Some Further Results on Modulated/Extended Lapped Transforms

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Abstract

Some further results on Modulated and Extended Lapped Transforms are reported here; primarily, sufficient conditions are derived to ensure perfect reconstruction for extended lapped transforms. These conditions are valid for transforms of length lM and place no restriction on l; in this respect they apply to a more general case than earlier derivations that constrained l to be even [1, 3]. Also, a new filter structure is reported for implementing the modulated lapped transform, that has a hardware complexity of O(M) as compared to O(MlogM) for other implementations.

1 Introduction

We report here, some further results on the modulated lapped transform, that has gained popularity recently on account of its low hardware complexity, and low blocking effect in transform coding systems. The MLT essentially corresponds to a maximally decimated multirate system (shown in Fig. 1), with the analysis filters being cosine modulated versions of a lowpass prototype of length $l\ M$, and with the synthesis filters being the time reverse of the analysis filters. In an earlier paper [3], sufficient conditions were derived for perfect reconstruction at the output of the multirate system, for the case $l = 2$, and in a subsequent paper [1], sufficient conditions were derived for l being any arbitrary even number. We present here, an alternative, simpler way of arriving at these sufficient conditions, that holds for any integer l, and that is based on a linear algebraic interpretation of the modulated filter-bank, and may be considered as an extension of the work of [3]. Also, we present an efficient implementation of the filter-bank based on some recently reported filter structures [7], that has a hardware complexity of O(M) as compared to O(Mlog M) for earlier implementations [2].

2 Linear Algebraic Interpretation of MLT/ELT

In this section, we present a linear algebraic interpretation of the MLT/ELT and derive sufficient conditions for perfect reconstruction, under the constraint that the synthesis filters are the time reverse of the analysis filters (this condition corresponds to imposing paraunitariness on the polyphase representation of the analysis and synthesis banks [4]). In the following, lower case letters indicate scalars, with the subscript denoting the time index, underlined lowercase letters indicate vectors, and underlined uppercase letters indicate matrices. Hence, x_i denotes the i^{th} input data, $\underline{x}_i$ is a vector denoting M samples of the input data from time iM to $iM + M - 1$ i.e. $\underline{x}_i = [x_{iM}\ x_{iM+1}\ \cdots\ x_{iM+M-2}\ x_{iM+M-1}]^T$. As there are M analysis filters, the output of the analysis-bank at any time has M components. Further, as we will deal with maximally decimated systems only, we need concern ourselves with the analysis-bank outputs only at times $iM - 1$. The vector $\underline{y}_i$ will be used to denote the M analysis-bank outputs at time iM-1. $\underline{0}$, $\underline{I}_M$ and $\underline{J}_M$ respectively denote the M by M zero, identity and counter-identity matrix, where the $(i, j)^{th}$ element of the conter-identity matrix is 1 if $i + j = M - 1$, and 0 otherwise $(i, j = 0, \ldots, M - 1)$. Further, we assume that the length of the impulse response is $l\ M$.

Due to the FIR nature of the analysis-bank, the analysis-bank outputs at any time will depend on the last lM samples of the input data. Hence, the output of the analysis-bank at time $l\ M - 1$ may be written in matrix notation as

$$\underline{y}_l = [\underline{A}] \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{lM-2} \\ x_{lM-1} \end{bmatrix} = [\underline{A}_0 \cdots \underline{A}_{l-1}] \begin{bmatrix} \underline{x}_0 \\ \underline{x}_1 \\ \vdots \\ \underline{x}_{l-2} \\ \underline{x}_{l-1} \end{bmatrix}$$
$$= \underline{A}_0\underline{x}_0 + \underline{A}_1\underline{x}_1 + \ldots + \underline{A}_{l-1}\underline{x}_{l-1} \quad (1)$$

where $\underline{A}$ is a $MxlM$ matrix comprised of the MxM

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matrices $\underline{A}_i$ as $\underline{A} = [\underline{A}_0\ \underline{A}_1\ \cdots\ \underline{A}_{l-2}\ \underline{A}_{l-1}]$. The rows of $\underline{A}$, in reverse, give the impulse response of the analysis-bank filters.

