TOPOLOGY OF TROPICAL MODULI SPACES OF WEIGHTED STABLE CURVES IN HIGHER GENUS

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Abstract. Given an integer $g \geq 0$ and a vector $w \in (\mathbb{Q} \cap (0,1])^n$ such that $2g-2+\sum w_i > 0$, we study the topology of the moduli space $\Delta_{g,w}$ of w-stable tropical curves of genus g with volume 1. The space $\Delta_{g,w}$ is the dual complex of the divisor of singular curves in Hassett's moduli space of w-stable genus g curves $\overline{\mathcal{M}}_{g,w}$. When $g \geq 1$, we show that $\Delta_{g,w}$ is simply connected for all values of w. We also give a formula for the Euler characteristic of $\Delta_{g,w}$ in terms of the combinatorics of w.

1. Introduction

Fix an integer $g \geq 0$ and a vector of rational weights $w \in (\mathbb{Q} \cap (0,1])^n$ satisfying

$$2g - 2 + \sum_{i=1}^{n} w_i > 0.$$

We study the topology of the moduli space $\Delta_{g,w}$ of w-stable tropical curves of genus g and volume 1. Following the work of Ulirsch [Uli15] and Chan, Galatius, and Payne [CGP18], the space $\Delta_{g,w}$ can be realized as the dual complex of the normal crossings divisor $\overline{\mathcal{M}}_{g,w} \smallsetminus \mathcal{M}_{g,w}$ on Hassett's moduli stack $\overline{\mathcal{M}}_{g,w}$; here $\mathcal{M}_{g,w}$ denotes the dense open substack of $\overline{\mathcal{M}}_{g,w}$ parameterizing smooth but not necessarily distinctly marked curves. Our first main theorem is that $\Delta_{g,w}$ is simply connected for $g \geq 1$.

Theorem 1.1. *For any $g \geq 1$ and $w \in (\mathbb{Q} \cap (0,1])^n$, the space $\Delta_{g,w}$ is simply connected.*

Our second result is a calculation of the Euler characteristic of $\Delta_{g,w}$ in terms of the *top weight* Euler characteristics of the moduli spaces $\mathcal{M}_{g,r}$ of smooth r-marked algebraic curves of genus g; see essential background on weight filtrations and mixed Hodge structures in Section 1.1. We call a partition $P_1 \sqcup \cdots \sqcup P_r \vdash [n]$ w-***admissible*** if $\sum_{i \in P_j} w_i \leq 1$ for all $1 \leq j \leq r$. Let $N_{r,w}$ denote the number of w-admissible partitions of $[n]$ with r parts.

Theorem 1.2. *Let $W = W_1 \subset \cdots \subset W_{6g-6+2r} \subseteq H^*(\mathcal{M}_{g,r}, \mathbb{Q})$ be the weight filtration of the rational singular cohomology of the moduli stack $\mathcal{M}_{g,r}$ and denote by $\chi^W_{6g-6+2r}$ the Euler characteristic of the top graded piece*

$$\mathrm{Gr}^W_{6g-g+2r} H^*(\mathcal{M}_{g,r}; \mathbb{Q}) = W_{6g-6+2r}/W_{6g-7+2r}$$

of the weight filtration. Then

$$\chi(\Delta_{g,w}) = 1 - \sum_{r=1}^{n} N_{r,w} \cdot \chi^W_{6g-6+2r}(\mathcal{M}_{g,r}).$$

A generating function for the numbers $\chi^W_{6g-6+2r}(\mathcal{M}_{g,r})$ is given in [CFGP19]; together with their result, Theorem 1.2 allows for the computer-aided calculation of $\chi(\Delta_{g,w})$ for arbitrary

g and w. In [CFGP19, Corollary 8.1], the authors give a closed form in the case when the number of marked points is large. For $r > g+1$,

$$\chi^W_{6g-6+2r}(\mathcal{M}_{g,r}) = (-1)^{r+1}\frac{(g+r-2)!}{g!}B_g,$$

where B_g is the g-th Bernoulli number, characterized by

$$\frac{t}{e^t-1} = \sum_{\ell=0}^{\infty} B_\ell \frac{t^\ell}{\ell!}.$$

Substituting into Theorem 1.2 yields the following closed form.

Corollary 1.3. *Given a weight vector w, such that $N_{r,w} = 0$ for $r \le g+1$, the Euler characteristic of $\Delta_{g,w}$ is*

$$\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{n} N_{r,w}(-1)^r \frac{(g+r-2)!}{g!}B_g$$

Let $S(m,r)$ denote the number of r-partitions of $[m]$ for $m \ge 0$ and $r \ge 0$; these are called *the Stirling numbers of the second kind.* Expanding the Bernoulli number B_g (see [Apo98]) in terms of Stirling numbers

$$B_g = \sum_{\ell=0}^{g}(-1)^\ell \frac{\ell!}{\ell+1}S(g,\ell),$$

we obtain the following closed form for the Euler characteristic of $\Delta_{g,w}$ for *heavy/light* weights.

Corollary 1.4. *Given a heavy/light weight vector $w = (1^{(n)}, \varepsilon^{(m)})$ where $n \ge g+1$, $m > 0$, and $\varepsilon < 1/m$,*

$$\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{m}\sum_{\ell=0}^{g}(-1)^{n+r+\ell}\frac{(g+n+r-2)!\ell!}{g!(\ell+1)}S(m,r)S(g,\ell).$$

Using this corollary above, we compute explicitly the Euler characteristics of $\Delta_{g,(1^{(n)},\varepsilon^{(m)})}$ in the following tables.

$g=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=2$	-	2	0	2
$n=3$	3	-3	9	-15
$n=4$	-5	19	-53	163
$n=5$	25	-95	385	-1535

$g=1$	$m=1$	$m=2$	$m=3$	$m=4$
$n=2$	2	-1	5	-7
$n=3$	-2	10	-26	82
$n=4$	13	-47	193	-767
$n=5$	-59	301	-1499	7501

$g=2$	$m=1$	$m=2$	$m=3$	$m=4$
$n=3$	3	-7	33	-127
$n=4$	-9	51	-249	1251
$n=5$	61	-359	2161	-12959
$n=6$	-419	2941	-20579	144061

$g=3$	$m=1$	$m=2$	$m=3$	$m=4$
$n=4$	1	1	1	1
$n=5$	1	1	1	1
$n=6$	1	1	1	1
$n=7$	1	1	1	1

Table 1: Euler characteristics of $\Delta_{g,(1^{(n)},\varepsilon^{(m)})}$ for $g = 0,1,2,3$ and some (n,m) where $n \ge g+1$ and $m > 0$. When $g = 0$, we start with $n = 2$ since the space $\Delta_{0,(1,\varepsilon^{(m)})}$ is empty. When $g = 0, n = 2, m = 1$, $\Delta_{0,(1,1,\varepsilon)}$ is also empty.

1.1. **Motivation.** Throughout the paper, we work over the complex numbers $\mathbb{C}$. The Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{g,n}$ is a toroidal compactification of the moduli stack $\mathcal{M}_{g,n}$; the toroidal structure comes from the fact that the boundary divisor $\overline{\mathcal{M}}_{g,n} \smallsetminus \mathcal{M}_{g,n}$ has normal crossings. As a Deligne-Mumford stack, the rational cohomology of $\mathcal{M}_{g,n}$ carries a mixed Hodge structure. That is, there is a weight filtration

$$W_1 \subset \cdots \subset W_{6g-g+2n} = H^*(\mathcal{M}_{g,n};\mathbb{Q})$$

such that, for each j, the quotient

$$\mathrm{Gr}_i^W H^j(\mathcal{M}_{g,n},\mathbb{Q}) = W_i \cap H^j(\mathcal{M}_{g,n};\mathbb{Q})/W_{i-1} \cap H^j(\mathcal{M}_{g,n};\mathbb{Q})$$

carries a pure Hodge structure of weight i. The top graded piece of the weight filtration of $\mathcal{M}_{g,n}$ can be identified with the reduced homology of the dual complex of the divisor $\overline{\mathcal{M}}_{g,n} \smallsetminus \mathcal{M}_{g,n}$, up to a degree shift. As discussed in [CGP18, CGP19], the dual complex of this divisor may be identified with the tropical moduli space $\Delta_{g,n}$, furnishing isomorphisms

$$\widetilde{H}_{j-1}(\Delta_{g,n};\mathbb{Q}) \cong \mathrm{Gr}^W_{6g-6+2n} H^{6g-6+2n-j}(\mathcal{M}_{g,n};\mathbb{Q}).$$

In [Has03], Hassett introduced the moduli stack $\overline{\mathcal{M}}_{g,w}$ as an alternative compactification of $\mathcal{M}_{g,n}$: in $\overline{\mathcal{M}}_{g,w}$, marked points are allowed to coincide if the sum of the corresponding entries of w is no greater than 1. Thus $\overline{\mathcal{M}}_{g,w}$ contains an open substack $\mathcal{M}_{g,w}$ parameterizing smooth, but not necessarily distinctly marked algebraic curves of genus g, and we have the containments $\mathcal{M}_{g,n} \subset \mathcal{M}_{g,w} \subset \overline{\mathcal{M}}_{g,w}$. Although the embedding $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,w}$ is no longer toroidal, $\overline{\mathcal{M}}_{g,w} \smallsetminus \mathcal{M}_{g,w}$ is still a normal crossings divisor, and the dual complex of this divisor has a natural modular interpretation as the moduli space $\Delta_{g,w}$ of tropical w-stable curves of volume 1 and genus g; see [Uli15, CHMR14]. Therefore, one has isomorphisms

$$\widetilde{H}_{j-1}(\Delta_{g,w};\mathbb{Q}) \cong \mathrm{Gr}^W_{6g-6+2n} H^{6g-6+2n-j}(\mathcal{M}_{g,w};\mathbb{Q}),$$

identifying the reduced rational homology of $\Delta_{g,w}$ with the top graded piece of the rational cohomology of $\mathcal{M}_{g,w}$.

