Metastability for expanding bubbles on a sticky substrate

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Jointed work with Hubert Lacoin (IMPA)

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Education

- March 2017-April 2021 Instituto de Matemática Pura e Aplicada
  Ph.D. student Adviser: Hubert Lacoin (2022 ICM speaker)

- March 2015-February 2017 Instituto de Matemática Pura e Aplicada
  Master student Adviser: Vladas Sidoravicius (2014 ICM speaker)

- September 2013-December 2014 Nankai University
  Master student (dropout) Adviser: Kainan Xiang

- September 2009-June 2013 Sun Yat-sen University
  Undergraduate student

Publications and preprints

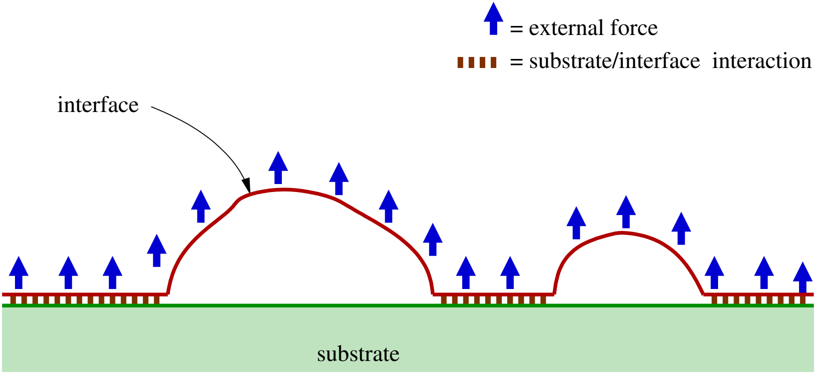
5. **N. Feldheim and S. Y,** Typical height of the (2+1)-D Solid-on-Solid surface with pinning above a wall in the delocalized phase
Stochastic Processes and their Applications. 165 (2023), 168-182.
4. **G. Amir and S.Yang,** The branching number of intermediate growth trees
Arxiv: 2205.14238 submitted.
3. **H. Lacoin and S.Y.,** Mixing time of the asymmetric simple exclusion process in a random environment Arxiv: 2102.02606
Accepted by Annals of Applied Probability
2. **H. Lacoin and S.Y.,** Metastability for expanding bubbles on a sticky substrate
Ann. Appl. Probab. 32(5): 3408-3449 (October 2022).
1. **S. Y.,** Cutoff for polymer pinning dynamics in the repulsive phase
Ann. Inst. H. Poincaré Probab. Statist. 57(3): 1306-1335 (2021).

Future work direction

Mixing time of irreversible Markov chains

- **S. Chatterjee**, Spectral gap of nonreversible Markov chains
- **A. Gaudilliere, C. Landim,** A Dirichlet principle for non reversible Markov chains and some recurrence theorems
- **C. Landim, D. Marcondes, I. Seo**, A Resolvent Approach to Metastability

Metastability for expanding bubbles on a sticky substrate
The physical situation we are considering



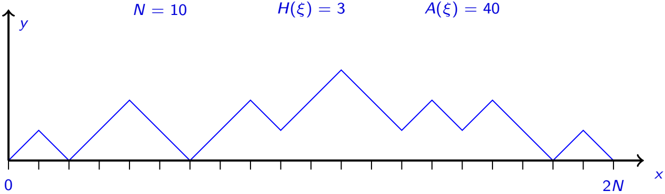
An interface is an element of

$$\Omega_N := \left\{ \xi \in \mathbb{Z}_+^{[\![0,2N]\!]} : \ \xi(0) = \xi(2N) = 0 \text{ and } \forall x \ , \ |\xi(x) - \xi(x-1)| = 1 \right\}.$$

The equilibrium measure

Given $\xi \in \Omega_N$,

- $H(\xi) := \sum_{x=1}^{2N-1} \mathbf{1}_{\{\xi(x)=0\}}$ (# contacts with $x-$axis),
- $A(\xi) := \sum_{x=1}^{2N-1} \xi(x)$: the area enclosed between ξ and the x-axis.



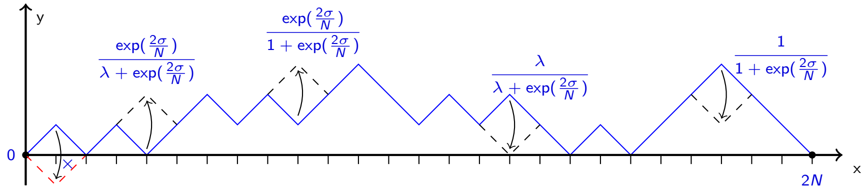
Given $\lambda \geq 0$ and $\sigma \geq 0$, define $\mu = \mu_N^{\lambda,\sigma}$ the probability on Ω_N:

$$\mu(\xi) = \frac{2^{-2N} \lambda^{H(\xi)} e^{\frac{\sigma}{N} A(\xi)}}{Z_N(\lambda,\sigma)} \quad ; \quad Z_N(\lambda,\sigma) := 2^{-2N} \sum_{\xi \in \Omega_N} \lambda^{H(\xi)} e^{\frac{\sigma}{N} A(\xi)}.$$

Corner-flip/Heat Bath dynamics $(\eta_t)_{t\geq 0}$ on Ω_N

Each coordinate is updated at rate one.

