Topology of tropical moduli spaces of weighted stable curves in higher genus

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Joint with Siddarth Kannan, Stefano Serpente, Claudia Yun
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5. Euler characteristics of $\Delta_{g,w}$.

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3. Examples, revealing topological properties.
4. Simple connectivity of $\Delta_{g,w}$ for $g > 0$.
5. Euler characteristics of $\Delta_{g,w}$.
6. Work in progress towards a proof for simple connectivity via geometric group theory.

History and Motivation

History

- Given $g \geq 0$, $n \geq 1$ and $w \in (\mathbb{Q} \cap (0,1])^n$ satisfying

  $$2g - 2 + \sum w_i > 0,$$

  Hassett defined the Deligne-Mumford (DM) stack $\overline{\mathcal{M}}_{g,w}$ as an alternate compactification of DM stack $\mathcal{M}_{g,n}$ of n-marked smooth curves of genus g in [Has03].
  The space $\overline{\mathcal{M}}_{g,w}$ parametrizes n-marked curves of genus g

  We have the containments

  $$\mathcal{M}_{g,n} \subset \mathcal{M}_{g,w} \subset \overline{\mathcal{M}}_{g,w}.$$

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$$\mathcal{M}_{g,n} \subset \mathcal{M}_{g,w} \subset \overline{\mathcal{M}}_{g,w}.$$
- In [Uli15], Ulirsch showed that the boundary divisor $\overline{\mathcal{M}}_{g,w} \setminus \mathcal{M}_{g,w}$ is a normal crossings divisor.

Motivation

The boundary complex, or the dual complex of the boundary divisor $\overline{\mathcal{M}}_{g,w} \smallsetminus \mathcal{M}_{g,w}$ is identified with $\Delta_{g,w}$.

- Shown by Harper in [Har17], homotopy types of the boundary complex is independent of the choice of compactification, for Deligne-Mumford stacks.

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- Shown by Harper in [Har17], homotopy types of the boundary complex is independent of the choice of compactification, for Deligne-Mumford stacks.
- As a DM stack, the rational cohomology of $\mathcal{M}_{g,w}$ carries a mixed Hodge structure, i.e. there is a weight filtration of the rational homology. The top graded piece of the weight filtration on $\mathcal{M}_{g,w}$ is isomorphic to the reduced rational homology of the dual complex $\Delta_{g,w}$; see work by Chan-Galatius-Payne in [CGP21].

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- Cerbu at al. also derived the homotopy types when w has at least two 1 entries.

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- Chan computed rational homology for $g = 2, n \leq 8$;
- when w is *heavy/light* or has at least two 1's, Cerbu et al. showed that $\Delta_{1,w}$ is homotopic to a wedge of spheres.
- when $w = (1^{(n)})$ and for $(g, n) \neq (0, 4), (0, 5)$, Allcock-Corey-Payne showed that $\Delta_{g,w}$ is simply connected .

What is $\Delta_{g,w}$

The category $\Gamma_{g,w}$

Given $g \geq 0, n \geq 1$, and a weight vector $w \in (\mathbb{Q} \cap (0, 1])^n$,

Definition
A ***w*-stable graph of genus *g*** is a tuple (G, h, m) such that

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   where $b^1(G)$ is the first betti number of G.

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3. (w-stability) $m : [n] \to V(G)$ is a (marking) function such that for all $v \in V(G)$

$$2h(v) - 2 + \mathrm{val}(v) + \sum_{i \in m^{-1}(v)} w_i > 2;$$

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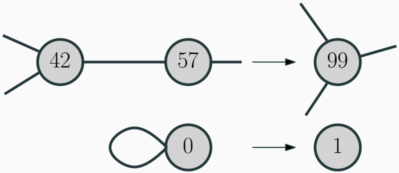

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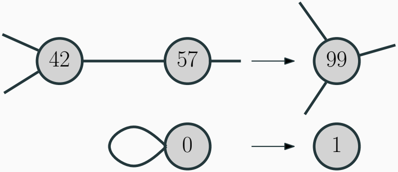

    2.2 graph isomorphisms that respect the vertex weights and markings.

Example

Let $0 < \varepsilon \ll 1$, $g = 1$, $w = (1, \varepsilon)$.

