Typical height of the (2+1)-D Solid-on-Solid surface with pinning above a wall in the delocalized phase

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Organization of the talk

1. Background and model
2. Our results
3. Intuition

Background and model

Background: qualitative approximation of Ising model

Three dimensional Ising model on a cube $[\![0, N+1]\!]^3$

- Spin value $\{1, -1\}$ on each site State space $\{-1, 1\}^{[\![0,N+1]\!]^3}$
- Boundary conditions: bottom face -1 and all the other face with $+1$ $\sigma \in \{-1, 1\}^{[\![0,N+1]\!]^3}$

$$\sigma(x, y, z) = \begin{cases} -1 & \text{if } z = 0, \\ 1 & \text{if } z = N+1 \cup x \in \{0, N+1\} \cup y \in \{0, N+1\}. \end{cases}$$

- Ising measure ($\beta = \frac{1}{T} \gg 1$ inverse temperature)

$$\mathbb{P}_{N,\beta}(\sigma) = \frac{1}{Z_{N,\beta}} \exp\left(\beta \sum_{\substack{i,j \in [\![0,N+1]\!]^3 \\ i \sim j}} \sigma_i \sigma_j\right)$$

- **Q:** the "-1" component incident to the bottom face?

$$f : [\![1, N]\!]^2 \to [\![0, N]\!]$$

The solid-on-solid model: a crystal surface model

Introduced by [Burton, Cabrera, Frank '51] [Temperley '52]

$(d+1)$-D SOS model on $\mathbb{Z}^d$:

- Box $\Lambda_N := [\![1, N]\!]^d$ External boundary $\partial\Lambda_N$

$$\partial\Lambda_N := \left\{ x \in \mathbb{Z}^d \setminus \Lambda_N : \ \exists y \in \Lambda_N, x \sim y \right\}$$

- State space $\phi \in \widetilde{\Omega}_{\Lambda_N} := \mathbb{Z}^{\Lambda_N} = \{f : \Lambda_N \to \mathbb{Z}\}$ Hamiltonian (0 b.c.)

$$\mathcal{H}_N(\phi) := \sum_{\substack{\{x,y\} \subset \Lambda_N \\ x \sim y}} |\phi(x) - \phi(y)| + \sum_{\substack{x \in \Lambda_N,\, y \in \partial\Lambda_N \\ x \sim y}} |\phi(x)|.$$

- SOS probability measure ($\beta > 0$ inverse temperature)

$$\forall \phi \in \widetilde{\Omega}_N, \qquad \mathbf{P}_N^{\beta}(\phi) := \frac{1}{\widetilde{\mathcal{Z}}_N^{\beta}} e^{-\beta \mathcal{H}_N(\phi)}$$

$$\widetilde{\mathcal{Z}}_N^{\beta} := \sum_{\psi \in \widetilde{\Omega}_N} e^{-\beta \mathcal{H}_N(\psi)} \leq \left(\frac{1 + e^{-d\beta}}{1 - e^{-d\beta}} \right)^{|\Lambda_N|}$$

SOS: rigid/rough

- $d = 1$: rough (delocalized) [Temperley '52, '56] [Fisher '84]
  for all $\beta > 0$, the expectation of the absolute value of the height at the center diverges in the thermodynamic limit.

- $d \geq 3$: rigid (localized) [Bricmont, Fontaine, Lebowitz '82]
  for all $\beta > 0$, the expectation of the absolute value of the height at the center is uniformly bounded (by Peierls argument).

- $d = 2$ a phase transition between rough and rigid
  - rough: for small β ([Fröhlich, Spencer '81, '83])
  - rigid: for large β ([Brandenberger, Wayne '82], [Gallavotti, Martin-Löf, Miracle-Solé '73]).
  - Numerical critical point: $\beta_c \approx 0.806$

(2+1)-D SOS above a hard wall

- Above a hard wall

$$\forall \phi \in \Omega_N := \left\{ \phi \in \widetilde{\Omega}_N : \phi \geq 0 \right\}, \qquad \mathbb{P}_N^{\beta}(\phi) := \mathbf{P}_N^{\beta}(\phi) / \mathbf{P}_N^{\beta}(\Omega_N).$$

- [Bricmont, Mellouki, and Fröhlich '86]: for large β, the average height H of the surface satisfies

$$\frac{1}{C\beta} \log N \leq H \leq \frac{C}{\beta} \log N.$$

- [Caputo, Lubetzky, Martinelli, Sly, Toninelli '14] for $\beta \geq 1$, the typical height of the surface concentrates at

$$H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$$

with fluctuations of order $O(1)$.

