Chapter 14

Advanced counting techniques 高级计数技术

§ 14.1 Recurrence relations 递推关系

14.1.1 The concept of recurrence relations

Exam： $a_n=2a_{n-1}-a_{n-2}$, $a_0=0, a_1=2$;

$\{a_n\}$:0,2,4,6,8,… $a_n=2n$

$2(3n-3)-(3n-6)=3n=a_n$

Exam： $a_n=a_{n-1}+2a_{n-2}$, $a_0=2, a_1=7$; $\{a_n\}$

Exam： $a_n=6a_{n-1}+9a_{n-2}$, $a_0=1, a_1=6$; $\{a_n\}$

§ 14.1 Recurrence relations

14.1.2 Modeling with recurrence relations

Example 1：Compound Interest

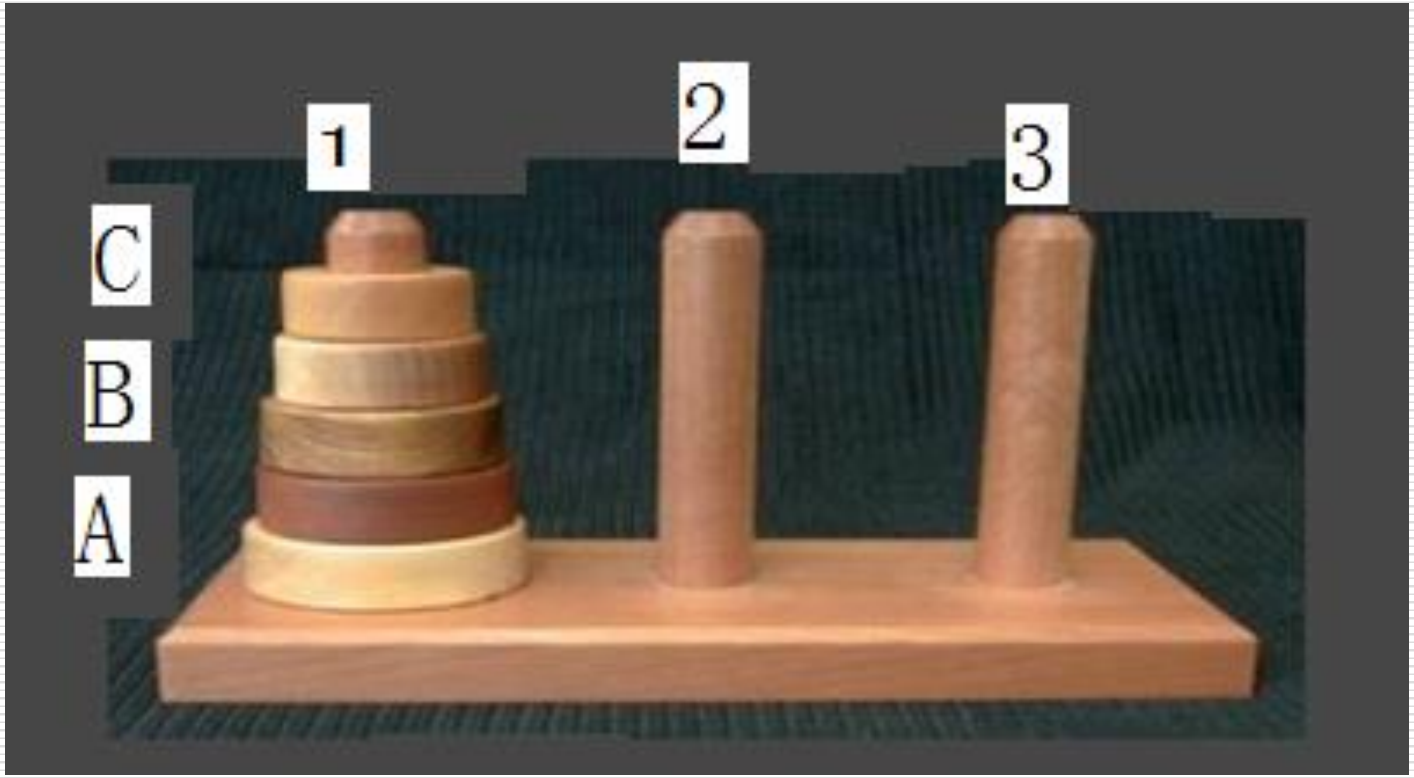
$$P_n = (1.11)P_{n-1}$$

Example 2：Fibonacci Numbers

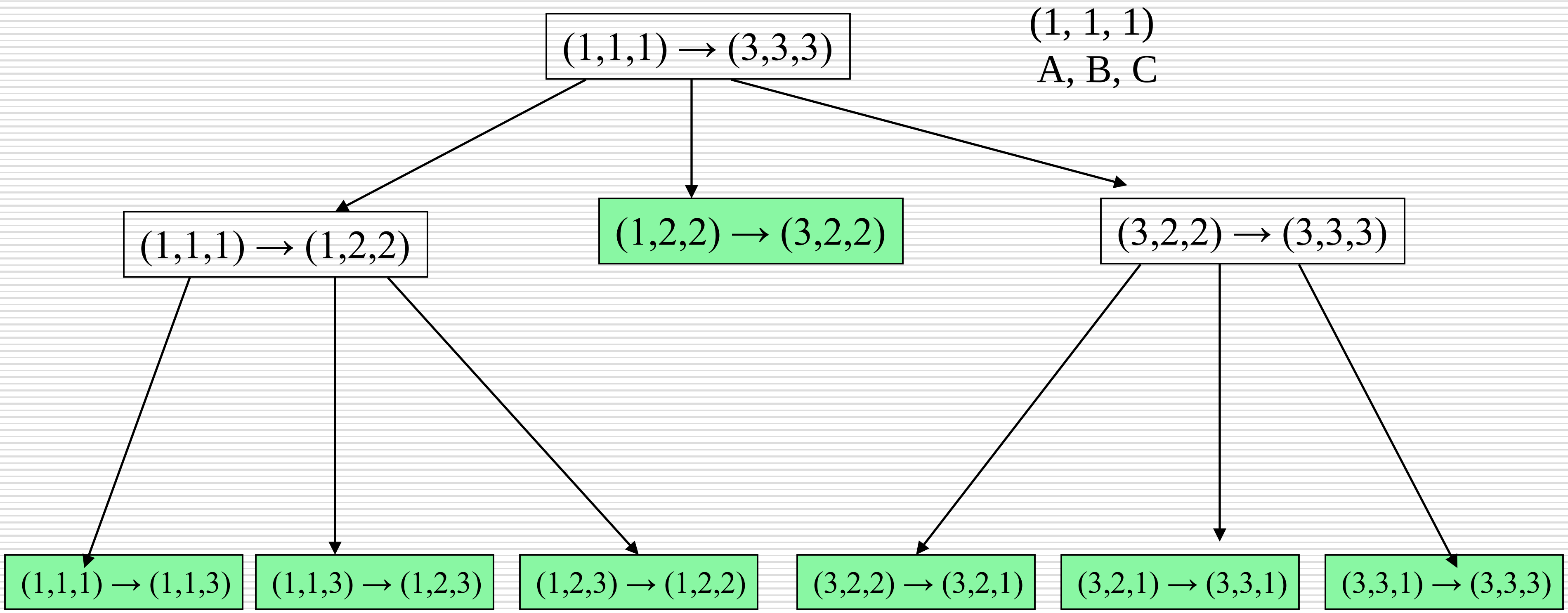
$$f_n = f_{n-1} + f_{n-2}$$

Example 3：The Tower Hanoi

(1, 1, 1)
A, B, C



Example 3: The Tower Hanoi



$$H_n = 2H_{n-1} + 1$$

Example 4: Codeword Enumeration

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits.Let a_n be the number of valid n-digit codewords.Find a recurrence relation and give initial conditions . 十进制数串中有偶数个0

Solution:

We conclude that

$$a_n = 9a_{n-1}+(10^{n-1} - a_{n-1})=8a_{n-1}+10^{n-1}$$

n位十进制数串中有偶数个0的个数为a_n，n−1位为a_{n-1}

n−1位末位加1，2，。。。9可以组成n位 $9a_{n-1}$

n−1位末位加0，n−1位必须有奇数个0， $10^{n-1} - a_{n-1}$

§ 14.2 Solving Recurrence Relations (1)

14.2.1 Linear homogeneous recurrence relation of degree k

k阶线性齐次递推关系

Definition:

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers , and $c_k \neq 0$.

§ 14.2 Solving Recurrence Relations (2)

14.2.2 Solving linear homogeneous recurrence relation with constant coefficients

Linear homogeneous recurrence relation with constant coefficients：

