On the Convergence Rate of Laplacian spectra from Point Clouds

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Abstract

The spectral structure of the Laplacian-Beltrami operator on manifold has been widely used in many applications, include spectral clustering, dimension reduction, mesh smoothing, compression, editing, shape segmentation, matching, parameterization, and so on. Typically, the underlying Riemannian manifold is unknown and often given by a set of sample points. The spectral structure of the Laplacian is estimated from some discrete Laplace operator constructed from this set of sample points, such as the weighted graph Laplacian. Then one fundmental issue involved in this kind of approach is the convergence of the eigen system of the discrete Laplacian to that of the continuous Laplacian. In this paper, for the compact manifold isometrically embedded in a Euclidean space with boundary, in the limit of the density of sample points tends to infinity, we give an estimate of the convergence rate of the eigenvalues and eigenvectors of the weighted graph Laplacian converges to the eigenvalues and eigenfunctions of the Laplace-Beltrami operator with the Neumann boundary. This result gives a solid mathematical foundation for the weighted graph Laplacian.

1 Introduction

The Laplace-Beltrami operator (LBO) is a fundamental object associated to Riemannian manifolds, which encodes all intrinsic geometry of the manifolds and has many desirable properties. It is also related to diffusion and heat equation on the manifold, and is connected to a large body of classical mathematics (see, e.g., [16]). In recent years, the Laplace-Beltrami operator has attracted much attention in many applied fields. The spectral structure of the Laplacian-Beltrami operator on manifold has been widely used in many applications, include spectral clustering, dimension reduction, mesh smoothing, compression, editing, shape segmentation, matching, parameterization, and so on [3, 9, 15, 14].

In this paper, we consider the following eigenproblem on a smooth, compact k-dimensional submanifold $\mathcal{M}$ with the Neumann boundary condition

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) = \lambda u(x), & \mathbf{x} \in \mathcal{M} \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M} \end{cases} \tag{1.1}$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$. Let g be the Riemannian metric tensor of $\mathcal{M}$, which is assumed to be inherited from the ambient space $\mathbb{R}^d$, that is, $\mathcal{M}$ isometrically embedded in $\mathbb{R}^d$ with the standard Euclidean metric. If $\mathcal{M}$ is an open set in $\mathbb{R}^d$, then $\Delta_{\mathcal{M}}$ becomes standard Laplace operator, i.e., $\Delta_{\mathcal{M}} = \sum_{i=1}^d \frac{\partial^2}{\partial {x^i}^2}$.

In [13], we have proposed a numerical method called the point integral method (PIM) to solve the above eigenproblem on manifolds. In the PIM, we only need a point cloud to discretize the manifold $\mathcal{M}$. In particular, we are given a set of points $P = (\mathbf{p}_1, \cdots, \mathbf{p}_n)$ sampling $\mathcal{M}$. In addition, we are given a vector $\mathbf{V} = (V_1, \cdots, V_n)^t$ where V_i is an volume weight of $\mathbf{p}_i$ in $\mathcal{M}$, so that for any Lipschitz function f on $\mathcal{M}$, $\int_{\mathcal{M}} f(\mathbf{x}) d\mu_{\mathbf{x}}$ can be approximated by $\sum_{i=1}^n f(\mathbf{p}_i) V_i$. Here $d\mu_{\mathbf{x}}$ is the volume form of $\mathcal{M}$.

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We consider the following discrete Laplace operator $L_{t,h}$

$$L_{t,h}u(\mathbf{p}_i) = \frac{1}{t}\sum_{\mathbf{p}_j\in P} R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right)(u(\mathbf{p}_i)-u(\mathbf{p}_j))V_j. \tag{1.2}$$

where $R:\mathbb{R}^+\to\mathbb{R}^+$ is a given kernel function. If set $V_j=\frac{1}{n}$, the discrete Laplace operator $L_{t,h}$ becomes the well-known weighted graph Laplacian [2]. Denote $\bar{R}$ the primitive function of $-R$, i.e. $\frac{d}{dr}\bar{R}(r)=-R(r)$. Now we consider the following generalized eigenproblem of $L_{t,h}$

$$-L_{t,h}u(\mathbf{p}_i) = \lambda\sum_{p_j\in P}\bar{R}\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right)u(\mathbf{p}_j)V_j \tag{1.3}$$

The purpose of the paper is to show the generalized eigenproblem (1.3) converges to the eigenproblem (1.1) at a rate $(t^{1/2}+\frac{h}{t^{k/4+3}})$ for the eigenvalues and $(t^{1/2}+\frac{h}{t^{k/4+2}})$ for the eigenfunctions, provided the input data $(P,\mathbf{V})$ is an h-integral approximation of $\mathcal{M}$ (see Section 2.2 for its definition). Note the rate reported in this paper depends on the dimension k of $\mathcal{M}$ and may not be optimal.

Following [], we bridge the LBO $\Delta_{\mathcal{M}}$ and the discrete Lapalce operator $L_{t,h}$ using the following integral Laplace operator

$$L_t u(\mathbf{x}) = \frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}. \tag{1.4}$$

We consider the solution operators associated with $\Delta_{\mathcal{M}}$, L_t and $L_{t,h}$, and show all solution operators are in fact compact and additionally the approximation errors of the solution operators in certain operator norms are bounded (see Theorem 4.3 and 4.4). This enables us to apply the results from spectral theory to obtain a convergence rate. Note that it is critical and also common in the numerical analysis to consider the solution operators [6], instead of the Laplacians themselves which are not even bounded. Comparing to [], we improve the error estimations for both the trancation error and the stability of L_t. In particular, in both Theorem 5.1 and 5.3, we bound the approximation errors using the Sobolev norms, instead of the infinite norm as in [].

1.1 Related work

The finite element method is popular in solving PDEs on manifolds. Dziuk [11] showed the FEM converges quadratically in L^2 norm and linearly in H^1 norm in solving the Poisson equations. The eigensystem of the LBO computed by the FEM converges linearly [19, 10, 20]. The FEM requires a mesh tesselating the domain and its performace depends heavily on the quality of meshes. However, it is not an easy task to generate a mesh of good quality, especially for an irregular domain [8], and becomes much more difficult for a curved submanifold [7].

We see that the discrete Laplace operator $L_{t,h}$ becomes the weighted graph Laplacian when the volume weight vector is constant. In the presence of no boundary, Belkin and Niyogi [4] showed that the spectra of the weighted graph Laplacian converges to the spectra of $\Delta_{\mathcal{M}}$. When there is a boundary, it was observed in [12, 5] that the integral Laplace operator L_t is dominated by the first order derivative and thus fails to be true Laplacian near the boundary. Recently, Singer and Wu [18] showed the spectral convergence in the presence of the Neumann boundary. In the previous approaches, the convergence analysis is based on the connection between the weighted graph Laplacian and the heat operator, and thus it is essential to use the Gaussian kernel in those approaches. The convergence analysis in this paper is very different from the previous ones. We consider this convergence problem from the point of view of solving the Poisson equation on submanifolds, which opens up many tools in the numerical analysis for studying the graph Laplacian.

The rest of the paper is organized as follows. In Section 2, we describe the basic assumptions and define the solutions operators we are working on. The main results are stated in Section 3. In Section 4, the proofs of the main results are given. The proofs depends on two theorems: one states that T_t converges to T in H^1 norm and the other states that $T_{t,h}$ converges to T_t in C^1 norm, whose proofs are given in Section 5 and 6 respectively. Finally, in Section 7, we conclude and point out a few direction for future research.

2 Assumptions and Notations

We follow the assumptions in [17] and use basically the same notations.

2.1 Assumptions

First we assume the function $R : \mathbb{R}^+ \to \mathbb{R}^+$ is C^2 and satisfies the following conditions:

(i) $R(r) = 0$ for $\forall r > 1$.

(ii) There exists a constant δ_0 so that $R(r) > \delta_0$ for $\forall r < \frac{1}{2}$.

Second, we assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are compact and C^∞ smooth. Consequently, it is well known that both $\mathcal{M}$ and $\partial\mathcal{M}$ have positive reaches.

Finally, we assume the input data $(P, \mathbf{V})$ is **h-integral approximation** of $\mathcal{M}$, i.e.,

For any function $f \in C^1(\mathcal{M})$, there is a constant C independent of h and f so that

$$\left| \int_{\mathcal{M}} f(\mathbf{y}) d\mu_{\mathbf{y}} - \sum_{i=1}^{n} f(\mathbf{p}_i) V_i \right| < Ch |\mathrm{supp}(f)| \|f\|_{C^1}, \quad \text{and} \tag{2.1}$$

where $\|f\|_{C^1} = \|f\|_\infty + \|\nabla f\|_\infty$ and $|X|$ denotes the volume of X.

Remark 2.1. *If the points in P are independent samples from uniform distribution on $\mathcal{M}$, then $\mathbf{V}$ can be taken as the constant vector. The integral of the functions on $\mathcal{M}$ can be estimated using Monte Carol method. In this case, from the central limit theorem, h is of the order of $1/\sqrt{n}$.*

2.2 Notations

To investigate the convergence of the problem 1.3 to the problem 1.1 of the Neumann boundary, we consider the solution operators T^N, T_t^N and $T_{t,h}^N$ as follows. Denote the operator $T^N : L^2(\mathcal{M}) \to H^2(\mathcal{M})$ to be the solution operator of the following problem

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M}. \end{cases} \tag{2.2}$$

where $\mathbf{n}$ is the out normal vector of $\mathcal{M}$. Namely, for any $f \in L^2(\mathcal{M})$, $u = T^N(f)$ with $\int_{\mathcal{M}} u = 0$ solves the problem (2.2).