1.2. **Previous work.** This work benefits from and builds on previous work of many authors on the topology of tropical moduli spaces, which we summarize here.

When $g = 0$, the complex $\Delta_{0,w}$ may be identified with various objects whose homotopy types are known.

(1) When $w = (1^{(n)})$, Vogtmann showed that $\Delta_{0,n}$ is homotopic to a wedge of $(n-2)!$ spheres of dimension $n-4$, by identifying it as the link of a vertex of a quotient simplicial complex by the outer automorphisms of a finitely generated free group; see [CV86, Vog90]. In [RW96], Robinson and Whitehouse gave a different proof of the same result, by contracting a large subcomplex $X_{0,n}$ of $\Delta_{0,n}$.
(2) When w is *heavy/light*, i.e. $w = (1^{(n)}, \varepsilon^{(m)})$ for $\varepsilon < 1/m$, Cavalieri, Hampe, Markwig, and Ranganathan in [CHMR14] identified $\Delta_{0,w}$ with the link at the origin of the Bergman fan of a graphic matroid, thereby deriving that $\Delta_{0,w}$ is homotopic to a wedge of $(n-2)!(n-1)^m$ spheres of dimension $n+m-4$, using [AK06]. In [CMP+20], Cerbu, Marcus, Peilen, Ranganathan, and Salmon rederived this result using Vogtmann's result on $\Delta_{0,n}$ and contracting a large subcomplex.
(3) When w has at least two weight-1 entries, Cerbu et al. in [CMP+20] showed that $\Delta_{0,w}$ is homotopic to a wedge of spheres of possibly varying dimensions, by identifying a large contractible subcomplex and using known results on homotopy types of

subspace arrangements. The authors also provided infinite families of w where $\Delta_{0,w}$ is disconnected, and examples where $\pi_1(\Delta_{0,w}) = \mathbb{Z}/2\mathbb{Z}$. In the latter scenario, the authors proved that the universal cover has the homotopy type of a wedge of spheres.

For higher values of g, the following results are known.

(1) When $w = (1^{(n)})$, Chan, Galatius, and Payne showed in [CGP19] that $\Delta_{1,n}$ is homotopic to $\frac{1}{2}(n-1)!$ spheres of dimension $n-1$. In [Cha15], Chan independently showed that the reduced rational homology $\widetilde{H}_*(\Delta_{2,n}, \mathbb{Q})$ is concentrated in the top two degrees and computed their ranks for $n \leq 8$. Chan also proved that $\widetilde{H}_*(\Delta_{2,n}, \mathbb{Z})$ has torsion in high degrees. For higher genera, Chan, Galatius, and Payne showed that $\Delta_{g,n}$ is at least $(n-3)$-connected [CGP19].
(2) When w has at least two weight-1 entries, [CMP+20] leveraged a relation between $\Delta_{0,w}$ and $\Delta_{1,w}$ to prove that $\Delta_{1,w}$ is homotopic to a wedge of spheres.
(3) When $w = (1^{(n)}, \varepsilon^{(m)})$ is heavy/light, the same authors showed that $\Delta_{1,w}$ is homotopic to $\frac{1}{2}(n-1)!n^m$ spheres of dimension $n+m-1$.

Most recently, in [ACP19], Allcock, Corey, and Payne showed that Δ_g and $\Delta_{g,n}$ are simply connected for $(g,n) \neq (0,4), (0,5)$. They give two proofs of this result; one relies on a spectral sequence associated with the filtration of $\Delta_{g,n}$ by its p-skeleta, and the other uses Harer's result [Har86] that Harvey's complex of curves $\mathcal{C}_{g,n}$ is simply connected, together with the fact that $\Delta_{g,n}$ is homeomorphic to the quotient of the complex $\mathcal{C}_{g,n}$ by the action of the pure mapping class group. The complexity of $\Delta_{g,w}$ grows dramatically with the genus and the combinatorics of w. The working framework in this paper is based on graph categories and symmetric Δ-complexes, heavily used in [CGP18, CGP19]. The main tools are the classical Seifert-van Kampen theorem from topology and the Grothendieck group of varieties.

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2. Background

2.1. **The graph categories $\Gamma_{g,w}$.** Given integers $n \geq 0, g \geq 0$ and a vector $w \in (\mathbb{Q} \cap (0,1])^n$ of rational numbers satisfying

$$2g - 2 + \sum_{i=1}^{n} w_i > 0, \tag{2.1}$$

we define a graph category $\Gamma_{g,w}$. All graphs considered in this paper allow loops and parallel edges. First, a weighted n-marked graph $\mathbf{G}$ is a triplet $\mathbf{G} = (G, m, h)$ consisting of a finite connected graph G together with a marking function $m : \{1, \ldots, n\} \to V(G)$ and a vertex weight function $h : V(G) \to \mathbb{Z}_{\geq 0}$. We say $\mathbf{G}$ is w-stable if it satisfies the stability condition

$$2h(v) - 2 + \mathrm{val}(v) + \sum_{i \in m^{-1}(v)} w_i > 0, \tag{2.2}$$

for all $v \in V(G)$, where $\mathrm{val}(v)$ denotes the valence of v in G, i.e., the number of half edges incident to v (a loop contributes twice to the valence of its vertex). The genus of $\mathbf{G}$ is defined as

$$\mathbf{g}(\mathbf{G}) := b^1(G) + \sum_{v \in V(G)} h(v),$$

where $b^1(G) = |E(G)| - |V(G)| + 1$ is the first Betti number of G. The objects in $\Gamma_{g,w}$ are w-stable weighted n-marked graphs of genus g; we sometimes refer to them as "combinatorial types."

The morphisms in $\Gamma_{g,w}$ are maps that factor as compositions of isomorphisms and edge contractions. To be precise, an *isomorphism* $\varphi : \mathbf{G} \to \mathbf{G}'$ where $\mathbf{G} = (G, m, h)$ and $\mathbf{G}' = (G', m', h')$ is an isomorphism $\varphi : G \to G'$ such that $m' = \varphi \circ m$ and $h' \circ \varphi = h$. An *edge contraction* $c : \mathbf{G} \to \mathbf{G}/e$ of an edge e in G is given by removing e and identifying its two endpoints if e is not a loop, and by removing e and increasing the weight of its base vertex by one if e is a loop. Formally, the edge-contracted object is $\mathbf{G}/e := (G/e, m', h')$ where G/e is obtained by removing e and identifying its endpoints as one vertex $[e]$. Suppose e has endpoints u and v in G, not necessarily distinct. Then the new marking function m' is defined by

$$m'(i) = \begin{cases} m(i) & \text{if } i \notin m^{-1}(\{u, v\}), \\ [e] & \text{otherwise,} \end{cases}$$

and the new vertex weight function h' is defined by

$$h'(w) = \begin{cases} h(w) & \text{if } w \neq [e], \\ h(w) + 1 & \text{if } w = [e] \text{ and } e \text{ is a loop,} \\ h(u) + h(v) & \text{if } w = [e] \text{ and } e \text{ is not a loop,} \end{cases}$$

where $w \in V(G/e) = V(G) \smallsetminus \{u, v\} \cup \{[e]\}$. See Figure 1 and 2. We will say that $\mathbf{G}'$ is an *uncontraction* of a graph $\mathbf{G}$ if $\mathbf{G}'$ contracts to $\mathbf{G}$ after a series of edge contractions. One can alternatively describe $\Gamma_{g,w}$ as the full subcategory of the category $\Gamma_{g,n}$ defined in [CGP19] whose objects are graphs in $\Gamma_{g,n}$ that satisfy (2.2); when $w = (1^{(n)})$, $\Gamma_{g,n} = \Gamma_{g,w}$. As to work with a small category, we hereafter tacitly replace $\Gamma_{g,n}$ with a choice of skeleton thereof. This also induces a choice of skeleton of $\Gamma_{g,w}$ for all weight vectors w. We will give two equivalent definitions of the tropical moduli space $\Delta_{g,w}$ using $\Gamma_{g,w}$.

2.2. **Direct description of $\Delta_{g,w}$ as a colimit.** The first description of $\Delta_{g,w}$ is as a colimit of a diagram of topological spaces indexed by $\Gamma_{g,w}$. For each $\mathbf{G} \in \Gamma_{g,w}$, we define

$$\sigma(\mathbf{G}) := \left\{ \ell : E(\mathbf{G}) \to \mathbb{R}_{\geq 0} \mid \sum_{e \in E(\mathbf{G})} \ell(e) = 1 \right\} \subset \mathbb{R}_{\geq 0}^{E(\mathbf{G})}.$$

If $|E(\mathbf{G})| = p + 1$ for $p \geq 0$, then any choice of ordering of the edges of $\mathbf{G}$ induces an identification of $\sigma(\mathbf{G})$ with the standard p-simplex σ^p in $\mathbb{R}_{\geq 0}^{p+1}$. If $|E(\mathbf{G})| = 0$, $\mathbf{G}$ indexes the empty set. Any $\Gamma_{g,w}$-morphism $\mathbf{G} \to \mathbf{H}$ induces an map $\sigma(\mathbf{H}) \to \sigma(\mathbf{G})$ which identifies

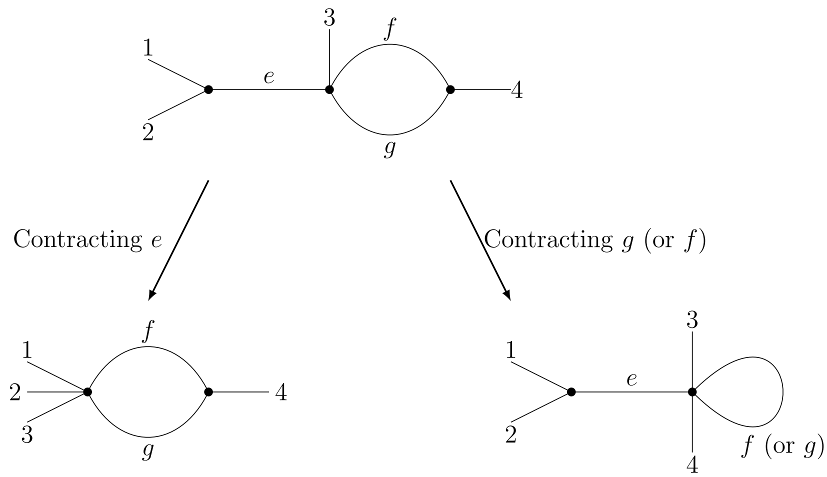

FIGURE 1. Contraction of non-loop edges of an object $\mathbf{G}$ in $\Gamma_{g,w} \subset \Gamma_{g,n}$. In the left contraction, the weight (not shown) of the vertex on the left is the sum of the endponts of e. In the right contraction, the weight on the right vertex is the sum of the endpoints of both f and g. Note that the genus $\mathbf{g}(\mathbf{G})$ remains constant and the stability condition is satisfied at all vertices after either contraction.

