

# Cutoff for polymer pinning dynamics in the repulsive phase

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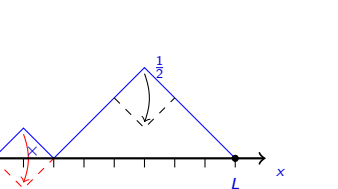


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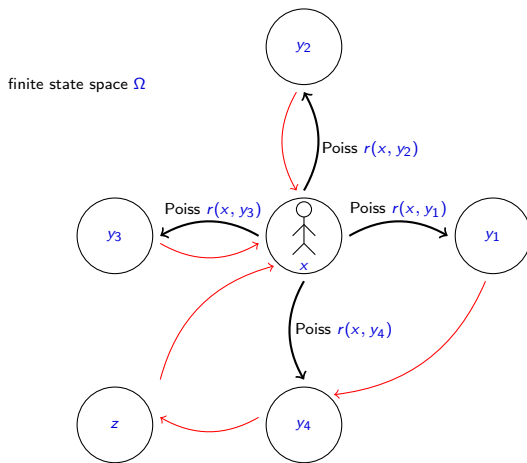
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Markov chains

pulsive phase



## Introduction to mixing for continuous-time Markov chains



# Setup

- Finite state space  $\Omega$ , elements  $x, y, z \dots$
- Generator:  $\mathcal{L} = (r(x, y))_{x, y \in \Omega}$  is an  $\Omega \times \Omega$  matrix:
  - ▶ Off diagonal elements are nonnegative;
  - ▶ Every row sum is equal to zero.

Homeomorphism  $\mathcal{L} : \mathbb{R}^\Omega \rightarrow \mathbb{R}^\Omega$  (for  $f \in \mathbb{R}^\Omega$ )

$$(\mathcal{L}f)(x) := \sum_{y \in \Omega} r(x, y) (f(y) - f(x)) .$$

- Markov semi-group  $(P_t)_{t \geq 0}$ :

$$P_t := e^{t\mathcal{L}} = \sum_{k=0}^{\infty} \frac{(t\mathcal{L})^k}{k!} ,$$

$$P_t(x, y) \geq 0, \quad \sum_{y \in \Omega} P_t(x, y) = 1 .$$

# Markov chain definition

The random process  $(X_t)_{t \geq 0}$  is a continuous-time Markov chain with generator  $\mathcal{L}$  and initial distribution  $\nu$  if it is càdlàg and



$$\forall x \in \Omega, \quad \mathbb{P}[X_0 = x] = \nu(x);$$

- Markov property: for  $0 \leq t_1 < \dots < t_n < s < s + t$ ,

$$\mathbb{P}[X_{s+t} = y | X_s = x; X_{t_k} = z_k, \forall k \leq n] = \mathbb{P}[X_{s+t} = y | X_s = x] = P_t(x, y).$$

# Invariant probability measure

- $\mu$  is an invariant probability measure if

$$(\forall t \geq 0, \mu P_t = \mu) \Leftrightarrow \mu \mathcal{L} = 0.$$

- Irreducible: for all  $x \neq y \in \Omega$ , there exists a path  $\Gamma_{xy} = (x, z_1, \dots, z_{\ell-1}, y)$  with  $r(z_{k-1}, z_k) > 0$  for all  $1 \leq k \leq \ell(x, y)$ .

## Theorem

*If  $(\Omega, \mathcal{L})$  is irreducible, there exists a unique invariant probability measure  $\mu$ , and the distribution  $\mathbb{P}^\nu$  of  $(X_t)_{t \geq 0}$  with initial distribution  $\nu$  converges to  $\mu$ , i.e.*

$$\lim_{t \rightarrow \infty} \sum_{y \in \Omega} \left| \mathbb{P}^\nu [X_t = y] - \mu(y) \right| = 0.$$

# Distance to equilibrium

- The total variation distance: two probability measures  $\alpha, \beta$  on  $\Omega$ ,

$$\|\alpha - \beta\|_{\text{TV}} := \sup_{A \subset \Omega} |\alpha(A) - \beta(A)| .$$

- The distance to equilibrium

$$d(t) := \max_{x \in \Omega} \|P_t(x, \cdot) - \mu\|_{\text{TV}} .$$

- Given  $\varepsilon \in (0, 1)$ , the  $\varepsilon$ -mixing time

$$t_{\text{mix}}(\varepsilon) := \inf \{t \geq 0 : d(t) \leq \varepsilon\} .$$

Notation:  $t_{\text{mix}} := t_{\text{mix}}(1/4)$ .

# Markov chain sequence and cutoff

- A sequence of Markov chains  $(\Omega_n, \mathcal{L}_n, \mu_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} |\Omega_n| = \infty$ :

$t_{\text{mix}}^{(n)}(\varepsilon)$ : the associated  $\varepsilon$ -mixing time.