When an update at x occurs at time t, η_t is sampled according to the conditional equilibrium measure $\mu_N^{\lambda,\sigma}(\cdot \mid \eta_{t-}(y), y \neq x)$.



The measure μ satisfies the detailed balance condition, *i.e.*

$$\mu(\xi)r(\xi,\xi^x) = \mu(\xi^x)r(\xi^x,\xi).$$

$\mathbf{P}^\xi$: the distribution of the Markov chain $(\eta_t)_{t\geq 0}$ starting from ξ.

$$T_N^{\lambda,\sigma} := \inf\left\{t > 0 : \max_{\xi\in\Omega_N} \|\mathbf{P}_t^\xi - \mu\|_{\mathrm{TV}} \leq 1/4\right\}$$

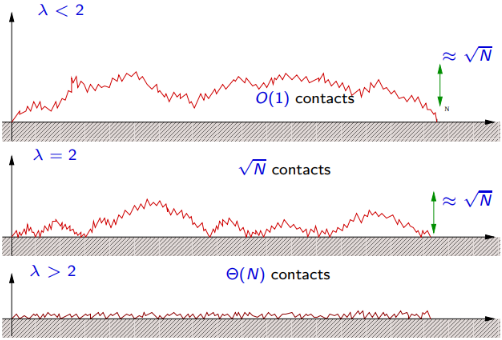
Presentation of our results for the interface model

(1) Previous results about related models

(2) Properties of the model at equilibrium

(3) Slow/fast mixing and metastability ($\sigma > 0$)

Equilibrium for $\sigma = 0$ [Fisher 1984 JSP]



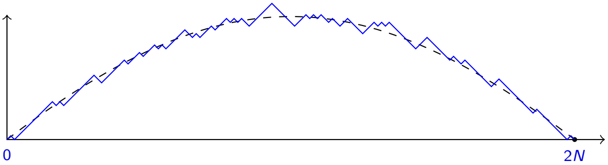
If $\sigma = 0$, the system undergoes a transition at $\lambda = 2$ between a pinned phase and an unpinned phase. This transition can be seen when looking at the free energy

$$\lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda, 0) = \log\left(\frac{\lambda}{2\sqrt{\lambda - 1}}\right)\mathbf{1}_{\{\lambda>2\}} =: F(\lambda).$$

No wall constraint / WASEP interfaces [Labbé '18 Prob. Surv.]

If there is no wall constraint ($\xi(x) < 0$ is allowed) and $\lambda = 1$, we have typically under the equilibrium measure ($u \in [0, 2]$)

$$\frac{\xi(\lceil uN \rceil)}{N} = \frac{1}{\sigma} \log \left(\frac{\cosh(\sigma)}{\cosh(\sigma(1-u))} \right) + o(1).$$



If $\widetilde{Z}_N(\sigma) := \frac{1}{2^{2N}} \sum_{\xi \in \widetilde{\Omega}_N} e^{\frac{\sigma}{N} A(\xi)}$ denotes the corresponding partition function, we have

$$\lim_{N \to \infty} \frac{1}{2N} \log \widetilde{Z}_N(\sigma) = G(\sigma) := \int_0^1 \log \cosh(\sigma(1-2u)) \mathrm{d}u.$$

Equilibrium behavior

The two strategies to take benefit of the wall interaction and of the external force are different and cannot be combined.

Proposition (Lacoin, Y. '22)

We have for any $\lambda \in (0, \infty)$ *and* $\sigma > 0$

$$\lim_{N \to \infty} \frac{1}{2N} \log Z_N(\lambda, \sigma) = F(\lambda) \vee G(\sigma).$$

