Lecture 11
Maximum Flow Problem

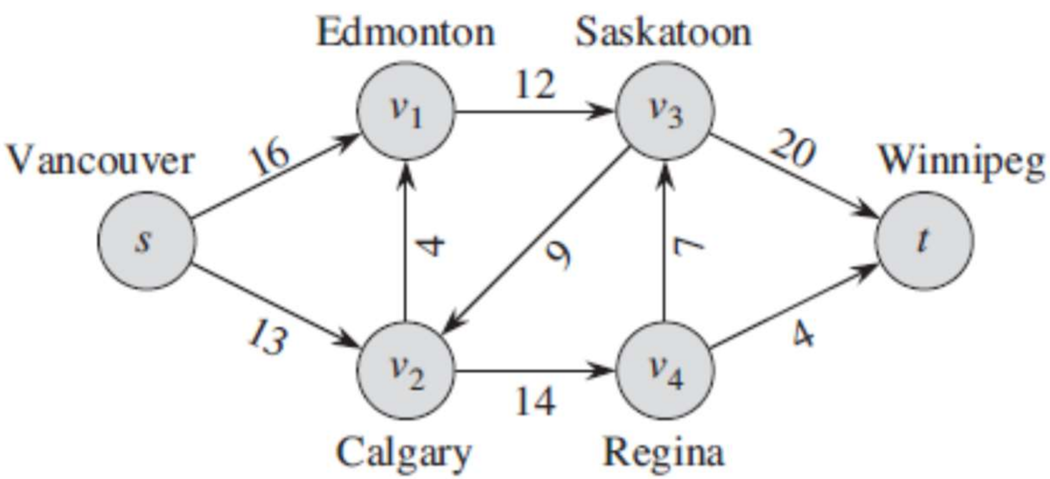
- The Ford-Fulkerson method

Maximum flow

- Liquids flow through pipes
- Current through electrical networks
- Information through communication networks
- … …

Flow Networks

- A flow network $G=(V,E)$ is a directed graph, where each edge $(u,v)\in E$ has a nonnegative capacity $c(u,v)>=0$.
- If $(u,v)\notin E$, we assume that $c(u,v)=0$.
- Requirement： if $(u,v)\in E$, then $(v,u)\notin E$
- two distinct vertices : **source *s*** and **sink *t*.**



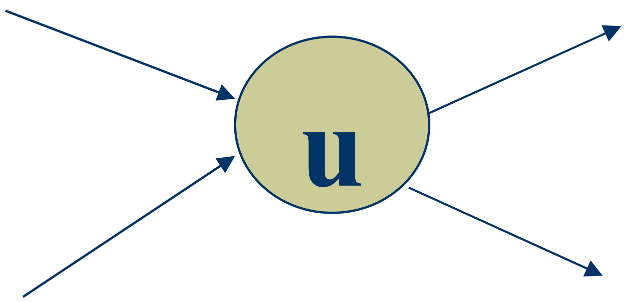
Flow

- Given a flow network G=(V,E) with capacity function c. Let s be the source and t the sink.
- A **flow** in G is a real-valued function f: $V \times V \rightarrow$ R satisfying the following three properties:

Capacity constraint: For all $u,v \in V$, $0 \le f(u,v) \le c(u,v)$.

If $(u,v) \notin E$, $f(u,v)=0$

Flow conservation: For all $u \in V-\{s,t\}$, $\sum_{v \in V} f(v,u) = \sum_{v \in V} f(u,v)$

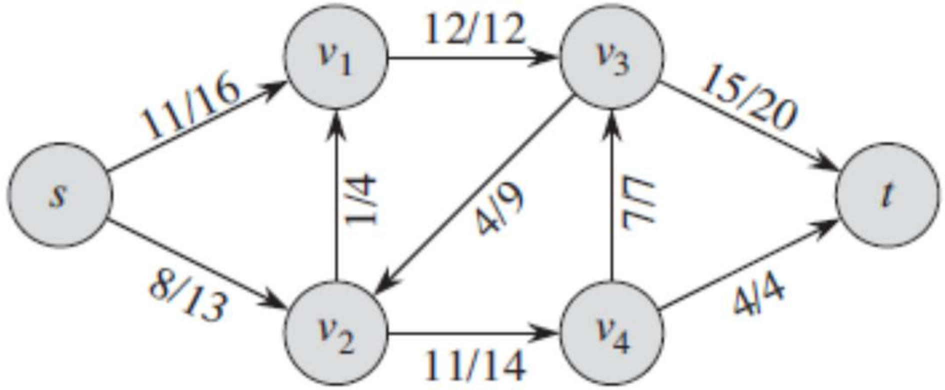


Value of a Flow

- The value of a flow is defined as

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$

- The total flow out of the source minus the flow into the source.

