第八章连通度和匹配

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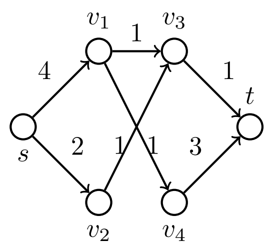
Maximum Flow:



Let $D = (V, A)$ be a flow network with a capacity function $c : A \to R^+$. Let s be the source of the network, and let t be the sink. We further assume that no edge enters the source s and no edge leaves the sink t.

A flow in D is a real-valued function $f : A \to R^+ \cup \{0\}$ that satisfies the following two properties:

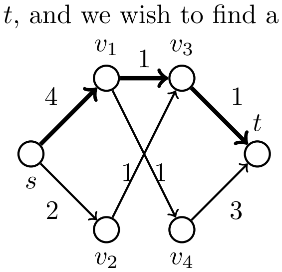
Capacity constraint: For all $(u, v) \in A$, $0 \leq f(u, v) \leq c(u, v)$.(Note that $f((u, v))$ and c((u,v)) are abbreviated as $f(u, v)$ and $c(u, v)$.)

Flow conservation: For all $u \in V \setminus \{s, t\}$, we require $\sum_{(v,u)\in A} f(v, u) = \sum_{(u,v)\in A} f(u, v)$.

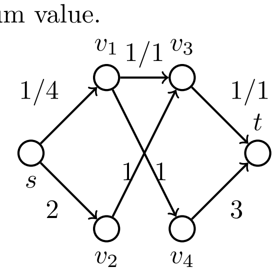
The value $|f|$ of a flow f is defined as

$$|f| = \sum_{(s,v)\in A} f(s, v)$$

In the maximum-flow problem, we are given a flow network D with source s and sink t, and we wish to find a flow of maximum value.



(a)



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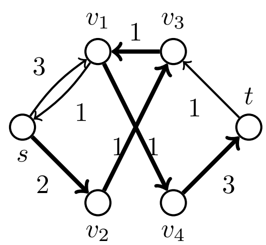
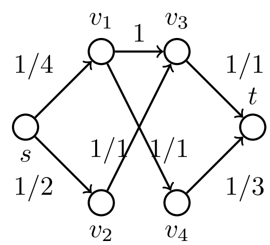
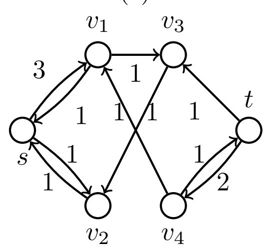


(c)



(d)



(e)

Given a flow network D, and a flow f on D, we define the residual graph $D_f = (V, A_f)$ of D with respect to f as follows.

- The node set of D_f is the same as that of D.
- For each edge (u, v) of D on which $f(u, v) < c(u, v)$, there are $c(u, v) - f(u, v)$ "leftover" units of capacity on which we could try pushing flow forward. So we include the edge (u, v) in D_f, with a capacity $c_f(u, v) = c(u, v) - f(u, v)$.
- For each edge (u, v) of D on which $f(u, v) > 0$, there are $f(u, v)$ units of flow that we can "undo" if we want to, by pushing flow backward. So we include the edge (v, u) in D_f, with a capacity $c_f(v, u) = f(u, v)$.

Given a flow network $D = (V, A)$ and a flow f, an augmenting path p is a path from s to t in the residual network D_f.

We call the maximum amount by which we can increase the flow on each edge in an augmenting path p the residual capacity of p, given by

$$c_f(p) = \min\{c_f(u, v) | (u, v) \text{is on p}\}$$

A cut (S, T) of flow network $D = (V, A)$ is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.

If f is a flow, then the net flow $f(S, T)$ across the cut (S, T) is defined to be

$$f(S, T) = \sum_{(u,v) \in A, u \in S, v \in T} f(u, v) - \sum_{(v,u) \in A, v \in T, u \in S} f(v, u)$$

The capacity of the cut (S, T) is

$$c(S,T) = \sum_{(u,v)\in A, u\in S, v\in T} c(u,v)$$

A minimum cut of a network is a cut whose capacity is minimum over all cuts of the network.

Theorem 1. *If f is a flow in a flow network $D = (V, A)$ with source s and sink t, then the following conditions are equivalent:*

1. f is a maximum flow in D.

2. The residual network D_f contains no augmenting paths.

3. $|f| = c(S,T)$ for some cut (S,T) of D.

Proof.

1⇒ 2: Suppose for the sake of contradiction that f is a maximum flow in D but that D_f has an augmenting path p. Then, the flow found by augmenting f with the flow in p is a flow in D with value strictly greater than $|f|$, contrdicting the assumption that f is a maximum flow.

