but cannot tolerate passband ripples, or in cases where Legendre filter produces very high group delay at the passband edge.

Recently published paper [8] describes the sum-of-squares Legendre polynomial approximation, which offers a smaller deviation of the amplitude characteristic in the passband, but its cutoff slope is equal to cutoff slope of the Optimal filters.

Amplitude and group delay response of these filters are shown in Figure 1.

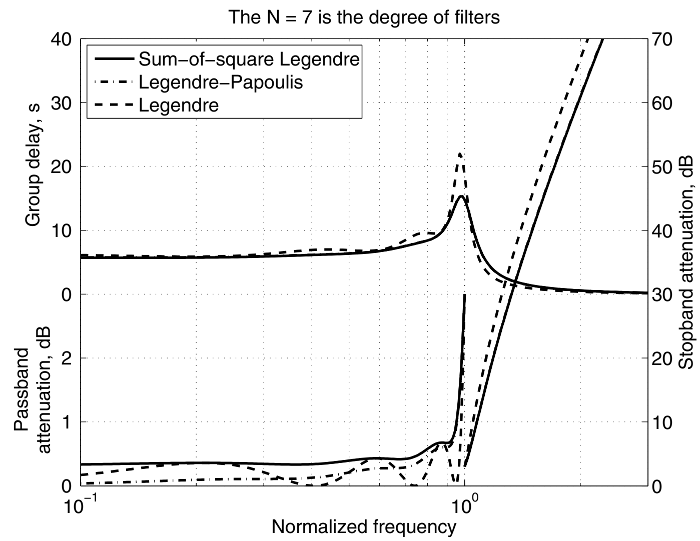

Fig. 1. Amplitude responses and group delay characteristic of the all-pole filters: Legendre, Legendre-Papoulis and sum-of-squares Legendre polynomials

Comparison of the cutoff steepness of filters considered here, can be performed by calculating the slopes of the characteristic function $\Psi(\omega^2)$

$$S = \frac{\mathrm{d}}{\mathrm{d}\,\omega}\Psi(\omega^2)\Big|_{\omega=1} \tag{1}$$

at the cutoff frequency, $\omega_c = 1$, for same attenuation in the passband, a_{max} [9]. $\Psi(\omega^2)$ is the characteristic function given by (3). These slopes are

- Legendre [4]:
$$S_L = n(n+1)$$
- Legender-Papoulis [5], [7]:
$$S_{LP} = \begin{cases} \dfrac{(n+1)^2}{2} & n \text{ is odd} \\ \dfrac{n(n+2)}{2} & n \text{ is even} \end{cases} \tag{2}$$
- Sum-of-squares Legendre polynomials [8]:

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$$S_{SoS} = \frac{n(n+2)}{2}.$$

This paper proposes a design methodology for the low-pass filters, based on sum-of-squares Legendre polynomials with one pair of zeros on the imaginary axis, in order to provide steeper slopes at the cutoff frequency. These filters are nearly monotonic in the pass-band and monotonic in the stopband.

II. Approximation

The magnitude response of any lowpass filter can be written in the form:

$$|H(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2\Psi^2(\omega)}} \tag{3}$$

where ω is frequency variable, ε^2 is constant which controls the maximum passband attenuation, a_{max} (in dB), as $\varepsilon^2 = 10^{0.1a_{max}} - 1$. The characteristic function $\Psi_n(\omega)$ is rational function normalized so that $\Psi_n(1) = 1$.

When the voltage or current source is not ideal and the load is finite and nonzero, it is the case of a doubly terminated lossless ladder network. The general form of the doubly resistively terminated two-port LC filter is shown in Fig. 2.

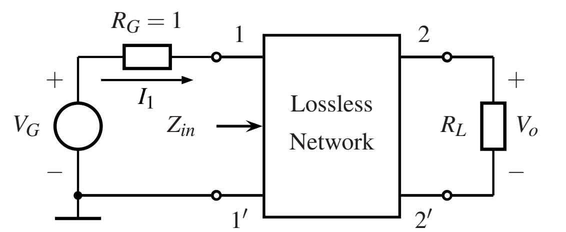

Fig. 2. A doubly terminated lossless network driven by voltage source.

