Pricing Structured Products

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Abstract

This project report is for structured products module. A structured product, also known as a market-linked investment, is a pre-packaged structured finance investment strategy based on a single security, a basket of securities, options, indices, commodities, debt issuance or foreign currencies, and to a lesser extent, derivatives. Structured products aspire to provide investors with highly targeted investments tied to their specific risk profiles, return requirements and market expectations. This project covers the valuation of various structured products in the financial markets, including SFD500, dual-purpose fund, convertible bond and callable bull/bear contract; the numerical methods and implementations for pricing these structured products mentioned above will be discussed, and numerical experiments results will be provided. In addition, in this report, the computation of VIX, and the pricing models for mortgage backed securities and guaranteed minimum withdrawal benefit will be discussed.

Keywords: VIX, SFD500, dual-purpose fund, convertible bond, callable bull/bear contract, mortgage backed securities, guaranteed minimum withdrawal benefit

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INTRODUCTION

This project report is for structured products module. A structured product, also known as a market-linked investment, is a pre-packaged structured finance investment strategy based on a single security, a basket of securities, options, indices, commodities, debt issuance or foreign currencies, and to a lesser extent, derivatives. Structured products aspire to provide investors with highly targeted investments tied to their specific risk profiles, return requirements and market expectations. There are several pros for structured products: i) Principal protection (depending on the type of structured product); ii) Tax-efficient access to fully taxable investments; iii) Enhanced returns within an investment (depending on the type of structured product); iv) Reduced volatility (or risk) within an investment (depending on the type of structured product); v) Ability to earn a positive return in low-yield or flat equity market environments; vi) Ability to minimize issuer risk by using collateral secured instruments (COSIs) backed with collateral in the form of securities or cash deposits. It is meaningful to study structured products.

This project report covers the pricing of various structured products in the financial markets, including SFD500, dual-purpose fund, convertible bond and callable bull/bear contract. The numerical methods, such as lattice method (binomial tree and finite difference method for solving the governing PDE) and Monte-Carlo method, are discussed; numerical results and relevant code are provided. In addition, in this report, the computation of VIX, and the pricing models for mortgage backed securities and guaranteed minimum withdrawal benefit will be discussed.

SPX-VIX

VIX is an important index to reflect the stock market volatility in the future (30 days). Moreover, to reflect the volatility, there are other two measures, which are implied volatility and local volatility.

Implied volatility

The following is the key steps to compute implied volatility.

Step 1: Black-Scholes Formula

In the risk neutral Black-Scholes world, option value is given by,

$$V_{call}(K, S, \sigma, T, r, q) = Se^{-qT}N(d+) - e^{-r}\ KN(d-)$$

$$V_{put}(K, S, \sigma, T, r, q) = -Se^{-qT}N(-d+) + e^{-rT}KN(-d-)$$

where

$$d_{\pm} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Step 2: Root finding of inverse Black-Scholes Formula

According to the BS formula above, one can obtain that this formula is monotone with respect to the variable sigma. Therefore, we can calculate the so-called implied volatility through the inverse function of BS formula given the spot option prices.

However, the inverse function cannot be explicit expressed, it is usually calculated by a root search process.

In our code, we use bisection method to find the root.

Step 3: Parameter specification

According to the put-call parity, we have

$$C - P = Se^{-qT} - Ke^{-rT}$$

We implement a linear regression of $C - P$ to K, the slope is $-e^{-rT}$, and the intercept is Se^{-qT}. Therefore, $r(T)$ and $q(T)$ are obtained.

Local volatility

A. Basic procedure

In the data, we have the option value of different strikes for 3 different maturities.

We denote three difference maturities as $T_1,\ T_2,\ T_3$. $T_1 < T_2 < T_3$.

We denote the strikes respectively corresponding to the tree maturities as:

For T_1: $K_{1,1} = K_{1min},\ K_{1,2},\ K_{1,3},\ \ldots,\ K_{1,N_1} = K_{1max}$;

For T_2: $K_{2,1} = K_{2min},\ K_{2,2},\ K_{2,3},\ \ldots,\ K_{2,N_2} = K_{2max}$;

For T_3: $K_{3,1} = K_{3min},\ K_{3,2},\ K_{3,3},\ \ldots,\ K_{3,N_3} = K_{3max}$;

We use interpolation to construct an option value function with input strike and time.

- For each maturity T_i, we construct an option value function with input only strike based on linear interpolation. For the strike which is out of the range $[K_{imin}, K_{imax}]$, we set the option value at this strike to be the option value of nearest strike from this strike, only the option value of K_{imin} and K_{imax} can be the possible value at this strike. We denote the function for this maturity as f_i.
- Now we get three option value function $f_1,\ f_2,\ f_3$. To find the option value with time t and strike K: if the time t is out of the range $[T_1, T_3]$, we set the option value at time t to be the option value of nearest time from the time t, only $f_1(K)$ and $f_3(K)$ can be the possible option value in this case. If the time t is within the

range $[T_1, T_3]$, we do the linear interpolation to find the value.

- Finally, we get the function $f(K,t)$

The Dupire equation:

$$\sigma_{loc}(K,T) = \sqrt{\frac{\frac{\partial V}{\partial T} + rK\frac{\partial V}{\partial K}}{K^2\frac{\partial^2 V}{\partial K^2}}}$$

We can use the $f(K,t)$ function to calculate the local volatility with finite difference form of the derivative (for example $dt = 0.01$, $dK = 100$):

$$\sigma_{loc}(K,t) = \sqrt{\frac{\frac{f(K,t+dt) - f(K,t-dt)}{2dt} + rK\frac{f(K+dK,t) - f(K-dK,t)}{2dK}}{K^2\frac{f(K+dK,t) + f(K-dK,t) - 2f(K,t)}{dK^2}}}$$

For example, we find local volatility with time 0.05 and strike 3000 using our code "$pro - vix.R$", the result: 0.126739388654558.

B. Failure case

Sometimes there will be calendar arbitrage with $\frac{\partial V}{\partial T} < 0$. In Dupire equation, this situation may make the number inside the square root less than 0, no solution in this case.

If $\frac{\partial^2 V}{\partial K^2} < 0$, there is also an arbitrage opportunity. There may not have a local volatility solution by Dupire equation.

VIX

To compute VIX, we follow the basic steps:

i) For a maturity T, compute V_T based on the following formula:

$$V_T = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2,$$

here K is strike price, F is forward price, $Q(K)$ is the midpoint of the bid-ask spread for the option with strike price K, $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$.

ii) Determine F

Choose a pair of put and call option with the same strike price and their prices being closest to each other, and use the following formula to compute the F:

$$F = e^{rT}[C(K) - P(K)] + K.$$

iii) Determine K_0

Use the first strike below F, namely $K_0 \leq F$.

iv) Choose options

Choose options which are currently out of money and both bid and ask price are nonzero. That means, for a call option, its strike should satisfy $K_i > K_0$; for a put option, its strike should satisfy $K_i < K_0$.

v) For a maturity T (30 days), choose existing maturities T_1 and T_2 (near-term and next-term respectively) which are nearest to T early and late respectively, compute VIX based on the following formula:

$$VIX = 100\sqrt{\frac{365}{30}\left[T_1 V_{T1}\frac{N_2 - 30}{N_2 - N_1} + T_2 V_{T2}\frac{30 - N_1}{N_2 - N_1}\right]}.$$

Some comments:

There might be some differences between our result and market value. And the reasons are as follow:

1) Risk-free interest rate, our r is derived from the above questions, might be different with the market real interest rate;

2) The real VIX is calculated by minutes while here our calculated one is based on daily data.

SFD500

The main difficulty for pricing SDF500 is that there are two barriers in SFD500. One is up barrier with discrete monitoring time, another is down barrier with continuous monitoring time.

About SFD500

Parameters for SFD500 in this project:

Underlying stock: ID 002027.SZ

Valuation time: 2019-01-04

Start date: 2019-01-04

End date: 2019-12-28

T: Maturity $T = 1$

T_i : Monitoring time for up barrier $T_0 = 0, T_1 = \frac{23}{365}, T_2 = \frac{57}{365}, T_3 = \frac{82}{365}, T_4 = \frac{110}{365}, T_5 = \frac{146}{365}, T_6 = \frac{174}{365},$

$$T_7 = \frac{203}{365}, T_8 = \frac{231}{365}, T_9 = \frac{259}{365}, T_{10} = \frac{294}{365}, T_{11} = \frac{322}{365}, T_{12} = T$$

t_i: Monitoring time for down barrier $t_i = \frac{i}{365}$, $\mathrm{i} = 0,1, \ldots, 365$, we discrete one year to be 365 grids, then we can treat the monitoring time of down barrier continuous in lattice and Monte-Carlo method.

S_0: Initial stock price $S_0 = 5.10$

B_u: Up barrier $B_u = 1.03 S_0$

B_d: Down barrier $B_d = 0.7 S_0$

R: Annualized coupon rate $R = 0.31$

N: The amount of the structured revenue security the investor has bought

σ: The volatility of the underlying stock. We can use historical data to calculate the volatility of the underlying stock.

r: Risk free interest rate

q: The dividend yield of the underlying stock

Basic information of SFD500:

i) Two barriers: up barrier B_u, down barrier B_d. If down barrier is touched, terminal payoff changes; if up barrier touched, this option is terminated and pays payoff immediately.

ii) Monitoring time for up barrier: $T_0 = 0,\ T_1,\ T_2, \ldots,\ T_{11},\ T_{12} = T$.

iii) Monitoring time for down barrier: continuous time.

Lattice method

First, we identify the model of SFD500 under the B-S framework.