Denote $T_t^N : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ to be the solution operator of the following problem

$$-\frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) \mathrm{d}\mathbf{y} = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{2.3}$$

Namely, $u = T_t^N(f)$ with $\int_{\mathcal{M}} u = 0$ solves the problem (2.3);

The last solution operator is $T_{t,h}^N : C(\mathcal{M}) \to C(\mathcal{M})$ which defined as follows.

$$T_{t,h}^N(f)(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \sum_j R_t(\mathbf{x}, \mathbf{x}_j) u_j V_j - \frac{t}{w_{t,h}(\mathbf{x})} \sum_j \bar{R}_t(\mathbf{x}, \mathbf{x}_j) f(\mathbf{x}_j) V_j \tag{2.4}$$

where $w_{t,h}(\mathbf{x}) = \sum_j R_t(\mathbf{x}, \mathbf{x}_j) V_j$ and $\mathbf{u} = (u_1, \cdots, u_n)^t$ with $\sum_i u_i V_i = 0$ solves the following linear system

$$-\frac{1}{t} \sum_j R_t(\mathbf{x}_i, \mathbf{x}_j)(u_i - u_j) V_j = \sum_j \bar{R}_t(\mathbf{x}_i, \mathbf{x}_j) f(\mathbf{x}_j) V_j \tag{2.5}$$

Proposition 2.1. *For any $t > 0$, $h > 0$,*

1. *T^N, T_t^N are compact operators on $H^1(\mathcal{M})$ into $H^1(\mathcal{M})$; $T_t^N, T_{t,h}^N$ are compact operators on $C^1(\mathcal{M})$ into $C^1(\mathcal{M})$.*

2. *All eigenvalues of $T^N, T_t^N, T_{t,h}^N$ are real numbers. All generalized eigenvectors of $T^N, T_t^N, T_{t,h}^N$ are eigenvectors.*

Proof. The proof of (1) is straightforward. First, it is well known that T^N is compact operator. $T_{t,h}^N$ is actually finite dimensional operator, so it is also compact. To show the compactness of T_t^N, we need the following formula,

$$T_t^N u = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) T_t^N u(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_t(\mathbf{x})}\int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}, \quad \forall u \in H^1(\mathcal{M}), \tag{2.6}$$

Using the assumption that $R \in C^2$, direct calculation would gives that that $T_t^N u \in C^2$. This would imply the compactness of T_t^N both in H^1 and C^1.

For the operator T^N, the conclusion (2) is well known. The proof of T_t^N and $T_{t,h}^N$ are very similar, so here we only present the proof for T_t^N.

Let λ be an eigenvalue of T_t^N and u is corresponding eigenfunction, then

$$L_t T_t^N u = \lambda L_t u$$

which implies that

$$\lambda = \frac{\int_M \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})u^*(\mathbf{x})u(\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}}{\int_{\mathcal{M}} u^*(\mathbf{x})(L_t u)(\mathbf{x})\mathrm{d}\mathbf{x}}$$

where u^* is the complex conjugate of u.

Using the symmetry of L_t and $\bar{R}(\mathbf{x},\mathbf{y})$, it is easy to show that $\lambda \in \mathbb{R}$.

Let u be a generalized eigenfunction of T_t^N with multiplicity $m > 1$ associate with eigenvalue λ. Let $v = (T_t^N - \lambda)^{m-1}u$, $w = (T_t^N - \lambda)^{m-2}u$, then v is an eigenfunction of T_t and

$$T_t^N v = \lambda v, \quad (T_t^N - \lambda)w = v$$

By applying L_t on both sides of above two equations, we have

$$\begin{aligned}
\lambda L_t v &= L_t(T_t^N v) = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y}, \\
L_t v &= L_t(T_t^N w) - \lambda L_t w = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})w(\mathbf{y})\mathrm{d}\mathbf{y} - \lambda L_t w
\end{aligned}$$

Using above two equations and the fact that L_t is symmetric, we get

$$\begin{aligned}
0 &= \left\langle w, \lambda L_t v - \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})v(\mathbf{y})\mathrm{d}\mathbf{y} \right\rangle_{\mathcal{M}} \\
&= \left\langle \lambda L_t w - \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})w(\mathbf{y})\mathrm{d}\mathbf{y}, v \right\rangle_{\mathcal{M}} \\
&= \langle L_t v, v\rangle_{\mathcal{M}} \geq C\,\|v\|_2^2
\end{aligned}$$

which implies that $(T_t^N - \lambda)^{m-1}u = v = 0$. This proves that u is a generalized eigenfunction of T_t^N with multiplicity $m-1$. Repeating above argument, we can show that u is actually an eigenfunction of T_t^N. $\square$

The following proposition proved in [17] tells us that the eigen problem $T_{t,h}^N(u) = \lambda u$ is equivalent to the eigen problems (1.3).

Proposition 2.2. *Let $\theta(u)$ denote the restriction of u to the sample points P, i.e., $\theta(u) = (u(\mathbf{p}_1), \cdots, u(\mathbf{p}_n))^t$.*

1. *If a function u is an eigenfunction of $T_{t,h}^N$ with the eigenvalue λ, then the vector $\theta(u)$ is an eigenvector of the eigenproblem* (1.3) *with eigenvalue* $1/\lambda$*.*

2. *If a vector $\mathbf{u}$ is an eigenvector of the eigenproblem* (1.3) *with the eigenvalue $\lambda \neq 0$, then $I_\lambda^N(\mathbf{u})$ is an eigenfunction of $T_{t,h}^N$ with eigenvalue $1/\lambda$, where*

$$I_\lambda^N(\mathbf{u})(\mathbf{x}) = \frac{\sum_{p_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j - \lambda t \sum_{p_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) u_j V_j}{\sum_{p_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j}, \text{ and}$$

3. *A function u is the eigenfunction of the eigenproblem* (1.1) *with the eigenvalue $\lambda \neq 0$ if and only if the function u is an eigenfunction of T^N with the eigenvalue* $1/\lambda$*.*

3 Main Results

Let X be a complex Banach space and $L : X \to X$ be a compact linear operator. The resolvent set $\rho(L)$ is given by the complex numbers $z \in \mathbb{C}$ such that $z - L$ is bijective. The spectrum of L is $\sigma(L) = \mathbb{C} \backslash \rho(L)$. It is well known that $\sigma(L)$ is a countable set with no limit points other than zero. All non-zero value s in $\sigma(L)$ are eigenvalues. If λ is a nonzero eigenvalue of L, the ascent multiplicity α of $\lambda - L$ is the smallest integer such that $\ker(\lambda - L)^\alpha = \ker(\lambda - L)^{\alpha+1}$.

Given a closed smooth curve $\Gamma \subset \rho(L)$ which encloses the eigenvalue λ and no other elements of $\sigma(L)$, the Riesz spectral projection associated with λ is defined by

$$E(\lambda, L) = \frac{1}{2\pi i} \int_\Gamma (z - L)^{-1} \mathrm{d}z, \tag{3.1}$$

where $i = \sqrt{-1}$ is the unit imaginary. The definition does not depend on the chosen of Γ. It is well known that $E(\lambda, L) : X \to X$ has following properties:

1. $E(\lambda, L) \circ E(\lambda, L) = E(\lambda, L), \quad L \circ E(\lambda, L) = E(\lambda, L) \circ L,\ E(\lambda, L) \circ E(\mu, L) = 0,$ if $\lambda \neq \mu$.
2. $E(\lambda, L)X = \ker(\lambda - L)^\alpha$, where α is the ascent multiplicity of $\lambda - L$.
3. If $\Gamma \subset \rho(L)$ encloses more eigenvalues $\lambda_1, \cdots, \lambda_n$, then
$$E(\lambda_1, \cdots, \lambda_n, L)X = \oplus_{i=1}^n \ker(\lambda_1 - L)^{\alpha_i}$$
where α_i is the ascent multiplicity of $\lambda_i - L$.

The properties (2) and (3) are of fundamental importance for the study of eigenvector approximation.

It is well-known that the eigenvalues of both T^N can be sorted as

$$\lambda_1 \leq \lambda_2 \leq \cdots, \leq \lambda_n \leq \cdots < 0,$$

where the same eigenvalue are repeated accroding to its multiplicity.

Now, we are ready to state the main theorems.

Theorem 3.1. *Assume the submanifold $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞ smooth and the input data $(P, \mathbf{V})$ is an h-integral approximation of $\mathcal{M}$, Let λ_i be the ith smallest eigenvalue of T^N counting multiplicity, and $\lambda_i^{t,h}$ be the ith smallest eigenvalue of $T_{t,h}$ counting multiplicity, then there exists a constant C such that*

$$|\lambda_i^{t,h} - \lambda_i| \leq C \left(t^{1/2} + \frac{h}{t^{d/4+3}} \right),$$

and there exist another constant C such that, for any $\phi \in E(\lambda_i, T)X$ and $X = H^1(\mathcal{M})$,

$$\|\phi - E(\sigma_i^{t,h}, T_{t,h}^N)\phi\|_{H^1(\mathcal{M})} \leq C \left(t^{1/2} + \frac{h}{t^{d/4+2}} \right).$$

where $\sigma_i^{t,h} = \{\lambda_j^{t,h} \in \sigma(T_{t,h}^N) : j \in I_i\}$ and $I_i = \{j \in \mathbb{N} : \lambda_j = \lambda_i\}$.

Our main convergence results described in the introduction follow easily from the above theorem and Proposition 2.2, which we state formally as follows.

Theorem 3.2. *Assume the submanifold $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞ smooth and the input data $(P, \mathbf{V})$ is an h-integral approximation of $\mathcal{M}$, Let λ_i be the ith largeest eigenvalue of eigenvalue problem* (1.1)*. And let $\lambda_i^{t,h}$ be the ith largest eigenvalue of discrete eigenvalue problem* (1.3)*, then there exists a constant C such that*

$$|\lambda_i^{t,h} - \lambda_i| \leq C\lambda_i^2 \left(t^{1/2} + \frac{h}{t^{d/4+3}}\right),$$

and there exist another constant C such that, for any $\phi \in E(\lambda_i, T)X$ and $X = H^1(\mathcal{M})$,

$$\|\phi - E(\sigma_i^{t,h}, T_{t,h})\phi\|_{H^1(\mathcal{M})} \leq C\left(t^{1/2} + \frac{h}{t^{d/4+2}}\right).$$

where $\sigma_i^{t,h} = \{\lambda_j^{t,h} \in \sigma(T_{t,h}) : j \in I_i\}$ and $I_i = \{j \in \mathbb{N} : \lambda_j = \lambda_i\}$.

