Optimal Investment and Consumption Decision with Behavioral Finance

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Abstract

Investors hold the portfolio of a risky stock and a risk-free bond, decide how to make the portfolio choice and consumption decision till the wealth goes to zero. Different investment and consumption strategies generate different value for investors. Under classical framework, we consider to maximize the total discounted expected utility of consumption with assumption that investors are rational. Under behavioral finance direction, behavioral pattern which is not considered under classical framework is the psychological aspect to influence the decision. In this paper, we consider the problem of investment and consumption decision with the psychological factors of loss-aversion and reference point adaptation. How investors' decision will be affected by these psychological factors. And what is the difference between the result which takes these factors into consideration and the result which does not. Numerical examples are given to illustrate the results.

Keywords: optimal control, loss-aversion, reference point adaptation, Hamilton-Jacobi-Bellman equation, constrained viscosity solution

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1. Introduction

This paper contributes to the problem of optimal investment and consumption decision under behavioral finance direction. The fundamental stochastic model of optimal investment and consumption decision was firstly introduced by Merton (1969). Many new works on this subject were initiated after then. Under behavioral finance direction, some works took risk aversion into consideration for optimal investment and consumption problem. In this paper, we extend this problem in behavioral finance direction, take the psychological effects of loss-aversion and reference point adaptation into consideration, and study how these psychological factors affect investors' choice.

Behavioral finance challenges some classical assumptions on the investors. Von Neumann and Morgenstern (1944) developed Expected Utility Theory (EUT) which assumes that: (1) investors evaluate wealth according to final asset position; (2) they are uniformly risk averse; (3) they objectively evaluate probabilities. But these assumptions are not realistic, for these reasons (see [3]): (1) investors evaluate wealth according to gains and losses with respect to a reference point; (2) investors are risk-averse on gains, risk-taking on losses and significantly more sensitive to losses than to gains; (3) investors overweight small probabilities and underweight large probabilities.

The rapid development of the field of behavioral finance shows that the effect of how investors behave is considerable. There have already some researches focused on financial problems with psychological factors (e.g. I. Fortin and J. Hlouskova (2010) studied optimal asset allocation under linear loss aversion; V. Zakamouline (2014) studied portfolio performance evaluation with loss aversion; Y. Shi, X. Cui, J. Yao and D. Li (2015) studied dynamic trading with reference point adaptation and loss aversion). In this paper, we assume investors are affected by loss aversion and reference point adaptation. And we extend the problem of investment and consumption decision with this psychological setting.

This paper is organized as follow: in the section 2, we show the basic model for optimal investment and consumption problem, remodel it under the assumption of loss-aversion and reference point adaptation, show the HJB equation for this problem and the candidates for optimal investment and consumption policies. In the section 3, we do numerical simulation to get the numerical solution, and analyze how the psychological factors affect the investors' decision. In the section 4, we draw a conclusion. In the section 5, we provide a detailed derivation for our problem.

2. The Problem

2.1 Problem for Rational Investors

Firstly, we consider the classical optimal investment and consumption decision model. We assume the dynamic of the bond price B_t and stock price S_t as:

$$dB_t = rB_t d\mathrm{t},\ \ B_0 = b$$

$$dS_t = \mu S_t d\mathrm{t} + \sigma S_t dW_t,\ \ S_0 = s$$

Here, $(W_t)_{t\geq 0}$ is standard one-dimensional Brownian motion defined on a probability space (Ω, F, P), r is the risk-free rate, μ is the expected excess return, σ is the volatility of the stock. Here r, μ and σ are all constants with $r < \mu$ and $\sigma > 0$. The (π_t^0, π_t) are fractions of wealth invested on the bond and stock respectively with equation $\pi_t^0 + \pi_t = 1$. The total wealth of investors X_t satisfied:

$$dX_t = X_t[(r + \pi_t(\mu - r))dt + \sigma\pi_t dW_t] - c_t dt,\ \ X_0 = x$$

The consumption rate c_t satisfied $c_t \geq 0$. The pair (π_t, c_t) represent the control variables. When X_t reaches zero, the process stops. For the rational investors who do not affected by their self's expectation, their purpose is to maximize this objective function:

$$J(x; \pi, c) = E\left[\int_0^\tau e^{-rs} l(c_t) ds\right],\ \ where\ \tau = inf\{t > 0: X_t = 0\}$$

Where $l(c)$ is the utility rate of consuming at rate $c \geq 0$.

2.2 Loss-Aversion

In this paper, we impose the effect of the loss-aversion to describe how investors evaluate their utility. Usually, the effect of investors' behavior pattern cannot be ignored.

For this problem, the investors who plan to maximize the total discounted expected utility based on EUT are treated as rational investors. But many investors are short-sighted, they have expectation for their investment and consumption at future time. And how they treat the value of their investment can be affected by the difference between the actual wealth and their expectation. The utility of consumption is also affected by investors' expectation, then the decision for consumption rate is affected. Even for the relatively rational investors, their utilities are hard to get rid of the effect of their expectation. In this paper, we assume that investors evaluate wealth according to gains and losses with respect to a reference point. The effect of loss-aversion is to say that the increment of value for gain is significantly less than the same size reduction of value for loss in investors' mind. We consider the problem of investment and consumption decision under this mindset of investors.

2.2.1 Reference Point Adaptation

Reference point is interpreted as the expectation of the investors. In our problem, the reference point at time t is the investors' expectation for their investment at time $t + \Delta t$. We assume that the reference point is updated with time.

HR. Arkes, D. Hirshleifer, D. Jiang and S. Lim (2008) proposed a partial and asymmetric rule for reference point updating (see [10]): (1) Investors on average tend to shift reference points upward after prior gains, and the updating is usually partial (i.e., the new reference point lies between the realized wealth and the previous reference point); (2) Investors on average tend to shift reference points downward after prior losses, and the updating is usually insufficient; (3) The magnitude of reference point adaptation is significantly greater following a gain than following a loss of equal size (asymmetric adaptation).