**Q:** How does  $t_{\text{mix}}^{(n)}(\varepsilon)$  grow in terms of  $n$  and  $\varepsilon$  ?

- Precutoff:

$$\sup_{\varepsilon \in (0, \frac{1}{2})} \limsup_{n \rightarrow \infty} \frac{t_{\text{mix}}^{(n)}(\varepsilon)}{t_{\text{mix}}^{(n)}(1 - \varepsilon)} < \infty.$$

- Cutoff: for all  $\epsilon \in (0, 1)$ ,

$$\lim_{n \rightarrow \infty} \frac{t_{\text{mix}}^{(n)}(\epsilon)}{t_{\text{mix}}^{(n)}(1 - \epsilon)} = 1. \quad \Leftrightarrow \quad \lim_{n \rightarrow \infty} d_n \left( c t_{\text{mix}}^{(n)} \right) = \begin{cases} 1 & \text{if } c < 1, \\ 0 & \text{if } c > 1. \end{cases}$$

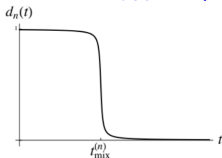


image from Levin and Peres



# Spectral gap of reversible chain

- The detailed balance condition: if for all  $x, y \in \Omega$

$$\mu(x)r(x, y) = \mu(y)r(y, x). \quad \text{Then } \mu\mathcal{L} = 0.$$

- Spectral gap: minimal nonzero eigenvalue of  $-\mathcal{L}$

$$\langle f, g \rangle_\mu := \sum_{x \in \Omega} \mu(x) f(x) g(x), \quad \text{Var}_\mu(f) := \langle f, f \rangle_\mu - \langle f, \mathbf{1} \rangle_\mu^2,$$

$$\text{gap} := \inf_{\text{Var}_\mu(f) > 0} \frac{-\langle f, \mathcal{L}f \rangle_\mu}{\text{Var}_\mu(f)}.$$

- Relaxation time:  $t_{\text{rel}} := \frac{1}{\text{gap}}.$

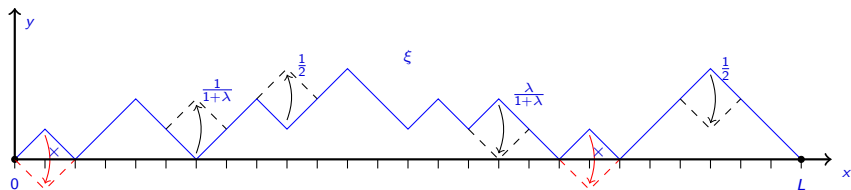
Letting  $\mu_{\min} := \min_{x \in \Omega} \mu(x)$ , for  $\varepsilon \in (0, 1)$  we have

$$t_{\text{rel}} \log \frac{1}{2\varepsilon} \leq t_{\text{mix}}(\varepsilon) \leq t_{\text{rel}} \log \frac{1}{2\varepsilon \mu_{\min}},$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log d(t) = -\text{gap}.$$

# Cutoff for polymer pinning dynamics in the repulsive phase

# The physical situation we are considering



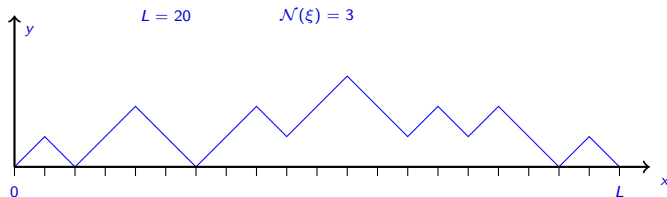
An interface is an element of

$$\Omega_L := \left\{ \xi \in \mathbb{Z}_+^{[0,L]} : \xi(0) = \xi(L) = 0 \text{ and } \forall x, |\xi(x) - \xi(x-1)| = 1 \right\} .$$

# The equilibrium measure

Given  $\xi \in \Omega_L$ ,

- $\mathcal{N}(\xi) := \sum_{x=1}^{L-1} \mathbf{1}_{\{\xi(x)=0\}}$  (# contacts with  $x$ -axis).



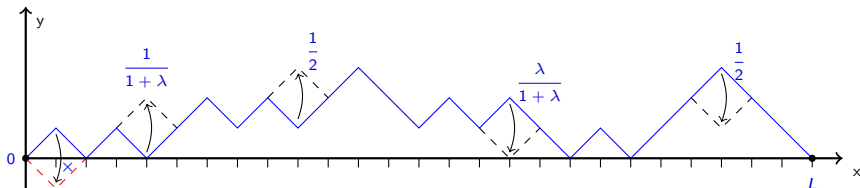
Given  $\lambda \geq 0$ , define  $\mu = \mu_L^\lambda$  the probability on  $\Omega_L$ :

$$\mu(\xi) = \frac{\lambda^{\mathcal{N}(\xi)}}{Z_L(\lambda)}, \quad Z_L(\lambda) := \sum_{\xi' \in \Omega_L} \lambda^{\mathcal{N}(\xi')}.$$

# Corner-flip/Heat Bath dynamics $(\eta_t)_{t \geq 0}$ on $\Omega_L$

Each coordinate is updated at rate one.

When an update at  $x$  occurs at time  $t$ ,  $\eta_t$  is sampled according to the conditional equilibrium measure  $\mu_L^\lambda(\cdot \mid \eta_{t-}(y), y \neq x)$ .