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \ldots - c_k a_{n-k} = 0$$

Characteristic equation（特征方程）：

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

§ 14.2 Solving Recurrence Relations (3)

14.2.2 Solving linear homogeneous recurrence relation with constant coefficients

（1）distinct root（不同根）

Theorem 1： Let c_1 and c_2 be real numbers. Suppose that $r^2-c_1r-c_2=0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ if and only if $a_n=b_1r_1^n+b_2r_2^n$ for $n=0,1,2, \ldots,$ where b_1 and b_2 are constants.

(1) distinct root

Show that: $r_1^2-c_1r_1-c_2=0$, $r_2^2-c_1r_2-c_2=0$

$c_1a_{n-1}+c_2a_{n-2}$

$=c_1(b_1r_1^{n-1}+b_2r_2^{n-1})+c_2(b_1r_1^{n-2}+b_2r_2^{n-2})$

$=b_1r_1^{n-2}(c_1 r_1+c_2) +b_2r_2^{n-2}(c_1 r_2+c_2)$

$= b_1r_1^{n-2}r_1^2 +b_2r_2^{n-2}r_2^2$

$= b_1r_1^n+b_2r_2^n$

$=a_n$

(1) distinct root

Exam: $a_{n}=a_{n-1}+2a_{n-2}$, $a_0=2, a_1=7$

$r^2-c_1r-c_2=0$, $r^2-r-2=0$

$r_1=2$, $r_2=-1$

$a_n=b_1r_1^n+b_2r_2^n=b_12^n+b_2(-1)^n$

$a_0=2=b_12^0+b_2(-1)^0$

$a_1=7=b_12^1+b_2(-1)^1$

$b_1=3$, $b_2=-1$

$a_n=b_12^n+b_2(-1)^n$

$=3\cdot 2^n+(-1)\cdot(-1)^n$

§ 14.2 Solving Recurrence Relations (5)

（2） Multiple root 多重根

Theorem 2： Let c_1 and c_2 be real numbers wth $c_2 \neq 0$. Suppose that $r^2-c_1r-c_2=0$ has only one root r_0.A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n=c_1a_{n-1}+c_2a_{n-2}$ if and only if $a_n=b_1r_0^n+b_2nr_0^n$ for $n=0,1,2,\ldots$, where b_1 and b_2 are constants.

