Lecture 12
Maximum Flow Continued

- Edmonds-Karp algorithm
- Maximum bipartite matching

Edmonds-Karp Algorithm

- The augmenting path is a shortest path from s to t in the residual network, where each edge has unit weight.
- Time complexity: $O(VE^2)$

Critical Lemma

- Lemma 26.8

  If the Edmonds-Karp is run on a flow network G=(V, E) with source s and sink t, then for all vertices $v \in V - \{s,t\}$, the shortest-path distance $\delta_f(s,v)$ in the residual network G_f increases monotonically with each flow augmentation.

Proof

- f : the flow before the first augmentation that decreases some $\delta_f(s,v)$.
- f': the flow after the augmentation.
- Let $v \in V-\{s,t\}$ be the vertex with the minimum $\delta_{f'}(s,v)$ whose distance was decreased by the augmentation, so that $\delta_{f'}(s,v) < \delta_f(s,v)$.
- p=s~→u→v be a shortest path from s to v in $G_{f'}$, so that $(u,v) \in E_{f'}$ and $\delta_{f'}(s,u) = \delta_{f'}(s,v)-1$.
- We have $\delta_{f'}(s,u) \geq \delta_f(s,u)$
- Now we prove that $(u,v) \notin E_f$ since otherwise,
- $\delta_f(s,v) \leq \delta_f(s,u)+1 \leq \delta_{f'}(s,u)+1 = \delta_{f'}(s,v)$, contradicts $\delta_{f'}(s,v) < \delta_f(s,v)$.
- Since $(u,v) \notin E_f$ and $(u,v) \in E_{f'}$, the augmentation must have increased the flow from v to u. As only flows on the shortest path can be increased, then (v,u) is on the shortest path in G_f, thus we have
- $\delta_f(s,v) = \delta_f(s,u)-1 \leq \delta_{f'}(s,u)-1 = \delta_{f'}(s,v)-2$, contradicts $\delta_{f'}(s,v) < \delta_f(s,v)$.
- Such vertex v can not exist.

Analyze its time complexity

- Theorem 26.9

  The total number of flow augmentations performed by the Edmonds-Karp algorithm is O(VE).

- critical edge: an edge (u,v) on an augmenting path p with

$$c_f(p) = c_f(u,v)$$

- There must be at least one critical edge on an augmenting path. After augment the flow, the critical edge disappears from the residual network.
- To prove the theorem, we will show that each edge can become critical at most |V|/2-1 times.

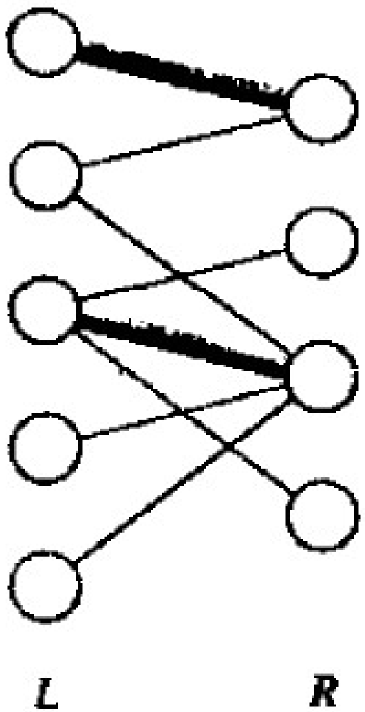
Proof of the Theorem

- $(u,v) \in E$, when (u,v) is critical on an augmenting path for the first time, we have, $\delta_f(s,v) = \delta_f(s,u)+1$.
- Then (u,v) disappears from the residual network. It can not reappear until the flow from u to v is decreased, which occurs only if (v,u) appears on an augmenting path. If f' is the flow in G when this event occurs, then we have $\delta_{f'}(s,u) = \delta_{f'}(s,v)+1$.
- Since $\delta_f(s,v) \le \delta_{f'}(s,v)$, then $\delta_{f'}(s,u) = \delta_{f'}(s,v)+1 \ge \delta_f(s,v)+1 = \delta_f(s,u)+2$.
- Consequently, from the time (u,v) becomes critical to the time it next becomes critical, the distance of u increases by at least 2. the distance of u is at most $|V|-2$. Thus, (u,v) can become critical at most $(|V|-2)/2 = |V|/2-1$ times.
- There are at most $O(E)$ edges and each augmenting path has at least one critical edge.

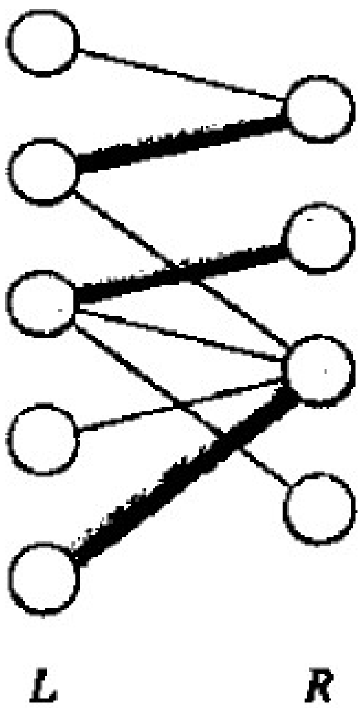
Maximum Matching in Bipartite Graphs

- Bipartite graph $G=(V,E)$: If $V=L\cup R$, and $L\cap R=\varnothing$, and $E=E(L, R)$, that is, each edge with one end in L and the other in R.
- Matching: $M\subseteq E$, such that no elements in M share common end points.
- Maximum matching M: for any other matching M', there is $|M|\geq|M'|$

Matching and Maximum Matching



(a)



