Formula derivation: 2406.11691v1

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1 Normal

Time of pulse to reach the Earth from a source at a distance d and group velocity v_g :

$$t_p = \int_0^d \frac{dS}{v_g} \tag{1}$$

Dispersion relation in plasma:

$$w^2 = w_p^2 + c^2 k^2 \tag{2}$$

Group velocity:

$$v_g(w) \equiv \frac{\partial \omega}{\partial k} = c \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \tag{3}$$

The plasma frequency (CGS or SI):

$$\omega_p^2 \equiv \left(\frac{4\pi n_e e^2}{m_e}\right) \equiv \left(\frac{n_e e^2}{\epsilon_0 m_e}\right) \tag{4}$$

If FRB is extragalactic, there has some changes:

- $w = (1+z) w_{obs}$
- n_e depends on redshift $n_e(z)$
- Eq.1 needs multiply by a term — (1+z)

  And note dS is the physical distance:

$$dS = \frac{c}{H(z)(1+z)} dz$$

So Eq.1 can be written:

$$\begin{aligned} t_p(w) &= \int_0^d \frac{dS}{v_g(w)}(1+z) \simeq \int_0^d \frac{dS}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)(1+z) \\ &= \int_0^z \frac{dz}{c}\frac{c}{H(z)(1+z)}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)(1+z) \\ &= \int_0^z \frac{dz}{H(z)}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \end{aligned} \tag{5}$$

The arrival time difference between two frequencies $w_2 > w_1$ can be expressed as:

$$\begin{aligned}
\Delta t = t_p(w_1) - t_p(w_2) &= \int_0^z \frac{dz}{2H(z)} \omega_p^2 (\frac{1}{w_1^2} - \frac{1}{w_2^2}) \\
&= \int_0^z \frac{dz}{2H(z)} \frac{4\pi n_e e^2}{m_e} \left(\frac{1}{w_1^2} - \frac{1}{w_2^2} \right) \\
&= \frac{2\pi e^2}{m_e c} \left(\frac{1}{w_{obs,1}^2} - \frac{1}{w_{obs,2}^2} \right) \int_0^z \frac{dz}{H(z)} \frac{n_z(z) c}{(1+z)^2}
\end{aligned} \tag{6}$$

So:

$$DM = \int_0^z \frac{dz}{H(z)} \frac{n_z(z) c}{(1+z)^2} \tag{7}$$

2 Fine structure constant

In runaway dilaton model, the time evolution of α is:

$$\frac{\Delta\alpha}{\alpha}(z) = -\gamma ln(1+z) \tag{8}$$

where γ is a constant.

The plasma frequency as a function of the fine-structure constant α can be written as ($\omega_p^2 \equiv \left(\frac{4\pi n_e e^2}{m_e}\right)$ and $\alpha = \frac{2\pi e^2}{hc}$):

$$\omega_p^2 = \frac{2n_e h\alpha c}{m_e} \tag{9}$$

Δt (Reference Eq.5)can be written as:

$$\begin{aligned}\Delta t = t_p(w_1) - t_p(w_2) &= \int_0^z \frac{dz}{2H(z)}\omega_p^2(\frac{1}{w_1^2} - \frac{1}{w_2^2}) \\ &= \int_0^z \frac{dz}{2H(z)}\frac{2n_e(z)h\alpha(z)c}{m_e}(\frac{1}{w_{obs,1}^2} - \frac{1}{w_{obs,2}^2})\frac{1}{(1+z)^2} \\ &= \frac{h\alpha_0}{m_e}(\frac{1}{w_{obs,1}^2} - \frac{1}{w_{obs,2}^2})\int_0^z \frac{dz}{H(z)}\frac{cn_e(z)(z)}{(1+z)^2}\frac{\alpha(z)}{\alpha_0}\end{aligned} \tag{10}$$

So :

$$\begin{aligned}DM &= \int_0^z \frac{dz}{H(z)}\frac{cn_e(z)}{(1+z)^2}\frac{\alpha(z)}{\alpha_0} \\ &= \int_0^z \frac{dz}{H(z)}\frac{cn_e(z)}{(1+z)^2}(\frac{\Delta\alpha(z)}{\alpha_0} + 1)\end{aligned} \tag{11}$$