4 Structure of the Proof

To prove Theorem 3.1, we need some results in perturbation theory for spectral projection. First, we need the following theorem to obtain the convergence rate of the eigenvalues.

Theorem 4.1. *Let $(X, \|\cdot\|_X)$ be an arbitrary Banach space. Let S and T be compact linear operators on X into X. Let $z \in \rho(T)$. Assume*

$$\|T - S\|_X \leq \frac{1}{2\|(z-T)^{-1}\|_X}. \tag{4.1}$$

Then $z \in \rho(S)$ and $(z-S)^{-1}$ has the bound

$$\|(z-S)^{-1}\|_X \leq 2\|(z-T)^{-1}\|_X. \tag{4.2}$$

Proof. For any $x \in X$

$$\begin{aligned}\|(z-S)x\|_X &\geq \|(z-T)x\|_X - \|(T-S)x\|_X \\ &\geq \left(\|(z-T)^{-1}\|_X - \|T-S\|_X\right)\|x\|_X \\ &\geq \frac{1}{2}\|(z-T)^{-1}\|_X\|x\|_X \end{aligned} \tag{4.3}$$

Using this inequality and the assumption that S is compact operator, we have $z \in \rho(S)$ and

$$\|(z-S)^{-1}\|_X \leq 2\|(z-T)^{-1}\|_X. \tag{4.4}$$

□

We also need the following theorem (e.g. [1]) to get the convergence rate of the eigenfunctions.

Theorem 4.2. *Let $(X, \|\cdot\|_X)$ be an arbitrary Banach space. Let S and T be compact linear operators on X into X. Let $z_0 \in \mathbb{C}$, $z_0 \neq 0$ and let $\epsilon > 0$ be less than $|z_0|$, denote the circumference $|z - z_0| = \epsilon$ by Γ and assume $\Gamma \subset \rho(T)$. Denote the interior of Γ by U. Let $\sigma_T = U \cap \sigma(T) \neq \emptyset$. $\sigma_S = U \cap \sigma(S)$. Let $E(\sigma_S, S)$ and $E(\sigma_T, T)$ be the corresponding spectral projections of S for σ_S and T for σ_T, i.e.*

$$E(\sigma_S, S) = \frac{1}{2\pi i}\int_\Gamma (z-S)^{-1}\mathrm{d}z, \quad E(\sigma_T, T) = \frac{1}{2\pi i}\int_\Gamma (z-T)^{-1}\mathrm{d}z. \tag{4.5}$$

Assume

$$\|(T-S)S\|_X \leq \min_{z\in\Gamma}\frac{|z|}{\|(z-T)^{-1}\|_X} \tag{4.6}$$

Then, we have

(1). Dimension $E(\sigma_S, S)X = E(\sigma_T, T)X$, thereby σ_S is nonempty and of the same multiplicity as σ_T.

(2). For every $x \in E(\sigma_T, T)X$,

$$\|x - E(\sigma_S, S)x\|_X \le \frac{M\epsilon}{c_0} \left(\|(T-S)x\|_X + \|x\|_X \|(T-S)S\|_X\right). \tag{4.7}$$

where $M = \max_{z\in\Gamma} \|(z-T)^{-1}\|_X$, $c_0 = \min_{z\in\Gamma} |z|$.

In order to utilize the above two theorems from pertubation theory, we show the following three theorems which bound the approximation errors of the solution operators $T^N, T_t^N, T_{t,h}^N$ in certain operator norms.

Theorem 4.3. *Under the assumptions in 2.1, for t small enough, there exists a constant $C > 0$ such that*

$$\|T - T_t\|_{H^1} \le Ct^{1/2} \tag{4.8}$$

Theorem 4.4. *Under the assumptions in 2.1, for t, h small enough, there exists a constant $C > 0$ such that*

$$\|T_{t,h} - T_t\|_{C^1} \le \frac{Ch}{t^{d/4+2}} \tag{4.9}$$

Theorem 4.5. *Under the assumptions in 2.1, for t, h small enough, there exists a constant C independent on t and h, such that*

$$\|T_t\|_{H^1} \le C, \quad \|T_{t,h}\|_\infty \le Ct^{-d/4}, \quad \|T_{t,h}\|_{C^1} \le Ct^{-(d+2)/4} \tag{4.10}$$

The proof of Theorem 4.5 can be found in [17], and Theorem 4.3 and Theorem 4.4 will be proved in Section 5 and Section 6 respectively.

Before we prove Theorem 3.1, we show the the estimates of $\|(z-T)^{-1}\|_{H^1(\mathcal{M})}$ and $\|(z-T_t)^{-1}\|_{C^1(\mathcal{M})}$ in the following two lemmas, which are needed to invoke the results from pertubation theory.

Lemma 4.1. *Let T be the solution operator of the Neumann problem* (2.2) *and $z \in \rho(T)$, then*

$$\|(z-T)^{-1}\|_{H^1(\mathcal{M})} \le \max_{n\in\mathbb{N}} \frac{1}{|z-\lambda_n|}, \tag{4.11}$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ is the set of eigenvalues of T.

Proof. Suppose $\phi_j,\ j \in \mathbb{N}$ be the normalized eigenfunction of T corresponding to $\lambda_j,\ j \in \mathbb{N}$. Then it is well known that $\{\phi_j\}_{j\in\mathbb{N}}$ is a orthonormal basis of $H^1(\mathcal{M})$.

For any $x \in H^1(\mathcal{M})$, $z \in \rho(T)$, first we can expand x over $\{\phi_j\}_{j\in\mathbb{N}}$ to obtain

$$x = \sum_{j=1}^\infty c_j \phi_j. \tag{4.12}$$

Then, we have

$$\begin{aligned}
\|(z-T)x\|_{H^1} &= \left\|\sum_{j=1}^\infty c_j (z-T)\phi_j\right\|_{H^1} = \left\|\sum_{j=1}^\infty c_j (z-\lambda_j)\phi_j\right\|_{H^1} \\
&= \left(\sum_{j=1}^\infty c_j^2 |z-\lambda_j|^2\right)^{1/2} \ge \min_{n\in\mathbb{N}} |z-\lambda_n| \left(\sum_{j=1}^\infty c_j^2\right)^{1/2} \\
&= \min_{n\in\mathbb{N}} |z-\lambda_n| \|x\|_{H^1}
\end{aligned} \tag{4.13}$$

□

Lemma 4.2. *Let T_t be the solution operator of the integral equation* (2.3) *and $z \in \rho(T_t)$, then*

$$\|(z-T_t)^{-1}\|_{C^1} \leq \left(\frac{|\mathcal{M}|}{|z|t^{(d+2)/4}}(\min_{n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right)^{-1} \tag{4.14}$$

Proof. For any $x \in H^1(\mathcal{M})$,

$$\begin{aligned}\|(z-T_t)x\|_{H^1} &\geq \|(z-T)x\|_{H^1} - \|(T-T_t)x\|_{H^1} \\ &\geq \left(\min_{n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right)\|x\|_{H^1}\end{aligned} \tag{4.15}$$

Then $(z-T_t)^{-1}$ exists and

$$\|(z-T_t)^{-1}\|_{H^1} \leq \left(\min_{n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right)^{-1}. \tag{4.16}$$

For any $u \in C^1(\mathcal{M})$,

$$\|(z-T_t)^{-1}u\|_{H^1} \leq C\left(\min_{n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right)^{-1}\|u\|_{C^1} \tag{4.17}$$

On the other hand, let $v = (z-T_t)^{-1}u$ which means $v = (u + T_t v)/z$

$$\begin{aligned}\|v\|_{C^1} &\leq \frac{1}{|z|}\left(\|u\|_{C^1} + \|T_t v\|_{C^1}\right) \\ &\leq \frac{1}{|z|}\left(\|u\|_{C^1} + t^{-(d+2)/4}\|v\|_{L^2}\right) \\ &\leq \left(\frac{|\mathcal{M}|}{|z|t^{(d+2)/4}}\left(\min_{n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right)^{-1} + 1\right)\|u\|_{C^1}\end{aligned}$$

$$\|(z-T_t)^{-1}\|_{C^1} \leq \max\left(\frac{2|\mathcal{M}|}{|z|t^{(d+2)/4}}\left(\min_{n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right)^{-1}, 2\right) \tag{4.18}$$

□

Using the above theorems and lemmas, we show that $\sigma(T_t)$ is close to $\sigma(T)$, which will be used in the proof of Theorem 3.1.

Proposition 4.1. *Let T_t be the solution operator of the integral equation* (2.3), *then*

$$\sigma(T_t) \subset \bigcup_{n\in\mathbb{N}} B\left(\lambda_n, 2\|T-T_t\|_{H^1(\mathcal{M})}\right) \tag{4.19}$$

Proof. Let $r_0 = \|T-T_t\|_{H^1(\mathcal{M})}$, $\mathcal{A} = \mathbb{C}\backslash\bigcup_{n\in\mathbb{N}} B(\lambda_n, 2r_0)$. For any $z \in \mathcal{A}$, using Lemma 4.1, we have

$$\|(z-T)^{-1}\|_{H^1(\mathcal{M})} \leq \max_{n\in\mathbb{N}}\frac{1}{|z-\lambda_n|} \leq \frac{1}{2r_0}$$

which implies that

$$\|T-T_t\|_{H^1(\mathcal{M})} = r_0 \leq \frac{1}{2\|(z-T)^{-1}\|_{H^1(\mathcal{M})}}.$$

Then using (3) of Lemma 4.1, we have $z \in \rho(T_t)$.

Since z is arbitrary in $\mathcal{A}$, we get $\mathcal{A} \subset \rho(T_t)$. This means that

$$\sigma(T_t) = \mathbb{C}\backslash\rho(T_t) \subset \mathbb{C}\backslash\mathcal{A} = \bigcup_{n\in\mathbb{N}} B(\lambda_n, 2\|T-T_t\|_{H^1(\mathcal{M})}).$$

□

Now we can show Theorem 3.1, i.e., the convergence of the eigenproblem with the Neumann boundary .