Typical height of (2+1)-D SOS above a wall

Theorem (Caputo, Lubetzky, Martinelli, Sly, Toninelli '14)

There exist two universal constants $C, K > 0$ such that for all $\beta \geq 1$ and all integer $k \geq K$, we have for all N,

$$\mathbb{P}_N^\beta\left(|\{x \in \Lambda_N : \ \phi(x) \geq H + k\}| > e^{-2\beta k} N^2\right) \leq e^{-Ce^{-2\beta k} N\left(1 \wedge e^{-2\beta k} N \log^{-8} N\right)}$$

and

$$\mathbb{P}_N^\beta\left(|\{x \in \Lambda_N : \ \phi(x) \leq H - k\}| > e^{-2\beta k} N^2\right) \leq e^{-e^{\beta k} N}.$$

Entropic repulsion: In the large β regime, the presence of an impenetrable wall pushes the surface up to the height of order $\frac{1}{4\beta} \log N$, instead of remaining uniformly bounded when no wall is present.

The $(2+1)$-D SOS surface with pinning above a wall

- State space $\Omega_N = \mathbb{Z}_+^{\Lambda_N}$
- Inverse temperature $\beta > 0$, pinning parameter $h \geq 0$
- Probability measure $\mathbb{P}_N^{\beta,h}$: above a wall, with 0 b.c., pinning reward h,

$$\mathbb{P}_N^{\beta,h}(\phi) := \frac{1}{\mathcal{Z}_N^{\beta,h}} e^{-\beta\mathcal{H}_N(\phi)+h|\{x\in\Lambda_N:\ \phi(x)=0\}|},$$

$$\mathcal{Z}_N^{\beta,h} := \sum_{\phi\in\Omega_N} e^{-\beta\mathcal{H}_N(\phi)+h|\{x\in\Lambda:\ \phi(x)=0\}|} \leq e^{h|\Lambda_N|}\left(\frac{1+e^{-2\beta}}{1-e^{-2\beta}}\right)^{|\Lambda_N|}.$$

- Free energy ($log\mathcal{Z}_\Lambda^{\beta,h}$ is superadditive for disjoint boxes $\Rightarrow \exists$ limit)

$$\mathrm{F}(\beta,h) := \lim_{N\to\infty}\frac{1}{N^2}\log\mathcal{Z}_N^{\beta,h}$$

$\mathrm{F}(\beta,h)$ is increasing and convex in h by Hölder's inequality: $\theta\in[0,1]$

$$\mathcal{Z}_N^{\beta,\theta h_1+(1-\theta)h_2} \leq \left(\mathcal{Z}_N^{\beta,h_1}\right)^{\theta}\cdot\left(\mathcal{Z}_N^{\beta,h_2}\right)^{1-\theta}.$$

The $(2+1)$-D SOS surface with pinning above a wall

- When $\mathrm{F}(\beta, h)$ is differentiable in h, the convexity allows us to exchange the order of limit and derivative to obtain the asymptotic contact fraction

$$\partial_h \mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N^2} \mathbb{E}_N^{\beta,h}\left[|\phi^{-1}(0)|\right].$$

- [Chalker '82]: Existence of criticality

$$h_w(\beta) := \sup\{h \in \mathbb{R}_+ : \mathrm{F}(\beta, h) = \mathrm{F}(\beta, 0)\} > 0 \quad \text{for all } \beta > 0$$

separates the delocalized phase $(\partial_h \mathrm{F}(\beta, h) = 0)$ from the localized phase $(\partial_h \mathrm{F}(\beta, h) > 0)$. Furthermore, for all $\beta > 0$

$$\log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right) \le h_w(\beta) \le \log\left(\frac{16(e^{4\beta}+1)}{e^{4\beta}-1}\right).$$

- [Alexander, Dunlop, Miracle-Solé, '11]: the lower bound above is asymptotically sharp, and when h decreases to h_w the system undergoes a sequence of layering transitions.

The $(2+1)$-D SOS surface with pinning above a wall

- [Lacoin '18]: for $\beta > \beta_1 \in (\log 2, \log 3)$

$$h_w(\beta) = \log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right)$$

and there exists a constant C_β such that

$$\forall u \in (0,1], \quad C^{-1}u^3 \leq \mathrm{F}(\beta, u + h_w(\beta)) - \mathrm{F}(\beta, h_w(\beta)) \leq Cu^3.$$

- [Lacoin '21]: when $h > h_w$, a complete picture of the typical height, the Gibbs states and regularity of the free energy.