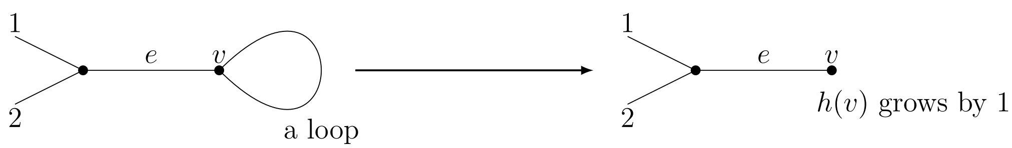

FIGURE 2. Contraction of a loop edge of an object in $\Gamma_{g,w} \subset \Gamma_{g,n}$. Note that $\mathbf{g}(\mathbf{G})$ remains constant and the stability condition is satisfied at v.

$\sigma(\mathbf{H})$ as a face of $\sigma(\mathbf{G})$. Then the tropical moduli space $\Delta_{g,w}$ can be defined as the colimit of the induced diagram of topological spaces:

$$\Delta_{g,w} := \operatorname{colim}_{\mathbf{G}\in\Gamma_{g,w}} \sigma(\mathbf{G}).$$

Using this description, we may refer to a point in $\Delta_{g,w}$ by specifying a pair $(\mathbf{G}, \ell)$ consisting of $\mathbf{G} \in \Gamma_{g,w}$ and ℓ a real, nonnegative length function on the edges of $\mathbf{G}$, such that the total edge length is equal to 1. When ℓ vanishes on an edge e, the point $(\mathbf{G}, \ell)$ is identified with the induced length function on $\mathbf{G}/e$, where $\mathbf{G}/e$ denotes the graph obtained from $\mathbf{G}$ by contracting e. Replacing $\sigma(\mathbf{G})$ with

$$C(\mathbf{G}) := (\mathbb{R}_{\geq 0} \sqcup \{\infty\})^{E(\mathbf{G})} = \{\ell : E(\mathbf{G}) \to \mathbb{R}_{\geq 0} \sqcup \{\infty\}\}$$

and taking an analogous colimit, we get instead the tropical moduli space $\overline{M}^{\mathrm{trop}}_{g,w}$, which arises as the skeleton of the Berkovich analytification of $\overline{M}_{g,w}$; see [Uli15]. This space lives in the category of *extended generalized cone complexes* as described by Abramovich, Caporaso, and

Payne in [ACP15]. The space $\Delta_{g,w}$ can then be identified with the link of $\overline{M}_{g,w}^{\mathrm{trop}}$ at the cone point, corresponding to the unique object of $\Gamma_{g,w}$ with no edges.

2.3. **Description of $\Delta_{g,w}$ as a symmetric Δ-complex.** The space $\Delta_{g,w}$ is also the geometric realization of a symmetric Δ-complex, in the sense of [CGP18]. Let I be the category having one object for each finite set

$$\overline{[p]} := \begin{cases} \{0,\dots,p\} & \text{for } p \geq 0, \\ \varnothing & p = -1. \end{cases}$$

and morphisms consisting of all injections. A symmetric Δ-complex X is a functor $X : I^{\mathrm{op}} \to \mathsf{Sets}$, and a morphism of symmetric Δ-complexes is a natural transformation of functors. For simplicity, we write $X(p)$ for $X(\overline{[p]})$. There is a geometric realization functor associating a topological space to each symmetric Δ-complex X, which we now describe. Each injection $\iota : \overline{[p]} \to \overline{[q]}$ induces a map on standard simplices $\iota_* : \sigma^p \to \sigma^q$, defined by

$$\iota_*\left(\sum_{i=0}^{p} t_i e_i\right) = \sum_{i=0}^{q}\left(\sum_{j\in\iota^{-1}(i)} t_j\right) e_i.$$

Then the geometric realization of X is defined as

$$|X| := \left(\coprod_{p\geq 0} X(p)\times\sigma^p\right) \Big/ \sim$$

where the equivalence relation $\sim$ is generated by relations of the form

$$(X(\iota)(x), a) \sim (x, \iota_*(a)),$$

whenever $\iota \in \mathrm{Hom}_I\left(\overline{[p]}, \overline{[q]}\right)$, $x \in X(p)$, and $a \in \sigma^p$. We will abuse notation and use $\Delta_{g,w}$ both for the functor and its geometric realization. We set

$$\Delta_{g,w}(p) = \{(\mathbf{G},\tau) \mid \mathbf{G} \in \Gamma_{g,w}, |E(\mathbf{G})| = p+1,\ \tau : E(\mathbf{G}) \to \overline{[p]} \text{ a bijection}\}/\sim,$$

where $(\mathbf{G},\tau) \sim (\mathbf{G}',\tau')$ if and only if there exists a $\Gamma_{g,w}$-isomorphism $\varphi : \mathbf{G} \to \mathbf{G}'$ such that the diagram

$$\begin{tikzcd} E(\mathbf{G}) \arrow[rr, "\varphi"] \arrow[dr, "\tau"'] & & E(\mathbf{G}') \arrow[dl, "\tau'"] \\ & \overline{[p]} & \end{tikzcd}$$

commutes. We put $[\mathbf{G},\tau]$ for the equivalence class of $(\mathbf{G},\tau)$. On morphisms, we define $\Delta_{g,w}$ as follows: given an injection $\iota : \overline{[p]} \to \overline{[q]}$ and $[\mathbf{G},\tau] \in \Delta_{g,w}(q)$, we set $\mathbf{H}$ to be the graph obtained from $\mathbf{G}$ by contracting all edges which are not labelled by $\tau^{-1}(\iota(\overline{[p]}))$, and $\pi : E(\mathbf{H}) \to \overline{[p]}$ to be the unique edge-labelling of $\mathbf{H}$ which preserves the order of the remaining edges. Then $\Delta_{g,w}(\iota)([\mathbf{G},\tau]) = [\mathbf{H},\pi]$.

Note that whenever $w, w' \in (\mathbb{Q}\cap(0,1])^n$ satisfy $w_i \leq w'_i$ for all i, we can identify $\Delta_{g,w}$ as a subcomplex (subfunctor) of $\Delta_{g,w'}$: there is an equality

$$\Delta_{g,w}(p) = \{[\mathbf{G},\tau] \in \Delta_{g,w'}(p) \mid \mathbf{G} \text{ is } w\text{-stable}\}.$$

For now, we illustrate this point via the following examples. In Section 3, we will use the symmetric Δ-complex description to make maps between various subcomplexes of $\Delta_{g,w}$ for comparable weight vectors.

Example 2.3. Let $g = 1$ and $w = (\varepsilon^{(3)})$ for $0 < \varepsilon < 1/3$. We describe $\Delta_{1,w}$ by its skeleta. The 0-skeleton of $\Delta_{1,w}$, i.e.

$$\Delta_{1,w}(0) = \{\mathbf{G} \in \Delta_{1,w} : |E(\mathbf{G})| = 1\}$$

has only a 0-simplex coming from the graph shown in Figure 3. The elements in the 1-skeleton $\Delta_{1,w}(1) = \{\mathbf{G} \in \Delta_{1,w} : |E(\mathbf{G})| = 2\}$ have three combinatorial types shown in Figure 4. Since contracting any edge in any graph in Figure 4 gives $\mathbf{G}_0$, the endpoints of all 1-simplices indexed by combinatorial types of $\Delta_{1,w}(1)$ are identified with the point corresponding to $\mathbf{G}_0$. Moreover, each combinatorial type of an element in $\Delta_{1,w}(1)$ admits a $\mathbb{Z}/2\mathbb{Z}$ automorphism induced by permuting the top and bottom edges. The resulting space $\Delta_{1,w}(1)$ is thus three half-edges glued at the point $\mathbf{G}_0$; see Figure 5. Lastly, there is only one combinatorial type for elements in $\Delta_{1,w}(2)$, yielding a 2-simplex $\mathbf{T}$; see Figure 6. The 2-simplex has no self-gluings in its interior since $\mathrm{Aut}(\mathbf{T})$ is trivial. The only gluings happen on the boundary of the 2-simplex: these are exactly the self-gluings of the 1-simplices seen before. The resulting space $\Delta_{1,w}$ after gluings is homeomorphic to a 2-sphere and indeed simply-connected; see Figure 7.

