Cross-sectional Asset Pricing

Empirical Asset Pricing

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Outcome of lecture

After the lecture, you should have

- knowlegde and understading of
  - Econometric methods and techniques for estimating and testing risk premia in the cross- section of asset returns
- and be able to
  - Discuss and estimate the SDF using common empirical methods, evaluate its ability to price the cross-section and conduct valid inference, and reflect on the findings and their implications

➜ The methodology to conduct a cross-sectional empirical asset pricing study!

Objective of today

- In other words:

Elephants and the Cross-Section of Expected Returns

Nora Laurinaityte[§] Christoph Meinerding[†] Christian Schlag[‡] Julian Thimme[*]

➜ You should be able to test whether any given factor is priced in a cross-section of assets!

The main idea of empirical asset pricing

- Recall from the last lecture that we have equivalence between SDF and β representations
- …given an SDF, we can always find a β representation and given a β representation, we can always find a linear factor model that defines the SDF
- Linear factor models of the SDF and, equivalently, β representations are by far the most popular in the empirical asset pricing literature (as you will see):

The main idea of empirical asset pricing

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Is there a risk-return tradeoff in the corporate bond market? Time-series and cross-sectional evidence☆

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the factor model of Bai et al. (2019) that introduces the downside risk, credit risk, and liquidity risk factors based on independently sorted bivariate portfolios of bond-level credit rating, value at risk (VaR), and illiquidity:[14]

$$R_{i,t} = \alpha_i + \beta_{1,i} \cdot MKT_t + \beta_{2,i} \cdot DRF_t + \beta_{3,i} \cdot CRF_t + \beta_{4,i} \cdot LRF_t + \epsilon_{i,t}, \quad (10)$$

where $R_{i,t}$ is the excess return on bond i in month t. Total risk of bond i is measured by the variance of $R_{i,t}$ in Eq. (9), denoted by σ_i^2. Idiosyncratic (or residual) risk of bond i is

The main idea of empirical asset pricing

Common Risk Factors in Currency Markets

Hanno Lustig
UCLA Anderson and NBER

Nikolai Roussanov
Wharton, University of Pennsylvania and NBER

Adrien Verdelhan
MIT Sloan and NBER

$$M_{t+1} = 1 - b(\Phi_{t+1} - \mu_\Phi),$$

where b is the vector of factor loadings and μ_Φ denotes the factor means. This linear factor model implies a beta pricing model: The expected excess return is equal to the factor price λ times the beta of each portfolio β^j:

$$E[Rx^j] = \lambda' \beta^j,$$

The main idea of empirical asset pricing

COMMON RISK FACTORS IN CRYPTOCURRENCY

Yukun Liu
Aleh Tsyvinski
Xi Wu

Table 9: Cryptocurrency Market and Size Factor Model

$$R_i - R_f = \alpha^i + \beta^i_{CMKT} CMKT + \beta^i_{CSMB} CSMB + \epsilon_i \quad (2)$$

where $CMKT$ is the cryptocurrency excess market returns and $CSMB$ is the cryptocurrency size factor.

The main idea of empirical asset pricing

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Are there common factors in individual commodity futures returns?

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3. Asset pricing models: Macro and equity-motivated tradable factors

In this section, we investigate whether models that include aggregate and equity-motivated tradable factors can explain the common variation of commodity futures returns. Let the beta formulation of a *K*-factor asset pricing model

$$E(r_i) = \beta_i' \lambda, \quad i = 1, 2, \ldots, N, \tag{1}$$

The main idea of empirical asset pricing

The main idea

The main approach for testing asset pricing models is the use of time-series regressions and cross-sectional regressions. Either:

- Conducted separately as in Fama-Macbeth two-stage regressions
- Jointly in a unified GMM framework.

The main idea of empirical asset pricing

- Either one postulates a β representation of a model (e.g. derived from CAPM) or one starts with a linear factor model for the SDF (often the case with CCAPM).
- In the latter case, one can estimate the SDF loadings (b) directly and the back out the risk premia (γ) in the β representation directly.
- Otherwise, one can always work with the (possibly implied) β representation.

  $\rightarrow$We will focus on this approach today and see an example with the SDF being starting point later.

Which factors to use?

- In empirical asset pricing an essential problem is the choice of factors (more about this in next (or end of the this) week).

Factor selection

There are, broadly speaking, three common ways to select factors for the asset pricing model:

1. Theoretical or economic intuition:
    - Factors can be directly derived in theoretical asset pricing models like CAPM or CCAPM
    - They can be motivated via the Intertemporal CAPM that allows for any state variable that predicts future investment opportunities (be aware of factor fishing!) for instance macroeconomic factors

Which factors to use?

