Topology of tropical moduli spaces of weighted stable curves in higher genus

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Joint with Siddarth Kannan, Stefano Serpente, Claudia Yun

Slides will be available at `http://www.shiyue.li`

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5. Simple connectivity of $\Delta_{g,w}$ for $g > 0$.
6. Euler characteristics of $\Delta_{g,w}$.
7. Future directions.

History and Motivation

History

- Given $g \geq 0, n \geq 1$ and $w \in (\mathbb{Q} \cap (0,1])^n$ satisfying

$$2g - 2 + \sum w_i > 0,$$

Hassett defined the Deligne-Mumford (DM) stack $\overline{\mathcal{M}}_{g,w}$ as an alternate compactification of DM stack $\mathcal{M}_{g,n}$ of n-marked smooth curves of genus g in [Has03].
The space $\overline{\mathcal{M}}_{g,w}$ parametrizes n-marked curves of genus g

We have the containments

$$\mathcal{M}_{g,n} \subset \mathcal{M}_{g,w} \subset \overline{\mathcal{M}}_{g,w}.$$

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- In [Uli15], Ulirsch gave that the boundary divisor $\overline{\mathcal{M}}_{g,w} \setminus \mathcal{M}_{g,w}$ is a normal crossings divisor.

Motivation

The boundary complex, or the dual complex of the boundary divisor $\overline{\mathcal{M}}_{g,w} \smallsetminus \mathcal{M}_{g,w}$ is identified with $\Delta_{g,w}$.

- Shown by Harper in [Har17], simple homotopy types of the boundary complex is independent of the choice of compactification, for Deligne-Mumford stacks.

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- Shown by Harper in [Har17], simple homotopy types of the boundary complex is independent of the choice of compactification, for Deligne-Mumford stacks.
- As a DM stack, the rational cohomology of $\mathcal{M}_{g,w}$ carries a mixed Hodge structure, i.e. there is a weight filtration of the rational homology. The top graded piece of the weight filtration on $\mathcal{M}_{g,w}$ is isomorphic to the reduced rational homology of the dual complex $\Delta_{g,w}$; see work by Deligne in [Del71].

What is $\Delta_{g,w}$

The category $\Gamma_{g,w}$

Given $g \geq 0, n \geq 1$, and a weight vector $w \in (\mathbb{Q} \cap (0,1])^n$,

Definition
A ***w*-stable graph of genus *g*** is a tuple (G, h, m) such that

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$$b^1(G) + \sum_{v \in G} h(v) = g,$$

   where $b^1(G)$ is the first betti number of G.

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3. (w-stability) $m : [n] \to V(G)$ is a (marking) function such that for all $v \in V(G)$
$$2h(v) + \mathrm{val}(v) + \sum_{i \in m^{-1}(v)} w_i > 2;$$

Example ($g = 1$)

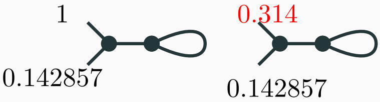

Figure 1: Okay (left), not okay (right).

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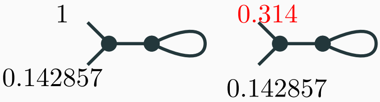

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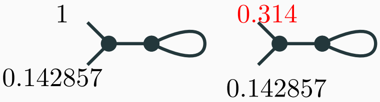

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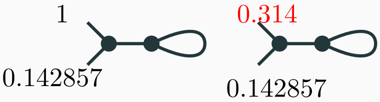

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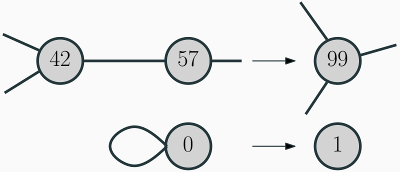

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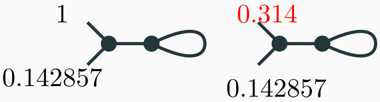

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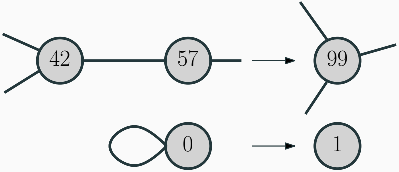

   2.2 graph isomorphisms that respect the vertex weights and markings.

