Multipole in FLASH 3
Internal report – Draft

ZL, PMR

1.   Intro

In the presence of gravity a simulation has to calculate gravitational acceleration on any mass element:

$$\mathbf{g} = -\nabla\phi(\mathbf{x}) \ . \tag{1}$$

For the self gravitating system, gravitational potential can be calculated through Poisson equation if the mass distribution is known:

$$\nabla^2\phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}) \tag{2}$$

Multipole method takes advantage of the fact that gravitational potential is expressible as Laplace series (generalized Fourier series), and can be written in integral form as:

$$
\begin{aligned}
\phi(\mathbf{x}) = \ & - \ \alpha\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\left[r^l\int_{r<r'}\frac{\rho(\mathbf{x})Y_{lm}^*(\theta',\phi')}{r'^{l+1}}d^3\mathbf{x}'\right. \\
& + \ \left.\frac{1}{r^{l+1}}\int_{r>r'}\rho(\mathbf{x}')Y_{lm}^*(\theta',\phi')r'^l d^3\mathbf{x}'\right] \ ,
\end{aligned}
\tag{3}
$$

where $Y_{lm}(\theta,\phi)$ are the spherical harmonics:

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\ P_{lm}(\cos\theta)e^{im\phi} \ , \tag{4}$$

and $P_{lm}(\cos\theta)$ are Legendre polynomials. Therefore, linearity of laplacian operator is used, and solution is searched as the sum of monopole moment ($l = 0$), dipole ($l = 1$), quadrupole ($l = 2$), sextupole ($l = 3$), and so on... However, since gravity sources have only one flavour (sign) – in contrast to electrostatic, all odd-l moments are zero here.

2.   Maclaurin Spheroid

As a test case, here is used oblate ($a_1 = a_2 > a_3$) Maclaurin spheroid, of a constant density $\rho = 1$ in the interoior, and $\rho = \epsilon \to 0$ outside (in FLASH3 small_rho is used). Spheroid is set motionless, and in a hydrostatic equilibrium. The gravitational potential of such object is analyticaly calculable, and is:

$$\phi(\mathbf{x}) = \pi G\rho\left[2A_1 a_1^2 - A_1(x^2 + y^2) + A_3(a_3^2 - z^2)\right] \ , \tag{5}$$