Consider $\Delta\alpha(z) = \alpha(z) - \alpha_0$ and $\gamma \to 0$:

$$\frac{\alpha(z)}{\alpha_0} = \frac{\alpha(z)}{\alpha(z) - \Delta\alpha(z)} = \frac{1}{\frac{\alpha(z)-\Delta\alpha(z)}{\alpha(z)}} = \frac{1}{1 - \frac{\Delta\alpha(z)}{\alpha(z)}} = \frac{1}{1+\gamma ln(1+z)} \simeq 1 - \gamma ln(1+z) \tag{12}$$

So the effect on DM is multiplied by a term $\frac{\Delta\alpha(z)}{\alpha_0} + 1$ or $\frac{\alpha(z)}{\alpha_0} = 1 - \gamma ln(1+z)$.

3 SNe observations–luminosity distance

DM_{IGM} contains cosmological information:

$$DM_{IGM}(z) = \frac{3c\Omega_b H_0^2}{8\pi G m_p} \int_0^z \frac{(1+z) f_{IGM}(z) \chi(z)}{H(z)} dz \tag{13}$$

where f_{IGM} a constant, and $\chi(z) = 7/8$.

If considering the fine structure constant, it makes sense to add $(\frac{\Delta\alpha(z)}{\alpha_0} + 1)$ to Eq.13.

$$\begin{aligned} DM_{IGM}(z) &= \frac{3c\Omega_b H_0^2}{8\pi G m_p} \int_0^z \frac{(1+z) f_{IGM}(z) \chi(z)}{H(z)} \left(\frac{\Delta\alpha(z)}{\alpha_0} + 1 \right) dz \\ &= \frac{21c\Omega_b H_0^2}{64\pi G m_p} \int_0^z \frac{(1+z)}{H(z)} \left(\frac{\Delta\alpha(z)}{\alpha_0} + 1 \right) dz \end{aligned} \tag{14}$$

And associating DM with luminosity distance of supernova, the luminosity distance is:

$$d_L(z) = c(1+z) \int_0^z \frac{dz'}{H(z')}$$

and

$$d_L(z=0) = 0$$

So the integral part is (Integration by parts):

$$\begin{aligned} &\int_0^z \frac{1}{H(z)} (\frac{\Delta\alpha(z)}{\alpha_0} + 1)(1+z) dz \\ =& \int_0^z \frac{1}{H(z)} (-\gamma \ln(1+z) + 1)(1+z) dz \\ =& \int_0^z (-\gamma \ln(1+z) + 1)(1+z) d[\int_0^z \frac{dz}{H(z)}] \\ =& [(-\gamma \ln(1+z) + 1)(1+z) \times \int_0^z \frac{dz}{H(z)}]|_0^z - \int_0^z (\int_0^z \frac{dz}{H(z)}) d[(-\gamma \ln(1+z) + 1)(1+z)] \\ =& \frac{d_L(z)}{c} (-\gamma \ln(1+z) + 1)|_0^z - \int_0^z \frac{d_L(z)}{c(1+z)} d[(-\gamma \ln(1+z) + 1)(1+z)] \\ =& \frac{d_L(z)}{c} (-\gamma \ln(1+z) + 1) - \int_0^z \frac{d_L(z)}{c(1+z)} \times [(-\gamma \ln(1+z) + 1) + (1+z)(-\gamma)\frac{1}{1+z}] dz \\ =& \frac{d_L(z)}{c} (-\gamma \ln(1+z) + 1) - \int_0^z \frac{d_L(z)}{c(1+z)} (-\gamma \ln(1+z) + 1) dz - \int_0^z \frac{d_L(z)}{c(1+z)} (-\gamma) dz \\ =& (-\gamma \ln(1+z) + 1) \frac{d_L(z)}{c} + (\gamma - 1) \int_0^z \frac{d_L(z)}{c(1+z)} dz + \gamma \int_0^z \frac{d_L(z)}{c(1+z)} ln(1+z) dz \end{aligned} \tag{15}$$

Same as Eq(3.7) in the [arXiv:2406.11691v1].

4 Reference

- A search for the fine-structure constant evolution from fast radio bursts and type Ia supernovae data.
- Constraints on a possible variation of the fine structure constant from galaxy cluster data. Eq(2.6) for the evolution of the α at low redshifts in runaway dilaton models.
- Cosmological model-independent constraints on the baryon fraction in the IGM from fast radio bursts and supernovae data. (Another paper on FRB and supernovae)