DSA3102 Essential Data Analytics Tools: Convex Optimisation

Semester II, AY 2024/2025

Assignment 1

Instructions.
Due date: <u>04 Feb 2025 (Tuesday) 11:59pm</u>
Electronic submission only: please upload a **single pdf file** that contains your solution to canvas.
Since the questions are challenging for new learners, it is strongly recommended that you collaborate with your classmates to complete all the questions.
Optional questions are not mandatory (do not need to submit); however, it is highly recommended to learn from the suggested solutions.
Late submission policy: The grace period ends on 05 Feb 2025 (Wednesday) 11:59 am. 20% of marks will be deducted for late submission (after the grace period), based on Canvas clock. No submission is accepted after 06 Feb 2024 11:59 am.

1. Determine and explain whether the following optimization problem has the global minimizer or not.

   (a)
   $$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 3)^2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 5 \\ & x_1 \ge 0 \\ & 0 \le x_2 \le 2 \end{aligned}$$

   (b)
   $$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 3)^2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 5 \end{aligned}$$

   (c)
   $$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(x_1, x_2) = (x_1 - 2)^2 - (x_2 - 3)^2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 5 \end{aligned}$$

   (d)
   $$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(x_1, x_2) = (x_1 - 2)^2 - (x_2 - 3)^2 \\ \text{s.t.} \quad & |x_1|^3 + |x_2|^5 \le 100 \end{aligned}$$

2. Prove that the set $\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^i \le 10\}$ is compact.

3. *Solution set of a quadratic inequality.* Let $\mathcal{C} \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$\mathcal{C} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + c \leq 0 \right\},$$

with $\boldsymbol{A} \in \mathbb{S}^n$ (i.e., matrix $\boldsymbol{A}$ is symmetric), $\boldsymbol{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then, $\mathcal{C}$ is convex if $\boldsymbol{A} \succeq 0$ (i.e., $\boldsymbol{A}$ is positive definite where all eigenvalues are nonnegative). While the general proof requires advanced math techniques, we consider a simpler case where $n = 1$. In this case, the set $\mathcal{C}$ reduces to:

$$\mathcal{C}^1 = \left\{ x \in \mathbb{R} \mid ax^2 + bx + c \leq 0 \right\}.$$

   (a) Prove that $\mathcal{C}^1$ is convex if $a \geq 0$.

   (b) Please provide an example showing that $\mathcal{C}^1$ with $a < 0$ is not convex.

4. Show that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are convex sets in $\mathbb{R}^{m+n}$, then so is their partial sum

$$\mathcal{S} = \{ (\boldsymbol{x}, \boldsymbol{y}_1 + \boldsymbol{y}_2) \mid \boldsymbol{x} \in \mathbb{R}^m,\ \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^n,\ (\boldsymbol{x}, \boldsymbol{y}_1) \in \mathcal{S}_1,\ (\boldsymbol{y}, \boldsymbol{y}_2) \in \mathcal{S}_2 \}.$$

   Hint. Using the definition of convex sets.

5. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous and convex function, show that for every line segment, its average value on the segment is less than or equal to the average of its values at the endpoints of the segment: For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$\int_0^1 f(\boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x}))\, d\lambda \leq \frac{f(\boldsymbol{x}) + f(\boldsymbol{y})}{2}.$$

   Hint. Using the definition of convex functions and calculating the integral of λ.

6. *Monotone mappings.* A function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone* if for all $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} \psi$,

$$(\psi(\boldsymbol{x}) - \psi(\boldsymbol{y}))^T (\boldsymbol{x} - \boldsymbol{y}) \geq 0.$$

   Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone.
   Hint: Using the first-order condition (twice) for convexity.

7. *Products and ratios of convex functions.* In general, the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbb{R}$. Prove the following:

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then $fg(x) := f(x)g(x)$ is convex.
Hint. Prove by the definition of convex functions. Without the loss of generality, you may let $x \le y$ and $\lambda \in [0,1]$. We may also need the inequality that $f(x)g(y)+f(y)g(x) \le f(x)g(x)+f(y)g(y)$ (you can verify it as both f and g are nondecreasing and positive).

(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then $fg(x) := f(x)g(x)$ is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then $f/g(x) := \frac{f(x)}{g(x)}$ is convex.
Hint. Prove by the definition that

$$\frac{f(\lambda x + (1-\lambda y))}{g(\lambda x + (1-\lambda y))} \le \lambda \frac{f(x)}{g(x)} + (1-\lambda)\frac{f(y)}{g(y)}.$$

Without the loss of generality, we assume that $x \le y$. By the property of the functions, you can verify that

$$f(x) + f(y) \le \frac{g(y)}{g(x)} f(x) + \frac{g(x)}{g(y)} f(y),$$

which is equivalent to

$$\frac{g(y)}{g(x)} \le 1 \le \frac{f(y)}{f(x)}.$$

8. (Optional) Prove that the union of finitely many closed sets (i.e., $\cup_{i=1}^k \mathcal{S}_i$ and k is finite) is closed, but the infinite union (i.e., $\cup_{i=1}^\infty \mathcal{S}_i$) may not be closed (provide a counterexample). Similarly, the intersection of finitely many open sets is open, but the infinite intersection may not be open.

9. (Optional) Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set, with $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in \mathcal{C}$, and let $\theta_1, \dots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \ge 0$ and $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 \boldsymbol{x}_1 + \cdots + \theta_k \boldsymbol{x}_k \in \mathcal{C}$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary integers $k \ge 2$.)
Hint. Use induction on k. You can write

$$\begin{aligned} &\theta_1 \boldsymbol{x}_1 + \cdots + \theta_k \boldsymbol{x}_k + \theta_{k+1} \boldsymbol{x}_{k+1} \\ &= \left(\sum_{i=1}^k \theta_i\right) \cdot \left(\frac{\theta_1}{\sum_{i=1}^k \theta_i} \boldsymbol{x}_1 + \cdots + \frac{\theta_k}{\sum_{i=1}^k \theta_i} \boldsymbol{x}_k\right) + \left(1 - \sum_{i=1}^k \theta_i\right) \boldsymbol{x}_{k+1}. \end{aligned}$$

10. (Optional) *Composition rules.* Show that the following functions are convex.