The measure  $\mu$  satisfies the detailed balance condition, i.e.

$$\mu(\xi)r(\xi, \xi^x) = \mu(\xi^x)r(\xi^x, \xi).$$

$\mathbf{P}^\xi$ : the distribution of the Markov chain  $(\eta_t^\xi)_{t \geq 0}$  starting from  $\xi$ .

$T_{\text{mix}}^{L, \lambda}(\varepsilon)$ : associated  $\varepsilon$ -mixing time.

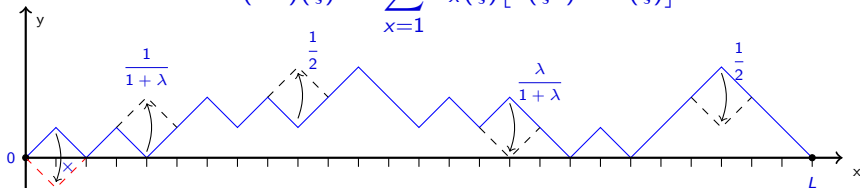
# Generator

- $\xi^x$ : obtained by flipping the corner at coordinate  $x$  of the path  $\xi$  provided there is a corner at  $x$  and  $\xi^x \in \Omega_L$ .
- jump rate

$$R_x(\xi) := \begin{cases} \frac{1}{2} & \text{if } \xi(x-1) = \xi(x+1) > 1, \\ \frac{\lambda}{1+\lambda} & \text{if } (\xi(x-1), \xi(x), \xi(x+1)) = (1, 2, 1), \\ \frac{1}{1+\lambda} & \text{if } (\xi(x-1), \xi(x), \xi(x+1)) = (1, 0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

- Generator: for  $f: \Omega_L \rightarrow \mathbb{R}$

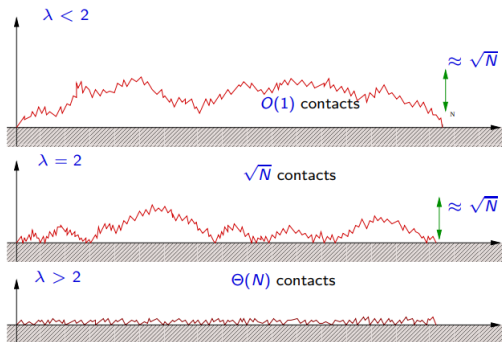
$$(\mathcal{L}f)(\xi) := \sum_{x=1}^{L-1} R_x(\xi) [f(\xi^x) - f(\xi)].$$



# Presentation of our results for the polymer pinning model

- (1) Properties of the model at equilibrium
- (2) Previous results for the polymer pinning dynamics
- (3) Our results for the polymer pinning dynamics in diffusive regime
- (4) Idea: Lower bound on the mixing time for  $\lambda \in [0, 2)$
- (5) Idea: Upper bound on the mixing time for  $\lambda \in [0, 1]$
- (6) Idea: Upper bound on the mixing time for  $\lambda \in (1, 2)$  concerning the extremal initial conditions

# Equilibrium Properties [Fisher 1984]



A transition at  $\lambda = 2$  between a pinned phase and an unpinned phase. This transition can be seen when looking at the free energy

$$\lim_{L \rightarrow \infty} \frac{1}{L} \log 2^{-L} Z_L(\lambda) = \log \left( \frac{\lambda}{2\sqrt{\lambda-1}} \right) \mathbf{1}_{\{\lambda > 2\}} =: F(\lambda).$$



# Details about the partition function

- Asymptotic of the partition function

$$2^{-L}Z_L(\lambda) \sim C(\lambda) \times \begin{cases} L^{-3/2} & \text{if } \lambda \in [0, 2), \\ L^{-1/2} & \text{if } \lambda = 2, \\ e^{LF(\lambda)} & \text{if } \lambda \in (2, \infty). \end{cases}$$

- How to calculate it? No positive constraint state space:

$$\tilde{\Omega}_L := \left\{ \xi \in \mathbb{Z}^{\llbracket 0, L \rrbracket} : \xi(0) = \xi(L) = 0 \text{ and } \forall x, |\xi(x) - \xi(x-1)| = 1 \right\}.$$

$$\tilde{Z}_L(\lambda) := \sum_{\xi \in \tilde{\Omega}_L} \lambda^{\mathcal{N}(\xi)}, \quad \tilde{Z}_L(\lambda) = 2Z_L(2\lambda),$$

- Renewal process viewpoint ( $\mathbf{P}$  : SRW     $\tilde{\mathbf{P}}$  : renewal law)  
 $K(n) := \mathbf{P}(S_1 \neq 0, S_2 \neq 0, \dots, S_{2n-1} \neq 0, S_{2n} = 0), \quad \forall n \geq 1.$

$$\tilde{K}(n) := \lambda e^{-2n\tilde{F}(\lambda)} K(n).$$

$$e^{-L\tilde{F}(\lambda)} 2^{-L} Z_L(\lambda) = e^{-L\tilde{F}(\lambda)} \sum_{k=1}^{L/2} \sum_{\substack{(n_1, \dots, n_k) \\ \sum_{i=1}^k n_i = L/2}} \prod_{i=1}^k K(n_i) \lambda$$

$$= \sum_{k=1}^{L/2} \sum_{\substack{(n_1, \dots, n_k) \\ \sum_{i=1}^k n_i = L/2}} \prod_{i=1}^k \tilde{K}(n_i) = \tilde{\mathbf{P}}(L \in \tau).$$