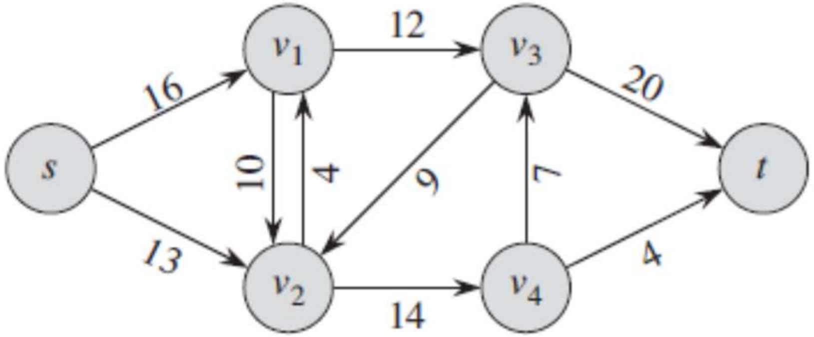


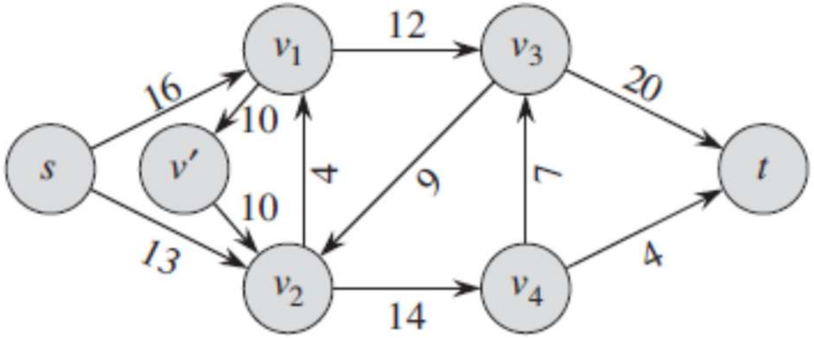
Maximum-flow Problem

- Input: a flow network G with capacity function c, source s and sink t
- Output: a flow of maximum value

How to solve it efficiently?

An example



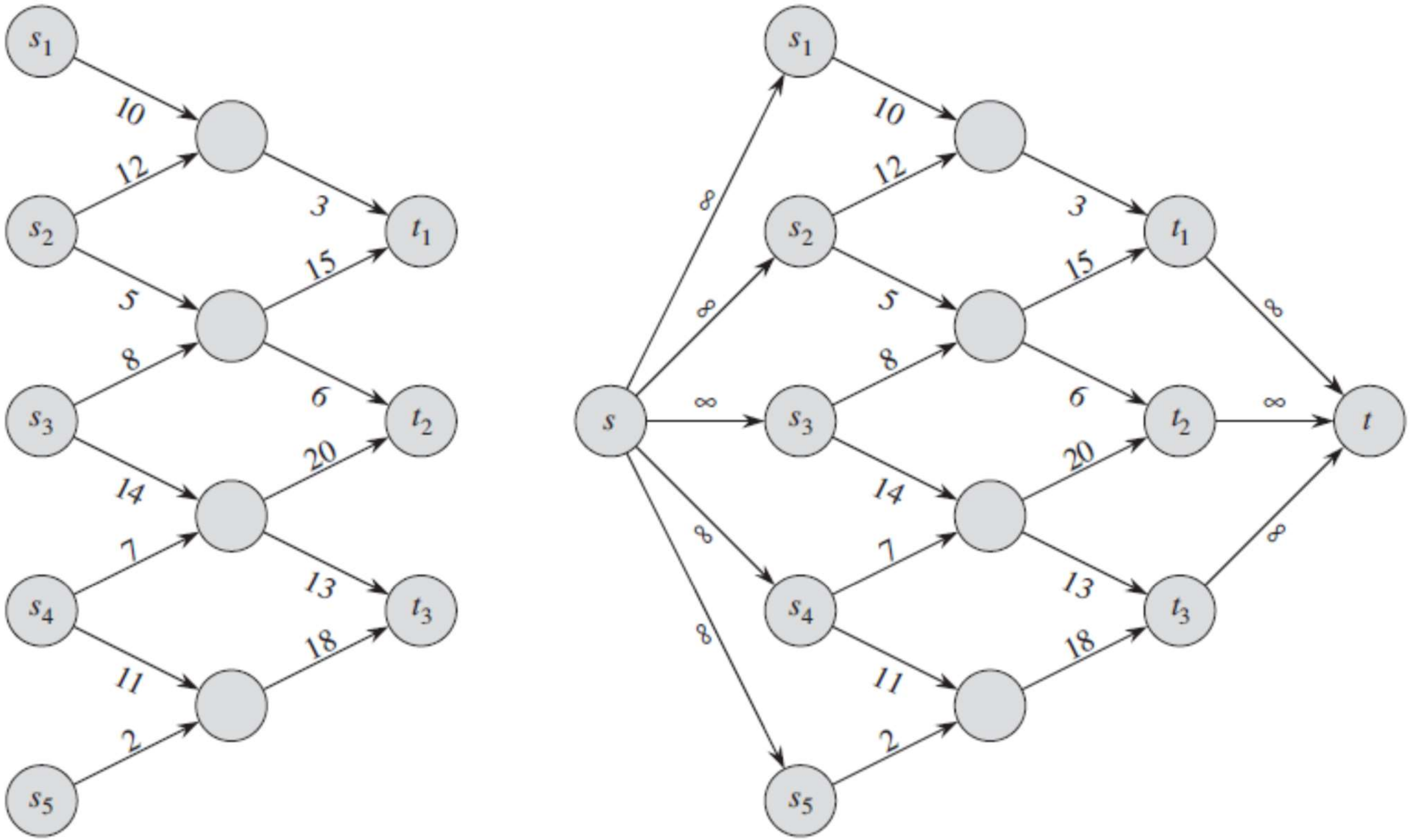


We can assume that there is no pair of mutually reverse edges in the flow network.