Multiple root

Show that:

$r_0^2 - c_1 r_0 - c_2 = 0$, $a_n = b_1 r_0^n + b_2 n r_0^n$

So $c_1 a_{n-1} + c_2 a_{n-2}$

$= c_1(b_1 r_0^{n-1} + b_2 n r_0^{n-1}) + c_2(b_1 r_0^{n-2} + b_2 n r_0^{n-2})$

$= c_1 b_1 r_0^{n-1} + c_2 b_1 r_0^{n-2} + c_1 b_2 n r_0^{n-1} + c_2 b_2 n r_0^{n-2}$

$= b_1 r_0^{n-2}(c_1 r_0^1 + c_2) + b_2 r_0^{n-2}(c_1 n r_0^1 + c_2 n))$

$= b_1 r_0^{n-2} r_0^2 + b_2 n r_0^{n-2} r_0^2 = b_1 r_0^n + b_2 n r_0^n = a_n$

（2） Multiple root

Exam： $a_n=6a_{n-1}-9a_{n-2}, a_0=1, a_1=6$

$r^2-c_1r-c_2=0$, $r^2-6r+9=0$

$r=3$

$a_n=b_1r_0^n+b_2nr_0^n=b_1 3^n+b_2\cdot n\cdot 3^n$

$a_0=1=b_1 3^0+b_2\cdot 0\cdot 3^0$

$a_1=6=b_1 3^1+b_2\cdot 1\cdot 3^1$

$b_1=1$, $b_2=1$

$a_n=3^n+n\cdot 3^n$

§ 14.2 Solving Recurrence Relations (5)

Exam:

The Tower Hanoi : $a_n=2a_{n-1}+1$

Codeword Enumeration : $a_n=8a_{n-1}+10^{n-1}$

§ 14.2 Solving Recurrence Relations (7)

14.2.3 Linear nonhomogeneous 线性非齐次 recurrence relations with constant coefficients 常系数

A linear nonhomogeneous recurrence relation with constant coefficients

is a recurrence relation of the form

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}+F(n)$$

where $c_1,c_2,\ldots c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

The recurrence relaton

$a_n=c_1a_{n-1}+c_2a_{n-2}+\ldots+c_ka_{n-k}$, is called the **associated homogeneous recurrence relation .**

相伴的线性齐次

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

§ 14.2 Solving Recurrence Relations (8)

Theorem 1：

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then every solution is of the form

$\boldsymbol{\{a_n^{(p)} + a_n^{(h)}\}}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

Show that:

particular solution: $\{a_n^{(p)}\}$

$a_n^{(p)}=c_1a_{n-1}^{(p)}+c_2a_{n-2}^{(p)}+\ldots+c_ka_{n-k}^{(p)}+F(n)$,

Other solution: $\{b_n\}$

$b_n=c_1b_{n-1}+c_2b_{n-2}+\ldots+c_kb_{n-k}+F(n)$,

$b_n-a_n^{(p)}=c_1(b_{n-1}-a_{n-1}^{(p)})+ c_2(b_{n-2}-a_{n-2}^{(p)})+\ldots c_k(b_{k-2}-a_{k-2}^{(p)})$

$\{b_n-a_n^{(p)}\}$ $\{a_n^{(h)}\}$: associated homogeneous recurrence relation

$a_n^{(h)}=b_n-a_n^{(p)}$

$b_n=$ $\mathbf{a_n^{(p)}+ a_n^{(h)}}$

Exam1

$$a_n=6a_{n-1}+8^{n-1},\ a_1=7$$

$a_n=6a_{n-1}+8^{n-1}$

$\{a_n^{(h)}\};\ a_n=6a_{n-1}$

$r-6=0\ ,r=6$

$a_n^{(h)}=\alpha\cdot 6^n$

$F(n)=8^{n-1}=1/8\cdot 8^n$ $a_n^{(p)}=a_1\cdot 8^n$ 代入$a_n=6a_{n-1}+8^{n-1}$

$a_1\cdot 8^n=6(a_1\cdot 8^{n-1})+8^{n-1}$

$a_1=1/2\ ;\ a_n^{(p)}=1/2\cdot 8^n$

$a_n=a_n^{(h)}+a_n^{(p)}=\alpha\cdot 6^n+1/2\cdot 8^n$

$a_1=7=\alpha\cdot 6+1/2\cdot 8$

$\alpha=1/2$

$A_n=1/2\cdot 6^n+1/2\cdot 8^n$

Exam2

$a_n=2a_{n-1}+1;\ a_1=1$

$a_n=2a_{n-1}+1$

$\{a_n^{(h)}\};\ a_n=2a_{n-1}$

$r-2=0\ ,r=2$

$a_n^{(h)}=\alpha\cdot 2^n$

$F(n)=1;\ a_n^{(p)}=c\cdot n+d$ 代入$a_n=2a_{n-1}+1$

$c\cdot n+d=2(c\cdot(n-1)+d)+1$

$c\cdot n+(d-2c+1)=0$

$c=0;d=-1$

$a_n^{(p)}=-1$

$a_n=a_n^{(h)}+a_n^{(p)}=\alpha\cdot 2^n-1$

$a_1=1=\alpha\cdot 2-1$

$\alpha=1$

$a_n=2^n-1$

§14.4 Generating Functions 生成函数

14.14.1 The concept of generating functions

（1）Ordinary generating function普通

Definition：

The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

§ 14.4 Generating Functions (2)

14.14.1 The concept of generating functions

（2）Exponential generating function 指数

Definition：

The exponential generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$f_e(x) = a_0 + a_1\frac{x^1}{1!} + a_2\frac{x^2}{2!} + \ldots + a_n\frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} a_n\frac{x^n}{n!}$$

14.14.2 Useful facts about power series

(2) The extended binomial theorem

Definition:

Let α be a real number and let k be a nonnegative integer. Then the extended binomial coefficient $C(\alpha,k)$ is defined by

$$\binom{\alpha}{k} = \begin{cases} \alpha(\alpha-1)\ldots(\alpha-k+1)/k! & if\ \ k>0 \\ 1 & if\ \ k=0 \end{cases}$$