## Previous results: Polymer pinning dynamics

[Caputo, Martinelli, Toninelli '08]:

- When  $\lambda \in [0, 2)$ ,  $\text{gap} \asymp L^{-2}$  and there is a precutoff, i.e.

$$\frac{1 + o(1)}{2\pi^2} L^2 \log L \leq T_{\text{mix}}^{L,\lambda}(\epsilon) \leq \frac{6 + o(1)}{\pi^2} L^2 \log L.$$

- When  $\lambda = 2$ ,  $\text{gap} \asymp L^{-2}$

$$cL^2 \leq T_{\text{mix}}^{L,\lambda}(1/4) \leq \frac{6 + o(1)}{\pi^2} L^2 \log L.$$

- When  $\lambda > 2$ ,  $\text{gap} \leq cL^{-1}$

$$T_{\text{mix}}^{L,\lambda}(1/4) \geq cL^2,$$

where  $c$  is independent of  $\lambda$ .

- When  $\lambda = \infty$ ,

$$T_{\text{mix}}^{L,\lambda}(1/4) \leq L^2.$$

[Lacoin '14] identified the constant in the mixing (hitting) time when  $\lambda = \infty$  for smooth initial profile.

# Our main result: cutoff

Understand the pattern of relaxation to equilibrium, and in particular identify the mixing time.

$$T_{\text{mix}}^{L,\lambda}(\varepsilon) := \inf \left\{ t : \forall \xi \in \Omega_N, \left\| \mathbf{P}_t^\xi - \mu \right\|_{\text{TV}} \leq \varepsilon \right\}.$$

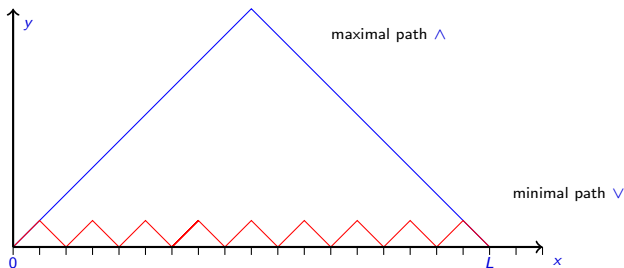
Theorem (Y, '21 (cutoff))

When  $\lambda \in [0, 1]$ , for all  $\varepsilon \in (0, 1)$  we have

$$\lim_{L \rightarrow \infty} \frac{\pi^2 T_{\text{mix}}^{L,\lambda}(\varepsilon)}{L^2 \log L} = 1.$$

# Our main result: partial cutoff

$$\check{T}_{\text{mix}}^{L,\lambda}(\varepsilon) := \inf \left\{ t : \max(\|\mathbf{P}_t^\wedge - \mu\|_{\text{TV}}, \|\mathbf{P}_t^\vee - \mu\|_{\text{TV}}) \leq \varepsilon \right\}.$$



Theorem (Y, '21 (Partial cutoff))

If  $\lambda \in (1, 2)$ , for all  $\varepsilon \in (0, 1)$  we have

$$\lim_{L \rightarrow \infty} \frac{\pi^2 \check{T}_{\text{mix}}^{L,\lambda}(\varepsilon)}{L^2 \log L} = 1.$$

# Polymer pinning dynamics

Idea: Lower bound on the mixing time for  $\lambda \in [0, 2)$

## Idea for the lower bound when $\lambda \in [0, 2)$

A weighted area function  $\Phi: \Omega_L \rightarrow \mathbb{R}$  [introduced by Wilson '04]:

$$\Phi(\xi) := \sum_{x=1}^{L-1} \xi(x) \sin\left(\frac{\pi x}{L}\right).$$

- Under equilibrium  $\mu$ ,  $\Phi$  is at most of order  $L^{3/2}$  since

$$\sup_{\lambda \geq 0, L \in 2\mathbb{N}} \sup_{x \in [1, L-1]} \mu_L^\lambda \left( \frac{(\xi(x))^k}{L^{k/2}} \right) < \infty.$$

- For the dynamics  $(\eta_t^\wedge)_{t \geq 0}$  starting from the highest path  $\wedge$ ,  $\Phi$  is initially of order  $L^2$ ;
- To show the time required by  $\Phi(\eta_t^\wedge)$  to become of order  $L^{3/2}$  is at least  $(1 - o(1)) \frac{1}{\pi^2} L^2 \log L$ , we estimate the mean  $\mathbf{E}[\Phi(\eta_t^\wedge)]$  and its fluctuation by building a Dynkin's martingale and controlling the martingale bracket.