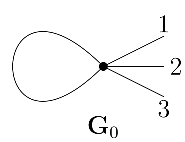

FIGURE 3. The combinatorial type of the only curve in $\Delta_{1,w}(0)$ for $w = (\varepsilon^{(3)})$.

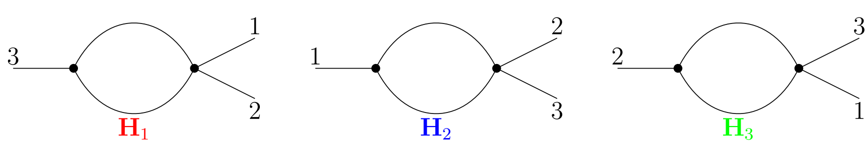

FIGURE 4. Combinatorial types in $\Delta_{1,w}(1)$, for $w = (\varepsilon^{(3)})$.

Example 2.4. Let $g = 1$, $w = (\varepsilon^{(3)})$, and $w' = (1, \varepsilon^{(2)})$. Since $w \leq w'$, $\Delta_{1,w'}$ contains $\Delta_{1,w}$. The space $\Delta_{1,w'}$ contains new simplices in its 0-skeleton,1-skeleton and 2-skeleton, corresponding to graphs shown in Figure 8, 9, 10 and 12, respectively. The boldface **1** at a vertex indicates that the vertex has weight 1. The 1-simplices have as endpoints the 0-simplices according to the following graph contractions: $\mathbf{E}_1$ contracts to $\mathbf{G}_1$ and $\mathbf{G}_2$, $\mathbf{E}_2$

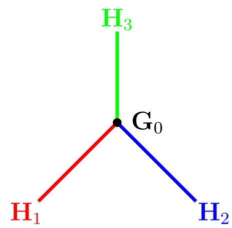

FIGURE 5. The geometric realization of the 1-skeleton of the symmetric Δ-complex $\Delta_{1,w}$ when $w = (\varepsilon^{(3)})$. We use colors to distinguish the simplexes.

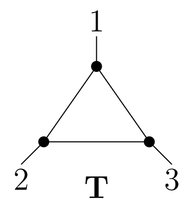

FIGURE 6. The only $(\varepsilon^{(3)})$-stable graph with 3 edges.

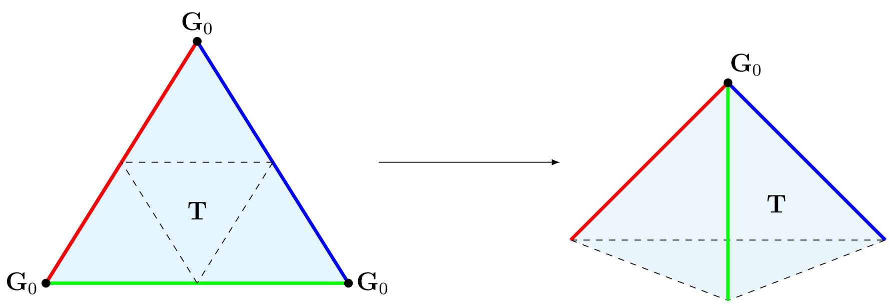

FIGURE 7. On the left, we have the 2-simplex before the gluings given by automorphisms of the combinatorial types in $\Delta_{1,w}(1)$. The three vertices of the simplex are glued together since they correspond to the same combinatorial types in $\Delta_{1,w}(0)$. The geometric realization of $\Delta_{1,w}$ is the (hollow) tetrahedron on the right, resulting from these gluings.

contracts to $\mathbf{G}_3$ and $\mathbf{G}_2$, $\mathbf{F}_1$ contracts to $\mathbf{G}_1$ and $\mathbf{G}_0$, $\mathbf{F}_2$ contracts to $\mathbf{G}_2$ and $\mathbf{G}_0$, $\mathbf{F}_3$ contracts to $\mathbf{G}_0$. Modding out by appropriate automorphisms and thus folding some 1-simplices into half-edges, the space $\Delta_{1,w'}(1)$ is shown in Figure 11. The 2-simplices corresponding to the graphs in the 2-skeleton have boundaries formed by 1-simplices corresponding to 1-edge graphs: $\mathbf{R}_1$ contracts to $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{E}_1$; $\mathbf{R}_2$ contracts to $\mathbf{F}_3$, $\mathbf{F}_2$ and $\mathbf{E}_2$; $\mathbf{P}_1$ contracts to $\mathbf{H}_1$ and to $\mathbf{F}_1$ on its parallel edges, and $\mathbf{P}_2$ contracts to $\mathbf{H}_2$ and to $\mathbf{F}_2$ on its parallel edges. The identifications on these simplices are described in Figuers 13 and 14. All the quotients of the two simplices are then glued together according to edge contractions. The resulting space $\Delta_{1,w'}$ is evidently simply connected and contains $\Delta_{1,w}$; see Figure 15.

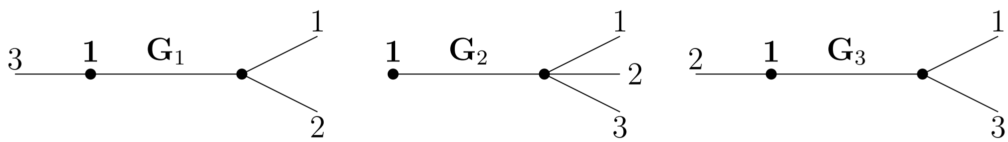

Figure 8. Combinatorial types in $\Delta_{1,w'}(1) \smallsetminus \Delta_{1,w}(1)$.

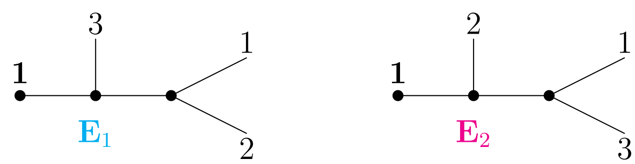

Figure 9. Combinatorial types in $\Delta_{1,w'}(1) \smallsetminus \Delta_{1,w}(1)$ that have trivial first Betti number.

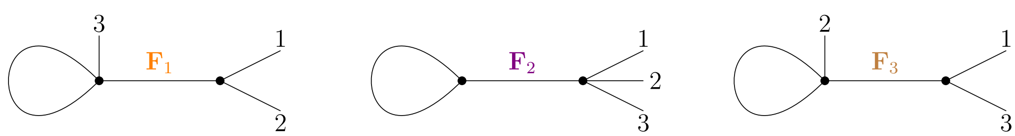

Figure 10. Combinatorial types in $\Delta_{1,w'}(1) \smallsetminus \Delta_{1,w}(1)$ that have non-trivial first Betti number.

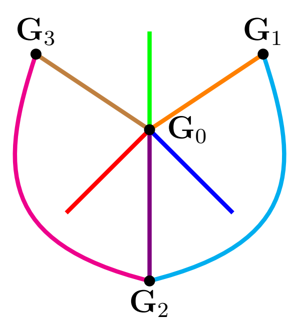

Figure 11. The geometric realization of the 1-skeleton of the symmetric Δ-complex $\Delta_{1,w'}$ when $w' = (1, \varepsilon, \varepsilon)$. The inclusion $\Delta_{1,w}(1) \subset \Delta_{1,w'}(1)$ is clearly visible in the picture.

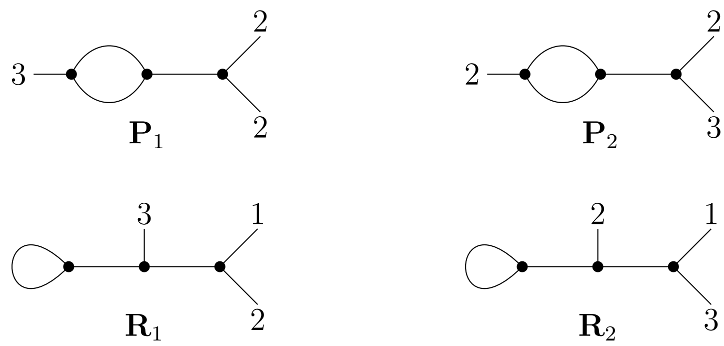

FIGURE 12. Combinatorial types in $\Delta_{g,w'}(2) \smallsetminus \Delta_{g,w}(2)$

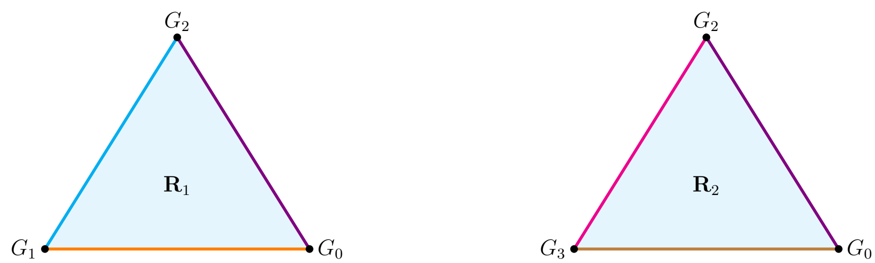

FIGURE 13. Each of $\mathbf{R}_1$ and $\mathbf{R}_2$ has a nontrivial automorphism which "flips" the loop but does not induce self-gluings. Moreover, there are no gluings of edges or vertices, so the 2-simplices in $\Delta_{1,w'}(2)$ remain solid triangles with edges being 1-simplices and with 3 distinct vertices.