§ 14.4 Generating Functions (4)

$$\binom{-2}{3} = \frac{(-2)(-2-1)(-2-2)}{3!} = -4$$

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!} = \frac{1}{16}$$

§ 14.4 Generating Functions (4)

$$\binom{-n}{r} = \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!}$$

$$= \frac{(-1)^r (n)(n+1)\ldots(n+r-1)}{r!}$$

$$= \frac{(-1)^r (n)(n+1)\ldots(n+r-1)(n-1)!}{r!(n-1)!}$$

$$= (-1)^r C(n+r-1, r) = C(-n, r)$$

14.14.2 Useful facts about power series

（2） The extended binomial theorem

The extended binomial theorem:

Let x be a real number with $|x| < 1$ and let α be a real number. Then

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{k}$$

§ 14.4 Generating Functions (6)

Find the generating functions for :

$(1+x)^{-n}$ and $(1-x)^{-n}$

$$(1+x)^{-n}=\sum_{k=0}^{\infty}\binom{-n}{k}x^k=\sum_{k=0}^{\infty}(-1)^k c(n+k-1,k)x^k$$

$$(1-x)^{-n}=\sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k=\sum_{k=0}^{\infty}c(n+k-1,k)x^k$$

§ 14.4 Generating Functions (6)

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise

§ 14.4 Generating Functions (6)

$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1+2x+3x^2+\cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1+C(n,1)x+C(n+1,2)x^2+\cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1-C(n,1)x+C(n+1,2)x^2-\cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1+C(n,1)ax+C(n+1,2)a^2x^2+\cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$	$(-1)^{k+1}/k$

14.14.3 Counting problems and generating functions 用生成函数解决计数问题

show：

0不选择 0不选择 ….. 0不选择

1选择 1选择 1选择

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (x^0 + x^1)(x^0 + x^1)\ldots(x^0 + x^1)$$

第1项 第2项 ….. 第n项

$C(\mathrm{n}, \mathrm{k})$ 从n个不同物体中无顺序的选取k个（不许重复）

§ 14.4 Generating Functions (6)

14.14.3 Counting problems and generating functions

$$(1-x)^{-n} = \frac{1}{(1\text{-x})^{n}} = (x^0 + x^1 + x^2 + x^3 \ldots +)^n = \sum_{r=0}^{\infty} c(\mathrm{n}+\mathrm{r}-1,\mathrm{r})\, x^{\mathrm{r}}$$

$$= (x^0 + x^1 + x^2 + x^3 \ldots +)\ldots(x^0 + x^1 + x^2 + x^3 \ldots +)$$

第1类 ..... 第n类

n类物体允许重复的选取r个的方案是： C(n+r-1,r)

§ 14.4 Generating Functions (6)

Example1:find the number of solutions of

$x_1+x_2+x_3=11$ x1 ≥ 0; x2 ≥ 0; x3 ≥ 0

方法1: C(3+11-1,11)=78

方法2:Using generating functions

$$(x^0+x^1+\ldots+x^{11})(x^0+x^1\ldots+x^{11})(x^0+x^1+\ldots+x^{11})$$

$$=1x^0+3x^1+\ldots+78x^{11}+?\,x^{12}+\ldots$$

§ 14.4 Generating Functions (6)

Example2:find the number of solutions of

$x_1+x_2+x_3=11$ x1 ≥ 1; x2 ≥ 2; x3 ≥ 3

组合法：C(3+11-6-1,11-6)=21

产生式法：C(3+11-6-1,11-6)=21

$$(x^1+\ldots+x^{11})(x^2\ldots+x^{11})(x^3+\ldots+x^{11})$$

$$=x^6+\ldots+21x^{11}+?\,x^{12}+\ldots$$

§ 14.4 Generating Functions (6)

Example3:find the number of solutions of

$x_1+x_2+x_3=11$ $\quad$ $\mathbf{5 \geq x_1 \geq 2; 6 \geq x_2 \geq 3; 7 \geq x_3 \geq 4}$

$$(x^2+\ldots+x^5)(x^3\ldots+x^6)(x^4+\ldots+x^7)$$

$$=+\ldots+?\,x^{11}+?\,x^{12}+\ldots$$

§ 14.4 Generating Functions (6)

Example4:find the number of solutions of

$x_1+2x_2=15$ **x1 ≥0; x2 ≥0**

$$(x^0+x^1+\ldots x^{15})(x^0+x^2+\ldots x^{16})=$$

$$(x^0+x^1+\ldots x^{15}+\ldots)(x^0+x^2+\ldots x^{16}+\ldots)=$$

$$=\frac{1}{1-x}\cdot\frac{1}{1-x^2}=\frac{1}{2(1-x)^2}+\frac{1}{4(1-x)}+\frac{1}{4(1+x)}$$

$$=\frac{1}{2}\sum_{r=0}^{\infty}(r+1)x^r+\frac{1}{4}\sum_{r=0}^{\infty}x^r+\frac{1}{4}\sum_{r=0}^{\infty}(-1)^r x^r$$

X^r的系数 $\frac{1}{2}(r+1)+\frac{1}{4}+\frac{1}{4}(-1)^r$ $\frac{1}{2}(15+1)+\frac{1}{4}+\frac{1}{4}(-1)^{15}=8$

Example5:利用生成函数求解n类物体允许重复的r组合数。

$$G(x)=\sum_{r=0}^{\infty} a_r x^r = (x^0+x^1+\ldots)\ldots(x^0+x^1+\ldots)=$$

$$=(x^0+x^1+x^2+x^3\ldots+)^n=(1-x)^{-n}=(1/(1-x))^n$$

x^r的系数 $\binom{-n}{r}(-1)^r=(-1)^r C(n+r-1,r)(-1)^r$

$$=C(n+r-1,r)$$

§ 14.4 Generating Functions (6)