Idea for Lower bound when  $\lambda \in [0, 2)$  [ $\kappa_L := 1 - \cos(\pi/L)$ ]

- [Caputo, Martinelli, Toninelli]

$$\mathbb{E}[\Phi(\eta_{t_0}^\wedge)] \geq \Phi(\eta_0^\wedge) e^{-\kappa_L t_0} - c(\lambda) L^{3/2} \geq 2C_\varepsilon L^{3/2}.$$

Notation:  $t_0 := \frac{1}{\pi^2} L^2 \log L - C_\varepsilon L^2$  ( $C_\varepsilon \gg 1$ .)

- Build a Dynkin's martingale:  $F(t, \xi) = \exp(\kappa_L(t - t_0))\Phi(\xi)$

$$M_t := F(t, \eta_t^\wedge) - F(0, \eta_0^\wedge) - \int_0^t (\partial_s + \mathcal{L})F(s, \eta_s^\wedge) ds.$$

$$\Psi(\xi) := \sum_{x=1}^{L-1} \sin\left(\frac{\pi x}{L}\right) \left[ \mathbf{1}_{\{\xi(x-1)=\xi(x+1)=0\}} - \left(\frac{\lambda-1}{\lambda+1}\right) \mathbf{1}_{\{\xi(x-1)=\xi(x+1)=1\}} \right].$$

$$(\partial_t + \mathcal{L})F(t, \eta_t^\wedge) = e^{\kappa_L(t-t_0)} \Psi(\eta_t^\wedge).$$

Each transition can change  $M_t$  in absolute value by at most  $2e^{\kappa_L(t-t_0)}$

$$\partial_t \langle M. \rangle_t \leq \sum_{x=1}^{L-1} 4e^{2\kappa_L(t-t_0)} \leq 4Le^{2\kappa_L(t-t_0)}.$$

Idea for Lower bound when  $\lambda \in [0, 2)$

$$\mathbb{E}[M_{t_0}^2] = \mathbb{E}[\langle M \cdot \rangle_{t_0}] \leq \int_0^{t_0} 4Le^{2\kappa_L(t-t_0)} dt \leq \frac{8L^3}{\pi^2}.$$

$$\bullet \quad \Psi(\xi) = \sum_{x=1}^{L-1} \sin\left(\frac{\pi x}{L}\right) \left[ \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} - \left(\frac{\lambda-1}{\lambda+1}\right) \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}} \right].$$

$$B(t) := \int_0^t e^{\kappa_L(s-t_0)} \Psi(\eta_s^\wedge) ds.$$

$$\begin{aligned} \mathbb{E}[|B(t_0)|] &\leq \mathbb{E}\left[\int_0^{t_0} e^{\kappa_L(t-t_0)} |\Psi(\eta_t^\wedge)| dt\right] \\ &\leq C(\lambda) \kappa_L^{-1} \sum_{x=1}^{L-1} \sin\left(\frac{\pi x}{L}\right) \frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}} \leq C(\lambda) L^{3/2}. \end{aligned}$$

$$\begin{aligned} \bullet \quad \mathbb{P}\left[|\Phi(\sigma_{t_0}^\wedge) - \mathbb{E}[\Phi(\sigma_{t_0}^\wedge)]| \geq C_\varepsilon L^{3/2}\right] &= \mathbb{P}\left[|M_{t_0} + B(t_0) - \mathbb{E}[B(t_0)]| \geq C_\varepsilon L^{3/2}\right] \\ &\leq \mathbb{P}\left[|M_{t_0}| \geq \frac{1}{3} C_\varepsilon L^{3/2}\right] + \mathbb{P}\left[|B(t_0)| \geq \frac{1}{3} C_\varepsilon L^{3/2}\right] \\ &\leq \frac{9\mathbb{E}[M_{t_0}^2]}{C_\varepsilon^2 L^3} + \frac{3\mathbb{E}[|B(t_0)|]}{C_\varepsilon L^{3/2}} \leq \varepsilon \text{ if } C_\varepsilon \text{ is sufficiently large.} \end{aligned}$$

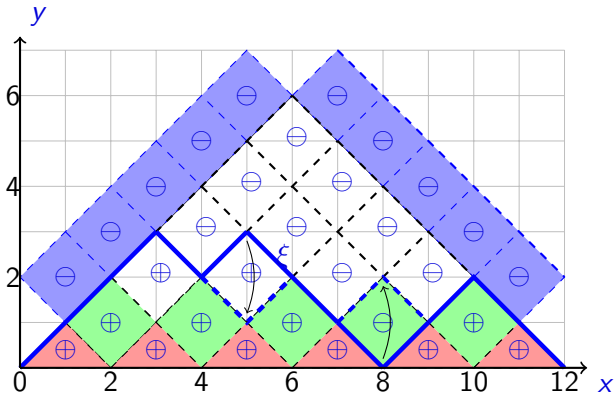


# Polymer pinning dynamics

Idea: Upper bound on the mixing time for  $\lambda \in [0, 1]$

# Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- A graphical construction preserves the monotonicity.



