1 Pauli Gate

$$XY = iZ \quad XZ = -iY \quad YZ = iX \tag{1}$$

2 UCC

$$U(\theta) = e^{\hat{T}(\theta) - \hat{T}(\theta)^\dagger} \tag{2}$$

$$\hat{T} = \theta \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \quad i > j > k > l \tag{3}$$

$$U(\theta) = e^{\theta(\hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l - \hat{a}_l^\dagger \hat{a}_k^\dagger \hat{a}_j \hat{a}_i)} \tag{4}$$

Now plug in the jordan wigner transofrmation.

$$\hat{a}_i^\dagger = I^{\otimes i} \otimes \frac{1}{2}(X - iY) \otimes Z^{\otimes N-i-1} \tag{5}$$

$$\hat{a}_i = I^{\otimes i} \otimes \frac{1}{2}(X + iY) \otimes Z^{\otimes N-i-1} \tag{6}$$

where N is the number of qbit.$i = [0, \cdots, N-1]$. Now we calculate

$$a_i^\dagger a_j^\dagger a_k a_l - a_l^\dagger a_k^\dagger a_j a_i \tag{7}$$

Let's from smaller term and assume $i > j$.

$$\begin{aligned}
a_i^\dagger a_j^\dagger &= [I^{\otimes i} \otimes \frac{1}{2}(X - iY) \otimes Z^{\otimes N-i-1}][I^{\otimes j} \otimes \frac{1}{2}(X - iY) \otimes Z^{\otimes N-j-1}] && (8)\\
&= [I^{\otimes j} \otimes \frac{1}{2}(X_j - iY_j) \otimes_{a=j+1}^{i-1} Z_a \otimes \frac{1}{2}(X_i - iY_i) Z_i \otimes I^{\otimes N-i-1}] && (9)\\
&= [I^{\otimes j} \otimes \frac{1}{2}(X_j - iY_j) \otimes_{a=j+1}^{i-1} Z_a \otimes \frac{1}{2}(-iY_i + X_i) \otimes I^{\otimes N-i-1}] && (10)\\
&= [I^{\otimes j} \otimes \frac{1}{2}(X_j - iY_j) \otimes_{a=j+1}^{i-1} Z_a \otimes \frac{1}{2}(X_i - iY_i) \otimes I^{\otimes N-i-1}] && (11)
\end{aligned}$$

Now annihilation term is $k > l$

$$\begin{aligned}
a_k a_l &= [I^{\otimes k} \otimes \frac{1}{2}(X_k + iY_k) \otimes Z^{\otimes N-k-1}][I^{\otimes l} \otimes \frac{1}{2}(X_l + iY_l) \otimes Z^{\otimes N-l-1}] && (12)\\
&= [I^{\otimes l} \otimes \frac{1}{2}(X_l + iY_l) \otimes_{a=l+1}^{k-1} Z_a \otimes \frac{1}{2}(X_k + iY_k) Z_k \otimes I^{\otimes N-k-1}] && (13)\\
&= [I^{\otimes l} \otimes \frac{1}{2}(X_l + iY_l) \otimes_{a=l+1}^{k-1} Z_a \otimes \frac{1}{2}(-iY_k - X_k) \otimes I^{\otimes N-k-1}] && (14)\\
&= -[I^{\otimes l} \otimes \frac{1}{2}(X_l + iY_l) \otimes_{a=l+1}^{k-1} Z_a \otimes \frac{1}{2}(X_k + iY_k) \otimes I^{\otimes N-k-1}] && (15)
\end{aligned}$$

Now we combine this two terms ,$i > j > k > l$

$$\begin{aligned}
a_i^\dagger a_j^\dagger a_k a_l &= [I^{\otimes j} \otimes \frac{1}{2}(X_j - iY_j) \otimes_{a=j+1}^{i-1} Z_a \otimes \frac{1}{2}(X_i - iY_i) \otimes I^{\otimes N-i-1}]\\
&\times -[I^{\otimes l} \otimes \frac{1}{2}(X_l + iY_l) \otimes_{a=l+1}^{k-1} Z_a \otimes \frac{1}{2}(X_k + iY_k) \otimes I^{\otimes N-k-1}]\\
&= -[I^{\otimes l} \otimes \frac{1}{2}(X_l + iY_l) \otimes_{a=l+1}^{k-1} Z_a \otimes \frac{1}{2}(X_k + iY_k) \otimes_{b=k+1}^{j} I_b \otimes \frac{1}{2}(X_j - iY_j) \otimes_{c=j+1}^{i-1} Z_c \otimes \frac{1}{2}(X_i - iY_i) \otimes I^{\otimes N-i-1}]\\
&= - \otimes_{a=l+1}^{k-1} Z_a \otimes_{b=j+1}^{i-1} Z_b \otimes \frac{1}{2}(X_l + iY_l) \otimes \frac{1}{2}(X_k + iY_k) \otimes \frac{1}{2}(X_j - iY_j) \otimes \frac{1}{2}(X_i - iY_i)\\
&= -\frac{1}{16} \otimes_{a=l+1}^{k-1} Z_a \otimes_{b=j+1}^{i-1} Z_b (X_l + iY_l)(X_k + iY_k)(X_j - iY_j)(X_i - iY_i)
\end{aligned}$$