Example7:利用生成函数求解n类物体允许重复的r组合数,且每类物体至少选取一个。

$$G(x)=\sum_{r=0}^{\infty} a_r x^r = (x^1+x^2+\ldots)\ldots(x^1+x^2+\ldots)=$$

$$=(x^1+x^2+x^3+\ldots+)^n = x^n(1-x)^{-n} = x^n(1/(1-x))^n$$

$$=x^n\sum_{r=0}^{\infty}\binom{-n}{r}(-x)^r = x^n\sum_{r=0}^{\infty}(-1)^r C(n+r-1,r)(-1)^r x^r$$

$$=\sum_{r=0}^{\infty} C(n+r-1,r)x^{n+r} \qquad =\sum_{t=n}^{\infty} C(t-1,t-n)x^t$$

$$=\sum_{r=n}^{\infty} C(r-1,r-n)x^r$$

x^r的系数 $C(r-1,r-n)$

§ 14.4 Generating Functions (6)

14.14.4 Using generating functions to solve recurrence relations

利用生成函数求解递推关系. $a_n = 2a_{n-1}+3, n\geq 1$， $a_0=3$

方法1: $$G(x)=a_0+a_1x+a_2x^2+\cdots$$

$$G(x)-2xG(x)=a_0+(a_1-2a_0)x+(a_2-2a_1)x^2+\cdots$$

$$=3+3x+3x^2+\cdots=3(1+x+x^2+\cdots)$$

$$(1-2x)G(x)=3(1+x+x^2+\cdots)$$

$$G(x)=3(1+x+x^2+\cdots)/(1-2x)$$

§ 14.4 Generating Functions (6)

$$G(x) = 3(1 + x + x^2 + \cdots) / (1 - 2x)$$

$$G(x) = 3\frac{1}{(1-x)(1-2x)} \qquad G(x) = 3(\frac{2}{1-2x} - \frac{1}{1-x})$$

$$G(x) = 3(2\sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n)$$

$$a_n = 3(2 \cdot 2^n - 1) \qquad a_n = 3 \cdot 2^{n+1} - 3$$

方法2:

$$a_n=2a_{n-1}+3;\ a_0=3$$

$a_n=2a_{n-1}+3$

$\{a_n^{(h)}\};\ a_n=2a_{n-1}$

$r-2=0\ ,r=2$

$a_n^{(h)}=\alpha\cdot 2^n$

$F(n)=3;\ a_n^{(p)}=c\cdot n+d$ 代入$a_n=2a_{n-1}+1$

$c\cdot n+d=2(c\cdot(n-1)+d)+3$

$c\cdot n+(d-2c+3)=0$

$c=0;d=-3$

$a_n^{(p)}=-3$

$a_n=a_n^{(h)}+a_n^{(p)}=\alpha\cdot 2^n-3$

$a_0=3=\alpha\cdot 2^0-3$

$\alpha=6$

$an=6\cdot 2^n-3=3\cdot 2^{n+1}-3$

§ 14.4 Generating Functions (6)

Example6:Using generating functions to solve the follow recurrence relations. CODEWORD problem

$$a_n=8a_{n-1}+10^{n-1}$$
$$a_0=1,\ a_1=9$$

§ 14.4 Generating Functions

方法1:

$$G(x)=a_0+a_1x+a_2x^2+\ldots$$

$a_n=8a_{n-1}+10^{n-1}$

$$-8xG(x)=-8a_0x-8a_1x^2-8a_2x^3+\ldots$$

$a_1-8a_0=10^0$

$$(1-8x)G(x)=a_0+(a_1-8a_0)x+(a_2-8a_1)x^2+\ldots$$

$$=1+10^0x+10^1x^2+10^2x^3+\ldots$$

$$=1+(10^0x^0+10^1x^1+10^2x^2+\ldots)x$$

$$=1+\frac{1}{1-10x}x=\frac{1-9x}{1-10x}$$

$$G(x)=\frac{1-9x}{(1-10x)(1-8x)} \qquad G(x)=\frac{1}{2}\left(\frac{1}{1-8x}+\frac{1}{1-10x}\right)$$

§ 14.4 Generating Functions

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n) x^n$$

$$a_n = \frac{1}{2} \cdot 8^n + \frac{1}{2} \cdot 10^n$$

§ 14.4 Generating Functions

方法2:

$a_n = 8 \cdot a_{n-1} + 10^{n-1}$

$a_0 = 1$,$a_1 = 9$

$\{a_n^{(h)}\}$ $a_n = 8a_{n-1}$

$r-8=0$ $r=8$

$$a_n^{(h)} = \partial \cdot 8^n$$

$$f(n) = 10^{n-1} = \frac{1}{10} 10^n \qquad a_n^{(p)} = p_0 \cdot 10^n$$

$$p_0 \cdot 10^n = 8 \cdot p_0 \cdot 10^{n-1} + 10^n \qquad p_0 = \frac{1}{2}$$

$$a_n^{(p)} = \frac{1}{2} 10^n \qquad a_n = a_n^{(h)} + a_n^{(p)} = \partial \cdot 8^n + \frac{1}{2} 10^n$$

§ 14.4 Generating Functions

$$a_0 = 1 = \partial \cdot 8^0 + \frac{1}{2} 10^0$$

$$\partial = \frac{1}{2}$$

$$a_n = \frac{1}{2} \cdot 8^n + \frac{1}{2} \cdot 10^n$$

§ 14.5 Inclusion-exclusion 容斥原理

The principle of inclusion-exclusion is useful in counting problems with the union or intersection of two finite sets .