The governing equation for SFD500:

$$L_{BS}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$\frac{\partial V}{\partial t} + L_{BS}V = 0,\ \ t_i \leq t < t_{i+1}\, for\ i = 0, 1, \ldots, 364$$

If we use lattice method: finite difference method or binomial tree method, we discretize one year into 365 grids, and stock price $S \in [S_{min}, S_{max}]$ with grid length ds.

The terminal and connection conditions:

$$V(S,T) = N(R+1),\ \ S \geq B_u$$

$$V(S,T) = N\left(R + \frac{367}{365}\right),\ \ B_d < S < B_u$$

$$V(S,T) = Nmin\left(\frac{S_T}{S_0}, 1\right),\ \ S \leq B_d$$

For $\mathrm{i} = 0,1,\dots,365$, if $t_i = T_k$ for some $\mathrm{i} = 1,2,\dots,12$:

$$V(S_-, t_i -) = V_d(S_+, t_i +),\ \ S \leq B_d$$

$$V(S, t_i) = N\left(R + \frac{t_i}{365}\right),\ \ S \geq B_u$$

For $\mathrm{i} = 0,1,\dots,365$, if $t_i \neq T_k$ for any $\mathrm{i} = 1,2,\dots,12$:

$$V(S_-, t_i -) = V_d(S_+, t_i +),\ \ S \leq B_d$$

When the down barrier is touched at some point (S,t), the derivative value at this point need a different PDE to solve. We denote the solution of this difference PDE as V_d.

Comparing to V, the terminal payoff of V_d has changed, the up barrier has been remained

V_d satisfies:

$$L_{BS}V_d = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_d}{\partial S^2} + rS\frac{\partial V_d}{\partial S} - rV_d$$

$$\frac{\partial V_d}{\partial t} + L_{BS}V_d = 0,\ \ T_i \leq t < T_{i+1}\, for\ i = 0, 1, \dots, 12$$

The terminal and connection conditions:

$$V_d(S,T) = N(R+1),\ \ S \geq B_u$$

$$V_d(S,T) = Nmin\left(\frac{S_T}{S_0}, 1\right),\ \ S < B_u$$

For $\mathrm{i} = 1,2,\dots,12$:

$$V_d(S,T_i) = N\left(R + \frac{T_i}{365}\right),\ \ S \geq B_u$$

Monte-Carlo method

τ: the first time when stock price touches the up barrier at the up-barrier monitoring dates.

η: the first time when stock price touches the down barrier at the down-barrier monitoring dates.

$$V(S,t) = E_t^*\left[e^{-r(T-t)}Nmin\left(\frac{S_T}{S_0},1\right)1_{\{\eta\leq T<\tau\}} + e^{-r(\tau-t)}N\left(R+\frac{\tau}{365}\right)1_{\{\tau\leq T\}} + e^{-r(T-t)}N\left(R+\frac{367}{365}\right)1_{\{T<\tau,T<\eta\}}\right]$$

We use the follow discretized form SDE to do Monte-Carlo simulation:

$$S_t = S_{t-\Delta t} * exp\ ((r-q-\frac{\sigma^2}{2}) * \Delta t + \sigma * \sqrt{\Delta t})$$

The basic procedure:

- Simulate a path of the underlying stock value based on above discretized SDE
- Identify if the path triggers the barrier on the monitoring date, determine the payoff of this path
- Simulate enough paths, and take the average of payoffs to be the result of Monte-Carlo method

Numerical results

Lattice (implicit): 0.967173006524236

Monte-Carlo (10000 times):

Median: 1.016968

Mean: 0.9633629

Standard deviation: 0.04322504

Expected survive time: 0.1591112

The difference between the results by lattice and Monte-Carlo method is about 0.00381.

Beyond the Black-Scholes world

Jump-diffusion model:

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t+\sum_{i=1}^{N_t} Q_i}$$

$\left\{\sum_{i=1}^{N_t} Q_i\right\}_{t=0}$ *is a compound Poisson process with normal distributed jumps* $N\left(\mu_j, \sigma_j^2\right)$ *and intensity* λ

We can use above SDE to generate Monte-Carlo path and price the product under this Jump-diffusion model with the same procedure of Monte-Carlo method mentioned before. Try different intensity parameter λ to see which one gives a better result.

Dual-Purpose Fund

Dual-purpose funds were introduced to China in 2010. For each dual-purpose fund, whose underlying asset is usually an open-end fund tracking some index, there are two classes of shares: the low-risk A shares which behave like a perpetual bond with periodical coupon payments, and the high-risk B shares which are essentially a closed-end fund magnifying the exposure of the underlying index. Simply put, the holders of B shares borrow money from the holders of A shares and make periodic interest payments to the latter, which leads the B shares to possess a continuum of leverage ratios. To reduce risk of both shares, a set of upward and downward reset clauses is imposed.

About dual-purpose fund

Parameters for dual-purpose fund in this project:

Dual-purpose fund: 银华锐进 (A shares) and 银华稳进 (B shares)

Underlying stock: SZSE 100 Price Index

Valuation time: year 2015 and year 2019

σ: The volatility of the underlying stock. We can use historical data to calculate the volatility of the underlying stock. Here, the σ is from historical data (last one year):

$$\sigma = \sqrt{\frac{1}{\delta t}\frac{1}{N}\sum_{i=1}^{N}(ln\frac{S_i}{S_{i-1}})^2}$$

H_u: the upper threshold for the underlying price to reach to trigger the upward reset

H_d: the lower threshold for the net asset value of B to reach to trigger the downward reset

r: Risk free interest rate

c: management fee

R: periodical coupon payments for A shares

α: pairing coefficient, fixed for each fund and specified in the contract. This way, each share of the underlying fund corresponds to α share of A plus $(1-\alpha)$ share of B

Tolerance: when the absolute difference of the two consecutive iteration values is less than the tolerance, stop the iteration.

	Fund 1 in 2015	Fund 2 in 2015	Fund 1 in 2019	Fund 2 in 2019
σ	0.179	0.199	0.235	0.237
Gridded S NO. (Ns)	50			
Gridded T NO. (Nt)	365			
H_u	2			
H_d	0.25			
r	0.04			
c	$log(1+0.0122) \approx 0.0121$			
R	0.0275+0.03	0.0275+0.035	0.015+0.03	0.015+0.035
α	0.5	0.4	0.5	0.4
tolerance	10^{-6}			

Basic information of dual-purpose fund:

i) There is A share and B share, and conversion clause for them. The conversion clause says investors can combine their positions in A and B with a ratio $\alpha:(1-\alpha)$ into positions in the underlying fund, here, $0 < \alpha < 1$. Under principle of no arbitrage, there is:

$$S = \alpha V_A + (1 - \alpha)V_B,$$

here, V_A and V_B are the values of A share and B share respectively. With the above equation, we only need to price A share, then simply get value of B share.

ii) Denote the net asset value of A as $NAV^A = 1 + Rv$, the net asset value of B as $NAV^B = \frac{S-\alpha NAV^A}{1-\alpha}$. There is $S = \alpha NAV^A + (1 - \alpha)NAV^B$. Here, v is the time since the last payment.

iii) Regular reset: since the last payment, if there is no irregular reset (upward or downward reset) within one year, there is regular reset at the one-year end. Holder of A shares receives payment R and NAV^A goes back to one.

iv) Downward reset (happens at η): when NAV^B goes below the lower threshold $H_d < 1$, equivalently when $S < H(v) = (1 - \alpha)H_d + \alpha(1 + Rv)$, there is downward reset. Holder of A shares receives payment $Rv + 1 - H_d$, and shares of A together with shares of B both reduce to $NAV^B_{\eta-}$%. Then, both NAV^A and NAV^B go back to one.

v) Upward reset (happens at τ): when S goes above $H_u > 1$, there is upward reset. Holder of A shares receives payment Rv and holder of B shares receives payment $\frac{S_{\tau-}-1-\alpha Rv}{1-\alpha}$. Then, both NAV^A and NAV^B go back to one.

Lattice method

There is a unique solution of V_A by the PDE (see QF5202 lecture notes 4 Theorem 2.1):

$$\frac{\partial V_A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_A}{\partial S^2} + (r - c)S\frac{\partial V_A}{\partial S} - rV_A = 0,\ \ 0 \le t < 1,\ \ H(t) < S < H_u$$

$$V_A(1, S) = R + V_A(0, S - \alpha R),\ \ H(t) < S < H_u$$

$$V_A(t, H_u) = Rt + V_A(0,1), \;\; 0 \le t \le 1$$

$$V_A\big(t, H(t)\big) = Rt + 1 - H_d + H_d V_A(0,1), \;\; 0 \le t \le 1.$$

To solve the above PDE numerically, we need to use iteration procedure (see QF5202 lecture notes 4 algorithm 1):

1. Set the initial guess $V_A^{(0)} = 0$;

2. For $i = 1, 2, \ldots$: Given $V_A^{(i-1)}$, solve for $V_A^{(i)}$, the solution to the equation

$$\frac{\partial V_A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_A}{\partial S^2} + (r-c)S\frac{\partial V_A}{\partial S} - rV_A = 0, \;\; 0 \le t < 1, \;\; H(t) < S < H_u$$