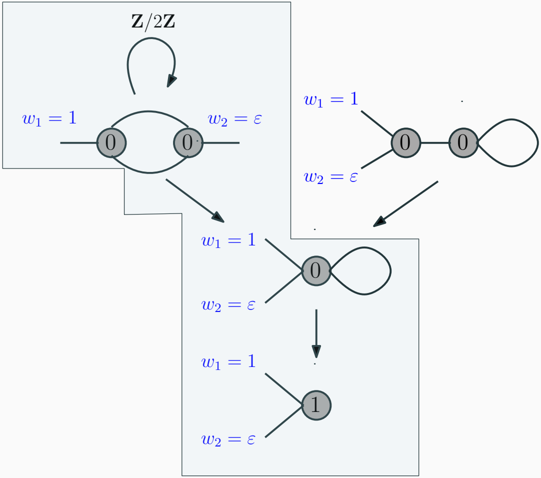

Figure 1: The category $\Gamma_{1,(1,\varepsilon)}$, containing the category $\Gamma_{1,(\varepsilon,\varepsilon)}$ (in vague blue shade) as a subcategory.

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Definition
The **volume** of $(\mathbf{G}, \ell)$ is

$$\mathrm{vol}((\mathbf{G}, \ell)) := \sum_{e \in E(G)} \ell(e).$$

A funtor $\Gamma_{g,w} \to$ Top

For each $\mathbf{G}$, define

$$\Delta(\mathbf{G}) := \left\{ \ell : E(G) \to \mathbb{R}_{\geq 0}, \sum_{e \in E(G)} \ell(e) = 1 \right\},$$

which can be identified with a $(|E(G)| - 1)$-simplex.

For each morphism $f : \mathbf{G} \to \mathbf{H}$ in $\Gamma_{g,w}$, there is an induced morphism of topological spaces

$$f^* : \Delta(\mathbf{H}) \to \Delta(\mathbf{G})$$

identifying $\Delta(\mathbf{H})$ as a face of $\Delta(\mathbf{G})$.

Then "Δ" gives a diagram $\Gamma_{g,w}^{\mathrm{op}} \to \mathrm{Top}$.

Definition
The moduli space $\Delta_{g,w}$ is the colimit of the diagram Δ:

$$\Delta_{g,w} := \operatorname{colim}_{\mathbf{G} \in \mathrm{Obj}(\Gamma_{g,w})} \Delta(\mathbf{G}).$$

It parametrizes w-stable tropical curves of genus g with unit volume.

Example

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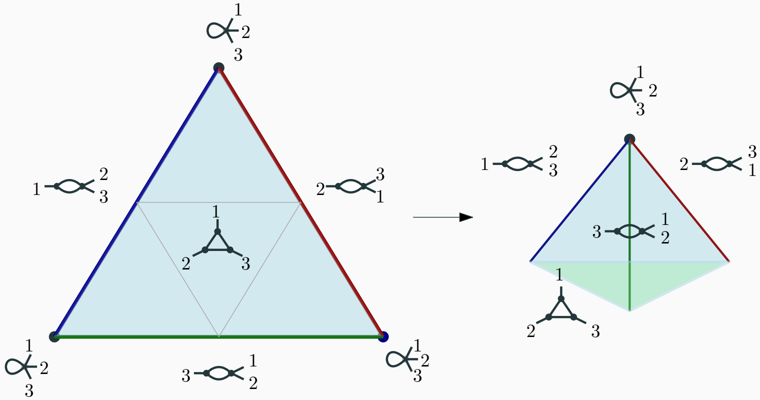

Figure 2: The space $\Delta_{1,(\varepsilon,\varepsilon,\varepsilon)}$ is homotopy equivalent to S^2.

The main theorems

Theorem (Kannan-L.-Serpente-Yun)
For any $g, n \geq 1$, and $w \in (\mathbb{Q} \cap (0,1])^n$, the space $\Delta_{g,w}$ is simply-connected.

Remark: $\Delta_{g,(1,\ldots,1)}$ is simply-connected for $(g,n) \neq (0,4),(0,5)$ by Allcock-Corey-Payne.

