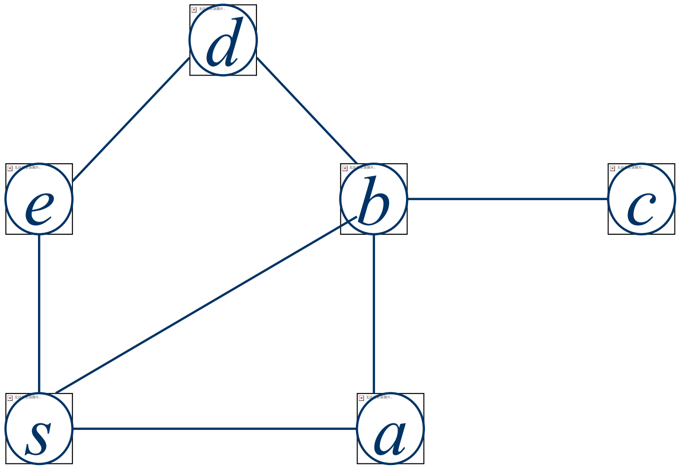
Lecture 2
Breadth-First Search

Breadth-first search (outline)

1. The single source shortest-paths problem for unweighted graph
2. The Breadth-first search algorithm
3. The correctness proof
4. The running time of BFS algorithm

- Note: *We only introduce BFS for undirected graphs. But it also works for directed graphs.*

Shortest paths



Example: There are three simple paths (distinct vertices) from source *s* to *b*: <*s*, *b*>, <*s*, *a*, *b*>, <*s*, *e*, *d*, *b*> of length 1, 2, 3, respectively.

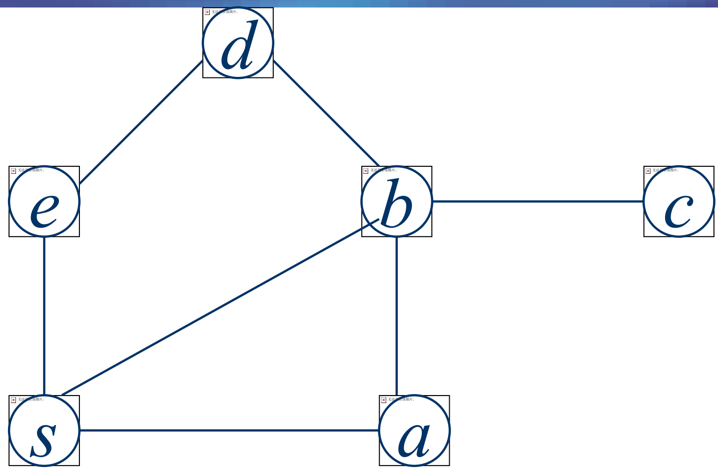
So the shortest path (a path containing the smallest number of edges) from *s* to *b* is <*s*, *b*>.

The shortest paths from *s* to other vertices *a*, *e*, *d*, *c* are:

<*s*, *a*>, <*s*, *e*>, <*s*, *e*, *d*>, <*s*, *b*, *c*>.

There are two shortest paths from *s* to *d*.

The shortest-paths problem



- Distance $\delta(s, v)$: The length of the shortest path from s to v. For example $\delta(s, c)$=2.
- The problem:
  - Input: A graph $G = (V, E)$ and a source vertex $s \in V$
  - Question: A shortest path from s to each vertex $v \in V$ and the distance $\delta(s, v)$.

The Breadth-First Search

- The idea of the BFS:
  - Each time, search as many vertices as possible.
  - Visit the vertices as follows:
    1. Visit all vertices at distance 1
    2. Visit all vertices at distance 2
    3. Visit all vertices at distance 3
    4. … …
- Initially, *s* is made GRAY, others are colored WHITE.
- After a gray vertex is processed, **its color is changed to black**, and the **color of all white neighbors is changed to gray.**
- Gray vertices are kept in a queue Q.

The Breadth-First Search (more details)

- *G* is given by its adjacency-lists.
- Initialization:
  - First Part: lines 1 – 4
  - Second Part: lines 5 - 9
- Main Part: lines 10 – 18
- Enqueue(*Q*, *v*): add a vertex *v* to the end of the queue *Q*
- Dequeue(*Q*): Extract the first vertex in the queue *Q*