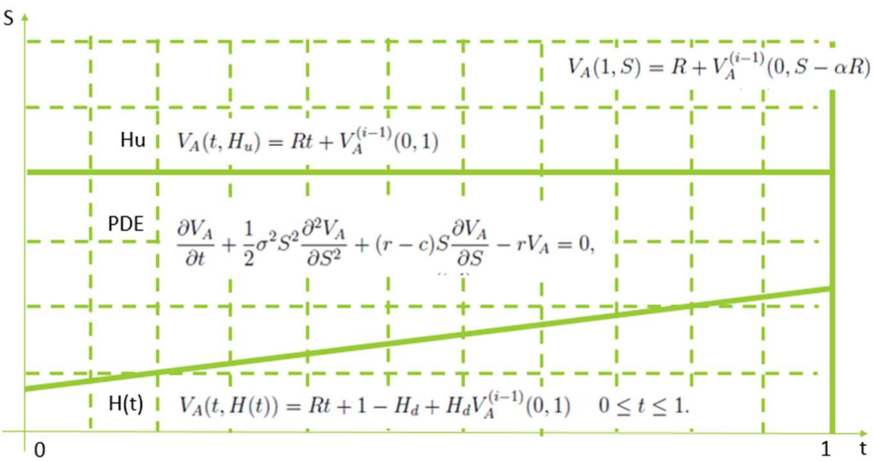
$$V_A(1, S) = R + V_A^{(i-1)}(0, S - \alpha R), \;\; H(t) < S < H_u$$

$$V_A(t, H_u) = Rt + V_A^{(i-1)}(0,1), \;\; 0 \le t \le 1$$

$$V_A\big(t, H(t)\big) = Rt + 1 - H_d + H_d V_A^{(i-1)}(0,1), \;\; 0 \le t \le 1.$$

3. If $\left\| V_A^{(i)} - V_A^{(i-1)} \right\| < tolerance$, stop and return $V_A^{(i)}$; otherwise set $i = i + 1$ and go to step 2.

Fig 3.2.1. Finite difference method for V_A



Monte-Carlo method

On the non-interest-payment dates, the net value of the underlying fund follows geometric Brownian motion under risk neutral probability, denote the SDE as SDE (1):

$$dS_t = (r-c)S_t dt + \sigma S_t dB_t$$

$$S_{i+1} = S_i * exp\left(\left(r - c - \frac{\sigma^2}{2}\right) * dt + \sigma * \sqrt{dt} * z\right),\ z \sim N(0,1)$$

Upon the irregular reset date, the net value of the underlying is set to be 1.

Upon the regular reset date, the net value of the underlying is reduced by αR.

Under risk neutral pricing framework, the market value is as follow, with current time $t \in [0,1]$.

$$V_A(t,S) = E_t^*\left[\sum_{1\le i<m} e^{-r(i-t)}R + \sum_{(\tau,\eta)} e^{-r(\tau-t)}\big(R(\tau - \lfloor\tau\rfloor) + V_A(0,1)\big)1_{\{\tau<\eta\}} + e^{-r(\eta-t)}\big(R(\eta - \lfloor\eta\rfloor) + 1 - H_d + H_d V_A(0,1)\big)1_{\{\eta<\tau\}}\right]$$

In the above formula, there is only one unknown value, that is the market value of the A share at time 0 and with underlying value 1.

We develop a method to get a number approximating this unknown value $V_A(0,1)$:

i) Initially, set the $V_A^0(0,1) = 0$, and set the tolerance as 0.01.

ii) For $i = 1, 2, \ldots$: Given $V_A^{i-1}(0,1)$.

iii) According to the SDE (1), simulate underlying fund value path from time 0 and with the underlying fund value 1. Set maturity to be 100 years to ensure that almost every path triggers the irregular reset event.

iv) By Monte-Carlo method, compute the $V_A^i(0,1)$:

$$V_A^i(0,1) = E_t^*\left[\sum_{1\le i<min(\tau,\eta)} e^{-r(i-t)}R + e^{-r(\tau-t)}\left(R(\tau-\lfloor\tau\rfloor)+V_A^{i-1}(0,1)\right)1_{\{\tau<\eta\}}\right.$$
$$\left. + e^{-r(\eta-t)}\left(R(\eta-\lfloor\eta\rfloor)+1-H_d+H_dV_A^{i-1}(0,1)\right)1_{\{\eta<\tau\}}\right]$$

v) The iteration-stop condition is $|V_A^i(0,1)-V_A^{i-1}(0,1)|<tolerance$. When this condition is met, we stop the iteration and treat $V_A^i(0,1)$ as $V_A(0,1)$. Otherwise, repeat the steps (ii) - (iv).

Refined model for two irregular resets happening at 2015 on share B:

After irregular reset, $t \in [t_{reset}, 1]$:

If $min(\tau,\eta) > 1$,

$$V_A(t,S) = E_t^*\left[e^{-r(1-t)R(1-t_{reset})} + \sum_{2\le i<min(\tau,\eta)} e^{-r(i-t)}R\right.$$
$$+ e^{-r(\tau-t)}\left(R(\tau-\lfloor\tau\rfloor)+V_A(0,1)\right)1_{\{\tau<\eta\}}$$
$$\left.+ e^{-r(\eta-t)}\left(R(\eta-\lfloor\eta\rfloor)+1-H_d+H_dV_A(0,1)\right)1_{\{\eta<\tau\}}\right].$$

If $min(\tau,\eta) \le 1$,

$$V_A(t,S) = E_t^*\left[e^{-r(min(\tau,\eta)-t)}\left(R(min(\tau,\eta)-t_{reset})+V_A(0,1)\right)1_{\{\tau<\eta\}}\right.$$
$$+ e^{-r(min(\tau,\eta)-t)}\left(R(min(\tau,\eta)-t_{reset})+1-H_d\right.$$
$$\left.\left.+ H_dV_A(0,1)\right)1_{\{\eta<\tau\}}\right].$$

The assumption is that the regular reset date is one year after irregular reset happens. But in the realistic, the regular reset date is fixed (at each year-end).

The purpose of modifying the model is to remove the influence of accrued interest which has already been pre-paid on the irregular reset date.

The modified model made result closer to the reality, but the modified model is still not realistic on the regular reset date assumption. The modified model assumes when the first irregular reset event happens, the upcoming regular reset date remains unchanged, if the irregular reset event happens again, the regular reset date is rescheduled to the same date in the coming year, which is also not realistic. However, regular reset date is fixed or random will not affect the result much. We tried this modified model to makes our result closer to the reality even only with a little improvement.