Density

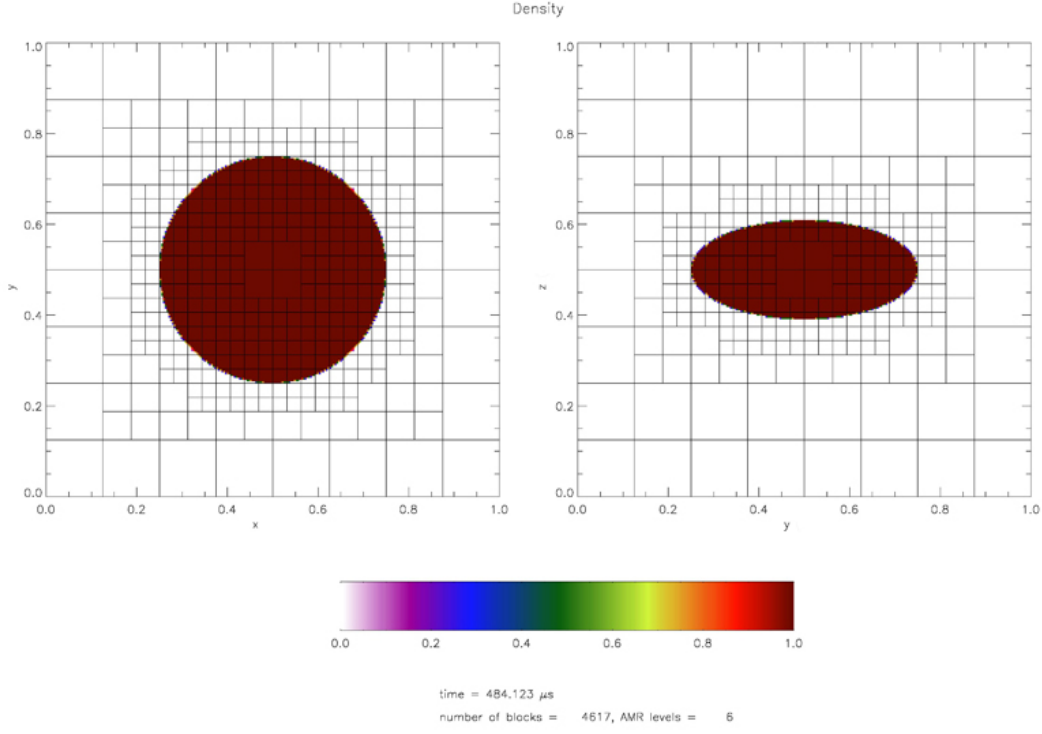time = 484.123 μs

number of blocks =   4617, AMR levels =   6

Fig. 1.— Density of Maclaurin spheroid (left: X–Y plane, right: Y–Z plane) with ellipticity $e = 0.9$. FLASH block structure is shown on top.

for a point inside the spheroid. Here

$$A_1 = A_2 \quad = \quad \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2} \ , \tag{6}$$

$$A_3 \quad = \quad \frac{2}{e^2} - \frac{2\sqrt{1 - e^2}}{e^3} \sin^{-1} e \ , \tag{7}$$

where e is the ellipticity of a spheroid:

$$e = \sqrt{1 - \left(\frac{a_3}{a_1}\right)^2} \ . \tag{8}$$

For a point outside the spheroid, potential is:

$$\phi(\mathbf{x}) = \frac{2a_3}{e^2} \pi G \rho \left[a_1 e \tan^{-1} h - \frac{1}{2} \left((x^2 + y^2) \left(\tan^{-1} h - \frac{h}{1 + h^2} \right) + 2z^2 (h - \tan^{-1} h) \right) \right] \ , \tag{9}$$

where

$$h = \frac{a_1 e}{\sqrt{a_3^2 + \lambda}} \ , \tag{10}$$

l_{max}	$\min(\epsilon)$	$\max(\epsilon)$	approx. time $[s]$
0	4.510^{-6}	0.154	9.8
1	4.510^{-6}	0.154	14.5
2	9.810^{-6}	0.066	34.7
4	1.010^{-8}	0.026	55.4
10	6.710^{-9}	0.004	609.7

Table 1: Minimal and maximal relative error in all zones of the simulation, calculated using equation 12. Last row is approximate time for one full timestep (gravity only).

and λ is the positive root of the equation

$$\frac{x^2}{a_1^2 + \lambda} + \frac{y^2}{a_2^2 + \lambda} + \frac{z^2}{a_3^2 + \lambda} = 1 \; . \tag{11}$$

Besides for knowing analytical solution, usefullness of the test is in having spherical symmetry in X–Y plane, which is the easiest geometry when the solution is looked for in the form of multipole expansion, but also lack of such symmetry in X–Z and Y–Z planes. The solution in those planes has infinite number of multipole moments, while the code calculates solution up to a certain l_{max}, specified by the user. The error is thus expected to be dominated by the first non-zero term in the remainder of expansion.

3. Discussion

Tests are done on a Macaurin spheroid with eccentricity $e = 0.9$, several other values for eccentricity were tried, and the results were all qualitatively the same. All tests were in 3D Cartesian coordinates. The code calculates potential on an adaptive mesh, and using formulas from previous section one can calculate analytical solution for the same block/zone structure, and look at relative error:

$$\epsilon = \left| \frac{\phi_{\text{analytical}} - \phi_{\text{FLASH}}}{\phi_{\text{analytical}}} \right| \tag{12}$$

from zone to zone.