In such passive network, the performance is measured by the ratio of power delivered to the load, P_L, and maximum power that can be delivered by the source, P_{max}. This ratio defines the transmission coefficient $H(j\omega)$ as

$$\frac{P_L}{P_{max}} = |H(j\omega)|^2 = 1 - |\Gamma(j\omega)|^2 \tag{4}$$

where $\Gamma(j\omega)$ is the reflection coefficient looking toward the input of the filter network,

$$|\Gamma(j\omega)|^2 = \Gamma(s)\Gamma(-s)\Big|_{s=j\omega} = \frac{|R_G - Z_{in}(j\omega)|^2}{|R_G + Z_{in}(j\omega)|^2} \tag{5}$$

where the reference impedance is the source resistance, R_G, and $Z_{in}(j\omega)$ is the input impedance looking toward the lossless network (Figure 2) at port 11'.

The transmission coefficient of the proposed n-th degree sum-of-square Legendre low-pass filter with single pair of zeros at $\pm j\omega_o$, where $\omega_o > 1$, is:

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 \dfrac{L_{2n}(\omega^2)}{L_{2n}(1)}\Big(\dfrac{\omega_o^2-1}{\omega^2-\omega_o^2}\Big)^2} \tag{6}$$

where $L_{2n}(\omega)$ is sum-of-squares orthonormal Legendre polynomials of the form:

$$L_{2n}(\omega) = \overline{p}_0^2(\omega) + \overline{p}_1^2(\omega) + \cdots + \overline{p}_n^2(\omega) \tag{7}$$

and $\overline{p}_i(\omega)$, $i = 1, \ldots .n$ are orthonormal Legendre polynomials of the first kind (entire even or odd) and of the n-th degree, and $L_{2n}(1) = (n+1)^2/2$. Applying Christoffel-Darboux formula [10], equation (7) is reduced to:

$$L_{2n}(\omega^2) = \frac{k_n}{k_{n+1}}\Big[\frac{\mathrm{d}\overline{p}_{n+1}}{\mathrm{d}\omega}\overline{p}_n(\omega) - \frac{\mathrm{d}\overline{p}_n(\omega)}{\mathrm{d}\omega}\overline{p}_{n+1}(\omega)\Big] \tag{8}$$

where k_n is leading coefficient of orthonormal Legendre polynomial $\overline{p}_n(\omega)$.

Performing analytic continuation, equation (6) gets form

$$H(s)H(-s) = \frac{(s^2+\omega_o^2)^2}{\dfrac{2\varepsilon^2(\omega_o^2-1)^2}{(n+1)^2}L_{2n}(s^2) + (s^2+\omega_o^2)^2}, \tag{9}$$

or in simpler form, $H(s)$ can be written as

$$H(s) = \frac{s^2+\omega_o^2}{D(s)} \tag{10}$$

where $D(s)$ represents the left half-plane roots of

$$\frac{2\varepsilon^2(\omega_o^2-1)^2}{(n+1)^2}L_{2n}(s^2) + (s^2+\omega_o^2)^2 = 0. \tag{11}$$

The magnitudes of the fifth order sum-of-squares-Legendre all-pole filter (without imaginary-axis zeros) and of the new function (with imaginary-axis zeros) for $n = 5$, $\omega_o = 1.8681$ and minimum stopband attenuation 50 dB, are compared in Figure 3. As it is illustrated, pair of zeros on the imaginary axis decrease passband attenuation, and the new filter performs monotonic attenuation characteristic in the passband (the first derivative of the transfer function in the passband is always greater than zero), while allpole function is nearly monotonic in the passband. On the other hand, the imaginary-axis zeros have small effect to group delay characteristic.