Numerical results

Fig 3.4.1. V_A by explicit finite difference method

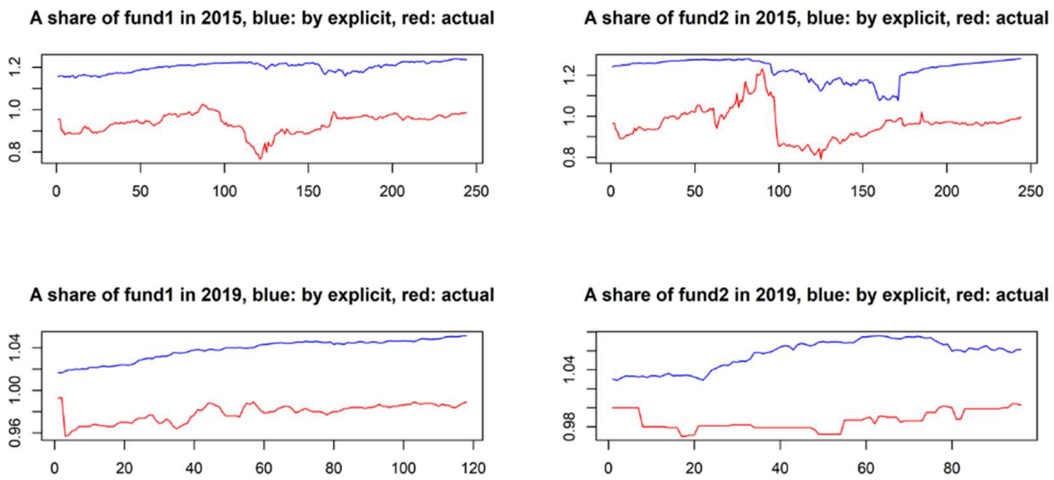


Fig 3.4.2. V_B by explicit finite difference method

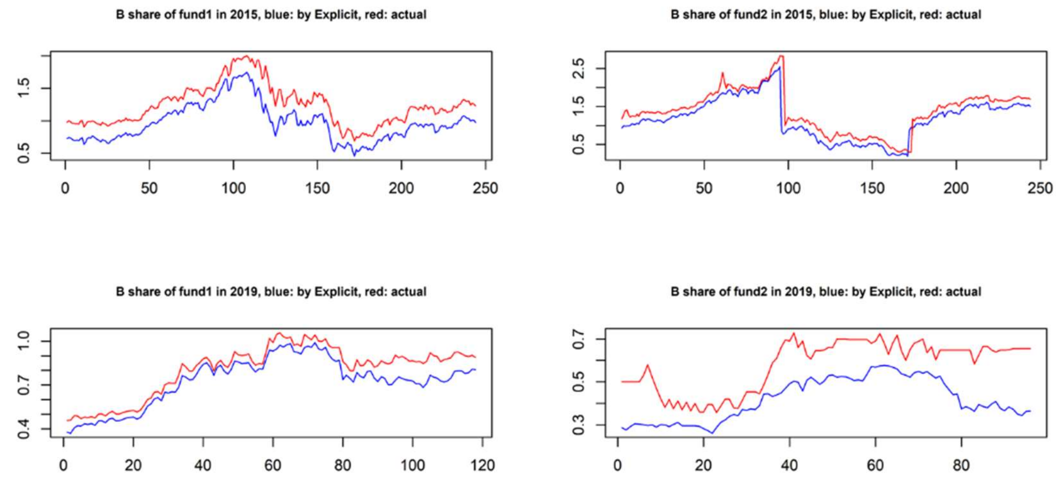


Fig 3.4.3. V_A by Monte-Carlo method

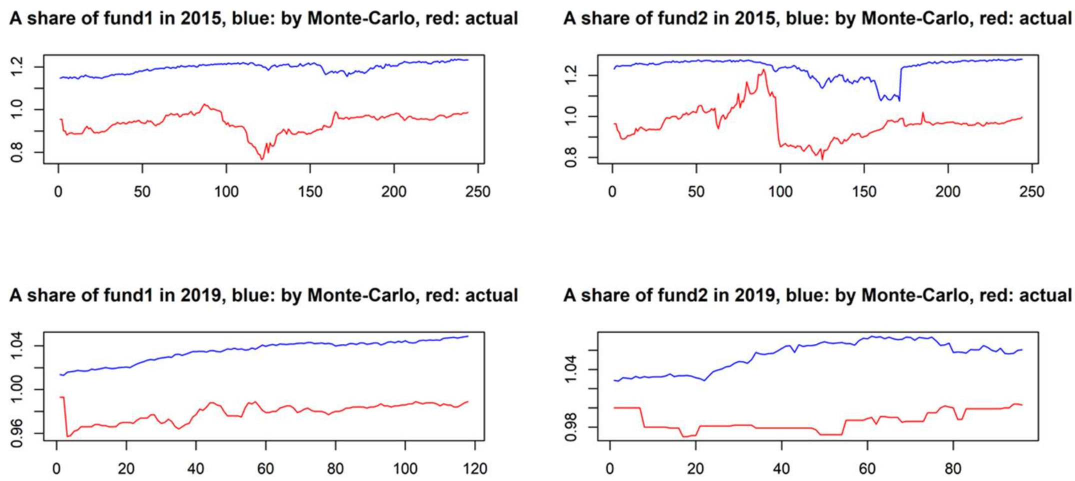


Fig 3.4.4. V_B by Monte-Carlo method

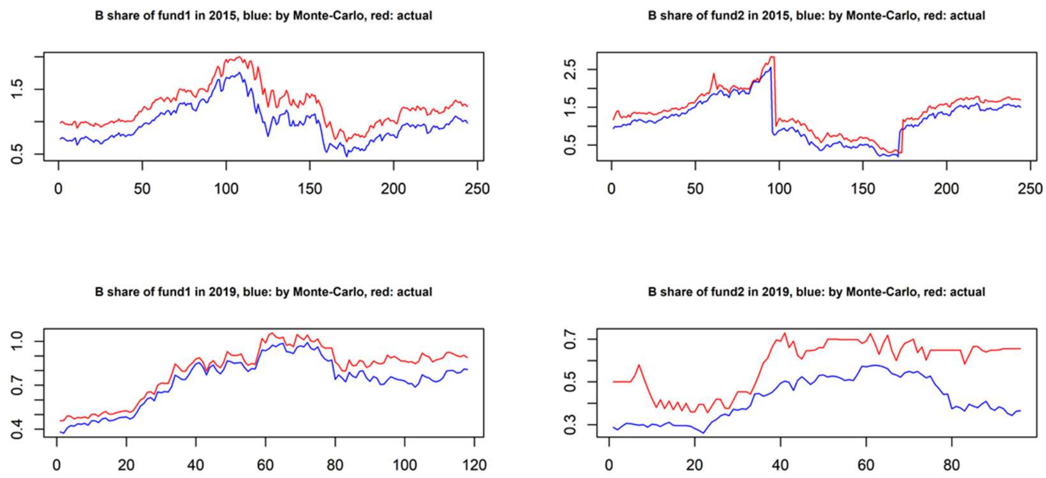


Fig 3.4.5. Comparison of V_A between implicit finite difference method and Monte-Carlo method

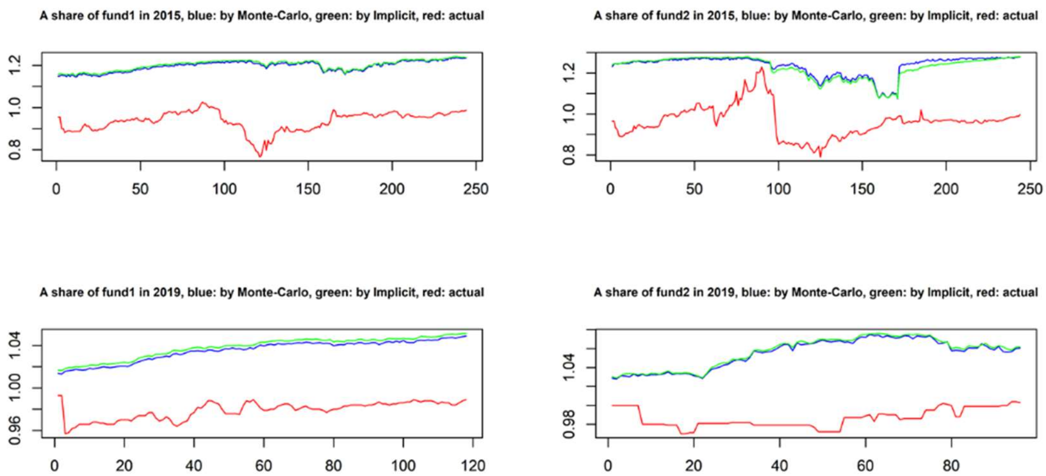
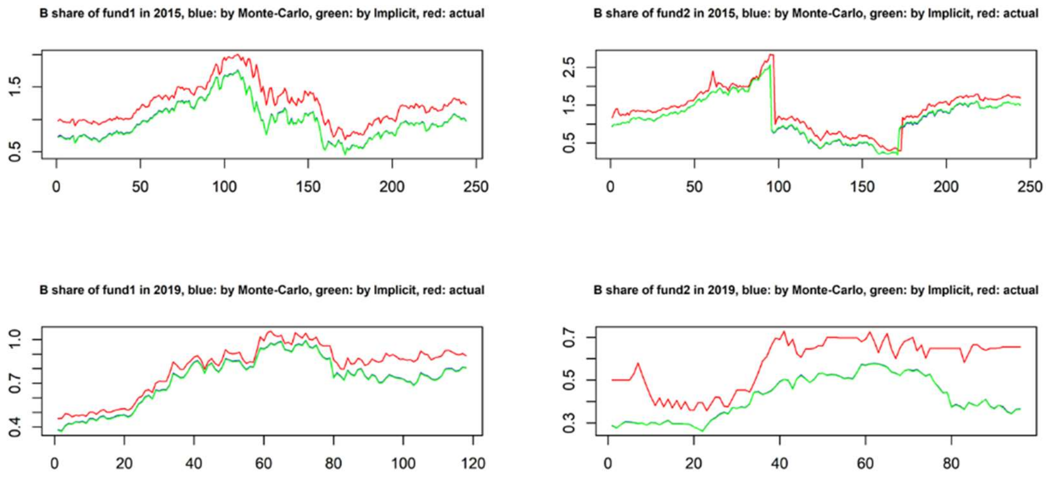


Fig 3.4.6. Comparison of V_B between implicit finite difference method and Monte-Carlo method



For A share, both values calculated by lattice method and M.C. method are close to each other but higher than actual value.

For B share, both values calculated by lattice method and M.C. method are lower than actual vales, but compare with share A, its degree of fitness with actual value is much better.

Closed-end fund (share A) usually has a discount for the following reasons: i) Liquidity deficiency; ii) Agency cost; iii) Investor sentiment.

Fig 3.4.7. Comparison on difference r (V_A in 2015)

