# Supplementary Material for re-submission of manuscript IEEE TMI-2023-2038

# Title: Blind CT Image Quality Assessment Using DDPM-derived Content and Transformer-based Evaluator

## I. The Derivation of Eq. (7)

DDPM starts with a forward process that gradually adds noise to the normal-dose CT image $\boldsymbol{x}_0 \sim q(\boldsymbol{x}_0)$ over the course of $T$ timesteps according to a variance schedule $\beta_1, \cdots, \beta_T$:

$$q(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}) = \mathcal{N}\left(\boldsymbol{x}_t; \sqrt{1-\beta_t}\boldsymbol{x}_{t-1}, \beta_t \boldsymbol{I}\right)$$

$$q(\boldsymbol{x}_{1:T}|\boldsymbol{x}_0) = \prod_{t=1}^{T} q(\boldsymbol{x}_t|\boldsymbol{x}_{t-1})$$

where $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_T$ are latent variables of the same dimensionality as the sample $\boldsymbol{x}_0 \sim q(\boldsymbol{x}_0)$.

According to the properties of the Gaussian distribution, the sampling result $\boldsymbol{x}_t$ at an arbitrary timestep $t$ can be written in the following closed form:

$$q(\boldsymbol{x}_t|\boldsymbol{x}_0) = \mathcal{N}\left(\boldsymbol{x}_t; \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0, (1-\bar{\alpha}_t)\boldsymbol{I}\right)$$

where $\alpha_t = 1 - \beta_t$ and $\bar{\alpha}_t = \prod_{i=1}^{t} \alpha_i$.

According to the Bayes theorem, the posterior $q(\boldsymbol{x}_{t-1}|\boldsymbol{x}_t, \boldsymbol{x}_0)$ can be calculated as follows:

$$q(\boldsymbol{x}_{t-1}|\boldsymbol{x}_t, \boldsymbol{x}_0) = \frac{q(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{x}_0)q(\boldsymbol{x}_{t-1}|\boldsymbol{x}_0)}{q(\boldsymbol{x}_t|\boldsymbol{x}_0)}$$

$$= \frac{q(\boldsymbol{x}_t|\boldsymbol{x}_{t-1})q(\boldsymbol{x}_{t-1}|\boldsymbol{x}_0)}{q(\boldsymbol{x}_t|\boldsymbol{x}_0)}$$

$$= \frac{\mathcal{N}\left(\boldsymbol{x}_t; \sqrt{1-\beta_t}\boldsymbol{x}_{t-1}, \beta_t\boldsymbol{I}\right)\mathcal{N}\left(\boldsymbol{x}_{t-1}; \sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0, (1-\bar{\alpha}_{t-1})\boldsymbol{I}\right)}{\mathcal{N}\left(\boldsymbol{x}_t; \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0, (1-\bar{\alpha}_t)\boldsymbol{I}\right)}$$

$$= \frac{\mathcal{N}\left(\boldsymbol{x}_t; \sqrt{\alpha_t}\boldsymbol{x}_{t-1}, (1-\alpha_t)\boldsymbol{I}\right)\mathcal{N}\left(\boldsymbol{x}_{t-1}; \sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0, (1-\bar{\alpha}_{t-1})\boldsymbol{I}\right)}{\mathcal{N}\left(\boldsymbol{x}_t; \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0, (1-\bar{\alpha}_t)\boldsymbol{I}\right)}$$