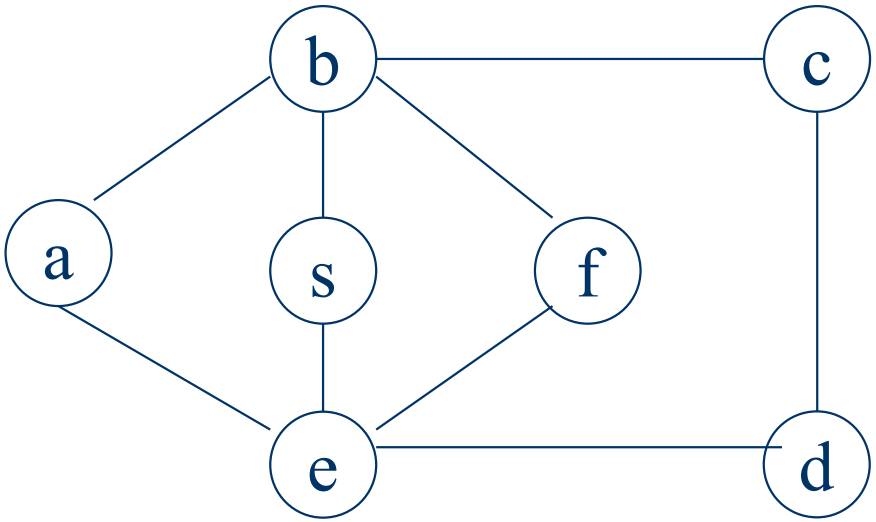
```
BFS(G, s)
1   for each vertex u∈V[G]-{s}
2          do color[u]<--WHITE
3               d[u]<--∞
4               π[u]<--NIL
5   color[s]<--GRAY
6   d[s]<--0
7   π[s]<--NIL
8   Q<--ϕ
9    ENQUEUE(Q, s)
10   while Q≠ϕ
11      do u<--DEQUEUE(Q)
12          for each v∈Adj[u]
13              do if color[v]=WHITE
14                  then color[v]<--GRAY
15                        d[v]<--d[u]+1
16                        π[v]<--u
17                        ENQUEUE(Q, v)
18          color[u]<--BLACK
```

What does the BFS do?

- Given a graph $G = (V, E)$, the BFS returns:
  - $d[v]$, proved to be the shortest distance from s to v
  - $\pi[v]$, the predecessor of v in the search, which can be used to derive a shortest path from s to vertex v.
- BFS actually returns a **shortest path tree** in which the unique simple path from s to node v is a shortest path from s to v in the original graph.
- In addition to the two arrays $d[v]$ and $\pi[v]$, BFS also uses another array $color[v]$, which has three possible values:
  - WHITE: represented "undiscovered" vertices;
  - GRAY: represented "discovered" but not "processed" vertices;
  - BLACK: represented "processed" vertices.

Example of Breadth-First Search

source vertex: *s*

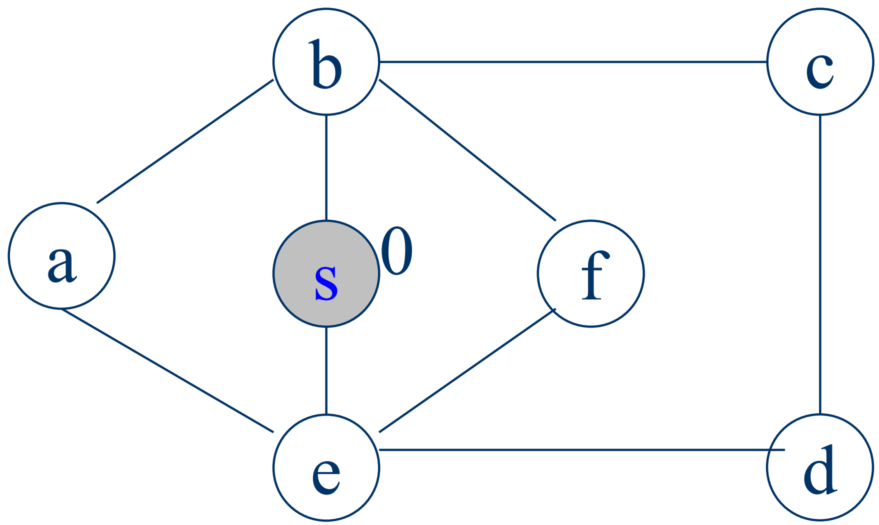


Example(Continued)

- Initialization

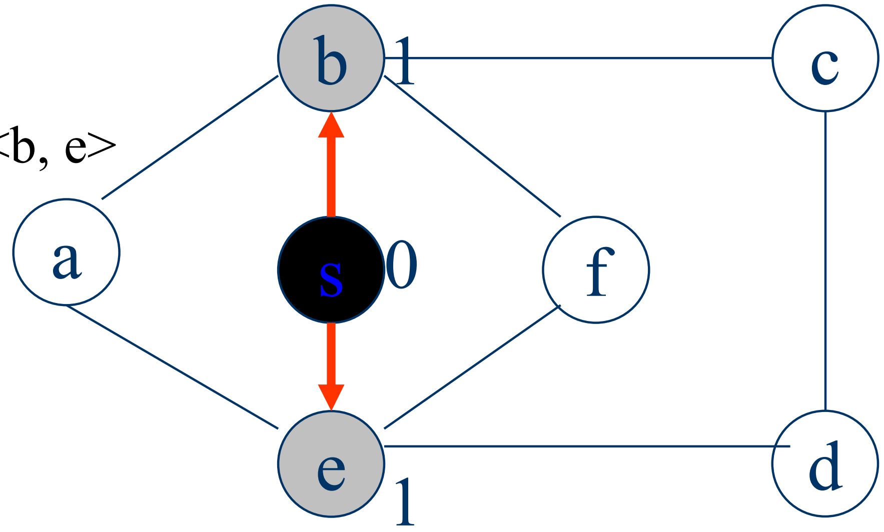
vertex u	s	a	b	c	d	e	f
color[u]	G	W	W	W	W	W	W
d[u]	0	∞	∞	∞	∞	∞	∞
π[u]	NIL	NIL	NIL	NIL	NIL	NIL	NIL

