

# Typical height of the $(2+1)$ -D Solid-on-Solid surface with pinning above a wall in the delocalized phase

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# Organization of the talk

1. Background and model
2. Our results
3. Ideas from the proof

## Background and model

# Background: qualitative approximation of Ising model

Three dimensional Ising model on a cube  $\llbracket 0, N+1 \rrbracket^3$

- Spin value  $\{1, -1\}$  on each site      State space  $\{-1, 1\}^{\llbracket 0, N+1 \rrbracket^3}$
- Boundary conditions: bottom face  $-1$  and all the other face with  $+1$   
 $\sigma \in \{-1, 1\}^{\llbracket 0, N+1 \rrbracket^3}$

$$\sigma(x, y, z) = \begin{cases} -1 & \text{if } z = 0, \\ 1 & \text{if } z = N+1 \cup x \in \{0, N+1\} \cup y \in \{0, N+1\}. \end{cases}$$

- Ising measure ( $\beta = \frac{1}{T} \gg 1$  inverse temperature)

$$\mathbb{P}_{N, \beta}(\sigma) = \frac{1}{Z_{N, \beta}} \exp \left( \beta \sum_{\substack{i, j \in \llbracket 0, N+1 \rrbracket^3 \\ i \sim j}} \sigma_i \sigma_j \right)$$

- **Q:** the " $-1$ " component incident to the bottom face?

$$f : \llbracket 1, N \rrbracket^2 \rightarrow \llbracket 0, N \rrbracket$$

# The solid-on-solid model: a crystal surface model

Introduced by [Burton, Cabrera, Frank '51] [Temperley '52]

$(d+1)$ -D SOS model on  $\mathbb{Z}^d$ :

- Box  $\Lambda_N := \llbracket 1, N \rrbracket^d$  External boundary  $\partial\Lambda_N$

$$\partial\Lambda_N := \left\{ x \in \mathbb{Z}^d \setminus \Lambda_N : \exists y \in \Lambda_N, x \sim y \right\}$$

- State space  $\phi \in \tilde{\Omega}_{\Lambda_N} := \mathbb{Z}^{\Lambda_N} = \{f : \Lambda_N \rightarrow \mathbb{Z}\}$  Hamiltonian (0 b.c.)

$$\mathcal{H}_N(\phi) := \sum_{\substack{\{x,y\} \subset \Lambda_N \\ x \sim y}} |\phi(x) - \phi(y)| + \sum_{\substack{x \in \Lambda_N, y \in \partial\Lambda_N \\ x \sim y}} |\phi(x)|.$$

- SOS probability measure ( $\beta > 0$  inverse temperature)

$$\forall \phi \in \tilde{\Omega}_N, \quad \mathbf{P}_N^\beta(\phi) := \frac{1}{\tilde{\mathcal{Z}}_N^\beta} e^{-\beta \mathcal{H}_N(\phi)}$$

$$\tilde{\mathcal{Z}}_N^\beta := \sum_{\psi \in \tilde{\Omega}_N} e^{-\beta \mathcal{H}_N(\psi)} \leq \left( \frac{1 + e^{-d\beta}}{1 - e^{-d\beta}} \right)^{|\Lambda_N|}$$

# SOS: rigid/rough

- $d = 1$ : rough (delocalized) [Temperley '52, '56] [Fisher '84]  
for all  $\beta > 0$ , the expectation of the absolute value of the height at the center diverges in the thermodynamic limit.
- $d \geq 3$ : rigid (localized) [Bricmont, Fontaine, Lebowitz '82]  
for all  $\beta > 0$ , the expectation of the absolute value of the height at the center is uniformly bounded (by Peierls argument).
- $d = 2$  a phase transition between rough and rigid
  - ▶ rough: for small  $\beta$  ([Fröhlich, Spencer '81, '83])
  - ▶ rigid: for large  $\beta$  ([Brandenberger, Wayne '82], [Gallavotti, Martin-Löf, Miracle-Solé '73]).
  - ▶ Numerical critical point:  $\beta_c \approx 0.806$

## (2+1)-D SOS above a hard wall

- Above a hard wall

$$\forall \phi \in \Omega_N := \left\{ \phi \in \tilde{\Omega}_N : \phi \geq 0 \right\}, \quad \mathbb{P}_N^\beta(\phi) := \mathbf{P}_N^\beta(\phi) / \mathbf{P}_N^\beta(\Omega_N).$$

- [Bricmont, Mellouki, and Fröhlich '86]: for large  $\beta$ , the average height  $H$  of the surface satisfies

$$\frac{1}{C\beta} \log N \leq H \leq \frac{C}{\beta} \log N.$$

- [Caputo, Lubetzky, Martinelli, Sly, Toninelli '14] for  $\beta \geq 1$ , the typical height of the surface concentrates at

$$H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$$

with fluctuations of order  $O(1)$ .

