

Spectral gap and cutoff of Simple Exclusion Process with IID conductances

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Seminário de sistema dinâmico da UFF

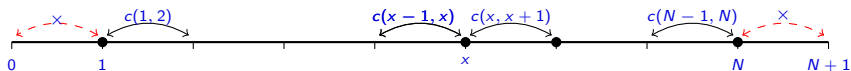
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Setup: Simple Exclusion Process with inhomogeneous conductance

Setup: Simple Exclusion Process (SEP)

Conductances: $(c(x, x+1))_{x \in \mathbb{N}}$ with values in $(0, \infty)$.

SEP with k particles in $\llbracket 1, N \rrbracket$ with conductances $c(x, x+1)_{x \in \mathbb{N}}$.



- (A) Each site is occupied by at most one particle (*the exclusion rule*).
- (B) At each edge $\{x, x+1\}$ with $1 \leq x < N$, we place a Poisson clock with rate $c(x, x+1) > 0$. When a clock rings, we swap the contents of the two sites.

Setup

- State space (1: particle 0: empty site.)

$$\Omega_{N,k} := \left\{ \xi : \llbracket 1, N \rrbracket \rightarrow \{0, 1\} \mid \sum_{i=1}^N \xi(i) = k \right\}.$$

- Generator: ($f : \Omega_{N,k} \mapsto \mathbb{R}$)

$$(\mathcal{L}_{N,k} f)(\xi) := \sum_{i=1}^{N-1} c(i, i+1) [f(\xi \circ \tau_{i,i+1}) - f(\xi)],$$

where $\tau_{i,j}$ is the transposition of the two elements i and j .

- Uniform prob measure $\mu_{N,k}$ satisfies the detailed balance condition:

$$\mu(\xi) \mathcal{L}_{N,k}(\xi, \eta) = \mu(\eta) \mathcal{L}_{N,k}(\eta, \xi),$$

then μ is the invariant prob measure.

Setup

- Distance to equilibrium

$$d_{N,k}(t) := \max_{\xi \in \Omega_{N,k}} \|P_t^\xi - \mu_{N,k}\|_{\text{TV}}.$$

P_t^ξ : marginal distribution at instant t of the chain starting with ξ .

- ε -mixing time

$$t_{\text{mix}}^{N,k}(\varepsilon) := \inf \{t \geq 0 : d_{N,k}(t) \leq \varepsilon\}.$$

- Cutoff: for all $\varepsilon \in (0, 1)$,

$$\lim_{N \rightarrow \infty} \frac{t_{\text{mix}}^{N,k}(\varepsilon)}{t_{\text{mix}}^{N,k}(1-\varepsilon)} = 1.$$

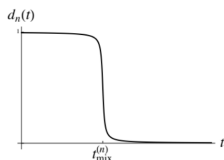


image from Levin and Peres

Setup

- : Spectral gap $\text{gap}_{N,k}$: minimal nonzero eigenvalue of $-\mathcal{L}_{N,k}$

$$\text{gap}_{N,k} := \inf_{f : \text{Var}_{\mu_{N,k}}(f) > 0} \frac{-\langle f, \mathcal{L}_{N,k} f \rangle_{\mu_{N,k}}}{\text{Var}_{\mu_{N,k}}(f)}$$

where $\text{Var}_{\mu_{N,k}}(f) := \langle f, f \rangle_{\mu_{N,k}} - \langle f, \mathbf{1} \rangle_{\mu_{N,k}}^2$.

- Relation between spectral gap and mixing time/distance to equilibrium:

$$\frac{1}{\text{gap}_{N,k}} \log \frac{1}{2\varepsilon} \leq t_{\text{mix}}^{N,k}(\varepsilon) \leq \frac{1}{\text{gap}_{N,k}} \log \frac{1}{2\varepsilon \mu_{\min}}$$

where $\mu_{\min} := \min_{\xi \in \Omega_{N,k}} \mu_{N,k}(\xi)$.

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log d_{N,k}(t) = -\text{gap}_{N,k}.$$

Question: How does the disordered setup (inhomogeneous conductances) affect the system in terms of spectral gap/mixing time?