Proof. Let $r_1 = \frac{2C}{t^{d/4+1}} \|T_{t,h} - T_t\|_{C^1} + \|T - T_t\|_{H^1(\mathcal{M})}$, $\mathcal{A} = \mathbb{C} \backslash \left[\bigcup_{n\in\mathbb{N}} B(\lambda_n, r_1) \bigcup B(0, t^{1/2})\right]$, For any $z \in \mathcal{A}$, using Lemma 4.1, we have

$$\begin{aligned}
\|(z - T_t)^{-1}\|_{C^1} &\leq \frac{C}{|z| t^{(d+2)/4}} \left(\min_{n\in\mathbb{N}} |z - \lambda_n| - \|T - T_t\|_{H^1}\right)^{-1} \\
&\leq \frac{C}{t^{d/4+1}} \left(r_1 - \|T - T_t\|_{H^1}\right)^{-1} \\
&= \left(2 \|T_{t,h} - T_t\|_{C^1}\right)^{-1}
\end{aligned}$$

which implies that

$$\|T_{t,h} - T_t\|_{C^1} \leq \frac{1}{2\|(z - T_t)^{-1}\|_{C^1}}.$$

Then using Lemma 4.1, we have $z \in \rho(T_{t,h})$.

Since z is arbitrary in $\mathcal{A}$, we get $\mathcal{A} \subset \rho(T_{t,h})$. This means that

$$\sigma(T_{t,h}) = \mathbb{C} \backslash \rho(T_{t,h}) \subset \mathbb{C} \backslash \mathcal{A} = \bigcup_{n\in\mathbb{N}} B(\lambda_n, r_1) \bigcup B(0, t^{1/2}). \tag{4.20}$$

Moreover, using Theorem 4.1 and the definition of r_1, we have

$$\sigma(T_t) \subset \bigcup_{n\in\mathbb{N}} B(\lambda_n, 2r_1). \tag{4.21}$$

On the other hand, using Theorem 4.3 and 4.4, we know that there exist $C > 0$ independent on t and h, such that

$$r_1 \leq C\left(t^{1/2} + \frac{h}{t^{d/2+3}}\right). \tag{4.22}$$

For any fixed eigenvalue $\lambda_n \in \sigma(T)$, let $\gamma_j = \min\limits_{\lambda\in\sigma(T)\backslash\{\lambda_j\}} |\lambda_j - \lambda|$, $j \in \mathbb{N}$ and $\gamma = \min\limits_{j\leq n} \gamma_j$. Using the structure of $\sigma(T)$, we know that $\gamma > 0$. Without loss of generality, we can assume t, h are small enough such that $r_1 \leq \gamma/6$.

Let $\Gamma_j = \{z \in \mathbb{C} : |z - \lambda_j| = \gamma/3\}$, U_j be the aera enclosed by Γ_j. Let

$$\sigma_{t,j} = \sigma(T_t) \bigcap U_j, \quad \sigma_{t,h,j} = \sigma(T_{t,h}) \bigcap U_j.$$

Using the definition of Γ_j, we know for any $j \leq n$, $\Gamma_j \subset \rho(T), \rho(T_t)$ and $\rho(T_{t,h})$.

In order to apply Theorem 4.2, we need to verify the conditions

$$\|(T - T_t)T_t\|_{H^1} \leq \min_{z\in\Gamma_j} \frac{|z|}{\|(z - T)^{-1}\|_{H^1}}, \tag{4.23}$$

$$\|(T_t - T_{t,h})T_{t,h}\|_{C^1} \leq \min_{z\in\Gamma_j} \frac{|z|}{\|(z - T_t)^{-1}\|_{C^1}}. \tag{4.24}$$

Using Lemma 4.1 and the choice of Γ_j, we have

$$\min_{z\in\Gamma_j} \frac{|z|}{\|(z - T)^{-1}\|_{H^1}} \geq \frac{\min_{z\in\Gamma_j} |z|}{\max_{z\in\Gamma_j} \|(z - T)^{-1}\|_{H^1}} \geq (|\lambda_j| - \gamma/3) \min_{z\in\Gamma_j, n\in\mathbb{N}} |z - \lambda_n| = (|\lambda_j| - \gamma/3)\gamma/3.$$

Then, using Theorem 4.3 and Lemma 4.5, condition 4.23 is true as long as t is small enough.

Using Lemma 4.2, we have

$$\begin{aligned}
\min_{z\in\Gamma_j} \frac{|z|}{\|(z-T_t)^{-1}\|_{C^1}} &\geq \frac{\min_{z\in\Gamma_j}|z|}{\max_{z\in\Gamma_j}\|(z-T_t)^{-1}\|_{C^1}} \\
&\geq \frac{(|\lambda_j|-\gamma/3)^2 t^{(d+2/4)}}{2|\mathcal{M}|}\left(\min_{z\in\Gamma_j,n\in\mathbb{N}}|z-\lambda_n| - \|T-T_t\|_{H^1}\right) \\
&\geq \frac{(|\lambda_j|-\gamma/3)^2 t^{(d+2/4)}\gamma}{12|\mathcal{M}|}. \qquad (4.25)
\end{aligned}$$

To get the last inequality, we use the fact that $\|T-T_t\|_{H^1} < r_1 \leq \gamma/6$ and $\min_{z\in\Gamma_j,n\in\mathbb{N}}|z-\lambda_n| = \gamma/3$.

Using Theorem 4.4 and Lemma 4.5, we can choose h small enough such that condition 4.24 is satisfied. Then using Theorem 4.2, we have

$$\dim(E(\lambda_j,T)) = \dim(E(\sigma_{t,j},T_t)) = \dim(E(\sigma_{t,h,j},T_{t,h})), \quad j=1,\cdots,n. \qquad (4.26)$$

Combining (4.20), above equality would imply that

$$|\lambda_j^{t,h}-\lambda_j| \leq C\left(t^{1/2}+\frac{h}{t^{d/2+3}}\right), \quad j\in\mathbb{N}. \qquad (4.27)$$

The convergence of eigenspace is also given by Theorem 4.2. For any $x\in E(\lambda_n,T)$, $\|x\|_{C^1}=1$,

$$\begin{aligned}
\|x-E(\sigma_{t,n},T_t)x\|_{H^1} &\leq C\|T-T_t\|_{H^1}\|x\|_{H^1} \leq Ct^{1/2}, \qquad (4.28)\\
\|E(\sigma_{t,n},T_t)x - E(\sigma_{t,h,n},T_{t,h})x\|_{C^1} &\leq C\left(\|T_t-T_{t,h}\|_{C^1}+\|(T_t-T_{t,h})T_{t,h}\|_{C^1}\right) \leq \frac{Ch}{t^{d/4+2}}. \qquad (4.29)
\end{aligned}$$

Finally, we have

$$\|x-E(\sigma_{t,h,n},T_{t,h})x\|_{H^1} \leq C\left(t^{1/2}+\frac{h}{t^{d/4+2}}\right). \qquad (4.30)$$

□

5 Proof of Theorem 4.3

We first introduce the local coordinate of the manifold $\mathcal{M}$. According to Proposition 6.1 of [17], $\mathcal{M}$ can be locally parametrized as follows.

$$\mathbf{x} = \Phi(\boldsymbol{\gamma}) : \Omega \subset \mathbb{R}^k \to \mathcal{M} \subset \mathbb{R}^d \qquad (5.1)$$

where $\boldsymbol{\gamma} = (\gamma^1,\cdots,\gamma^k)^t \in \mathbb{R}^k$ and $\mathbf{x} = (x^1,\cdots,x^d)^t \in M$.

Let $\partial_{i'} = \frac{\partial}{\partial\gamma^{i'}}$ be the tangent vector along the direction $\gamma^{i'}$. Since $\mathcal{M}$ is a submanifold in $\mathbb{R}^d$ with induced metric, $\partial_{i'} = (\partial_{i'}\Phi^1,\cdots,\partial_{i'}\Phi^d)$ and the metric tensor

$$g_{i'j'} = <\partial_{i'},\partial_{j'}> = \partial_{i'}\Phi^l\partial_{j'}\Phi^l.$$

Let $g^{i'j'}$ denote the inverse of $g_{i'j'}$, i.e.,

$$g_{i'l'}\,g^{l'j'} = \delta_{ij} = \begin{cases} 1, & i=j, \\ 0, & i\neq j. \end{cases}$$

For any function f on $\mathcal{M}$, $\nabla f = g^{i'j'}\partial_{j'}f\partial_{i'}$ denotes the gradient of f. For convenience, let $\nabla^j f$ denote the x^j component of the gradient ∇f, i.e.,

$$\nabla^j f = \partial_{i'}\Phi^j g^{i'j'}\partial_{j'}f \quad \text{and} \quad \partial_{i'}f = \partial_{i'}\Phi^j\nabla^j f. \qquad (5.2)$$

Then Theorem 4.3 is an easy corollary of following three theorems.

Theorem 5.1. *Assume $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. Let $u(\mathbf{x})$ be the solution of the problem (2.2) and $u_t(\mathbf{x})$ be the solution of corresponding problem (2.3). If $f \in C^\infty(\mathcal{M}), g \in C^\infty(\partial\mathcal{M})$ in both problems, then there exists constants C, T_0 depending only on $\mathcal{M}$ and $\partial\mathcal{M}$, so that for any $t \le T_0$,*

$$\left\| L_t(u-u_t) - \int_{\partial\mathcal{M}} n_j(\mathbf{y})\eta_i(\mathbf{x},\mathbf{y})\nabla_i\nabla_j u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \right\|_{L^2(\mathcal{M})} \le Ct^{1/2}\|u\|_{H^3(\mathcal{M})}, \tag{5.3}$$

$$\left\| \nabla\left(L_t(u-u_t) - \int_{\partial\mathcal{M}} n_j(\mathbf{y})\eta_i(\mathbf{x},\mathbf{y})\nabla_i\nabla_j u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \right) \right\|_{L^2(\mathcal{M})} \le C\|u\|_{H^3(\mathcal{M})}. \tag{5.4}$$

where $\eta(\mathbf{x},\mathbf{y}) = \xi^i(\mathbf{x},\mathbf{y})\partial_i\Phi(\alpha)$ and $\alpha = \Phi^{-1}(\mathbf{x}), \xi(\mathbf{x},\mathbf{y}) = \Phi^{-1}(\mathbf{x}) - \Phi^{-1}(\mathbf{y})$.