The idea for continuous reference point adaptation comes from a discretized form of reference point adaptation proposed by Y. Shi, X. Cui, J. Yao and D. Li (see [9]). The change of the reference point at time $t + \Delta t$ is based on the difference between the actual wealth and the expected wealth.

The reference point X^{rf} updated follow this dynamic:

$$dX_t^{rf} = \begin{cases} \alpha_g(X_t - X_t^{rf})dt - c_t dt, & X_t \geq X_t^{rf} \\ \alpha_l(X_t - X_t^{rf})dt - c_t dt, & X_t < X_t^{rf} \end{cases}, \quad X_0^{rf} = x^{rf}$$

Here the parameters satisfy $1 > \alpha_g > \alpha_l > 0$ corresponding to the partial and asymmetric rule for reference point updating.

Equivalently, the dynamic of X^{rf} is as follow:

$$dX_t^{rf} = [(\alpha_g - \alpha_l)I\{X_t \geq X_t^{rf}\} + \alpha_l](X_t - X_t^{rf})dt - c_t dt, \quad X_0^{rf} = x^{rf}$$

The X_t^{rf} being larger means the investors being more optimistic with their investment at time *t*. And the optimistic or pessimistic emotion for the investment will affect the expectation for consumption. More precisely, being more (less) optimistic will lead to larger (smaller) expectation for consuming. We denote the reference point for consuming rate as C^{rf}, and suppose there is a relationship between X^{rf} and C^{rf}:

$$C^{rf} = kX^{rf}, \quad k \in [0, +\infty)$$

The parameter k reflects the preference of investors to consume how much from their wealth. Note that when k equals to zero, the problem is reduced to the problem of optimal investment and consumption decision with investors' wealth evaluation based on asset position, not gains and losses.

We can also assume other rational relationship between X^{rf} and C^{rf} with the general form: $C^{rf} = f(X^{rf})$. Here function $f(\cdot)$ is increasing with $f(0) = 0$. That may improve the model to be closer to reality.

2.2.2 Utility Rate of Consuming

We use the utility rate of consuming $l(c)$ explicitly as following form with $\lambda \in (1, +\infty]$:

$$l = \begin{cases} c - C^{rf}, & c \geq C^{rf} \\ -\lambda(C^{rf} - c), & c < C^{rf} \end{cases} = [(1+\lambda)I\{c \geq C^{rf}\} - \lambda]\left|c - C^{rf}\right|$$

Here λ is the parameter to characterize loss-aversion. The λ being larger means the investors averse loss more compared to the same size gain. The parameter λ being one means there is no loss-aversion on investors.

For control variables $(\pi.,c.)$, we restrict them as $\pi_{min} \leq \pi \leq \pi_{max}$ and $0 \leq c \leq c_{max}$. We denote the set of all admissible controls as $A(x, x^{rf})$ which satisfies:

$$A(x, x^{rf}) = \{(\pi., c.): F_t\ adapted;\ (\pi., c.) \in [\pi_{min}, \pi_{max}] \times [0, c_{max}];\ X_t \in (0, +\infty),\ \ P - a.s.\}$$

Then the problem is called a state constraints problem:

$$v(x, x^{rf}) = max_{(\pi.,c.)\in A} E\left[\int_0^\tau e^{-rs}[(1+\lambda)I\{c_s \geq C_s^{rf}\} - \lambda]\left|c_s - C_s^{rf}\right| ds\right]$$

The Hamilton-Jacobi-Bellman equation (HJB equation) for this problem is:

$$\begin{aligned} -rv + xrv_x + \left[(\alpha_g - \alpha_l)I\{x \geq x^{rf}\} + \alpha_l\right](x - x^{rf})v_{x^{rf}} \\ + max_{\pi\in[\pi_{min},\pi_{max}]}\left\{\pi x(\mu - r)v_x + \frac{1}{2}\pi^2\sigma^2x^2v_{xx}\right\} \\ + max_{c\in[0,c_{max}]}\{[(1+\lambda)I\{c \geq kx^{rf}\} - \lambda]\left|c - kx^{rf}\right| \\ - c\left(v_x + v_{x^{rf}}\right)\} = 0 \end{aligned}$$

With the Dirichlet boundary condition:

$$v|_{x=0} = 0$$

We have the candidates for optimal investment and consumption policies as following:

$$\pi = arg_{\pi_{min},\pi_{max},\frac{-(\mu-r)v_x}{\sigma^2 x v_{xx}}\in[\pi_{min},\pi_{max}]} \left\{max\left[\pi x(\mu - r)v_x + \frac{1}{2}\pi^2\sigma^2 x^2 v_{xx}\right]\right\}$$

$$c = arg_{0,c_{max},kx^{rf}\in[0,c_{max}]} \left\{max\left[[(1+\lambda)\mathrm{I}\{c \geq kx^{rf}\} - \lambda]\left|c - kx^{rf}\right| - c\left(v_x + v_{x^{rf}}\right)\right]\right\}$$

3. Numerical Results

The domain for numerical scheme is $[0, x_{max}] \times [x_{min}^{rf}, x_{max}^{rf}]$. In our numerical simulation, we set $x_{max} = x_{max}^{rf}$. Denote V as the numerical solution. When $x^{rf} = x_{min}^{rf}$, the investors are extremely pessimistic initially. The actual initial wealth shows striking contrast to the expectation. In this case, we assume investors consume with max consuming rate $c_{\cdot} = c_{max}$ and put the wealth on risk-free asset with maximum $\pi_{\cdot}^{0} = \pi_{max}^{0}$, since investors are risk-averse on gains. When $x = x_{max}$ with a large enough x_{max}, investors' wealth is big enough. In this case, we assume investors consume with max consuming rate $c_{\cdot} = c_{max}$ and put the wealth on risk-free asset with maximum $\pi_{\cdot}^{0} = \pi_{max}^{0}$, since investors are risk-taking on losses and risk-averse on gains, same with previous case:

$$V|_{x=0} = 0$$

$$V|_{x=x_{max}} = v_1(x = x_{max}, x^{rf}; \pi_{\cdot} = \pi_{min})$$

$$V|_{x^{rf}=x_{min}^{rf}} = v_1\left(x, x^{rf} = x_{min}^{rf}; \pi_{\cdot} = \pi_{min}\right)$$

