组合数学

Discrete　Mathematics

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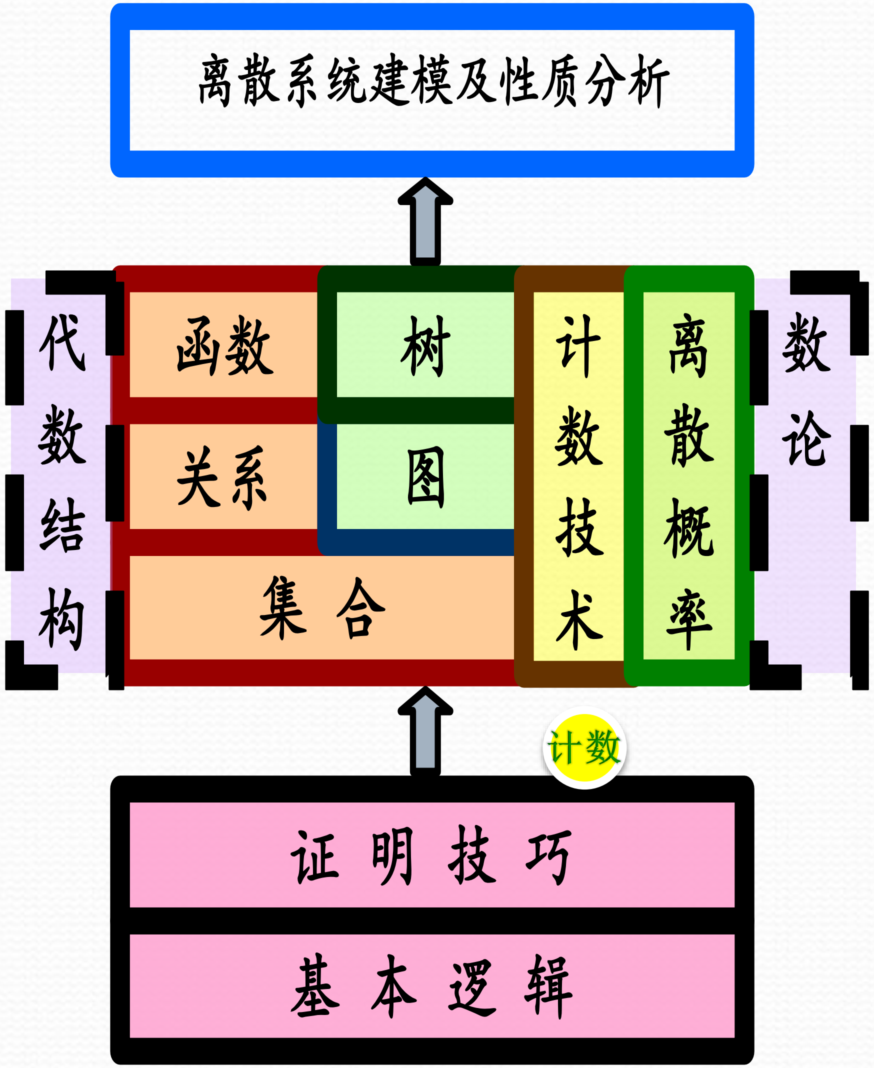
离散数学的知识图谱