$Q = \langle s \rangle$
(put s into Q (discovered),
mark s gray (unprocessed))



Example(continued)

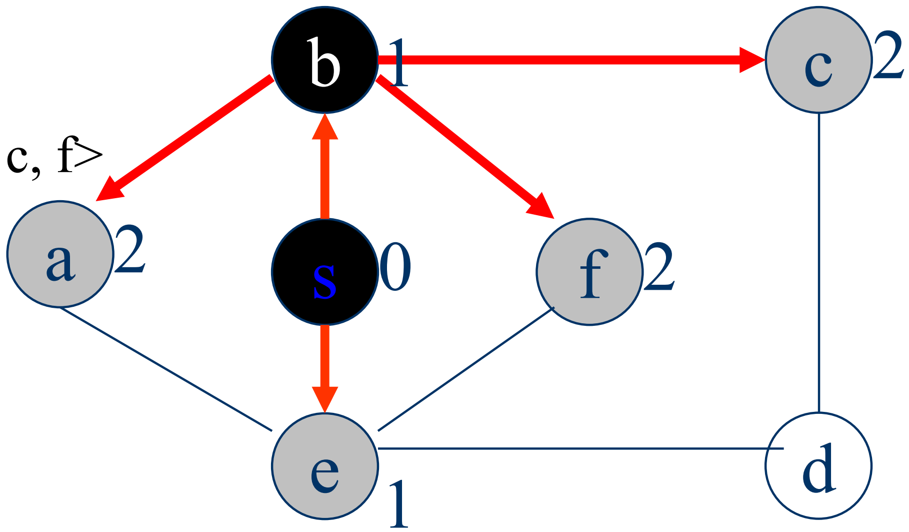
- While loop, first iteration
  - Dequeue s from Q. Find Adj[s]=<b, e>
  - Mark b,e as “G”
  - Update d[b], d[e], π[b], π[e]
  - Put b, e into Q
  - Mark s as “B”
- Q=<b, e>



vertex u	s	a	b	c	d	e	f
color[u]	B	W	G	W	W	G	W
d[u]	0	∞	1	∞	∞	1	∞
π[u]	NIL	NIL	s	NIL	NIL	s	NIL

Example(continued)

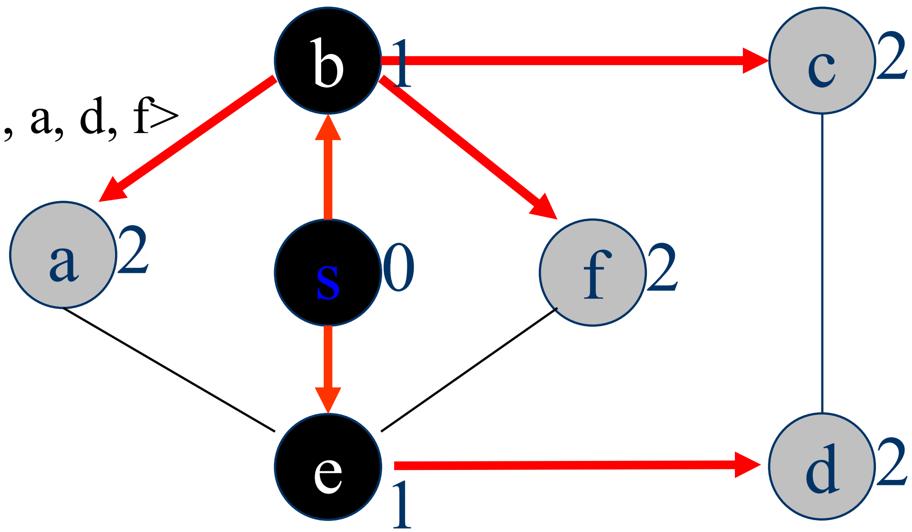
- While loop, second iteration
  - Dequeue b from Q, find Adj[b]=<s, a, c, f>
  - Mark a, c, f as “G”,
  - Update d[a], d[c], d[f], π[a], π[c],π[f]
  - Put a, c, f into Q
  - Mark b as “B”
- Q=<e, a, c, f>



vertex u	s	a	b	c	d	e	f
color[u]	B	G	B	G	W	G	G
d[u]	0	2	1	2	∞	1	2
π[u]	NIL	b	s	b	NIL	s	b

Example(continued)

