The Structure, Stability, and Dynamics of Self-Gravitating Systems

Maclaurin Spheroids

As was pointed out earlier, there is no particular reason why one should guess ahead of time that the equilibrium properties of *any* rotating, self-gravitating configuration should be describable in terms of analytical functions. As luck would have it, however, the gravitational potential at the surface of and inside an homogeneous spheroid is expressible in terms of analytical functions. (The potential is constant on concentric spheroidal surfaces that generally have a different axis ratio from the spheroidal mass distribution.) Furthermore, the gradient of the gravitational potential is separable in cylindrical coordinates, proving to be a simple *linear* function of both ϖ and z.

If the spheroid is uniformly rotating, this behavior conspires nicely with the behavior of the centrifugal acceleration -- which also will be a linear function of ϖ (see, in particular, Eq. [I.A.20]) -- to permit an analytical (and integrable) prescription of the pressure gradient. Not surprisingly, it resembles the functional form of the pressure gradient that is required to balance the gravitational force in uniform-density spheres.

As a consequence of this good fortune, the equilibrium structure of a uniformly rotating, uniform-density (n = 0), axisymmetric configuration can be shown to be precisely an oblate spheroid whose internal properties are describable in terms of analytical expressions. These expressions were first derived by Colin Maclaurin (1742) in *A Treatise on Fluxions*, and have been enumerated in many subsequent publications (*cf.* Tassoul 1978; Chandrasekhar 1987).

Boundary (Surface) Definition

Let a_1 be the equatorial radius and a_3 the polar radius of a uniform-density object whose surface is defined precisely by an oblate spheroid. The degree of flattening of the object is conveniently parameterized in terms of the object's eccentricity,

$$e \equiv [1 - (a_3/a_1)^2]^{1/2}.$$

[Equation III.G.1]
=
EFE, Chapter 3, Eq. (37) and Chapter 5, Eq. (5)
Tassoul '78, § 4.5, Eq. (50)

(For an oblate spheroid, $a_3 \leq a_1$; hence, the eccentricity is restricted to the range $0 \leq e \leq 1$.) Of course, the meridional cross-section of such a spheroid is an ellipse with the same eccentricity. The foci of this ellipse lie in the equatorial plane of the spheroid at a distance $\varpi = ea_1$ from the minor (z) axis.

The total mass of such a spheroid is,

$$M = (4/3) \pi a_1^2 a_3 \rho = (4/3) \pi \rho a_1^3 (1-e^2)^{1/2}.$$

[Equation III.G.2]

For the purposes of normalization, we will find it useful to define,

$$a_{mean} \equiv (a_1^2 a_3)^{1/3} = a_1 (1-e^2)^{1/6},$$

[Equation III.G.3]

which is the radius of a spherical, n = 0 polytrope that has the same density and mass as the spheroid.

The Gravitational Potential

In an accompanying discussion entitled, "Properties of Homogeneous Ellipsoids," an expression [III.R.1] is given for the gravitational potential $\Phi(\mathbf{x})$ at an internal point or on the surface of an homogeneous ellipsoid with semi-axes $(x,y,z) = (a_1,a_2,a_3)$. For an homogeneous, oblate spheroid in which $a_1 = a_2 > a_3$, this analytical expression defining the potential reduces to the form,

$$\Phi(\varpi,z) = - \pi\, G\rho\, \{\, I_{BT}a_1{}^2 - [\, A_1\varpi^2 + A_3z^2\,]\, \},$$

[Equation III.G.4]

where, as defined on the accompanying page, the coefficients A_1, A_3, and I_{BT} are functions only of the spheroid's eccentricity.

Q

According to the accompanying introductory discussion, for axisymmetric configurations the Poisson equation takes the following form in cylindrical coordinates:

$$(1/\varpi)\partial_\varpi(\varpi\, \partial_\varpi\Phi) + \partial_z(\partial_z\Phi) = 4\pi G\rho\, .$$

[Equation III.F.13]

Show that the above analytical expression [III.G.4] for the gravitational potential of a uniform-density spheroid satisfies this form of the Poisson equation.