- Partial order “ $\leq$ ” on  $\Omega_L$ :  $(\xi \leq \xi') \Leftrightarrow (\forall x \in \llbracket 0, L \rrbracket, \xi(x) \leq \xi'(x))$ .
- The graphical construction preserves monotonicity

$$\xi \leq \xi' \Rightarrow \eta_t^\xi \leq \eta_t^{\xi'}, \quad \forall t \geq 0.$$

$$\eta_t^\vee \leq \eta_t^\xi \leq \eta_t^\wedge, \quad \forall t \geq 0, \forall \xi \in \Omega_L.$$

## Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- Reduce the problem to estimate the coalescing time

$$\|\mathbf{P}_t^\xi - \mathbf{P}_t^\mu\|_{\text{TV}} \leq \sum_{\xi' \in \Omega_L} \mu(\xi') \|\mathbf{P}_t^\xi - \mathbf{P}_t^{\xi'}\|_{\text{TV}} \leq \max_{\xi' \in \Omega_L} \|\mathbf{P}_t^\xi - \mathbf{P}_t^{\xi'}\|_{\text{TV}}.$$

$$\|\mathbf{P}_t^\xi - \mathbf{P}_t^{\xi'}\|_{\text{TV}} \leq \mathbb{P}[\eta_t^\xi \neq \eta_t^{\xi'}] \leq \mathbb{P}[\eta_t^\wedge \neq \eta_t^\vee].$$

Coalescing times

$$\begin{aligned}\tilde{\tau} &:= \inf \{t > 0 : \eta_t^\wedge = \eta_t^\vee\}, \\ \tau' &:= \inf \{t > 0 : \eta_t^\vee = \eta_t^\mu\}, \\ \tau &:= \inf \{t > 0 : \eta_t^\wedge = \eta_t^\mu\}, \\ \tilde{\tau} &= \max(\tau, \tau').\end{aligned}$$

It is more practical to deal with  $\tau', \tau$  than  $\tilde{\tau}$ .

## Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- An area function  $\bar{\Phi}: \Omega_L \rightarrow [0, \infty)$  given by

$$\bar{\Phi}(\xi) := \sum_{x=1}^{L-1} \xi(x) \bar{\cos}_\beta(x) \quad \bar{\cos}_\beta(x) := \cos(\beta(x - L/2)/L)$$

where  $\beta < \pi$  and  $\beta$  is chosen sufficiently close to  $\pi$ .

$$\xi \leq \xi' \quad \Rightarrow \quad \bar{\Phi}(\xi) \leq \bar{\Phi}(\xi'),$$

$$\delta_{\min} := \min_{\xi \leq \xi', \xi \neq \xi'} (\bar{\Phi}(\xi') - \bar{\Phi}(\xi)) = 2 \cos\left(\frac{\beta(L/2 - 1)}{L}\right) \geq \frac{1}{2}(\pi - \beta).$$

- The area function between the paths  $A_t := \delta_{\min}^{-1} [\bar{\Phi}(\eta^\wedge) - \bar{\Phi}(\eta^\mu)]$   
 $A_t - A_0 - \int_0^t \mathcal{L}A_s ds$  is a Dynkin martingale  
 $A_t$  is a supermartingale when  $\lambda \in [0, 1]$ .

$$\mathcal{L}\xi_x = (\Delta\xi)_x + \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} + \left(\frac{1-\lambda}{1+\lambda}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}.$$

For  $\lambda \in [0, 1]$ , if  $\xi \leq \xi'$ ,

$$\mathcal{L}\xi_x - (\Delta\xi)_x \geq \mathcal{L}\xi'_x - (\Delta\xi')_x, \quad \forall x \in \llbracket 1, L-1 \rrbracket.$$

$$\sum_{x=1}^{L-1} \cos(x) ((\Delta\xi')_x - (\Delta\xi)_x) = -(1 - \cos(\beta/L)) \sum_{x=1}^{L-1} \cos(x) (\xi'_x - \xi_x).$$

$$\begin{aligned} (\mathcal{L}\bar{\Phi})(\xi') - (\mathcal{L}\bar{\Phi})(\xi) &= \sum_{x=1}^{L-1} \overline{\cos}(x) ((\Delta\xi')_x - (\Delta\xi)_x + \mathcal{L}\xi'_x - (\Delta\xi')_x - (\mathcal{L}\xi_x - (\Delta\xi)_x)) \\ &\leq \sum_{x=1}^{L-1} \overline{\cos}(x) ((\Delta\xi')_x - (\Delta\xi)_x) \\ &= -(1 - \cos(\beta/L)) \sum_{x=1}^{L-1} \overline{\cos}(x) (\xi'_x - \xi_x), \\ &\Rightarrow A_t \text{ is a supermartingale.} \end{aligned}$$

Idea: Upper bound on the mixing time for  $\lambda \in [0, 1]$

(1) The decay rate of  $\mathbb{E}[A_t]$  is at least  $1 - \cos(\beta/L) \Rightarrow$

$$t_{\delta/2} := \frac{1 + \delta/2}{\pi^2} L^2 \log L, \quad A_{t_{\delta/2}} \ll L^{3/2}.$$