Theorem 5.2. *Assume $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. Let $u(\mathbf{x})$ solves the integral equation*

$$L_t u = r(\mathbf{x}) - \bar{r} \tag{5.5}$$

where $r \in H^1(\mathcal{M})$ and $\bar{r} = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} r(\mathbf{x})\mathrm{d}\mathbf{x}$. Then, there exist a constant $C > 0, T_0 > 0$ independent on t, such that

$$\|u\|_{H^1(\mathcal{M})} \le C\left(\|r\|_{L^2(\mathcal{M})} + t\|\nabla r\|_{L^2(\mathcal{M})}\right) \tag{5.6}$$

as long as $t \le T_0$.

Theorem 5.3. *Assume $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. Let $u(\mathbf{x})$ solves the integral equation*

$$L_t u = \int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta_i(\mathbf{x},\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} - \bar{b} \tag{5.7}$$

where η is same as that in Theorem 5.1 and

$$\bar{b} = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}}\int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta_i(\mathbf{x},\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mathbf{x}.$$

Then, there exist constant $C > 0, T_0 > 0$ independent on t, such that

$$\|u\|_{H^1(\mathcal{M})} \le C\sqrt{t}\,\|\mathbf{b}\|_{H^1(\mathcal{M})}. \tag{5.8}$$

as long as $t \le T_0$.

The proof of Theorem 5.2 can be found in [17]. In next two subsections, we will prove Theorem 5.3 and Theorem 5.1 sequentially.

5.1 Proof of Theorem 5.3

Lemma 5.1. *For any function $u \in L^2(\mathcal{M})$, there exist a constant $C > 0$ independent on t and u, such that*

$$\langle u, L_t u\rangle_{\mathcal{M}} \ge C\int_{\mathcal{M}} |\nabla v|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{5.9}$$

where $\langle f, g\rangle_{\mathcal{M}} = \int_{\mathcal{M}} f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}$ for any $f, g \in L_2(\mathcal{M})$ and

$$v(\mathbf{x}) = \frac{C_t}{w_t(\mathbf{x})}\int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \tag{5.10}$$

and $w_t(\mathbf{x}) = C_t\int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\mathrm{d}\mu_{\mathbf{y}}$.

Lemma 5.2. *Assume $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. There exist a constant $C > 0$ independent on t so that for any function $u \in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u = 0$ and for any sufficient small t*

$$\langle L_t u, u\rangle_{\mathcal{M}} \geq C\|u\|^2_{L_2(\mathcal{M})} \tag{5.11}$$

Proof. of Theorem 5.3

The key point is to show that

$$\left|\int_{\mathcal{M}} u(\mathbf{x}) \left(\int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta^j \bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} - \bar{b}\right)\mathrm{d}\mu_{\mathbf{x}}\right| \leq C\sqrt{t}\ \|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})} \tag{5.12}$$

Notice that

$$|\bar{b}| = \frac{1}{|\mathcal{M}|}\left|\int_{\mathcal{M}}\int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta_i(\mathbf{x},\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mathbf{x}\right| \leq C\sqrt{t}\ \|\mathbf{b}(\mathbf{y})\|_{L^2(\partial\mathcal{M})} \leq C\sqrt{t}\ \|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}$$

. Then it is enough to show that

$$\left|\int_{\mathcal{M}} u(\mathbf{x}) \left(\int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta^j \bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right| \leq C\sqrt{t}\ \|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})} \tag{5.13}$$

First, we have

$$|2t\nabla^j \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) + \eta^j \bar{R}_t(\mathbf{x},\mathbf{y})| \leq C|\xi|^2 \bar{R}_t(\mathbf{x},\mathbf{y}) \tag{5.14}$$

where $\bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) = C_t\bar{\bar{R}}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right)$ and $\bar{\bar{R}}(r) = \int_r^\infty \bar{R}(s)\mathrm{d}s$.

This would gives us that

$$\begin{aligned}
&\left|\int_{\mathcal{M}} u(\mathbf{x})\int_{\partial\mathcal{M}} b_i(\mathbf{y})\left(\eta^j\bar{R}_t(\mathbf{x},\mathbf{y}) + 2t\nabla_i\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\right)\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}\right| \\
\leq\ & C\int_{\mathcal{M}} |u(\mathbf{x})|\int_{\partial\mathcal{M}} |b_i(\mathbf{y})||\xi|^2\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
\leq\ & Ct\int_{\mathcal{M}} |u(\mathbf{x})|\int_{\partial\mathcal{M}} |b_i(\mathbf{y})|\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
\leq\ & Ct\int_{\partial\mathcal{M}} |b_i(\mathbf{y})|\int_{\mathcal{M}} |u(\mathbf{x})|\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \\
\leq\ & Ct\|\mathbf{b}(\mathbf{y})\|_{L^2(\partial\mathcal{M})}\left(\int_{\partial\mathcal{M}}\left(\int_{\mathcal{M}}\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\right)\left(\int_{\mathcal{M}}|u(\mathbf{x})|^2\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2} \\
\leq\ & Ct\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\int_{\mathcal{M}}|u(\mathbf{x})|^2\left(\int_{\partial\mathcal{M}}\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2} \\
\leq\ & Ct^{3/4}\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{L^2(\mathcal{M})}
\end{aligned} \tag{5.15}$$

On the other hand, using Guass integral formula, we have

$$\begin{aligned}
&\int_{\mathcal{M}} u(\mathbf{x})\int_{\partial\mathcal{M}} b_i(\mathbf{y})\nabla_i\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
=\ & \int_{\partial\mathcal{M}} b_i(\mathbf{y})\int_{\mathcal{M}} u(\mathbf{x})\nabla_i\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} \\
=\ & \int_{\partial\mathcal{M}} b_i(\mathbf{y})\int_{\partial\mathcal{M}} n_i(\mathbf{x})u(\mathbf{x})\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} - \int_{\partial\mathcal{M}}\int_{\mathcal{M}} \mathrm{div}_{\mathbf{x}}[b_i(\mathbf{y})u(\mathbf{x})]\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}.
\end{aligned} \tag{5.16}$$

For the first term, we have

$$\begin{aligned}
&\left|\int_{\partial\mathcal{M}} b_i(\mathbf{y})\int_{\partial\mathcal{M}} n_i(\mathbf{x})u(\mathbf{x})\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\right| \\
\leq\ & \int_{\partial\mathcal{M}} |b_i(\mathbf{y})|\int_{\partial\mathcal{M}} |u(\mathbf{x})|\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} \\
\leq\ & C\|\mathbf{b}(\mathbf{y})\|_{L^2(\partial\mathcal{M})}\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}} |u(\mathbf{x})|\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)^2\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2} \\
\leq\ & C\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}} \bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)\left(\int_{\partial\mathcal{M}} |u(\mathbf{x})|^2\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2} \\
\leq\ & Ct^{-1/4}\,\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}} \bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)|u(\mathbf{x})|^2\mathrm{d}\tau_{\mathbf{x}}\right)^{1/2} \\
\leq\ & Ct^{-1/2}\,\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{L^2(\partial\mathcal{M})} \\
\leq\ & Ct^{-1/2}\,\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})}.
\end{aligned}\tag{5.17}$$

We can also bound the second term of (5.16)

$$\begin{aligned}
&\left|\int_{\partial\mathcal{M}}\int_{\mathcal{M}} \mathrm{div}_{\mathbf{x}}[b_i(\mathbf{y})u(\mathbf{x})]\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\right| \\
\leq\ & C\|\mathbf{b}(\mathbf{y})\|_{L^2(\partial\mathcal{M})}\int_{\partial\mathcal{M}}\int_{\mathcal{M}} |\nabla u(\mathbf{x})|\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} \\
\leq\ & C\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\int_{\partial\mathcal{M}}\left(\int_{\mathcal{M}} \bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\right)\left(\int_{\mathcal{M}} |\nabla u(\mathbf{x})|^2\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2} \\
\leq\ & C\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\int_{\mathcal{M}} |\nabla u(\mathbf{x})|^2\left(\int_{\partial\mathcal{M}} \bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2} \\
\leq\ & Ct^{-1/4}\,\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})}.
\end{aligned}\tag{5.18}$$

Then, the inequality (5.13) is obtained by (5.15), (5.16), (5.17) and (5.18).

Now, using Lemma 5.2, we have

$$\|u\|^2_{L^2(\mathcal{M})} \leq C\left\langle u, L_t u\right\rangle = C\left|\int_{\mathcal{M}} u(\mathbf{x})\int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta^j\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}\right| \leq C\sqrt{t}\,\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})}. \tag{5.19}$$

Denote $p(\mathbf{x}) = \int_{\partial\mathcal{M}} b_i(\mathbf{y})\eta^j\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}$, then direct calculation would give us that

$$\|p(\mathbf{x})\|_{L^2(\mathcal{M})} \leq Ct^{1/4}\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}, \tag{5.20}$$

$$\|\nabla p(\mathbf{x})\|_{L^2(\mathcal{M})} \leq Ct^{-1/4}\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}. \tag{5.21}$$

The integral equation $L_t u = p$ gives that

$$u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})}\,p(\mathbf{x}) \tag{5.22}$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \tag{5.23}$$

Then by Lemma 5.1, we have

$$\begin{aligned}
\|\nabla u\|_{L^2(\mathcal{M})}^2 &\le 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2 \left\|\nabla\left(\frac{p(\mathbf{x})}{w_t(\mathbf{x})}\right)\right\|_{L^2(\mathcal{M})}^2 \\
&\le C\left\langle u, L_t u\right\rangle + Ct\|p\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla p\|_{L^2(\mathcal{M})}^2 \\
&\le C\sqrt{t}\,\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})} + Ct\|p\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla p\|_{L^2(\mathcal{M})}^2 \\
&\le C\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right).
\end{aligned} \tag{5.24}$$

Using (5.19) and (5.24), we have

$$\|u\|_{H^1(\mathcal{M})}^2 \le C\|\mathbf{b}(\mathbf{y})\|_{H^1(\mathcal{M})}\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right) \tag{5.25}$$

which proves the theorem. □

5.2 Proof of Theorem 5.1

Proof. Let $r(\mathbf{x}) = L_t u - L_t u_t$, then we have

$$\begin{aligned}
r(\mathbf{x}) &= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + 2\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})g(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} - \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}} u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{t}\int_{\mathcal{M}} (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}} u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}
\end{aligned} \tag{5.26}$$

Here we use that fact that u is the solution of the Poisson equation with Newmann boundary condition (2.2), such that

$$\begin{aligned}
\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})g(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} &= \int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \\
&= \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}} u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \nabla_{\mathbf{y}}\bar{R}_t(\mathbf{x},\mathbf{y})\nabla u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}} u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \frac{1}{t}\int_{\mathcal{M}} (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathrm{d}\mu_{\mathbf{y}},
\end{aligned}$$

and

$$\int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}} u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.$$

Denote

$$\mathcal{B}_{\mathbf{x}}^r = \{\mathbf{y}\in\mathcal{M} : |\mathbf{x}-\mathbf{y}| \le r\}, \quad \mathcal{M}_{\mathbf{x}}^t = \left\{\mathbf{y}\in\mathcal{M} : |\mathbf{x}-\mathbf{y}|^2 \le 32t\right\} \tag{5.27}$$

where $r > 0$ is a positive number which is small enough.