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From the above expression [III.G.4] for the gravitational potential, we can deduce immediately that the gradient of the gravitational potential at any point on the surface or inside a uniform-density spheroid is:

$$\nabla\Phi = \mathbf{e}_\varpi\, (2\pi\, G\rho A_1)\, \varpi + \mathbf{e}_z\, (2\pi\, G\rho A_3)\, z.$$

[Equations III.G.5]
=
Tassoul '78, § 4.5, Eq. (47)

Q

Show that if we choose to express the properties of Maclaurin spheroids in terms of the system of T1 Coordinates instead of the system of cylindrical coordinates, and define the T1 Coordinate parameter q such that,

$$q^2 \equiv A_3/A_1,$$

[Equation III.G.6]

the gradient in the gravitational potential takes the relatively simple form:

$$\nabla\Phi = \mathbf{e}_1 \, (1/h_1) \, (2\pi \, G\rho A_3) \, \xi_1.$$

[Equation III.G.7]

In other words,

$$\partial_{\xi_1} \Phi = 2\pi \, G\rho A_3 \, \xi_1,$$

[Equation III.G.8]

and,

$$\partial_{\xi_2} \Phi = 0.$$

[Equation III.G.9]

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The Equilibrium Pressure and Angular Velocity

In addition to deriving a gravitational potential that satisfies the Poisson equation for our chosen density distribution, we also must determine a pressure distribution $P(\varpi,z)$ and a (constant) angular velocity of rotation ω that satisfies the conditions derived earlier for hydrostatic equilibrium in uniformly rotating, axisymmetric configurations, namely:

By following a procedure analogous to the one that is used in numerical self-consistent-field algorithms, derive analytical expressions for the equilibrium angular velocity ω and equilibrium pressure distribution inside a Maclaurin spheroid. Specifically,

- Choose two points on the surface of the oblate spheroid where H = 0 [for example, $(\varpi,z) = (0,a_3)$ and $(\varpi,z) = (a_1,0)$];
- Determine the value of Φ at these two points via the analytical expression for the gravitational potential [III.G.4];
- Determine the values of the two constants C_o and ω in the algebraic equation that defines the equilibrium structure [III.F.24'];
- Plug the expressions for Φ, C_o, and ω back into the equilibrium structure equation [III.F.24'], and combine terms to derive a general expression for $P = H\rho$ throughout the spheroid.

Via this procedure, you should derive the following expressions for Maclaurin spheroids of arbitrary flattening:

$$\omega^2 = 2\pi\, G\, \rho\, [A_1 - (1-e^2)A_3],$$

[Equation III.G.10]

and,

$$P(\varpi,z) = P_0\, (3A_3/2)\, (1 - e^2)^{2/3}\, [1 - (\varpi/a_1)^2 - (z/a_3)^2\,],$$

[Equation III.G.11]

where,

$$P_0 \equiv (2/3)\,\pi\, G\rho^2\, (a_{mean})^2.$$

[Equation III.G.12]
=
Tassoul '78, § 4.5, Eq. (51)[1]

Demonstrate by simple substitution that these expressions for ω and $P(\varpi,z)$ [III.G.10 & III.G.11] combine to effectively balance $\nabla\Phi$ throughout the configuration, according to the above coupled force-balance equations [III.F.14 & III.F.15].

By setting a_{mean} equal to the radius of a spherical, $n = 0$ polytrope as derived elsewhere [III.B.7] (*i.e.,* $a_{mean} = \sqrt{6}\, a_{n\,=\,0}$), show that P_0 is precisely the central pressure of an equilibrium, uniform-density sphere. Show, furthermore, that the derived prescription [III.G.11] for $P(\varpi, z)$ inside a Maclaurin spheroid properly reduces to the equation [III.B.4] derived earlier for the pressure throughout a spherically symmetric, $n = 0$ polytrope when one sets $a_1 = a_3 = \sqrt{6}\, a_{n\,=\,0}$.

Notice that in terms of the dimensionless angular velocity

$$\Omega^2 \equiv \omega^2 / (\pi\, G\, \rho) = 2\, [A_1 - (1-e^2)A_3]\ ,$$

[Equation III.G.13]
=
EFE, Chapter 5, Eqs. (4) & (6)
Tassoul '78, § 4.5, Eq. (52) and § 10.2, Eq. (12)

there is a unique relationship between the star's eccentricity and its equilibrium angular velocity. In the accompanying figure taken directly from EFE, $\omega^2/\pi G\rho$ is plotted as a function of e. A figure taken from Tassoul (1978) illustrates how $\omega^2/2\pi G\rho$ varies as a function of τ_{rot}, the ratio of rotational to gravitational potential energy (derived below). From either of these two figures, it is clear that Ω does not increase without limit. Instead, it reaches a maximum $\Omega = \Omega_{max} \approx 0.45$ at a value of $e \approx 0.9$ and $\tau_{rot} \approx 0.25$. (One application below provides a means by which equation [III.G.13] can be evaluated numerically for any choice of equilibrium eccentricity, and the following homework exercise provides a precise determination of Ω_{max}.)