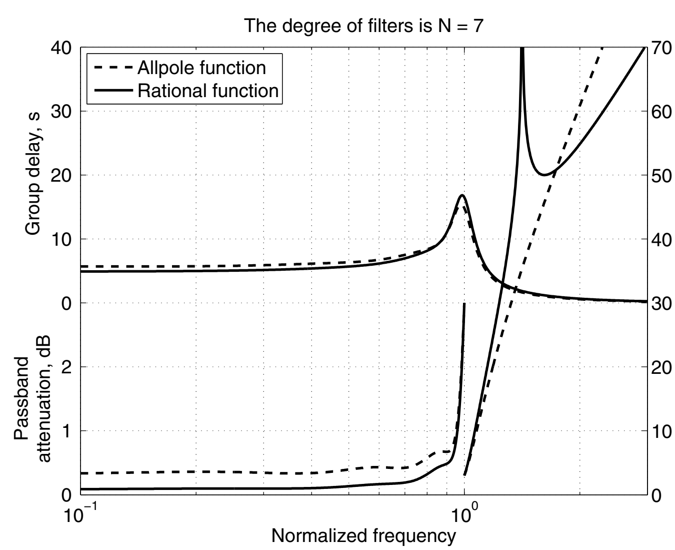

Fig. 3. Amplitude responses and group delay characteristic of the all-pole and rational Legendre filters with $n = 5$ and $\omega_o = 1.4147$.

A comparison of the cutoff sharpness of the various filters to the proposed filter can be made by calculating the cutoff slopes (1) where

$$\Psi(\omega^2) = \frac{2L_{2n}(\omega^2)}{(n+1)^2}\Big(\frac{\omega_o^2-1}{\omega^2-\omega_o^2}\Big)^2. \tag{12}$$

By differentiating equation (12) with respect to ω, cutoff slope at the frequency $\omega = 1$ is obtained as

$$S_L = \frac{n(n+2)}{2} + \frac{4}{\omega_o^2 - 1}, \tag{13}$$

and it decreases if ω_o increases. If $\omega_o \to \infty$ allpole filter function is obtained.

III. Realization

Transfer functions of doubly terminated passive LC ladder filters have low sensitivities to component variations in the passband [11]. The all-pole low-pass prototypes synthesis, given a prescribed insertion-loss between a resistive source and a resistive load, is a classical procedure presented in many textbooks on network synthesis [12]–[14]. Modifications of these prototypes enable the design of other filter types. Moreover, some methods transform the LC ladder network directly into digital filter [15], by first representing the ladder filter branch equations in flowgraph form and then using the bilinear z-transformation in order to transform the ladder elements into digital filter elements.

The LC ladder network design technique adjusts source impedance to unity, $R_G = 1$, and pass-band edge to unity, $\omega_c = 1$, in order to transform specifications to the low-pass domain and to determine the filter prototype. Transformations of the low-pass prototype into the needed domain lead to the final network.

Since $H(0) < 1$, the Legendre filter cannot be realized as LC lader network with equal terminations. If the passband ripple ε and filter degree are known, for R_L we get:

$$R_L = 1 + 2\rho^2 \pm 2\rho\sqrt{1+\rho^2} \tag{14}$$

where

$$\rho = \frac{\varepsilon\sqrt{2L_{2n}(0)}}{n+1}\,\frac{\omega_o^2 - 1}{\omega_o^2}.$$

Proof: Substituting $\omega = 0$ in the equation (31), we get

$$\frac{P_{max}}{P_o} = 1 + \frac{2\varepsilon^2 L_{2n}(0)}{(n+1)^2}\frac{(\omega_o^2-1)^2}{\omega_o^4}. \tag{15}$$

On the other hand

$$\frac{P_{max}}{P_o} = \frac{V_G^2}{4}\frac{R_L}{V_o^2} = \frac{(1+R_L)^2}{4R_L}. \tag{16}$$

After some algebraic manipulation one obtains

$$\frac{P_{max}}{P_o} = 1 + \frac{(1-R_L)^2}{4R_L}. \tag{17}$$

From (15) and (17) following that

$$\frac{(1-R_L)^2}{4R_L} = \Big(\frac{\varepsilon\sqrt{2L_{2n}(0)}}{n+1}\,\frac{\omega_o^2-1}{\omega_o^2}\Big)^2 \tag{18}$$

from where R_L is obtained.