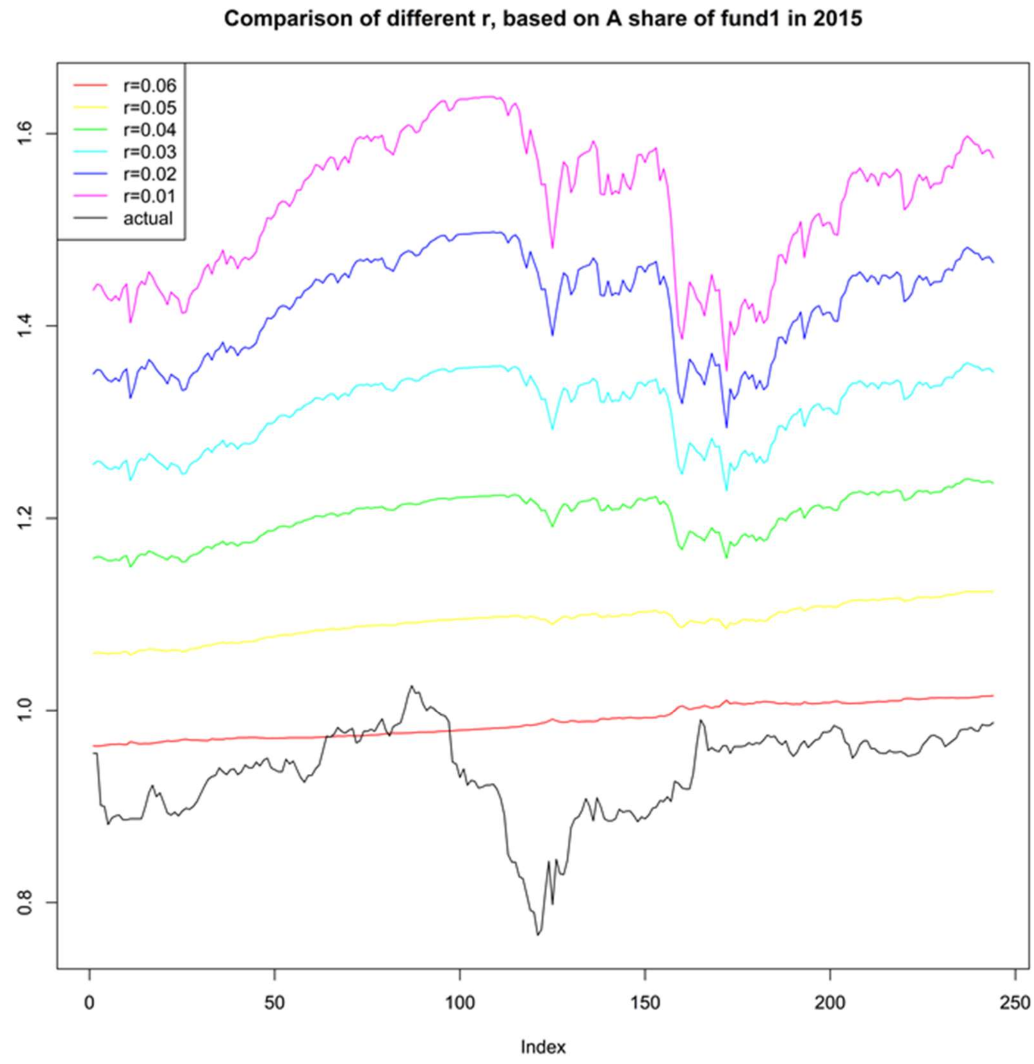
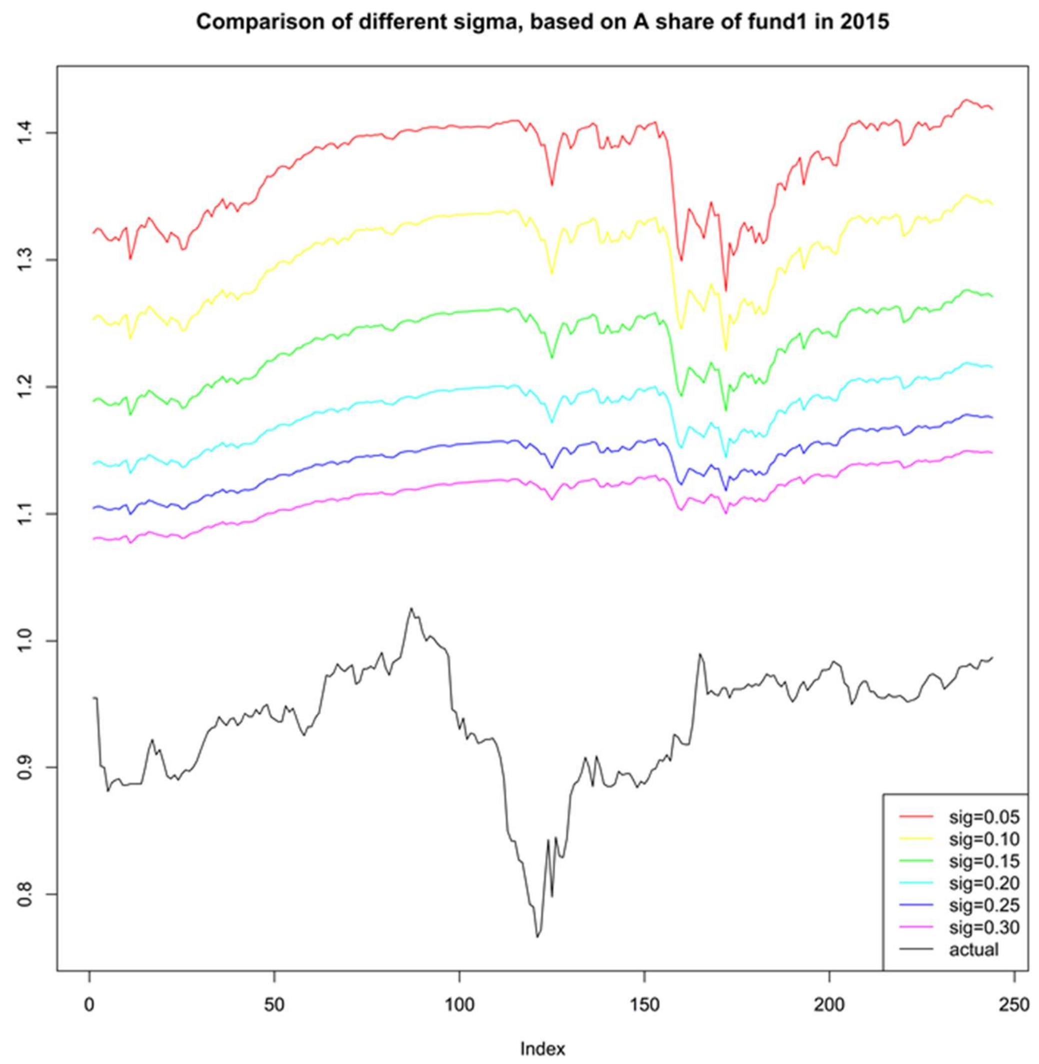


Fig 3.4.8. Comparison on difference σ (V_A in 2015)



Summary

For the time-consuming, Explicit (1.153592 mins) > Implicit (5.886479 mins) > Monte Carlo (5.081755 hours).

The estimated value of our two dual funds has little difference between using Lattice Method and Monte Carlo method in both share A and B. The estimate values are always higher than actual value no matter what method we choose in share A but lower to the actual values in share B. Moreover, share B is highly correlated with actual values than share A. Close-end fund discount may affect the estimate value of share A, moreover, some other factors may affect the result of our model, like r and σ.

Convertible Bond

A convertible bond has the characteristics of a regular bond but with the extra feature that the bond may, at a time of the holders choosing, be exchanged for a specified asset. This exchange is called “conversion”. Since the conversion feature makes the convertible bond similar mathematically to American options, we expect that its pricing model is analogous to that of American options.

About convertible bond

Parameters for convertible bond in this project:

T: Maturity $T = 1.5$, also $T = T_{2019}$, here, T_{2019} is coupon payment date in 2019 (also the end of year 2019)

T_{2018}: Coupon payment date in 2018, also the end of year 2018, $T_{2018} = 0.5$

T_s: Conversion start date, comparing to current time 2018-06-20, $T_s = -3$

cop_{2018}: Coupon rate in 2018, $cop_{2018} = 1.5\%$

cop_{2019}: Coupon rate in 2019, without bonus $cop_{2019no-bonus} = 2\%$, with bonus $cop_{2019bonus} = 6\%$

F: Face value, $F = 100$

X: Current conversion price $X = 7.24$

K: Conversion ratio, with continuous dividend yield q, $K_{cont-div} = \frac{F}{Xexp(-q(t-T_s))}$, without dividend $K_{no-div} = \frac{F}{X}$

B_c: Barrier for call, $B_c = 1.3X$

B_p: Barrier for put, $B_p = 0.7X$

B_{sc}: Barrier for soft call:

$$B_{sc} = \hat{E}[S_u: u = inf\{\tau: S_\tau \geq B_c \; satisfying \; 15 \; out \; of \; 30, \; S_t = S\}]$$

B_{sp}: Barrier for soft put:

$$B_{sp} = \hat{E}\left[S_u: u = inf\left\{\tau: S_\tau \leq B_p \; satisfying \; 30 \; out \; of \; 30, \; S_t = S\right\}\right]$$

r: Risk free rate $r = 3\%$

q: Dividend yield

$AccI(t)$: Accrued interest $AccI(t) = (t + 0.5 - \lfloor t + 0.5 \rfloor) * cop_{\lfloor 2018.5+t \rfloor}$

P_p: Prescribed price for holder to sell this bond when soft put condition is met, $P_p = 103$

S_0: Initial stock price, $S_0 = 5.28$

B_u: Up barrier

B_d: Down barrier

Basic information of convertible bond:

i) Time: denote current time as $T = 0$ (2018/6/20), the end of year 2018 and 2019 as T_{2018} and T_{2019} respectively when there will be coupon payment. Note that the maturity of the convertible bond is $T = T_{2019}$, and denote the conversion start date as $T_s = -3$.

ii) Dividend: denote the face value of the convertible bond as F. Continuous dividend yield q is applied, the conversion ratio is $\frac{F}{Xexp(-q(t-T_s))}$.

iii) Soft call clause: when there are 15 days out of past consecutive 30 days satisfying that the stock price is equal to or larger than barrier B_c, soft call clause is triggered. We use the barrier for soft call B_{sc} triggered to approximate the soft call clause triggered. The B_{sc} can be calculated as:

$$B_{sc} = \hat{E}[S_u: u = inf\{\tau: S_\tau \geq B_c\ satisfying\ 15\ out\ of\ 30,\ \ S_t = S\}].$$

Call clause is simpler than soft call clause. When the stock price is equal to or larger than barrier B_c, call clause is triggered.

When call clause or soft call clause is trigger, issuer has the right to purchase back the bond for a specified amount M_C. In our numerical experiments, $M_C = F + AccI(t)$.

However, holders' conversion right has priority.

iv) Soft put clause: when there are 30 days out of past consecutive 30 days satisfying that the stock price is equal to or smaller than barrier B_p, soft put clause is triggered. We use the barrier for soft put B_{sp} triggered to approximate the soft put clause triggered. The B_{sp} can be calculated as:

$$B_{sp} = \hat{E}\left[S_u: u = inf\left\{\tau: S_\tau \leq B_p\ satisfying\ 30\ out\ of\ 30,\ \ S_t = S\right\}\right].$$

Put clause is simpler than soft put clause. When the stock price is equal to or smaller than barrier B_p, put clause is triggered.

