Green's Function Theory

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1 Defination of one-body Green's Function

$$G_{s,s'}(r,t;r',t') = -\frac{i}{\hbar}\langle\Psi_0^N|T[\hat{\psi}_s(r,t)\hat{\psi}_{s'}^{\dagger}(r',t')]|\Psi_0^N\rangle \tag{1}$$

where s, r, t are spin ,spatila and time coordinates. Ψ_0^N is the ground state wave function of N electrons system. $\hat{\psi}_s(r,t)$ and $\hat{\psi}_{s'}^{\dagger}(r',t')$ are the field operators in Heisenberg picture.

$$\hat{\psi}_s(r,t) = e^{i\hat{H}t/\hbar}\hat{\psi}_s(r)e^{-i\hat{H}t/\hbar} \tag{2}$$

$$\hat{\psi}_{s'}^{\dagger}(r',t') = e^{i\hat{H}t'/\hbar}\hat{\psi}_{s'}^{\dagger}(r')e^{-i\hat{H}t'/\hbar} \tag{3}$$

$\hat{H}$ is total time-independent hamimltonian of system. T is time-ordering operator defined as:

$$T[\hat{A}(t)\hat{B}(t')] = \begin{cases} \hat{A}(t)\hat{B}(t') & t > t' \\ \pm\hat{B}(t')\hat{A}(t) & t \leq t' \end{cases} \tag{4}$$

the time lines goes from right to left. The + sign is for bosons and - sign is for fermions. Change time-order operator to step function:

$$T[\hat{A}(t)\hat{B}(t')] = \theta(t-t')\hat{A}(t)\hat{B}(t') \pm \theta(t'-t)\hat{B}(t')\hat{A}(t) \tag{5}$$

where

$$\theta(t-t') = \begin{cases} 1 & t > t' \\ 0 & t \leq t' \end{cases} \tag{6}$$

$$G_{s,s'}(r,t;r',t') = -\frac{i}{\hbar}\left[\theta(t-t')\langle\Psi_0^N|\hat{\psi}_s(r,t)\hat{\psi}_{s'}^{\dagger}(r',t')|\Psi_0^N\rangle \pm \theta(t'-t)\langle\Psi_0^N|\hat{\psi}_{s'}^{\dagger}(r',t')\hat{\psi}_s(r,t)|\Psi_0^N\rangle\right] \tag{7}$$

Now operate the hamiltonian on the ground state wavefunction :

$$\hat{H}|\Psi_0^N\rangle = E_0^N|\Psi_0^N\rangle \tag{8}$$

$$\begin{aligned} G_{s,s'}(r,t;r',t') = &-\frac{i}{\hbar}[\theta(t-t')\langle\Psi_0^N|e^{i\hat{H}t/\hbar}\hat{\psi}_s(r)e^{-i\hat{H}(t-t')/\hbar}\hat{\psi}_{s'}^{\dagger}(r')e^{-i\hat{H}t'/\hbar}|\Psi_0^N\rangle \\ &\pm\theta(t'-t)\langle\Psi_0^N|e^{i\hat{H}t'/\hbar}\hat{\psi}_{s'}^{\dagger}(r')e^{-i\hat{H}(t'-t)/\hbar}\hat{\psi}_s(r)e^{-i\hat{H}t/\hbar}|\Psi_0^N\rangle] \\ = &-\frac{i}{\hbar}[\theta(t-t')\langle\Psi_0^N|e^{iE_0^Nt/\hbar}\hat{\psi}_s(r)e^{-i\hat{H}(t-t')/\hbar}\hat{\psi}_{s'}^{\dagger}(r')e^{-iE_0^Nt'/\hbar}|\Psi_0^N\rangle \\ &\pm\theta(t'-t)\langle\Psi_0^N|e^{iE_0^Nt'/\hbar}\hat{\psi}_{s'}^{\dagger}(r')e^{-i\hat{H}(t'-t)/\hbar}\hat{\psi}_s(r)e^{-iE_0^Nt/\hbar}|\Psi_0^N\rangle] \\ = &-\frac{i}{\hbar}[\theta(t-t')e^{iE_0^N(t-t')/\hbar}\langle\Psi_0^N|\hat{\psi}_s(r)e^{-i\hat{H}(t-t')/\hbar}\hat{\psi}_{s'}^{\dagger}(r')|\Psi_0^N\rangle \\ &\pm\theta(t'-t)e^{iE_0^N(t'-t)/\hbar}\langle\Psi_0^N|\hat{\psi}_{s'}^{\dagger}(r')e^{-i\hat{H}(t'-t)/\hbar}\hat{\psi}_s(r)|\Psi_0^N\rangle] \end{aligned}$$