The synthesis-bank reconstructs the input data vector $\underline{x}_{l-1}$ by filtering the vectors $\underline{y}_l, \underline{y}_{l+1}, \cdots, \underline{y}_{2l-2}, \underline{y}_{2l-1}$ interpolated with zeros (M zeros are interpolated between consecutive received samples at the synthesis-bank) through the synthesis filters. This may be written in matrix notation as

$$\hat{\underline{x}}_l = [\underline{A}_{l-1}^T\ \underline{A}_{l-2}^T\ \cdots\ \underline{A}_1^T\ \underline{A}_0^T] \begin{bmatrix} \underline{y}_l \\ \underline{y}_{l+1} \\ \vdots \\ \underline{y}_{2l-2} \\ \underline{y}_{2l-1} \end{bmatrix} \qquad (2)$$

Expressing $\underline{y}_l, \underline{y}_{l+1}, \cdots, \underline{y}_{2l-2}, \underline{y}_{2l-1}$ as in (1), and substituting in (2), we see that in order to make $\hat{\underline{x}}_l$ equal to $\underline{x}_l$, the necessary and sufficient conditions are [3]

$$\sum_{i=1}^{l} \underline{A}_{l-i}{}^T \underline{A}_{k-i} = \underline{0} \qquad k = 1, \ldots, l-1 \quad (3)$$
$$\qquad l+1, \ldots, 2l-1$$

$$\sum_{i=1}^{l} \underline{A}_{l-i}{}^T \underline{A}_{l-i} = \underline{I}_M \qquad (4)$$

where $\underline{A}_i = \underline{0}$ if $i < 0$ or if $i \geq l$.

Consider now the special case of the MLT/ELT [1]. The samples of the impulse response of the k^{th} analysis filter are given by

$$h_k(n) = \sqrt{\frac{2}{M}}\ w(lM-1-n) \cos\left[\frac{\pi}{M}\left(k+\frac{1}{2}\right)\left(lM-1-n+\frac{M+1}{2}\right)\right] \qquad (5)$$
$$n = 0, \ldots, lM-1$$

where $w(n)$ represents the window function. Recalling that the rows of $\underline{A}$ in (1) in reverse give the impulse response of the analysis filters, we may now express $\underline{A}$ in terms of the impulse response samples as follows:

$$\underline{A} = \left[\hat{\underline{A}}_0\ \hat{\underline{A}}_1\ \cdots\ \hat{\underline{A}}_{l-2}\ \hat{\underline{A}}_{l-1}\right] \begin{bmatrix} \underline{W}_0 & & \\ & \ddots & \\ & & \underline{W}_{l-1} \end{bmatrix} \qquad (6)$$

where the $(k,m)^{th}$ element of $\hat{\underline{A}}_i$ is given by

$$\hat{\underline{A}}_i|_{(k,m)} = \sqrt{\frac{2}{M}} \cos\left[\frac{\pi}{M}(k+\frac{1}{2})(iM+m+\frac{M+1}{2})\right] \qquad (7)$$

and $\underline{W}_i$ is a diagonal matrix, with the diagonal elements being M consecutive samples of the window function, starting from time iM i.e. $\underline{W}_i = diag\ (w_{iM}, w_{iM+1}, \cdots, w_{iM+M-2}, w_{iM+M-1})$. Using (7), it can be shown that

$$\hat{\underline{A}}_p^T \hat{\underline{A}}_q = \left\{ \begin{array}{ll} \underline{I}_M + \underline{J}_M e^{j\pi p} & p = q \\ \underline{0} & (p+q) \text{ odd} \\ \underline{I}_M e^{j\pi(p-q)/2} - & \\ \quad \underline{J}_M e^{j\pi(p+q+2)/2} & (p+q) \text{ even} \end{array} \right\} \qquad (8)$$

Substituting (8) and (6) in (3) and (4), we get

$$\sum_{i=1}^{l} \underline{W}_{l-i}\ \underline{W}_{k-i}\ e^{j\pi(l-k)/2} - \sum_{i=1}^{l} \underline{W}_{l-i}\ \underline{J}_M\ \underline{W}_{k-i}\ e^{j\pi(l+k+2-2i)/2} = \underline{0} \qquad (9)$$

for $k \in [1, \ldots, l-1]$ and such that $(l-k)$ is even

$$\sum_{i=1}^{l} \underline{W}_{l-i}^2 + \sum_{i=1}^{l} \underline{W}_{l-i}\ \underline{J}_M\ \underline{W}_{l-i}\ e^{j\pi i} = \underline{I}_M \qquad (10)$$