Here v_1 is the solution of this problem, we solve it by Monte-Carlo method later:

$$v_1(x, x^{rf}; \pi_{\cdot} = \pi_*) = E\left[\int_0^{\tau} e^{-rs}\left[(1+\lambda)I\left\{c_{max} \geq kX_s^{rf}\right\} - \lambda\right]\left|c_{max} - kX_s^{rf}\right| ds\right],$$

$$where\ \tau = inf\{t > 0: X_t = 0\}$$

$$dX_t = X_t[(r + \pi_*(\mu - r))dt + \sigma\pi_* dW_t] - c_{max}dt,\ \ X_0 = x$$

$$dX_t^{rf} = \left[(\alpha_g - \alpha_l)I\left\{X_t \geq X_t^{rf}\right\} + \alpha_l\right](X_t - X_t^{rf})dt - c_{max}dt,\ \ X_0^{rf} = x^{rf}$$

Substitute the candidates for optimal investment and consumption policies, we write

HJB equation in this form:

$$
\begin{aligned}
-rv + xrv_x &+ \left[(\alpha_g - \alpha_l) I\{x \geq x^{rf}\} + \alpha_l\right](x - x^{rf}) v_{x^{rf}} \\
&+ max_{\pi \in [\pi_{min}, \pi_{max}]} \Big\{ \pi_{min} x(\mu - r) v_x \\
&+ \frac{1}{2} {\pi_{min}}^2 \sigma^2 x^2 v_{xx},\ \ \pi_{max} x(\mu - r) v_x \\
&+ \frac{1}{2} {\pi_{max}}^2 \sigma^2 x^2 v_{xx},\ \ \frac{-(\mu - r)^2 {v_x}^2}{2\sigma^2 v_{xx}} \Big\} \\
&+ max_{c \in [0, c_{max}]} \Big\{ -\lambda\left|-kx^{rf}\right|,\ \ \left[(1+\lambda) I\{c_{max} \geq kx^{rf}\} - \lambda\right] \Big| c_{max} \\
&- kx^{rf} \Big| - c_{max}\left(v_x + v_{x^{rf}}\right),\ \ -kx^{rf}\left(v_x + v_{x^{rf}}\right) \Big\} = 0
\end{aligned}
$$

We use finite difference method to solve the HJB equation numerically, use upwind technique to discretize first order derivative of v with respect to x and x^{rf}, and use iteration to deal with the nonlinear part.

For the iteration procedure, we use an approximation algorithm (see [12]) proposed by HP. Peyrl, FA. Herzog and HP. Geering (2005). First, we guess the control variables $(\pi., c.)^{(0)} = (\pi_{min}, c_{max})$, denote the *n*th control variables as $(\pi., c.)^{(n)}$, fix them and solve the HJB equation $v^{(n)}$, then calculate the new control variables $(\pi., c.)^{(n+1)}$ with:

$$
\begin{aligned}
\pi^{(n+1)} = arg_{\pi_{min}, \pi_{max}, \frac{-(\mu - r) {v_x}^{(n)}}{\sigma^2 x {v_{xx}}^{(n)}} \in [\pi_{min}, \pi_{max}]} &\Big\{ max \Big[\pi x(\mu - r) {v_x}^{(n)} \\
&+ \frac{1}{2} \pi^2 \sigma^2 x^2 {v_{xx}}^{(n)} \Big] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
c^{(n+1)} = arg_{0, c_{max}, kx^{rf} \in [0, c_{max}]} &\Big\{ max \Big[\left[(1+\lambda) \mathrm{I}\{c \geq kx^{rf}\} - \lambda\right] \left|c - kx^{rf}\right| \\
&- c\left({v_x}^{(n)} + {v_{x^{rf}}}^{(n)}\right) \Big] \Big\}
\end{aligned}
$$

For $i = 1, \dots, N_x - 1$ with $N_x dx = x_{max}$ and $j = 1, \dots, N_{x^{rf}} - 1$ with $N_{x^{rf}} dx^{rf} = x^{rf}_{max} - x^{rf}_{min}$, the *n*th discretized HJB equation $v^{(n)}$ is:

$$
\begin{aligned}
-rV^{i,j^{(n)}} &+ xr\frac{V^{i+1,j^{(n)}} - V^{i,j^{(n)}}}{dx} \\
&+ \left[I\{x \geq x^{rf}\}\alpha_g \frac{V^{i,j+1^{(n)}} - V^{i,j^{(n)}}}{dx^{rf}} \right. \\
&\left. + I\{x < x^{rf}\}\alpha_l \frac{V^{i,j^{(n)}} - V^{i,j-1^{(n)}}}{dx^{rf}} \right](x - x^{rf}) \\
&+ \pi^{(n)}x(\mu - r)\frac{V^{i+1,j^{(n)}} - V^{i,j^{(n)}}}{dx} \\
&+ \frac{1}{2}\pi^{(n)^2}\sigma^2 x^2 \frac{V^{i+1,j^{(n)}} + V^{i-1,j^{(n)}} - 2V^{i,j^{(n)}}}{(dx)^2} \\
&+ \left[(1+\lambda)I\left\{c^{(n)} \geq kx^{rf}\right\} - \lambda\right]\left|c^{(n)} - kx^{rf}\right| \\
&- c^{(n)}\left(\frac{V^{i,j^{(n)}} - V^{i-1,j^{(n)}}}{dx} + \frac{V^{i,j^{(n)}} - V^{i,j-1^{(n)}}}{dx^{rf}}\right) = 0
\end{aligned}
$$

When $\left\|V^{(n+1)} - V^{(n)}\right\|_\infty < threhshold$, accept $V^{(n+1)}$ to be the numerical solution of the original HJB equation.