When put clause or soft put clause is trigger, holder has the right to return the bond to the issuer for an amount M_P. In our numerical experiments, $M_P = P_p$.

v) Coupon: coupon rate at T_{2018} is $cop_{2018} = 1.5\%$, coupon rate at T_{2019} is $cop_{2019} = 2\%$.

vi) Bonus coupon clause: if there is no conversion occurring, at maturity $T = T_{2019}$, there is bonus coupon $cop_{2019} = 6\%$.

Lattice method

A. Basic procedure

- we need to compute volatility σ through stock price data to derive up-movement parameter and down-movement parameter.

$$\sigma = \sqrt{\frac{1}{\delta t}\frac{1}{N}\sum_{i=1}^{N}(ln\frac{S_i}{S_{i-1}})^2}$$

$$u = e^{\sigma\sqrt{\delta t}} \ and \ d = e^{-\sigma\sqrt{\delta t}}$$

- And then risk neutral probability should be computed through the following formula:

$$p = \frac{e^{(r-q)\delta t} - d}{u - d}$$

- We model the stock price up to maturity 2019-12-25 using binominal tree method. We do initialization at maturity, compute the maturity price through:

$$V(S,T) = max(1, KS)$$

- we use the backward recursive to get the current convertible bond price. In different case, the backward recursive formula needs to be changed. The following formula is one of possible formulas:

$$V(S, t - \delta t) = max(e^{-r\delta t}[pV(Su,t) + (1-p)V(Sd,t)], KS)$$

B. Models

Case 1: no dividend, no put clause, no call clause, no bonus coupon clause

Case 2: no dividend, no put clause, no call clause, with bonus coupon clause

Case 3: no dividend, no put clause, with call clause, with bonus coupon clause

Case 4: no dividend, no put clause, with soft call clause, with bonus coupon clause

Case 5: no dividend, with soft put clause, with soft call clause, with bonus coupon clause

Case 6: with dividend, no put clause, no call clause, with bonus coupon clause

Case 7: with dividend, with soft put clause, with soft call clause, with bonus coupon clause

- Case 1: If no dividend, no put clause, no call clause, no bonus coupon clause:

$$K = K_{no-div} = \frac{F}{X}$$

The governing equation:

$$L_{BS}V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$-L_{BS}V = 0, \ \ if\ nS < V$$

$$-L_{BS}V \geq 0, \ \ if\ V = nS$$

The terminal condition:

$$V(S,T) = max(F, KS)$$

Check for early conversion:

$$V(S_-, t-) = max(V(S_+, t+), KS)$$

The connection condition:

$$cop_{2019} = cop_{2019no-bonu} \ = 2\%$$

$$V(S_-, T_{2018}-) = V(S_+, T_{2018}+) + cop_{2018} * F$$

$$V(S_-, T_{2019}-) = V(S_+, T_{2019}+) + cop_{2019} * F$$

- We discuss Case 2 and Case 6 together:

Case 2: If no dividend, no put clause, no call clause, with bonus coupon clause:

$$K = K_{no-div} = \frac{F}{X}$$

Case 6: If with dividend, no put clause, no call clause, with bonus coupon clause:

$$K = K_{cont-div} = \frac{F}{Xexp(-q(t - T_s))}$$

The governing equation:

$$L_{BS}V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$-L_{BS}V = 0, \ \ if \ nS < V$$

$$-L_{BS}V \geq 0, \ \ if \ V = nS$$

The terminal condition:

$$cop_{2019} = cop_{2019bonus} = 6\%$$

$$V(S,T) = max(F(1 + cop_{2019}), KS)$$

Check for early conversion:

$$V(S_-, t-) = max(V(S_+, t+), KS)$$

The connection condition:

$$V(S_-, T_{2018}-) = V(S_+, T_{2018}+) + cop_{2018} * F$$

- We discuss Case 3 and Case 4 together:

Case 3: If no dividend, no put clause, with call clause, with bonus coupon clause:

$$B_u = B_c, \ \ K = K_{no-div} = \frac{F}{X}$$

Case 4: If no dividend, no put clause, with soft call clause, with bonus coupon clause:

$$B_u = B_{sc},\ K = K_{no-div} = \frac{F}{X}$$

For Case 3, this question can be derived through modifying the procedure of Case 2 by adding a call provision to each step. Namely, if the stock price is larger than B_c, then the issuer can redeem the bond at the price of its face value plus accrued interest.

For Case 4:

1. To begin with, our main logic is to convert soft call clause though the following formula:

$$B_{sc} = \hat{E}[S_u: u = inf\{\tau: S_\tau \geq B_c\ satisfying\ 15\ out\ of\ 30,\ \ S_t = S\}].$$

2. Use Monte Carlo method to simulate 100000 times stock price paths, and find the S which satisfies the soft call. Then we take the average of these S.

3. Now we have B_{sc} for a once-touch call clause, which is the same case as Case 3. We do the same algorithm in Case 3.

The governing equation:

$$L_{BS}V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$-L_{BS}V = 0,\ \ if\ nS < V < B_u$$

$$-L_{BS}V \geq 0,\ \ if\ V = nS$$

$$-L_{BS}V \leq 0,\ \ if\ V = B_u$$

The terminal condition:

$$cop_{2019} = cop_{2019bonus} = 6\%$$

$$V(S,T) = max(F(1 + cop_{2019}), KS)$$

Check for early conversion:

$$V(S_-, t-) = max\big(min\big(V(S_+, t+), F + AccI(t)\big), KS\big),\ \ S \geq B_u$$

$$V(S_-, t-) = max(V(S_+, t+), KS),\ \ S < B_u$$

The connection condition:

$$V(S_-, T_{2018}-) = V(S_+, T_{2018}+) + cop_{2018} * F$$

- We discuss Case 5 and Case 7 together:

Case 5: If no dividend, with soft put clause, with soft call clause, with bonus coupon clause:

$$B_u = B_{sc},\ B_d = B_{sp},\ K = K_{no-div} = \frac{F}{X}$$

Case 7: If with dividend, with soft put clause, with soft call clause, with bonus coupon clause:

$$B_u = B_{sc},\ B_d = B_{sp},\ K = K_{cont-div} = \frac{F}{Xexp(-q(t-T_s))}$$

The governing equation:

$$L_{BS}V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$-L_{BS}V = 0,\ \ if\ max(B_d, nS) < V < B_u$$

$$-L_{BS}V \geq 0,\ \ if\ V = max(B_d, nS)$$

$$-L_{BS}V \leq 0,\ \ if\ V = B_u$$

The terminal condition:

$$cop_{2019} = cop_{2019bonus} = 6\%$$

$$V(S,T) = max(F(1 + cop_{2019}), KS)$$

Check for early conversion:

$$V(S_-, t-) = max\big(min\big(V(S_+, t+), F + AccI(t)\big), KS\big),\ \ S \geq B_u$$

$$V(S_-, t-) = max\big(V(S_+, t+), P_p, KS\big),\ \ S \leq B_d$$

$$V(S_-, t-) = max(V(S_+, t+), KS),\ \ B_d < S < B_u$$

The connection condition:

$$V(S_-, T_{2018}-) = V(S_+, T_{2018}+) + cop_{2018} * F$$

Numerical results

Lattice method	Case 1: no dividend, no put clause, no call clause, no bonus coupon clause 101.8331
	Case 2: no dividend, no put clause, no call clause, with bonus coupon clause 105.2065
	Case 3: no dividend, no put clause, with call clause, with bonus coupon clause 98.90793
	Case 4: no dividend, no put clause, with soft call clause, with bonus coupon clause 100.5002
	Case 5: no dividend, with soft put clause, with soft call clause, with bonus coupon clause 103
	Case 6: with dividend, no put clause, no call clause, with bonus coupon clause 105.4495
	Case 7: with dividend, with soft put clause, with soft call clause, with bonus coupon clause 100.9804

Callable Bull/Bear Contract

A callable bull/bear contract, or CBBC in short form, is a derivative financial instrument that provides investors with a leveraged investment in underlying assets, which can be a single stock, or an index. CBBC is usually issued by third parties, mostly investment banks, but neither by stock exchanges nor by asset owners. It was first introduced in Europe and Australia in 2001, and it is now popular in United Kingdom, Germany, Switzerland, Italy, and Hong Kong. CBBC is actively traded among investors in Europe and Hong Kong, which is partially since it can cater to individual investors' behavioral biases (like lottery preferences).

About callable bull/bear contract

Parameters for callable bull/bear contract in this project:

Underlying: CNOOC (883.hk)

Strike price (K): 11.8

Maturity (T): 20/12/2019

Call price (C): 12

Current time ($t = 0$): 21/06/2019

Current underlying price (S_0): 13.48

Current quoted price of CBBC: 1.64

Entitlement ratio: 10,000

No Mandatory Call Event (MCE) occurs until t_0

Partition for trading sessions (T_i)

The volatility of the underlying stock (σ)

Risk free interest rate (r)

The dividend yield of the underlying stock (q)

Basic information of callable bull/bear contract:

i) Partitions for trading session: the nodes are T_i with $i = 0, \ldots, N$. $T_0 = 0$, $T_N = T$, here T is maturity.

ii) Call price: the underlying price initially is lower than call price, once it reaches the call price, the mandatory call event occurs, the contract becomes a lookback call option with maturity at the end of the next trading section.

Lattice method

The given CBBC is a callable bull contract (R type). It is similar with a knock-out call option except that the knock-out boundary condition is not simply zero, but a fixed lookback call option.