Here, we have ignored the conditions imposed by (3) for $k = l+1, \ldots, 2l-1$ because they are identical to the conditions for $k = 1, \ldots, l-1$. For the special case of $l = 2$ (MLT) and $l = 4$ (ELT), these conditions reduce to

$$\left\{ \begin{array}{l} \underline{W}_0^2 + \underline{W}_1^2 = \underline{I}_M \\ \underline{W}_0 \underline{J}_M \underline{W}_0 = \underline{W}_1 \underline{J}_M \underline{W}_1 \end{array} \right\} \quad \text{for } l\text{=}2 \quad (11)$$

$$\left\{ \begin{array}{l} \underline{W}_1 \underline{W}_3 + \underline{W}_0 \underline{W}_2 = \underline{0} \\ \underline{W}_0^2 + \underline{W}_1^2 + \underline{W}_2^2 + \underline{W}_3^2 = \underline{I}_M \\ \underline{W}_0 \underline{J}_M \underline{W}_0 - \underline{W}_1 \underline{J}_M \underline{W}_1 + \\ \underline{W}_2 \underline{J}_M \underline{W}_2 - \underline{W}_3 \underline{J}_M \underline{W}_3 = \underline{0} \end{array} \right\} \quad \text{for } l\text{=}4 \quad (12)$$

Now, if we further impose the condition that the window is symmetric, i.e. $\underline{W}_1 = \underline{W}_0 \underline{J}_M \underline{W}_0$ for the case $l = 2$, and $\underline{W}_3 = \underline{W}_0 \underline{J}_M \underline{W}_0$, $\underline{W}_2 = \underline{W}_1 \underline{J}_M \underline{W}_1$ for the case $l = 4$, we obtain

$$\left\{ \underline{W}_0^2 + \underline{W}_1^2 = \underline{I}_M \right\} \quad \text{for } l\text{=}2 \quad (13)$$

$$\left\{ \begin{array}{l} \underline{W}_1 \underline{W}_3 + \underline{W}_0 \underline{W}_2 = \underline{0} \\ \underline{W}_0^2 + \underline{W}_1^2 + \underline{W}_2^2 + \underline{W}_3^2 = \underline{I}_M \end{array} \right\} \quad \text{for } l\text{=}4 \quad (14)$$

which are seen to be identical to the conditions derived in [1]. Furthermore, the conditions (9, 10) hold for odd l also, unlike those in [1]. Also, (9, 10) were derived without making any assumptions about symmetry in the window function.

3 Efficient Implementation

One of the main reasons for the sustained interest in modulated lapped transforms is the possibility of implementing them efficiently. Earlier implementations [2] were based on using order recursive fast DCT algorithms to implement the transform. Here, we propose a filter-bank implementation on the lines of [7]. The hardware complexity of this filter-bank is O(M) as compared to O(M log M) for earlier implementations, hence making it suitable for implementation on general purpose signal processors. It must be mentioned however, that the filter-bank is required to operate at the same rate as the input data, rather than at the decimated rate.

From (1) and (6), we may write

$$\underline{y}_l == \left[\underline{\hat{A}}_0\ \underline{\hat{A}}_1\ \cdots\ \underline{\hat{A}}_{l-2}\ \underline{\hat{A}}_{l-1}\right] \begin{bmatrix} \underline{W}_0\underline{x}_0 \\ \underline{W}_1\underline{x}_1 \\ \vdots \\ \underline{W}_{l-2}\underline{x}_{l-2} \\ \underline{W}_{l-1}\underline{x}_{l-1} \end{bmatrix} \quad (15)$$

i.e $\underline{y}_l$ is obtained by windowing the data and then applying it to a filter-bank whose impulse responses are given by the reversed rows of $\underline{\hat{A}} = [\underline{\hat{A}}_0\ \underline{\hat{A}}_1\ \cdots\ \underline{\hat{A}}_{l-2}\ \underline{\hat{A}}_{l-1}]$. The windowing may be implemented simply with a 1-tap linear periodically time varying filter whose periodic impulse response is given by the samples of the window function. So we need to concentrate primarily on the implementation of the unwindowed filter-bank. Setting $w(i) = 1$ in (5), the transfer function of the k^{th} analysis filter is given by

$$H_k(z) = z^{-1}\sum_{n=0}^{lM-1} h_k(n)z^{-n}$$
$$= \sqrt{\frac{2}{M}}\left[1 - z^{-lM}e^{j\pi l/2}\right]\frac{2a_k z^{-1} - 2b_k z^{-2}}{1 - 2c_k z^{-1} + z^{-2}} \quad (16)$$

$$\text{where} \quad c_k = \cos\left[\frac{\pi}{M}(k + \frac{1}{2})\right]$$
$$a_k = \sqrt{\frac{2}{M}}\cos\left[\frac{\pi}{M}(k+\frac{1}{2})(lM - 1 + \frac{M+1}{2})\right]$$
$$\text{and} \quad b_k = \sqrt{\frac{2}{M}}\cos\left[\frac{\pi}{M}(k+\frac{1}{2})(lM + \frac{M+1}{2})\right]$$
$$k = 0, \ldots, M-1$$