Example

Let $0 < \varepsilon \ll 1$, $g = 1$, $w = (1, \varepsilon)$.

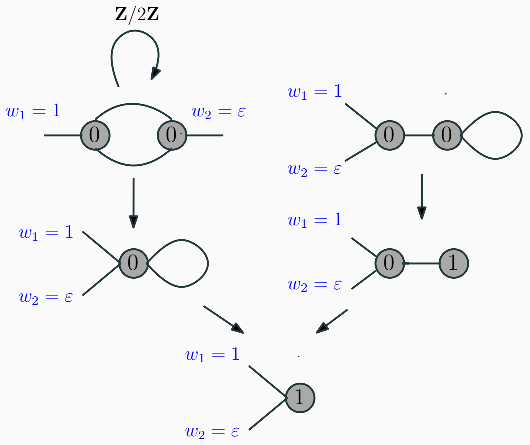

Figure 2: The category $\Gamma_{1,(1,\varepsilon)}$, containing the category $\Gamma_{1,(\varepsilon,\varepsilon)}$ as a subcategory.

Upgrade: abstract w-stable genus-g tropical curves

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Definition
The **volume** of (G, ℓ) is

$$\mathrm{vol}((\mathsf{G}, \ell)) := \sum_{e \in E(G)} \ell(e).$$

A funtor $\Gamma_{g,w} \to$ Top

For each G, define

$$\Delta(\mathrm{G}) := \left\{ \ell : E(G) \to \mathbb{R}_{\geq 0}, \sum_{e \in E(G)} \ell(e) = 1 \right\},$$

which can be identified with a $(|E(G)| - 1)$-simplex.

For each morphism $f : \mathrm{G} \to \mathrm{H}$ in $\Gamma_{g,w}$, there is an induced morphism of topological spaces

$$f^* : \Delta(\mathrm{H}) \to \Delta(\mathrm{G})$$

identifying $\Delta(\mathrm{H})$ as a face of $\Delta(\mathrm{G})$.

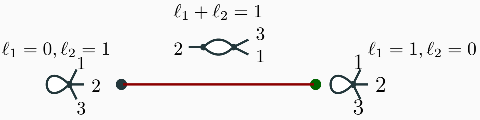

Figure 3: The graph that is a loop is a face of the 1-simplex whose graph is the “fish”.

Then "Δ" gives a diagram $\Gamma_{g,w}^{\mathrm{op}} \to \mathrm{Top}$.

Definition
The moduli space $\Delta_{g,w}$ is the colimit of the diagram Δ:

$$\Delta_{g,w} := \mathrm{colim}_{G \in \mathrm{Obj}(\Gamma_{g,w})} \Delta(G).$$

It parametrizes w-stable tropical curves of genus g with unit volume.

Warning: this is not in general a simplicial complex, but a "geometric realization of symmetric Δ-complex" which morally speaking, allows half edges, folded 2-simplexes, etc..

Motivation II: related work

Conventions:

- When $w = (1, \ldots, 1)$, we call $\Delta_{g,w}$ as $\Delta_{g,n}$.

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- When w has at least two 1 entries, Cerbu et al. derived the homotopy types of $\Delta_{g,w}$.

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- When w is *heavy/light* or has at least two 1's, Cerbu et al. showed that $\Delta_{1,w}$ is homotopic to a wedge of spheres.

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- When w is *heavy/light* or has at least two 1's, Cerbu et al. showed that $\Delta_{1,w}$ is homotopic to a wedge of spheres.
- For $(g, n) \neq (0, 4), (0, 5)$, Allcock-Corey-Payne showed that $\Delta_{g,n}$ is simply connected.

Speculation: $\Delta_{g,w}$'s probably have trivial lower homotopy groups, for higher g and general w.

Example (Haven't shown up above)
Let $0 < \varepsilon \ll 1$, $g = 1$ and $w = (\varepsilon, \varepsilon, \varepsilon)$.