[DeMorgan law] domain U，complement sets $\overline{A}$

$$\overline{A} = \{x \in U \wedge x \notin A\}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

DeMogan定理的推广：设 $A_1, A_2, \ldots, A_n$ 是 U 的子集

则 (a) $\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$

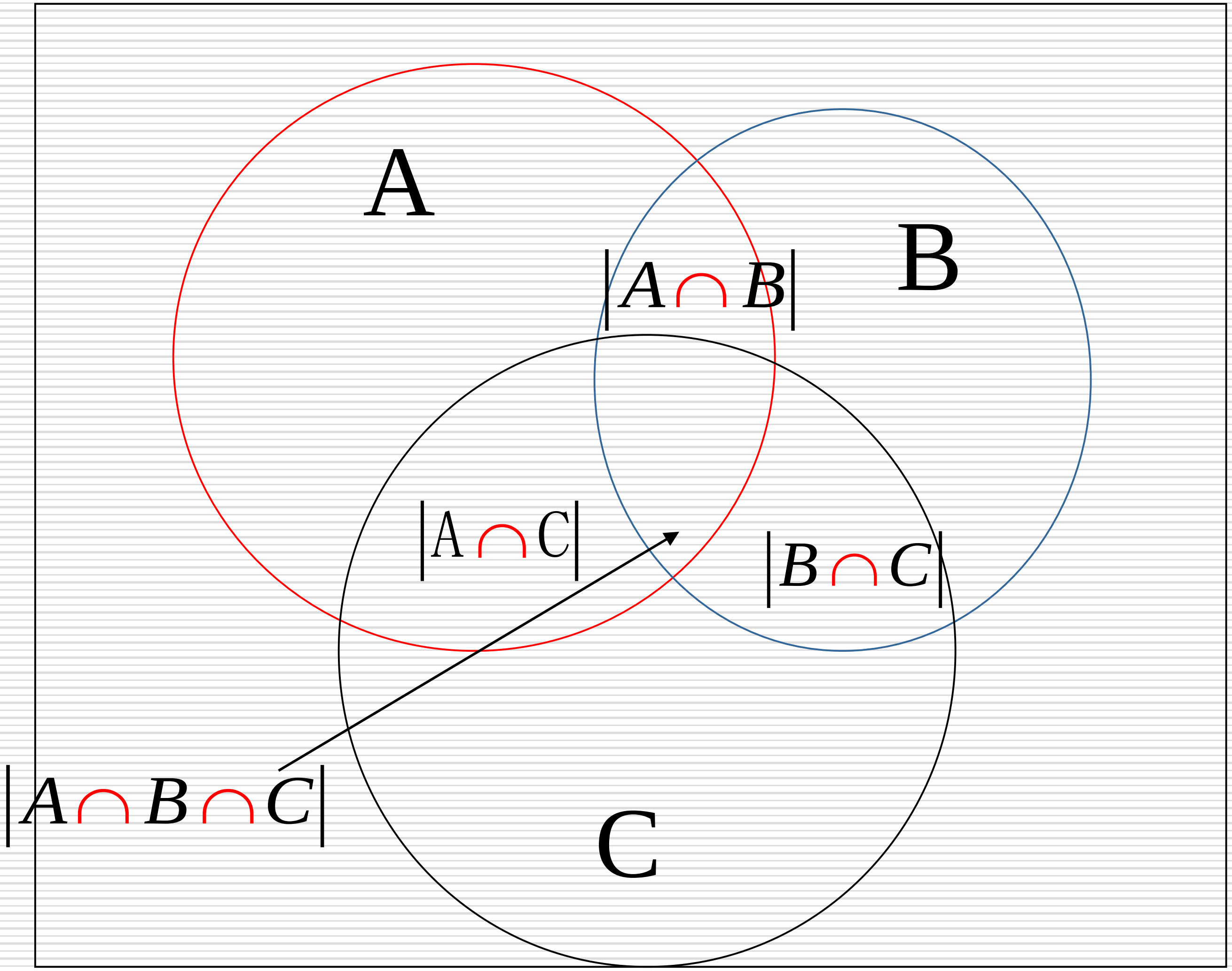
(b) $\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$

最简单的计数问题是求有限集合A和B的并的元素数目.

$$|A \cup B| = |A| + |B| - |A \cap B| \qquad (1)$$

即具有性质A或B的元素的个数等于具有性质A和B的元素个数。

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |\mathrm{A} \cap \mathrm{C}| - |B \cap C| + |A \cap B \cap C| \qquad (2)$$


A
B
C
$|A \cap B|$
$|A \cap C|$
$|B \cap C|$
$|A \cap B \cap C|$

Exam: 一个学校只有三门课程：数学、物理、化学。已知修这三门课的学生分别有170、130、120人；同时修数学、物理两门课的学生45人；同时修数学、化学的20人；同时修物理、化学的22人。同时修三门的3人。问这学校共有多少学生？

令： M 为修数学的学生集合；
P 为修物理的学生集合；
C 为修化学的学生集合；

$$|\mathrm{M}| = 170, |P| = 130, |C| = 120, |M \cap P| = 45$$

$$|M \cap C| = 20, |P \cap C| = 22, |M \cap P \cap C| = 3$$

$$\therefore \quad |M \cup P \cup C| = |M| + |P| + |C| - |M \cap P|$$

$$- |M \cap C| - |P \cap C| + |M \cap P \cap C|$$

$$= 170 + 130 + 120 - 45 - 20 - 22 + 3$$

$$= 336$$

即学校学生数为336人。

同理可推出：

$$
\begin{aligned}
|A\cup B\cup C\cup D| &= |A|+|B|+|C|+|D|-|A\cap B| \\
&\quad -|A\cap C|-|B\cap C|-|A\cap D|+|A\cap B\cap C| \\
&\quad +|A\cap B\cap D|+|B\cap C\cap D|-|A\cap B\cap C\cap D|
\end{aligned}
$$