Since the manifold $\mathcal{M}$ is compact, there exists $\mathbf{x}_i \in \mathcal{M},\ i = 1, \cdots, N$ such that

$$\mathcal{M} \subset \bigcup_{i=1}^{N} \mathcal{B}_{\mathbf{x}_i}^r \tag{5.28}$$

By Proposition 6.1 in [17], we have there exist a parametrization $\Phi_i : \Omega_i \subset \mathbb{R}^k \to U_i \subset \mathcal{M}, ; i = 1, \cdots, N$, such that

1. $\mathcal{B}_{\mathbf{x}_i}^{2r} \subset U_i$ and Ω_i is convex.

2. $\Phi \in C^3(\Omega)$;

3. For any points $\mathbf{x}, \mathbf{y} \in \Omega$, $\frac{1}{2}|\mathbf{x}-\mathbf{y}| \le \|\Phi_i(\mathbf{x}) - \Phi_i(\mathbf{y})\| \le 2|\mathbf{x}-\mathbf{y}|$.

Moreover, we denote $\Phi(\beta) = \mathbf{x}, \Phi(\alpha) = \mathbf{y}, \xi = \beta - \alpha, \eta = \xi^i \partial_i \Phi(\alpha)$, and

$$\begin{aligned}
r_1(\mathbf{x}) &= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\left(u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y}) - \frac{1}{2}\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y}))\right)\mathrm{d}\mu_{\mathbf{y}} \\
r_2(\mathbf{x}) &= \frac{1}{2t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} - \int_{\mathcal{M}} \eta^i(\nabla^i\nabla^j u(\mathbf{y})\nabla^j \bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
r_3(\mathbf{x}) &= \int_{\mathcal{M}} \eta^i(\nabla^i\nabla^j u(\mathbf{y})\nabla^j \bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \mathrm{div}\ \left(\eta^i(\nabla^i\nabla^j u(\mathbf{y})\right)\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
r_4(\mathbf{x}) &= \int_{\mathcal{M}} \mathrm{div}\ \left(\eta^i(\nabla^i\nabla^j u(\mathbf{y})\right)\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}} u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned} \tag{5.29}$$

then

$$r(\mathbf{x}) = r_1(\mathbf{x}) - r_1(\mathbf{x}) - r_3(\mathbf{x}) + r_4(\mathbf{x}). \tag{5.30}$$

Next, we will prove the theorem by estimating above four terms one by one.

First, let us consider r_1. To simplify the notation, let

$$d(\mathbf{x},\mathbf{y}) = u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y}) - \frac{1}{2}\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y})).$$

Then, we have

$$\begin{aligned}
\int_{\mathcal{M}} |r_1(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} &= \int_{\mathcal{M}} \left|\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) d(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}} \\
&\le \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right)\left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})|d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}} \\
&\le C\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})|d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
&\le C\sum_{i=1}^N \int_{\mathcal{M}}\int_{B_{\mathbf{x}_i}^r} R_t(\mathbf{x},\mathbf{y})|d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
&= C\sum_{i=1}^N \int_{B_{\mathbf{x}_i}^{2r}}\int_{B_{\mathbf{x}_i}^r} R_t(\mathbf{x},\mathbf{y})|d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
&= C\sum_{i=1}^N \int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned} \tag{5.31}$$

Using the fact that Ω_i is convex and the Newton-Leibniz formula, we can get

$$\begin{aligned}
d(\mathbf{x},\mathbf{y}) &= u(\mathbf{x})-u(\mathbf{y})-(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})-\frac{1}{2}\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y}))\\
&= \xi^i\xi^{i'}\int_0^1\int_0^1\int_0^1 s_1\frac{d}{ds_3}\left(\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\right)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
&= \xi^i\xi^{i'}\xi^{i''}\int_0^1\int_0^1\int_0^1 s_1^2s_2\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i''}\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
&\quad+\xi^i\xi^{i'}\xi^{i''}\int_0^1\int_0^1\int_0^1 s_1^2\partial_{i''}\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
&\quad+\xi^i\xi^{i'}\xi^{i''}\int_0^1\int_0^1\int_0^1 s_1^2s_2\partial_i\Phi^j(\alpha+s_3s_2s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\partial_{i''}\Phi^{j''}(\alpha+s_3s_2s_1\xi)\\
&\qquad\qquad\nabla^{j''}\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1
\end{aligned}$$

Using this equality and $\Phi\in C^3(\Omega)$, it is easy to show that

$$\begin{aligned}
&\int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t}R_t(\mathbf{x},\mathbf{y})|d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}\\
\le\ & Ct^3\int_0^1\int_0^1\int_0^1\int_{B_{\mathbf{x}_i}^r}\int_{\mathcal{M}_{\mathbf{y}}^t}R_t(\mathbf{x},\mathbf{y})\left|D^{2,3}u(\Phi_i(\alpha+s_3s_2s_1\xi))\right|^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\\
\le\ & Ct^3\max_{0\le s\le 1}\int_{B_{\mathbf{x}_i}^r}\int_{\mathcal{M}_{\mathbf{y}}^t}R_t(\mathbf{x},\mathbf{y})\left|D^{2,3}u(\Phi_i(\alpha+s\xi))\right|^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \qquad (5.32)
\end{aligned}$$

where

$$\left|D^{2,3}u(\mathbf{x})\right|^2=\sum_{j,j',j''=1}^n|\nabla^{j''}\nabla^{j'}\nabla^j u(\mathbf{x})|^2+\sum_{j,j'=1}^n|\nabla^{j'}\nabla^j u(\mathbf{x})|^2.$$

Let $\mathbf{z}_i=\Phi_i(\alpha+s\xi),\ 0\le s\le 1$, then for any $\mathbf{y}\in B_{\mathbf{x}_i}^r$ and $\mathbf{x}\in\mathcal{M}_{\mathbf{y}}^t$,

$$|\mathbf{z}_i-\mathbf{y}|\le 2s|\xi|\le 4s|\mathbf{x}-\mathbf{y}|\le 8s\sqrt{t},\quad |\mathbf{z}_i-\mathbf{x}_i|\le|\mathbf{z}_i-\mathbf{y}|+|\mathbf{y}-\mathbf{x}_i|\le r+8s\sqrt{t}.$$

We can assume that t is small enough such that $8\sqrt{t}\le r$, then we have

$$\mathbf{z}_i\in B_{\mathbf{x}_i}^{2r}.$$

After changing of variable, we obtain

$$\begin{aligned}
&\int_{B_{\mathbf{x}_i}^r}\int_{\mathcal{M}_{\mathbf{y}}^t}R_t(\mathbf{x},\mathbf{y})\left|D^{2,3}u(\Phi_i(\alpha+s_3s_2s_1\xi))\right|^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}\\
\le\ & \frac{C}{\delta_0}\int_{B_{\mathbf{x}_i}^r}\int_{B_{\mathbf{x}_i}^{2r}}\frac{1}{s^k}R\left(\frac{|\mathbf{z}_i-\mathbf{y}|^2}{128s^2t}\right)\left|D^{2,3}u(\mathbf{z}_i)\right|^2\mathrm{d}\mu_{\mathbf{z}_i}\mathrm{d}\mu_{\mathbf{y}}\\
=\ & \frac{C}{\delta_0}\int_{B_{\mathbf{x}_i}^r}\frac{1}{s^k}R\left(\frac{|\mathbf{z}_i-\mathbf{y}|^2}{128s^2t}\right)\mathrm{d}\mu_{\mathbf{y}}\int_{B_{\mathbf{x}_i}^{2r}}\left|D^{2,3}u(\mathbf{z}_i)\right|^2\mathrm{d}\mu_{\mathbf{z}_i}\\
\le\ & C\int_{B_{\mathbf{x}_i}^{2r}}\left|D^{2,3}u(\mathbf{x})\right|^2\mathrm{d}\mu_{\mathbf{x}} \qquad (5.33)
\end{aligned}$$

This estimate would give us that

$$\|r_1(\mathbf{x})\|_{L^2(\mathcal{M})}\le Ct^{1/2}\|u\|_{H^3(\mathcal{M})} \qquad (5.34)$$

Now, we turn to estimate the gradient of r_1.

$$\int_{\mathcal{M}} |\nabla_{\mathbf{x}} r_1(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \;\le\; C\int_{\mathcal{M}}\left|\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) d(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}} + C\int_{\mathcal{M}}\left|\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}}.$$

Using the same techniques in the calculation of $\|r_1(\mathbf{x})\|_{L^2(\mathcal{M})}$, we can get that the first term of right hand side can bounded as follows

$$\int_{\mathcal{M}}\left|\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) d(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}} \le C\|u\|^2_{H^3(\mathcal{M})}. \tag{5.35}$$