Of course, increasing spatial (force) resolution improves quality of the solution, and that was converging well. Here, I'm showing how solution depends on the choice of l_{max}, the cutoff l in eqn. 3. On figures 2-6, shown are gravitational potential for Maclaurin spheroid, FLASH solution and relative errors for several l_{max}'s. If we compare figures 2 and 3, where the last terms in multipole expansion are monopole and dipole, we see no difference whatsoever, as it should be. Similarly, result of $l_{max} = 3$ is indistinguishable from $l_{max} = 2$, and so on. Thus, the reasonable, physically motivated values for maximum multipole should be even number. Specifying odd number is not improving the solution, but increases calculation time as can be seen in table 1.

In X–Y plane, where the solution is radially symmetric, monopole term is enough to qulitatively capture the right potential. Further moments are just reducing errors. As expected, the error is the biggest on boundary, and decreases both outwards and inwards. However, in other planes, where solution does not have such symmetry, truncating potential to certain l_{max} leads to an error whose leading term will be the spherical harmonic of order $l_{max} + 2$, and that can be nicely seen in lower right sections of figures 2-6. Increasing l_{max} reduces the error, but also increases the required time for computation, and that increase is not linear because of the double sum in equation 3. Luckily, convergence is rather fast, and already for $l_{max} = 4$, there are only few zones with relative error bigger than 1%, while for the most of the computational domain the error is several orders of magnitude less.