应用

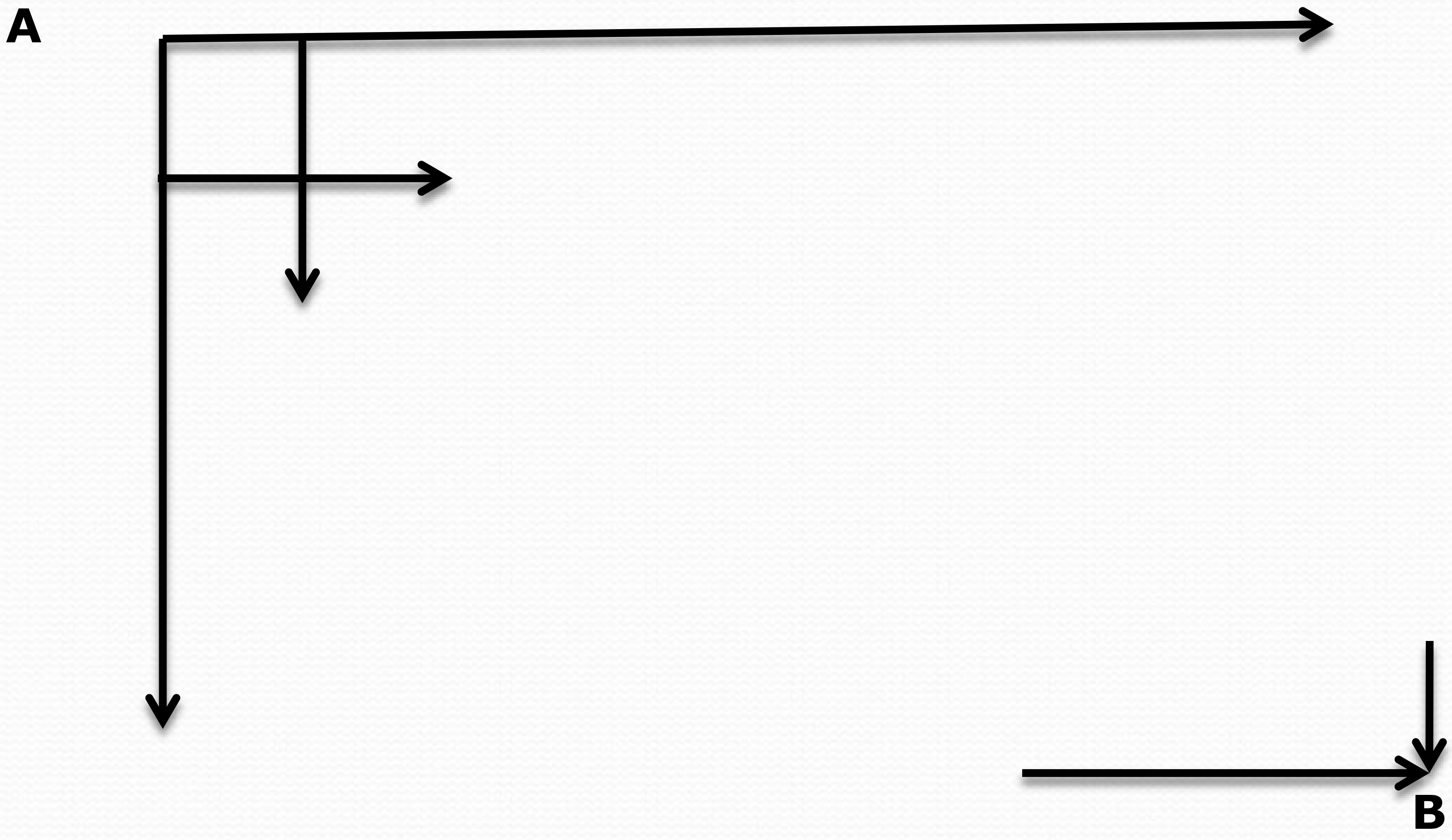
概念

工具

方法

基础

计数问题？



计数问题？

十进制数串中有偶数个**0**的数串个数。

。。。

关于计数问题，你能想到的什么？

排列、组合

容斥原理，鸽巢原理

递推关系，生成函数

Ramsey数(拉姆齐数），Stirling数（斯特林数）

组合计数

组合设计，组合优化，组合矩阵等

Chapter 13

Counting
计数

§ 13.1 The basics of counting (1)