Explore the Properties

- Capacity constraint: Flow must not exceed capacity.
- Flow-conservation: total flow out of (or into) a vertex is 0.
  - For all $v \in V - \{s,t\}$, "flow in equals flow out"
- $(u,v) \notin$ E and $(v,u) \notin$ E, $f(u,v)=f(v,u)=0$.

Networks with multiple sources and sinks



(a) (b)

The Ford-Fulkerson Method

- Why call it a "method" rather than an "algorithm"?

  Because it encompasses several implementations with different running times.

- The Ford-Fulkerson method depends on *three important ideas*:

  residual networks, augmenting paths, and cuts.

- These ideas are essential to the important max-flow min-cut theorem, which characterizes the value of maximum flow in terms of cuts of the flow network.

The Ford-Fulkerson Method

- FORD-FULKERSON-METHOD(G,s,t)
  1. initialize flow f to 0
  2. while there exists an *augmenting path p*
  3. do *augment* flow f along p
  4. return f

Residual Networks

- Given a flow network and a flow, the **residual network** consists of edges that can admit more flow.
- More formally, given:

  $G=(V,E)$ --a flow network with source s and sink t

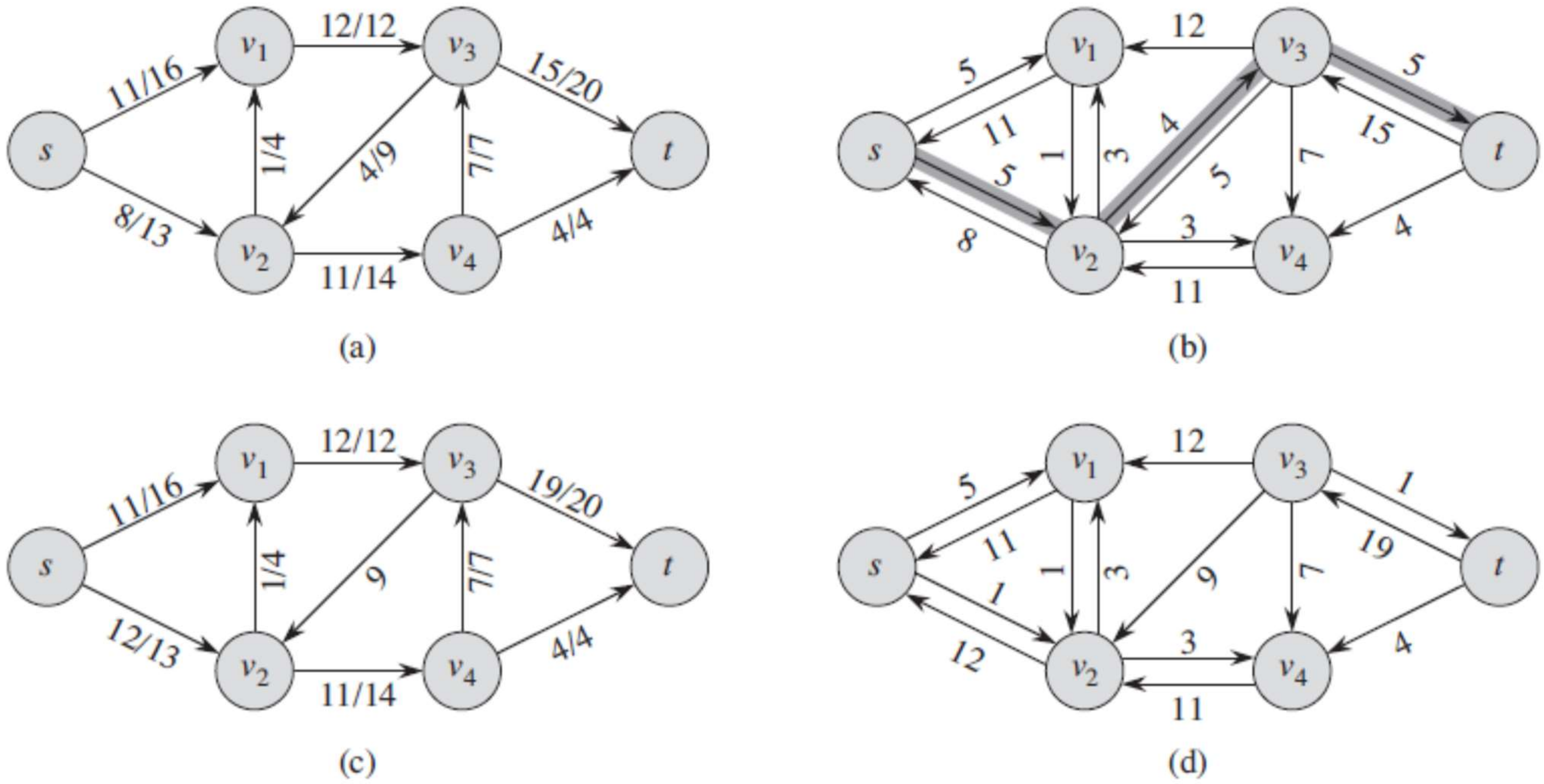
  f: a flow in G.
- The amount of *additional* flow can be pushed form u to v before exceeding the capacity $c(u,v)$ is the residual capacity of (u,v), given by:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v), & (u,v) \in E \\ f(v,u), & (v,u) \in E \\ 0, & \text{otherwise} \end{cases}$$