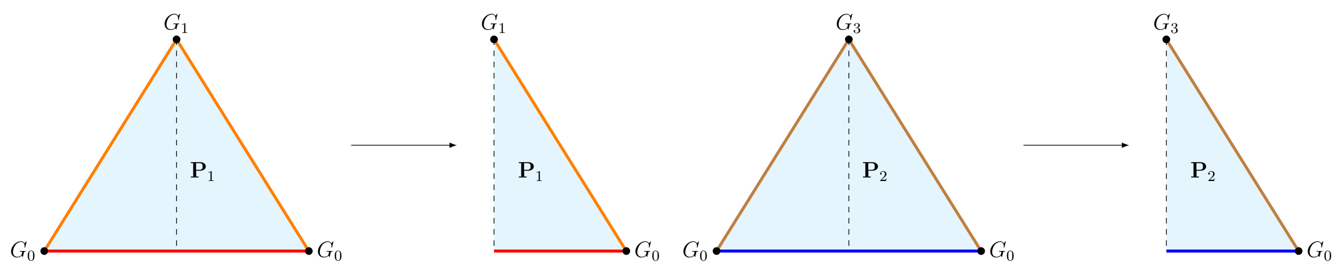

FIGURE 14. Each of $\mathbf{P}_1$ and $\mathbf{P}_2$ has a nontrivial automorphism which induces a self-gluing on the corresponding simplex. The resulting 2-simplex has an edge corresponding to a simplex with two distinct vertices, an edge corresponding to the self-gluing of a 1-simplex, and an edge which is not a simplex.

3. $\Delta_{g,w}$ IS SIMPLY CONNECTED

Let $g \geq 1$, and fix a weight vector $w \in (\mathbb{Q} \cap (0,1])^n$. We will prove that $\pi_1(\Delta_{g,w})$ is trivial. Our proof relies on the fact that $\pi_1(\Delta_{g,w})$ is trivial when $w = (1^{(n)})$, shown by Allcock, Corey, and Payne [ACP19].

3.1. **Subcomplexes of $\Delta_{g,w}$.** Recall the notation $[n] = \{1, \ldots, n\}$. For each subset $S \subseteq [n]$, we define a subcomplex $\Delta_{g,w}(S) \subseteq \Delta_{g,w}$ as follows. Given a w-stable weighted n-marked graph $\mathbf{G} = (G, m, h)$ and a subset $S \subseteq [n]$, we call $v \in V(\mathbf{G})$ an S***-vertex*** if $S \subseteq m^{-1}(v)$.

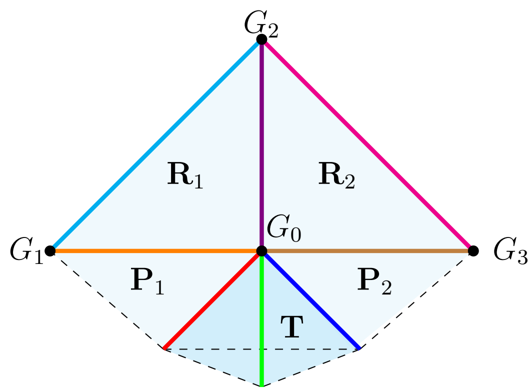

FIGURE 15. The geometric realization of the space $\Delta_{1,w'}$, for $w' = (1, \varepsilon^{(2)})$. Again we can see $\Delta_{1,w} \subset \Delta_{1,w'}$ and that $\Delta_{1,w'}$ is simply connected.

Definition 3.1. Given a $S \subseteq [n]$, we define a subcategory $\Gamma_{g,w}(S)$ of $\Gamma_{g,w}$ by setting the objects of $\Gamma_{g,w}(S)$ to be

$$\{\mathbf{G} \in \Gamma_{g,w} : \mathbf{G} \text{ has an } S\text{-vertex}\}.$$

In the same way, we define a subcomplex $\Delta_{g,w}(S)$ of $\Delta_{g,w}$ by defining $\Delta_{g,w}(S)(p)$ as

$$\{[\mathbf{G}, \tau] \in \Delta_{g,w}(p) : \mathbf{G} \text{ has an } S\text{-vertex}\}.$$

for each $p \geq -1$.

As defined, $\Delta_{g,w}(S)$ is a subcomplex of $\Delta_{g,w}$ because the property of having an S-vertex is closed under edge contractions. The following lemma follows from the definition of an S-vertex.

Lemma 3.2. *If a collection of subsets $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ of $[n]$ satisfies*

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha \neq \varnothing,$$

then

$$\bigcap_{\alpha \in \mathcal{A}} \Delta_{g,w}(S_\alpha) = \Delta_{g,w}\left(\bigcup_{\alpha \in \mathcal{A}} S_\alpha\right).$$

Define a weight vector $w^S \in (\mathbb{Q} \cap (0,1])^{n-|S|+1}$ by removing from w in order those entries indexed by S, and then appending an entry of weight

$$\min\left(\sum_{i \in S} w_i, 1\right).$$

As an example, if we take $w = (1/4, 2/3, 1/2, 1)$, $S = \{1, 3\}$, and $T = \{2, 3\}$, then $w^S = (2/3, 1, 3/4)$, while $w^T = (1/4, 1, 1)$.

Proposition 3.3. *Let $S \subseteq [n]$. If $\sum_{i \in S} w_i \leq 1$, then we have an isomorphism*

$$\Delta_{g,w}(S) \cong \Delta_{g,w^S}.$$

Otherwise, there is a homeomorphism

$$|\Delta_{g,w}(S)| \cong \mathrm{Cone}(|\Delta_{g,w^S}|),$$

where $|\cdot|$, as we recall, denotes the geometric realization of a symmetric Δ-complex functor.

Proof. For the first part, suppose $\sum_{i\in S} w_i \le 1$. We define a natural transformation of functors

$$\eta : \Delta_{g,w^S} \longrightarrow \Delta_{g,w},$$

by defining its component at $\overline{[p]} \in I^{op}$,

$$\eta_p : \Delta_{g,w^S}(p) \to \Delta_{g,w}(p),$$

as follows. For every $[\mathbf{G},\tau] \in \Delta_{g,w^S}(p)$, the image $\eta_p([\mathbf{G},\tau])$ is the graph obtained replacing the last marking by the set of markings indexed by S (nothing changes on τ). This is a natural transformation since for every $i : \overline{[p]} \to \overline{[q]}$, the following diagram is commutative:

$$\begin{tikzcd} \Delta_{g,w^S}(q) \arrow[r, "\eta_q"] \arrow[d, "\Delta_{g,w^S}(i)"] & \Delta_{g,w}(p) \arrow[d, "\Delta_{g,w}(i)"] \\ \Delta_{g,w^S}(p) \arrow[r, "\eta_p"] & \Delta_{g,w}(p). \end{tikzcd}$$

Indeed, the relative positions of the markings with respect to the last marking (or on the right hand side of the diagram, the S-indexed markings) in each graph involved undergo the same changes during edge contractions and relabellings of edges involved in $\Delta_{g,w}(i)$ and $\Delta_{g,w^S}(i)$. Moreover, the stability at each vertex persists in all the involved graphs under η_p and η_q by the construction of w^S. Therefore, $\eta_p \circ \Delta_{g,w^S}(i) = \Delta_{g,w}(i) \circ \eta_q$.

Furthermore, η is an isomorphism of subfunctors (subcomplexes) between Δ_{g,w^S} and $\Delta_{g,w}(S)$. This amounts to checking that all the components $\eta_p : \Delta_{g,w^S}(p) \to \Delta_{g,w}(S)(p)$ for all $p \ge -1$ are isomorphisms. Indeed, given $[\mathbf{H},\pi] \in \Delta_{g,w}(S)(p)$, the morphism that replaces the set of markings indexed by S by a marking of weight $\sum_{i\in S} w_i$ is the inverse of η_p (again, nothing changes in the edge label function).

For the second part, where $\sum w_i > 1$, we use the colimit description of these tropical moduli spaces in Section 2.2. The goal is to construct a homeomorphism of topological spaces from the cone over Δ_{g,w^S} to $\Delta_{g,w}(S)$

$$f : \left(\Delta_{g,w^S} \times [0,1]\right) / \left(\Delta_{g,w^S} \times \{1\}\right) \cong \Delta_{g,w}(S),$$

where both Δ_{g,w^S} and $\Delta_{g,w}(S)$ are equipped with the final topology. Recall that each point in Δ_{g,w^S} is represented by a pair $(\mathbf{G},\ell)$ where $\mathbf{G} \in \Gamma_{g,w^S}$ and $\ell : E(\mathbf{G}) \to \mathbb{R}_{\ge 0}$. Similarly for each point in $\Delta_{g,w}(S)$. We let w^S be indexed by $([n] \smallsetminus S) \cup \{n+1\}$.

Now to construct the homeomorphism f, for each $\mathbf{G} = (G,m,h) \in \Gamma_{g,w^S}$, we set $f(\mathbf{G})$ to be the graph $\mathbf{G}' = (G',m',h') \in \Gamma_{g,w}(S)$ that is the new graph obtained by adding a new vertex with zero weight, connecting it to the vertex that houses $n+1$, and replacing the marking $n+1$ with S-indexed markings; see Figure 16. Formally, $\mathbf{G}'$ is defined as follows. Let v be the vertex where the marking $n+1$ is attached; that is, $v = m(n+1) \in V(G)$. Let v_0 be a new vertex and set

$$V(G') = V(G) \cup \{v_0\},$$
$$E(G') = E(G) \cup \{v, v_0\}.$$

Set the new marking function $m' : [n] \to V(G')$ to be

$$m'(i) = \begin{cases} m(i) & i \in [n] \smallsetminus S, \\ v_0 & i \in S, \end{cases}$$

and finally, set the new weight function $h' : V(G') \to \mathbb{Z}_{\geq 0}$ to be

$$h'(u) = \begin{cases} h(u) & u \in V(G), \\ 0 & u = v_0. \end{cases}$$

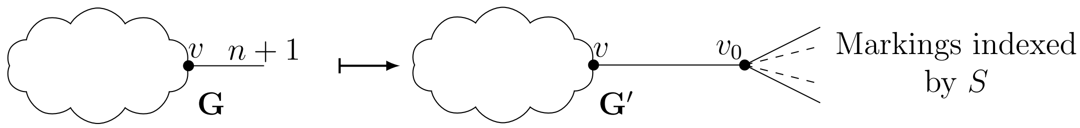

FIGURE 16. A graph $\mathbf{G} \in \Gamma_{g,w^S}$ and $f(\mathbf{G}) = \mathbf{G}' \in \Gamma_{g,w}(S)$. The cloud shapes represent the parts of the graphs that are unchanged under f.