Factor selection (cont'd)

[...]

2. Statistical: From APT we can extract factors from a large data set of asset returns, using e.g. Principal Component Analysis.
3. Firm characteristics: Creating facors based on firm characteristics, motivated by return anomalies. Most prominent example are the SMB and HML of Fama and French (1993)

- We will continue as if K factors have been chosen, for whatever of above reasons
- We will focus on unconditional asset pricing models

Fama-Macbeth

Fama-Macbeth regressions

- A pioneering approach to **estimating** **asset pricing models** and conducting **inference** is through Fama-Macbeth two-stage regressions.

Fama-MacBeth procedure

The Fama and MacBeth (1973) methodology is a **cross-sectional regression method** that consists of **two-steps**.

1. In the **first step**, we obtain time-series **betas** β_i for all assets from time series regressions of excess returns onto risk factors. This step is necessary as β_i is **not directly observable** and, thus, needs to be estimated.
2. In the **second step**, we obtain an estimate of the **risk premia** γ through a series of cross-sectional regressions using the estimated β_i, $\hat{\beta}_i$, from the first step as input.

Fama-MacBeth regressions (cont'd)

Fama-MacBeth procedure: First-stage regression

For each asset $i = 1, \ldots, N$, we estimate β_i using a single full sample **time-series OLS regression** of the form

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_i f_t + \varepsilon_{i,t}, \tag{1}$$

where $\varepsilon_{i,t}$ is a zero-mean error term.

$\ldots$, obtaining $\hat{\beta}_i$ for $i, \ldots, N$.

- We use a **full-sample estimation** in obtaining $\hat{\beta}_i$, effectively assuming a **constant factor loading** (risk exposure). This can be extended to **rolling** βs which we will see below.

Fama-MacBeth regressions (cont'd)

Fama-MacBeth procedure: Second-stage regression(s)

For each time-period $t = 1, \ldots, T$, we run **cross-sectional regressions** of all assets against **the estimated betas**, i.e.

$$R_{i,t} - R_{f,t} = \gamma_{0t} + \gamma_t \hat{\beta}_i + \eta_{i,t}, \tag{2}$$

where the estimated value $\hat{\beta}_i$ is obtained from the **first-stage time series regressions** and η_{it} is a mean-zero error term.
Estimating these T cross-sectional regressions provides us with a **time series of estimates** of $\{\hat{\gamma}_{0,t}, \hat{\gamma}_t\}$, which can be used to form estimates of $\hat{\gamma}_0$ and $\hat{\gamma}$ as follows

$$\hat{\gamma}_j = \frac{1}{T} \sum_{t=1}^{T} \hat{\gamma}_{j,t}, \tag{3}$$

where $j = \{0, 1, \ldots, K\}$.

Fama-MacBeth regressions (cont'd)

- Now we have an **estimate** for the **risk premia** of the factors.
- Fama and MacBeth (1973) suggest the following expression for computing an **estimate of the variance** of each risk premium estimate

$$\text{Var}[\hat{\gamma}_j] = \frac{1}{T^2} \sum_{t=1}^{T} \left(\hat{\gamma}_{jt} - \hat{\gamma}_j\right)^2 . \tag{4}$$

- One can also get the entire covariance matrix for risk premia (useful later in this lecture), here assuming a constant is subsumed in γ, by

$$\text{Var}[\hat{\gamma}] = \frac{1}{T^2} \sum_{t=1}^{T} \left(\hat{\gamma}_t - \hat{\gamma}\right) \left(\hat{\gamma}_t - \hat{\gamma}\right)' . \tag{5}$$

Fama-MacBeth regressions (cont'd)

Fama-MacBeth procedure: Hypothesis testing

With an estimate $\hat{\gamma}_j$ and an associated standard error $\sqrt{\text{Var}[\hat{\gamma}_j]}$ in hand, our interest is in testing the following null hypothesis

$$H_0 : \gamma_j = 0. \tag{6}$$

We can test this hypothesis using the following (conventional) t-statistic

$$t(\gamma_j) = \frac{\hat{\gamma}_j}{\sqrt{\text{Var}[\hat{\gamma}_j]}} \xrightarrow{d} N(0,1), \quad \text{as } T \to \infty. \tag{7}$$

The test statistic $t(\gamma_j)$ follows a student-t distribution with $T - K$ degrees of freedom in finite samples and a standard normal distribution asymptotically (why?).

…One can naturally entertain many different hypotheses using this framework.

Rolling βs as an alternative first-stage

- In fact, the original Fama-Macbeth methodology was introduced with **rolling window** βs.