The Green's function only depends on time difference:

$$G_{s,s'}(r,t;r',t') = G_{s,s'}(r,r';t-t') \tag{9}$$

$$G_{s,s'}(r,r';t-t') = -\frac{i}{\hbar} \tag{10}$$

Now we insert a complete set of eigenstates of N+1 and N-1 electrons system:

$$\hat{H}|\Psi_n^{N+1}\rangle = E_n^{N+1}|\Psi_n^{N+1}\rangle \tag{11}$$

$$\hat{H}|\Psi_n^{N-1}\rangle = E_n^{N-1}|\Psi_n^{N-1}\rangle \tag{12}$$

$$G_{s,s'}(r,r';t-t') = -\frac{i}{\hbar}[\theta(t-t')e^{iE_0^N(t-t')/\hbar}\sum_n\langle\Psi_0^N|\hat{\psi}_s(r)|\Psi_n^{N+1}\rangle e^{-iE_n^{N+1}(t-t')/\hbar}\langle\Psi_n^{N+1}|\hat{\psi}_{s'}^\dagger(r')|\Psi_0^N\rangle$$
$$\pm\theta(t'-t)e^{iE_0^N(t'-t)/\hbar}\sum_n\langle\Psi_0^N|\hat{\psi}_{s'}^\dagger(r')|\Psi_n^{N-1}\rangle e^{-iE_n^{N-1}(t'-t)/\hbar}\langle\Psi_n^{N-1}|\hat{\psi}_s(r)|\Psi_0^N\rangle]$$

Now define the following quantities:

$$A_{n,s}(r) = \langle\Psi_0^N|\hat{\psi}_s(r)|\Psi_n^{N+1}\rangle \tag{13}$$

$$A_{n,s}^*(r') = \langle\Psi_n^{N+1}|\hat{\psi}_s^\dagger(r')|\Psi_0^N\rangle \tag{14}$$

$$B_{n,s'}(r') = \langle\Psi_0^N|\hat{\psi}_{s'}^\dagger(r')|\Psi_n^{N-1}\rangle \tag{15}$$

$$B_{n,s'}^*(r) = \langle\Psi_n^{N-1}|\hat{\psi}_{s'}(r)|\Psi_0^N\rangle \tag{16}$$

and are commonly referred to as the Lehmann amplitudes or Dyson orbital Now we have:

$$G_{s,s'}(r,r';t-t') = -\frac{i}{\hbar}[\theta(t-t')\sum_n A_{n,s}(r)A_{n,s}^*(r')e^{-i(E_n^{N+1}-E_0^N)(t-t')/\hbar}$$
$$\pm\theta(t'-t)\sum_n B_{n,s'}^*(r)B_{n,s'}(r')e^{-i(E_n^{N-1}-E_0^N)(t'-t)/\hbar}]$$

Doing Fourier transform to energy domain:

$$G_{s,s'}(r,r';\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega(t-t')}G_{s,s'}(r,r';t-t') \tag{17}$$

$$G_{s,s'}(r,r';\omega) = \sum_n\left[\frac{A_{n,s}(r)A_{n,s}^*(r')}{\omega-(E_n^{N+1}-E_0^N)/\hbar+i\eta}\pm\frac{B_{n,s'}^*(r)B_{n,s'}(r')}{\omega-(E_0^N-E_n^{N-1})/\hbar-i\eta}\right]$$

where η is an infinitesimal positive number.

