Surface evolving with tangential velocity field based on surface diffusion

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Abstract

In this note, we present several models to introduce tangential velocity field in surface evolving flow based on surface diffusion. The purpose of introducing tangential velocity is to preserve the distribution of the node points during the evolution of the surface.

1 Introduction

We consider the evolution of a closed surface $\Gamma(t)$ in $\mathbb{R}^3$ under a given velocity field.

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t), \quad X(\boldsymbol{x},0) = \boldsymbol{x}, \quad \boldsymbol{x} \in \Gamma_0 \subset \mathbb{R}^3, \quad t \geq 0 \tag{1.1}$$

with $\boldsymbol{v}$ is a given velocity field in $\mathbb{R}^3 \times [0,T]$, Γ_0 is a closed surface embedded in $\mathbb{R}^3$. $\Gamma(t)$ is the image of the flow $X(\cdot,t)$.

One difficulty in the computing of above surface evolution problem is that the mesh may be distorted during the evolution. To improve the quality of the mesh, many methods have been proposed in the literature, such as BGN method[1, 2, 3]. In this note, we propose several models to introduce tangential velocity field based on the surface diffusion of density function.

2 Tangential velocity induced by surface diffusion

Suppose we add a tangential velocity field $\boldsymbol{v}_T$, then the surface evolution becomes

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t) + \boldsymbol{v}_T(\boldsymbol{x},t), \quad X(\boldsymbol{x},0) = \boldsymbol{x}, \quad \boldsymbol{x} \in \Gamma_0 \subset \mathbb{R}^3, \quad t \geq 0 \tag{2.1}$$

where $\boldsymbol{v}_T(\boldsymbol{x},t) \in \mathcal{T}_{\Gamma(t)}(X(\boldsymbol{x},t))$, $\mathcal{T}_{\Gamma(t)}(X(\boldsymbol{x},t))$ denotes the tangential space of $\Gamma(t)$ at $X(\boldsymbol{x},t)$ and $\Gamma(t)$ is the surface corresponding to $X(\cdot,t)$.

Let $\rho(X(\boldsymbol{x},t),t)$ be the density function in $\Gamma(t)$. With the given velocity field in (2.1), ρ is evolved by the Fokker-Planck type equation

$$\frac{d}{dt}\rho(X(\boldsymbol{x},t),t) = -\rho\, \mathrm{div}_{\Gamma(t)} \left(\boldsymbol{v}(X(\boldsymbol{x},t),t) + \boldsymbol{v}_T(\boldsymbol{x},t)\right), \quad \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_0, \quad t \geq 0 \tag{2.2}$$

where $\mathrm{div}_{\Gamma(t)}$ is the divergence operator in $\Gamma(t)$. Dividing by ρ in both sides, we have

$$\frac{d}{dt}\log\rho(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)} \left(\boldsymbol{v}(X(\boldsymbol{x},t),t) + \boldsymbol{v}_T(\boldsymbol{x},t)\right), \quad \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_0, \quad t \geq 0 \tag{2.3}$$

Let

$$s(X(\boldsymbol{x},t),t) = \log\left(\frac{\rho(X(\boldsymbol{x},t),t)}{\rho_0(\boldsymbol{x})}\right),$$

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then s satisfies

$$\frac{d}{dt}s(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t) + \boldsymbol{v}_T(\boldsymbol{x},t)\right), \quad s(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0. \tag{2.4}$$

The key idea is to choose $\boldsymbol{v}_T$ as

$$\boldsymbol{v}_T(\boldsymbol{x},t) = -\mu\nabla_{\Gamma(t)}s(X(\boldsymbol{x},t),t), \quad \mu>0 \tag{2.5}$$

where $\nabla_{\Gamma(t)}$ is the gradient operator in $\Gamma(t)$ and $\mu>0$ is a parameter. With above tangential velocity field, the equation of s becomes

$$\frac{d}{dt}s(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) + \mu\Delta_{\Gamma(t)}s, \quad s(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0. \tag{2.6}$$

where $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator in $\Gamma(t)$.

Now, s obeys a diffusion equation. The solution of diffusion equation tends to a constant which means that the density also tends to constant. So the velocity field given in (2.5) is capable to preserve the distribution and prevent large deformation .

In summary, the model we propose is as following

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t)) - \mu\nabla_{\Gamma(t)}s(X(\boldsymbol{x},t),t), \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.7}$$

$$\frac{d}{dt}s(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) + \mu\Delta_{\Gamma(t)}s, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.8}$$

$$X(\boldsymbol{x},0)=\boldsymbol{x}, \quad s(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\Gamma_0. \tag{2.9}$$

More generally, we can choose different $s(\boldsymbol{x},0)$ to change the target distribution.

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t)) - \mu\nabla_{\Gamma(t)}s(X(\boldsymbol{x},t),t), \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.10}$$

$$\frac{d}{dt}s(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) + \mu\Delta_{\Gamma(t)}s, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.11}$$

$$X(\boldsymbol{x},0)=\boldsymbol{x}, \quad s(\boldsymbol{x},0)=s_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma_0. \tag{2.12}$$

Then the underlying density ρ satisfies

$$\frac{d}{dt}\log\rho(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) + \mu\Delta_{\Gamma(t)}s, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.13}$$

$$\log\rho(\boldsymbol{x},0) = \log\rho_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma_0. \tag{2.14}$$

It is easy to verify that

$$s(X(\boldsymbol{x},t),t) - s_0(\boldsymbol{x}) = \log\rho(X(\boldsymbol{x},t),t) - \log\rho_0(\boldsymbol{x}) \tag{2.15}$$

So the density becomes

$$\rho(X(\boldsymbol{x},t),t) = \exp(s)\rho_0(\boldsymbol{x})\exp(-s_0(\boldsymbol{x}))$$

If s tends to constant, then

$$\rho \propto \rho_0(\boldsymbol{x})\exp(-s_0(\boldsymbol{x}))$$

Hence, we can choose s_0 to control the distribution in $\Gamma(t)$.