We simulate with this shared parameters setting: $\mu = 0.1$, $r = 0.05$, $\sigma = 0.3$, $x_{max} = 10$, $x_{max}^{rf} = 10$, $x_{min}^{rf} = -5$, $\pi_{min} = 0$, $\pi_{max} = 1$, $c_{max} = 10$, $threhshold = 5 \times 10^{-2}$.

3.1 Effect of Investors' Expectation

First, we see the effect of investors' expectation.

Fig 1. Value surface with psychological factors ($\lambda = 1.5$, $\alpha_g = 0.8$, $\alpha_l = 0.6$, $k = 0.5$)

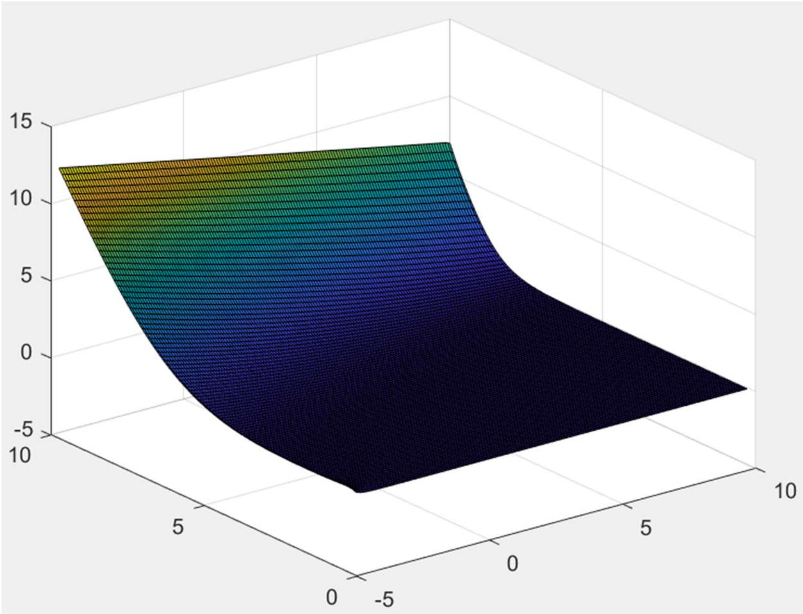
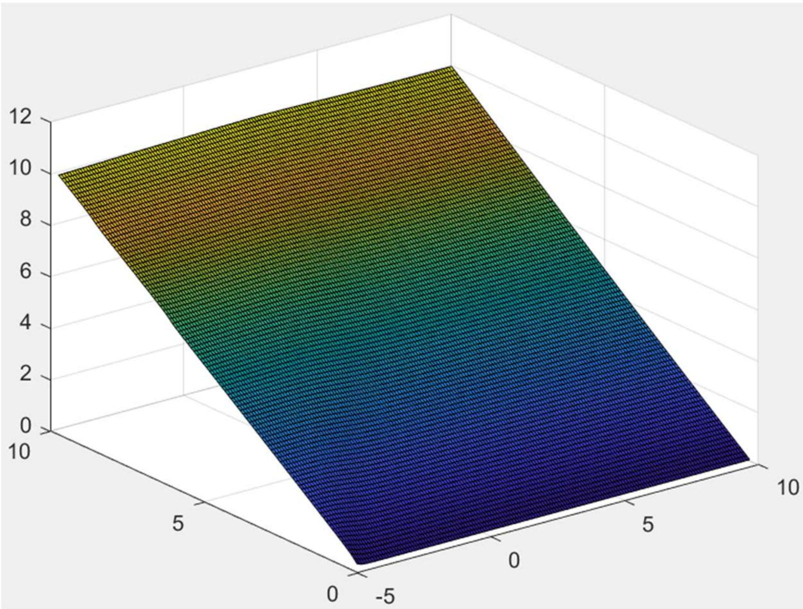


Fig 2. Value surface without Psychological factors ($\lambda = 1$, $\alpha_g = 0$, $\alpha_l = 0$, $k = 0$)



The figure 1 and figure 2 show the value surface with and without considering psychological factors respectively. In figure 1, when the reference point increases, the value decreases. In figure 2, there will be no influence on the value when reference point increases and wealth is fixed.

Fig 3. Value's change with respect to reference point adaptation parameter α_l ($\lambda = 1.5$, $\alpha_g = 0.8$, $k = 0.5$, $x = 1$, $x^{rf} = 1$)

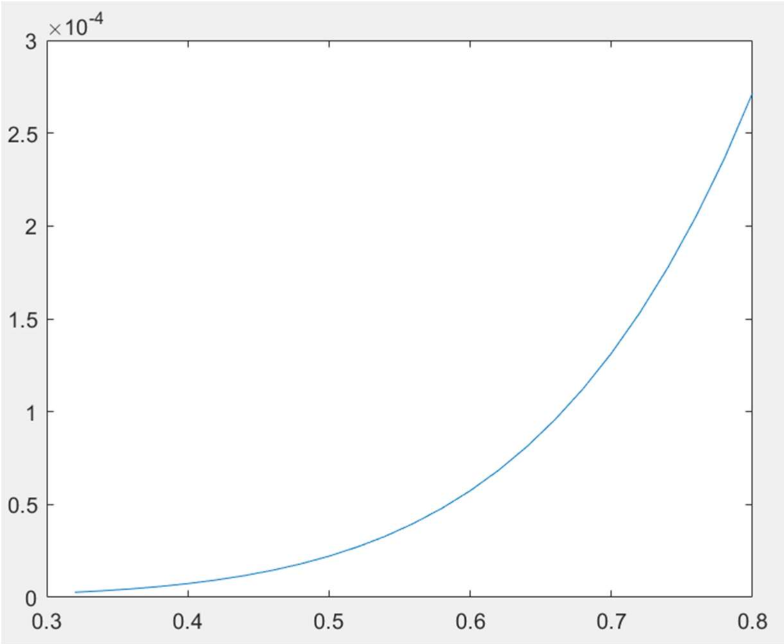


Figure 3 shows the relation between the value and the reference point adaptation parameter α_l. When α_l increase which means investors' expectation changes more drastically in losses, the value increases.