(2) For  $t \geq t_{\delta/2}$ , applying the supermartingale approach [Labbé, Lacoïn, '20] to show: it only takes an extra amount of time of order  $L^2$  for  $A_t$  to shrink from  $L^{3/2}$  to zero. Idea:

$\eta > 0$ : sufficiently small,  $K := \lceil 1/(2\eta) \rceil > 1/(2\eta)$ . Define  $(\mathcal{T}_i)_{i=2}^K$  by

$$\mathcal{T}_2 := \inf \left\{ t \geq t_{\delta/2} : A_t \leq L^{\frac{3}{2}-2\eta} \right\},$$

$$\mathcal{T}_i := \inf \left\{ t \geq \mathcal{T}_{i-1} : A_t \leq L^{\frac{3}{2}-i\eta} \right\}, \text{ for } i \in \llbracket 3, k \rrbracket,$$

$$\mathcal{T}_\infty := \max(\mathcal{T}_1, t_{\delta/2}).$$

To show:

$$\lim_{L \rightarrow \infty} \mathbb{P} \left[ \{ \mathcal{T}_2 = t_{\delta/2} \} \cap \left( \bigcap_{i=3}^K \{ \Delta \mathcal{T}_i \leq 2^{-i} L^2 \} \right) \cap \{ \mathcal{T}_\infty - \mathcal{T}_K \leq L^2 \} \right] = 1.$$

## Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- During the time interval  $[\mathcal{T}_{i-1}, \mathcal{T}_i]$  for  $3 \leq i \leq K$ , apply the surpermartingale approach ([Labbé, Lacoïn '20]) to show w.h.p.

$$\langle A. \rangle_{\mathcal{T}_i} - \langle A. \rangle_{\mathcal{T}_{i-1}} \leq L^{3-2(i-1)\eta + \frac{1}{2}\eta},$$

$$\langle A. \rangle_{\mathcal{T}_\infty} - \langle A. \rangle_{\mathcal{T}_K} \leq L^2.$$

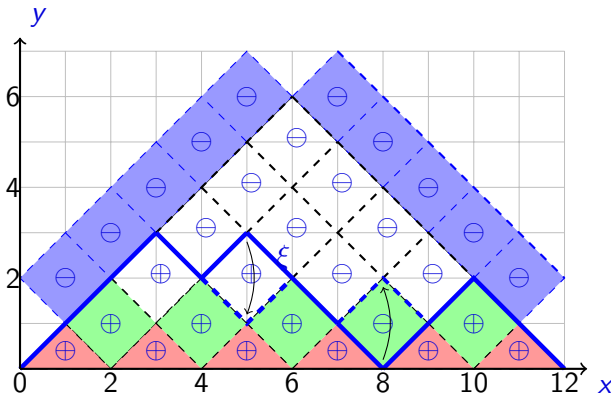
- Compare  $\mathcal{T}_\infty - \mathcal{T}_K$  with  $\langle A. \rangle_{\mathcal{T}_\infty} - \langle A. \rangle_{\mathcal{T}_K}$ . As  $\partial_t \langle A. \rangle \geq 1$  for all  $t < \mathcal{T}_\infty$ , we have

$$\mathcal{T}_\infty - \mathcal{T}_K \leq \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} \partial_t \langle A. \rangle dt = \langle A. \rangle_{\mathcal{T}_\infty} - \langle A. \rangle_{\mathcal{T}_K}.$$

- For  $3 \leq i \leq K$ , to compare  $\langle A. \rangle_{\mathcal{T}_i} - \langle A. \rangle_{\mathcal{T}_{i-1}}$  with  $\mathcal{T}_i - \mathcal{T}_{i-1}$ , we provide a better lower bound on  $\partial_t \langle A. \rangle$  in terms of the highest point of  $\eta_t^\wedge$  and the maximal length of a monotone segment of  $\eta_t^\mu$ . We use induction method to show that  $\mathcal{T}_i - \mathcal{T}_{i-1} \leq 2^{-i} L^2$  for all  $i \in \llbracket 3, K \rrbracket$ , arguing by contradiction.

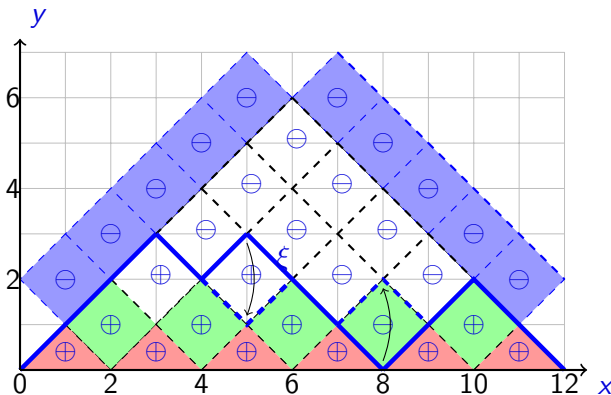
## Upper bound on the mixing time for $\lambda \in (1, 2)$ concerning extremal initial conditions

- When  $\lambda \in (1, 2)$ ,  $A_t$  is not a super-martingale due to the entropic repulsion of the hard wall.