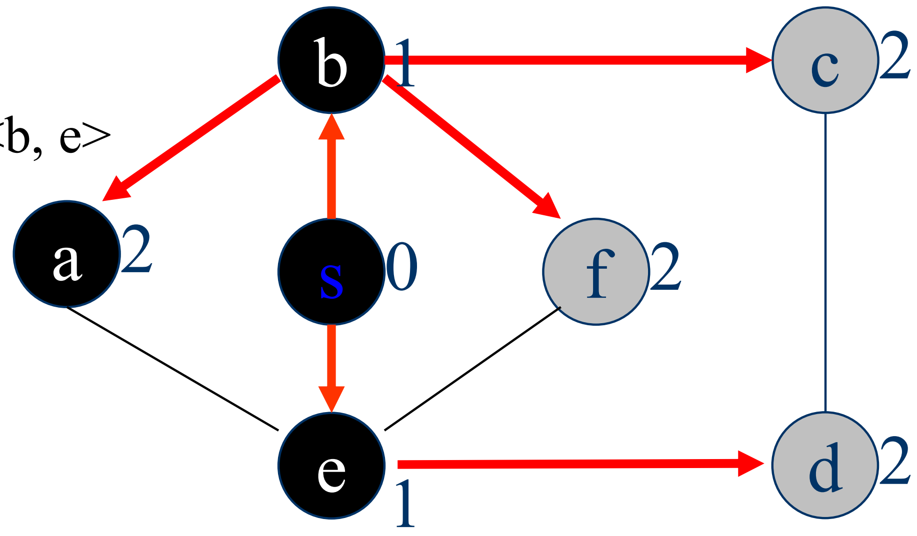
- While loop, third iteration
  - Dequeue e from Q, find Adj[e]=<s, a, d, f>
  - Mark d as “G”, mark e as “B”
  - Update d[d], π[d],
  - Put d into Q
- Q=<a,c,f,d>



vertex u	S	a	b	c	d	e	f
color[u]	B	G	B	G	G	B	G
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

Example(continued)

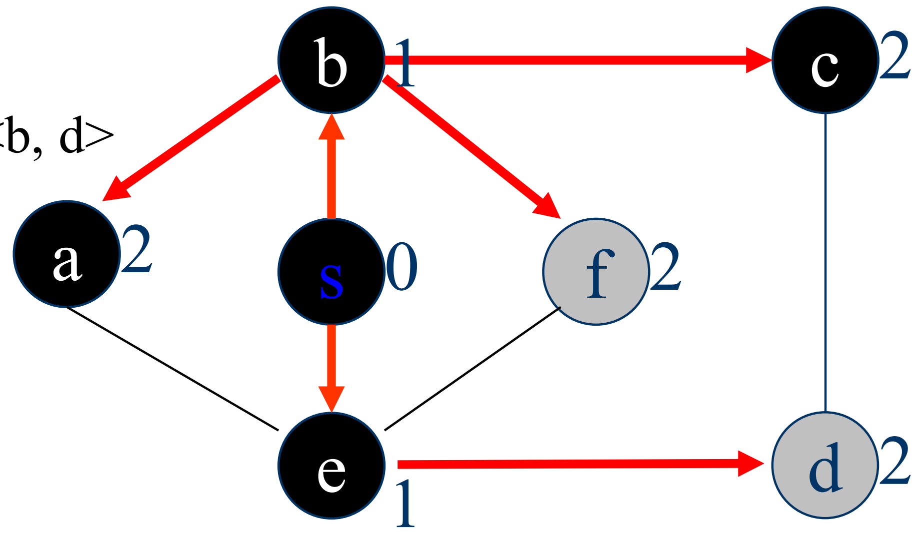
- While loop, fourth iteration
  - Dequeue a from Q, find Adj[a]=<b, e>
  - mark a as “B”
- Q=<c, f, d>



vertex u	s	a	b	c	d	e	f
color[u]	B	B	B	G	G	B	G
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

Example(continued)

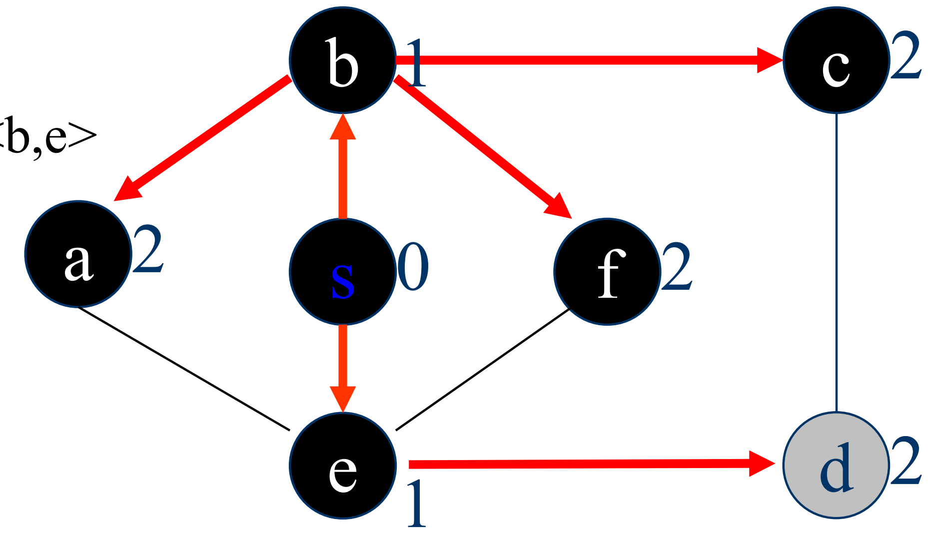
- While loop, fifth iteration
  - Dequeue c from Q, find Adj[c]=<b, d>
  - mark c as "B"
- Q=<f, d>