利用数学归纳法可得一般的定理：

设 $A_1,A_2,\ldots,A_n$ 是有限集，则

$$|A_1 \cup A_2 \cup \ldots \cup A_n|$$

$$= \sum_{i=1}^{n}|A_i| - \sum_{i=1}^{n}\sum_{j>i}|A_i \cap A_j|$$

$$+\sum_{i=1}^{n}\sum_{j>i}\sum_{k>j}|A_i \cap A_j \cap A_k| - \ldots$$

$$+(-1)^{n-1}|A_1 \cap A_2 \cap \ldots \cap A_n| \quad (4)$$

§14.6 Applications Inclusion-exclusion 容斥原理的应用

Let A_i be the subset containing the elements that have property Pi, the number of elements with all the properties $P_1, P_2, P_3 \ldots P_k$ will be denoted by $N(P_1P_2P_3\ldots P_k)$

$$|A_1 \cap A_2 \cap \ldots \cap A_k| = N(P_1P_2\ldots P_k)$$

§ 14.6 Applications Inclusion-exclusion (2)

If the number of elements with none of the properties $P_1, P_2, P_3 \ldots P_k$ is denoted by $N(P'_1 P'_2 P'_3 \ldots P'_k)$ and the number of elements in set is denoted by N,then

$$N(P_1' P_2' \ldots P_k') = N - \left| A_1 \cup A_2 \cup \ldots \cup A_k \right|$$

From the inclusion-exclusion principle,we see that:

The Form of Applications Inclusion-exclusion

$$N(P_1' P_2' \ldots P_k') = \left| \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n} \right|$$

$$= N - \left| A_1 \cup A_2 \cup \ldots \cup A_{n-1} \cup A_n \right|$$

$$= N - \sum_{i=1}^{n} |A_i| + \sum_{i=1}^{n} \sum_{j>i} |A_i \cap A_j|$$

$$- \sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} |A_i \cap A_j \cap A_k| + \ldots$$

$$+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|$$

§14.6 Applications Inclusion-exclusion

另一种形式

$$N(P_1'P_2'\ldots P_k') =$$

$$N - \sum_{i=1}^{n} N(P_i) + \sum_{i=1}^{n}\sum_{j>i} N(P_iP_j)$$

$$-\sum_{i=1}^{n}\sum_{j>i}\sum_{k>j} N(P_iP_jP_k) + \ldots$$

$$+(-1)^n N(P_1P_2\ldots P_n)$$

exam1:方程 $x_1 + x_2 + x_3 = 11$

满足x_1<=3, x_2<=4, 且 x_3<=6 不同正整数解的个数?

用生成函数求解

$x_1+x_2+x_3=11$ $\quad$ $3 \geq x_1 \geq 0$; $4 \geq x_2 \geq 0$; $5 \geq x_3 \geq 0$

$G(x)=(1+X+X^2+X^3)(1+X+X^2+X^3+X^4)(1+X+X^2+X^3+X^4+X^5+X^6)=$

$\ldots.+6X^{11}\ldots.$

exam1:方程 $x_1 + x_2 + x_3 = 11$

满足x_1<=3, x_2<=4, 且 x_3<=6 不同正整数解的个数？

用容斥原理求解

解：令 P1表示性质X1>3 ，令 P2表示性质X2>4 ，令 P3表示性质X1>6

***满足* x_1<=3, x_2<=4, and x_3<=6解的个数是:**

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) +$$

$$N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

其中，N =解的总数=*C*(3+11-1,11)=78;

N(P1)=满足 X1>3 的解的数目= *C*(3+7-1,7)=36;

N(P2)=满足 X2>4 的解的数目= *C*(3+6-1,6)=28;

N(P3)=满足 X1>6 的解的数目= *C*(3+4-1,4)=15;

N(P1P2)=满足 X1>3 and X2>4 的解的数目= *C*(3+2-1,2)=6;

N(P1P3)=满足 X1>3 and X3>6 的解的数目
= *C*(3+0-1,0)=1;
N(P2P3)=满足 X2>4 and X3>6 的解的数目= *0*
N(P1P2P3)=满足 X1>3 and X2>4 and X3>6 的解的数目
= *0*

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) +$$

$$N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

$$= 78 - 36 - 28 - 15 + 6 + 1 + 0 + 0 = 6$$

Exam2: 求由a,b,c,d四个字母构成的n位符号串中，a,b,c,d至少出现一次的符号串数目。

solution：令A、B、C分别为n位符号串中不出现a，b，c符号的集合。

由于n位符号串中每一位都可取a，b，c，d四种符号中的一个，故不允许出现a的n位符号串的个数应是 3^n，即

$$|A| = |B| = |C| = 3^n$$

$$|A \cap B| = |A \cap C| = |B \cap C| = 2^n$$

$$\left|A\cap B\cap C\right|=1$$

a，b，c至少出现一次的n位符号串集合即为 $\overline{A}\cap\overline{B}\cap\overline{C}$

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right|=4^n-(|A|+|B|+|C|)+(|A\cap B|$$

$$+|A\cap C|+|C\cap B|)-|A\cap B\cap C|$$

$$=4^n-3\bullet 3^n+3\bullet 2^n-1$$

容斥原理　　映上函数的个数问题

Example: How many onto functions are there from a set with **six** element to a set with **three** elements?
6→3；6个元素到3个元素的集合可以构成多少种不同的映上函数?