Previous Results

Homogeneous conductances $c(x, x+1) \equiv 1$ for one particle ($k = 1$)

- Spectral gap

$$\text{gap}_{N,1} = 2(1 - \cos(\pi/N)) = (1 + o(1))\pi^2/N^2.$$

- Eigenfunctions

$$g_i^{(N)}(x) := \cos(i\pi(x - 1/2)/N), \quad 0 \leq i < N.$$

- Eigenvalues

$$-\lambda_i^{(N)} = -2(1 - \cos(i\pi/N)) \quad \mathcal{L}_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)}.$$

Homogeneous conductances $c(x, x+1) \equiv 1$ for many particles

- [Aldous] [Wilson] [Lacoin] Assuming $\liminf_{N \rightarrow \infty} \min(k, N - k) = \infty$,

$$t_{\min}^{N,k}(\varepsilon) = (1 + o(1)) \frac{N^2}{2\pi^2} \log k, \quad \text{gap}_{N,k} = \text{gap}_{N,1} = (1 + o(1)) \frac{\pi^2}{N^2}.$$

Previous results

Inhomogeneous conductance $c(x, x+1) > 0$

- Aldous' spectral gap conjecture (Proved by [Caputo, Liggett, Richthammer, JAMS '10]):

$$\text{gap}_{N,k} = \text{gap}_{N,1}, \quad \forall k \in \llbracket 1, N-1 \rrbracket.$$

- A function $f : \llbracket 1, N \rrbracket \rightarrow \mathbb{R}$ for $2 \leq b \leq c \leq N-1$
Local maximum at $\llbracket b, c \rrbracket$ if f is constant on $\llbracket b, c \rrbracket$, $f(b-1) < f(b)$ and $f(c) > f(c+1)$.
Analogous definition holds for a local minimum.
 f is j -monotone if it displays exactly $(j-1)$ distinct local extrema in $\llbracket 2, N-1 \rrbracket$.
Nodal domains:

#connected components of $\{x \in \llbracket 1, N \rrbracket, f(x) \neq 0\}$.

- [Miclo]: $L_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)}$ with $0 = \lambda_0^{(N)} < \lambda_1^{(N)} < \dots < \lambda_{N-1}^{(N)}$
 $g_i^{(N)}$ is i -monotone and has $i+1$ nodal domains.

Our results

Proposition (Y. '25)

For any positive conductances $(c(x, x+1))_{x \in \mathbb{N}}$, $g_1^{(N)}$ is strictly monotone.

Write $r(x, x+1) := 1/c(x, x+1)$ and $r(n, m) := \sum_{x=n}^{m-1} r(x, x+1)$.

Assume (LLN) condition

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sup_{2 \leq m \leq N} \left| (r^{(N)}(1, m) - (m-1)) \right| = 0. \quad (\text{LLN})$$

When $(r^{(N)}(x-1, x))_{2 \leq x \leq N}$ is IID with expectation $\mathbb{E}[r(x, x+1)] = 1$, by the strong LLN we have

$$\mathbb{P} \left(\lim_{N \rightarrow \infty} \frac{1}{N} \max_{2 \leq m \leq N} |r(1, m) - (m-1)| = 0 \right) = 1.$$

Our results

Theorem (Y. '25)

If the (LLN) condition on the resistances holds, we have

$$\lim_{N \rightarrow \infty} \frac{N^2 \text{gap}_N}{\pi^2} = 1.$$

Furthermore, concerning the shape and (weighted) derivative of the eigenfunction g_1 with $g_1(1) := 1$ corresponding to the spectral gap, i.e. $\mathcal{L}_{N,1}g_1 = -\text{gap}_N \cdot g_1$ and setting

$$h(x) := \cos\left(\frac{\pi(x - 1/2)}{N}\right), \quad \forall x \in \llbracket 1, N \rrbracket,$$

we have $((c\nabla f)(x) := c(x-1, x)[f(x) - f(x-1)])$

$$\lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} |g_1(x) - h(x)| = 0,$$

$$\lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} |N(c\nabla g_1)(x) - N(\nabla h)(x)| = 0.$$

Our results

Remark

The method in the forementioned theorem also works for the other j -monotone eigenfunctions under the (LLN) assumption, i.e. with $K_0 \in \mathbb{N}$ being any prefixed constant, for all $1 \leq i \leq K_0$,

$$\begin{aligned}\lim_{N \rightarrow \infty} |\lambda_i N^2 / \pi^2 - i^2| &= 0, \\ \lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} \left| g_i(x) - \cos \left(\frac{i\pi(x - 1/2)}{N} \right) \right| &= 0, \\ \lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} |N(c\nabla g_i)(x) - N(\nabla h_i)(x)| &= 0,\end{aligned}$$

where $g_i(1) = 1$.