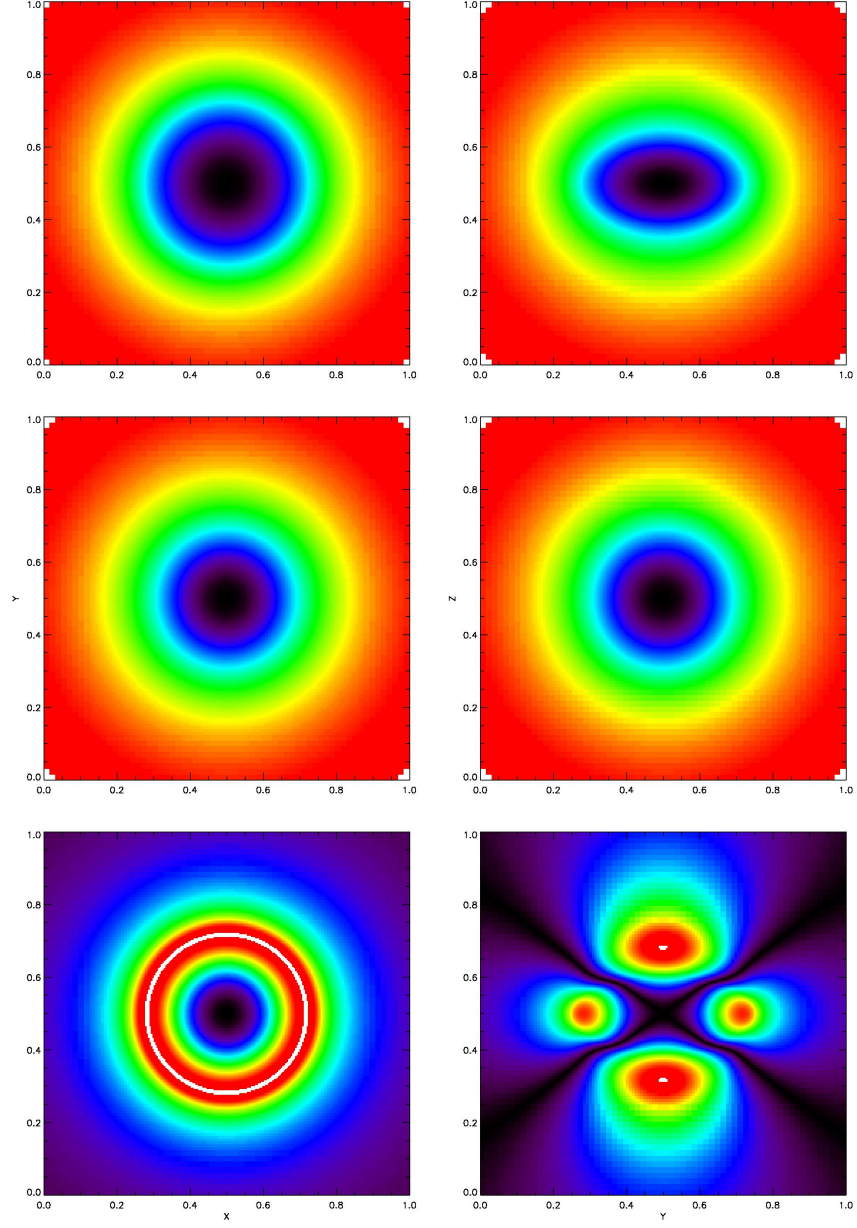
Fig. 2.— Maclaurin spheroid: $l_{max} = 0$, 6 refinement levels. Left coulumn is X–Y plane, cut through z=0.5, right column is Y–Z plane cut through x=0.5 . From top to bottom: analytical solution for the gravitational potential introduced on FLASH grid; solution of FLASH multipole solver; relative error.
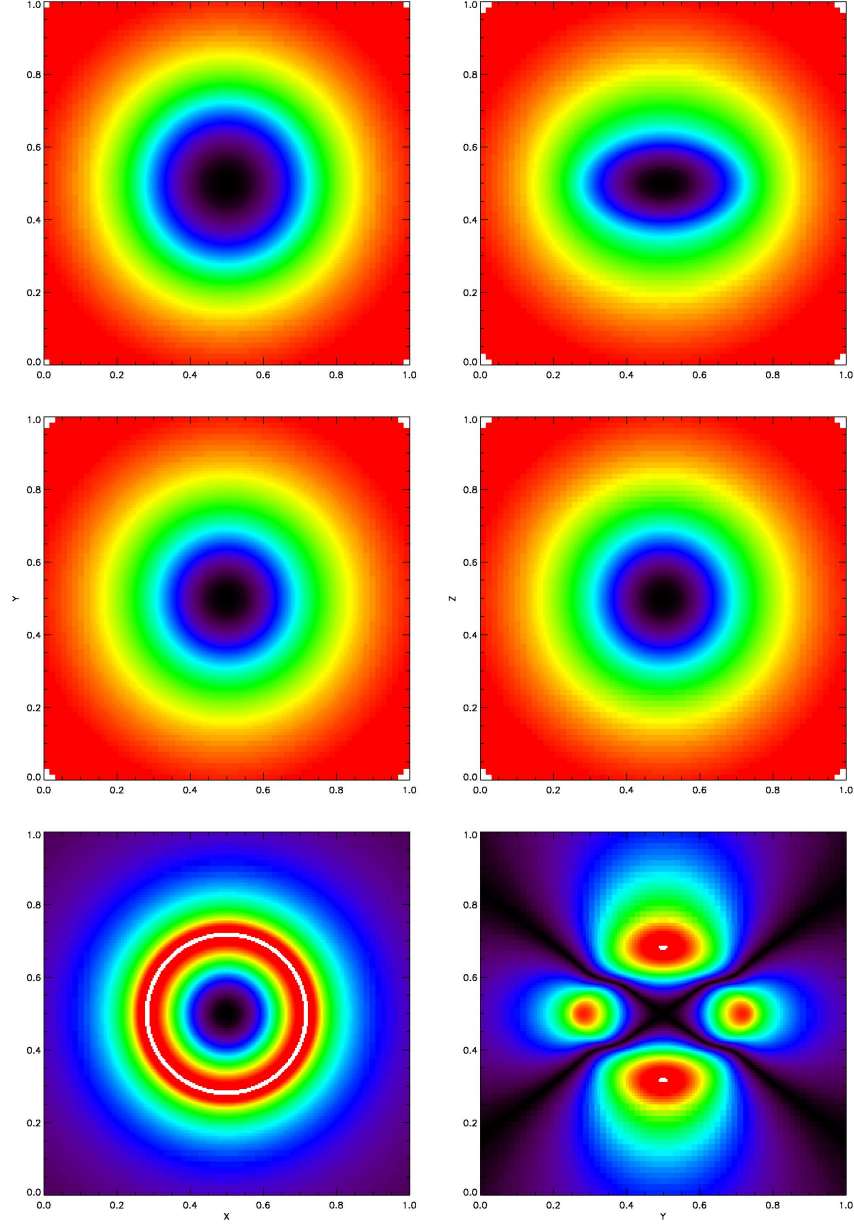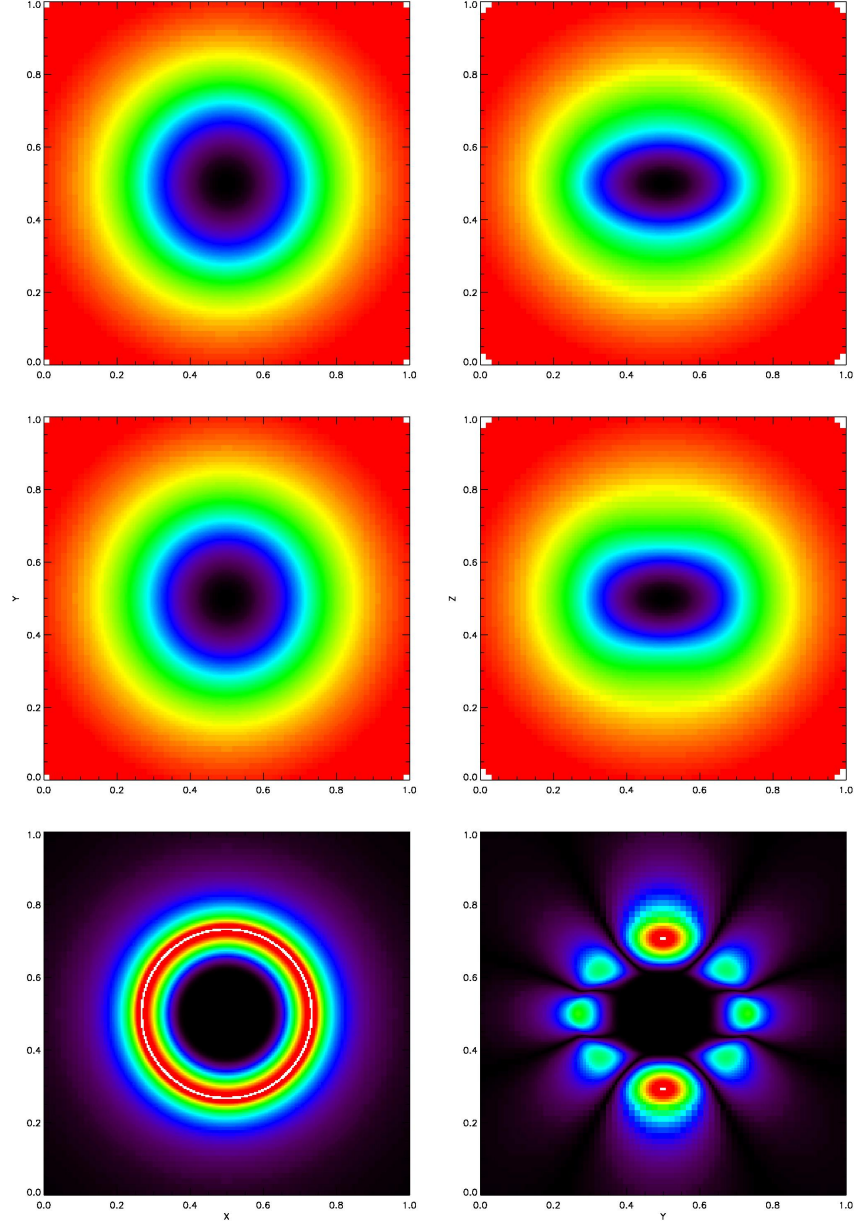
Fig. 3.— Maclaurin spheroid: $l_{max} = 1$, 6 refinement levels. Left coulumn is X–Y plane, cut through z=0.5, right column is Y–Z plane cut through x=0.5 . From top to bottom: analytical solution for the gravitational potential introduced on FLASH grid; solution of FLASH multipole solver; relative error.