The PDE of such CBBC is as follows:

Governing equation:

$$L_{BS}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

$$\frac{\partial V}{\partial t} + L_{BS}V = 0$$

The terminal condition:

$$V(S, T) = max(S - K, 0)$$

The boundary condition:

$$V(C, t) = V_{lb}(C, t, t, K, T_{i+2}), \ \ T_i \le t < T_{i+1}$$

where, t: time when MCE occurs.

The V_{lb} is a lookback call option with strike K, time to maturity at T_{i+2}, and satisfies the following governing equation:

$$L_{BS}V_{lb} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{lb}}{\partial S^2} + rS\frac{\partial V_{lb}}{\partial S} - rV_{lb}$$

$$\frac{\partial V_{lb}}{\partial t} + L_{BS}V_{lb} = 0, \quad t \le t_* < T_{i+2}$$

$$\frac{\partial V_{lb}}{\partial m}|_{S=m} = 0$$

The terminal condition:

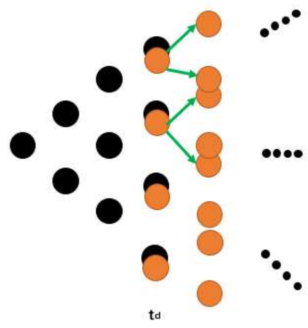
$$V_{lb}(S, T_{i+2}, t, K, T_{i+2}) = max(m - K, 0), \quad m\ is\ minimum\ S\ during\ t\ to\ T_{i+2}$$

Such fixed lookback call option has analytical solution derived from its corresponding floating lookback put option with put-call parity.[1] Therefore, with the solution for the fixed lookback call option, using finite difference method to solve the above PDE, then the value of the CBBC is found.

Binomial tree method

We use CRR method to construct a binomial tree, but a significant change is the constant dividend payment, after that, the nodes in the tree will shift down with a same distance, resulting into the tree to be exponential. As shown in the following diagram, after the shift at the time of dividend payment date, the next upside and downside node will not meet, then the tree is exponent.

[1] John C Hull, Options, futures, and other derivatives, 10th edition



In order to solve this problem, we use forward shooting grid method (FSGM) to calculate the value after dividend paid.

Another issue is the node value after barrier hit, which is the fixed lookback call option value. We can construct another binomial tree to solve the value with initial underlying price C between time t to time T_{i+2}. Also, the close-form solution of the fixed lookback call option can also be used for simplification.

The pseudo code for the CBBC pricing with Binomial tree method is as follows:

- Initialize the input parameters:

  sig(sigma), $r, S_0, T, D,\ t_d$

  $K,\ C,\ T_{i+2}\ for\ T_i \leq t < T_{i+1}$

  Nt (steps of T), $\Delta t = \frac{T}{Nt}$, to make sure that t_d and T_{i+2} is right on the node.

  V (vector of final option value)

  Calculate u, d, df (discount factor), p, q

- Construct the binomial tree
  - Before the time t_d, a usual binomial tree;
  - After the time t_d, a shift down of the nodes and FSGM is developed.

- Get the fixed lookback call option value when the node of the underlying price is no higher than the barrier. (value from a close-form or another normal binomial tree)
- Get the final payoff of all the nodes;
- Backward all the option values for nodes;
- Until the first node is calculated, and return the value;

Still one thing to mention, for barrier option, it's better to make the node just on the barrier and use interpolation to get the initial value. However, in our case, owing to the exponential binomial tree, it is not easy to ensure the barrier is just on the node before and after the dividend payment. Thus, we do not develop such improvement.

Monte-Carlo method

Besides PDE, CBBC can also be priced with risk neutral valuation, as follows:

$$V(S,t) = E_t^*\Big[e^{-r(T_{i+2}-t)}\, max\big(min_{[t,T_{i+2}]}(S) - K, 0\big)\, 1_{\{T_{i+2}\leq T\}} \\ + e^{-r(T-t)} max(S_T - K, 0) 1_{\{T_{i+2}>T\}}\Big],\ \ T_i \leq t \leq T_{i+1}$$

(*)

τ: the first time when stock price touches the call price.

Hence, in terms of the above formula, Monte-Carlo method and binomial tree method can be developed to price the CBBC. This report mainly focuses on such two methods in details.

This is a path-dependent option with a discrete dividend payment during the process, we assume the underlying price follow geometric Brownian motion, and a dividend payment on the time t_d. That is,

$$S_t = S_{t-\Delta t} exp\ ((r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t})$$

$$S_{t_d-} = S_{t_d+} + D$$

Where,

D (dividend payment) = 0.4 per share in our case;

t_d (time to pay dividend) = 23/08/2019 in our case;

There are two kinds of situations:

1) When the call price is hit, then the option value is just the discounted value of the fixed lookback call option (which can also be simulated together or using the analytical solution);

2) When the call price is not hit during the whole process, the option value is the discounted value of the terminal condition (payoff).

The pseudo code for the CBBC pricing with Monte-Carlo method is as follows:

- Initialize the input parameters:

  sig(sigma), r, S_0, T, D, t_d

  K, C, $T_{i+2}\ for\ T_i \leq t < T_{i+1}$

  Nt (steps of T), $\Delta t = \frac{T}{Nt}$

  Ns (number of samples)

  V (vector of final option value)

- For loop Ns

  - Generate an underlying price vector $(Nt + 1)$ one by one, if $t \geq t_d$, dividend D should be subtracted first;

- For loop Nt
  - ✓ If underlying price hit C (callable price) at time $i\Delta t$, then continue to T_{i+2} the payoff is $max(min(S_{i*\Delta t} \sim S_{T_{i+2}}) - K, 0)$, and discount it to initial time;
  - ✓ If underlying price never hit, then the final payoff is $max(S_T - K, 0)$, and discount it to initial time;
- Store the result into vector V;

➢ Return the mean of the vector V;

Numerical results

Lattice (implicit): 1.7642

Monte-Carlo (10000 times): 1.7882

The difference between the results of lattice and Monte-Carlo method is about 0.024.

Some comments

Input parameters:

Before explain the methods in detail, some vital input parameters should be discussed first.

1) Volatility

Both the two methods require volatility. There are many kinds of approaches to derive different kinds of volatility. Here, implied volatility derived from Black-Scholes Merton Model is used. Basically, there are two approaches:

Firstly, as known that N-type CBBC is just like barrier option which has a close-form solution under the scheme of Black-Scholes Merton Model, then we can get the quoted price of N-type CBBC (same underlying) with different strike prices but similar time to maturity with the given CBBC. Using the quoted price with the close-form solution to derive the implied volatility. The quoted data could be found here.[2]

Secondly, the underlying stock (883.hk) not only has derivatives like CBBC, but also warrant, similar with options. Then using quoted price of the warrant with the same underlying to derive the implied volatility is also a choice. The quoted data could be found here.[3]

2) Dividend payment

Dividend payment impacts the pricing result, we refer to the history of dividend payment of the underlying stock, and find that there will probably a dividend payment with around 0.4 per share around 23/08/2019[4]. Thus, such discrete dividend payment should be considered in Monte-Carlo method and binomial tree method.

A connection condition should be added to the above PDE, as follows:

$$V(S, t_d -) = V(S - D, t_d +)$$

Moreover, when doing Monte-Carlo simulation and binomial tree, such discrete dividend payment should also be considered.

3) Interest rate

Interest rate is also an important input parameter for our model. Here, a T-bill rate for 26 weeks is regarded as interest rate.

Risk management:

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[2] http://www.quamnet.com/Quote.action?quoteSectionCode=warrants&stockCode=883
[3] http://www.quamnet.com/Quote.action?quoteSectionCode=warrants&stockCode=883
[4] http://www.aastocks.com/tc/stocks/analysis/dividend.aspx?symbol=00883

Since such CBBC is likely a knock out call option, then we can use underlying product to make a dynamic delta hedging, say we short delta share of underlying stock. Such approach is better for N-type CBBC, but still works for R-type CBBC when the underlying price is far from the call price, however, when the underlying price is near call price, it may not work. Besides, we can also use call-put parity and static hedging.

Other CBBCs:

CBBC on HSI is almost similar with the above one. However, there are still several things different:

1) CBBC on HSI is considered to have a continuous dividend payment, then the risk-neutral valuation formula should be modified;

2) Since there are more derivatives on HSI, then the implied volatility from its quoted price is more reliable;

3) Also, thanks to the above reason, there are more approaches to choose to hedge the exposure of the long position of CBBC on HSI.

Mortgage Backed Securities

Most mortgage loans contain the embedded prepayment privilege that gives the mortgagor the right to terminate the contract prematurely by paying the remaining principal plus any applicable transaction costs. Mortgage-backed securities (MBS) are created by pooling together many individual mortgages. Investors then buy a piece of this pool and in return get the sum of all the interest and principal payments. These MBSs can often then be bought and sold through a secondary market. By buying into this pool of mortgages the investor gets a stake in the housing-loan market, but with less of a prepayment risk. Most prepayers do not act "rationally" on an individual basis, but when there are a lot of them the "average" prepayment can be considered. This is rather like diversification. Thus, we can consider the average behavior of the borrower.