伴域 codomain $B=\{b_1,b_2,b_3\}$,let P_1,P_2,P_3 be the properties that b_1,b_2 and b_3 are not in the range（值域） of the function.

$N(P_1)$=值域中不含b_1 的函数个数

$N(P_1' P_2' P_3')$=值域中包含b_1,b_2 和 b_3 的函数个数（满射）.

容斥原理　　映上函数的个数问题

$N=3^6$

$N(P_i)$值域中不含b_i 的函数个数.

$N(P_i)=2^6$; $N(P_iP_j)=1^6$; $N(P_1P_2P_3)=0$

$N(P_1' P_2' P_3')=N-(N(P_1)+N(P_2)+N(P_3))$

$+(N(P_1P_2)+N(P_3P_3)+N(P_2P_3))$

$-N(P_1P_2P_3)=3^6-(2^6+2^6+2^6)+(1^6+1^6+1^6)-0=$

$3^6-C(3,1)2^6+C(3,2)1^6-0=540$

Theorem：

Let m and n be positive integers with m ≥ n.Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)1^m$$

onto functions from a set with m elements to a set with n elements. (m ≥ n) exam:5种不同的工作分别分给4个人去完成，问有多少种分配方案?

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)1^m$$
$$= 4^5 - C(4,1)(4-1)^5 + C(4,2)(4-2)^5 - C(4,4-1)1^5 = 240$$

n个球	m个盒子	是否允许空盒	计数方案	备注
有区别	有区别	允许空盒	m^n	全排列
无区别	有区别	允许空盒	C(m+n-1,n)	m个有区别的元素,取n个作允许重复的组合
无区别	有区别	不允许空盒	C(n-1,m-1)	(1)选取m个球每盒一个 (2)n-m有区别的球放入m个有区别盒子中,允许某盒不放 C(n-m+m-1,n-m)=C(n-1,m-1)
无区别	无区别	允许空盒		一本书的6本复印件放入4个相同的箱子中
无区别	无区别	不允许空盒		n-m个无区别物体允许为空的放入无区分m盒子
有区别	有区别	不允许空盒		映上函数的个数
有区别	无区别	允许空盒	(集合的划分)	4人分配完全相同的3间办公室
有区别	无区别	不允许空盒		Stirling数

有区别	无区别	不允许空盒	S(n,m)

n个有区别的球放到m个相同的盒子中，要求无空盒，其不同的分配方案数用S(n,m)表示，称为第二类Stirling数（斯特林数）.

例如红，黄，蓝，白四种颜色的球，放到两个无区别的盒子里，不允许有空盒，其方案有7种：

	1	2	3	4	5	6	7
第1盒子	r	y	b	w	ry	rb	rw
第2盒子	ybw	rbw	ryw	ryb	bw	yw	Yb

∴ S(4,2)=7

定理

第二类Stirling $S(n,k)$有下列性质:

(a) $S(n,0)=0$,

(b) $S(n,1)=1$,

(c) $S(n,n)=1$,

(d) $S(n,2)=2^{n-1}-1$,

(e) $S(n,n-1)=C(n,2)$.

证明:(d)设有n个不相同的球b1,b2,...bn,从中取出一球b1,对其余的n-1个球,每个都有与b1同盒,或不与b1同盒两种选择,

n-1个球入盒方案是2^{n-1},

但其余的n-1个球全部与b1同盒不许出现(出现空盒),所以$S(n,2)=2^{n-1}-1$

定理

第二类Stirling S(n,k)有下列性质:

(e) $S(n,n-1)=C(n,2)$.

证明:(e) n个球放到n-1个盒子里,不允许有一空盒,故必有一盒有两个球,

从n个有区别的球中取2个共有C(n,2)种组合方案。

定理

第二类Stirling数满足下面的递推关系，

$$S(n,m)=mS(n-1,m)+S(n-1,m-1),(n\geq 1,m\geq 1)$$

证明：设有n个不相同的球b1,b2,...bn,从中取出一球b1,入盒方案分为两类， b1独占一个盒子，或者b1不独占一个盒子，

（1）b1独占一个盒子，入盒方案S(n-1,m-1)

（2）b1不独占一个盒子，m个盒子中任选一个放入b1，对其余的n-1个小球放入m个盒子中，方案数S(n-1,m)，所以该类总方案数：mS(n-1,m)

$$S(n,m)=mS(n-1,m)+S(n-1,m-1)$$

$$S(n,m)=mS(n-1,m)+S(n-1,m-1)$$

例如：$S(5,2)=2S(4,2)+S(4,1)=2\times 7+1=15$

例如：$S(5,4)=4S(4,4)+S(4,3)=4\times 1+6=10$

n \ m	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	05	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

有区别	有区别	不允许空盒	映上函数的个数

Theorem：

Let m and n be positive integers with m ≥ n.Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)1^m$$

onto functions from a set with m elements to a set with n elements. (m>n)

映上函数的个数的另一种表示形式：m!S(n,m), m≥n

映上函数的个数:

映上函数：有区别的物体放入有区别的盒子中，且不允许空盒。

有区别的物体放入无区别的盒子中，且不允许空盒。

计数方案是S(n,m)，

盒子有区别，进行全排列，m!

所以有, S(n,m) m!

exam:How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job? 5种不同的工作分配给4人去完成，每人至少有一种工作，问有多少种分配方案？

方法1：

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)1^m$$
$$= 4^5 - C(4,1)(4-1)^5 + C(4,2)(4-2)^5 - C(4,4-1)1^5 = 240$$

方法2：

S(5,4) 4!=10*24=240

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