- Censoring: canceling any prescribed updates in any given spin positions and any chosen time intervals.



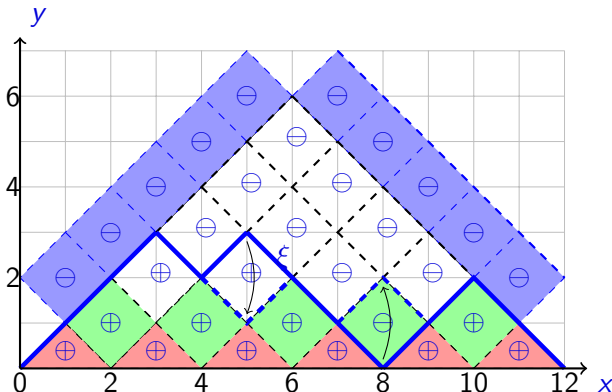


- Peres-Winkler inequality: for monotone spin systems, censoring delays mixing for dynamics starting with extremal initial condition.  
For any prescribed censoring scheme  $\mathcal{C}$ , for all  $\lambda \in [0, \infty)$ , all  $t \geq 0$  and  $\xi \in \{\wedge, \vee\}$ , we have

$$\|\mathbf{P}_t^\xi - \mu\|_{\text{TV}} \leq \|\mathbf{P}_t^{\xi, \mathcal{C}} - \mu\|_{\text{TV}}.$$

## Idea for the upper bound concerning the maximal initial condition when $\lambda \in (1, 2)$

- We censor updates in those spin positions colored green for  $t < t_{\delta/2}$ , and don't censor any update for  $t \geq t_{\delta/2}$ .



Therefore, the dynamics  $(\eta_t^{\wedge, \mathcal{C}})_{0 \leq t < t_{\delta/2}}$  does not touch the  $x$ -axis except at the two coordinates  $x = 0, L$ .

- By the cutoff theorem for  $\lambda = 1$ , the distribution of  $\eta_{t_{\delta/2}}^{\wedge, \mathcal{C}}$  is close to  $\mu_L^0$  in total variation distance.
- 1) The Radon-Nikodym derivative of  $\mu_L^0$  with respect to  $\mu_L^\lambda$  is bounded by a constant, by

$$2^{-L} Z_L(\lambda) \sim C(\lambda) \times \begin{cases} L^{-3/2} & \text{if } \lambda \in [0, 2), \\ L^{-1/2} & \text{if } \lambda = 2, \\ e^{LF(\lambda)} & \text{if } \lambda \in (2, \infty). \end{cases}$$

2)

$$\text{gap}_{L,\lambda} \geq \kappa_L = 1 - \cos\left(\frac{\pi}{L}\right).$$

3) Combining Cauchy-Schwarz inequality and the reversibility of the Markov chain, for any probability distribution  $\nu$  on  $\Omega_L$ , [Caputo, Lacoïn, Martinelli, Simenhaus, Toninelli '12] proves that

$$\|\nu P_t - \mu\|_{\text{TV}} \leq \frac{1}{2} e^{-t \cdot \text{gap}_{L,\lambda}} \sqrt{\text{Var}_\mu(\rho)},$$

where  $\rho := \frac{d\nu}{d\mu}$  and  $\text{Var}_\mu(\rho) := \mu(\rho^2) - \mu(\rho)^2$ .

Therefore, the distribution of  $\eta_{t_{\delta/2} + C_\varepsilon L^2}^{\wedge}$  is close to  $\mu$ .

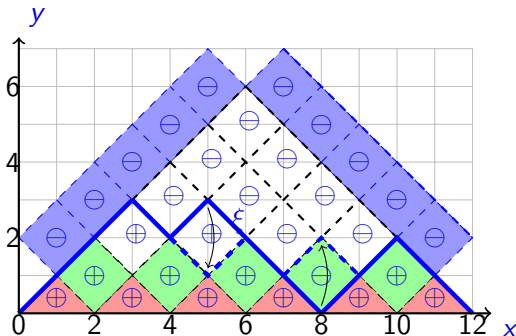
# Idea for the upper bound concerning the minimal initial condition when $\lambda \in (1, 2)$

- (i) Run the dynamics  $(\eta_t^\vee)_{0 \leq t < s_0(L)}$  without censoring, where

$$s_0(L) := 10L^{16/9} \log L \ll L^2 \log L.$$

W.h.p.  $\eta_{s_0(L)}^\vee$  does not touch the  $x$ -axis in the interval  $\llbracket M, L - M \rrbracket$  for some  $M$  sufficiently large.

- (ii) In the time interval  $[s_0(L), s_0(L) + t_{\delta/2})$ , censor updates in those spin positions colored green.





## Open question

To understand the effect of entropic repulsion on the dynamics. In particular, to prove

For  $\lambda \in (1, 2)$ , we have

$$T_{\text{mix}}^{L,\lambda}(\varepsilon) \leq \frac{1 + o(1)}{\pi^2} L^2 \log L.$$

Thank you for your attention!