By equating (4) and (6) we get:

$$|\Gamma(\omega)|^2 = 1 - \cfrac{1}{1 + \cfrac{2\varepsilon^2(\omega_o^2-1)^2}{(n+1)^2}\cfrac{L_{2n}(\omega^2)}{(\omega^2-\omega_o^2)^2}} \tag{19}$$

The reflection coefficient $\Gamma(s)$ can be derived from $\Gamma(s)\Gamma(-s)\big|_{\omega=-js} = |\Gamma(\omega)|^2$. Function $\Gamma(s)$ can be extracted from $\Gamma(s)\Gamma(-s)$ considering following properties: $\Gamma(s)$ and $\Gamma(-s)$ have opposite poles and opposite zeros, and the poles of $\Gamma(s)$ lie in the left half of the s-plane, i.e. the denominator polynomial of $\Gamma(s)$ is a Hurwitz polynomial. For minimum phase system all zeros of $\Gamma(s)$ lie also in the left half of the s-plane.

Equation (5) may be rearranged as

$$Z_{in}(s) = R_G\frac{1-\Gamma(s)}{1+\Gamma(s)} = R_G\frac{D_e + D_o}{N_e + N_o}. \tag{20}$$

Since $Z_{in}(s)$ is driving point of an LC network,

$$Z_{in} = \cfrac{\cfrac{z_{11}z_{22} - z_{12}^2}{R_L} + z_{11}}{\cfrac{z_{22}}{R_L} + 1} \tag{21}$$

equation (20) can be written as

$$Z_{in}(s) = \cfrac{\cfrac{Do}{No} + \cfrac{De}{No}}{\cfrac{N_e}{N_o} + 1} \tag{22}$$

where $R_G = 1$. The z_{11} and z_{22} parameters are identified as

$$z_{11}(s) = \frac{D_e}{N_o}, \qquad z_{22}(s) = R_L\frac{N_e}{N_o}. \tag{23}$$

IV. An Example

The fifth order rational transfer function with single pair of zeros at $\pm j1.8680664$ is

$$H(s) = \frac{0.0922800(s^2 + 3.4896722)}{\begin{array}{c} s^5 + 1.7294853s^4 + 2.4804080s^3 \\ +2.0588346s^2 + 1.1555074s + 0.3299317 \end{array}}. \tag{24}$$

Then, the reflection coefficient has form

$$\Gamma(s) = \frac{\begin{array}{c} s^5 + 0.9181766s^4 + 1.4063725s^3 \\ +0.7616094s^2 + 0.3813457s + 0.0717884 \end{array}}{\begin{array}{c} s^5 + 1.7294853s^4 + 2.4804080s^3 \\ +2.0588346s^2 + 1.1555074s + 0.3299317 \end{array}} \tag{25}$$

and input impedance is

$$Z_{in}(s) = \frac{\begin{array}{c} 0.8112479\ s^4 + 1.0739305\ s^3 + 1.2970999\ s^2 \\ +0.7740808\ s + 0.2581179 \end{array}}{\begin{array}{c} 2\ s^5 + 2.6476011\ s^4 + 3.8866756\ s^3 \\ +2.8203187\ s^2 + 1.5367720\ s + 0.4016947 \end{array}}. \tag{26}$$

Factoring the numerator and denominator of $\Gamma(s)$ into even and odd parts and putting them in equation (23), one gets

$$\begin{aligned} z_{11}(s) =& \frac{0.8113086s^4 + 1.2972251s^2 + 0.2581433}{2s^5 + 3.8867807s^3 + 1.5368531s} \\ =& \frac{0.4056262(s^2 + 1.3659999)(s^2 + 0.2329293)}{s(s^2 + 1.3909384)(s^2 + 0.5524519)}. \end{aligned} \tag{27}$$