Our results: mixing time

Assumption

Exist constants $\nu \in (0, 1)$ and $C_{\mathbb{P}} > 0$, a sequence of positive numbers $(\bar{\Upsilon}_N)_N > 0$ with $\lim_{N \rightarrow \infty} \bar{\Upsilon}_N = 0$ and $\lim_{N \rightarrow \infty} \bar{\Upsilon}_N \log N = \infty$ such that

$$\begin{aligned} \max_{1 \leq x < N} r(x, x+1) &\leq C_{\mathbb{P}} \exp((\log N)^\nu), \\ \min_{1 \leq x < N} r(x, x+1) &\geq \bar{\Upsilon}_N. \end{aligned}$$

Exists $\varrho \in (0, 1]$ and $c_\varrho > 0$ such that

$$c_\varrho N^\varrho \leq k_N \leq N/2.$$

Theorem (Y. '25)

Under (LLN) and the assumption above, for all $\varepsilon \in (0, 1)$ we have

$$\lim_{N \rightarrow \infty} \frac{2\pi^2 t_{\text{mix}}^{N,k}(\varepsilon)}{N^2 \log k_N} = 1.$$

Outline

- Idea for the j -monotonicity of eigenfunctions
- Idea for the spectral gap
- Idea for the shape & derivative of eigenfunction
- Idea for the lower bound on the mixing time
- Idea for the upper bound on the mixing time

Idea: j —monotonicity of eigenfunctions

Idea: j -monotonicity of eigenfunctions

$\mathcal{L}_{N,1}$ is a symmetric matrix. Then it is diagonalizable: $g_0 = \mathbf{1}$ and

$$\begin{cases} 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}, \\ \mathcal{L}_{N,1} g_i = -\lambda_i g_i \text{ and } g_i(1) = 1, & \forall i \in \llbracket 0, N-1 \rrbracket, \\ \frac{1}{N} \sum_{x=1}^N g_i(x) g_j(x) = C_{i,j} \delta_{i,j}, & \forall i, j \in \llbracket 0, N-1 \rrbracket. \end{cases}$$

$\delta_{i,j}$: Kronecker delta $(C_{i,j})_i$ are some positive constants. Observe:

$$(c\nabla g_i)(x+1) - (c\nabla g_i)(x) = -\lambda_i g_i(x) \Rightarrow$$

$$(c\nabla g_i)(x+1) = -\lambda_i \sum_{y=1}^x g_i(y).$$

Given $c(x, x+1)_x$, $g_i(1) = 1$ and λ_i together determine g_i , implying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}.$$

Assuming g_i is i -monotone, use the variational formula to show

g_1 is strictly monotone.

$F_i(\xi) := \sum_{x=1}^N g_i(x) \xi(x)$ is an eigenfunction of $\mathcal{L}_{N,k}$ with eigenvalue $-\lambda_i$.

F_1 is monotone in the natural partial order $\Rightarrow \text{gap}_{N,k} = \lambda_1$.

Idea: j -monotonicity of eigenfunctions

Setting $c(N, N+1) = 1$, for $\lambda > 0$, define $f^\lambda : \llbracket 0, N+1 \rrbracket \mapsto \mathbb{R}$ by $f^\lambda(0) = f^\lambda(1) = 1$ and for $x \in \llbracket 1, N \rrbracket$,

$$f^\lambda(x+1) = f^\lambda(x) + \frac{1}{c(x, x+1)} \left[(c\nabla f^\lambda)(x) - \lambda f^\lambda(x) \right].$$

Note that (the restriction to $\llbracket 1, N \rrbracket$ of) f^λ is an eigenfunction of $\mathcal{L}_{N,1}$ if and only if

$$f^\lambda(N+1) = f^\lambda(N).$$

There is no eigenfunction satisfying $f^\lambda(1) = 0$ or $f^\lambda(N) = 0$.