$$= \exp\left\{-\left[\frac{\left(\boldsymbol{x}_t - \sqrt{\alpha_t}\boldsymbol{x}_{t-1}\right)^2}{2(1-\alpha_t)} + \frac{\left(\boldsymbol{x}_{t-1} - \sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0\right)^2}{2(1-\bar{\alpha}_{t-1})} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{2(1-\bar{\alpha}_t)}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{\left(\boldsymbol{x}_t - \sqrt{\alpha_t}\boldsymbol{x}_{t-1}\right)^2}{1-\alpha_t} + \frac{\left(\boldsymbol{x}_{t-1} - \sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{\boldsymbol{x}_t^2 - 2\sqrt{\alpha_t}\boldsymbol{x}_t\boldsymbol{x}_{t-1} + \alpha_t\boldsymbol{x}_{t-1}^2}{1-\alpha_t} + \frac{\boldsymbol{x}_{t-1}^2 - 2\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_{t-1}\boldsymbol{x}_0 + \bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{-2\sqrt{\alpha_t}\boldsymbol{x}_t\boldsymbol{x}_{t-1} + \alpha_t\boldsymbol{x}_{t-1}^2}{1-\alpha_t} + \frac{\boldsymbol{x}_{t-1}^2 - 2\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_{t-1}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}} + \frac{\boldsymbol{x}_t^2}{1-\alpha_t} + \frac{\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{-2\sqrt{\alpha_t}\boldsymbol{x}_t\boldsymbol{x}_{t-1}}{1-\alpha_t} + \frac{\alpha_t\boldsymbol{x}_{t-1}^2}{1-\alpha_t} + \frac{\boldsymbol{x}_{t-1}^2}{1-\bar{\alpha}_{t-1}} + \frac{2\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_{t-1}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}} + \frac{\boldsymbol{x}_t^2}{1-\alpha_t} + \frac{\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\left(\frac{\alpha_t}{1-\alpha_t} + \frac{1}{1-\bar{\alpha}_{t-1}}\right)\boldsymbol{x}_{t-1}^2 - 2\left(\frac{\sqrt{\alpha_t}\boldsymbol{x}_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}}\right)\boldsymbol{x}_{t-1} + \frac{\boldsymbol{x}_t^2}{1-\alpha_t} + \frac{\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{\alpha_t - \bar{\alpha}_t + 1 - \alpha_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\boldsymbol{x}_{t-1}^2 - 2\left(\frac{\sqrt{\alpha_t}\boldsymbol{x}_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}}\right)\boldsymbol{x}_{t-1} + \frac{\boldsymbol{x}_t^2}{1-\alpha_t} + \frac{\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\boldsymbol{x}_{t-1}^2 - 2\left(\frac{\sqrt{\alpha_t}\boldsymbol{x}_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}}\right)\boldsymbol{x}_{t-1} + \frac{\boldsymbol{x}_t^2}{1-\alpha_t} + \frac{\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_{t-1}} - \frac{\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{1-\bar{\alpha}_t}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\right)\left[\boldsymbol{x}_{t-1}^2 - 2\frac{\left(\frac{\sqrt{\alpha_t}\boldsymbol{x}_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}}\right)}{\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}}\boldsymbol{x}_{t-1} + \frac{(1-\bar{\alpha}_{t-1})\boldsymbol{x}_t^2}{1-\bar{\alpha}_t} + \frac{(1-\alpha_t)\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_t} - \frac{(1-\alpha_t)(1-\bar{\alpha}_{t-1})\left(\boldsymbol{x}_t - \sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\right)^2}{(1-\bar{\alpha}_t)^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\right)\left[\boldsymbol{x}_{t-1}^2 - 2\frac{\left(\frac{\sqrt{\alpha_t}\boldsymbol{x}_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}}\right)}{\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}}\boldsymbol{x}_{t-1} + \frac{(1-\bar{\alpha}_{t-1})\boldsymbol{x}_t^2}{1-\bar{\alpha}_t} + \frac{(1-\alpha_t)\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_t} - \frac{(1-\alpha_t)(1-\bar{\alpha}_{t-1})\left(\boldsymbol{x}_t^2 - 2\sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\boldsymbol{x}_t + \bar{\alpha}_t\boldsymbol{x}_0^2\right)}{(1-\bar{\alpha}_t)^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\right)\left[\boldsymbol{x}_{t-1}^2 - 2\frac{\left(\frac{\sqrt{\alpha_t}\boldsymbol{x}_t}{1-\alpha_t} + \frac{\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_{t-1}}\right)(1-\alpha_t)(1-\bar{\alpha}_{t-1})}{1-\bar{\alpha}_t}\boldsymbol{x}_{t-1} + \frac{(1-\bar{\alpha}_{t-1})\boldsymbol{x}_t^2}{1-\bar{\alpha}_t} + \frac{(1-\alpha_t)\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{1-\bar{\alpha}_t} - \frac{(1-\alpha_t)(1-\bar{\alpha}_{t-1})\left(\boldsymbol{x}_t^2 - 2\sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\boldsymbol{x}_t + \bar{\alpha}_t\boldsymbol{x}_0^2\right)^2}{(1-\bar{\alpha}_t)^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\right)\left[\boldsymbol{x}_{t-1}^2 - 2\frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}\boldsymbol{x}_{t-1} + \frac{(1-\bar{\alpha}_t)^2\alpha_t\boldsymbol{x}_t^2 + (1-\bar{\alpha}_{t-1})(1-\alpha_t)\sqrt{\bar{\alpha}_t}\boldsymbol{x}_0\boldsymbol{x}_t + (1-\alpha_t)^2\bar{\alpha}_{t-1}\boldsymbol{x}_0^2}{(1-\bar{\alpha}_t)^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\right)\left[\boldsymbol{x}_{t-1}^2 - 2\frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}\boldsymbol{x}_{t-1} + \left(\frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}\right)^2\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{1-\bar{\alpha}_t}{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}\right)\left(\boldsymbol{x}_{t-1}^2 - \frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}\right)^2\right\}$$

$$= \exp\left\{-\frac{\left(\boldsymbol{x}_{t-1}^2 - \frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}\right)^2}{2\left(\frac{(1-\alpha_t)(1-\bar{\alpha}_{t-1})}{1-\bar{\alpha}_t}\right)}\right\}$$

$$= \mathcal{N}\left(\boldsymbol{x}_{t-1}; \frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}, \frac{(1-\bar{\alpha}_{t-1})\sqrt{\alpha_t}\boldsymbol{x}_t + (1-\alpha_t)\sqrt{\bar{\alpha}_{t-1}}\boldsymbol{x}_0}{1-\bar{\alpha}_t}\boldsymbol{I}\right)$$

$= \mathcal{N}(\boldsymbol{x}_{t-1}; \tilde{\boldsymbol{\mu}}_t(\boldsymbol{x}_t, \boldsymbol{x}_0), \sigma_t^2 \boldsymbol{I})$

That is:

$$q(\boldsymbol{x}_{t-1}|\boldsymbol{x}_t, \boldsymbol{x}_0) = \mathcal{N}(\boldsymbol{x}_{t-1}; \tilde{\boldsymbol{\mu}}_t(\boldsymbol{x}_t, \boldsymbol{x}_0), \sigma_t^2 \boldsymbol{I})$$

where

$$\tilde{\boldsymbol{\mu}}_t(\boldsymbol{x}_t, \boldsymbol{x}_0) = \frac{\sqrt{\alpha_t}(1 - \bar{\alpha}_{t-1})}{1 - \bar{\alpha}_t} \boldsymbol{x}_t + \frac{\sqrt{\bar{\alpha}_{t-1}}(1 - \alpha_t)}{1 - \bar{\alpha}_t} \boldsymbol{x}_0$$

$$\sigma_t^2 = \frac{(1 - \bar{\alpha}_{t-1})(1 - \alpha_t)}{1 - \bar{\alpha}_t}$$