- Residual network: $G_f = (V,E_f)$, where

  $\mathbf{E_f=\{(u,v) \in V \times V: c_f(u,v) > 0\}}$**: residual edge.**

Residual Networks (example)



Augmentation and Cancellation

Given a flow network G=(V,E), if f is a flow in G and f' is a flow in the corresponding residual network G_f, $f \uparrow f'$, the ***augmentation*** of flow f by f', is a function from V×V to R, defined by

$$f \uparrow f'(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u), & (u,v) \in E \\ 0, & \text{otherwise} \end{cases}$$

The relationship between a flow in a residual network and one in the original network

- Lemma 26.1
- Let G=(V,E) be a flow network with source s and sink t ,and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f .Then the function $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.

Proof of Lemma 26.1

- Verify that $f \uparrow f'$ obeys the capacity constraint for each edge in E and flow conservation at each vertex in V-{s, t}.
- Nonnegative.
- For the capacity constraint, first observe that if (u, v) $\in$E, then $c_f(v,u)=f(u,v)$
- Therefore, $f'(v,u) \le c_f(v,u) = f(u,v)$
- Hence,

$$\begin{aligned}
(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) \\
&\geq f(u,v) + f'(u,v) - f(u,v) \\
&= f'(u,v) \\
&\geq 0 .
\end{aligned}$$

Proof continued

- capacity constraint

$$
\begin{aligned}
& (f \uparrow f')(u, v) \\
= \;& f(u, v) + f'(u, v) - f'(v, u) && \text{(by equation (26.4))} \\
\le \;& f(u, v) + f'(u, v) && \text{(because flows are nonnegative)} \\
\le \;& f(u, v) + c_f(u, v) && \text{(capacity constraint)} \\
= \;& f(u, v) + c(u, v) - f(u, v) && \text{(definition of } c_f\text{)} \\
= \;& c(u, v) .
\end{aligned}
$$

Proof continued

- flow conservation, for all u∈V-{s, t}

$$\sum_{(u,v)\in E} f \uparrow f'(u,v)$$

$$= \sum_{(u,x)\in E} [f(u,x) + f'(u,x) - f'(x,u)]$$

$$= \sum_{(u,x)\in E} f(u,x) + \sum_{\substack{(u,x)\in E \\ (u,x)\in E_f}} f'(u,x) - \sum_{\substack{(u,x)\in E \\ (x,u)\in E_f}} f'(x,u) \qquad (1)$$

根据剩余网络中，点u的流守恒。

$$\sum_{(u,x)\in E_f} f'(u,x) = \sum_{(x,u)\in E_f} f'(x,u)$$

$$左 = \sum_{\substack{(u,x)\in E \\ (u,x)\in E_f}} f'(u,x) + \sum_{\substack{(x,u)\in E \\ (u,x)\in E_f}} f'(u,x) = 右 = \sum_{\substack{(x,u)\in E \\ (x,u)\in E_f}} f'(x,u) + \sum_{\substack{(u,x)\in E \\ (x,u)\in E_f}} f'(x,u)$$

$$(1) = \sum_{(x,u)\in E} f(x,u) + \sum_{\substack{(x,u)\in E \\ (x,u)\in E_f}} f'(x,u) - \sum_{\substack{(x,u)\in E \\ (u,x)\in E_f}} f'(u,x)$$

$$= \sum_{(x,u)\in E} [f(x,u) + f'(x,u) - f'(u,x)]$$

$$= \sum_{(x,u)\in E} f \uparrow f'(x,u)$$

Proof continued

- $|f \uparrow f'| = |f| + |f'|$ V_1: each of these vertices has an edge to s.
- V_2: s has an edge to each of these vertices.
- $|f \uparrow f'|$

$$
\begin{aligned}
&= \sum_{v \in V_1} (f(s,v) + f'(s,v) - f'(v,s)) - \sum_{v \in V_2} (f(v,s) + f'(v,s) - f'(s,v)) \\
&= \sum_{v \in V_1} f(s,v) + \sum_{v \in V_1} f'(s,v) - \sum_{v \in V_1} f'(v,s) - \sum_{v \in V_2} f(v,s) - \sum_{v \in V_2} f'(v,s) + \sum_{v \in V_2} f'(s,v) \\
&= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) + \sum_{v \in V_1} f'(s,v) + \sum_{v \in V_2} f'(s,v) - \sum_{v \in V_1} f'(v,s) - \sum_{v \in V_2} f'(v,s) \\
&= \sum_{v \in V_1} f(s,v) - \sum_{v \in V_2} f(v,s) + \sum_{v \in V_1 \cup V_2} f'(s,v) - \sum_{v \in V_1 \cup V_2} f'(v,s) \\
&= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{v \in V} f'(s,v) - \sum_{v \in V} f'(v,s) \\
&= |f| + |f'|
\end{aligned}
$$