The impedance function z_{11} must be performed to set the zeros of transfer function at $s = \pm j1.8680664$. The admittance function $Y_1 = 1/z_{11}$ has a pole at infinity. Part of the residue of this pole can be removed to create zero at $s = \pm j1.8680664$. The necessary residue is shunt capacitor

$$C_2 = \frac{Y_1}{s}\Big|_{s=j1.8680664} = 2.1971860. \tag{28}$$

The remain admittance Y_2 has a pair of zeros at $s = \pm j1.8680664$

$$\begin{aligned} Y_2 =& Y_1 - C_2 s \\ =& \frac{0.2174041s(s^2+3.4896722)(s^2+1.2781123)}{0.811308(s^2+1.3659999)(s^2+0.2329293)}. \end{aligned} \tag{29}$$

The impedance $Z_2 = 1/Y_2$ has a pair of poles at $s = \pm j1.8680664$, which are removed by the partial fraction expansion

$$\begin{aligned} Z_2 =& \frac{k_1 s}{s^2+3.4896722} + Z_3 \\ =& \frac{3.7317996(s^2+1.3659999)(s^2+0.2329293)}{s(s^2+3.4896722)(s^2+1.2781123)}, \end{aligned} \tag{30}$$

where

$$k_1 = Z_2\,\frac{s^2+3.4896722}{s}\Big|_{s=j1.8680664} = 3.3443055. \tag{31}$$

In the above expression for Z_2, the term

$$\frac{3.3443055\ s}{s^2+3.4896722} = \cfrac{1}{0.2990157\ s + \cfrac{1}{0.9583438\ s}} \tag{32}$$

can be realized as parallel resonant circuit, as shown in Figure 3, which consists of $C_1 = 0.2990157$ and $L_1 = 0.9583438$.

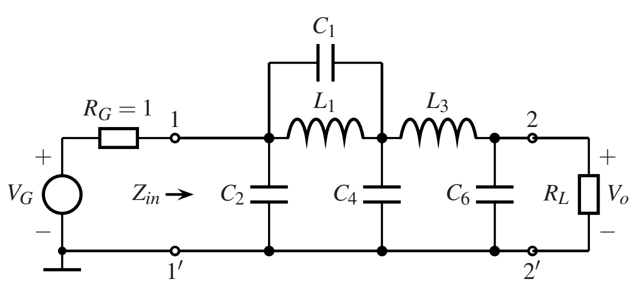

Fig. 4. A doubly terminated LC ladder network driven by voltage source.

The remain impedance Z_3 has form

$$Z_3 = \frac{0.3874941\ s^2 + 0.3402581}{s^3 + 1.2781123\ s}. \tag{33}$$

The remainder part of LC ladder network can be realized by continued partial fraction expansion of $Y_3 = 1/Z_3$ as

$$Y_3 = 2.5806844\ s + \cfrac{1}{0.9687024\ s + \cfrac{1}{1.1756182\ s}}. \tag{34}$$

Thus, $C_4 = 2.5806844$, $L_3 = 0.9687024$ and $C_6 = 1.175618$. Finally, the equation (14) is used to determine $R_L = 0.6425722$. The complete circuit is shown in Fig. 4.

V. Conclusion

A new method for generating the transfer function which results in monotonic response in the passband, has been proposed. The stopband performance of the filter can be improved by adding one simple or multiple pair of transmission zeros of real frequency. For any prescribed minimum stopband attenuation, the location of these zeros can be determined.

Comparison of the proposed transfer function with all-pole transfer functions have been presented. As revised by the results of the comparison, the approximation with one pair of zeros on the imaginary axis offers, in many cases, the best solution regarding the passband loss and the cutoff characteristic.

Numerical example, including the element values for an unequally terminated LC ladder realization of the proposed filter, is given to illustrate this method.

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