It remains to check that $\mathbf{G}'$ is an object in $\Gamma_{g,w}(S)$. First, notice that v_0 is an S-vertex and that $\mathbf{g}(\mathbf{G}') = \mathbf{g}(\mathbf{G}) = g$. It now suffices to check that v and v_0 satisfy the stability condition. For v we have

$$\begin{aligned} 2h'(v) - 2 + \mathrm{val}_{\mathbf{G}'}(v) + \sum_{i \in (m')^{-1}(v)} w_i &= 2h(v) - 2 + \mathrm{val}_G(v) + 1 + \left(\sum_{i \in m^{-1}(v)} w_i \right) - 1 \\ &= 2h(v) - 2 + \mathrm{val}_G(v) + \sum_{i \in m^{-1}(v)} w_i \\ &> 0, \end{aligned}$$

by stability of v in $\mathbf{G}$. For v_0, we use the assumption that $\sum_{i \in S} w_i > 1$ to obtain

$$2h'(v_0) - 2 + \mathrm{val}_{G'}(v_0) + \sum_{i \in} m'^{-1}(v_0) = 0 - 2 + 1 + \sum_{i \in S} w_i > 0.$$

Now we define the morphism $f : \left(\Delta_{g,w^S} \times [0,1]\right) / \left(\Delta_{g,w^S} \times \{1\}\right) \to \Delta_{g,w}(S)$ by sending

$$((\mathbf{G}, \ell), t) \mapsto (\mathbf{G}', \ell'),$$

where

$$\ell'(e) = \begin{cases} t & e = \{v, v_0\}, \\ (1-t)\ell(e) & e \in E(G). \end{cases}$$

When $t = 0$, $f(-, 0)$ coincides with the map η_p for all $p > 0$. When $t = 1$, $((\mathbf{G}, \ell), 1)$ is sent to the cone point in $\Delta_{g,w}(S)$ represented by the graph consisting of a weight-g vertex marked by $[n] \smallsetminus S$ and a weight-0 vertex v_0 marked by S; see Figure 17. Since both Δ_{g,w^S} and $\Delta_{g,w}(S)$ can be given the structure of finite CW-complexes, they are compact and Hausdorff. Since the map f is a continuous bijection, it is a homeomorphism.

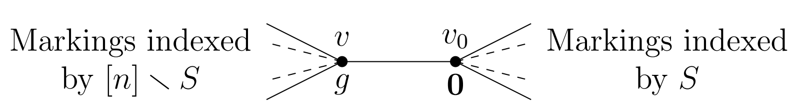

FIGURE 17. The graph representing the cone point in $\Delta_{g,w}(S)$.

□

3.2. **Proof that $\Delta_{g,w}$ is simply connected.** Before we prove that $\Delta_{g,w}$ is simply connected, i.e. it is nonempty, path connected and has trivial fundamental group, we need some auxiliary definitions.

Definition 3.4. Given a weight vector $w \in (\mathbb{Q} \cap (0,1])^n$, we associate an abstract simplicial complex $K(w)$ with vertex set equal to $[n]$ by declaring that $S \subseteq [n]$ belongs to $K(w)$ if and only if $\sum_{i \in S} w_i \leq 1$.

Observe that the association of $K(w)$ to the vector w is order-reversing: if $w, w' \in (\mathbb{Q} \cap (0,1])^n$ with $w_i \leq w'_i$ for all i, then $K(w')$ is a subcomplex of $K(w)$.

Definition 3.5. Let $S \subseteq [n]$ and let $\mathbf{G} = (G, m, h) \in \Gamma_{g,w}$. A vertex $v \in V(\mathbf{G})$ is called an S-***antenna*** if $h(v) = 0$, $\mathrm{val}(v) = 1$, and $m^{-1}(v) = S$.

By the definition of w-stability, there exist graphs with S-antennas in $\Gamma_{g,w}$ if and only if $S \notin K(w)$. We will also require the following basic topological lemma obtained via the Seifert-van Kampen theorem (see [Hat02]) and induction.

Lemma 3.6. *Let X be a path-connected CW-complex, and suppose that $X = \cup_{i=1}^N U_i$ where each U_i is a simply connected CW-subcomplex. Suppose further that for any $1 \leq i_1, \ldots, i_k \leq N$, the intersection $\cap_{j=1}^k U_{i_j}$ is simply connected. Then X is simply connected.*

Lemma 3.7. *Let $g \geq 1$ and $w \in (\mathbb{Q} \cap (0,1])^n$, with $n \geq 1$. Then $\Delta_{g,w}$ is path-connected.*

Proof. It is enough to show that points corresponding to $\Delta_{g,w}(0)$ are path-connected to each other, because each point $(\mathbf{G}, \ell) \in \Delta_{g,w}$ is path-connected to $(\mathbf{G}/I, \ell')$ for any edge set $I \subset E(G)$ and length function ℓ', and every graph can be contracted to a graph with only one edge. If a w-stable graph $\mathbf{G}$ with genus g has only one edge, then $\mathbf{G}$ is either a loop at a weight-$(g-1)$ vertex supporting all the markings, or a bridge connecting two vertices with weights summing up to g.

Suppose $\mathbf{G}$ is the former and $\mathbf{G}'$ is the latter. Further suppose that $\mathbf{G}'$ has one vertex of weight $\mathbf{k} \geq 1$ and supports markings indexed by some subset $S \subset [n]$. Note that such vertex always exists because $\mathbf{G}'$ has genus greater than 0. Since $\mathbf{G}'$ is stable, we have $2(\mathbf{g} - \mathbf{k}) + \sum_{i \notin S} w_i > 1$. Then $\mathbf{G}$ and $\mathbf{G}'$ are vertices of the 1-simplex corresponding to the w-stable genus g graph $\mathbf{H}$ consisting of two vertices, with weights $\mathbf{k-1}$ and $\mathbf{g-k}$, respectively, an edge that joins them and a loop at the vertex with weight $\mathbf{k-1}$. See Figure 18. Therefore $\mathbf{G}$ and $\mathbf{G}'$ are path-connected to each other.

□

We now have the necessary framework to prove that $\pi_1(\Delta_{g,w})$ is trivial. Let us denote by $\ell(w)$ the length of w, which is n for $w \in (\mathbb{Q} \cap (0,1])^n$, and by $j(w)$ the number of entries of w which are strictly less than 1, i.e. $j(w) := |\{i \in [n] \mid w_i < 1\}|$.

Theorem 3.8. *Let $g \geq 1$ and $w \in (\mathbb{Q} \cap (0,1])^n$. Then $\Delta_{g,w}$ is simply connected.*

Proof. Fix $g \geq 1$. We will proceed by induction on the pair $(\ell(w), j(w))$, where $\mathbb{Z}^2$ is given the lexicographic order. In the base case when $\ell(w) = 1, j(w) = 0$, we have $\Delta_{g,w} \cong \Delta_{g,1}$, so $\pi_1(\Delta_{g,1})$ is trivial by [ACP19]. Suppose $\ell(w) \geq 2$, and the statement is true for all w' such that $(\ell(w'), j(w'))$ is strictly less than $(\ell(w), j(w))$ with respect to the lexicographic order. Reordering the entries of w if necessary, we can assume that $w_1 < 1$. Now denote by $\overline{w}$ the weight vector obtained from w by changing w_1 to 1. So in the lexicographic order,

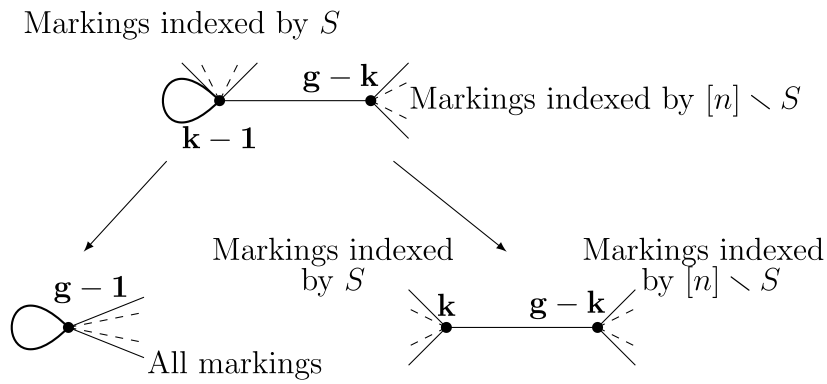

FIGURE 18. A w-stable graph $\mathbf{H}$ contracts to a graph with one vertex supporting all the markings and a loop, and a w-stable graph with two connected vertices.

$(\ell(\overline{w}), j(\overline{w})) < (\ell(w), j(w))$. We also have an embedding $\Delta_{g,w} \hookrightarrow \Delta_{g,\overline{w}}$ of topological spaces, as $w_i \leq \overline{w}_i$ for all i. Now we analyze the locus

$$\Sigma_{g,w} := \overline{\Delta_{g,\overline{w}} \smallsetminus \Delta_{g,w}} \subseteq \Delta_{g,\overline{w}}.$$

We suppose that $\Sigma_{g,w}$ is nonempty; otherwise $\Delta_{g,w} \cong \Delta_{g,\overline{w}}$, and we have that $\pi_1(\Delta_{g,\overline{w}})$ is trivial by assumption.