Outline of proof

Double induction on

$$\ell(w) := \text{length of } w$$

and

$$j(w) := \#\{w_i : w_i < 1\}.$$

For each w,

1. Reordering w s.t. $w_1 < 1$ and define $\bar{w} = (1, w_2, w_3, \ldots)$. Define

$$\Sigma_{g,w} = \overline{\Delta_{g,\bar{w}} \smallsetminus \Delta_{g,w}} \subseteq \Delta_{g,\bar{w}}.$$

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2. Prove that both $\Sigma_{g,w}$ and $\Delta_{g,w} \cap \Sigma_{g,w}$ are simply-connected.
3. Use Seifert-van Kampen Theorem on the diagram

$$\begin{tikzcd} 0 = \pi_1(\Delta_{g,w} \cap \Sigma_{g,w}) \arrow[r] \arrow[d] & \pi_1(\Sigma_{g,w}) = 0 \arrow[d] \\ \pi_1(\Delta_{g,w}) \arrow[r] & \text{by I.H. } \pi_1(\Delta_{g,\overline{w}}) = 0. \end{tikzcd}$$

Unpack definition of $\Sigma_{g,w}$, and show that

$$\Sigma_{g,w} = \overline{\Delta_{g,\bar{w}} \smallsetminus \Delta_{g,w}} = \bigcup_{S \in K(w) \smallsetminus K(\bar{w})} \Delta_{g,\bar{w}}(S),$$

where

$$K(w) := \{S \subseteq [n] : \sum_{i \in S} w_i \leq 1\}$$

and $\Delta_{g,\bar{w}}(S)$ is the subcomplex of $\Delta_{g,\bar{w}}$ represeting tropical curves with a vertex supporting markings indexed by S.

Figure 3: Underlying graphs in the boundary of $\Sigma_{g,w}$ (left) and interior of $\Sigma_{g,w}$ (right) for some $S \in K(w) \smallsetminus K(\bar{w})$.

The main theorems

A partition $P_1 \sqcup \cdots \sqcup P_r$ of $[n]$ is w-admissible if for all $1 \leq j \leq r$,

$$\sum_{i \in P_j} w_i \leq 1.$$

Let $N_{r,w}$ denote the number of w-admissible $[n]$-partitions with r parts.

Let B_g be the g-th Bernoulli numbers ($B_1 = \pm\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30} \dots$)

Theorem (Kannan-L.-Serpente-Yun)

$$\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{n} N_{r,w} (-1)^r \frac{(g+r-2)!}{g!} B_g.$$

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Proof.
Analyze the stratification of the coarse moduli scheme $M_{g,w}$ of $\mathcal{M}_{g,w}$, write $[M_{g,w}]$ in the Grothedieck group of varieties as a decomposition into $[M_{g,r}]$. Then use results on top weight Euler characteristics of $M_{g,1^{(r)}}$ by Chan-Faber-Galatius-Payne in [CFGP20]. □

Work in progress

1. Use an approach by Hatcher in [Hat95, HV04] to define $S(M, w)$ parametrizing w-stable genus-g graphs as the subcomplex of the space $S(M)$ of embedded 2-spheres for the 3-manifold M which is the connected sum of n copies of $S^1 \times S^2$ deleting g open disks.

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4. Since the group $\mathbb{G}_{g,n}$ has generating sets with fixed points in $S(M, w)$, by [Arm68, Theorem 4], $\Delta_{g,w} = S(M, w)/\mathbb{G}_{g,n}$ has trivial fundamental group.
   (Thanks to Sam Payne for pointing out this GGT approach after writing of the manuscript.)

Thank you!

Ask me questions.

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Each $\Delta_{g,\bar{w}}(S)$ is simply connected

A technical lemma: If $\sum_{i\in S} w_i \leq 1$, then

$$\Delta_{g,w}(S) = \Delta_{g,w^S};$$

Otherwise

$$\Delta_{g,w}(S) \cong \mathrm{Cone}(\Delta_{g,w^S}).$$

Here, w^S is the weight vector removing weights indexed by S and appending

$$\min\left(\sum_{i\in S} w_i, 1\right).$$

Example

$w = \left(\frac{1}{4}, \frac{2}{3}, \frac{1}{2}, 1\right).$

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Example

$w = \left(\frac{1}{4}, \frac{2}{3}, \frac{1}{2}, 1\right)$.

- $S = \{1, 3\}$, $w^S = \left(\frac{2}{3}, 1, \frac{3}{4}\right)$;

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- $T = \{2, 3\}$, $w^T = \left(\frac{1}{4}, 1, 1\right)$.