Fig 4. Value's change with respect to loss-aversion parameter λ ($\alpha_l = 0.6$, $\alpha_g = 0.8$, $k = 0.5$, $x = 1$, $x^{rf} = 1$)

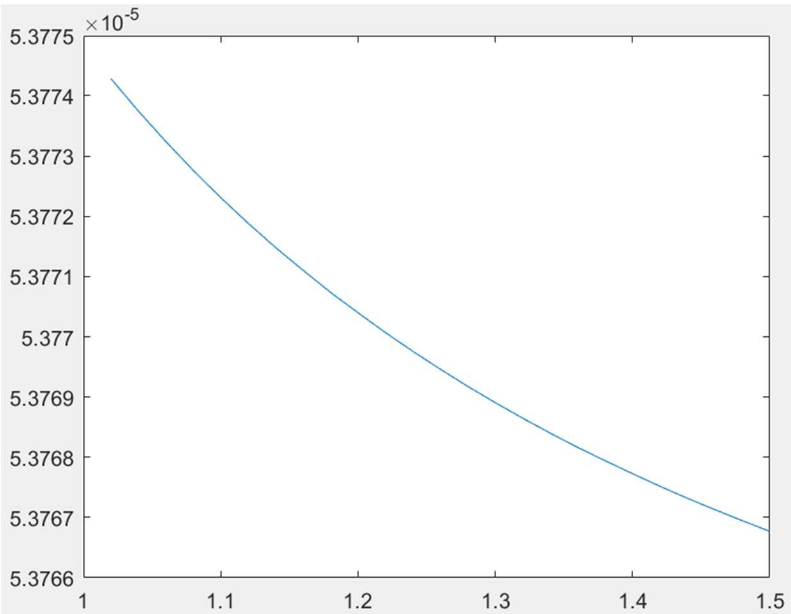
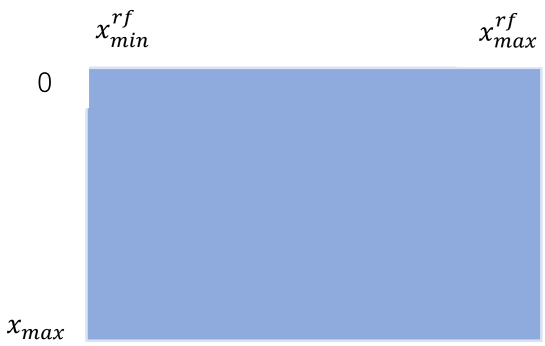


Figure 4 shows the relation between value and the loss-aversion parameter λ. When λ increases, which means investors averse losses more, the value decreases.

For the heatmap figures, the axes are as follow:



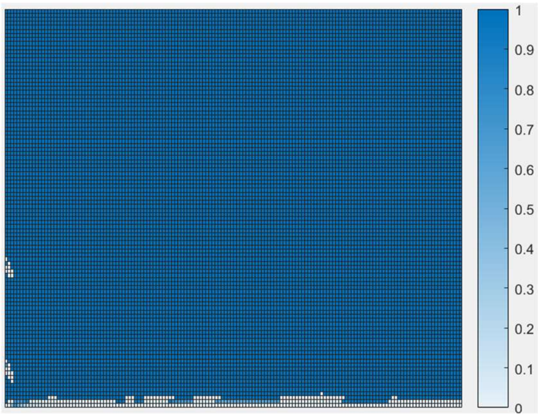


Fig 5. Optimal investment decision without psychological effect

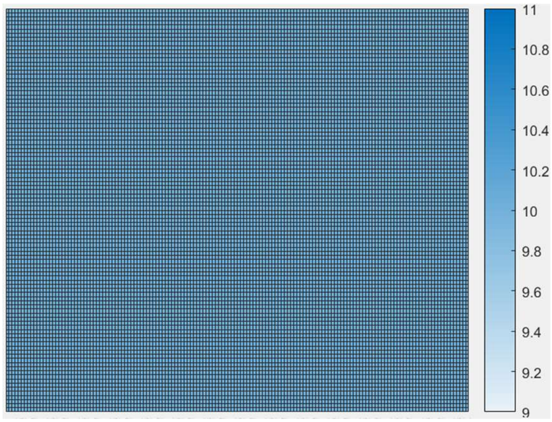


Fig 6. Optimal consumption decision without psychological effect

Figure 5 and 6 show that without psychological effects of loss-aversion and reference point adaptation, the optimal investment and consumption decision is relatively stable in our simulation. When the wealth is extremely large, investors tend to invest risk-free asset corresponding to the assumption that investors are risk-averse on gains.

3.2 Effect of Loss-Aversion

Fix the parameters as $\alpha_g = 0.8$, $\alpha_l = 0.6$, $k = 0.5$, change the loss-aversion parameter λ.