vertex u	s	a	b	c	d	e	f
color[u]	B	B	B	B	G	B	G
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

Example(continued)

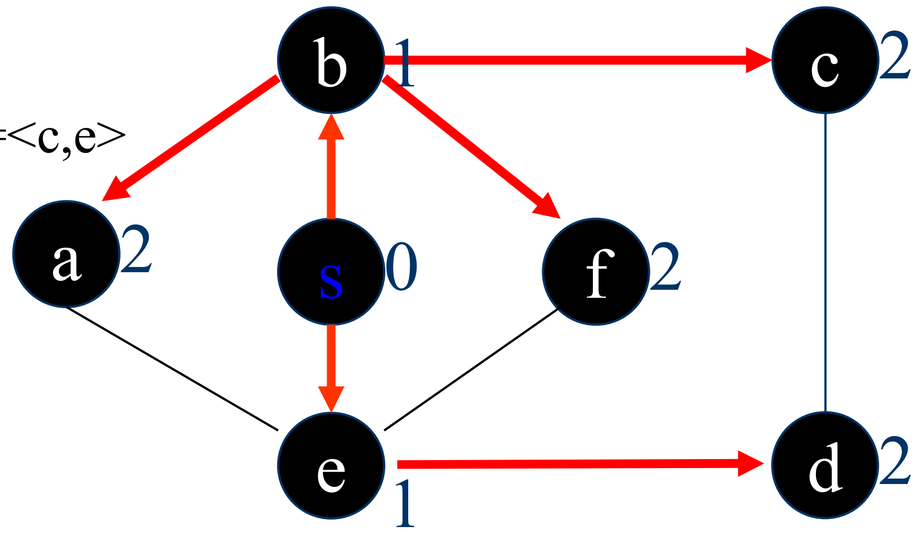
- While loop, sixth iteration
  - Dequeue f from Q, find Adj[f]=<b,e>
  - mark f as “B”
- Q=<d>



vertex u	s	a	b	c	d	e	f
color[u]	B	B	B	B	G	B	B
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

Example(continued)

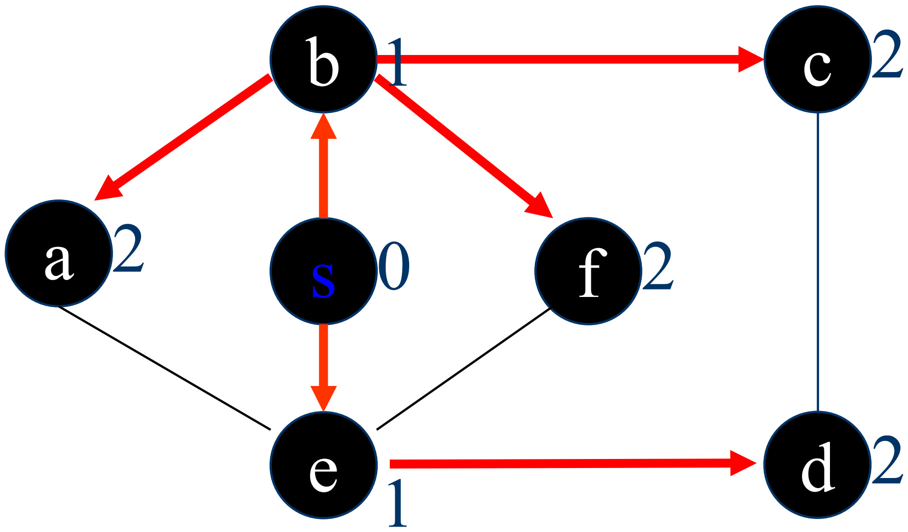
- While loop, seventh iteration
  - Dequeque d from Q, find Adj[d]=<c,e>
  - mark d as “B”
- Q is empty



vertex u	s	a	b	c	d	e	f
color[u]	B	B	B	B	B	B	B
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

Example(continued)