The exact expression for G as given can be approximated within the framework of the quasiparticle picture,

$$G_{s,s'}(r,r';\omega) = \sum_n\frac{\phi_{n,s}(r)\phi_{n,s'}^*(r')}{\omega-\epsilon_n/\hbar+i\eta\,\mathrm{sgn}(\epsilon_n-\mu/\hbar)} \tag{18}$$

where $\phi_{n,s}(r)$ are the quasiparticle wavefunctions, ϵ_n are the quasiparticle energies, and μ is the chemical potential.

2 Interaction Picture

Many quantum system have Hamiltonian of the from

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{19}$$

where $\hat{H}_0$ is *free* Hamiltonian with known spectrum which is used to classify the states of the system, and $\hat{V}$ is *perturbation* which cause transitions between eigenstates of the $\hat{H}_0$.

To study the transition (scattering, making new particles,etc.) casused by $\hat{V}$ we want to use a fixed basis of $\hat{H}_0$ eigenstates, but we ant to keep the transitions seperate from wave-packet spreding and other effects due to Schödinger phases e^{-iEt} of $\hat{H}_0$ itself.

2.1 Schrödinger Picture

In the Schrödinger picture, the operators are time-independent, while the state vectors evolve in time according to the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = \hat{H}|\Psi_S(t)\rangle \tag{20}$$

$$|\Psi_S(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi_S(0)\rangle \tag{21}$$

2.2 Heisenberg Picture

In the Heisenberg picture, the state vectors are time-independent, while the operators evolve in time according to the Heisenberg equation of motion:

$$|\Psi_H\rangle = |\Psi_S(0)\rangle \tag{22}$$

$$i\hbar\frac{\partial}{\partial t}\hat{O}_H(t) = [\hat{O}_H(t), \hat{H}] \implies \hat{O}_H(t) = e^{i\hat{H}t/\hbar}\hat{O}_S e^{-i\hat{H}t/\hbar} \tag{23}$$

The relationship between Schrödinger and Heisenberg pictures is given by:

$$\langle\Psi_S(t)|\hat{O}_S|\Psi_S(t)\rangle = \langle\Psi_H|\hat{O}_H(t)|\Psi_H\rangle \tag{24}$$

$$|\Psi_H\rangle = e^{i\hat{H}t/\hbar}|\Psi_S(t)\rangle \tag{25}$$

$$\hat{O}_H(t) = e^{i\hat{H}t/\hbar}\hat{O}_S e^{-i\hat{H}t/\hbar} \tag{26}$$

2.3 Interaction Picture

In the interactoin picture, the field operators depends on time as if they were free fields.

$$\hat{O}_I(t) = e^{i\hat{H}_0 t/\hbar}\hat{O}_S e^{-i\hat{H}_0 t/\hbar} \implies i\hbar\frac{\partial}{\partial t}\hat{O}_I(t) = [\hat{O}_I(t), \hat{H}_0] \tag{27}$$

This is differenct from Heisenberg picture where the operators evolve according to the full Hamiltonian $\hat{H}$,and are too complicated to solve. Then the state vectors in interaction picture evolve according to the perturbation $\hat{V}$ only:

$$i\hbar\frac{\partial|\Psi_I(t)\rangle}{\partial t} = i\hbar\frac{\partial}{\partial t}e^{i\hat{H}_0 t/\hbar}|\Psi_S(t)\rangle = -\hat{H}_0|\Psi_I(t)\rangle + e^{iH_0 t/\hbar}\hat{H}|\Psi_S(t)\rangle \tag{28}$$

$$= -\hat{H}_0|\Psi_I(t)\rangle + e^{iH_0 t/\hbar}(\hat{H}_0 + \hat{V})e^{-i\hat{H}_0 t/\hbar}|\Psi_I(t)\rangle \tag{29}$$

$$= \hat{V}_I(t)|\Psi_I(t)\rangle \tag{30}$$

The evolution operator in interaction picture is defined as:

$$|\Psi_I(t)\rangle = \hat{U}_I(t, t_0)|\Psi_I(t_0)\rangle \tag{31}$$

The evolution operator satisfies the equation):

$$i\hbar\frac{\partial}{\partial t}\hat{U}_I(t, t_0) = \hat{V}_I(t)\hat{U}_I(t, t_0) \tag{32}$$

with the initial condition:

$$\hat{U}_I(t_0, t_0) = 1 \tag{33}$$

The formal solution of this equation is given by the Dyson series: we integrate the differential equation from t_0 to t:

$$\hat{U}_I(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1 \hat{V}_I(t_1)\hat{U}_I(t_1, t_0) \tag{34}$$

By iterating this equation, we obtain the Dyson series expansion:

$$\hat{U}_I(t, t_0) = 1 + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \tag{35}$$

$$\times \hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n) \tag{36}$$

Note the time-ordering of the integrals ensures that $t \geq t_1 \geq t_2 \geq \cdots \geq t_n$.

2.3.1 Hypercube and Simplex

the n-dimensional hypercube $[t_0, t]^n$ can be decomposed into n! simplexes, each corresponding to a particular ordering of the time variables. For example, in two dimensions, the square $[t_0, t]^2$ can be divided into two triangles: one where $t_1 \geq t_2$ and another where $t_2 \geq t_1$. In three dimensions, the cube $[t_0, t]^3$ can be divided into six tetrahedra, each corresponding to a different ordering of t_1, t_2, t_3. In general, the n-dimensional hypercube can be decomposed into n! simplexes, each corresponding to a different permutation of the time variables. This decomposition is crucial for understanding the structure of time-ordered integrals in quantum field theory and many-body physics.

2.3.2 Time-Ordering Operator

The $V_I(t)$ operators in the integral should be arranged from to left in the order of decreasing time variables. To achieve this, we define an timeordering operator T, which arranges operators in order of decreasing time arguments.

$$T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{37}$$

are symemetric with respect to permutations of the time variables.

we want to make the term more symmetric by permutations of integral variables. There are n! permutations of the time variables $t_1, t_2, \ldots, t_n$. Each permutation corresponds to a different ordering of the time variables, and thus to a different simplex in the n-dimensional hypercube. By summing over all n! permutations, we can rewrite the Dyson series as:

$$\hat{U}_I(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \tag{38}$$

$$\times T\left[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)\right] \tag{39}$$

To express the Dyson series in a more compact form, we introduce the time-ordering operator T, which arranges operators in order of decreasing time arguments. Using the time-ordering operator, we can rewrite the Dyson series as:

$$\hat{U}_I(t,t_0) = T\left[\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} dt'\hat{V}_I(t')\right)\right] \tag{40}$$

2.4 Adiabatic Switching

In order to connect the ground state of the non-interacting system to the ground state of the interacting system, we introduce a time-dependent Hamiltonian that gradually turns on the interaction:

$$\hat{H}(t) = \hat{H}_0 + ge^{-\epsilon|t|}\hat{V} \tag{41}$$

where ϵ is a small positive parameter that controls the rate of switching. At $t = -\infty$, the Hamiltonian reduces to the non-interacting Hamiltonian $\hat{H}_0$, and at $t = 0$, the interaction is fully turned on.

Appliying the interaction picture and Dyson series to this time-dependent Hamiltonian, we can study the evolution of the system from the non-interacting ground state to the interacting ground state.

$$|\Psi_I(t)\rangle = \hat{U}_\epsilon(t,t_0)|\Psi_I(t_0)\rangle \tag{42}$$

with the evolution operator given by:

$$\hat{U}_\epsilon(t,t_0) = T\left[\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} dt' ge^{-\epsilon|t'|}\hat{V}_I(t')\right)\right] \tag{43}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{i}{\hbar}g)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \tag{44}$$

$$\times T\left[e^{-\epsilon|t_1|}\hat{V}_I(t_1)e^{-\epsilon|t_2|}\hat{V}_I(t_2)\cdots e^{-\epsilon|t_n|}\hat{V}_I(t_n)\right] \tag{45}$$