Rolling βs

- One way to obtain (some) **time-variation** into β is to run a **first-stage type regression** at every time point t for $t = M, \ldots, T$, where M is the length of the rolling window.
- Typically, the **window length is fixed**, discarding the oldest time point whenever data in the most recent one gets available.
- This generates $\hat{\beta}_{it}$.

- Inclusion in Fama-Macbeth regressions is simple, as it amounts to using $\hat{\beta}_{it}$ for the t'th cross-sectional regression instead of always $\hat{\beta}_i$.

A single cross-sectional regression

- It can suffice to run a **single cross-sectional regression** in the **second stage**, which is the general approach explained in Goyal (2012) (cf. his eq. (18)).

Single cross-sectional regression

The single cross-sectional regression approach estimates

$$\overline{R_{it} - R_{ft}} = \gamma_0 + \gamma \hat{\beta}_i + \eta_i, \tag{8}$$

$\overline{R_{it} - R_{ft}} = T^{-1} \sum_{t=1}^{T} R_{it} - R_{ft}$ is the sample average of excess return to asset i.

- This is motivated from **rational expectations of investors** (or that $\mathbb{E}[R_{it} - R_{ft}]$ can be consistently estimated by its sample average, given stationarity of data).
- This will provide us with estimates of $\hat{\gamma}_0$ and $\hat{\gamma}$ that are identical to those from the t-by-t procedure.
- The usual **OLS standard errors** will be **very very** wrong.

Example: Linear consumption-based asset pricing

- One can obtain an **implementable, linear consumption-based factor model** without the need for specifying a certain utility function as per

$$\mathbb{E}[R_{it} - R_{ft}] = \gamma_c \beta_{ic}, \tag{9}$$

where "c" indicates consumption growth, denoted $\tilde{c}_t$, and

$$\beta_{ic} = \frac{\text{Cov}[R_{it} - R_{ft}, \tilde{c}_t]}{\text{Var}[\tilde{c}_t]}. \tag{10}$$

- The ($\approx$ 2 page) derivations can be found in my lecture notes.
- Let us consider an implementation in the Matlab live script *famaMacbeth.mlx*.

Two (big) issues in Fama-Macbeth regressions

Drawback of the Fama-MacBeth approach

* The Fama-MacBeth approach, while simple and highly useful, does have several problems

1. The inputs, $\mathbb{E}\left[\tilde{r}_i\right]$, $\mathbb{E}\left[\tilde{r}_M\right]$ and β_{iM}, are unobservable and have to be estimated
2. This gives rise to a so-called errors-in-variables problem as we are using estimated β_{iM}s in the second-stage cross-sectional regression
    - The errors-in-variables problem biases standard errors and biases λ_M towards zero
    - To reduce the problem, one can either group stocks into portfolio to get better β_{iM} estimates
    - or we can explicitly adjust standard errors to account for the bias introduced by the errors-in-variables problem (outside the scope of this course)
3. The approach does not account for autocorrelation and heteroskedasticity
    - One can solve this by using GMM instead (outside the scope of this course)

Errors-in-variables

- A major problem with the conventional Fama-Macbeth regression analysis is a **generated regressor** or **errors-in-variables** (EIV) issue.
- In the **cross-sectional stage**, **explanatory variables** are themselves **estimates** and contain, therefore, **estimation error**
- This estimation error, $\nu_i = \hat{\beta}_i - \beta_i$, will cause an **overstated precision** of the **risk premia estimates** if one uses the classical Fama-Macbeth standard errors from (4)

Errors-in-variables

- The **overstated precision** is directly a function of the **precision** in $\hat{\beta}$s. As such:
  1. Macroeconomic data are typically **measured with error** and often **weakly related to returns**, causing β to be imprecisely estimated
  2. If **risk factors are returns** themselves, this will generally reduce estimation error in $\hat{\beta}$
  3. The **larger time-series** the less estimation error in $\hat{\beta}$. For instance, monthly frequency tends to deliver less problems with EIV than annual data
  4. **Portfolios** of returns typically **average out noise** from individual assets, hence using those as test assets (LHS in first stage regression) improves precision of $\hat{\beta}$

Shanken correction for errors-in-variables

- Shanken (1992) provides a **solution** to the **EIV problem**
- OLS standard errors are scaled upwards to reflect this overstated precision of $\hat{\gamma}$.
- The **correction term** depends on which version of the Fama-Macbeth regression analyses is applied

Shanken corrections for EIV

Let $\text{Var}[\hat{\gamma}]$ be the Fama-Macbeth covariance matrix given in (4), thus including the intercept. Then the Shanken-corrected covariance matrices are as follows:

1. If one uses full-sample (constant) βs from the first stage in a t-by-t cross-sectional stage,

$$\text{Var}_{\text{EIV}}[\hat{\gamma}] = T^{-1}\left((1+c)\left(T\text{Var}[\hat{\gamma}] - \widetilde{\text{Var}}[f_t]\right) + \widetilde{\text{Var}}[f_t]\right) \tag{11}$$

[...]