About mortgage backed securities

Parameters for mortgage backed securities (MBS):

r: interest rate with the risk neutral process: $dr = k(\mu - r)dt + \sigma\sqrt{r}dB_t^*$

$V(r,t)$: value of the mortgagor's liability

$M(r,t)$: value of a mortgage backed securities

$P(t)$: the remaining outstanding principal of the mortgage loan

$\varphi(t)$: payout upon prepayment, $\varphi(t) = P(t)(1+X)$, here, X is the proportional factor of transaction cost

C: payment in continuous-time version

dJ_t: models the exogenous prepayment:

$$dJ_t = \begin{cases} 0, & if\ exogenous\ prepayment\ does\ not\ occur, with\ prob\ 1-\lambda dt \\ 1, & if\ exogenous\ prepayment\ occurs, with\ prob\ \lambda dt \end{cases}$$

dL_t: models the endogenous prepayment:

$$dL_t = \begin{cases} 0, & if\ endogenous\ prepayment\ does\ not\ occur, with\ prob\ 1 - \rho dt \\ 1, & if\ endogenous\ prepayment\ occurs, with\ prob\ \rho dt \end{cases}$$

Model for pricing MBS

A. Price $V(r,t)$

i) **With optimal payment:**

$$E^*[dV] \geq rVdt,$$

then

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2} + k(\mu - r)\frac{\partial V}{\partial r} - rV + C \geq 0.$$

Denote

$$LV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2} + k(\mu - r)\frac{\partial V}{\partial r} - rV.$$

We have:

$$max\{-LV - C, V - \varphi\} = 0$$

$$subject\ to\ V(r,T) = 0.$$

ii) **With average behavior:**

We present Stanton's model for MBS (Stanton, 1995).

In the risk neutral world:

$$dV = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2} + k(\mu - r)\frac{\partial V}{\partial r} - rV + C\right]dt + \sigma\sqrt{r}\frac{\partial V}{\partial r}dB_t^*$$

$$+ [\varphi(t) - V]dJ_t - [V - \varphi(t)]^+ dL_t.$$

By $E^*[dV] = rVdt$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 V}{\partial r^2} + k(\mu - r)\frac{\partial V}{\partial r} - rV + C + \lambda[\varphi(t) - V] - \rho[V - \varphi(t)]^+ = 0$$

$$subject\ to\ V(r, T) = 0\ and\ \lim_{r\to\infty} V(r, T) = 0.$$

B. Price $M(r, t)$ with $V(r, t)$ by average behavior

In the risk neutral world:

$$dM = \left[\frac{\partial M}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 M}{\partial r^2} + k(\mu - r)\frac{\partial M}{\partial r} - rM + C\right] dt + \sigma\sqrt{r}\frac{\partial M}{\partial r} dB_t^* + [P(t) - M]dJ_t + [P(t) - M]I_{\{V>\varphi\}}dL_t.$$

By $E^*[dM] = rMdt$:

$$\frac{\partial M}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 M}{\partial r^2} + k(\mu - r)\frac{\partial M}{\partial r} - rM + C + \lambda[P(t) - M] + \rho[P(t) - M]I_{\{V>\varphi\}} = 0$$

$$subject\ to\ M(r, T) = 0\ and\ \lim_{r\to\infty} M(r, T) = 0.$$

Guaranteed Minimum Withdrawal Benefit

The following is the basic function for guaranteed minimum withdrawal benefit (GMWB). The policyholder of the GMWB initially pays w_0 (e.g. \$100) to the issuer for the benefit with maturity T (e.g. 10 years). Then the holder gets an investment account W_t with the initial endowment w_0, investing in some asset S_t. At the end of each year the policyholder will withdraw \$10 (for example) that will be deducted from the account if there is still balance in the account. If $W_t \le 0$ for some t, the account will be closed, but the withdrawal will be guaranteed by the policy issuer until the maturity. If $W_T > 0$ (after final withdrawal), then the balance will be returned to the policyholder. The issuer will charge some insurance fee for taking a risk.

About guaranteed minimum withdrawal benefit

We present the model for dynamic product with continuous withdrawal (Dai, Kwok and Zong, 2008).

Parameters for guaranteed minimum withdrawal benefit:

S_t: invested asset with the process: $dS_t = \mu S_t dt + \sigma S_t dB_t$

G: a given withdrawal rate

$\gamma(t)$: the policyholder chooses to withdraw at rate $\gamma(t)$. There is a penalty charge of k when $\gamma(t) > G$. Having the penalty, during $[t, t+dt]$, the policyholder can only get $[G + (1-k)(\gamma - G)]dt$.

Γ_t: the accumulative withdrawal amounts.

$F(d\Gamma_t)$: the amount the policyholder can receive:

$$F(d\Gamma_t) = \begin{cases} d\Gamma_t, & if\ d\Gamma_t \le Gds \\ Gds + (1-k)(d\Gamma_t - Gds), & if\ d\Gamma_t > Gds \end{cases}$$

α: the proportion of the insurance charge

W_t: investment account. With continuous withdrawal, under risk-neutral world, $dW_t = (r-\alpha)W_t dt + \sigma W_t dB_t^* - d\Gamma_t$ if $W_t > 0$

dA_t: the guaranteed balance with $dA_t = -d\Gamma_t$ and $A_0 = w_0$

$V(W_t, A_t, t)$: price of the dynamic GMWB

Model for pricing GMWB

$$V(W_t, A_t, t) = \max_{\Gamma_s} E_t^*[e^{-r(T-t)}W_T^+ + \int_t^T e^{-r(s-t)}F(d\Gamma_s)]$$

Consider $\gamma \in [0, \lambda]$ for λ big enough, denote the associated value function as $\bar{V}(W_t, A_t, t)$, then:

$$dW_t = (r-\alpha)W_t dt + \sigma W_t dB_t^* - \gamma dt \text{ if } W_t > 0$$

$$\bar{V}(W_t, A_t, t) = \max_{\gamma_s} E_t^*[e^{-r(T-t)}W_T^+ + \int_t^T e^{-r(s-t)}f(\gamma_s)\, ds]$$

here

$$f(\gamma) = \begin{cases} \gamma, & if\ \gamma \le G \\ G + (1-k)(\gamma - G), & if\ \gamma > G \end{cases}.$$

By Feynman-Kac formula, the $\bar{V}(W_t, A_t, t)$ follows:

$$\max_{\gamma} \left\{ \frac{\partial \bar{V}}{\partial t} + \frac{1}{2}\sigma^2 W^2 \frac{\partial^2 \bar{V}}{\partial W^2} + (r-\alpha)W\frac{\partial \bar{V}}{\partial W} - r\bar{V} - \gamma\frac{\partial \bar{V}}{\partial W} - \gamma\frac{\partial \bar{V}}{\partial A} + f(\gamma) \right\} = 0$$

Denote

$$L\bar{V} = \frac{\partial \bar{V}}{\partial t} + \frac{1}{2}\sigma^2 W^2 \frac{\partial^2 \bar{V}}{\partial W^2} + (r-\alpha)W\frac{\partial \bar{V}}{\partial W} - r\bar{V},$$

$$g(\gamma) = -\gamma\frac{\partial\bar{V}}{\partial W} - \gamma\frac{\partial\bar{V}}{\partial A} + f(\gamma) = \begin{cases} \left(1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)\gamma, & if\ \gamma \le G \\ kG + \left(1 - k - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)\gamma, & if\ \gamma > G \end{cases}.$$

Then

$$L\bar{V} + \max_{\gamma} g(\gamma) = 0.$$

We have

$$\max_{\gamma} g(\gamma) = \begin{cases} kG + \left(1 - k - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)\lambda, & when\ 1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A} > k \\ \left(1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)G, & when\ 0 \le 1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A} \le k \\ 0, & when\ 1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A} < 0 \end{cases}$$

$$= \min\left\{\left(1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)^{+}, k\right\}G + \lambda\left(1 - k - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)^{+},$$

then

$$L\bar{V} + \min\left\{\left(1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)^{+}, k\right\}G + \lambda\left(1 - k - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)^{+} = 0.$$

Let λ go to infinity. Finally, we arrive at

$$\min\left\{-LV - \left(1 - \frac{\partial\bar{V}}{\partial W} - \frac{\partial\bar{V}}{\partial A}\right)^{+} G, \frac{\partial\bar{V}}{\partial W} + \frac{\partial\bar{V}}{\partial A} - (1-k)\right\} = 0,$$

$$W > 0,\ \ t \in [0,T),\ \ A \in (0, w_0)$$

$$subject\ to\ V(W,A,T) = max(W, (1-k)A),\ \ V(W,0,t) = e^{-\alpha(T-t)}W,$$

$$V(W,A,t) \to e^{-\alpha(T-t)}W\ as\ W \to \infty,\ \ V(0,A,t) = V_0(A,t).$$

Here, for $V_0(A,t)$, we have

$$min\left\{-\frac{\partial V_0}{\partial t}+rV_0-\left(1-\frac{\partial V_0}{\partial A}\right)^+ G,\frac{\partial V_0}{\partial A}-(1-k)\right\}=0\,,\quad t\in[0,T),\quad A\in(0,w_0)$$

$$subject\ to\ V_0(A,T)=(1-k)A,\quad V_0(0,t)=0.$$

CONCLUSIONS

In this project report, we discuss the pricing of various structured products in the financial markets, including SFD500, dual-purpose fund, convertible bonds and callable bull/bear contract. With more details about the structured products mentioned above, the discussion of the pricing models and numerical methods, numerical results and relevant code are provided. In addition, we discuss the computation of VIX, and the pricing models for mortgage backed securities and guaranteed minimum withdrawal benefit. The pricing models in this report are mainly under Black-Scholes framework. For extension, the pricing models beyond Black-Scholes framework could be considered.

APPENDIX

Code (R language) for computing SPX-VIX and pricing SFD500, dual-purpose fund, convertible bond and callable bull/bear contract (based on the methods mentioned in this report) is available at https://shiyu23.github.io.

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