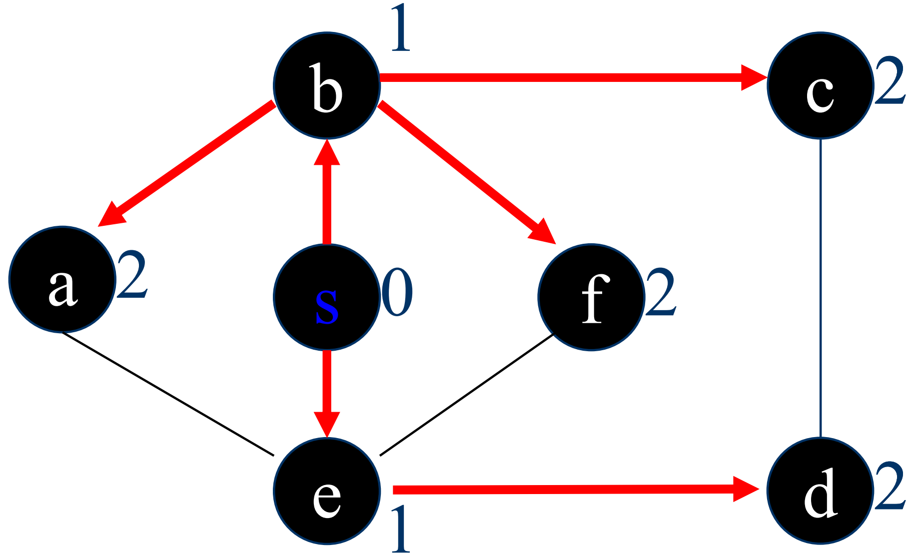
- While loop, eighth iteration
- Since Q is empty, stop



vertex u	s	a	b	c	d	e	f
color[u]	B	B	B	B	B	B	B
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

What does BFS produce

- BFS creates a prodecessor subgraph $G_\pi = (V_\pi, E_\pi)$, where
  - $V_\pi = \{u \in V \mid \pi[u] \neq \text{NIL}\} \cup \{s\}$
  - $E_\pi = \{(\pi[u], u) \mid$ where $\pi[\ ]$ is computed in the BFS(G, s) calls$\}$
  - prodecessor subgraph is a tree since it is connect and $|E_\pi| = |V_\pi| - 1$



The red edges form a tree //note there are |V|-1 edges

Our Goal

- Shortest-path distance $\delta(s, v)$:
  - the minimum number of edges in any path from vertex s to vertex v; if there is no path from s to v, then $\delta(s, v)=\infty$.
- Shortest path:
  - A path of length $\delta(s, v)$: $s \rightarrow v$.

Recall what does the BFS return?

- The BFS returns:
  - $d[v]$, proved to be the shortest distance from s to v
  - $\pi[v]$, the predecessor of v in the search, which can be used to derive a shortest path from s to vertex v.
- What we need to prove:
  - $\delta(s, v) = d[v]$
  - $s \rightarrow v = s \rightarrow \pi[v] + (\pi[v] , v)$

An Important Property

- **Lemma 22.1**: Let $G=(V, E)$ be a directed or undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for any edge $(u, v) \in E$, $\delta(s, v) \leq \delta(s, u)+1$.
- Proof:
  - If u is reachable from s, then so is v. In this case, the shortest path from s to v can not be longer than the shortest path from s to u followed by the edge (u, v) and thus the inequality holds.
  - If u is not reachable from s, then $\delta(s, u)=\infty$, and the inequality holds.

A Property of the BFS

- **Lemma 22.2:** Let $G=(V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then upon termination, for each vertex $v \in V$, the value $d[v]$ computed by BFS satisfies $d[v] \geq \delta(s, v)$.

Proof of Lemma 22.2

- Proof:
  - We use induction on the number of ENQUEUE operations. Our inductive hypothesis is that $d[v] \geq \delta(s, v)$ for all $v \in V$.
  - The basis of the induction is the situation immediately after enqueuing s in line 9 of BFS. The inductive hypothesis holds here, because $d[s] = 0 = \delta(s, s)$ and $d[v] = \infty \geq \delta(s, v)$, $v \in V - \{s\}$.
  - For the inductive step, consider a white vertex v that is discovered during the search from a vertex u.
  - We obtain

$$\begin{aligned} d[v] &= d[u] + 1 \text{(line 15)} \\ &\geq \delta(s, u) + 1 \text{(inductive hypothesis)} \\ &\geq \delta(s, v) \text{ (Lemma 22.1)} \end{aligned}$$

Proof of Lemma 22.2

- Vertex *v* is then enqueued, and it is never enqueued again because it is also grayed and the **then** clause of lines 14–17 is executed only for white vertices.
- Thus, the value of $d[v]$ never changes again, and the inductive hypothesis is maintained.

A property of the Queue

- **Lemma 22.3**
- Suppose that during the execution of BFS on a graph $G=(V, E)$, the queue Q=$<v_1, v_2, \ldots, v_r>$, where v_1 is the head of Q and v_r is the tail.
- Then $d[v_r] \leq d[v_1] + 1$ and $d[v_i] \leq d[v_{i+1}]$, for $1 \leq i \leq r-1$.