Now we can study the evolution of the system as the interaction is gradually turned on from the non-interacting ground state at $t_0 = -\infty$.

$$|\Psi_S(t_0) = e^{-iE_0t/\hbar}|\Phi_0\rangle \tag{46}$$

where $\hat{H}_0|\Phi_0\rangle = E_0|\Phi_0\rangle$. and the corresponding state in interaction picture is:

$$|\Psi_I(t_0)\rangle = e^{i\hat{H}_0t_0/\hbar}|\Psi_S(t_0)\rangle = |\Phi_0\rangle \tag{47}$$

Thus $|\Psi_I(t_0)$ becomes time independent as $t_0 \to -\infty$. The same conclusion follows from

$$i\hbar\frac{\partial|\Psi_I(t)\rangle}{\partial t} = ge^{-\epsilon|t|}\hat{V}_I(t)|\Psi_I(t)\rangle \tag{48}$$

From definition we known

$$|\Psi_H\rangle = |\Psi_S(0)\rangle = |\Psi_I(0)\rangle = \hat{U}_\epsilon(0,-\infty)|\Phi_0\rangle \tag{49}$$

which expresse and exact eighenstate of the interacting system in terms of the non-interacting ground state $|\Phi_0\rangle$ and the evolution operator $\hat{U}_I(0,-\infty)$.

2.5 Gell-Mann and Low Theorem

Consider the expression

$$(\hat{H}_0 - E_0)|\Psi_H\rangle = (\hat{H}_0 - E_0)\hat{U}_\epsilon(0,-\infty)|\Phi_0\rangle = [\hat{H}_0, \hat{U}_\epsilon(0,-\infty)]|\Phi_0\rangle \tag{50}$$

We shall explicitly evaluate the commutator on the right-hand side. Using the Dyson series for $\hat{U}_\epsilon(0,-\infty)$

$$[\hat{H}_0, \hat{U}_\epsilon(0,-\infty)] = \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{i}{\hbar}g)^n \int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}[\hat{H}_0, T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)]] \tag{51}$$

take an arbitary time ordering of the n time indices.

$$[\hat{H}_0, \hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] = [\hat{H}_0, \hat{V}_I(t_1)]\hat{V}_I(t_2)\cdots\hat{V}_I(t_n) \tag{52}$$

$$+ \hat{V}_I(t_1)[\hat{H}_0, \hat{V}_I(t_2)]\cdots\hat{V}_I(t_n) \tag{53}$$

$$+ \cdots \tag{54}$$

$$+ \hat{V}_I(t_1)\hat{V}_I(t_2)\cdots[\hat{H}_0, \hat{V}_I(t_n)] \tag{55}$$

The evolution of $\hat{V}_I(t)$ is given by

$$i\hbar\frac{\partial}{\partial t}\hat{V}_I(t) = [\hat{V}_I(t), \hat{H}_0] \implies [\hat{H}_0, \hat{V}_I(t)] = -i\hbar\frac{\partial}{\partial t}\hat{V}_I(t) \tag{56}$$

Substitue into the commutator expression:

$$[\hat{H}_0, \hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] = -i\hbar\frac{\partial}{\partial t_1}\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n) \tag{57}$$

$$- i\hbar\hat{V}_I(t_1)\frac{\partial}{\partial t_2}\hat{V}_I(t_2)\cdots\hat{V}_I(t_n) \tag{58}$$

$$+ \cdots \tag{59}$$

$$- i\hbar\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\frac{\partial}{\partial t_n}\hat{V}_I(t_n) \tag{60}$$

$$= (-i\hbar)(\sum_{j=1}^{n}\frac{\partial}{\partial t_j})\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n) \tag{61}$$