The estimation of second term is a little involved. First, we have

$$\begin{aligned}
&\int_{\mathcal{M}}\left|\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}} \\
\le\;& \int_{\mathcal{M}}\left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right)\left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}} \\
\le\;& C\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
\le\;& C\sum_{i=1}^N\int_{\mathcal{M}}\int_{B^r_{\mathbf{x}_i}} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
=\;& C\sum_{i=1}^N\int_{B^{2r}_{\mathbf{x}_i}}\int_{B^r_{\mathbf{x}_i}} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \\
=\;& C\sum_{i=1}^N\int_{B^r_{\mathbf{x}_i}}\left(\int_{\mathcal{M}^t_{\mathbf{y}}} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned} \tag{5.36}$$

Using Newton-Leibniz formula, we have

$$\begin{aligned}
d(\mathbf{x},\mathbf{y}) \;=\;& \xi^i\xi^{i'}\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\alpha+s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_2s_1\xi))\right)\mathrm{d}s_2\mathrm{d}s_1 \\
&-\xi^i\xi^{i'}\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\alpha)\partial_{i'}\Phi^{j'}(\alpha)\nabla^{j'}\nabla^j u(\Phi(\alpha))\right)\mathrm{d}s_2\mathrm{d}s_1
\end{aligned} \tag{5.37}$$

Then the gradient of $d(\mathbf{x},\mathbf{y})$ has following representation,

$$\begin{aligned}
\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y}) \;=\;& \xi^i\xi^{i'}\nabla_{\mathbf{x}}\left(\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\alpha+s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_2s_1\xi))\right)\mathrm{d}s_2\mathrm{d}s_1\right) \\
&+\nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\int_0^1\int_0^1\int_0^1 s_1\frac{d}{ds_3}\left(\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\right)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\
=\;& d_1(\mathbf{x},\mathbf{y})+d_2(\mathbf{x},\mathbf{y}).
\end{aligned} \tag{5.38}$$

For d_1, we have

$$\begin{aligned}
&\int_{B^r_{\mathbf{x}_i}}\left(\int_{\mathcal{M}^t_{\mathbf{y}}} R_t(\mathbf{x},\mathbf{y})|d_1(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \\
\le\;& Ct^2\int_0^1\int_0^1\int_{B^r_{\mathbf{x}_i}}\left(\int_{\mathcal{M}^t_{\mathbf{y}}} R_t(\mathbf{x},\mathbf{y})|D^{2,3}u(\Phi(\alpha+s_2s_1\xi))|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}s_2\mathrm{d}s_1 \\
\le\;& Ct^2\max_{0\le s\le 1}\int_{B^r_{\mathbf{x}_i}}\left(\int_{\mathcal{M}^t_{\mathbf{y}}} R_t(\mathbf{x},\mathbf{y})|D^{2,3}u(\Phi(\alpha+s\xi))|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}
\end{aligned} \tag{5.39}$$

which means that

$$\int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|d_1(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \le C\int_{B_{\mathbf{x}_i}^{2r}}|D^{2,3}u(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}} \tag{5.40}$$

For d_2, we have

$$\begin{aligned}
& d_2(\mathbf{x},\mathbf{y}) \\
= & \nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\int_{[0,1]^3} s_1\frac{d}{ds_3}\left(\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\right)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\
= & \nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\xi^{i''}\int_{[0,1]^3} s_1^2s_2\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i''}\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\
& +\nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\xi^{i''}\int_{[0,1]^3} s_1^2\partial_{i''}\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\
& +\nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\xi^{i''}\int_{[0,1]^3} s_1^2s_2\partial_i\Phi^j(\alpha+s_2s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\partial_{i''}\Phi^{j''}(\alpha+s_3s_2s_1\xi) \\
& \qquad\qquad \nabla^{j''}\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1
\end{aligned}$$

This formula tells us that

$$\begin{aligned}
& \int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|d_2(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \\
\le\ & Ct^2\int_0^1\int_0^1\int_0^1\int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|D^{2,3}u(\Phi(\alpha+s_2s_1\xi))|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\
\le\ & Ct^2\max_{0\le s\le 1}\int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|D^{2,3}u(\Phi(\alpha+s\xi))|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}
\end{aligned} \tag{5.41}$$

Using the same arguments as that in the calculation of $\|r_1\|_{L^2(\mathcal{M})}$, we have

$$\int_{B_{\mathbf{x}_i}^r}\left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|d_2(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \le C\int_{B_{\mathbf{x}_i}^{2r}}|D^3u(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}} \tag{5.42}$$

Combining (5.40) and (5.42), we have

$$\|\nabla r_1(\mathbf{x})\|_{L^2(\mathcal{M})} \le C\|u\|_{H^3(\mathcal{M})} \tag{5.43}$$

The estimates of r_2, r_3 and r_4 are similar as those in our previous paper [17]. In order to make this proof self-consistent, we also give a complete proof of this part.

For r_2, first, notice that

$$\begin{aligned}
\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y}) &= \frac{1}{2t}\partial_{m'}\Phi^j(\alpha)g^{m'n'}\partial_{n'}\Phi^i(\alpha)(x^i-y^i)R_t(\mathbf{x},\mathbf{y}), \\
\frac{\eta^j}{2t}R_t(\mathbf{x},\mathbf{y}) &= \frac{1}{2t}\partial_{m'}\Phi^j(\alpha)g^{m'n'}\partial_{n'}\Phi^i(\alpha)\xi^{i'}\partial_{i'}\Phi^i R_t(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \nabla^j\bar{R}_t(\mathbf{x},\mathbf{y}) - \frac{\eta^j}{2t}R_t(\mathbf{x},\mathbf{y}) \\
= & \frac{1}{2t}\partial_{m'}\Phi^i g^{m'n'}\partial_{n'}\Phi^j(x^j-y^j) - \eta^j R_t(\mathbf{x},\mathbf{y}) \\
= & \frac{1}{2t}\partial_{m'}\Phi^i g^{m'n'}\partial_{n'}\Phi^j\left(x^j-y^j-\xi^{i'}\partial_{i'}\Phi^j\right)R_t(\mathbf{x},\mathbf{y}) \\
= & \frac{1}{2t}\xi^{i'}\xi^{j'}\partial_{m'}\Phi^i g^{m'n'}\partial_{n'}\Phi^j\left(\int_0^1\int_0^1 s\partial_{j'}\partial_{i'}\Phi^j(\alpha+\tau s\xi)\mathrm{d}\tau\mathrm{d}s\right)R_t(\mathbf{x},\mathbf{y})
\end{aligned}$$

Thus, we get

$$\left|\nabla^j \bar{R}_t(\mathbf{x},\mathbf{y}) - \frac{\eta^j}{2t} R_t(\mathbf{x},\mathbf{y})\right| \leq \frac{C|\xi|^2}{t} R_t(\mathbf{x},\mathbf{y})$$
$$\left|\nabla_{\mathbf{x}} \left(\nabla^j \bar{R}_t(\mathbf{x},\mathbf{y}) - \frac{\eta^j}{2t} R_t(\mathbf{x},\mathbf{y})\right)\right| \leq \frac{C|\xi|}{t} R_t(\mathbf{x},\mathbf{y}) + \frac{C|\xi|^3}{t^2} |R_t'(\mathbf{x},\mathbf{y})|$$

Then, we have following bound for r_2,

$$\begin{aligned}
\int_{\mathcal{M}} |r_2(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} &\leq Ct \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) |D^2 u(\mathbf{y})| \mathrm{d}\mu_{\mathbf{y}}\right)^2 \mathrm{d}\mu_{\mathbf{x}} \\
&\leq Ct \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right) \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \\
&\leq Ct \max_{\mathbf{y}} \left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}}\right) \int_{\mathcal{M}} |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \\
&\leq Ct \|u\|^2_{H^2(\mathcal{M})}. \qquad (5.44)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_{M} |\nabla r_2(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} &\leq Ct \int_{\mathcal{M}} \left(\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right) \int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \\
&\leq C\sqrt{t} \max_{\mathbf{y}} \left(\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}}\right) \max_{\mathbf{x}} \left(\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right) \int_{\mathcal{M}} |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \\
&\leq C \|u\|^2_{H^2(\mathcal{M})}. \qquad (5.45)
\end{aligned}$$

r_3 is relatively easy to estimate by using the well known Gauss formula.

$$r_3(\mathbf{x}) = \int_{\partial \mathcal{M}} n^j \eta^i (\nabla^i \nabla^j u) \bar{R}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \qquad (5.46)$$

Now, we turn to bound the last term r_4. Notice that

$$\begin{aligned}
\nabla^j \left(\nabla^j u\right) &= (\partial_{k'} \Phi^j) g^{k'l'} \partial_{l'} \left((\partial_{m'} \Phi^j) g^{m'n'} (\partial_{n'} u)\right) \\
&= (\partial_{k'} \Phi^j) g^{k'l'} \left(\partial_{l'} (\partial_{m'} \Phi^j)\right) g^{m'n'} (\partial_{n'} u) + (\partial_{k'} \Phi^j) g^{k'l'} (\partial_{m'} \Phi^j) \partial_{l'} \left(g^{m'n'} (\partial_{n'} u)\right) \\
&= \frac{1}{\sqrt{g}} (\partial_{m'} \sqrt{g}) g^{m'n'} (\partial_{n'} u) + \partial_{m'} \left(g^{m'n'} (\partial_{n'} u)\right) \\
&= \frac{1}{\sqrt{g}} \partial_{m'} \left(\sqrt{g} g^{m'n'} (\partial_{n'} u)\right) \\
&= \Delta_{\mathcal{M}} u. \qquad (5.47)
\end{aligned}$$