The other way to control the distribution is to set the tangential velocity as

$$\boldsymbol{v}_T(\boldsymbol{x},t) = \mu\nabla_{\Gamma(t)}\log p(X(\boldsymbol{x},t),t) - \mu\nabla_{\Gamma(t)}s(X(\boldsymbol{x},t),t), \quad \mu>0 \tag{2.16}$$

where p is the target distribution.

Then, the whole model becomes

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t)) + \mu\nabla_{\Gamma(t)}\log p(X(\boldsymbol{x},t),t) - \mu\nabla_{\Gamma(t)}s(X(\boldsymbol{x},t),t), \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.17}$$

$$\frac{d}{dt}s(X(\boldsymbol{x},t),t) = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) - \mu\Delta_{\Gamma(t)}\log p + \mu\Delta_{\Gamma(t)}s, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{2.18}$$

$$X(\boldsymbol{x},0)=\boldsymbol{x}, \quad s(\boldsymbol{x},0)=\log\rho_0, \quad \boldsymbol{x}\in\Gamma_0. \tag{2.19}$$

3 Tangential velocity field based on the estimation of ρ

Using equation (2.11) and (2.15), we have

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t)) - \mu\nabla_{\Gamma(t)}\left(\log\rho(X(\boldsymbol{x},t),t) - \log\rho_0(\boldsymbol{x},t)\right) - \mu\nabla_{\Gamma(t)}s_0, \tag{3.1}$$

Since ρ is the density function, if the surface is discretized to a triangular mesh, it is easy to estimate the density at each node. One simple way is to set

$$\rho(X(\boldsymbol{x}_i,t),t) \propto 1/dS_i$$

dS_i is the surface area element at $X(\boldsymbol{x}_i,t)$ and can be estimated as

$$dS_i = \frac{1}{3}\sum_{T_j\sim i}|T_j|$$

T_j is planar triangle of the mesh, $T_j \sim i$ means $X(\boldsymbol{x}_i,t)$ is vertice of T_j, $|T_j|$ denotes the area of T_j.

4 Tangential velocity field based on the explicit solution of ρ

Notice that $\rho(X(\boldsymbol{x},t),t)$ has an explicit representation.

$$\rho(X(\boldsymbol{x},t),t)|\det\boldsymbol{J}(X(\boldsymbol{x},t),t)| = \rho_0(\boldsymbol{x}). \tag{4.1}$$

The flow map $X(\cdot,t)$ induces a map between the tangent space of $\Gamma(0)$ and $\Gamma(t)$. $\boldsymbol{J}(X(\boldsymbol{x},t),t)$ denotes the Jacobi matrix of the map between the tangent spaces.

Then (2.15) implies that

$$s(X(\boldsymbol{x},t),t) = s_0(\boldsymbol{x}) - |\det\boldsymbol{J}(X(\boldsymbol{x},t),t)| \tag{4.2}$$

The equation of s can be eliminated and we get a model of $X(\boldsymbol{x},t)$ only.

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t)) + \mu\nabla_{\Gamma(t)}\log|\det\boldsymbol{J}(X(\boldsymbol{x},t),t)| - \mu\nabla_{\Gamma(t)}s_0, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{4.3}$$

$$X(\boldsymbol{x},0) = \boldsymbol{x}, \quad \boldsymbol{x}\in\Gamma_0. \tag{4.4}$$

5 Divergence constant velocity field

Another way to introduce the tangential velocity field such that the right hand side of eq. (2.3) is a constant over $\Gamma(t)$.

Let $\boldsymbol{v}_T = \nabla_{\Gamma(t)}s$ and

$$\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v} + \nabla_{\Gamma(t)}s\right) = \mathrm{const} \tag{5.1}$$

Then s can be solved by

$$\Delta_{\Gamma(t)}s = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) + \bar{v}(t) \tag{5.2}$$

$\bar{v}(t)$ is a constant such that the average of right hand side of (5.2) is zero.

With this choice of the tangential velocity field, the density is

$$\rho(X(\boldsymbol{x},t),t) = \rho_0(\boldsymbol{x})\exp(\bar{v}(t)),$$

which means that the density does not change up to a constant factor.

In this case, the model is

$$\frac{dX(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(X(\boldsymbol{x},t),t)) + \nabla_{\Gamma(t)}s, \quad \boldsymbol{x}\in\Gamma_0, \quad t\geq 0 \tag{5.3}$$

$$\Delta_{\Gamma(t)}s = -\mathrm{div}_{\Gamma(t)}\left(\boldsymbol{v}(X(\boldsymbol{x},t),t)\right) + \bar{v}(t) \tag{5.4}$$

$$X(\boldsymbol{x},0) = \boldsymbol{x}, \quad \boldsymbol{x}\in\Gamma_0. \tag{5.5}$$

6 Conclusion

We propose several surface evolution models with artificial tangential velocity field. The numerical methods and analysis will be developed based on these models.

References

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