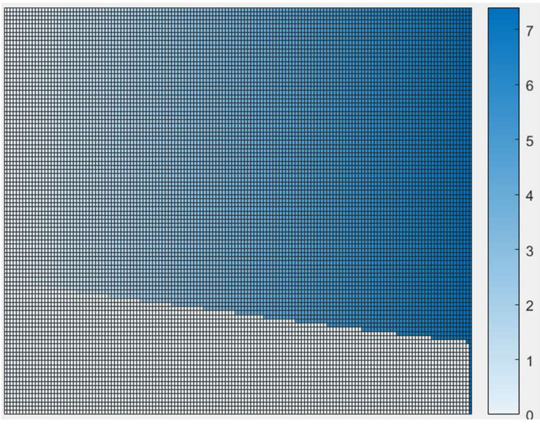


Fig 7. Optimal consumption decision with $\lambda = 1.5$

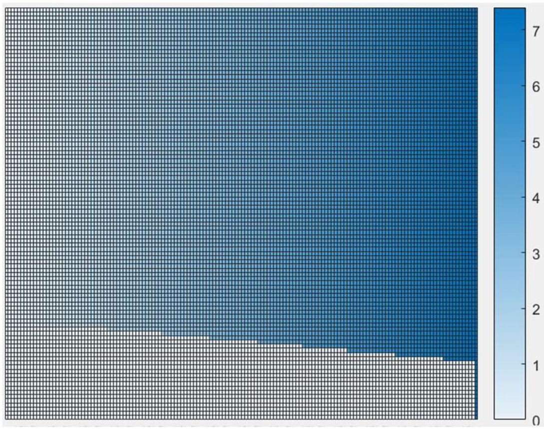


Fig 8. Optimal consumption decision with $\lambda = 2$

Figure 7 and 8 show that when the investors' expectation increases, the optimal consumption increases. That means when the expectation is large, investors tend to spend their wealth in consumption, rather than in investment. When investors' wealth is large enough to an amount, investors stop consume. That means for a fixed expectation, when the wealth is exceeding the expectation, investors tend to put their wealth to invest rather than to consume. When the loss-aversion parameter λ increases, the amount to stop consume increases.

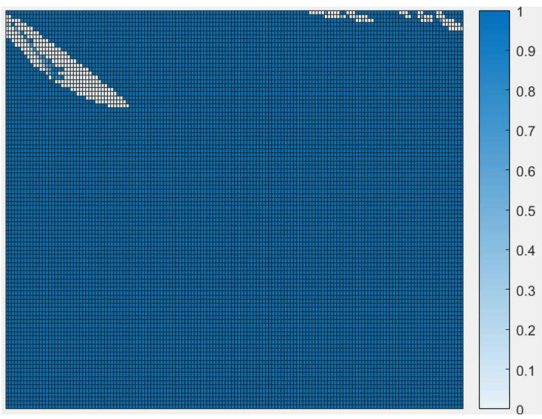


Fig 9. Optimal investment decision with $\lambda = 1.5$

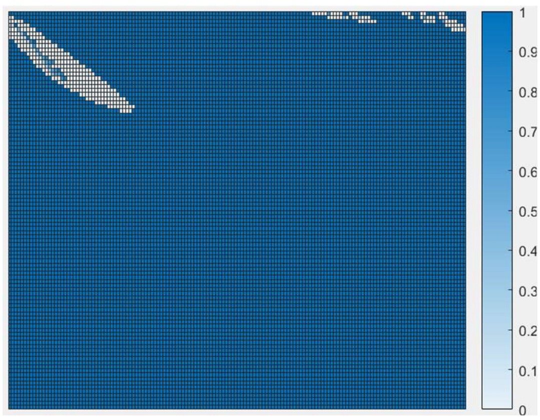


Fig 10. Optimal investment decision with $\lambda = 2$

There is no significant difference in the optimal investment decision when the loss-aversion parameter λ changes. Investors put their wealth on risky asset with maximum proportion in most cases. When it is the case that the wealth is relatively small and is linear with the expectation or the case that the wealth is close to zero and significantly smaller than the expectation, investors tend to invest risk-free asset.

3.3 Effect of Reference Point Adaptation

Fix the parameters as $\lambda = 1.5$, $\alpha_g = 0.8$, $k = 0.5$, change the reference point adaptation parameter α_l.

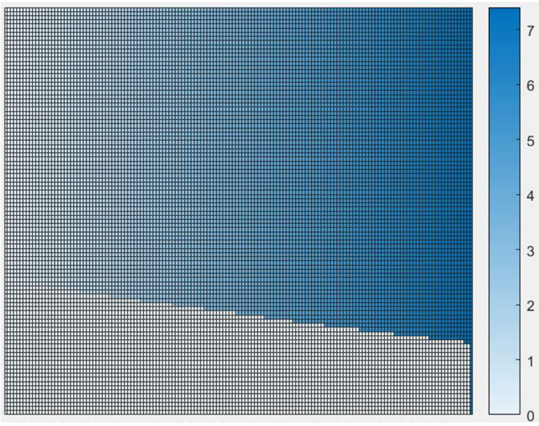


Fig 11. Optimal consumption decision with $\alpha_l = 0.4$

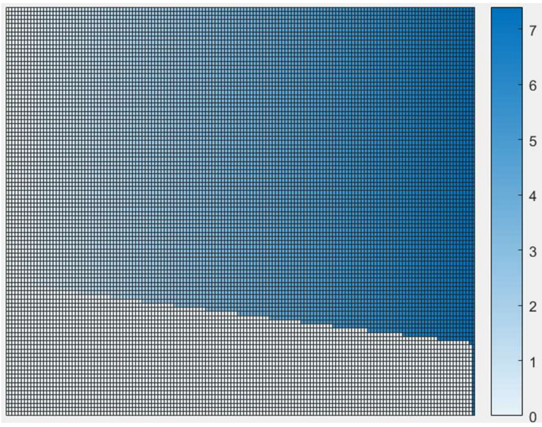


Fig 12. Optimal consumption decision with $\alpha_l = 0.7$

Figure 11 and 12 show that there is no significant difference between the optimal consumption decision when the reference point adaptation parameter α_l increases.