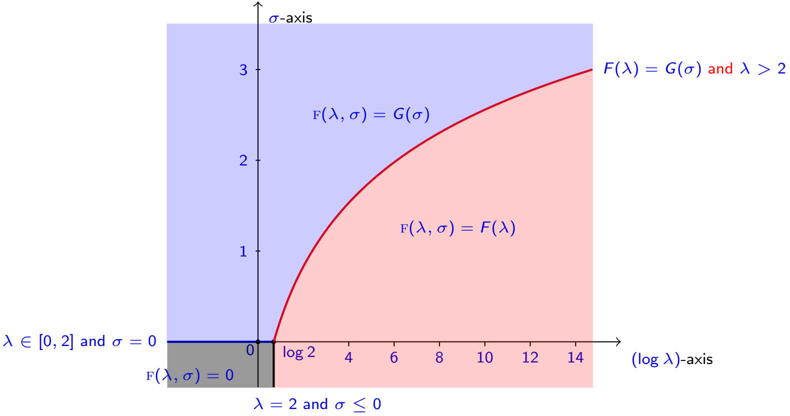
(A) *If* $G(\sigma) > F(\lambda)$, *then* $Z_N(\lambda, \sigma) \asymp \frac{1}{\sqrt{N}} e^{2NG(\sigma)}$.

(B) *If* $F(\lambda) \geq G(\sigma)$, *then* $Z_N(\lambda, \sigma) \asymp e^{2NF(\lambda)}$.

From this result we derive the detailed behavior of the paths.

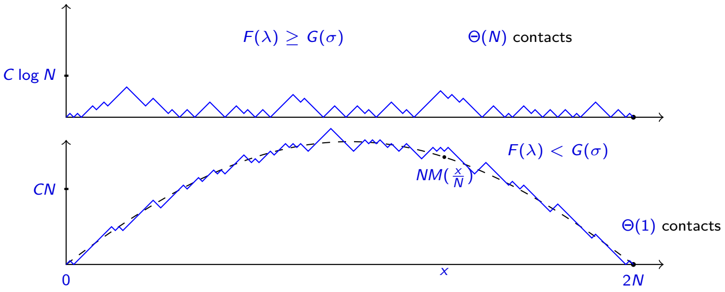
Free energy

$$\mathrm{F}(\lambda, \sigma) := \lim_{N \to \infty} \frac{1}{2N} \log Z_N(\lambda, \sigma).$$



Theorem: macroscopic shape

$$M_\sigma(u) = \frac{1}{\sigma}\log\left(\frac{\cosh(\sigma)}{\cosh(\sigma(1-u))}\right).$$



Dynamical polymer pinning model/WASEP

The problem of mixing time for interface with pinning or WASEP has been studied in previous works.

- When $\sigma = 0$, the mixing time is at most of order $N^2 \log N$ [Caputo, Martinelli, Toninelli '08 EJP]:

$$e.g. \quad T_N^{\lambda,0} \asymp N^2 \log N, \text{ and } \mathrm{gap} \asymp N^{-2} \text{ for } \lambda \in [0,2).$$

- Without wall and pinning, [Levin, Peres '16 JSP] [Labbé, Lacoin '20 AAP]

$$T_N^{\sigma} \asymp N^2 \log N.$$

Our main result: $\lambda > 2$ and $\sigma \geq 0$

Theorem (Lacoin, Y. '22)

When $\lambda > 2$ and $\sigma \geq 0$, then there exists $\sigma_c(\lambda) > 0$ such that

$$\begin{cases} T_N^{\lambda,\sigma} \leq N^C & \text{if } \sigma \leq \sigma_c(\lambda), \\ T_N^{\lambda,\sigma} = e^{2NE(\lambda,\sigma)} N^{O(1)} & \text{if } \sigma > \sigma_c(\lambda), \end{cases}$$

where $\sigma_c(\lambda)$ and $E(\lambda,\sigma) > 0$ are explicit.

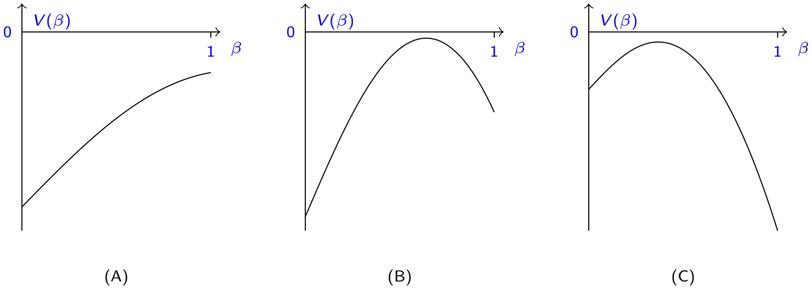
We believe when $\lambda \in [0,2]$ and $\sigma \geq 0$, there exists some constant C

$$T_N^{\lambda,\sigma} \leq N^C.$$

Heuristic for $\lambda > 2$ and $\sigma \geq 0$

β: fraction of the largest excursion $\quad V(\beta) := -(1-\beta)F(\lambda) - \beta G(\beta\sigma)$

(paths with only one large excursion of size $2\beta N$: $e^{-2NV(\beta)}$.)