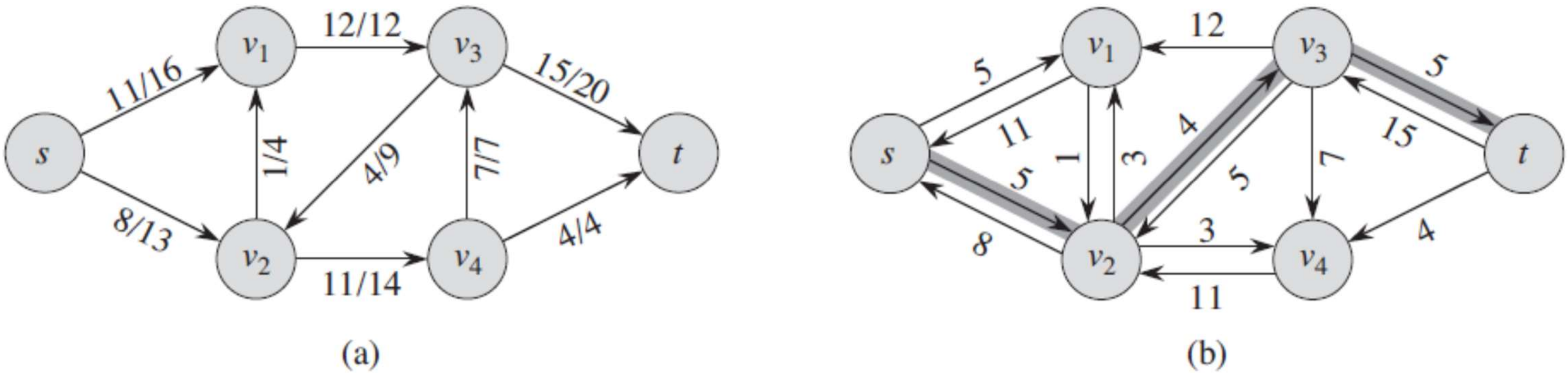
Augmenting paths

- Given a flow network $G=(V,E)$ and a flow f, an augmenting path is a simple path from s to t in the residual network G_f
- Residual capacity of p : the maximum amount of flow that we can ship along the edges of an augmenting path p, i.e.,

$$c_f(p) = \min \{ c_f(u,v) : (u,v) \text{ is on } p \}.$$

The residual capacity is 1.

Augmenting Path (example)



Augmenting Path

- Lemma 26.2

Let G=(V,E) be a flow network, Let f be a flow in G, and let p be an augmenting path in G_f .Let f_p :V×V→R by

$$f_p(u,v) = \begin{cases} c_f(p) \text{ if } (u,v) \text{ is on } p \\ 0 \quad \text{ otherwose.} \end{cases}$$

Then f_p is a flow in G_f with value $|f_p|=c_f(p)>0$.

Augmenting Path

- Corollary 26.3

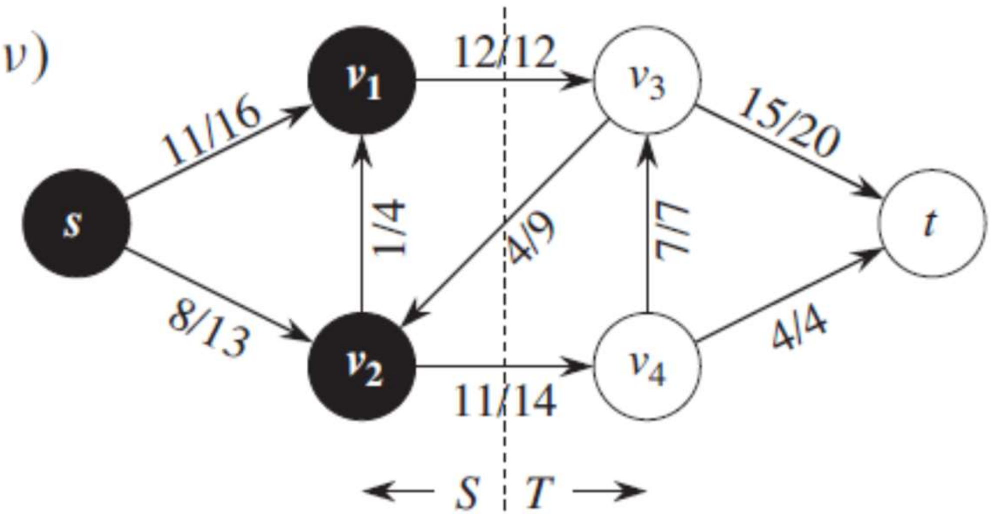
Let G=(V,E) be a flow network, Let f be a flow G, and let p be an augmenting path in G_f. Let f_p be defined as in equation (26.8). Define a function $f'=f\uparrow f_p$. Then f' is a flow in G with value $|f'|=|f|+|f_p|>|f|$.

Cut

- Cut (S, T): S is a subset of V, $T = V\text{-}S$, $s \in S$ and $t \in T$.
- Net flow across the cut (S,T): $f(S,T)$
- Capacity of the cut (S,T): $c(S,T)$
- Minimum cut of a network: a cut with minimum capacity.