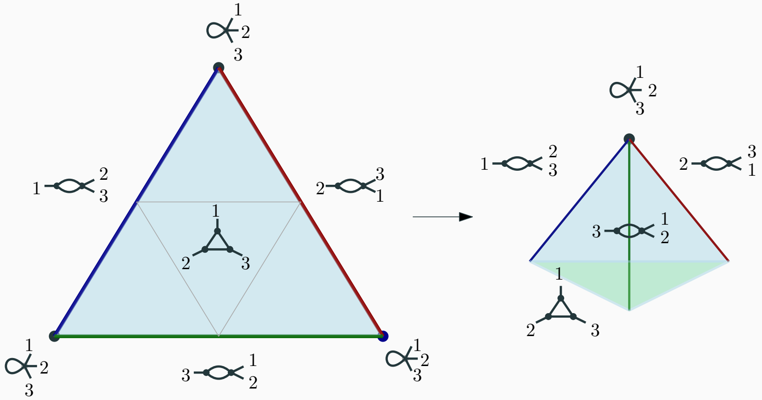

Figure 4: The space $\Delta_{1,(\varepsilon,\varepsilon,\varepsilon)}$ is homotopy equivalent to S^2.

The main theorems

Theorem (Kannan-L.-Serpente-Yun)
For any $g, n \geq 1$, and $w \in (\mathbb{Q} \cap (0,1])^n$, the space $\Delta_{g,w}$ is simply-connected.

Recall $\Delta_{g,n}$ is simply-connected for $(g,n) \neq (0,4),(0,5)$ by Allcock-Corey-Payne.

Proof: a double induction

Double induction on

$$\ell(w) := \text{length of } w$$

and

$$j(w) := \#\{w_i : w_i < 1\}.$$

Example
Let $\varepsilon < 1/3$.
For $w = (1, \varepsilon, \varepsilon)$, $\ell(w) = 3$, $j(w) = 2$.
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Base case: $\ell(w) = 1, j(w) = 0$, Allcock-Corey-Payne showed $\Delta_{g,1}$ is s.c.
Inductive step:
(1) For each w, reorder w s.t. $w_1 < 1$ and define $\bar{w} = (1, w_2, w_3, \ldots)$.

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Example
For $w = (\varepsilon, \varepsilon, \varepsilon)$, $\bar{w} = (1, \varepsilon, \varepsilon)$.
Notice that $\Delta_{g,\bar{w}}$ contains $\Delta_{g,w}$ as a subcomplex.

Example

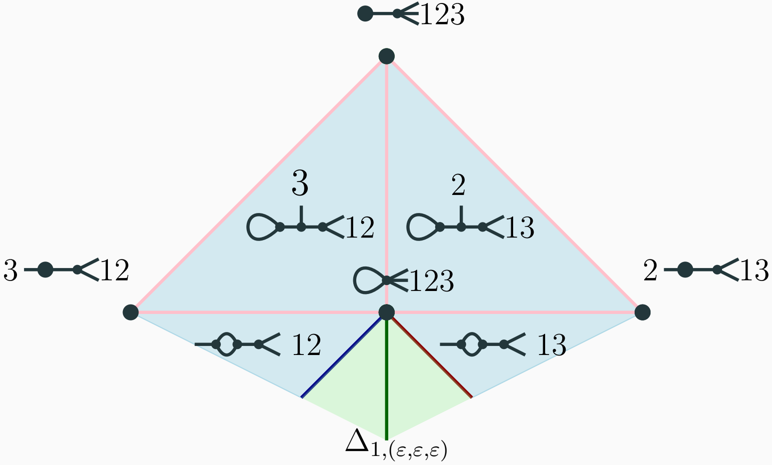

Figure 5: $\Delta_{1,(1,\varepsilon,\varepsilon)} \subset \Delta_{1,\varepsilon,\varepsilon}$

Crux of induction: study a subcomplex

(2) Define the subcomplex

$$\Sigma_{g,w} = \overline{\Delta_{g,\bar{w}} \setminus \Delta_{g,w}} \subset \Delta_{g,\bar{w}}.$$

(3) Decompose $\Sigma_{g,w}$.