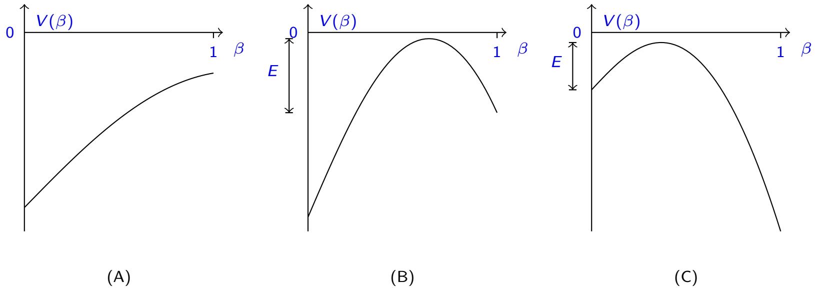


(A) If $G(\sigma) + \sigma G'(\sigma) \leq F(\lambda)$, then the pinned region can grow without obstruction and the system should mix in polynomial time.

(B) If $G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma)$, then the system starting from the fully unpinned state takes a long time to reach the fully pinned equilibrium state.

(C) If $F(\lambda) < G(\sigma)$, then the system starting from the fully pinned state takes a long time to reach the fully unpinned equilibrium state.

$V(\beta) = -(1-\beta)F(\lambda) - \beta G(\beta\sigma)$



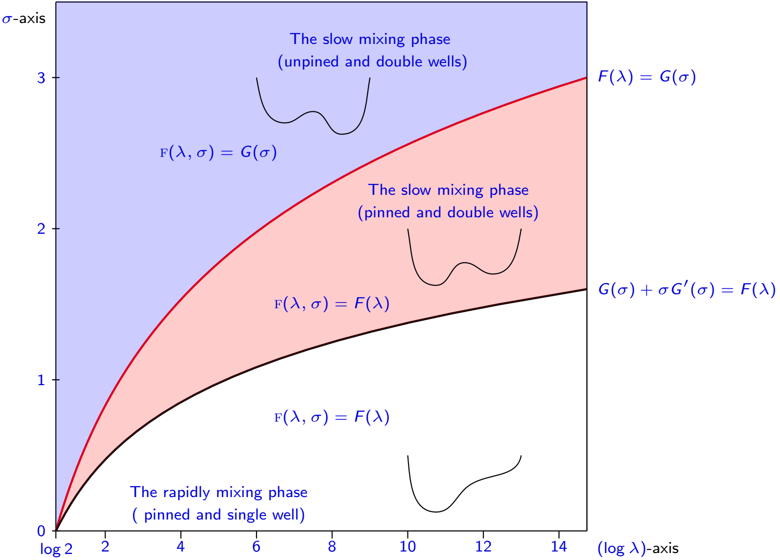
Activation Energy

The size of the effective potential barrier to be overcome in case (B) and (C) is equal to

$$E(\lambda, \sigma) := F(\lambda) \wedge G(\sigma) - [(1-\beta^*)F(\lambda) + \beta^* G(\beta^*\sigma)]$$

with β^* such that $V(\beta^*) = \max_{\beta \in [0,1]} V(\beta)$.

Our result: phase diagram (for $\lambda > 2$ and $\sigma \geq 0$)



Metastability

Assuming $E(\lambda, \sigma) > 0$, let $\mathcal{H}_N$ denote the domain of attraction of the unstable local equilibrium of the dynamics:

$$\mathcal{H}_N := \begin{cases} \{\xi \in \Omega_N : L_{\max}(\xi) > \beta^* N\} & \text{if } G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma), \\ \{\xi \in \Omega_N : L_{\max}(\xi) \leq \beta^* N\} & \text{if } F(\lambda) < G(\sigma), \end{cases}$$

where

$$L_{\max}(\xi) := \max\{y - x \; : \; \xi_{2x} = 0, \xi_{2y} = 0, \forall z \in [\![x, y]\!], \xi_{2z} > 0\}.$$

Theorem (Lacoin, Y. '22)

We have

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)} \left(\eta_{t T^N_{\mathrm{rel}}(\lambda,\sigma)} \in \mathcal{H}_N \right) = \exp(-t),$$

where $T^N_{\mathrm{rel}}(\lambda, \sigma) = e^{2NE(\lambda,\sigma)} N^{O(1)}$ *is the relaxation time of the system.*

Proof ingredients

- Lower bound on mixing time follows directly from the heuristics using bottleneck arguments.
- For the upper bound, the hard part is to show that the system always mixes fast within $\mathcal{H}_N$ and $\mathcal{H}_N^{\complement}$. The proof is intricate and relies on chain decomposition argument [Jerrum *et al.* '04 AAP].
- Once fast mixing in each potential well is proved, the metastability statement follows from a general meta-theorem [Beltran and Landim '15 PTRF].

Thank you for your attention