Idea: j -monotonicity of eigenfunctions

For $\lambda > 0$ and $x \in \llbracket 1, N+1 \rrbracket$, we set

$$b(\lambda, x) := -\frac{(c\nabla f^\lambda)(x)}{f^\lambda(x-1)}$$

convention: $b(\lambda, x) = \overline{\infty}$ if $f^\lambda(x-1) = 0$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\overline{\infty}\}$. We have

$$b(\lambda, x+1) = \frac{b(\lambda, x)}{1 - c(x-1, x)^{-1}b(\lambda, x)} + \lambda.$$

Given a fixed $c > 0$, define $\Xi^{(c)} : \mathbb{R} \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ as

$$\Xi^{(c)}(\lambda, b) = \frac{b}{1 - c^{-1}b} + \lambda.$$

The function $b \mapsto \Xi^{(c)}(\lambda, b)$ may have zero, one or two fixed points depending on the values of λ and c , see the following figure.

Idea: j -monotonicity of eigenfunctions

$$\Xi^{(c)}(\lambda, b) = \frac{b}{1 - c^{-1}b} + \lambda.$$

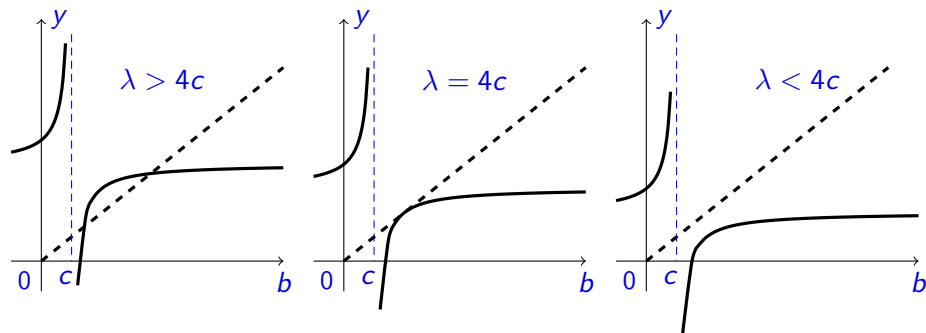


Figure: Solid lines: $b \mapsto \Xi^{(c)}(b, \lambda)$ with $\lambda > 0$ fixed. Black dashed lines: $y = b$. Blue dashes lines: $b = c$.

If $b \mapsto \Xi^{(c)}(\lambda, b)$ has fixed points b_1 and b_2 (not necessarily distinct) such that $b_1 \leq b_2$, we define $I(\lambda, c) = [b_1, b_2]$. Otherwise, set $I(\lambda, c) = \emptyset$.

Idea: j –monotonicity of eigenfunctions

Define the “angle mapping” function

$$\varphi(c, \lambda, \theta) := \inf\{\theta' \geq \theta + \pi \mathbf{1}_{I(\lambda, c)}(\tan \theta) : \tan \theta' = \Xi^{(c)}(\lambda, \tan \theta)\}.$$

Recursively define an “angle” $\theta(\lambda, x)$: $\theta(\lambda, 1) = 0$ and for $x \in \llbracket 1, N \rrbracket$,

$$\theta(\lambda, x+1) := \varphi(c(x-1, x), \lambda, \theta(\lambda, x))$$

with convention: $\tan(\pi/2 + k\pi) = \overline{\infty}$ for $k \in \mathbb{Z}$.

Lemma

For fixed $c, \lambda > 0$, the map $\theta \mapsto \varphi(c, \lambda, \theta)$ is continuous and strictly increasing.

Lemma

For fixed $c, \theta > 0$, the map $\lambda \mapsto \varphi(c, \lambda, \theta)$ is strictly increasing and uniformly continuous in θ .