This transfer function may be optimally implemented (optimal in the sense of low sensitivity and roundoff noise) with a resonator-in-feedback-loop kind of structure [7, 8]. The structure is shown in Fig. 2 for the case $l = 2$ with $V_{fb,k}$ being the filter-bank outputs. Here,

$$\frac{V_{fbc,k}}{V_e} = H_{fbc,k}(z) = \frac{2c_k z^{-1} - 2z^{-2}}{1 - 2c_k z^{-1} + z^{-2}}$$
$$k = 0.\ldots, M-1$$
$$\frac{V_{fb,k}}{V_e} = H_{fb,k}(z) = \frac{2a_k z^{-1} - 2b_k z^{-2}}{1 - 2c_k z^{-1} + z^{-2}}$$
$$\frac{V_e}{V_{in}} = 1 + z^{-2M}$$

where a_k, b_k, c_k are as given above. Assuming that direct form biquads are used to implement the above transfer functions, 3 multipliers are needed for each biquad, hence the filter-bank needs 3M+1 multipliers.

For the case $l = 2$, the lapped nature of the transform makes it necessary to have two filter-banks in parallel as shown in Fig. 3. These two filter-banks are preceded by different LPTV filters, the impulse response of the first LPTV filter is given by the diagonal entries of $diag(\underline{W}_0, \underline{W}_1)$ while those of the second LPTV filter are given by the diagonal entries of $diag(\underline{W}_1, \underline{W}_0)$. Hence, the overall system requires 6M+2 multipliers and two additional multipliers to implement the two LPTV filters. The synthesis part of the system may be implemented simply as the transpose of the analysis-bank and will not be considered further here.

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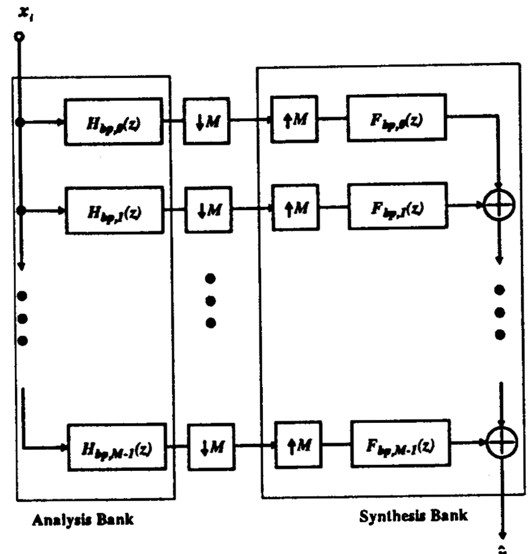

Fig. 1 Maximally decimated multirate system

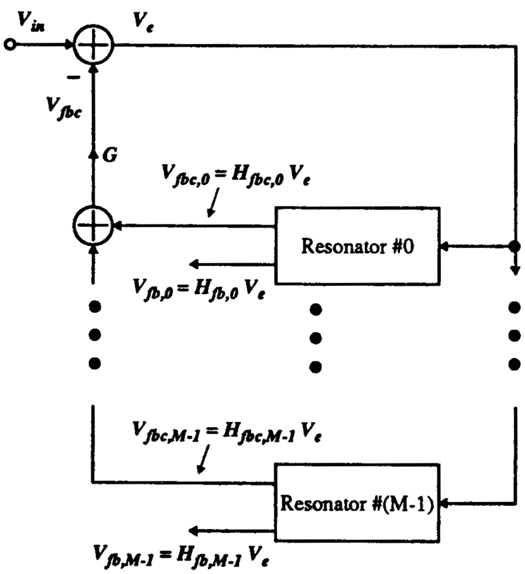

Fig. 2 Analysis-Bank Structure

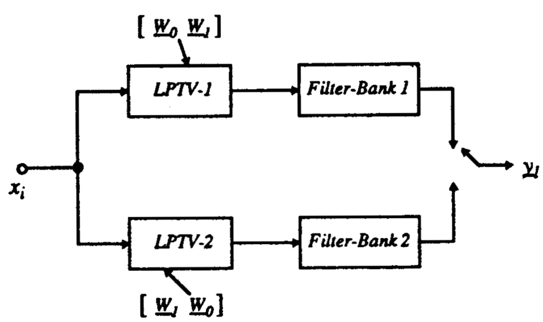

Fig. 3 Analysis-Bank for Lapped Transforms (l=2)