Another important property of time-ordering operator T is that the time derivatives can be taken outside the time-ordering operator:

$$T\left[(\sum_{j=1}^{n}\frac{\partial}{\partial t_j})\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)\right] = (\sum_{j=1}^{n}\frac{\partial}{\partial t_j})T\left[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)\right] \tag{62}$$

This can be shown by explicitly using properties of the step functions that define the time-ordering operator.

$$(\sum_{j=1}^{n}\frac{\partial}{\partial t_j})\theta(t_{P_1}-t_{P_2})\theta(t_{P_2}-t_{P_3})\cdots\theta(t_{P_{n-1}}-t_{P_n}) = \delta(t_{P_1}-t_{P_2})\theta(t_{P_2}-t_{P_3})\cdots\theta(t_{P_{n-1}}-t_{P_n}) \tag{63}$$

$$+ \theta(t_{P_1}-t_{P_2})\delta(t_{P_2}-t_{P_3})\cdots\theta(t_{P_{n-1}}-t_{P_n}) + \cdots \tag{64}$$

$$+ \theta(t_{P_1}-t_{P_2})\theta(t_{P_2}-t_{P_3})\cdots\delta(t_{P_{n-1}}-t_{P_n}) \tag{65}$$

However,the delta functions only contribute when two time variables are equal, which is a set of measure zero in the time integrals. We assume time order that $t_{P_\mu} \neq t_{P_\nu}$. Thus we have:

$$[\hat{H}_0, T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)]] = (-i\hbar)(\sum_{j=1}^{n}\frac{\partial}{\partial t_j})T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{66}$$

All time derivativs make the same contributisons under permutations ; we therefore retain just one,says $\frac{\partial}{\partial t_1}$ adn multiply by n.Let's substitute this result back into the commutator expression:

$$[\hat{H}_0, \hat{U}_\epsilon(0,-\infty)] = \sum_{n=1}^{\infty} \frac{1}{n!}(-\frac{i}{\hbar}g)^n \int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)} \tag{67}$$

$$\times (-i\hbar)n\frac{\partial}{\partial t_1}T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{68}$$

Now we can perform the integral over t_1 by parts:

$$[\hat{H}_0, \hat{U}_\epsilon(0,-\infty)] = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}(-g)\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \int_{-\infty}^{0} dt_1 \tag{69}$$

$$\times \frac{\partial}{\partial t_1}\left[e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)]\right] \tag{70}$$

$$-\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}(-g)\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \int_{-\infty}^{0} dt_1 \tag{71}$$

$$\times \left[\frac{\partial}{\partial t_1} e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}\right] T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{72}$$

The first term is simply the intergrand evaluated at the end of points:
if $t_1 \to -\infty$

$$e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] = 0 \tag{73}$$

if $t_1 = 0$

$$e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] = e^{-\epsilon(|t_2|+\cdots+|t_n|)}\hat{V}_I(0)T\left[\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)\right] \tag{74}$$

And the first term becomes:

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}(-g)\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \tag{75}$$

$$\times\, e^{-\epsilon(|t_2|+\cdots+|t_n|)}\hat{V}_I(0)T\left[\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)\right] \tag{76}$$

$$= g\hat{V}_I(0)\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \tag{77}$$

$$\times\, e^{-\epsilon(|t_2|+\cdots+|t_n|)}T\left[\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)\right] \tag{78}$$

$$= -g\hat{V}_I(0)\hat{U}_\epsilon(0,-\infty) \tag{79}$$

Now because $t_n \leq 0$, we have $e^{-\epsilon|t_n|} = e^{\epsilon t_n}$. The second term is