Idea: j -monotonicity of eigenfunctions

Lemma

For fixed $c > 0$, the map $(\lambda, \theta) \mapsto \varphi(c, \lambda, \theta)$ is jointly continuous.

f^λ is an eigenfunction if and only if $\theta(\lambda, N+1)$ is a multiple of π .

$$f^\lambda \text{ is an eigenfunction} \quad \Leftrightarrow \quad \theta(\lambda, N+1) = k\pi \text{ for } k \in \llbracket 0, N-1 \rrbracket.$$

Let $\lambda_k > 0$ denote the unique number satisfying $\theta(\lambda_k, N+1) = k\pi$ and set $f_k := f^{\lambda_k}$. Let $x_i \in \llbracket 1, N \rrbracket$ such that $\theta(\lambda_k, x_i) \leq i\pi < \theta(\lambda_k, x_i + 1)$ for $i \in \llbracket 1, k-1 \rrbracket$.

Lemma

For λ_k mentioned above and the associated sequence $(x_i)_i$, we have that $\# \{(x_i)_i\} = k-1$, $1 < x_i < N$ are the local extrema of f_k (or the pair $\{x_i-1, x_i\}$ when $\theta(\lambda_k, x_i) = i\pi$) and no any other local extrema.

Idea: the spectral gap

Idea for the spectral gap

Setting $B^{(N)}(x) := b(\lambda, x)N$ and $\lambda := \alpha/N^2$, we have

$$B^{(N)}(x+1) = \frac{B^{(N)}(x)}{1 - N^{-1}r^{(N)}(x-1, x)B^{(N)}(x)} + \frac{\alpha}{N},$$

which starts from $B^{(N)}(1) := 0$.

$$N[B(x+1) - B(x)] = \frac{r(x-1, x)B(x)^2}{1 - B(x)r(x-1, x)N^{-1}} + \alpha$$

Intuition: the asymptotic ODE

$$\begin{cases} \frac{dy}{dx} = y^2 + \alpha, & x \in (0, 1) \\ y(0) = 0. \end{cases}$$

Its unique solution:

$$y(x) = \sqrt{\alpha} \tan(\sqrt{\alpha} \cdot x).$$

Therefore $\alpha = i^2\pi^2$.

Idea: shape/derivative of the
eigenfunction

Idea for shape of the eigenfunction

Eigenfunction corresponding to the spectral gap when $r(j-1, j) \equiv 1$

$$h(x) = h_N(x) = \cos\left(\frac{\pi(x-1/2)}{N}\right).$$

The spectral gap is

$$\bar{\lambda} := 2 \left[1 - \cos\left(\frac{\pi}{N}\right) \right] = \frac{\pi^2}{N^2} + O\left(\frac{1}{N^4}\right).$$

By $b(\lambda, x) = -\frac{(c\nabla g)(x)}{g(x-1)}$ and $b(\lambda, x) = B(x)/N$, for $x \geq 2$ we have

$$g(x) = [1 - r(x-1, x)N^{-1}B(x)] g(x-1).$$

Writing $u(x) := h(x) - g(x)$, we have

$$\begin{aligned} u(x) &= u(x-1) \left[1 - \frac{r(x-1, x)B(x)}{N} \right] \\ &\quad + \frac{h(x-1)}{N} [r(x-1, x)B(x) - \bar{B}(x)]. \end{aligned}$$

Iterate the equation above to conclude the proof for $x \leq N/3$.

Idea for the derivative of the eigenfunction

For $x \leq N/3$: by

$$(c\nabla g_i)(x+1) = -\lambda_i \sum_{y=1}^x g_i(y),$$

we have

$$\begin{aligned} & |N(c\nabla g)(x) - N(\nabla h)(x)| \\ &= \left| -N\lambda_1 \sum_{k=1}^{x-1} g(k) + N\bar{\lambda} \sum_{k=1}^{x-1} h(k) \right| \\ &\leq \left| -N\lambda_1 \sum_{k=1}^{x-1} [g(k) - h(k)] \right| + \left| N(\bar{\lambda} - \lambda_1) \sum_{k=1}^{x-1} h(k) \right| \\ &\leq N\lambda_1 \sum_{k=1}^{x-1} |g(k) - h(k)| + N|\bar{\lambda} - \lambda_1|(x-1). \end{aligned}$$