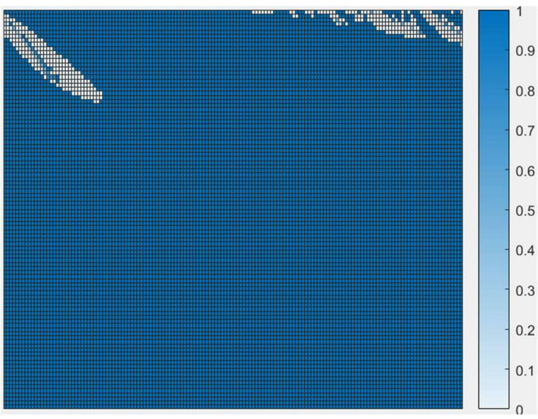


Fig 13. Optimal investment decision with $\alpha_l = 0.4$

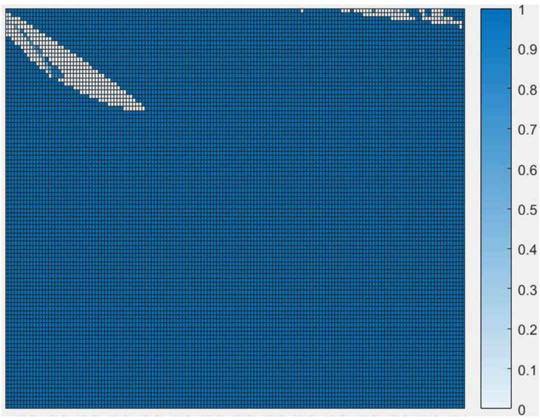


Fig 14. Optimal investment decision with $\alpha_l = 0.7$

Figure 13 and 14 show that in most cases, investors put their wealth on risky asset with maximum proportion. When it is the first case that the wealth is relatively small and is linear with the expectation or the second case that the wealth is close to zero and significantly smaller than the expectation, investors tend to invest risk-free asset. This phenomenon is inconsistent with the assumption that investors are risk-taking on losses and risk-averse on gains. When the reference point adaptation parameter α_l increases, the condition for investors to invest risk-free asset gets relaxed in the first case and gets strict in the second case.

3.4 Effect of Consumption Preference Rate

Fix the parameters as $\lambda = 1.5$, $\alpha_l = 0.6$, $\alpha_g = 0.8$, change the consumption preference rate k which means how much the investors prefer to consume from their wealth.

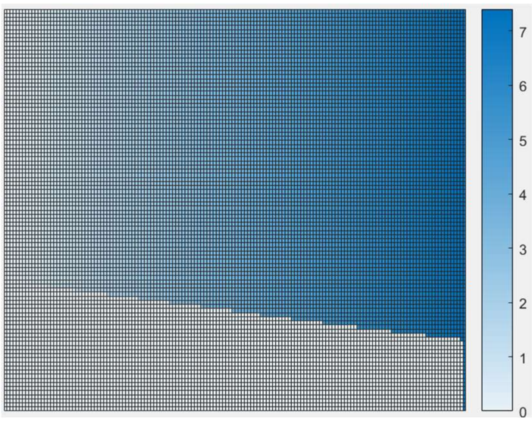


Fig 15. Optimal consumption decision with $k = 0.5$

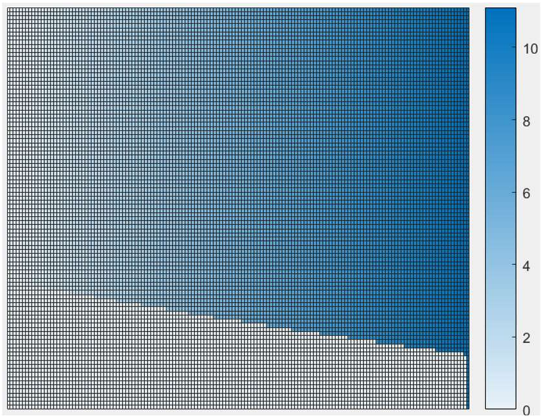


Fig 16. Optimal consumption decision with $k = 0.75$

Figure 15 and 16 show that when investors' wealth is exceeding the expectation and reaches an amount, investors stop consume. When the consumption preference rate k increases, the required wealth for investors to stop consume increases.

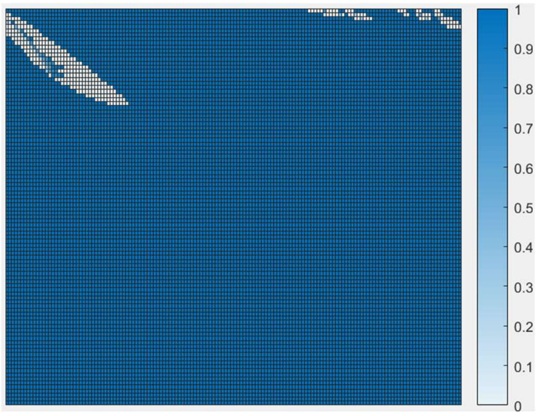


Fig 17. Optimal investment decision with $k = 0.5$

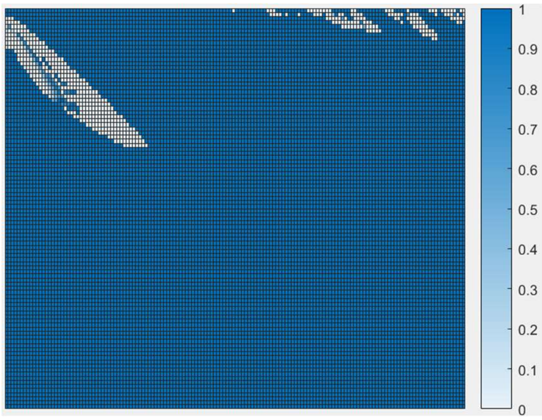


Fig 18. Optimal investment decision with $k = 0.75$

Figure 17 and 18 show that also in most cases, investors put their wealth on risky asset with maximum proportion. When it is the case that the wealth is relatively small and is linear with the expectation or the case that the wealth is close to zero and significantly smaller than the expectation, investors tend to invest risk-free asset. When the consumption preference rate k increases, the condition for investors to put their wealth on risk-free asset gets relaxed.