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}(-g)\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \int_{-\infty}^{0} dt_1 \tag{80}$$

$$\times \left[\frac{\partial}{\partial t_1} e^{\epsilon(t_1+t_2+\cdots+t_n)}\right] T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{81}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}(-g)(\epsilon)\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \int_{-\infty}^{0} dt_1 \tag{82}$$

$$\times\, e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{83}$$

$$= -g\epsilon\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\frac{i}{\hbar}g)^{n-1}\int_{-\infty}^{0} dt_2 \cdots \int_{-\infty}^{0} dt_n \int_{-\infty}^{0} dt_1 \tag{84}$$

$$\times\, e^{-\epsilon(|t_1|+|t_2|+\cdots+|t_n|)}T[\hat{V}_I(t_1)\hat{V}_I(t_2)\cdots\hat{V}_I(t_n)] \tag{85}$$

$$= -g\epsilon\frac{\partial}{\partial(\frac{-i}{\hbar}g)}\hat{U}_\epsilon(0,-\infty) \tag{86}$$

$$= -i\hbar g\epsilon\frac{\partial}{\partial g}\hat{U}_\epsilon(0,-\infty) \tag{87}$$

The final reuslt for the commutator is:

$$[\hat{H}_0, \hat{U}_\epsilon(0,-\infty)] = -g\hat{V}_I(0)\hat{U}_\epsilon(0,-\infty) + i\hbar g\epsilon\frac{\partial}{\partial g}\hat{U}_\epsilon(0,-\infty) \tag{88}$$

Substituting this result back into the expression for $(\hat{H}_0 - E_0)|\Psi_H\rangle$, we obtain:

$$(\hat{H}_0 - E_0)|\Psi_H\rangle = \left[-g\hat{V}_I(0) + i\hbar g\epsilon\frac{\partial}{\partial g}\right]\hat{U}_\epsilon(0,-\infty)|\Phi_0\rangle \tag{89}$$

$$\hat{H} - E_0|\Psi_H\rangle = i\hbar g\epsilon\frac{\partial}{\partial g}|\Psi_H\rangle \tag{90}$$

If we Multiply both sides by $\langle\Phi_0|$ and divide by $\langle\Phi_0|\Psi_H\rangle$,and because of $\frac{\partial}{\partial g}|\Phi\rangle = 0$ we get:

$$\frac{\langle\Phi_0|\hat{H} - E_0|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle} = i\hbar g\epsilon\frac{\partial}{\partial g}\ln(\langle\Phi_0|\Psi_H\rangle) = E - E_0 = \Delta E \tag{91}$$

If ϵ allowd to vanish at this point ,it would be tempting to conclude that $\Delta E = 0$, which is clearly not true.In fact ,the amplitude $\langle\Phi_0|\Psi_H\rangle$ must acquire an infitite phase proborttional to $i\epsilon^{-1}$.so $\epsilon\frac{\partial}{\partial g}\ln(\langle\Phi_0|\Psi_H\rangle)$ remains finite as $\epsilon \to 0$. We mannipulate the expression to obtain the final form of the Gell-Mann and Low theorem:

$$\left(\hat{H} - E_0 - i\hbar\epsilon g\frac{\partial}{\partial g}\right)\frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle} = i\hbar g\epsilon\frac{\frac{\partial}{\partial g}|\Psi_H\rangle}{\langle\Phi_0|\Psi_h\rangle} - i\hbar\epsilon g\frac{\partial}{\partial g}\frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle} = \frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle}[i\hbar\epsilon g\frac{\partial}{\partial g}\ln(\langle\Phi_0|\Psi_H\rangle)] \tag{92}$$

$$\left(\hat{H} - E_0 - i\hbar\epsilon g\frac{\partial}{\partial g}\right)\frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle} = \frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle}(E - E_0) \tag{93}$$

we finally have:

$$\left(\hat{H} - E\right)\frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle} = i\hbar\epsilon g\frac{\partial}{\partial g}\frac{|\Psi_H}{\langle\Phi_0|\Psi_H\rangle} \tag{94}$$

This is the Gell-Mann and Low theorem, which states that if the limit $\epsilon \to 0$ exists, then the state $\frac{|\Psi_H\rangle}{\langle\Phi_0|\Psi_H\rangle}$ is an exact eigenstate of the full Hamiltonian $\hat{H}$ with eigenvalue E. The state from Adiabatic switching need note to be ground state; it can be any eigenstate of the interacting system.

3 Wick's Theorem