(i) Define

$$K(w) := \{S \subseteq [n] : \sum_{i \in S} w_i \leq 1\}$$

Example

$K((\varepsilon, \varepsilon, \varepsilon)) = \{1, 2, 3, 12, 23, 13, 123\}$.

$K((1, \varepsilon, \varepsilon)) = \{2, 3, 23\}$.

Crux of induction: study a subcomplex

(ii) For a $S \subseteq [n]$, define $\Delta_{g,\bar{w}}(S)$ to be the subcomplex in $\Delta_{g,\bar{w}}$ representing tropical curves with a vertex supporting (at least) markings indexed by S.

Example

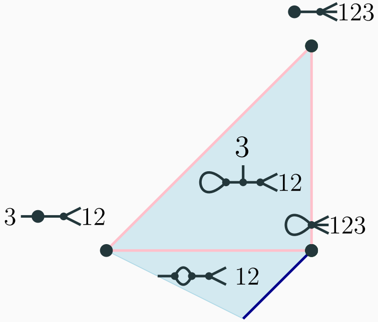

Figure 6: For $w = (\varepsilon, \varepsilon, \varepsilon), \bar{w} = (1, \varepsilon, \varepsilon)$, $S = \{12\}$, then $\Delta_{1,\bar{w}}(S)$ is as above.

Crux of induction: study a subcomplex

(iii) Decompose $\Sigma_{g,w}$ as

$$\Sigma_{g,w} = \overline{\Delta_{g,\bar{w}} \smallsetminus \Delta_{g,w}} = \bigcup_{S \in K(w) \smallsetminus K(\bar{w})} \Delta_{g,\bar{w}}(S).$$

Example

Recall: for $w = (\varepsilon, \varepsilon, \varepsilon)$ and $\bar{w} = (1, \varepsilon, \varepsilon)$,
$K((\varepsilon, \varepsilon, \varepsilon)) = \{1, 2, 3, 12, 23, 13, 123\}$,
$K((1, \varepsilon, \varepsilon)) = \{2, 3, 23\}$,
Thus $\Sigma_{1,w} = \Delta_{1,\bar{w}}(1) \cup \Delta_{1,\bar{w}}(2) \cup \Delta_{1,\bar{w}}(12) \cup \Delta_{1,\bar{w}}(13) \cup \Delta_{1,\bar{w}}(123)$.

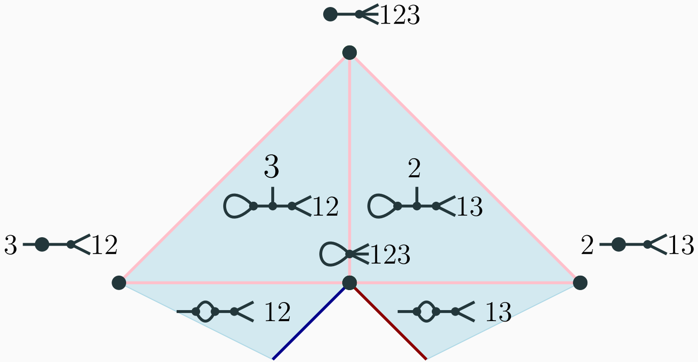

Figure 7: Visualizing the decomposition of $\Sigma_{1,(\varepsilon,\varepsilon,\varepsilon)}$ in pink/light blue.

Crux of induction: study each sub-subcomplex

(4) Each $\Delta_{g,\bar{w}}(S)$ is simply-connected.
(i) Define w^S as the weight vector removing weights indexed by S and appending

$$\min(\sum_{i \in S} w_i, 1).$$

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$w = \left(\frac{1}{4}, \frac{2}{3}, \frac{1}{2}, 1\right)$.