4. Conclusions

In this paper, we consider the optimal investment and consumption decision problem under behavioural finance's assumption. More specifically, we impose the effect of loss-aversion and reference point adaptation to solve this stochastic optimal control problem. Compare the numerical result under our setting to the result without considering psychological factors. And do the simulation with difference parameters which related to the psychological factors to see how these factors affect investors. Note that we assume the investors are risk-taking on losses and risk-averse on gains when we impose the boundary conditions for numerical scheme, but the numerical result shows that the investors tend to put wealth on risky asset with maximum proportion in most cases, and invest risk-free asset when wealth is close to zero and the expectation is extremely large or when the wealth is relatively small and is linear with the expectation. This is inconsistent with the assumption that investors are risk-taking on losses and risk-averse on gains. This inconsistency provides a direction for further model refinement to make the model more realistic.

5. Appendix

In this section, we derive the solution for the stochastic optimal control problem. First, we derive the HJB for the $v(x, x^{rf})$. By Itö lemma, we have:

$$dv(X_t, X_t^{rf})$$

$$= v_x dX_t + v_{x^{rf}} dX_t^{rf} + v_{xx^{rf}} dX_t dX_t^{rf} + \frac{1}{2}(v_{xx}(dX_t)^2$$

$$+ v_{x^{rf}x^{rf}}\left(dX_t^{rf}\right)^2)$$

$$= v_x(X_t[(r + \pi_t(\mu - r))dt + \sigma\pi_t dW_t] - c_t dt) + v_{x^{rf}}\{[(\alpha_g$$

$$- \alpha_l)I\{X_t \geq X_t^{rf}\} + \alpha_l](X_t - X_t^{rf})dt - c_t dt\} + \frac{1}{2}\pi_t^2\sigma^2 X_t^2 v_{xx} dt$$

By Dynamic Programming Principle (DPP), we derive:

$$0 = max_{(\pi.,c.)\in A}\Big\{-rv + [(1 + \lambda)I\{c \geq kx^{rf}\} - \lambda]\left|c - kx^{rf}\right| + v_x\{x[r + \pi(\mu - r)]$$

$$- c\} + v_{x^{rf}}\left\{\left[(\alpha_g - \alpha_l)I\{x \geq x^{rf}\} + \alpha_l\right](x - x^{rf}) - c\right\}$$

$$+ \frac{1}{2}\pi^2\sigma^2 x^2 v_{xx}\Big\}$$

$$= rv + xrv_x + \left[(\alpha_g - \alpha_l)I\{x \geq x^{rf}\} + \alpha_l\right](x - x^{rf})v_{x^{rf}}$$

$$+ max_{\pi\in[\pi_{min},\pi_{max}]}\left\{\pi x(\mu - r)v_x + \frac{1}{2}\pi^2\sigma^2 x^2 v_{xx}\right\}$$

$$+ max_{c\in[0,c_{max}]}\Big\{[(1 + \lambda)I\{c \geq kx^{rf}\} - \lambda]\left|c - kx^{rf}\right|$$

$$- c\left(v_x + v_{x^{rf}}\right)\Big\}$$

Thus, we get the HJB equation for our problem:

$$-rv + xrv_x + \left[(\alpha_g - \alpha_l)I\{x \geq x^{rf}\} + \alpha_l\right](x - x^{rf})v_{x^{rf}}$$
$$+ max_{\pi\in[\pi_{min},\pi_{max}]}\left\{\pi x(\mu - r)v_x + \frac{1}{2}\pi^2\sigma^2x^2v_{xx}\right\}$$
$$+ max_{c\in[0,c_{max}]}\{[(1+\lambda)I\{c \geq kx^{rf}\} - \lambda]\left|c - kx^{rf}\right|$$
$$- c\left(v_x + v_{x^{rf}}\right)\} = 0$$

The problem $max_\pi\{\cdot\}$ have quadratic form with respect to π. When it achieves its maximum, π must be end point or the critical point of this quadratic function. The problem $max_c\{\cdot\}$ is to choose *c* which maximizes a continuous piecewise linear function with only one discontinuous point, *c* must be the end point or the discontinuous point. Recall that $\pi_{min} \leq \pi \leq \pi_{max}$ and $0 \leq c \leq c_{max}$. We have the candidates for optimal investment and consumption policies:

$$\pi = arg_{\pi_{min},\pi_{max},\frac{-(\mu-r)v_x}{\sigma^2xv_{xx}}\in[\pi_{min},\pi_{max}]}\left\{max\left[\pi x(\mu - r)v_x + \frac{1}{2}\pi^2\sigma^2x^2v_{xx}\right]\right\}$$

$$c = arg_{0,c_{max},kx^{rf}\in[0,c_{max}]}\left\{max\left[[(1+\lambda)\mathrm{I}\{c \geq kx^{rf}\} - \lambda]\left|c - kx^{rf}\right|\right.\right.$$
$$\left.\left.- c\left(v_x + v_{x^{rf}}\right)\right]\right\}$$

The Dirichlet boundary condition:

$$v|_{x=0} = 0$$

Our problem is a state constraints problem, which means DPP is hard to prove under complex state constraints and measurability problems introduced by stochasticity. On the other hand, $v(x, x^{rf})$ is not necessarily smooth. We need to verify that if $v(x, x^{rf})$ is an appropriate weak solution namely viscosity solution of the HJB equation.

It is standard to show that $v(x, x^{rf})$ is state constrained viscosity solution of the HJB equation in the state space $(0, +\infty) \times (-\infty, +\infty)$. The uniqueness can be verified from [11].

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