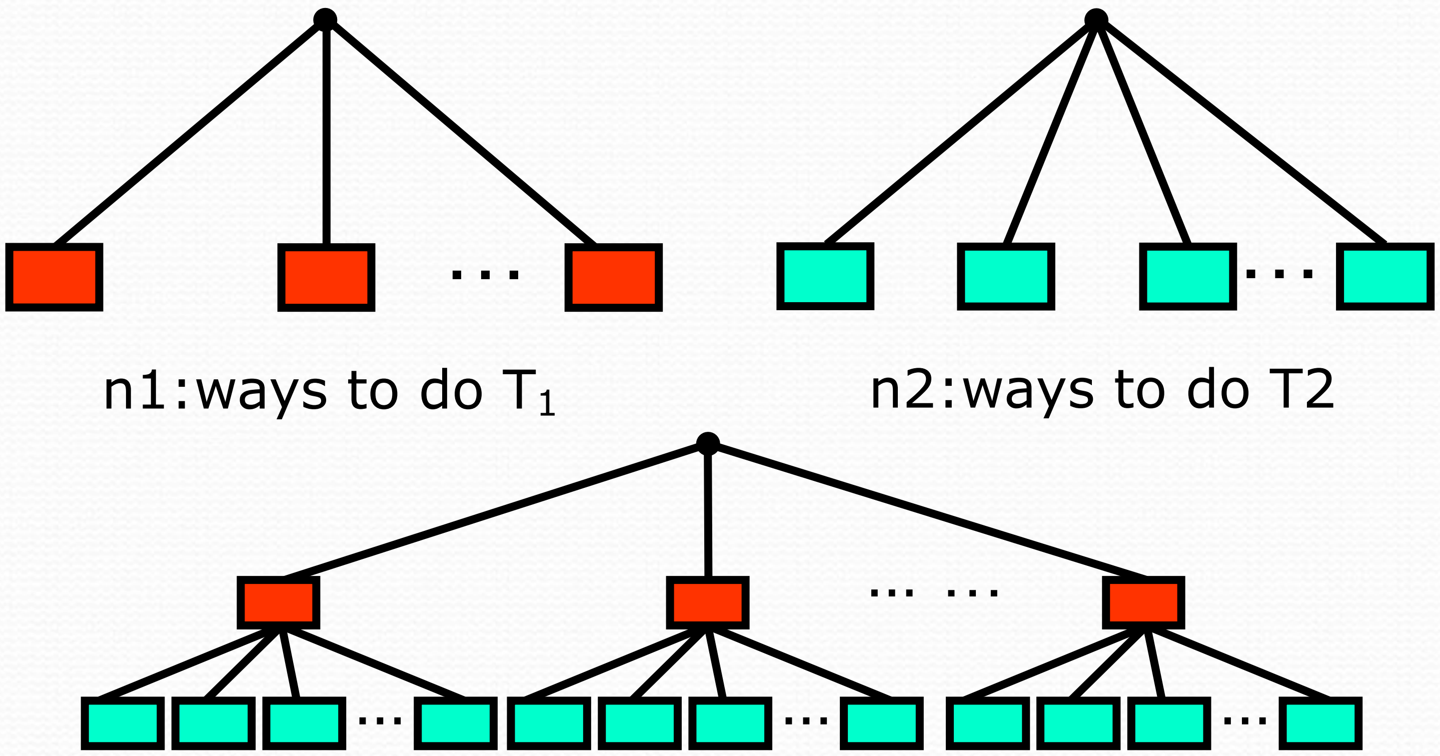
13.1.1 Basic counting principles

（1）The product rule 乘法规则

Definition:

Suppose that a procedure can be broken down into a sequence of two tasks.If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done,then there are $n_1 \times n_2$ ways to do the procedure.