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(ii) Prove a technical lemma: If $\sum_{i \in S} \bar{w}_i \leq 1$, then

$$\Delta_{g,\bar{w}}(S) = \Delta_{g,\bar{w}^S};$$

Otherwise

$$\Delta_{g,\bar{w}}(S) \cong \mathrm{Cone}(\Delta_{g,\bar{w}^S}).$$

Example (continued with old friends $w = (\varepsilon, \varepsilon, \varepsilon)$ and $\bar{w} = (1, \varepsilon, \varepsilon)$)
When $S = 23$, $\bar{w}^S = (1, \varepsilon)$,

$$\Delta_{1,\bar{w}}(23) = \Delta_{1,(1,\varepsilon)} \cong \Delta_{1,2}.\text{s.c. by I.H.};$$

when $S = 12$ or 13, $\bar{w}^S = (1, 1)$,

$$\Delta_{1,\bar{w}}(13) \cong \mathrm{Cone}(\Delta_{1,(1,1)}).\text{s.c.}.$$

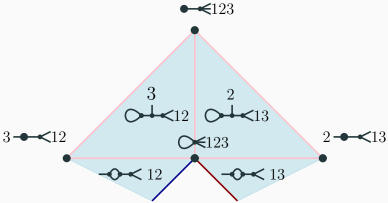
Crux of induction: study each sub-subcomplex $\Delta_{g,\bar{w}}(S)$

(iii) Seifert-van Kampen revisited: Let X be a path-connected CW-complex, and suppose that $X = \cup_{i=1}^{N} U_i$ where each U_i is a simply connected CW-subcomplex. Suppose further that for any $1 \leq i_1, \ldots, i_k \leq N$, the intersection $\cap_{j=1}^{k} U_{i_j}$ is simply connected. Then X is simply connected.

For any $S_1, \ldots, S_N \in K(w) \smallsetminus K(\overline{w})$, we have $1 \in \cap_{i=1}^{N} S_i$,

$$\bigcap_{i=1}^{N} \Delta_{g,\overline{w}}(S_i) = \Delta_{g,\overline{w}}\left(\bigcup_{i=1}^{N} S_i\right), \text{s.c. for same reasons before}$$

Therefore, $\Sigma_{g,w}$ is simply connected.



Finally

$$\begin{tikzcd} 0 = \pi_1(\Delta_{g,w} \cap \Sigma_{g,w}) \arrow[r] \arrow[d] & \pi_1(\Sigma_{g,w}) = 0 \arrow[d] \\ \pi_1(\Delta_{g,w}) \arrow[r] & \text{by I.H. } \pi_1(\Delta_{g,\overline{w}}) = 0. \end{tikzcd}$$

Figure 8: Use van Kampen.

Euler characteristics computation

For a heavy/light weight vector $w = (1^{(n)}, \varepsilon^{(m)})$ where

- $m > 0$,

we have the following computation results for $\chi(\Delta_{g,w})$ for some (n, m).

$g=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=2$	-	2	0	2
$n=3$	3	-3	9	-15
$n=4$	-5	19	-53	163
$n=5$	25	-95	385	-1535

$g=1$	$m=1$	$m=2$	$m=3$	$m=4$
$n=2$	2	-1	5	-7
$n=3$	-2	10	-26	82
$n=4$	13	-47	193	-767
$n=5$	-59	301	-1499	7501

$g=2$	$m=1$	$m=2$	$m=3$	$m=4$
$n=3$	3	-7	33	-127
$n=4$	-9	51	-249	1251
$n=5$	61	-359	2161	-12959
$n=6$	-419	2941	-20579	144061

$g=3$	$m=1$	$m=2$	$m=3$	$m=4$
$n=4$	1	1	1	1
$n=5$	1	1	1	1
$n=6$	1	1	1	1
$n=7$	1	1	1	1

Euler characteristics computation

For a heavy/light weight vector $w = (1^{(n)}, \varepsilon^{(m)})$ where

- $m > 0$,
- $0 < \varepsilon < 1/m$;

we have the following computation results for $\chi(\Delta_{g,w})$ for some (n, m).

$g=0$	$m=1$	$m=2$	$m=3$	$m=4$
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Euler characteristics computation

For a heavy/light weight vector $w = (1^{(n)}, \varepsilon^{(m)})$ where

- $m > 0$,
- $0 < \varepsilon < 1/m$;
- $n \geq g + 1$,

we have the following computation results for $\chi(\Delta_{g,w})$ for some (n, m).