As the figure from EFE illustrates, Ω does not increase without limit but, instead, reaches a maximum, $\Omega = \Omega_{max}$, at a value of $e \approx 0.9$. At what precise value of the eccentricity e_{max} (and at what corresponding maximum τ_{rot}) does $\Omega = \Omega_{max}$? What is the precise value of Ω_{max}?

Allowed Equilibrium Eccentricities

To date, no one has inverted equation [III.G.13] in closed form to derive an analytical expression for e as a function of Ω, but $e(\Omega)$ can be determined numerically. Obviously from the accompanying figure, it is not possible to construct an equilibrium star with $\Omega^2 > \Omega^2_{max}$, and **at** Ω_{max} there is a single value of the eccentricity, e_{max}, for which an equilibrium exists. But, as was first noticed by Simpson (1743), at any value of Ω selected in the range $0 \le \Omega < \Omega_{max}$, two possible equilibrium figures of different eccentricities can be found. (One application below provides a means by which equation [III.G.13] can be evaluated numerically to determine what pair of equilibrium eccentricities are permitted for any choice of angular velocity $\Omega < \Omega_{max}$, and the preceeding homework exercise provides a precise determination of e_{max}.)

Other Properties

Drawing on the general definition of the moment of inertia tensor of a uniform-density ellipsoid [III.R.6], we know that the moment of inertia of a uniform-density, oblate spheroid about its symmetry axis is,

$$I = I_{11} + I_{22} = (2/5) \, M \, a_1^2.$$

[Equation III.G.14]

For purposes of normalization, define:

$$J_{norm} \equiv (G \, M^3 \, a_{mean})^{1/2},$$

[Equation III.G.15]

which, dimensionally, has units of angular momentum; and

$$E_{norm} \equiv G \, M^2 \, a_{mean}^{\;2}/a_1^{\;3},$$

[Equation III.G.16]

which has units of energy. Then the total angular momentum of a Maclaurin spheroid is,

$$J / J_{norm} = I\omega/J_{norm} = (6^{1/2}/5) \, (1-e^2)^{-1/3} \, [A_1 - (1-e^2)A_3]^{1/2};$$

[Equation III.G.17]

its rotational kinetic energy is,

$$E_{rot}/E_{norm} = (1/2)I\omega^2/E_{norm} = (3/10) \, \Omega^2(1 - e^2)^{-1/3};$$

[Equation III.G.18]

and its gravitational potential energy is,

$$E_{grav}/E_{norm} = - (3/5) \, (1 - e^2)^{-1/3}[2A_1 + (1 - e^2)A_3].$$

[Equation III.G.19]

Hence, the ratio of rotational to gravitational potential energy in a Maclaurin spheroid is,

$$\tau_{rot} \equiv E_{rot} / |E_{grav}| = [A_1 - (1-e^2)A_3] / [2A_1 + (1-e^2)A_3].$$

[Equation III.G.20]

In many of the classical discussions of ellipsoidal figures of equilibrium, the behavior of the function J(e) also usually is illustrated

APPLICATIONS		
III.G.1	Table III.F.1 illustrates how various physical parameters vary with e and τ_{rot} along the Maclaurin sequence. Through the accompanying application, numerical values of the same set of physical parameters can be calculated for any choice of e or any selected range of eccentricities.	
III.G.2	If you would like to determine the pair of eccentricities at which an equilibrium configuration can be constructed for a given choice of Ω (or for a range of angular velocities), select Application 2. [NOT YET AVAILABLE]	

Footnotes

[1]Utilizing the definitions of P_0 and a_{mean}, it is clear that the leading coefficient in the pressure function $P(\varpi,z)$ (*i.e.*, the pressure at the center of the spheroid) can be written in the form,

$$P_0 \, (3A_3/2) \, (1 - e^2)^{2/3} = \pi G \rho^2 \, a_3^2 \, A_3.$$

[Equation III.G.12']

Tassoul (1978) incorrectly states that this coefficient is $2\pi G \rho^2 \, a_3^2 \, A_3$; too large by a factor of 2.

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