Proof of Lemma 22.3(1)

- The proof is by induction on the number of queue operations. Initially, when the queue contains only s, the lemma certainly holds.
- For the inductive step, we must prove that the lemma holds after both dequeuing and enqueuing a vertex. If the head v_1 of the queue is dequeued, v_2 becomes the new head. (If the queue becomes empty, then the lemma holds vacuously.)
- By the inductive hypothesis, $d[v_1] \leq d[v_2]$,

$d[v_r] \leq d[v_1] + 1 \leq d[v_2] + 1$, $d[v_i] \leq d[v_{i+1}]$, $2 \leq i \leq r\text{-}1$.

- Thus, the lemma follows with v_2 as the head.

Proof of Lemma 22.3(2)

- In order to understand what happens upon enqueuing a vertex, we need to examine the code more closely. When we enqueue a vertex v in line 17 of BFS, it becomes v_{r+1}. At that time, we have already removed vertex u, whose adjacency list is currently being scanned, from the queue Q.
- By the inductive hypothesis, the new head v_1 has $d[v_1] \geq d[u]$. Thus, $d[v_{r+1}]= d[v]=d[u]+1\leq d[v_1]+1$.
- From the inductive hypothesis, we also have $d[v_r] \leq d[u]+1$,

  or from line 15 we have $d[v_r] = d[u]+1$,

  and so $d[v_r] \leq d[u]+1 = d[v]=d[v_{r+1}]$.
- The remaining inequalities are unaffected. Thus, the lemma follows when v is enqueued.

Proof of Lemma 22.3

- Proof:
  - 对入队和出队操作次数进行归纳证明。我们证明入队或出队操作之后，引理总成立。
  - 当Q=<s>时, 引理中的性质成立。
  - v_1出队，v_2成为队首。则依据归纳假设

    $d[v_1] \le d[v_2]$, $d[v_r] \le d[v_1]+1 \le d[v_2]+1$, $d[v_i] \le d[v_{i+1}]$, $2 \le i \le r-1$。
  - v入队，成为队尾v_{r+1}。之前存在u出队，此时算法正在搜索u的邻接链表, 发现白色顶点v。

    $d[v_1] \ge d[u]$，$d[v_r] \le d[u]+1$ （归纳假设）

    $d[v_{r+1}] = d[v] = d[u]+1$ （算法第15行）

    $d[v_{r+1}] = d[u]+1 \le d[v_1]+1$; $d[v_r] \le d[u]+1 = d[v_{r+1}]$。

A Corollary（推论）

- **Corollary 22.4:**
- Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j.
- Then $v_i.d \leq v_j.d$ at the time that v_j is enqueued.
- **Proof:**
- Immediate from Lemma 22.3 and the property that each vertex receives a finite d value at most once during the course of BFS.

A Corollary（推论）

- 推论22.4: v_i先于v_k入队，则 $d[v_i] \le d[v_k]$。
- 先入队的距离小。
- 一旦入队，$d[]$值就不发生变化。

Correctness of BFS

- 定理22.5： $\delta(s, v)=d[v]$

  $s \to v = s \to \pi[v] + (\pi[v] , v)$

- 证明： （part 1）
  - 反证法。假设存在一些顶点，它们的$d[]$值不等于最短路径距离。设v是这些顶点中，最短路径距离最小的。（$v \neq s$）
  - $d[v] \geq \delta(s, v)$ （引理22.2），因此$d[v] > \delta(s, v)$ 。v和s之间 必然存在一条最短路径 $s \to v$ ，否则 $\delta(s, v) =\infty \geq d[v]$ 。设u是这条最短路径上v的前驱，即$s \to u — v$。因此, $\delta(s, v) = \delta(s, u) +1$。这意味着$\delta(s, u) < \delta(s, v)$, 根据v的选择方式， $d[u] =\delta(s, u)$

    $d[v] > \delta(s, v)= \delta(s, u) +1= d[u] +1$。
  - 接下来分析当u出队时，v可能的三种颜色：W，G，B。

    我们证明每一种情况下,上式都不可能成立。

Correctness of BFS

- 定理22.5：$\delta(s, v) = d[v]$

  $$s \to v = s \to \pi[v] + (\pi[v], v)$$

- 证明： （part 2）
  - v是白色的, $d[v] = d[u] + 1$。(算法15行)
  - v是黑色的, v已经出队, $d[v] \le d[u]$ 。(推论22.4)
  - v是灰色的, v在顶点w的邻接链表里, w先于u出队.