§ 13.1 The basics of counting (2)



n1*n2: ways to do the procedure

Basic Counting Principles: The Product Rule

Example: How many bit strings of length seven are there? 7位二进制有多少不同的数？

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

§13.1 The basics of counting

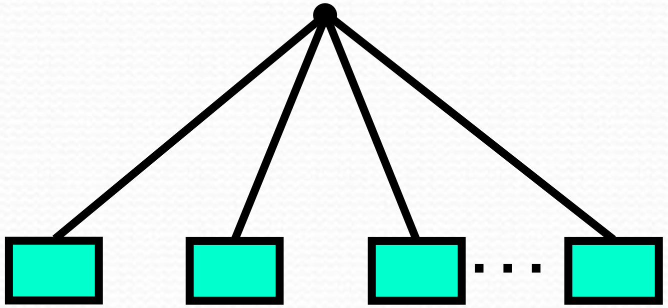
13.1.1 Basic counting principles

（2）The sum rule 加法规则

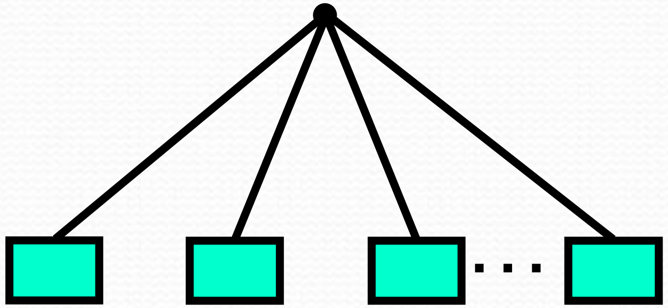
Definition：

If a first task can be done in n_1 ways and a second task in n_2 ways,and if these tasks cannot be done at the same time, then there are n_1+n_2 ways to do one of these tasks.

§ 13.1 The basics of counting



n1:ways to do T1



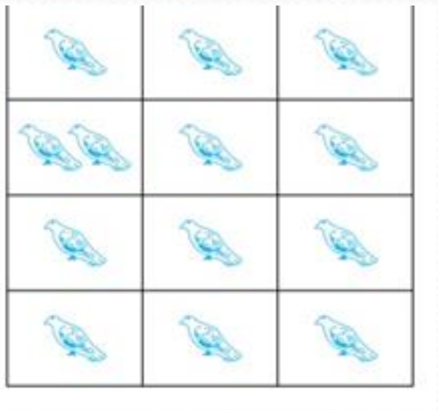
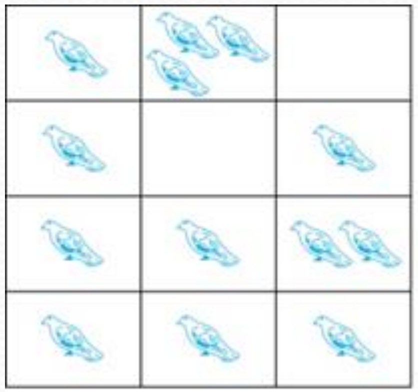
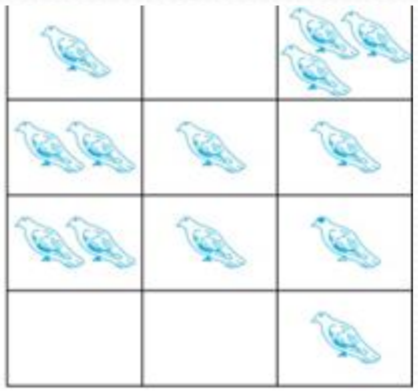
n2:ways to do T2

n1+n2: ways to do these tasks

§ 13.2 The Pigeonhole principle
（鸽巢原理）

13.2.1 Introduction

- If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.
- 13只鸽子要入住12个鸽巢，每个鸽子要有一个鸽巢。

The Pigeonhole Principle

Pigeonhole Principle（鸽巢原理）: If k is a positive integer and k + 1 objects are placed into *k* boxes, then at least one box contains two or more objects.

把K+1个物体放入K个箱子中，至少有一个箱子中至少有两个物体.

