Final Examination

FE621 Sections A and WS

from Wednesday May 12, 2021 to Sunday May 16, 2021

Name:

- This is an online examination. Some problems require use of mathematical derivations, some require use of a computer program.
- For the written parts please scan or take pictures of the derivations and upload them.
- For the computer parts please upload the source files used. Also detail your results and comment in a separate report or the main report. The problems you are supposed to use a computer for are accompanied by text in italics.
- There are 2 problems worth a total of 100 points. Please double check you solved all problems.
- Be very specific with your definitions and derivations. Showcase your work.
- Communication with other students either physical or virtual is strictly forbidden.

Problem 1. The payoff of an arithmetic Asian call option is:

$$\left(\frac{1}{N+1}\sum_{i=0}^{N} S_{t_i} - K\right)_+ .$$

Its value may be computed using straight Monte Carlo simulations. However, in order to obtain a small standard error, the number of simulations must be very high. To solve this computationally expensive problem, we will use the payoff of a geometric Asian call option as the control variate:

$$\left(\left(\prod_{i=0}^{N} S_{t_i}\right)^{\frac{1}{N+1}} - K\right)_+$$

The idea is to use the known analytic price of the geometric Asian option and the distance between MC simulations to obtain an approximate value for the arithmetic Asian option.

In this problem we consider $r = 3\%,\ \sigma = 0.3,\ S_0 = 100$, and assume the goal is to price an arithmetic Asian call option with strike $K = 100$ and maturity $T = 5$.

We assume the asset follows the standard log-normal/geometric Brownian motion model:

$$S(\Delta t) = S(0)e^{((\mu-\frac{\sigma^2}{2})\Delta t+(\sigma\sqrt{\Delta t})\epsilon)}$$

(a) The price of a geometric Asian option in the Black-Scholes model is given by:

$$P_g = e^{-rT}\left(S_0 e^{\rho T} N(d_1) - K N(d_2)\right)$$

where:

$$\rho = \frac{1}{2}\left(r - \frac{1}{2}\sigma^2 + \hat{\sigma}^2\right)$$

$$\hat{\sigma} := \sigma\sqrt{\frac{2N+1}{6(N+1)}}$$

such that $\hat{\sigma}$ is adjusted sigma and N is the total number of trading days $(T * 252)$.

$$d_1 := \frac{1}{\sqrt{T}\hat{\sigma}}\left(\ln\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\hat{\sigma}^2\right)T\right)$$

$$d_2 := \frac{1}{\sqrt{T}\hat{\sigma}}\left(\ln\left(\frac{S_0}{K}\right) + \left(\rho - \frac{1}{2}\hat{\sigma}^2\right)T\right)$$

Use the above formula to price this geometric Asian call option.

(b) Implement a Monte Carlo scheme to price an arithmetic Asian call option (P_a^{sim}). Use $M = 1,000,000$ simulations. Record the answer, a 95% confidence interval and the time it takes to obtain the result.

(c) Implement a Monte Carlo scheme to price a geometric Asian Call option (P_g^{sim}).

(d) Using $M = 10,000$ simulations and the same exact random variables create:

- numbers X_i which represent the price of the geometric Asian Option price for path i
- numbers Y_i which represent the price of the arithmetic Asian Option price for path i

Finally calculate b^* such that:

$$b^* = \frac{\sum_{i=1}^{M}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{M}(X_i - \overline{X})^2}$$

Note that b^* is actually the slope of a regression line $Y = a + bX + \varepsilon$. Please also record the price of the arithmetic P_a^{sim} and the geometric P_g^{sim}.

(e) Calculate the error of pricing for the geometric Asian option:

$$E_g = P_g^{sim} - P_g$$

(f) Calculate the modified arithmetic Asian option price (P_a^*) as:

$$P_a^* = P_a^{sim} - b^* E_g$$

Compare with the results in (b). Comment. Vary the value of M in part (d). What do you observe.

Problem 2. We consider the following stochastic volatility model, i.e. Heston model:

$$
\begin{aligned}
dS_t &= (r-q)S_t dt + \sqrt{V_t} S_t dW_t^{(1)}, \\
dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^{(2)},
\end{aligned} \tag{1}
$$

where $E[dW_t^{(1)} dW_t^{(2)}] = \rho dt$ with $-1 \leq \rho \leq 1$.

Consider a derivative security whose value is given by $U(S, v, t)$, where S represents the spot stock price, v denotes the volatility and the maturity is t. The corresponding PDE is given by

$$
\begin{aligned}
\frac{\partial U}{\partial t} =& \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} \\
&- rU + (r-q)S\frac{\partial U}{\partial S} + \kappa(\theta - v)\frac{\partial U}{\partial v}
\end{aligned} \tag{2}
$$

(a) Consider uniform grids, and assume that the maximum values of (S, v, t) as $(S_{\max}, v_{\max}, t_{\max})$, and the minimum values are $(0, 0, 0)$. Assume that you discretize the stock price range into N_S equal intervals, the volatility range into N_V equal intervals, and the maturity range into N_T equal intervals. Write down the explicit finite difference approximation of the derivatives involved in the PDE (2) using central differences as much as possible.

(b) Plug in the above finite difference expressions into the PDE (2) and simplify it. Write down the final equation.

(c) Consider an European call option with strike K, and write down the boundary condition at maturity, the boundary condition for $S = S_{\min}$, the boundary condition for $S = S_{\max}$, the boundary condition for $v = v_{\max}$, the boundary condition for $v = v_{\min}$

(d) Given the following model parameters: $S_0 = 100$, $K = 100$, $T = 1$, $\kappa = 6.21$, $\theta = 0.36$ $\sigma = 0.2$, $V_0 = 0.3$, $\rho = -0.7$. Compute the European call option price under the Heston model using the above explicit finite difference scheme.

(e) Using the above parameters, use Monte Carlo simulation to price an European call option. Use $2,000,000$ simulations and at least 100 discretization time intervals. Compare the price you obtain with the result in part (d). What do you find?