- **Q:** When $0 \leq h \leq h_w$, how does the surface look like?

Our results: typical height in delocalized phase

Our result (Subcritical regime: $h \in (0, h_w)$)

Pinning does not change the typical height ($h = 0$).

Theorem (N. Feldheim, Y. '23)

Fix $\beta \geq 1$, $h \in (0, h_w)$ *and* $N \geq 1$. *Let* $H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$.

1. *There exist universal constants* $C, K > 0$ *s.t. for all integer* $m \geq K$,

$$\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([H+m,\infty))\right| > e^{-2\beta m} N^2\right) \leq e^{-Ce^{-2\beta m}N\left(1 \wedge e^{-2\beta m} N \log^{-8} N\right)}.$$

2. *For* $\delta > 0$ *and* $m \in \mathbb{N}$ *we have*

$$\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([0,H-m])\right| > 2e^{-2\beta m} N^2\right) \leq 3e^{-\min\left(\frac{1}{2}e^{2\beta m} - 4\beta(1+\kappa),\, \delta\right)N}.$$

where (for $h \in (0, h_w)$, $e^{-h} + e^{-4\beta} > 1$)

$$\kappa(\beta, h, \delta) := \frac{4\beta + \delta}{\log\left(e^{-h} + e^{-4\beta}\right)}.$$

At criticality: conjecture and result

At $h = h_w$, Lacoin conjectured: the surface height concentrates around

$$H_w := \left\lfloor \frac{1}{6\beta} \log N \right\rfloor ,$$

with fluctuations similar to the subcritical regime.

Isolated and non-isolated zeros

$$q_1(\phi) := \{x \in \Lambda_N : \ \phi(x) = 0, \forall y \in \Lambda_N, y \sim x, \phi(y) \geq 1\} ,$$
$$q_{2+}(\phi) := \{x \in \Lambda_N : \ \phi(x) = 0, \exists y \in \Lambda_N, y \sim x, \phi(y) = 0\} .$$

Theorem (N. Feldheim, Y. '23)

For $\beta \geq 1$ and $h = h_w$, we have for all $N \in \mathbb{N}$ and $C > 0$:

$$\mathbb{P}_N^{\beta, h_w} \left(\phi \in \Omega_N : \ |q_{2+}(\phi)| \geq CN\right) \leq e^{-N\left(\frac{C}{20} e^{-6\beta} - 4\beta\right)} .$$

At criticality: lower bound on the height

Proposition

For all $\beta \geq 1$, $C > 0$, $h = h_w$, $N \in \mathbb{N}$ *and* $m \in \mathbb{N}$, *letting* $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$ *we have*

$$\mathbb{P}_N^{\beta, h_w}\left(\left\{|\phi^{-1}(0)| \leq CN^{\frac{4}{3}}\right\} \bigcap \left\{\left|\phi^{-1}([1, H_w - m])\right| \geq 2e^{-2\beta m} N^2\right\}\right) \leq 2\exp\left(4\beta N + 4\beta C N^{\frac{4}{3}} - \frac{1}{2} e^{2\beta m} N^{\frac{4}{3}}\right).$$

It suffices to prove that for large enough $C > 0$, we have

$$\mathbb{P}_N^{\beta, h_w}\left(|q_1(\phi)| > CN^{4/3}\right) = o(1)$$

in order to obtain a lower bound on the typical height of the surface at criticality, matching the conjectured height H_w.

Intuition

Subcritical regime: typical height $H = \lfloor \frac{1}{4\beta} \log N \rfloor$

When $\beta \gg 1$, the surface is roughly flat and repelled to height H by lifting the inner boundary sites to H. By spatial mixing property, the isolated zeros are roughly IID with cardinality $N^2 \exp(-4\beta H)$. At equilibrium the penalty (lifting the boundaries up) and reward from the pinning balance,

$$\exp(-4\beta NH) \exp\left(hN^2 \exp(-4\beta H)\right) \asymp 1.$$

At criticality: typical height $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$

The penalty for lifting the surface up to H_w is $\exp(-4\beta NH_w)$, and the reward is mainly from the zeros of size two with cardinality $N^2 \exp(-6\beta H_w)$. At equilibrium, the two effects balance,

$$\exp\left(-4\beta NH_w\right) \exp\left(N^2 \exp(-6\beta H_w)\right) \asymp 1.$$

The isolated zeros is roughly IID with cardinality $N^2 e^{-4\beta H_w} = N^{4/3}$.

Thank you for your attention!