For $N/3 \leq x \leq 2N/3$: use $A(x) = 1/B(x)$

$$A(x) \left[1 - \frac{g(x)}{g(x-1)} \right] = r(x-1, x)N^{-1}.$$

Idea: the lower bound on the mixing time

Idea for the lower bound on the mixing time

$$F(\xi) = F_1(\xi) = \sum_{1 \leq x \leq N} \xi(x) g_1(x)$$

is an eigenfunction satisfying $\mathcal{L}_{N,k} F = -\text{gap}_N \cdot F$. For $t_0 > 0$, define

$$F(t, \xi) := e^{\lambda_1(t-t_0)} F(\xi), \quad \forall \xi \in \Omega_{N,k},$$

and study the Dynkin martingale

$$M_t := F(t, \eta_t^\nu) - F(0, \eta_0^\nu) - \int_0^t (\partial_s + \mathcal{L}_{N,k}) F(s, \eta_s^\nu) ds.$$

$$\mathbf{E}[F(\eta_{t_0}^\nu)] = \mathbf{E}[F(t_0, \eta_{t_0}^\nu)] = \mathbf{E}[F(0, \eta_0^\nu)] = e^{-\lambda_1 t_0} \mathbf{E}[F(\eta_0^\nu)].$$

$$\mathbf{E}[M_{t_0}^2] = \mathbf{E}\left[\int_0^{t_0} \partial_s \langle M \rangle_s ds\right]$$

$$\bar{\eta}_t^\nu(x, x+1) := \eta_t^\nu(x)(1 - \eta_t^\nu(x+1)) + \eta_t^\nu(x+1)(1 - \eta_t^\nu(x))$$

$$\partial_t \langle M \cdot \rangle_t = e^{2\lambda_1(t-t_0)} \sum_{x=1}^{N-1} \bar{\eta}_t^\nu(x, x+1) r(x, x+1) [c(x, x+1)(g(x) - g(x+1))]^2.$$

Idea for the lower bound on the mixing time

At equilibrium

$$\mathbf{E} [F(\eta_{t_0}^\mu)] = \mu_{N,k}(F) = \frac{k}{N} \sum_{1 \leq x \leq N} g_1(x) = 0, \quad \text{Var}_\mu(F) \asymp k.$$

- ① If ν concentrates at one configuration, then

$$\mathbf{E} [F(\eta_{t_0}^\nu) - \mathbf{E} [F(\eta_{t_0}^\nu)]]^2 = \mathbf{E} [F(t_0, \eta_{t_0}^\nu) - F(0, \eta_0^\nu)]^2 = \mathbf{E} [M_{t_0}^2].$$

- ② If ν is non-degenerated, we have

$$\begin{aligned} & \mathbf{E} [F(\eta_{t_0}^\nu) - \mathbf{E} [F(\eta_{t_0}^\nu)]]^2 \\ &= \mathbf{E} [F(t_0, \eta_{t_0}^\nu) - F(0, \eta_0^\nu) + F(0, \eta_0^\nu) - \mathbf{E} [F(0, \eta_0^\nu)]]^2 \\ &\leq 2\mathbf{E} [M_{t_0}^2] + 2\mathbf{E} [F(0, \eta_0^\nu) - \mathbf{E} [F(0, \eta_0^\nu)]]^2. \end{aligned}$$

If $N/64 \leq k \leq N/2$, take $\nu = \delta_\wedge$.

If $(\log N)^{1+\gamma} \leq k < N/64$, take ν as follows: sample a configuration according to $\mu_{N,2k}$, keep the first k particles and project the rest to be empty sites.

