Solutions to 1.1 Exercises

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1 exercise 1.1

Exercise 1: Problem:

An boat is traveling at a constant speed v_0 towards the bank of a river vertically. The flow speed of the river is on the x axis and the absolute value is Proportional to the distance from both banks. The proportionality constant is k. the width of the river is a. Find the equation of the trajectory of the boat.

Solution:

Let the position of the boat be (x, y), where y is the distance from the bottom bank. The speed of the boat relative to the ground is

$$\frac{dx}{dt} = ky(a - y), \quad \frac{dy}{dt} = v_0.$$

Thus, we have

$$\frac{dx}{dy} = \frac{ky(a - y)}{v_0}.$$

Integrating both sides, we get

$$x = \frac{ka}{2v_0}y^2 + \frac{ky^3}{3v_0} + C.$$

If we assume the boat starts from the bottom bank at $y = 0$ and $x = 0$, then $C = 0$. Therefore, the equation of the trajectory is

$$x = \frac{ka}{2v_0}y^2 + \frac{ky^3}{3v_0}.$$

Exercise 2: Problem:

Determine the differential equation of a curve in the Cartesian Oxt-plane such that, at each point of the curve, the tangent line together with the radius vector to that point and the x-axis forms an isosceles triangle.

Solution:

- First scenario: the absolute value of radius vector is the same as the length of the tangent line segment from the point of tangency to the x-axis.

Here, the length of radius vector is $\sqrt{x^2+y^2}$, and the length of the tangent line segment from the point of tangency to the x-axis is $\sqrt{(t\frac{dx}{dt})^2+y^2}$. Applying the isosceles triangle condition, we have

$$\sqrt{x^2+y^2}=\sqrt{(t\frac{dx}{dt})^2+y^2}.$$

Squaring both sides and simplifying, we get

$$x^2=(t\frac{dx}{dt})^2.$$

Taking the square root of both sides, we have

$$x=t\frac{dx}{dt} \quad \text{or} \quad x=-t\frac{dx}{dt}.$$

Rearranging, we obtain the differential equations

$$\frac{dx}{dt}=\frac{x}{t} \quad \text{or} \quad \frac{dx}{dt}=-\frac{x}{t}.$$

- Second scenario: the absolute value of radius vector is the same as the length of the x-axis segment from the origin to the intersection of the tangent line with the x-axis.

  Here, the length of radius vector is $\sqrt{x^2+t^2}$, and the length of the x-axis segment from the origin to the intersection of the tangent line with the x-axis is $x-t\frac{dx}{dt}$
  Applying the isosceles triangle condition, we have

$$\sqrt{x^2+t^2}=x-t\frac{dx}{dt}.$$

  Squaring both sides and simplifying, we get

$$x^2+t^2=x^2-2xt\frac{dx}{dt}+(t\frac{dx}{dt})^2.$$

  Rearranging, we obtain the differential equation

$$t^2=-2xt\frac{dx}{dt}+(t\frac{dx}{dt})^2.$$

  Dividing both sides by t^2 (assuming $t\neq 0$), we have

$$1=-2\frac{x}{t}\frac{dx}{dt}+(\frac{dx}{dt})^2.$$

  Rearranging, we get

$$(\frac{dx}{dt})^2-2\frac{x}{t}\frac{dx}{dt}-1=0.$$

  Solving this quadratic equation for $\frac{dx}{dt}$, we find

$$\frac{dx}{dt}=\frac{x}{t}\pm\sqrt{(\frac{x}{t})^2+1}.$$

- Third scenario: the length of the tangent line segment from the point of tangency to the x-axis is the same as the length of the x-axis segment from the origin to the intersection of the tangent line with the x-axis.

  Here, the length of the tangent line segment from the point of tangency to the x-axis is $\sqrt{(t\frac{dx}{dt})^2 + y^2}$, and the length of the x-axis segment from the origin to the intersection of the tangent line with the x-axis is $x - t\frac{dx}{dt}$.

  Applying the isosceles triangle condition, we have

  $$\sqrt{(t\frac{dx}{dt})^2 + t^2} = x - t\frac{dx}{dt}.$$

  Squaring both sides and simplifying, we get

  $$(t\frac{dx}{dt})^2 + t^2 = x^2 - 2xt\frac{dx}{dt} + (t\frac{dx}{dt})^2.$$

  Rearranging, we obtain the differential equation

  $$y^2 = x^2 - 2xt\frac{dx}{dt}.$$

  Dividing both sides by t^2 (assuming $t \neq 0$), we have

  $$1 = (\frac{x}{t})^2 - 2\frac{x}{t}\frac{dx}{dt}.$$

  Rearranging, we get

  $$2\frac{x}{t}\frac{dx}{dt} = (\frac{x}{t})^2 - 1.$$

  Thus, the differential equation is

  $$\frac{dx}{dt} = \frac{(\frac{x}{t})^2 - 1}{2\frac{x}{t}}.$$