Shanken correction for errors-in-variables

Shanken corrections for EIV (cont'd)

[...]

2. If one uses **full-sample** (constant) βs from the first stage in a **single** cross-sectional stage,

$$\mathrm{Var}_{\mathrm{EIV}}[\hat{\gamma}] = T^{-1}\left((1+c)T\mathrm{Var}[\hat{\gamma}] + \widetilde{\mathrm{Var}}[f_t]\right) \tag{12}$$

3. If one uses **rolling** βs, estimated over y years with m data points per year, from the first stage in a t-by-t cross-sectional stage,

$$\mathrm{Var}_{\mathrm{EIV}}[\hat{\gamma}] = T^{-1}\left((1+c^*)T\mathrm{Var}[\hat{\gamma}] + \widetilde{\mathrm{Var}}[f_t]\right) \tag{13}$$

where $c = \hat{\gamma}'\widetilde{\mathrm{Var}}[f_t]^{-1}\hat{\gamma}$, $\mathrm{Var}[f_t]$ the sample covariance matrix of the risk factors, $\widetilde{\mathrm{Var}}[f_t]$ the $(K+1)\times(K+1)$ matrix with zeros in the first row and column, corresponding to the places of the intercept, and $\mathrm{Var}[f_t]$ in the lower right block, and

$$c^* = c\left(1 - \frac{(y-1)(y+1)}{3yT/m}\right). \tag{14}$$

Shanken correction for errors-in-variables

- Be **careful** in using the **correct formulas** - this is not always acknowledged in the empirical literature ((13) is hidden in a footnote ... in an Appendix!)
- The effect of the Shanken correction can be substantial and impact conclusions severely, see e.g. Table II of Kan et al. (2013).

Shanken correction for errors-in-variables

Panel A: OLS

	CAPM		C-LAB				FF3			
	$\hat{\gamma}_0$	$\hat{\gamma}_{vw}$	$\hat{\gamma}_0$	$\hat{\gamma}_{vw}$	$\hat{\gamma}_{lab}$	$\hat{\gamma}_{prem}$	$\hat{\gamma}_0$	$\hat{\gamma}_{vw}$	$\hat{\gamma}_{smb}$	$\hat{\gamma}_{hml}$
Estimate	1.61	−0.46	1.77	−0.90	0.21	0.45	1.94	−0.95	0.16	0.41
t-ratio$_{fm}$	3.48	−1.19	4.16	−2.48	1.76	3.53	5.64	−3.00	1.18	3.41
t-ratio$_{s}$	3.46	−1.18	2.63	−1.70	1.12	2.25	5.45	−2.93	1.18	3.41
t-ratio$_{jw}$	3.39	−1.17	2.79	−1.79	1.20	2.46	5.53	−2.93	1.19	3.44
t-ratio$_{pm}$	3.12	−1.11	2.78	−1.76	0.99	2.71	5.17	−2.75	1.19	3.42

	ICAPM						CCAPM	
	$\hat{\gamma}_0$	$\hat{\gamma}_{vw}$	$\hat{\gamma}_{term}$	$\hat{\gamma}_{def}$	$\hat{\gamma}_{div}$	$\hat{\gamma}_{rf}$	$\hat{\gamma}_0$	$\hat{\gamma}_{cg}$
Estimate	1.14	−0.15	0.20	−0.14	−0.02	−0.44	0.96	0.18
t-ratio$_{fm}$	2.61	−0.47	2.50	−2.69	−1.32	−3.13	2.57	0.75
t-ratio$_{s}$	1.69	−0.33	1.62	−1.75	−0.89	−2.03	2.51	0.73
t-ratio$_{jw}$	1.76	−0.35	1.56	−1.55	−0.91	−1.84	2.53	0.76
t-ratio$_{pm}$	1.56	−0.32	1.38	−1.50	−0.85	−1.85	2.14	0.65