2⇒ 3: Suppose that D_f has no augmenting path, that is, that D_f contains no path from s to t. Define

$$S = \{v \in V | \text{there exists a path from } s \text{ to } v \text{ in } D_f\}$$

and $T = V \setminus S$. The partition (S,T) is a cut: we have $s \in S$ trivially and $t \notin S$ because there is no path from s to t in D_f. Now consider a pair of vertices $u \in S$ and $v \in T$. If $(u,v) \in A$, we must have $f(u,v) = c(u,v)$, since otherwise $(u,v) \in A_f$, which would place v in set S. If $(v,u) \in A$, we must have $f(v,u) = 0$, because otherwise $c_f(u,v) = f(v,u)$ would be positive and we would have $(u,v) \in A_f$, which would place v in S.

We thus have

$$\begin{aligned} f(S,T) &= \sum_{(u,v)\in A, u\in S, v\in T} f(u,v) - \sum_{(v,u)\in A, v\in T, u\in S} f(v,u) \\ &= \sum_{(u,v)\in A, u\in S, v\in T} c(u,v) \\ &= c(S,T) \end{aligned}$$

Therefore, $|f| = f(S,T) = c(S,T)$.

3⇒ 1: $|f| \le c(A,B)$ for all cuts (A,B). The conditon $|f| = c(S,T)$ thus implies that f is a maximum flow.

□

定义1. 图G的**顶点连通度**是指为了产生一个不连通图或平凡图所需要从G中去掉的最少顶点数目，记为$\kappa(G)$。

定义2. 图G的**边连通度**是指为了产生一个不连通图或平凡图所需要从G中去掉的最少边的数目，记为$\lambda(G)$。

定理1. 对任一图G，有$\kappa(G) \le \lambda(G) \le \delta(G)$。

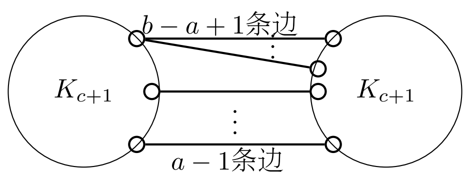
证明. 先证$\lambda(G) \le \delta(G)$。如果$\delta(G) = 0$，则G不连通或者为平凡图，此时$\lambda(G) = 0$，$\lambda(G) \le \delta(G)$成立。如果$\delta(G) > 0$，不妨设$\deg v = \delta(G)$,从G中去掉与v关联的$\delta(G)$条边之后，得到的图中v为孤立顶点，所以$\lambda(G) \le \delta(G)$。因此，对任意的图G，$\lambda(G) \le \delta(G)$。

接下来证明$\kappa(G) \le \lambda(G)$。如果G不连通或者为平凡图，则$\kappa(G) = \lambda(G) = 0$。如果$G$是连通的且有一座桥$x$，则$\lambda(G) = 1$。因为在这种情况下$G$或者有一个割点关联于$x$或者$G$为$K_2$，所以$\kappa(G) = 1$。最后假定$\lambda(G) \ge 2$，则$G$中有$\lambda(G)$条边，移去它们后所得到的图不连通。显然，移去这些边中的$\lambda(G) - 1$条边后得到一个图，它有一条桥$x = uv$。对于这$\lambda(G) - 1$条边中每一条，选取一个关联于它但与u和v都不同的顶点。移去这些顶点之后就移去了这$\lambda(G) - 1$条边。如果这样产生的图是不连通的，则$\kappa(G) < \lambda(G)$。否则，x是这样产生的图的一条桥，从而移去u或v就产生了一个不连通图或平凡图。所以，在任何情况下，$\kappa(G) \le \lambda(G)$。 □

定理2. 对任何整数a, b, c， $0 < a \le b \le c$， 存在一个图G使得

$$\kappa(G) = a, \lambda(G) = b, \delta(G) = c$$

证明.



□

定理3. 设$G = (V, E)$有p个顶点且$\delta(G) \ge [\frac{p}{2}]$，则$\lambda(G) = \delta(G)$。

（同学们先思考一下，看看自己能不能证出这个定理。）

证明. $\lambda(G) \le \delta(G)$显然成立，只需要证明$\lambda(G) \ge \delta(G)$。

因为$\delta(G) \ge [\frac{p}{2}]$，所以$G$是连通的。如果$G$为平凡图， 则$\lambda(G) = \delta(G) = 0$。如果$G$不是平凡图，则$\lambda(G) > 0$，从而存在$V$的真子集$A$使得$G$中联结$A$中的一个顶点与$V \setminus A$中的一个顶点的边恰有$\lambda(G)$条。所有这些边的集合记为$F$。