# Typical height of (2+1)-D SOS above a wall

Theorem (Caputo, Lubetzky, Martinelli, Sly, Toninelli '14)

*There exist two universal constants  $C, K > 0$  such that for all  $\beta \geq 1$  and all integer  $k \geq K$ , we have for all  $N$ ,*

$$\mathbb{P}_N^\beta \left( |\{x \in \Lambda_N : \phi(x) \geq H + k\}| > e^{-2\beta k} N^2 \right) \leq e^{-C e^{-2\beta k} N (1 \wedge e^{-2\beta k} N \log^{-8} N)}$$

*and*

$$\mathbb{P}_N^\beta \left( |\{x \in \Lambda_N : \phi(x) \leq H - k\}| > e^{-2\beta k} N^2 \right) \leq e^{-e^{\beta k} N}.$$

Entropic repulsion: In the large  $\beta$  regime, the presence of an impenetrable wall pushes the surface up to the height of order  $\frac{1}{4\beta} \log N$ , instead of remaining uniformly bounded when no wall is present.



# The $(2 + 1)$ -D SOS surface with pinning above a wall

- State space  $\Omega_N = \mathbb{Z}_+^{\Lambda_N}$
- Inverse temperature  $\beta > 0$ , pinning parameter  $h \geq 0$
- Probability measure  $\mathbb{P}_N^{\beta, h}$ : above a wall, with 0 b.c., pinning reward  $h$ ,

$$\mathbb{P}_N^{\beta, h}(\phi) := \frac{1}{\mathcal{Z}_N^{\beta, h}} e^{-\beta \mathcal{H}_N(\phi) + h |\{x \in \Lambda_N: \phi(x)=0\}|},$$

$$\mathcal{Z}_N^{\beta, h} := \sum_{\phi \in \Omega_N} e^{-\beta \mathcal{H}_N(\phi) + h |\{x \in \Lambda_N: \phi(x)=0\}|} \leq e^{h|\Lambda_N|} \left( \frac{1 + e^{-2\beta}}{1 - e^{-2\beta}} \right)^{|\Lambda_N|}.$$

- Free energy ( $\log \mathcal{Z}_\Lambda^{\beta, h}$  is superadditive for disjoint boxes  $\Rightarrow \exists$  limit)

$$F(\beta, h) := \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathcal{Z}_N^{\beta, h}$$

$F(\beta, h)$  is increasing and convex in  $h$  by Hölder's inequality:  $\theta \in [0, 1]$

$$\mathcal{Z}_N^{\beta, \theta h_1 + (1-\theta)h_2} \leq \left( \mathcal{Z}_N^{\beta, h_1} \right)^\theta \cdot \left( \mathcal{Z}_N^{\beta, h_2} \right)^{1-\theta}.$$

## The $(2 + 1)$ -D SOS surface with pinning above a wall

- When  $F(\beta, h)$  is differentiable in  $h$ , the convexity allows us to exchange the order of limit and derivative to obtain the asymptotic contact fraction

$$\partial_h F(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \mathbb{E}_N^{\beta, h} [|\phi^{-1}(0)|] .$$

- [Chalker '82]: Existence of criticality

$$h_w(\beta) := \sup \{ h \in \mathbb{R}_+ : F(\beta, h) = F(\beta, 0) \} > 0 \quad \text{for all } \beta > 0$$

separates the delocalized phase ( $\partial_h F(\beta, h) = 0$ ) from the localized phase ( $\partial_h F(\beta, h) > 0$ ). Furthermore, for all  $\beta > 0$

$$\log \left( \frac{e^{4\beta}}{e^{4\beta} - 1} \right) \leq h_w(\beta) \leq \log \left( \frac{16(e^{4\beta} + 1)}{e^{4\beta} - 1} \right) .$$

- [Alexander, Dunlop, Miracle-Solé, '11]: the lower bound above is asymptotically sharp, and when  $h$  decreases to  $h_w$  the system undergoes a sequence of layering transitions.