Here we use the fact that

$$\begin{aligned}
(\partial_{k'} \Phi^j) g^{k'l'} \left(\partial_{l'} (\partial_{m'} \Phi^j)\right) &= (\partial_{k'} \Phi^j) g^{k'l'} \left(\partial_{m'} (\partial_{l'} \Phi^j)\right) \\
&= (\partial_{m'} (\partial_{k'} \Phi^j)) g^{k'l'} (\partial_{l'} \Phi^j) \\
&= \frac{1}{2} g^{k'l'} \partial_{m'} (g_{k'l'}) \\
&= \frac{1}{\sqrt{g}} (\partial_{m'} \sqrt{g}) \qquad (5.48)
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
& g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\xi^l)(\partial_l\Phi^i)(\nabla^i\nabla^j u) \\
= \; & -g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\Phi^i)(\nabla^i\nabla^j u) \\
= \; & -g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\Phi^i)(\partial_{m'}\Phi^i)g^{m'n'}\partial_{n'}\left(\nabla^j u\right) \\
= \; & -g^{i'j'}(\partial_{j'}\Phi^j)\partial_{i'}\left(\nabla^j u\right) \\
= \; & -\nabla^j\left(\nabla^j u\right).
\end{aligned} \tag{5.49}
$$

where the first equalities are due to that $\partial_{i'}\xi^l = -\delta^l_{i'}$. Then we have

$$
\begin{aligned}
& \text{div}\;\left(\eta^i(\nabla^i\nabla^j u(\mathbf{y}))\right) + \Delta_{\mathcal{M}} u \\
= \; & \frac{1}{\sqrt{|g|}}\partial_{i'}\left(\sqrt{|g|}g^{i'j'}(\partial_{j'}\Phi^j)\xi^l(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y}))\right) - g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\xi^l)(\partial_l\Phi^i)(\nabla^i\nabla^j u) \\
= \; & \frac{\xi^l}{\sqrt{|g|}}\partial_{i'}\left(\sqrt{|g|}g^{i'j'}(\partial_{j'}\Phi^j)(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y}))\right)
\end{aligned} \tag{5.50}
$$

Here we use the equalities (5.47), (5.49), $\eta^i = \xi^l\partial_{i'}\Phi^l$ and the definition of div,

$$
\text{div}X = \frac{1}{\sqrt{|g|}}\partial_{i'}(\sqrt{|g|}g^{i'j'}\partial_{j'}\Phi^k X^k). \tag{5.51}
$$

where X is a smooth tangent vector field on $\mathcal{M}$ and $(X^1, \ldots, X^d)^t$ is its representation in embedding coordinates.

Hence,

$$
r_4(\mathbf{x}) = \int_{\mathcal{M}} \frac{\xi^l}{\sqrt{|g|}}\partial_{i'}\left(\sqrt{|g|}g^{i'j'}(\partial_{j'}\Phi^j)(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y}))\right)\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}
$$

Then it is easy to get that

$$
\|r_4(\mathbf{x})\|_{L^2(\mathcal{M})} \;\le\; Ct^{1/2}\|u\|_{H^3(\mathcal{M})}, \tag{5.52}
$$

$$
\|\nabla r_4(\mathbf{x})\|_{L^2(\mathcal{M})} \;\le\; C\|u\|_{H^3(\mathcal{M})}. \tag{5.53}
$$

The proof is complete by combining (5.34),(5.43),(5.44),(5.45),(5.46),(5.52),(5.53). □

6 Proof of Theorem 4.4

First, we need the following two theorems which have been proved in [17].

Theorem 6.1. *Assume $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞ and the input data $(P, \mathbf{V})$ is an h-integral approximation of $\mathcal{M}$. Let $f \in C(\mathcal{M})$ in both problems, then there exists constants C, T_0, r_0 depending only on $\mathcal{M}$ and $\partial\mathcal{M}$ so that*

$$
\|T_{t,h}f - T_t f\|_{H^1(\mathcal{M})} \;\le\; \frac{Ch}{t^{3/2}}\|f\|_\infty, \tag{6.1}
$$

as long as $t \le T_0$ and $\frac{h}{\sqrt{t}} \le r_0$.

Then the main idea to prove Theorem 4.4 is to lift the covergence from H^1 to C^1 by using the fact that $T_t u$ and $T_{t,h}u$ have higher order regularity for any $u \in C(\mathcal{M})$. The details are given as following.

Proof. **of Theorem 4.4**:

First, for any $u \in C^1(\mathcal{M})$, we know that $T_t u$ and $T_{t,h}u$ have following representations

$$T_t u = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})T_t u(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_t(\mathbf{x})}\int_{\mathcal{M}} \bar{R}(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{6.2}$$

$$T_{t,h} u = \frac{1}{w_{t,h}(\mathbf{x})}\sum_i R_t(\mathbf{x},\mathbf{x}_i)T_{t,h} u(\mathbf{x}_i)V_i + \frac{t}{w_{t,h}(\mathbf{x})}\sum_i \bar{R}(\mathbf{x},\mathbf{x}_i)u(\mathbf{x}_i)V_i. \tag{6.3}$$

Denote

$$T_t^1 u = \frac{1}{w_{t,h}(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})T_t u(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_{t,h}(\mathbf{x})}\int_{\mathcal{M}} \bar{R}(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{6.4}$$

$$T_t^2 u = \frac{1}{w_{t,h}(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})T_{t,h} u(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_{t,h}(\mathbf{x})}\int_{\mathcal{M}} \bar{R}(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}, \tag{6.5}$$

$$\tag{6.6}$$

Direct calculation would give that

$$\|w_t(\mathbf{x}) - w_{t,h}(\mathbf{x})\|_\infty \le \frac{Ch}{t^{1/2}}, \quad \|\nabla w_t(\mathbf{x}) - \nabla w_{t,h}(\mathbf{x})\|_\infty \le \frac{Ch}{t} \tag{6.7}$$

and

$$\begin{aligned}
\left|\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})T_t u(\mathbf{y})\mathrm{d}\mathbf{y}\right| &\le Ct^{-d/4}\|T_t u\|_{L^2} \le Ct^{-d/4}\|u\|_{H^1} \le Ct^{-d/4}\|u\|_{C^1}, \\
\left|\nabla_{\mathbf{x}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})T_t u(\mathbf{y})\mathrm{d}\mathbf{y}\right| &\le Ct^{-(d+2)/4}\|T_t u\|_{L^2} \le Ct^{-(d+2)/4}\|u\|_{C^1}
\end{aligned} \tag{6.8}$$

and

$$\left|\int_{\mathcal{M}} \bar{R}(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right| \le C\|u\|_\infty, \quad \left|\nabla_{\mathbf{x}}\int_{\mathcal{M}} \bar{R}(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right| \le Ct^{-1/2}\|u\|_\infty \tag{6.9}$$

Using above inequalites, we have

$$\left|T_t u - T_t^1 u\right| \le \frac{Ch}{t^{(d+2)/4}}\|u\|_\infty, \tag{6.10}$$

$$\left|\nabla(T_t u - T_t^1 u)\right| \le \frac{Ch}{t^{d/4+1}}\|u\|_\infty, \tag{6.11}$$

which proves that

$$\left\|T_t u - T_t^1 u\right\|_{C^1} \le \frac{Ch}{t^{d/4+1}}\|u\|_\infty. \tag{6.12}$$

Secondly, we have

$$\begin{aligned}
\left|T_t^1 u - T_t^2 u\right| &= \left|\frac{1}{w_{t,h}(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\left(T_t u(\mathbf{y}) - T_{t,h}u(\mathbf{y})\right)\mathrm{d}\mathbf{y}\right| \\
&\le Ct^{-d/4}\left\|T_t u - T_{t,h}u\right\|_{L^2} \le \frac{Ch}{t^{d/4+3/2}}\|u\|_\infty.
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
\left|\nabla\left(T_t^1 u - T_t^2 u\right)\right| &= \left|\nabla_{\mathbf{x}}\left(\frac{1}{w_{t,h}(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\left(T_t u(\mathbf{y}) - T_{t,h}u(\mathbf{y})\right)\mathrm{d}\mathbf{y}\right)\right| \\
&\le Ct^{-d/4+1/2}\left\|T_t u - T_{t,h}u\right\|_{L^2} \le \frac{Ch}{t^{d/4+2}}\|u\|_\infty.
\end{aligned} \tag{6.14}$$

This implies that

$$\left\|T_t^1 u - T_t^2 u\right\|_{C^1} \;\leq\; \frac{Ch}{t^{d/4+2}}\|u\|_\infty \tag{6.15}$$

Using Theorem 4.5, we have

$$\begin{aligned} |T_{t,h}u| \;&\leq\; Ct^{-d/4}\|u\|_\infty, &(6.16)\\ |\nabla T_{t,h}u| \;&\leq\; Ct^{-(d+2)/4}\|u\|_\infty. &(6.17)\end{aligned}$$

Thus,

$$\begin{aligned} \left|T_{t,h}u - T_t^2 u\right| \;&\leq\; \frac{Ch}{t^{(d+2)/4}}\|u\|_\infty, &(6.18)\\ \left|\nabla\left(T_{t,h}u - T_t^2 u\right)\right| \;&\leq\; \frac{Ch}{t^{d/4+1}}\|u\|_\infty. &(6.19)\end{aligned}$$

which also reads

$$\left\|T_{t,h}u - T_t^2 u\right\|_{C^1} \leq \frac{Ch}{t^{d/4+1}}\|u\|_\infty \tag{6.20}$$

The proof is complete by combining (6.12), (6.15) and (6.20). □

7 Conclusion and Future Work

In this paper, we proved that the convergence of the point integral method for the spectra of the Laplace-Beltrami operator on point cloud. And the rate of convergence is also obtained. This work builds a solid mathematical foundation for many Laplace spectra based algorithm.

In many applications, the sample points of the manifold are draw according to some probability distribution. Then one interesting problem is to study the performance of the point integral method on the random samplesas the number of sample points tends to infinity. Based on the results reported in this paper We can show that with overwhelming probability, the spectra given by the point integral method converge to the spectra of following eigen problem

$$\begin{cases} -\frac{1}{p^2(\mathbf{x})}\nabla\cdot(p^2(\mathbf{x})\nabla u(\mathbf{x})) = \lambda u(x), \;\; \mathbf{x}\in\mathcal{M} \\ \hfill \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, \;\; \mathbf{x}\in\partial\mathcal{M}, \end{cases} \tag{7.1}$$

where p is the probability distribution. The rate of convergence can also be obtained. This result will be reported in our subsequent paper.

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