    $d[v] = d[w]+1$, $d[w] \le d[u]$, 从而 $d[v] \le d[u] + 1$。
  - 综上，假设的v是不可能存在的。因此$d[v] = \delta(s, v)$.
  - 所有s不可达的顶点的$d[]$值都是∞。

  （part 3）
  - 如果$\pi[v]=u$，则$d[v] = d[u] + 1$，$s \to u + (u, v)$ 是s到v的最短路径。

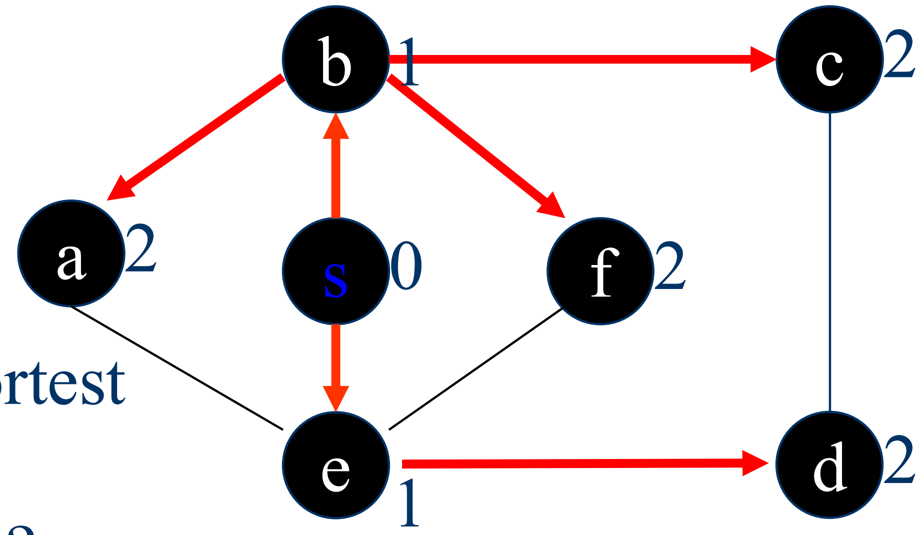
Predecessor subgraph

- BFS creates a predecessor subgraph $G_\pi = (V_\pi, E_\pi)$, where
  - $V_\pi = \{u \in V | \pi[u] \neq \text{NIL}\} \cup \{s\}$
  - $E_\pi = \{(\pi[u], u) |$ where $\pi[\ ]$ is computed in the BFS(G, s) calls$\}$
- The predecessor subgraph $G_\pi = (V_\pi, E_\pi)$ is a shortest path tree if
  - V_π consists of vertices reachable from s。
  - For all $v \in V_\pi$, there is a unique simple path form s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called tree edges (树边)。

Breadth-first Tree

- 引理22.6: BFS得到前驱图$G_\pi = (V_\pi, E_\pi)$是shortest path tree。
- 证明：
  - 算法16行$\pi[v]=u$ 当且仅当 $(u, v) \in E$且$\delta(s, v)<\infty$（意味着v是从s可达的）。因此V_π中包含了所有从s可以到达的顶点。G_π是树，每个顶点到s都有唯一的一条路径。根据定理22.5，$s \to \pi[v] +(\pi[v] , v)$ 是s到v的最短路径。从而树中每个顶点到s的路径都是最短路径。
  - 如何归纳的？
  - $\delta(s, v) = d[v] = d[\pi[v]] +1= \delta(s, \pi[v]) +1$，这意味着，$s \to \pi[v] +(\pi[v] , v)$ 是s到v的最短路径。

Example of Breadth-first Tree



Question:
How do you construct a shortest path from s to any vertex by using the following table?

vertex u	s	a	b	c	d	e	f
color[u]	B	B	B	B	B	B	B
d[u]	0	2	1	2	2	1	2
π[u]	NIL	b	s	b	e	s	b

The Answer

```
PRINT-PATH(G, s, v)
if v = s
    then print s
    else  if π[v] = NIL
              then print "no path from" s "to" v "exists"
              else  PRINT-PATH(G, s, π[v])
                    print v
```

BFS for disconnected graph

- We can modify BFS so that it returns a forest.
- More specifically, if the original graph is composed of connected components C_1, C_2, …, C_k, then BFS will return a tree corresponding to each C_i.

BFS for disconnected graph

```
BFS_GENERAL(G)
for each vertex u ∈ V
    color[u] ← WHITE;
    d[u] ← ∞;
    π[u] ← NIL;
for each vertex u ∈ V
    if d[u] = ∞
        then BFS(G, u);
```

Analysis of the Breadth-First Search Algorithm

- We assume that it takes one unit time to test the color of a vertex, or to update the color of a vertex, or to compute $d[v] = d[u] + 1$, or to set $\pi[v] = u$, or to en-queue, or to de-queue.
- The following analysis is valid for connected graphs.
  - The initialization requires $O(V)$ time units.
  - In the while loop, each vertex u is en-queued and de-queued **exactly once**. So, each adjacency list Adj[u] is scanned **exactly once** after u is de-queued. So scanning the adjacency lists needs time $O(E)$.
- Total time is: $T(V, E) = O(\mathrm{V}+E)$.

Conclusion

- Content
  - BFS： algorithm， proof of its correctness， cases， analysis of its time complexity
- Homework
  - 22.2-2 and 22.2-7