Now consider the decomposition

$$\Delta_{g,\overline{w}} = \Delta_{g,w} \cup \Sigma_{g,w}.$$

Recall that the Seifert-van Kampen theorem for CW-complexes expresses $\pi_1(X)$ for a path-connected CW-complex X as the amalgamated free product

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V),$$

where U and V are path-connected CW-subcomplexes that cover X, such that $U \cap V$ is path-connected. We will show that $\Sigma_{g,w}$ is a subcomplex of $\Delta_{g,\overline{w}}$ and that both $\Sigma_{g,w}$ and $\Delta_{g,w} \cap \Sigma_{g,w}$ are simply connected, so that Seifert-van Kampen applies and we have the following pushout diagram of groups:

$$\begin{tikzcd} \pi_1(\Delta_{g,w} \cap \Sigma_{g,w}) \arrow[r] \arrow[d] & \pi_1(\Sigma_{g,w}) \arrow[d] \\ \pi_1(\Delta_{g,w}) \arrow[r] & \pi_1(\Delta_{g,\overline{w}}) \end{tikzcd}, \tag{3.9}$$

where all arrows are induced by inclusions. In particular, the two groups in the top row in Diagram 3.9 will be trivial. Since $\pi_1(\Delta_{g,\overline{w}})$ is trivial by induction, it will follow that $\pi_1(\Delta_{g,w})$ must be trivial.

To show that $\Sigma_{g,w}$ is a simply connected subcomplex of $\Delta_{g,\overline{w}}$, we first establish the equality

$$\Sigma_{g,w} = \bigcup_{S \in K(w) \smallsetminus K(\overline{w})} \Delta_{g,\overline{w}}(S). \tag{3.10}$$

Indeed, the graphs parameterized by the subspace $\Delta_{g,\overline{w}} \smallsetminus \Delta_{g,w}$ are precisely those that are $\overline{w}$-stable but not w-stable. Any such graph must have an S-antenna for some $S \in K(w) \smallsetminus K(\overline{w})$;

since S-antennas are also S-vertices, we have the containment

$$\Delta_{g,\overline{w}} \smallsetminus \Delta_{g,w} \subseteq \bigcup_{S \in K(w) \smallsetminus K(\overline{w})} \Delta_{g,\overline{w}}(S).$$

The object on the right is a closed subcomplex of $\Delta_{g,\overline{w}}$, so we must have

$$\overline{\Delta_{g,\overline{w}} \smallsetminus \Delta_{g,w}} = \Sigma_{g,w} \subseteq \bigcup_{S \in K(w) \smallsetminus K(\overline{w})} \Delta_{g,\overline{w}}(S).$$

For the reverse containment, simply note that any metric graph in $\Delta_{g,\overline{w}}(S)$ for $S \in K(w) \smallsetminus K(\overline{w})$ either contains an S-antenna, or has a $\overline{w}$-stable uncontraction with an S-antenna. As such, $\Delta_{g,\overline{w}}(S) \subseteq \Sigma_{g,w}$ and the equality is proven; in particular, $\Sigma_{g,w}$ is a subcomplex of $\Delta_{g,\overline{w}}$. We now argue that $\Sigma_{g,w}$ is simply connected using the criterion of Lemma 3.6 and Equation 3.10. Indeed, for all $S \in K(w) \smallsetminus K(\overline{w})$, we have $\sum_{i \in S} \overline{w}_i > 1$ by the definition of $K(\overline{w})$. Therefore for each such S, we have

$$\Delta_{g,\overline{w}}(S) \cong \mathrm{Cone}(\Delta_{g,\overline{w}^S})$$

by Lemma 3.3, so each $\Delta_{g,\overline{w}}(S)$ appearing in the union of Equation 3.10 is contractible. Moreover, given any $S_1, \ldots, S_N \in K(w) \smallsetminus K(\overline{w})$, we have $1 \in \cap_{i=1}^N S_i$, so Lemma 3.2 implies that

$$\bigcap_{i=1}^{N} \Delta_{g,\overline{w}}(S_i) = \Delta_{g,\overline{w}}\left(\bigcup_{i=1}^{N} S_i\right),$$

which is again contractible by Lemma 3.3 and hence is simply connected. Thus Lemma 3.6 gives that the subcomplex $\Sigma_{g,w}$ is simply connected.

We will now prove that $\Delta_{g,w} \cap \Sigma_{g,w}$ is simply connected, again using the criterion of Lemma 3.6. Identifying $\Delta_{g,w}(S)$ with its image under the embedding $\Delta_{g,w} \hookrightarrow \Delta_{g,\overline{w}}$, we have

$$\Delta_{g,w} \cap \Sigma_{g,w} = \bigcup_{S \in K(w) \smallsetminus K(\overline{w})} \Delta_{g,w}(S).$$

For each $S \in K(w) \smallsetminus K(\overline{w})$, we have $\sum_{i \in S} w_i \leq 1$, so $\Delta_{g,w}(S) \cong \Delta_{g,w^S}$ by Lemma 3.3. Since $|S| \geq 2$ for any such S, we have $\ell(w^S) = \ell(w) - |S| + 1 < \ell(w)$, so by induction, each $\Delta_{g,w}(S)$ appearing in the union above is simply connected. Given $S_1, \ldots, S_N \in K(w) \smallsetminus K(\overline{w})$, we have

$$\bigcap_{i=1}^{N} \Delta_{g,w}(S_i) = \Delta_{g,w}\left(\bigcup_{i=1}^{N} S_i\right),$$

again by Lemma 3.2. Setting $\mathcal{S} = \cup_{i=1}^N S_i$, Lemma 3.3 gives that $\Delta_{g,w}(\mathcal{S})$ is either isomorphic to $\Delta_{g,w^{\mathcal{S}}}$, or the cone over it. In the first case $\Delta_{g,w}(\mathcal{S})$ is simply connected by induction, and in the second it is simply connected because it is contractible. Hence Lemma 3.6 gives that $\Delta_{g,w} \cap \Sigma_{g,w}$ is simply connected. As already discussed, that $\pi_1(\Delta_{g,w})$ is trivial now follows from the fact that Diagram 3.9 is a pushout square and the inductive assumption that $\pi_1(\Delta_{g,\overline{w}})$ is trivial. $\square$

Remark 3.11. We remark that the simply-connectedness of $\Delta_{g,w}$ breaks down in $g = 0$ for some special weights. In particular, by [CMP+20, Theorem A], when $g = 0$,

(1) when $w = (1^{(n)}, \varepsilon^{(m)})$ where $n + m = 5$ and $n \geq 2$, the fundamental group $\pi_1(\Delta_{0,w})$ are free groups of ranks $1, 4, 6$, and 6 respectively;

(2) when $w = \big((1/m)^{(2m)}, \varepsilon^{(3)}\big)$, the fundamental group $\pi_1(\Delta_{0,w})$ is a free group of rank $\frac{1}{2}\binom{2m}{m}$;
(3) when $w = \big((1/k)^{(2k+2+m)}\big)$, $\pi_1(\Delta_{0,w}) = \mathbb{Z}/2\mathbb{Z}$.

4. The Euler characteristic of $\Delta_{g,w}$

Let $M_{g,w}$ be the coarse moduli space of $\mathcal{M}_{g,w}$. In this section we exhibit a useful decomposition of the class $[M_{g,w}]$ in terms of classes $[M_{g,r}]$ in the Grothendieck group of varieties. Using the fact that the virtual Poincaré polynomial is an Euler-Poincaré characteristic, this allows us to deduce the formula of Theorem 1.2.

4.1. **The Grothendieck group of varieties and Euler-Poincaré characteristics.** We denote by $K_0(\mathrm{Var}/k)$ the *Grothendieck group of varieties*. This group is the quotient of the free abelian group on k-varieties by relations of the form

$$[X] = [X \smallsetminus Y] + [Y],$$

when Y is a closed subvariety of X. Such relation are called the *cut-and-paste* relations. The additive identity is $[\varnothing]$. An *Euler-Poincaré characteristic* of $K_0(\mathrm{Var}/k)$ is a group homomorphism

$$\chi : K_0(\mathrm{Var}/k) \to A$$

to an abelian group A. That is, for any closed subvariety Y of X,

$$\chi([X]) = \chi([Y]) + \chi([X \smallsetminus Y]).$$

See [Cra04, Loe09]. Specializing to $k = \mathbb{C}$, one example of an Euler-Poincaré characteristic is given by the *virtual Poincaré polynomial*, which is the group homomorphism $K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[t]$ defined by the formula

$$P_X(t) = \sum_{m=0}^{2d} (-1)^m \chi_c^m(X) t^m,$$

where $d = \dim X$ and

$$\chi_c^m(X) := \sum_{j=0}^{2d} (-1)^j \dim \mathrm{Gr}_m^W H_c^j(X;\mathbb{Q}).$$

4.2. **The stratification of $M_{g,w}$.** Let $g \geq 0$, $n \geq 0$ and $w \in (\mathbb{Q} \cap (0,1])^n$ such that

$$2g - 2 + \sum_{i=1}^n w_i > 0.$$

To describe a stratification of $M_{g,w}$, we say that a set partition of $[n]$

$$\mathcal{P} = P_1 \sqcup \cdots \sqcup P_r \vdash [n]$$

is w-***admissible*** if

$$\sum_{i \in P_j} w_i \leq 1$$

for all $1 \leq j \leq r$. Given such a partition, we write $\mathcal{P} \vdash_w [n]$. We set $N_{r,w}$ to be the number of w-admissible partitions of $[n]$ with r parts.