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

§13.2 The Pigeonhole principle

例:假如6人中任意2人都认识,或任意2人都不认识.证明在这6人组中或者3人彼此认识,或者3人互不认识.

解:设A是6人中的其中一人.则其他5人与A认识的组成一组,与A不认识的组成一组

根据鸽巢原理其中一组中至少有3人

假如该3人组是与A认识的,如3人中有2人认识

则该2人与A组成3人认识组;如3人中没有2人认识,即3人彼此互不认识.

§13.2 The Pigeonhole principle

假如该3人组是与A不认识的,如3人中有2人不认识,则该2人与A组成3人不认识组;如3人中没有2人不认识,即3人彼此互相认识.

§ 13.2 The Pigeonhole principle

13.2.2 The generalized Pigeonhole principle

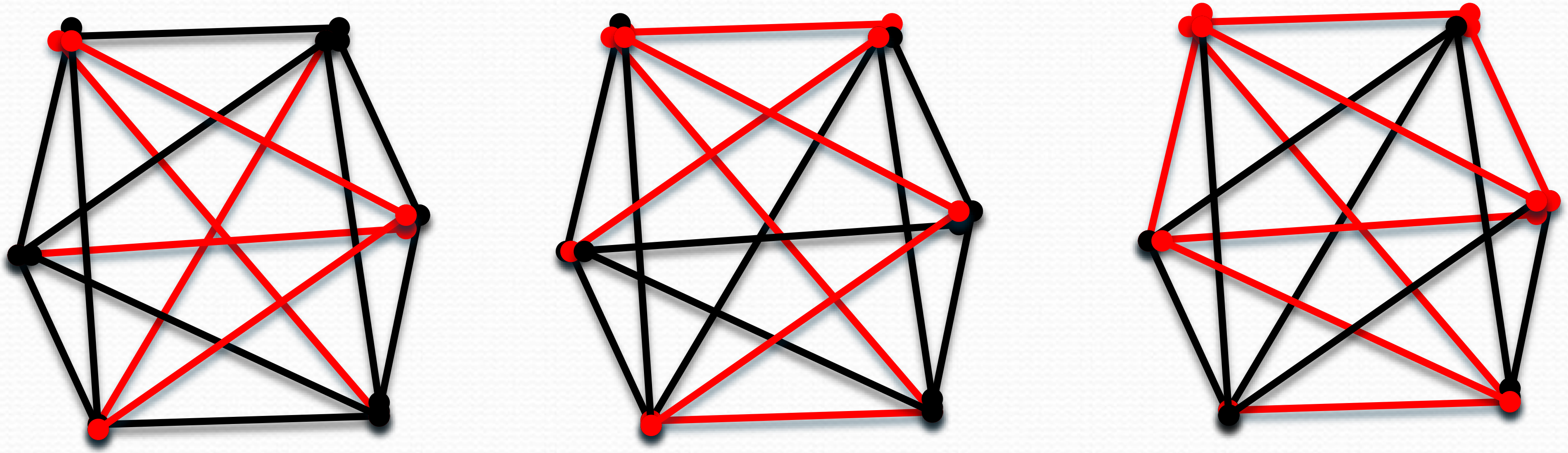
定义：假设p, q为正整数， p, q≧2 , 则存在最小正整数R(p, q), 使得n≧R(p, q)时, 或者有p人是彼此相识, 或者有q人彼此不相识， 称R(p, q)为Ramsey数(拉姆齐数)。

Ramsey Numbers 拉姆齐数

q / p	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25	35 41	49 61	56 84	69 115	80 149	96 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	95 216	121 316	141 442	153	181	193	221	242
6				102 165	111 298	127 495	153 780	177 1171	253	262	278	292	374
7					205 540	216 1031	7 1713	7 2826	322	416	511		
8						282 1870	8 3583	316 6090			635		703
9							565 6588	580 12677					
10								798 23556					

$$R(p, q) = R(q, p)$$

拉姆齐数的应用图中边着色



k6完全图,对他的边用红，黑两种颜色任意涂色，则必存在同色边的三角形。

§ 13.3 Permutations and Combinations (1)