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$



Cut

- Lemma 26.4

Let f be a flow in a flow network G with source s and sink t , and let (S,T) be a cut of G. Then the net flow across (S,T) is $f(S,T)= |f|$.

Proof of lemma 26.4

For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) = 0$

$$
\begin{aligned}
|f| &= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right) \\
&= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \sum_{v \in V} f(u,v) - \sum_{u \in S-\{s\}} \sum_{v \in V} f(v,u) \\
&= \sum_{v \in V} \left(f(s,v) + \sum_{u \in S-\{s\}} f(u,v) \right) - \sum_{v \in V} \left(f(v,s) + \sum_{u \in S-\{s\}} f(v,u) \right) \\
&= \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u) \\
&= \sum_{v \in S} \sum_{u \in S} f(u,v) + \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\
&= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) + \left(\sum_{v \in S} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) \right) \\
&= f(S,T)
\end{aligned}
$$

Cut

- Corollary 26.5

  The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.

- Proof
  - Let (S,T) be any cut of G and let f be any flow. By Lemma 26.4 and capacity constraints,

$$|f| = f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \leq \sum_{u\in S}\sum_{v\in T} c(u,v) = c(S,T)$$

Cut

- Theorem 26.7(Max-flow Min-cut Theorem)

  If f is a flow in a flow network G=(V,E) with source s and sink t then the following conditions are equivalent:

  1. f is a maximum flow in G.
  2. The residual network G_f contains no augmenting paths.
  3. $|f|=c(S,T)$ for some cut (S,T) of G

Proof

- (1)$\Rightarrow$(2): Suppose for the sake of contradiction that f is a maximum flow in G but G_f has an augment path p. Then, by Corollary 26.3, the flow sum $f \uparrow f_p$, where f_p is given by equation (26.8), is a flow in G with value strictly greater than $|f|$, contradicting the assumption that f is a maximum flow.
- (2)$\Rightarrow$(3): Suppose that G_f has no augmenting path, that is, that G_f contains no path from s to t. Define

  S={ v $\in$ V : there exists a path from s to v in G_f }

  and T=V-S. The partition (S,T) is a cut, s$\in$S and t$\notin$S. For each pair of vertices u and v such that u$\in$S and v$\in$T, if (u,v)$\in$E, we have $f(u,v)=c(u,v)$, since otherwise (u,v)$\in E_f$, which would place v in the set S. If (v,u)$\in$E, we have $f(v,u)=0$ for the same reason. By lemma 26.4, therefore, $|f|=f(S,T)=c(S,T)$.
- (3)$\Rightarrow$(1): By Corollary 26.5, $|f| \le c(S,T)$ for all cuts (S,T).

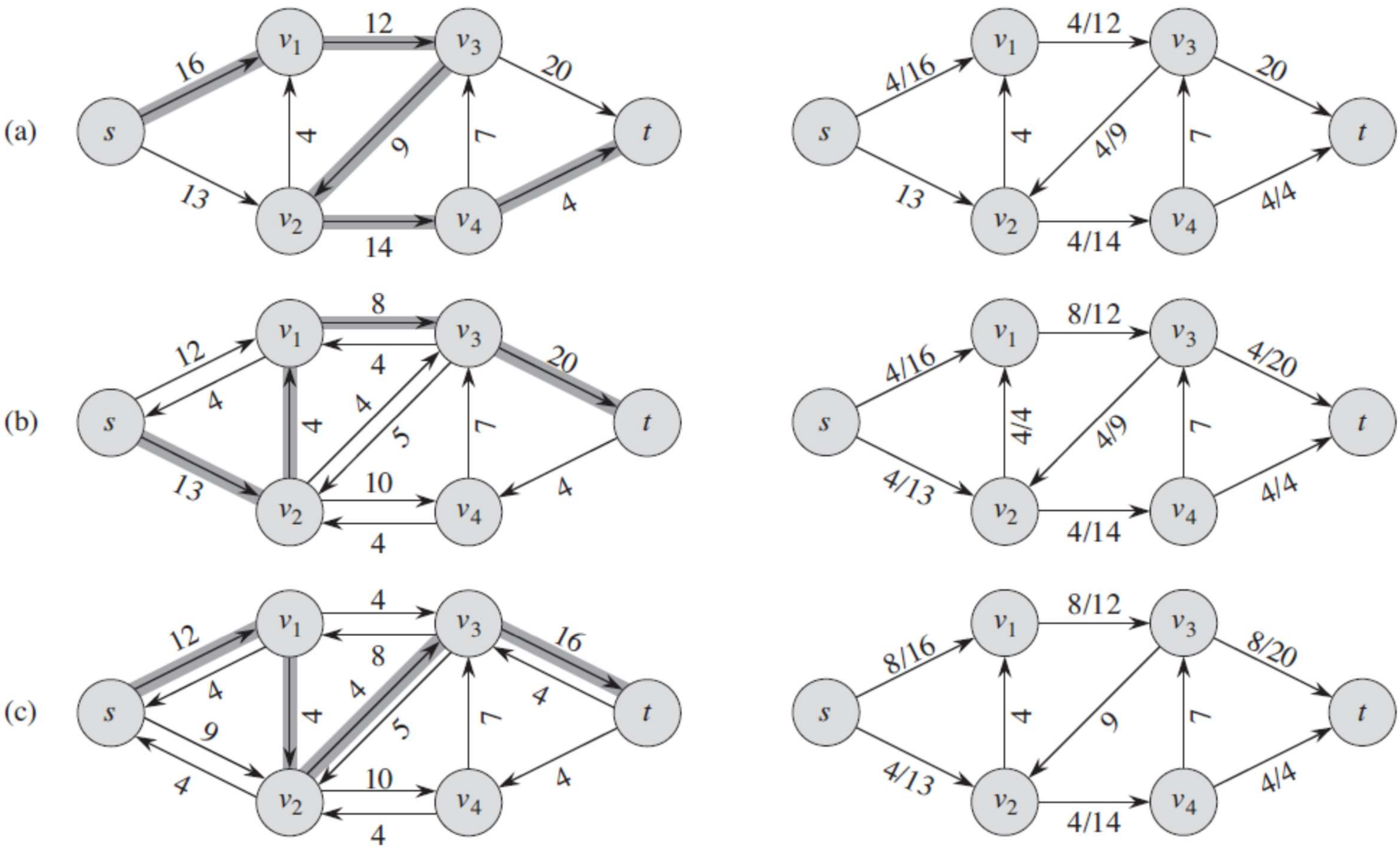
  Then $|f| = c(S,T)$ implies that f is a maximum flow.