An explanation for the lower bound $t_{\text{mix}} \geq \frac{1}{2\lambda_1} \log k - \frac{C}{\lambda_1}$:

$$ke^{-\lambda_1 \cdot (\frac{1}{2\lambda_1} \log k - \frac{C}{\lambda_1})} = e^C \sqrt{k}.$$

Idea: the upper bound on the mixing time

Idea: the upper bound on the mixing time

Height function:

$$\xi \in \Omega_{N,k} \quad \rightarrow \quad h^\xi(x) := \sum_{y=1}^x \xi(y) - \frac{k}{N}x.$$

A partial order:

$$(\xi \leq \xi') \quad \Leftrightarrow \quad \left(h^\xi(x) \leq h^{\xi'}(x), \forall x \in \llbracket 1, N \rrbracket \right).$$

Attractive:

$$\left(h^\xi \leq h^{\xi'} \right) \quad \Rightarrow \quad \left(\forall t \geq 0, h_t^\xi \leq h_t^{\xi'} \right).$$

Coalescing time:

$$\begin{aligned} T_1 &:= \inf \left\{ t \geq 0 : h_t^\wedge = h_t^\mu \right\}, \\ T_2 &:= \inf \left\{ t \geq 0 : h_t^\vee = h_t^\mu \right\}. \end{aligned}$$

Idea: the upper bound on the mixing time

Construct a supermartingale: inspired by [Wilson, '04], embed the segment $\llbracket 1, N \rrbracket$ in $\llbracket -\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor \rrbracket$ and place conductance $(c(x, x+1) = 1)_{x \notin [1, N-1]}$. The principle eigenfunction satisfies:

$$\lim_{N \rightarrow \infty} \sup_{x \in \llbracket -\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor \rrbracket} \left| G(x) - \cos \left(\frac{\pi(x + \lfloor \delta N \rfloor + 1/2)}{\bar{N}} \right) \right| = 0.$$

Define $\bar{G}(x) := G(x) - G(x+1) > 0$ and

$$\mathbf{F}(\xi) := \sum_{x=1}^{N-1} h^\xi(x) \bar{G}(x).$$

For $\xi, \xi' \in \Omega_{N,k}$ with $\xi \leq \xi'$, since $h^\xi(x) \leq h^{\xi'}(x)$ and $\bar{G}(x) > 0$,

$$\mathbf{F}(\xi) \leq \mathbf{F}(\xi').$$

Furthermore, if $\xi \leq \xi'$ with $\xi \neq \xi'$, we have $\mathbf{F}(\xi) < \mathbf{F}(\xi')$.

Idea: upper bound on the mixing time

Using $h^\xi(0) = h^\xi(N) = 0$ and for $x \in \llbracket 1, N-1 \rrbracket$

$$\begin{aligned}(\mathcal{L}_{N,k} h^\xi)(x) &= c(x, x+1) [\xi(x+1) - \xi(x)] \\ &= c(x, x+1) \left[\left(h^\xi(x+1) - h^\xi(x) \right) - \left(h^\xi(x) - h^\xi(x-1) \right) \right]\end{aligned}$$

we obtain

$$\begin{aligned}(\mathcal{L}_{N,k} \mathbf{F})(\xi) &= \sum_{x=1}^{N-1} \bar{G}(x) (\mathcal{L}_{N,k} h^\xi)(x) \\ &= -\bar{\lambda}_1 \mathbf{F}(\xi) - h^\xi(1) c(0, 1) \bar{G}(0) - h^\xi(N-1) c(N, N+1) \bar{G}(N)\end{aligned}$$

where $\bar{\lambda}_1$ is the spectral gap of the system in the longer line segment.

Idea: upper bound on the mixing time

For $\xi \leq \xi'$,

$$\begin{aligned} (\mathcal{L}_{N,k} \mathbf{F})(\xi') - (\mathcal{L}_{N,k} \mathbf{F})(\xi) &= -\bar{\lambda}_1 [\mathbf{F}(\xi') - \mathbf{F}(\xi)] \\ &\quad - \left[h^{\xi'}(1) - h^{\xi}(1) \right] c(0,1) \bar{G}(0) - \left[h^{\xi'}(N-1) - h^{\xi}(N-1) \right] c(N, N+1) \bar{G}(N) \\ &\leq -\bar{\lambda}_1 [\mathbf{F}(\xi') - \mathbf{F}(\xi)] . \end{aligned}$$

Then $(\mathbf{F}(h_t^\wedge) - \mathbf{F}(h_t^\mu))_{t \geq 0}$ is a supermartingale with decay rate $\bar{\lambda}_1$.

Combine the approaches in [Lacoin AOP'16] and [Labbé, Lacoin AAP'20] to adapt to the disordered setup to conclude the proof.