13.13.1 Permutations 排列

（1）r-permutation

An ordered arrangement of r elements of a set is called an r-permutation.The number of r-permutations of a set with n elements is denoted by P(n,r).

$$P(n,r)=\begin{cases}1 & n\ge r=0\\ 0 & n<r\end{cases}$$

§ 13.3 Permutations and Combinations (2)

13.13.1 Permutations

（1） r-permutation

Theorem:

The number of r-permutations of a set with n distinct elements is

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1).$$

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

§ 13.3 Permutations and Combinations (2)

13.13.1 Permutations

（1） r-permutation

$$P(n,r)=n(n-1)(n-2)\ldots(n-r+1)=\frac{n!}{(n-r)!}$$

$$P(n,n)=n(n-1)(n-2)\ldots 2\cdot 1=n!$$

§ 13.3 Permutations and Combinations (5)

13.13.1 Permutations

（2）r-permutation with repetition 允许重复选取的排列

Theorem：

The number of r-permutations of a set of n distinct objects with repetition allowed is n^r.

13.13.2 Combinations 组合

（1）r-combination

An r-combination of elements of a set is an unordered selection of r elements from the set.Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combination of a set with n distinct elements is denoted by $C(n,r)$.

13.13.2 Combinations

（1） r-combination

Note that C(n,r) is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

$$C(n,r) = \begin{cases} 1 & n \geq r = 0 \\ 0 & n < r \end{cases}$$

§ 13.3 Permutations and Combinations

13.13.2 Combinations

（1） r-combination

Theorem：

The number of r-combinations of a set with n elements,where n is a nonnegative integer and r is an integer with $0 \le r \le n$,equals

$$C(n,r)=\binom{n}{r}=\frac{P(n,r)}{r!}=\frac{n!}{r!(n-r)!}$$

§13.3 Permutations and Combinations (

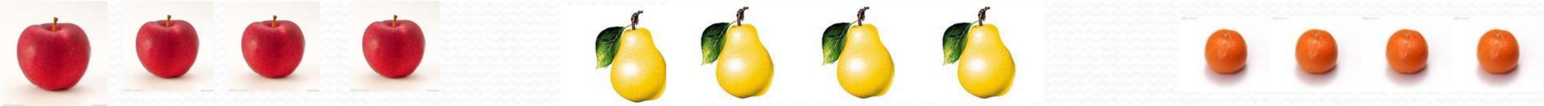
13.13.2 Combinations

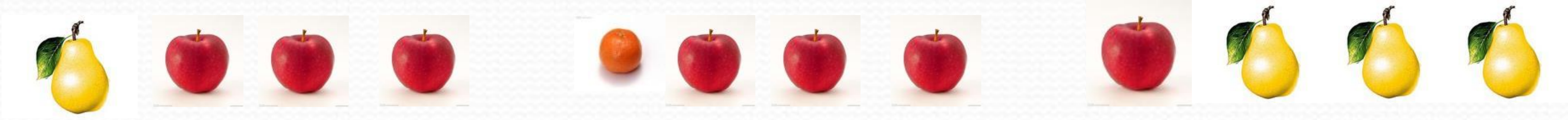
（2）Combinations with repetition 允许重复的组合

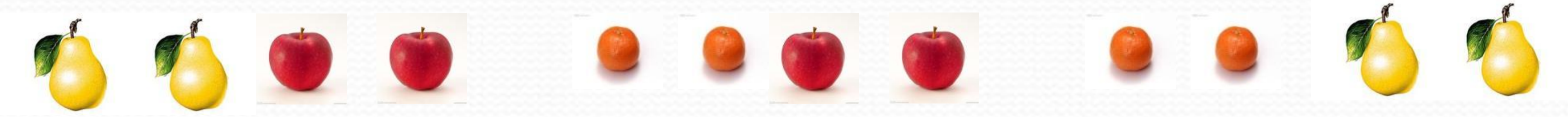
Theorem1：

There are **C(n+r-1,r)** r-combinations from a set with n elements when repetition of elements is allowed.