Proposition 4.1. *In the Grothendieck group of k-varieties $K_0(\mathrm{Var}/k)$,*

$$[M_{g,w}] = \sum_{r=1}^{n} N_{r,w}[M_{g,r}].$$

Proof. The locus $M_{g,w}$ parameterizes irreducible smooth curves of genus g with n markings, such that whenever $\sum_{i\in S} w_i \leq 1$ for some $S \subseteq [n]$, the markings indexed by S are allowed to coincide. Given a w-admissible partition

$$\mathcal{P} = P_1 \sqcup \cdots \sqcup P_r \vdash_w [n],$$

we define

$$Z_{\mathcal{P}} := \{(C, p_1, \ldots, p_n) \in M_{g,w} \mid p_i = p_j \text{ if and only if } i, j \in P_s \text{ for some } s \in [r]\}.$$

Then $Z_{\mathcal{P}} \cong M_{g,r}$. As $\mathcal{P}$ ranges over all w-admissible partitions of $[n]$, the loci $Z_{\mathcal{P}}$ form a locally closed stratification of $M_{g,w}$. Hence in the Grothendieck group, by [Mus, Proposition 1.1], we have

$$[M_{g,w}] = \sum_{\mathcal{P}\vdash_w [n]} [Z_{\mathcal{P}}] = \sum_{r=1}^{n} \sum_{\substack{\mathcal{P}\vdash_w [n]\\ |\mathcal{P}|=r}} [Z_{\mathcal{P}}] = \sum_{r=1}^{n} N_{r,w}[M_{g,r}],$$

as claimed. □

Remark 4.2. In [BH05, Section 4], Bini and Harer computed the Euler characteristics of $\mathcal{M}_{g,n}$ for $2g-2+n>0$. Our decomposition in Proposition 4.1 can be used to give an Euler characteristic of $\mathcal{M}_{g,w}$ in terms of those of $\mathcal{M}_{g,r}$ for $0 < r \leq n$.

We now record two corollaries which amount to the calculation of the numbers $N_{r,w}$ for special values of w.

Corollary 4.3. *Let w be heavy/light, i.e. $w = (1^{(n)}, \varepsilon^{(m)})$ for $m \geq 2$ and $\varepsilon < 1/m$ satisfying $2g-2+n \geq 0$. Then*

$$[M_{g,w}] = \sum_{r=1}^{m} S(m,r)[M_{g,n+r}],$$

where $S(m,r)$ is the Stirling number of the second kind, or the number of r-partitions of a m-set.

Furthermore, the m-restricted Stirling number of the second kind for $n, r \geq 0$ is defined to be the number of partitions of an n-set into r nonempty subsets, each of which has at most m elements, and is denoted by

$$\left\{ {n \atop r} \right\}_{\leq m}.$$

For the generating function and other recurrence relations of the m-restricted Sterling numbers, see [KLM16]. Then we have the following.

Corollary 4.4. *Let $w = ((1/m)^{(n)})$ for $n > m > 1$ such that $2g-2+n/m > 0$. Then*

$$[M_{g,w}] = \sum_{r=\lceil \frac{n}{m} \rceil}^{n} \left\{ {n \atop r} \right\}_{\leq m} [M_{g,r}].$$

In particular when $m = 2$, we have

$$\left\{ {n \atop r} \right\}_{\leq 2} = \frac{\prod_{i=0}^{n-r-1} \binom{n-2i}{2}}{(n-r)!},$$

since a $(1/2^{(n)})$-admissible partition having r parts must consist of exactly $(n-r)$ subsets of size two and singletons otherwise.

Corollary 4.5. *Let $w = ((1/2)^{(n)})$ for $n \geq 0$ such that $2g - 2 + n/2 > 0$. Then*

$$[M_{g,w}] = \sum_{r=\lceil \frac{n}{2} \rceil}^{n} \frac{\prod_{i=0}^{n-r-1} \binom{n-2i}{2}}{(n-r)!} [M_{g,r}].$$

4.3. **The Euler characteristics of $\Delta_{g,w}$ and $\mathcal{M}_{g,w}$.** We can now exploit the additivity of Euler-Poincaré characteristics and the connection between $\mathcal{M}_{g,w}$ and $\Delta_{g,w}$ to prove Theorem 1.2. For a complex algebraic variety (or stack) X of dimension d, let χ^{tw} be the top weight Euler characteristic, defined as

$$\chi^{\text{tw}}(X) := \sum_{i=0}^{2d} (-1)^i \dim \mathrm{Gr}^W_{2d} H^i(X; \mathbb{Q}),$$

and for any space Y, let $\widetilde{\chi}(Y)$ be the reduced Euler characteristic. Recall from [CGP18, Theorem 5.8] that the isomorphism

$$\mathrm{Gr}^W_{6g-6+2n} H^{6g-6+2n-k}(\mathcal{M}_{g,n}; \mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta_{g,n}; \mathbb{Q})$$

is a special case of the isomorphism

$$\mathrm{Gr}^W_{2d} H^{2d-k}(\mathcal{X}; \mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}); \mathbb{Q})$$

whenever $\mathcal{X}$ is a smooth and separated d-dimensional DM stack over $\mathbb{C}$, $\overline{\mathcal{X}}$ is a smooth normal crossings of $\mathcal{X}$, and $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$ is the dual complex of the normal crossings divisor $\overline{\mathcal{X}} \smallsetminus \mathcal{X}$.

Lemma 4.6. *Let $\mathcal{X}$ be a smooth, separated DM stack over $\mathbb{C}$ and let $\overline{\mathcal{X}}$ be a smooth normal crossings compactification of $\mathcal{X}$. Then*

$$\chi^{\text{tw}}(\mathcal{X}) = -\widetilde{\chi}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}})).$$

Proof. Let $d = \dim \mathcal{X}$. We have

$$\begin{aligned}
\chi^{\text{tw}}(\mathcal{X}) &= \sum_{i=0}^{2d} (-1)^i \dim \mathrm{Gr}^W_{2d} H^i(\mathcal{X}; \mathbb{Q}) \\
&= \sum_{i=0}^{2d} (-1)^i \dim \widetilde{H}_{2d-i-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}); \mathbb{Q}) \\
&= \sum_{i=0}^{2d} (-1)^{i+1} \dim \widetilde{H}_i(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}); \mathbb{Q}) \\
&= -\widetilde{\chi}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}})).
\end{aligned}$$

□

We now prove the additivity of the top-weight Euler characteristic for complex algebraic varieties.

Lemma 4.7. *The top weight Euler characteristic*

$$\chi^{\mathrm{tw}} : K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}$$

is an Euler-Poincaré characteristic.

Proof. Fix a complex algebraic variety X and set $d = \dim X$. We use the fact that the virtual Poincaré polynomial

$$P_X(t) = \sum_{m=0}^{2d} (-1)^m \chi_c^m(X) t^m$$

defines a group homomorphism $K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[t]$. The Poincaré duality pairing

$$H_c^j(X;\mathbb{Q}) \times H^{2d-j}(X;\mathbb{Q}) \to \mathbb{Q}$$

induces a perfect pairing of graded pieces

$$\mathrm{Gr}_m^W H_c^j(X;\mathbb{Q}) \times \mathrm{Gr}_{2d-m}^W H^{2d-j}(X;\mathbb{Q}) \to \mathbb{Q}$$

for $0 \le m \le 2j$; see [PS08, Theorem 6.23]. Thus we can write

$$\begin{aligned} \chi_c^0(X) &= \sum_{j=0}^{2d} (-1)^j \dim \mathrm{Gr}_{2d}^W H^{2d-j}(X;\mathbb{Q}) \\ &= \sum_{j=0}^{2d} (-1)^{2d-j} \dim \mathrm{Gr}_{2d}^W H^{2d-j}(X;\mathbb{Q}). \end{aligned}$$

In particular, it follows that

$$\chi_c^0(X) = \chi^{\mathrm{tw}}(X),$$

so for any X,

$$\chi^{\mathrm{tw}}(X) = P_X(0),$$

completing the proof. □

We can now prove Theorem 1.2, restated here.

Theorem 1.2. *Let $W = W_1 \subset \cdots \subset W_{6g-6+2r} \subseteq H^*(\mathcal{M}_{g,r};\mathbb{Q})$ be the weight filtration of the rational singular cohomology of the moduli stack $\mathcal{M}_{g,r}$ and denote by $\chi^W_{6g-6+2r}$ the Euler characteristic of the top graded piece*

$$\mathrm{Gr}^W_{6g-g+2r} H^*(\mathcal{M}_{g,r};\mathbb{Q}) = W_{6g-6+2r}/W_{6g-7+2r}$$

of the weight filtration. Then

$$\chi(\Delta_{g,w}) = -\left(\sum_{r=1}^{n} N_{r,w} \cdot \chi^W_{6g-6+2r}(\mathcal{M}_{g,r})\right) + 1.$$

Proof of Theorem 1.2. By Proposition 4.1 and Lemma 4.7, we have

$$\chi^{\mathrm{tw}}(M_{g,w}) = \sum_{r=1}^{n} N_{r,w} \chi^{\mathrm{tw}}(M_{g,r}).$$

By [Beh04, Proposition 36] and [Edi10, Theorem 4.40], the coarse moduli scheme $\mathcal{X} \to X$ of a DM stack $\mathcal{X}$ induces an isomorphism of rational cohomology $H^*(\mathcal{X};\mathbb{Q}) \cong H^*(X;\mathbb{Q})$, which is an isomorphism of mixed Hodge structures. Therefore,

$$\chi^{\mathrm{tw}}(\mathcal{X}) = \chi^{\mathrm{tw}}(X).$$

Since $\Delta_{g,w} = \Delta(\mathcal{M}_{g,w} \subset \overline{\mathcal{M}}_{g,w})$, the result now follows from Lemma 4.6 and the fact that $\widetilde{\chi}(\Delta_{g,w}) = \chi(\Delta_{g,w}) - 1$. □

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