	CC-CAY				U-CCAPM		D-CCAPM			
	$\hat{\gamma}_0$	$\hat{\gamma}_{cay}$	$\hat{\gamma}_{cg}$	$\hat{\gamma}_{cg\text{-}cay}$	$\hat{\gamma}_0$	$\hat{\gamma}_{cg36}$	$\hat{\gamma}_0$	$\hat{\gamma}_{vw}$	$\hat{\gamma}_{cg}$	$\hat{\gamma}_{cgdur}$
Estimate	1.46	−1.46	−0.02	0.00	0.68	3.46	2.20	−1.18	0.45	1.84
t-ratio$_{fm}$	3.81	−2.42	−0.13	0.99	1.15	3.39	5.82	−3.48	2.26	3.25
t-ratio$_{s}$	2.62	−1.67	−0.09	0.69	0.80	2.36	4.41	−2.81	1.72	2.48
t-ratio$_{jw}$	3.26	−2.07	−0.10	0.78	0.93	2.66	5.22	−3.27	1.69	2.42
t-ratio$_{pm}$	2.86	−1.21	−0.06	0.30	0.95	2.34	5.10	−3.22	1.25	2.30

Shanken correction for errors-in-variables

- Let us consider some examples by continuing our example in the Matlab live script *famaMacbeth.mlx*.

Shanken correction for errors-in-variables

- Note that in the *famaMacbeth.mlx* example where we included **firm characteristics**, the Shanken correction **needs** a slight augmentation to function properly
- The typical argument is that **firm characteristics** are **directly observed** and **does not contribute** to **EIV problems** in the cross-sectional stage
- For that reason, we (still) only need to incorporate the EIV issues coming from estimated βs which influence the risk premia associated with characteristics
- This is achieved by augmenting $\widetilde{\text{Var}}[f_t]$ as to **include zero rows and zero columns** at the **place of the characteristics**, similarly to what we did for the the intercept

GMM

GMM approach to cross-sectional asset pricing

- The Fama-Macbeth regression analysis, with or without Shanken corrections, still **does not account for the presence of autocorrelation**
- Moreover, they tend to **assume normaility of regression errors** as well
- Lastly, while the approach by Shanken (1992) appears manageable to correct for EIV, an **easier** and **much more elegant** approach exists

$\Rightarrow$ map the whole thing into **GMM**!

GMM approach to cross-sectional asset pricing

GMM approach (intuitively)

- The main idea is to define two sets of moments:
  1. The first set matches the time series stage for obtaining β
  2. The second set matches the cross-sectional stage for obtaining γ
- Merging those two sets of moments **“internalizes"** any EIV, as β and γ are essentially estimated jointly
- The long-run covariance matrix of moments S will then **capture** directly the **effect of generated regressors**
- …and we already know how to amend S (nonparametrically) to account for autocorrelation and heteroskedasticity through HAC

GMM approach to cross-sectional asset pricing

Definition: Beauty

The beauty of this approach is that it captures (almost) all issues in one set of moments, yet it is essentially **still a Fama-Macbeth regression** analysis but with an additional layer that provides proper and accurate standard errors accounting for both EIV, autocorrelation, and heteroskedasticity and is much more mild in assumptions.

GMM approach to cross-sectional asset pricing

- For simplicity, let us define excess returns for the i'th asset as $R^e_{it} \equiv R_{it} - R_{ft}$, $i = 1, \ldots, N$.
- We may write the time series stage in vector form as follows

$$R^e_t = \alpha + \beta f_t + \varepsilon_t, \tag{15}$$

  where $R_t, \alpha, \varepsilon_t$ are all $N \times 1$, β is $N \times K$ and f_t is $K \times 1$.
- Now, recall that the **identifying moment conditions** of OLS are, intuitively, that the error term is mean zero and that it is uncorrelated with the regressors.

GMM approach to cross-sectional asset pricing

- Formally, the **OLS moment conditions** for a univariate dependent variable y, regressors x, and coefficients θ are

$$\mathbb{E}[y - \theta x] = 0 \quad \text{and} \quad \mathbb{E}[(y - \theta x)x] = 0. \tag{16}$$

Time-series stage moment conditions

Thus, OLS estimation of the time-series stage maps into the following set of moments

$$\mathbb{E}\begin{bmatrix} R_t^e - \alpha - \beta f_t \\ (R_t^e - \alpha - \beta f_t) \otimes f_t \end{bmatrix} = \mathbb{E}\begin{bmatrix} \varepsilon_t \\ \varepsilon_t \otimes f_t \end{bmatrix} = \begin{bmatrix} 0_{N\times 1} \\ 0_{NK\times 1} \end{bmatrix}, \tag{17}$$

equalling $N + NK$ moment conditions.