# The $(2 + 1)$ -D SOS surface with pinning above a wall

- [Lacoin '18]: for  $\beta > \beta_c \in (\log 2, \log 3)$

$$h_w(\beta) = \log \left( \frac{e^{4\beta}}{e^{4\beta} - 1} \right)$$

and there exists a constant  $C_\beta$  such that

$$\forall u \in (0, 1], \quad C^{-1}u^3 \leq F(\beta, u + h_w(\beta)) - F(\beta, h_w(\beta)) \leq Cu^3.$$

- [Lacoin '21]: when  $h > h_w$ , a complete picture of the typical height, the Gibbs states and regularity of the free energy.
- **Q:** When  $0 \leq h \leq h_w$ , how does the surface look like?

Our results: typical height in delocalized phase

Our result (Subcritical regime:  $h \in (0, h_w)$ )

Pinning does not change the typical height ( $h = 0$ ).

Theorem (N. Feldheim, Y. '23)

Fix  $\beta \geq 1$ ,  $h \in (0, h_w)$  and  $N \geq 1$ . Let  $H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$ .

- ❶ There exist universal constants  $C, K > 0$  s.t. for all integer  $m \geq K$ ,  
$$\mathbb{P}_N^{\beta, h} \left( |\phi^{-1}([H + m, \infty))| > e^{-2\beta m} N^2 \right) \leq e^{-C e^{-2\beta m} N (1 \wedge e^{-2\beta m} N \log^{-8} N)}.$$
- ❷ For  $\delta > 0$  and  $m \in \mathbb{N}$  we have  
$$\mathbb{P}_N^{\beta, h} \left( |\phi^{-1}([0, H - m])| > 2e^{-2\beta m} N^2 \right) \leq 3e^{-\min(\frac{1}{2}e^{2\beta m} - 4\beta(1+\kappa), \delta)N}.$$

where (for  $h \in (0, h_w)$ ,  $e^{-h} + e^{-4\beta} > 1$ )

$$\kappa(\beta, h, \delta) := \frac{4\beta + \delta}{\log(e^{-h} + e^{-4\beta})}.$$

## At criticality: conjecture and result

At  $h = h_w$ , Lacoin conjectured: the surface height concentrates around

$$H_w := \left\lfloor \frac{1}{6\beta} \log N \right\rfloor ,$$

with fluctuations similar to the subcritical regime.

Isolated and non-isolated zeros

$$\begin{aligned} q_1(\phi) &:= \{x \in \Lambda_N : \phi(x) = 0, \forall y \in \Lambda_N, y \sim x, \phi(y) \geq 1\}, \\ q_{2+}(\phi) &:= \{x \in \Lambda_N : \phi(x) = 0, \exists y \in \Lambda_N, y \sim x, \phi(y) = 0\}. \end{aligned}$$

Theorem (N. Feldheim, Y. '23)

For  $\beta \geq 1$  and  $h = h_w$ , we have for all  $N \in \mathbb{N}$  and  $C > 0$ :

$$\mathbb{P}_N^{\beta, h_w}(\phi \in \Omega_N : |q_{2+}(\phi)| \geq CN) \leq e^{-N(\frac{C}{20}e^{-6\beta} - 4\beta)}.$$

# At criticality: lower bound on the height

## Proposition

For all  $\beta \geq 1$ ,  $C > 0$ ,  $h = h_w$ ,  $N \in \mathbb{N}$  and  $m \in \mathbb{N}$ , letting  $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$  we have

$$\begin{aligned} \mathbb{P}_N^{\beta, h_w} \left( \left\{ |\phi^{-1}(0)| \leq CN^{\frac{4}{3}} \right\} \cap \left\{ |\phi^{-1}([1, H_w - m])| \geq 2e^{-2\beta m} N^2 \right\} \right) \\ \leq 2 \exp \left( 4\beta N + 4\beta CN^{\frac{4}{3}} - \frac{1}{2} e^{2\beta m} N^{\frac{4}{3}} \right). \end{aligned}$$

It suffices to prove that for large enough  $C > 0$ , we have

$$\mathbb{P}_N^{\beta, h_w} \left( |q_1(\phi)| > CN^{4/3} \right) = o(1)$$

in order to obtain a lower bound on the typical height of the surface at criticality, matching the conjectured height  $H_w$ .