The Basic Ford-Fulkerson Algorithm

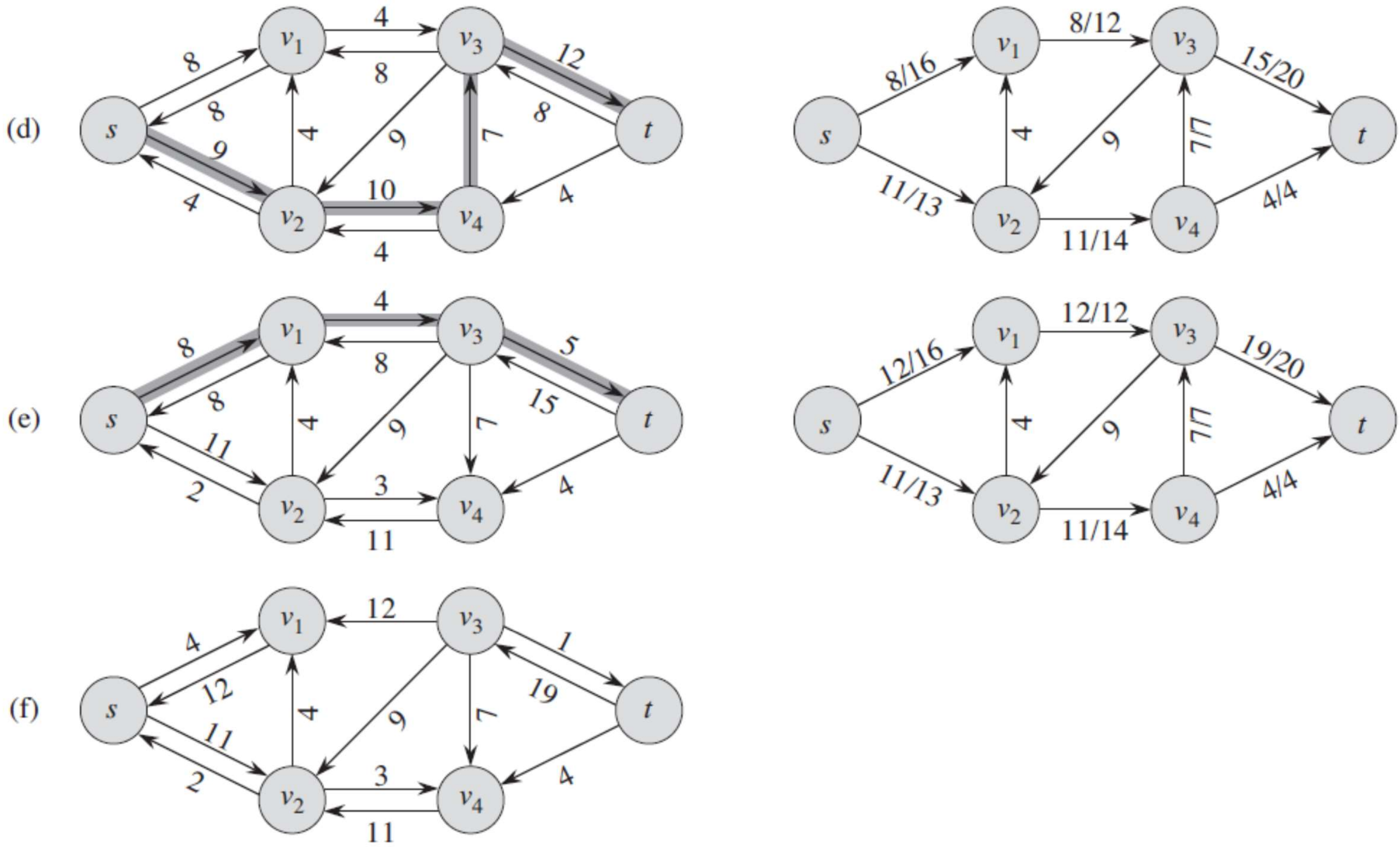
- Ford-Fulkerson(G,s,t)