- Since we have N many αs and NK many βs to estimate, the system is exactly identified and **reduces to the analytical solution of OLS regressions** for each i
- The Kronecker product ensures that all errors terms are uncorrelated with all risk factors

GMM approach to cross-sectional asset pricing

- Recall that the β representation of the K-factor asset pricing model delineates

$$\mathbb{E}[R_t^e] = \gamma_0 + \gamma\beta. \tag{18}$$

- Strictly speaking, in many cases there is **no constant** γ_0 and the restriction $\gamma_0 = 0$ could be imposed
- We will focus on the case where we include the constant and consider $\gamma_0 = 0$ a **testable restriction**
- This defines the second, **cross-sectional stage of the Fama-Macbeth analysis**
- As such, it will naturally also define the second set of moments…

GMM approach to cross-sectional asset pricing

Cross-sectional stage moment conditions

Thus, the implication from any K-factor asset pricing model expressed in β representation is the following moment conditions

$$\mathbb{E}[R_t^e - \gamma_0 - \gamma\beta] = 0_{N\times 1}. \tag{19}$$

Joint moment conditions

Thus, the joint moment conditions are given by

$$\mathbb{E}\begin{bmatrix} R_t^e - \alpha - \beta f_t \\ (R_t^e - \alpha - \beta f_t) \otimes f_t \\ R_t^e - \gamma_0 - \gamma\beta \end{bmatrix} = \mathbb{E}\begin{bmatrix} \varepsilon_t \\ \varepsilon_t \otimes f_t \\ \eta_i \end{bmatrix} = \begin{bmatrix} 0_{N\times 1} \\ 0_{NK\times 1} \\ 0_{N\times 1} \end{bmatrix}, \tag{20}$$

equalling $N(K+2) = NK + 2N$ moment conditions.

- Note that the system is now **overidentified**, as we added N moments and only need to estimate $K+1$ additional parameters.

GMM approach to cross-sectional asset pricing

- The joint system in (20) will generally **not** reproduce OLS estimates. In order to achieve those (and be consistent with the Fama-Macbeth approach), we need to amend the moments slightly.
- Similarly to the time-series stage, we need mean zero errors and them being uncorrelated with regressors.
- To achieve that, define the following $(N+1)(K+1) \times N(K+2)$ matrix

$$e = \begin{bmatrix} I_{N(K+1)} & 0_{N(K+1)\times N} \\ 0_{(K+1)\times N(K+1)} & \chi' \end{bmatrix}, \tag{21}$$

  where $\chi = (\iota_{N\times 1}, \beta)$ is $N \times (K+1)$ and ι is a N-vector of ones.
- Note that $(N+1)(K+1) = NK + N + K + 1$.

GMM approach to cross-sectional asset pricing

- The neat thing here is that when **pre-multiplying** e onto the joint moment conditions in (20), we maintain the first $N + NK$ conditions as is (the time series stage moments), but weight each of the last N moments conditions (the cross-sectional stage moments) by χ
- $\ldots\chi$ is indeed containing the regressors used in the cross-sectional stage
- $\ldots$so it fits the natural OLS type of identifying moments!

Joint OLS-type moment conditions

The OLS type moment conditions

$$\begin{bmatrix} I_{N(K+1)} & 0_{N(K+1)\times N} \\ 0_{(K+1)\times N(K+1)} & \chi' \end{bmatrix} \mathbb{E}\begin{bmatrix} R_t^e - \alpha - \beta f_t \\ (R_t^e - \alpha - \beta f_t) \otimes f_t \\ R_t^e - \gamma_0 - \gamma\beta \end{bmatrix} = \begin{bmatrix} 0_{N\times 1} \\ 0_{NK\times 1} \\ 0_{(K+1)\times 1} \end{bmatrix},$$

which is equivalent to $eg = 0_{(N+1)(K+1)}$ and where g is defined as (20), reproduces the two-pass Fama-Macbeth estimates.

GMM approach to cross-sectional asset pricing

- So **why bother** doing all this if we might as well just run Fama-Macbeth regressions? **standard errors**!

- Of course, we could also estimate parameters directly via the standard GMM recipe, yet why not keep it simple?

The GMM recipe for asset pricing

In order to **estimate and evaluate asset pricing models** in a β representation, the following approach will be used:

1. Estimate β_i for $i = 1, \ldots, N$ via standard Fama-Macbeth time series stage regressions.
2. Estimate γ_j for $j \in \{0, 1, \ldots, K\}$ via a standard Fama-Macbeth single cross-sectional regression.
3. Use the definition of e in (21) and the joint moment condition system in (20) to obtain standard errors that are robust to EIV, autocorrelation, and heteroskedasticity.

GMM approach to cross-sectional asset pricing

- To compute those standard errors, define

$$\theta' = (\alpha', \text{vec}(\beta)', \gamma_0, \gamma)', \tag{22}$$

  which contains all $N + NK + 1 + K = (N+1)(K+1)$ parameters to be estimated.
- The vectorization defines the transformation of any matrix of dimension, say, $N \times K$, into a column vector of dimension $NK \times 1$ simply by stacking all columns of the matrix on top of another.
- We are interested in obtaining $\text{Var}[\hat{\theta}]$ for which we need the ingredients of e, S, and yet another matrix (the gradient) defined on the next slide.