由$|A| + |V \setminus A| = p$知必有$|A| \le [\frac{p}{2}]$或者$|V \setminus A| \le [\frac{p}{2}]$。不妨设$|A| \le [\frac{p}{2}]$。由于$\delta(G) \ge [\frac{p}{2}]$，$A$中的每个顶点至少与$V \setminus A$中的一个顶点邻接。否则，如果$A$中的某个顶点$u$只与$A$中的顶点邻接，则$\deg u \le |A| - 1 \le [\frac{p}{2}] - 1 < \delta(G)$，矛盾。设$v$为$A$中的任一顶点，$v$与$V \setminus A$中的$x$个顶点邻接，与$A$中的$y$个顶点邻接，则$\deg v = x + y$。$v$与$V \setminus A$中的$x$个顶点邻接，所对应的边的集合记为$F_1$，则$F_1 \subseteq F$；$v$与$A$中的$y$个顶点邻接，而这$y$个顶点中的每个顶点都至少与$V \setminus A$中的一个顶点邻接，所对应的边的集合记为$F_2$，则$F_2 \subseteq F$ 并且$F_1 \cap F_2 = \phi$，从而

$$\lambda(G) \ge |F_1| + |F_2| = x + y = \deg v \ge \delta(G)$$

□

定义3. 设G为一个图，如果$\kappa(G) \geq n$，则称G为n**-顶点连通**的，简称n-连通;如果$\lambda(G) \geq n$，则称G为n**-边连通**的。

定理4. 设$G = (V, E)$为有p个顶点的图，$p \geq 3$，则G为2-连通的，当且仅当G的任意两个不同的顶点在G的同一个圈上。

定义4. 设u与v为图G中的两个不同的顶点。两条联结u与v的路，如果除了u与v外没有公共顶点，则称这两条路为联结u与v的**不相交路**；如果联结u与v的两条路上没有公共边，则称这两条路为联结u与v的**边不相交路**。

定义5. 图G的顶点集S称为分离G的两个不邻接的顶点u和v，如果u和v分别在$G - S$的两个不同的支中。图G的边集$F \subseteq E$称为分离G的两个不同的顶点u和v，如果u和v分别在$G - F$的两个不同的支中。

定理5. 分离图G的两个不邻接的顶点s和t的顶点最少数目等于联结s和t的不相交路的最多数目。

定理6. 图G为$n-$连通的当且仅当每一对不同顶点间至少有n条不相交路。

定理7. 分离图G的两个不同的顶点s和t的边的最少数目等于边不相交$s - t$路的最多数目。

定理8. 图G为$n-$边连通的当且仅当G的任一对不同的顶点间至少有n条边不相交路。

定义6. 设$G = (V, E)$为一个图，G的任意两条不邻接的边x与y称为**互相独立**的边。G的边集E的子集Y称为G的一个**匹配**，如果Y中任意两条边都是互相独立的。

定义7. 设Y为图$G = (V, E)$的一个匹配，如果$2|Y| = |V|$，则称Y为G的一个**完美匹配**。

定义8. 设Y为图$G = (V, E)$的一个匹配，如果对于G的任一匹配Y'，恒有$|Y'| \leq |Y|$, 则称Y为G的一个**最大匹配**。

定义9. 设$G = (V, E)$为一个偶图且$V = V_1 \cup V_2$,$V_1 \cap V_2 = \phi$，$\forall x \in E$，x为联结V_1的一个顶点与V_2的一个顶点的边。如果存在G的一个匹配Y使得$|Y| = min\{|V_1|, |V_2|\}$，则称$Y$是偶图$G$的一个**完全匹配**。

定理9. 设$G = ((V_1, V_2), E)$为偶图，存在G的一个完全匹配Y且$|Y| = |V_1|$的充分必要条件是对V_1的任意子集A，$|N(A)| \geq |A|$，其中

$$N(A) = \{y \in V_2 | \exists x \in A \{x, y\} \in E\}$$

定义10. 设X为一个有穷集合，$A_1, A_2, \ldots, A_n$为X的子集的一个序列，由X的互不相同的元素构成的集合$\{s_1, s_2, \ldots, s_n\}$称为系统

$$T : A_1, A_2, \ldots, A_n$$

的**相异代表系**，如果$s_i \in A_i$，$i = 1, 2, \ldots, n$。

定理10. 设X为一个有限集，系统$T : A_1, A_2, \cdots, A_n$为$X$的一些子集组成的，则$T$有相异代表系的充分必要条件是$\forall I \subseteq \{1, 2, \cdots, n\}$有

$$|\bigcup_{i \in I} A_i| \geq |I|$$