  1. **For** each edge $(u, v) \in E$
  2.   **do** $f[u,v] \leftarrow 0$
  3. **While** there exists a path p from s to t in the residual network G_f
  4. $c_f(p) \leftarrow \min\{c_f(u,v)\text{: } (u,v) \text{ is in } p\}$
  5.   **for** each edge (u,v) in p
  6.   if $(u,v) \in E$
  7. $f[u,v] \leftarrow f[u,v] + c_f(p)$
  8.   else
  9. $f[v,u] \leftarrow f[v,u] - c_f(p)$

The Basic Ford-Fulkerson Algorithm



The Basic Ford-Fulkerson Algorithm



Analysis of Ford-Fulkerson Method

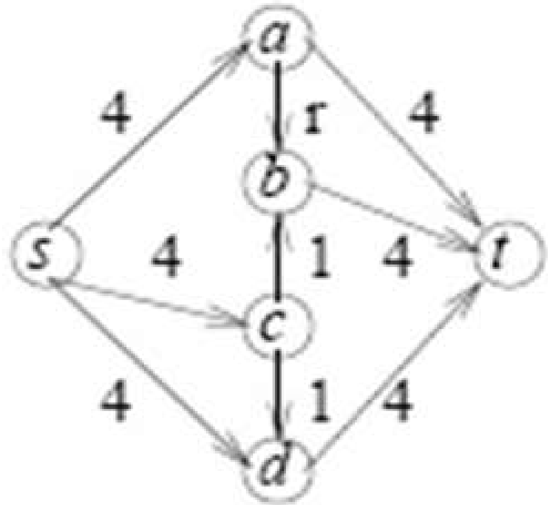
- The selection of the augmenting path in this method is arbitrary.
- If the capacities are irrational, the method may not terminate. If the flow does converge, however, it may converge to a value that is not necessarily maximum.
- If the capacities are integers, the While loop is executed at most $|f^*|$ times, since the flow value increases by at least 1 in each iteration.
- If we use either DFS or BFS, the time to find an augmenting path takes $O(E)$ time, i.e. each iteration of While loop the takes $O(E)$ time.
- Thus, the total running time of the method is $O(E\,|f^*|)$.

A Bad Example for Ford Fulkerson

[U. Zwick, TCS 148, p. 165–170, 1995]

Let $r = \dfrac{\sqrt{5}-1}{2}$.

Consider the graph



And the augmenting paths

$p_0 = \langle s, c, b, t \rangle$

$p_1 = \langle s, a, b, c, d, t \rangle$

$p_2 = \langle s, c, b, a, t \rangle$

$p_3 = \langle s, d, c, b, t \rangle$

The sequence of augmenting paths $p_0(p_1, p_2, p_1, p_3)^*$ is an infinite sequence of positive flow augmentations.

The flow value does not converge to the maximum value 9.

Analysis of Ford-Fulkerson Method

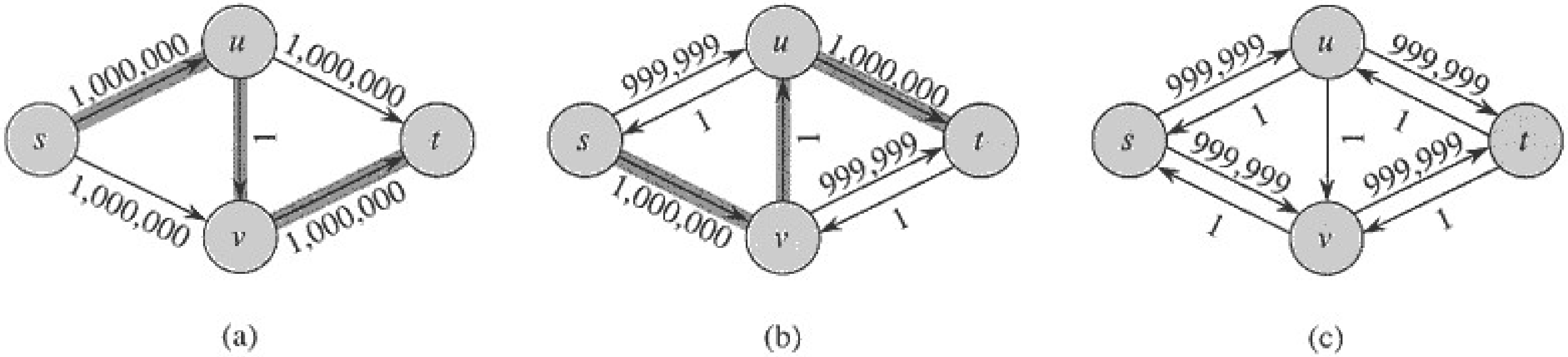


Figure 26.6 **(a)** A flow network for which FORD-FULKERSON can take $\Theta(E\ |f^*|)$ time, where f^* is a maximum flow, shown here with $|f^*| = 2{,}000{,}000$. An augmenting path with residual capacity 1 is shown. **(b)** The resulting residual network. Another augmenting path with residual capacity 1 is shown. **(c)** The resulting residual network.