GMM approach to cross-sectional asset pricing

Covariance of $\hat{\theta}$

The $(N+1)(K+1) \times (N+1)(K+1)$ covariance of $\hat{\theta}$ is

$$\text{Var}[\hat{\theta}] = T^{-1}(eD)^{-1}eSe'(eD)^{-1\prime}, \tag{23}$$

where $D = \mathbb{E}[\partial g(\theta)/\partial\theta]$ is equal to

$$D = -\begin{bmatrix} \begin{bmatrix} 1 & \mathbb{E}[f_t'] \\ \mathbb{E}[f_t] & \mathbb{E}[f_t f_t'] \end{bmatrix} \otimes I_N & 0_{N(K+1)\times(K+1)} \\ \begin{bmatrix} 0 & \gamma' \end{bmatrix} \otimes I_N & \chi \end{bmatrix} \tag{24}$$

and S is the long-run covariance matrix equal to

$$S = \sum_{s=-\infty}^{\infty} \mathbb{E}\left[\begin{bmatrix} R_t^e - \alpha - \beta f_t \\ (R_t^e - \alpha - \beta f_t) \otimes f_t \\ R_t^e - \gamma_0 - \gamma\beta \end{bmatrix} \begin{bmatrix} R_{t-s}^e - \alpha - \beta f_{t-s} \\ (R_{t-s}^e - \alpha - \beta f_{t-s}) \otimes f_{t-s} \\ R_{t-s}^e - \gamma_0 - \gamma\beta \end{bmatrix}' \right] \tag{25}$$

(Note that there is an error in D in Goyal (2012))

GMM approach to cross-sectional asset pricing

- Note that the formula is just the formula shown in past lectures with a certain weighting matrix e.
- …and S is defined similarly as earlier, so is D.
- It is neat that D has an analytical form, and $\mathbb{E}[f_t]$ and $\mathbb{E}[f_t f_t']$ can easily be estimated by their sample counter parts as

$$T^{-1} \sum_{t=1}^{T} f_t \xrightarrow{p} \mathbb{E}[f_t] \tag{26}$$

and

$$T^{-1} \sum_{t=1}^{T} f_t f_t' \xrightarrow{p} \mathbb{E}[f_t f_t']. \tag{27}$$

GMM approach to cross-sectional asset pricing

- The last ingredient is an estimate of S.
- We discussed this during the second double lectures, including how to construct a nonparametric HAC we did this for any generic set of moment conditions, in which (20) naturally falls.
- Let us consider how all this can be implemented, using the Matlab live script *GMM_crossSectionalAssetPricing.mlx*.

A comment on traded vs. non-traded factors

- Some factors are returns themselves, e.g. the market risk premium, SMB, or HML.
- In those cases, we do not need to estimate their risk premium in a cross-sectional stage, but can suffice by taking its sample mean.
- When factors are non-traded, we need to estimate their risk premia using the cross-sectional stage.

Why would one do the cross-sectional stage with traded factors then?

Lewellen et al. (2010) emphasize that one useful diagnostic test of an asset pricing model with traded factors (e.g. CAPM or the Fama-French three factor model) is that the estimated risk premia from the cross-sectional stage should be statistically indistinguishable from their sample mean taken over the time series dimension.

Estimating SDF loadings

- Instead of estimating the β representation, we could also have estimated the parameters of the SDF (called SDF loadings) denoted by b.
- We can make inference on them, and back out risk premia directly through the covariance matrix of factors.
- But be aware that the statement by Goyal (2012) that

  “Which method one uses [TSR+CSR approach or SDF approach] is, therefore, largely a matter of individual preference."

  is at best very imprecise and slightly wrong.
- Even though risk premia and SDF loading are directly related, they differ in interpretation!

Estimating SDF loadings

- For now, we will state the approach, assuming an SDF of the form

$$M_t = 1 - b' f_t. \tag{28}$$

- We may then form standard GMM moment conditions from the fundamental equation of asset pricing as

$$\mathbb{E}[(1 - b' f_t) R_t^e] = 0_{N \times 1}. \tag{29}$$

- We also have that (be careful with the dimensions in Goyal (2012) here)

$$D = \mathbb{E}\left[R_t^e f_t'\right], \tag{30}$$

which can be estimated simply as

$$T^{-1} \sum_{t=1}^{T} R_t^e f_t'. \tag{31}$$

Estimating SDF loadings

- Note also that

$$S = \sum_{s=-\infty}^{\infty} \mathbb{E}[(R_t^e - b' f_t R_t^e)(R_{t-s}^e - b' f_{t-s} R_{t-s}^e)'], \tag{32}$$

where Goyal (2012) has a minor typo as well in his eq. (40).