$g=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=2$	-	2	0	2
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Euler characteristics computation

A partition $P_1 \sqcup \cdots \sqcup P_r$ of $[n]$ is w-admissible if for all $1 \leq j \leq r$,

$$\sum_{i \in P_j} w_i \leq 1.$$

Let $N_{r,w}$ denote the number of w-admissible $[n]$-partitions with r parts.

Example

For $w = (1, 1, \frac{3}{4}, \frac{1}{2})$, the partitian $\{1, 2\} \cup \{3, 4\}$ is not w-admissible but $\{1\} \cup \{2\} \cup \{3\} \cup \{4\}$ is.

In particular, $N_{r,w} = 0$ for all $r \neq 4$ and $N_{4,w} = 1$.

Let B_g be the g-th Bernoulli numbers.

Euler characteristics computation

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Let B_g be the g-th Bernoulli numbers.

Example

$B_1 = \pm\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}, \ldots$

A formula for Euler characteristics of $\Delta_{g,w}$

Theorem (Kannan-L.-Serpente-Yun)

$$\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{n} N_{r,w}(-1)^r \frac{(g+r-2)!}{g!} B_g.$$

A corollary for heavy/light Hassett spaces

Let $S(m, r)$ denote the number of r-partitions of $[m]$ for $m \geq 1$ and $r \geq 0$; these are called the Stirling numbers of the second kind.

Example ($m = 4$)

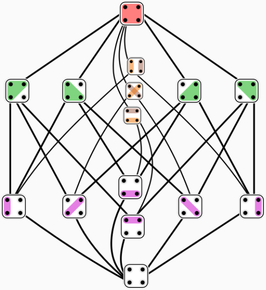

Figure 9: The lattice showing r partitions of $[4]$ for $1 \leq r \leq 4$. Source: wikipedia. In particular, $S(4,1) = 1$, $S(4,2) = 7$, $S(4,3) = 5$, $S(4,4) = 1$.

Corollary

Given a heavy/light weight vector $w = (1^{(n)}, \varepsilon^{(m)})$ *where* $n \geq g + 1$, $m > 0$, *and* $0 < \varepsilon < 1/m$,

$$\chi(\Delta_{g,w}) = 1 + \sum_{r=1}^{m} \sum_{\ell=0}^{g} (-1)^{n+r+\ell} \frac{(g+n+r-2)!\ell!}{g!(\ell+1)} S(m,r) S(g,\ell).$$

How to study Euler characteristics of $\Delta_{g,w}$

- Analyze the stratification of the coarse moduli scheme $M_{g,w}$ of $\mathcal{M}_{g,w}$.

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How to study Euler characteristics of $\Delta_{g,w}$

- Analyze the stratification of the coarse moduli scheme $M_{g,w}$ of $\mathcal{M}_{g,w}$.
- Write $[M_{g,w}]$ in the Grothedieck group of varieties as a decomposition into $[M_{g,r}]$.
- Then use results on top weight Euler characteristics of $M_{g,r}$ by Chan-Faber-Galatius-Payne in [CFGP20].

Future directions

1. Use Harvey's curve complexes or Hatcher's sphere systems to identify $\Delta_{g,w}$ as a quotient complex by some group action (on-going with the same co-authors).

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2. What is the geometric significance of the simple connectivity of $\Delta_{g,w}$ (as a dual complex of boundary divisor of $\mathcal{M}_{g,w}$)?
3. Study homotopy types of $\Delta_{g,w}$ for higher genus and general w.

Thank you!

Ask me questions.

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Decomposition of $\Sigma_{g,w}$

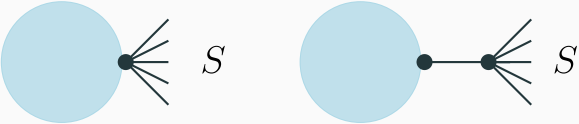

Figure 10: Underlying graphs in the interior of $\Sigma_{g,w}$ (right) and in the boundary of $\Sigma_{g,w}$ (left) and for some $S \in K(w) \setminus K(\bar{w})$.