There are 16 pauli string , but we calculate the conjugate term Now with $i > j > k > l$

$$a_l^\dagger a_k^\dagger a_j a_i = a_k^\dagger a_l^\dagger a_i a_j \tag{16}$$

$$\begin{aligned}
a_l^\dagger a_k^\dagger a_j a_i &= [I^{\otimes l} \otimes \frac{1}{2}(X_l - iY_l) \otimes_{a=l+1}^{k-1} Z_a \otimes \frac{1}{2}(X_k - iY_k) \otimes I^{\otimes N-k-1}] && (17)\\
&\times -[I^{\otimes j} \otimes \frac{1}{2}(X_j + iY_j) \otimes_{a=j+1}^{i-1} Z_a \otimes \frac{1}{2}(X_i + iY_i) \otimes I^{\otimes N-i-1}] && (18)\\
&= -\frac{1}{16} \otimes_{a=l+1}^{k-1} Z_a \otimes_{b=j+1}^{i-1} Z_b (X_l - iY_l)(X_k - iY_k)(X_j + iY_j)(X_i + iY_i) && (19)
\end{aligned}$$

	X_jX_i(1)	X_jY_i(-i)	Y_jX_i(-i)	Y_jY_i(-1)
X_lX_k (1)	1	-i	-i	-1
X_lY_k (i)	i	1	1	-i
Y_lX_k (i)	i	1	1	-i
Y_lY_k (-1)	-1	i	i	1

	X_jX_i(1)	X_jY_i(i)	Y_jX_i(i)	Y_jY_i(-1)
X_lX_k (1)	1	i	i	-1
X_lY_k (-i)	-i	1	1	i
Y_lX_k (-i)	-i	1	1	i
Y_lY_k (-1)	-1	-i	-i	1

Finally we want to compute $a_i^\dagger a_j^\dagger a_k a_l - a_l^\dagger a_k^\dagger a_j a_i$

We just find the coeff that are differenct. So the final result is

$$\begin{aligned}
a_i^\dagger a_j^\dagger a_k a_l - a_l^\dagger a_k^\dagger a_j a_i &= -\frac{1}{8}[-iX_lX_kX_jY_i - iX_lX_kY_jX_i && (20)\\
&iX_lY_kX_jX_i - iX_lY_kY_jY_i && (21)\\
&iY_lX_kX_jX_i - iY_lX_kY_jY_i && (22)\\
&iY_lY_kX_jY_i + iY_lY_kY_jX_i] && (23)\\
&= \frac{i}{8}[X_lX_kX_jY_i + X_lX_kY_jX_i && (24)\\
&- X_lY_kX_jX_i + X_lY_kY_jY_i && (25)\\
&- Y_lX_kX_jX_i + Y_lX_kY_jY_i && (26)\\
&- Y_lY_kX_jY_i - Y_lY_kY_jX_i] && (27)
\end{aligned}$$

3 Commutator

all of those eight term have the form of same two and diff two . if they diff by diff two ,that is two .if they diff by same two ,that is also two. if all diff then two by two also two.

$$[X \otimes Y, Y \otimes X] = (XY) \otimes (YX) - (YX) \otimes (XY) = (iZ) \otimes (-iZ) - (-iZ) \otimes (iZ) = 0 \tag{28}$$

so all those eight terms commute with each other.

4 CNOT -staircase construction

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{29}$$

$$HZH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X \tag{30}$$

$$HH = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = I \tag{31}$$

$$e^X = e^{HZH} = He^Z H \tag{32}$$

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{33}$$

$$R_x(-\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad R_x(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{34}$$

$$R_x(-\pi/2)R_x(\pi/2) = I \tag{35}$$

$$\begin{aligned}
R_x(-\pi/2)ZR_x(\pi/2) &= \frac{1}{2}(1+iX)Z(1-iX) = \frac{1}{2}(1+iX)(Z-iZX) = \frac{1}{2}(1+iX)(Z-i(iY)) && (36)\\
&= \frac{1}{2}(1+iX)(Z+Y) = \frac{1}{2}(Z+iXZ+Y+iXY) = Z+i(-iY)+Y+i(iZ) = Y && (37)
\end{aligned}$$

Now what is the quantum circuit for $i > j > k > l$

$$U(\theta) = e^{\frac{i\theta}{8} \otimes_{a=l+1}^{k-1} Z_a \otimes_{j+1}^{i-1} Z_b X_l X_k X_j Y_i} \tag{38}$$

$$U(\theta) = (HHHR_x(-\pi/2))e^{\frac{i\theta}{8}(\otimes_{a=l}^{k} Z_a \otimes_{b=j}^{i} Z_b)}(HHHR_x(\pi/2)) \tag{39}$$