class A 

class B 

class C 

class D 

15=3+6+3+3 solutions

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

n个元素（类物体）允许重复的r组合。

XXXX|XX|X **XX|XXX|XX**

XXXXXX||X ？ ？ ？ ？ ？ ？ ？ ？ ？

3类物体允许重复的7组合。

$C(n + r - 1, r)$ 选定物体 $C(3 + 7 - 1, 7)$

$C(n + r - 1, n - 1)$ 选定“隔板” $C(3 + 7 - 1, 3 - 1)$

§ 13.4 Binomial coefficients (1)

13.4.1 The binomial theorem

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$
$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

§ 13.4 Binomial coefficients (2)

13.4.1 The binomial theorem

The binomial theorem: Let x and y be variables,and let n be a nonnegative integer.Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

§ 13.4 Binomial coefficients (3)

13.4.1 The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k$$

C(n,r) = C(n,n-r).

§ 13.4 Binomial coefficients (4)

13.4.1 The binomial theorem

if y=1 then

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k}$$

$$= \sum_{k=0}^{n} \binom{n}{n-k} x^{k}$$

§ 13.4 Binomial coefficients (6)

13.4.2 Pascal's identity and triangle

Pascal's identity： Let n and k be positive integers with $n \geq k$ .Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Show： based on SET and SUBSET

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1\quad 1$$
$$1\quad 2\quad 1$$
$$1\quad 3\quad 3\quad 1$$
$$1\quad 4\quad 6\quad 4\quad 1$$
$$1\quad 5\quad 10\quad 10\quad 5\quad 1$$
$$1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1$$
$$1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1$$
$$1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1$$
$$\cdots$$

(b)

§13.5 Generalized permutation and combinations (1)

13.5.1 combinations with repetition

1、 n个无区别的小球放入m个有区别的箱子里的方案数?

相当于从m类物体中允许重复的选取n个物体的方案数:

C(m+n-1,n)、 C(m+n-1,m-1)

2、 生活中的什么情况下可以使用该模型。选男女代表等

3、 $(x+y+z)^4$展开式有多少项?

C(3+4-1,4), C(3+4-1,3-1)

那$(x+y+z)^n$ 展开式有多少项?

C(3+n-1,3-1)

4、$x_1+x_2+x_3=11(n)$ 其中x1≥0,x2≥0,x3≥0 正整数解的个数?

§ 13.5 Generalized permutation and combinations (2)

13.5.2 Distributing objects into boxes

n个球	m个盒子	是否允许空盒	计数方案	备注
有区别	有区别	允许空盒	m^n	全排列
无区别	有区别	允许空盒	C(m+n-1,n)	m个有区别的元素,取n个作允许重复的组合
无区别	有区别	不允许空盒	C(n-1,m-1)	(1)选取m个球每盒一个 (2)n-m有区别的球放入m个有区别盒子中,允许某盒不放 C(n-m+m-1,n-m)=C(n-1,m-1)
无区别	无区别	允许空盒		一本书的6本复印件放入4个相同的箱子中
无区别	无区别	不允许空盒		n-m个无区别物体允许为空的放入无区分m盒子

§13.5 Generalized permutation and combinations (2)

13.5.2 Distributing objects into boxes

n个球	m个盒子	是否允许空盒	计数方案	备注
有区别	有区别	不允许空盒		映上函数的个数
有区别	无区别	允许空盒	(集合的划分)	4人分配完全相同的3间办公室
有区别	无区别	不允许空盒		Stirling数