SDF loading estimates and their variance

If the weighting matrix is the identity matrix (yielding $\hat{b}_1$) or the optimal S^{-1} matrix (yielding $\hat{b}_2$), we find

$$\hat{b}_1 = (D'D)^{-1} D' \overline{R^e}, \tag{33}$$

$$\hat{b}_2 = (D'S^{-1}D)^{-1} D' S^{-1} \overline{R^e}. \tag{34}$$

with

$$\text{Var}[\hat{b}_1] = T^{-1}(D'D)^{-1} D'SD(D'D)^{-1}, \tag{35}$$

$$\text{Var}[\hat{b}_2] = T^{-1}(D'SD)^{-1}. \tag{36}$$

Skeptical appraisal

A skeptical appraisal of asset pricing tests

- While our approach to **cross-sectional asset pricing** is **intriguing** and deals with numerous econometric issues, it is still subject to critique.

A skeptical appraisal

In many situations encountered in practice, it may be easy to find factors that explain the cross-section of expected returns. **Finding a high cross-sectional R^2 and small pricing errors often has little economic meaning** and, in the authors' view, does not, by itself, provide much support for a proposed model. The problem is not just a sampling issue - it cannot be solved by getting standard errors right - though sampling issues exacerbate the problem.

- Lewellen et al. (2010) delineate several **"prescriptions"** as to how to conduct proper asset pricing analyses.
- Note, however, the paper is from 2010 and many improvements have since then been developed several of which we will see in Weeks 8 and 9.

A skeptical appraisal of asset pricing tests

Prescription 1

Expand the set of test portfolios beyond size-value portfolios of Fama and French (1993), for instance using industry portfolios, statistical portfolios, additional asset classes (bonds, currencies, etc.) or simply use individual stocks.

- In many, many applications the only set of assets used in testing a given asset pricing model or whether a given factor is priced is the 25 size-value portfolios of Fama and French (1993)
- However, their cross-sectional variation exhibit a very strong factor structure by **by construction**, well explained by a few factors (SMB and HML)
- It is, as such, not very challenging to find any other factor that explains the cross-sectional variation in those assets.
- …a related solution is to add SMB and HML (or ME and BM characteristics) to your model and test whether they drive out your new factor(s)/model

A skeptical appraisal of asset pricing tests

Prescription 2

Take the magnitude, sign, and significance of the cross-sectional coefficients seriously.

- If a model implies $\gamma_0 = 0$ (like the CAPM), **make sure that you test** this and comment on the results, even though it provides a high R^2.
- If a model implies that γ is equal to **average factor excess returns** (for traded factors), make sure you evaluate this. Otherwise, it is **indication of model misspecification**

A skeptical appraisal of asset pricing tests

Prescription 3

Report confidence intervals for the cross-sectional R^2.

- While we will not deal with asymptotic distributions of R^2 in this course, the prescription is still important for the interpretation of R^2.
- …that is, R^2 is also an **estimated** metric and should be considered as such
- It typically has very wide confidence bands, representing this kind of uncertainty on the pricing ability, see e.g. Kan et al. (2013)

Other good habits

- Make a **sufficient amount of robustness checks** using, e.g., other sample periods or test assets and generally challenge the subjective choices you have made in the analysis.

A (repeated) message

…most importantly, **always** have a strong economic motivation for why the models works/makes sense.

Potential projects

About potential projects

- Cross-sectional asset pricing deals with questions like:
  1. What explains the cross-sectional variation in expected returns?
  2. Which risk factors matter? What are their (required) compensation in the financial market?
  3. Is a certain risk factor priced in the financial markets?
- Of course, all analyses have a certain focus, for instance the paper by Menkhoff et al. (2012) that posed the question as to whether FX volatility risk explained the cross-sectional variation in average return on carry trades.
- Or Fama and French (1993) that addressed the puzzle that CAPM fails by proposing two new factors.

About potential projects

- Test the cross-sectional asset pricing abilities of a certain risk factor or model, preferably a *new* risk factor.
- Test an existing model on *new* asset classes, like currencies, bonds, cryptocurrencies, commodities etc.
- Re-evaluate an existing and important, yet likely outdated model or risk factor proposed in the literature with new, up-to-date data, more subsamples, etc.
- Test a conditional model via scaled factors, like Lettau and Ludvigson (2001). For instance, recession attention from Bybee et al. (2019), EPU Baker et al. (2016), climate policy uncertainty Gavriilidis (2021), etc...
- …many of these ideas can be merged with those coming from the next topic

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