Fig. 4.— Maclaurin spheroid: $l_{max} = 2$, 6 refinement levels. Left coulumn is X–Y plane, cut through z=0.5, right column is Y–Z plane cut through x=0.5 . From top to bottom: analytical solution for the gravitational potential introduced on FLASH grid; solution of FLASH multipole solver; relative error.
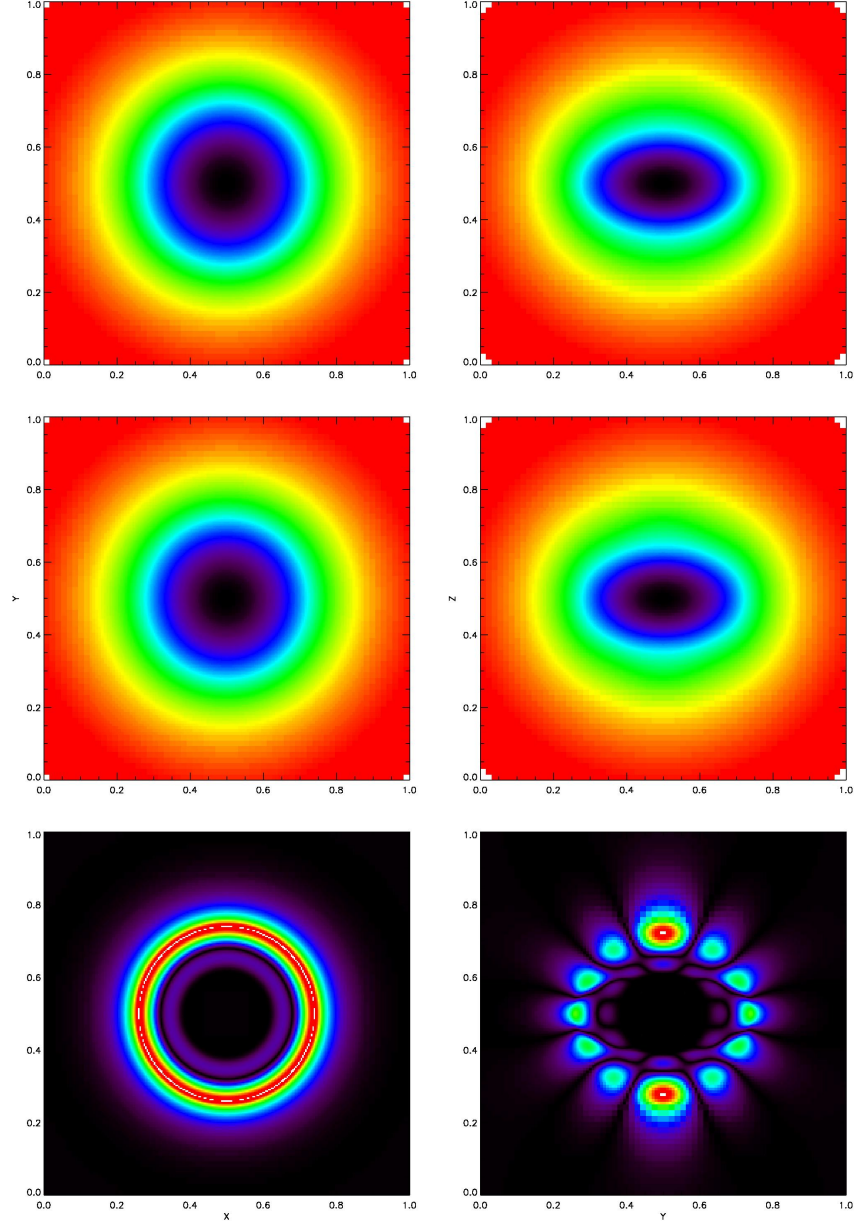
Fig. 5.— Maclaurin spheroid: $l_{max} = 4$, 6 refinement levels. Left coulumn is X–Y plane, cut through z=0.5, right column is Y–Z plane cut through x=0.5 . From top to bottom: analytical solution for the gravitational potential introduced on FLASH grid; solution of FLASH multipole solver; relative error.
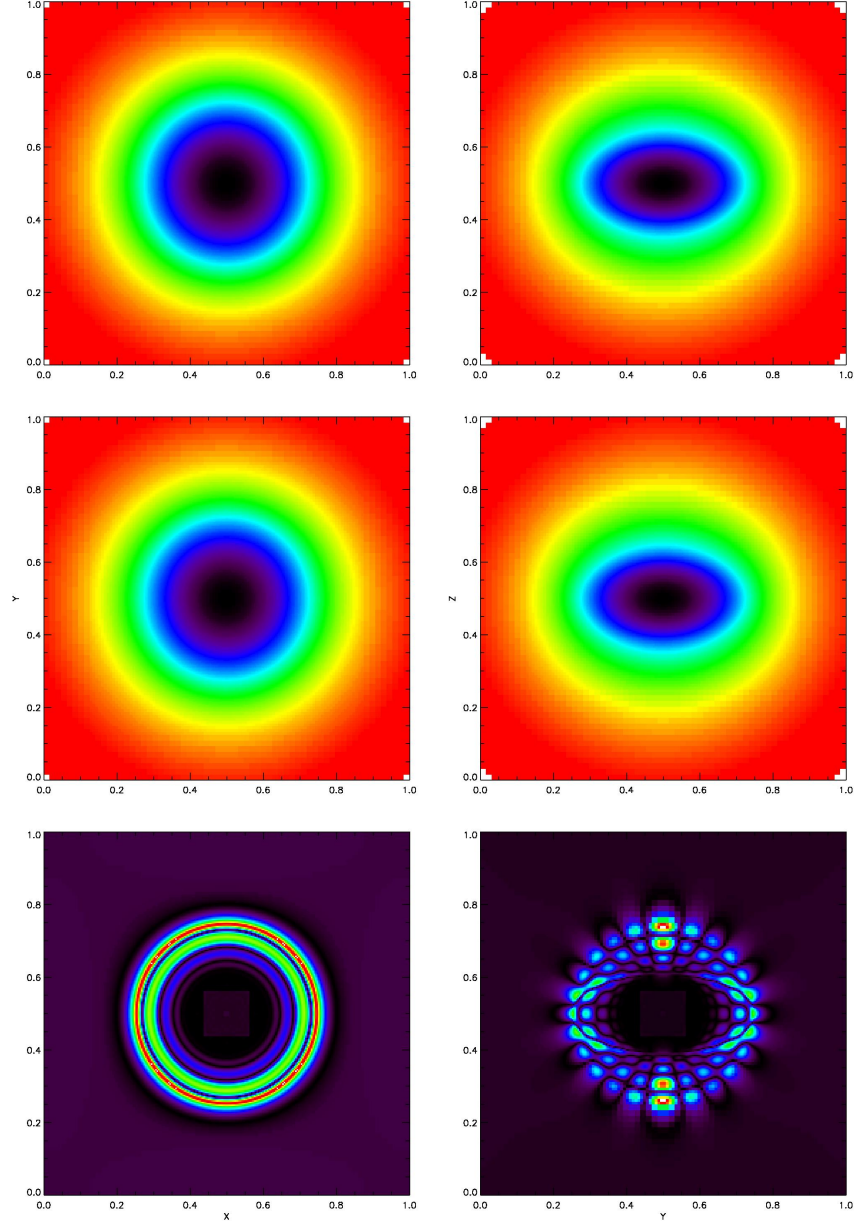
Fig. 6.— Maclaurin spheroid: $l_{max} = 10$, 6 refinement levels. Left coulumn is X–Y plane, cut through z=0.5, right column is Y–Z plane cut through x=0.5 . From top to bottom: analytical solution for the gravitational potential introduced on FLASH grid; solution of FLASH multipole solver; relative error.