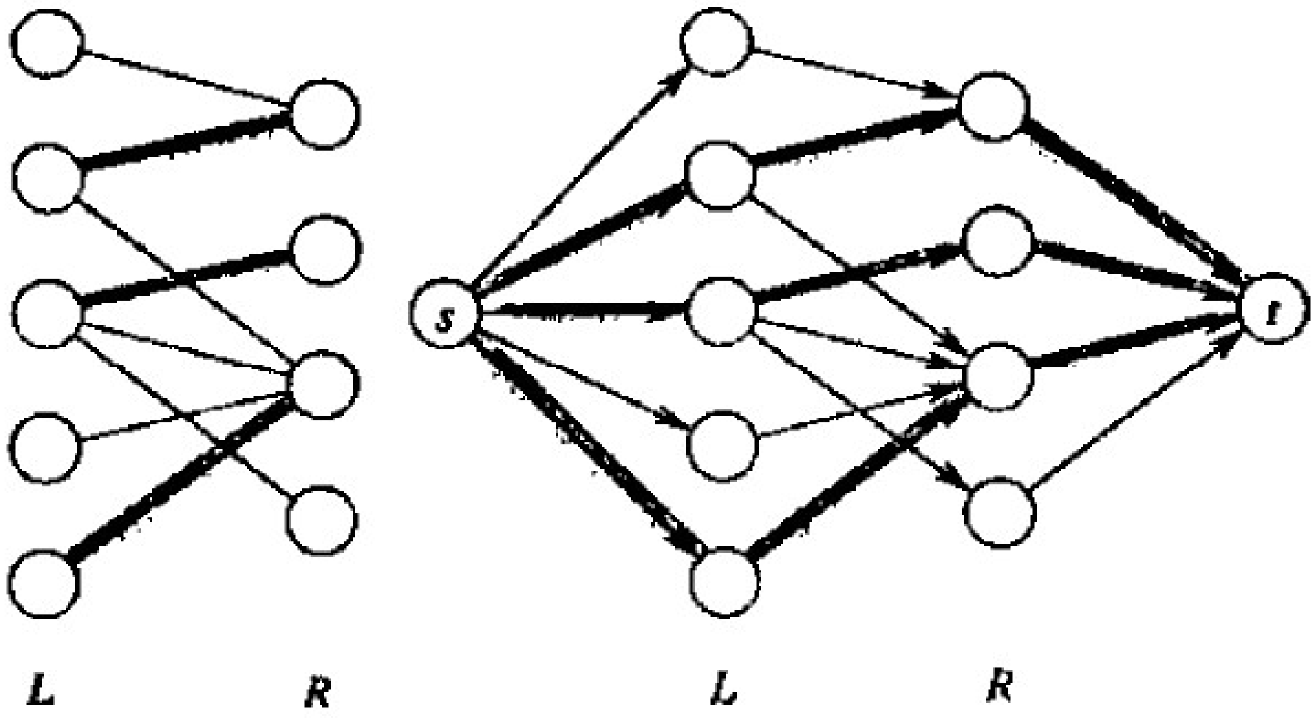
(b)

Maximum matching in bipartite graph

- Problem:
  - Input: a bipartite undirected graph $G=(L \cup R, E)$
  - Output: a maximum matching M of G.

How?

- Solution: By using maximum flow algorithm.
- Directed graph $G'=(V',E')$:
  - $V'=\{s,t\} \cup V$
  - $E'=\{(s,u)|u\in L\} \cup\{(v,t)|v \in R\} \cup\{(u,v)| u\in L, v \in R, (u,v) \in E\}$
- Network: G', with source s and sink t, and capacity function f:
  - $c(s,u)=1$
  - $c(v,t)=1$
  - $c(u,v)=1$



(a) (b)

Max. matching VS Max. flow

- Lemma 26.10

  Let G=(V=L∪R, E) be a bipartite graph and G'=(V',E') be its corresponding flow network.

  If M is a matching in G, then there is an integer-valued flow f in G' with value $|f|=|M|$.

  Conversely, if f is an integer-valued flow in G' , then there is a matching M in G with cardinality $|M|=|f|$.

Proof of the Lemma

- (→) Define f: if $(u,v) \in M$, then $f(s,u)= f(u,v)=f(v,t)=1$ and $f(u,s)= f(v,u)=f(t,v)=-1$, otherwise $f(u,v)=0$. *f* is a flow.
- Each edge $(u,v) \in M$ corresponds to 1 unit of flow in G' that traverses the path $s \to u \to v \to t$, and the paths are disjoint, except for s and t.

Then $|f|=f(L \cup \{s\}, R \cup \{t\})=|M|$

Proof of the Lemma

- (←) Define $M=\{(u,v):u\in L, v\in R, \text{ and } f(u,v)>0\}$. M is a matching.
- Each vertex u has at most one entering edge (s,u), c(s,u)=1. If one unit positive flow does enter, then one unit positive flow must leave. Since *f* is integer-valued, the one unit flow can enter and leave on at most one edge. Thus if $f(s,u)=1$, there is exactly one vertex v such that $f(u,v)=1$, and at most one edge leaving u carries positive flow.
- For every matched vertex $u\in L$, $f(s,u)=1$, and for every edge $(u,v)\in E-M$, $f(u,v)=0$.
- $|M|= f(L,R) = f(L,V') - f(L,L) - f(L,s) - f(L,t) = f(s,L) = f(s,V')=|f|$

Why integer-valued flow

- Theorem 26.11

  If the capacity function *c* takes on integral values, then the maximum flow produced by Ford-Fulkerson method has the property that $|f|$ is integer-valued. Moreover all $f(u,v)$ is an integer.

Correctness and Time Complexity

- Corollary 26.12: $|M|=|f|$. From Lemma 26.10.
- Time complexity: O(VE) why?