# Ideas from the proof



## Subcritical regime ( $h \in (0, h_w)$ ): Upward fluctuation

- Partial order " $\leq$ " on  $\Omega_N \times \Omega_N$ :

$$\phi \leq \psi \quad \Leftrightarrow \quad \forall x \in \Lambda_N, \phi(x) \leq \psi(x).$$

- Function  $f : \Omega_N \mapsto \mathbb{R}$  is increasing if  $\phi \leq \psi \Rightarrow f(\phi) \leq f(\psi)$ .
- Event  $\mathcal{A} \subset \Omega_N$  is increasing if  $\mathbf{1}_{\mathcal{A}}$  is increasing.
- $(\mu_1, \mu_2$  on  $\Omega_N)$   $\mu_2$  dominates  $\mu_1$  ( $\mu_1 \preceq \mu_2$ ) if for any bounded increasing function  $f : \Omega_N \mapsto \mathbb{R}$ ,

$$\mu_1(f) \leq \mu_2(f).$$

### Lemma

For all  $\beta > 0$  and  $0 \leq h_1 \leq h_2$ , we have

$$\mathbb{P}_N^{\beta, h_2} \preceq \mathbb{P}_N^{\beta, h_1}.$$

Proof: Verify Holley's condition  $\mathbb{P}^{h_1}(\phi \vee \psi) \mathbb{P}^{h_2}(\phi \wedge \psi) \geq \mathbb{P}^{h_1}(\phi) \mathbb{P}^{h_2}(\psi)$ .

## Subcritical regime ( $h \in (0, h_w)$ ): Upward fluctuation

- The following event is increasing

$$\left\{ \phi \in \Omega_N : |\{x \in \Lambda_N : \phi(x) \geq H + m\}| > e^{-2\beta m} N^2 \right\}$$

- [Caputo, Lubetzky, Martinelli, Sly, Toninelli '14] There exist two universal constants  $C, K > 0$  such that for all  $\beta \geq 1$  and all integer  $k \geq K$ , we have for all  $N$ ,

$$\begin{aligned} \mathbb{P}_N^{\beta,0} \left( |\{x \in \Lambda_N : \phi(x) \geq H + k\}| > e^{-2\beta k} N^2 \right) \\ \leq e^{-C e^{-2\beta k} N (1 \wedge e^{-2\beta k} N \log^{-8} N)} \end{aligned}$$

- $\mathbb{P}_N^{\beta,h} \preceq \mathbb{P}_N^{\beta,0}$

Combining the three items, we obtain the desired upper bound on the upward fluctuation.

## Subcritical regime ( $h \in (0, h_w)$ ) : Downward fluctuation

### Lemma

For all  $\beta \geq 1$ ,  $h \in [0, h_w)$ ,  $\delta > 0$  and  $N \geq 1$ , we have

$$\mathbb{P}_N^{\beta, h} (|\phi^{-1}(0)| \geq \kappa N) \leq e^{-\delta N},$$

where  $\kappa = \kappa(\beta, h, \delta) = \frac{4\beta + \delta}{\log(e^{-h} + e^{-4\beta})}$ .

Idea: Lift the surface up by one

For  $\phi \in \Omega_N$  and each  $A \subseteq \phi^{-1}(0)$ , we define  $U_A \phi : \Lambda_N \mapsto \mathbb{Z}_+$  as follows

$$(U_A \phi)(x) := \begin{cases} \phi(x) + 1, & \text{if } x \notin A, \\ 0, & \text{if } x \in A. \end{cases}$$

The action  $U_A$  increases the height of each site in  $\Lambda_N \setminus A$  by one, we have

$$\begin{aligned} \mathcal{H}_N(U_A \phi) &\leq \mathcal{H}_N(\phi) + 4|A| + 4N, \\ |\phi^{-1}(0)| - |(U_A \phi)^{-1}(0)| &= |\phi^{-1}(0) \setminus A|. \end{aligned}$$

$$\mathbb{P}_N^{\beta,h}(U_A\phi) \geq \mathbb{P}_N^{\beta,h}(\phi) \cdot \exp(-h|\phi^{-1}(0) \setminus A| - 4\beta|A| - 4\beta N),$$

$$\begin{aligned} & \sum_{A \subseteq \phi^{-1}(0)} \mathbb{P}_N^{\beta,h}(U_A\phi) \\ & \geq e^{-4\beta N} \cdot \mathbb{P}_N^{\beta,h}(\phi) \sum_{A \subseteq \phi^{-1}(0)} \exp(-h|\phi^{-1}(0) \setminus A| - 4\beta|A|) \\ & = e^{-4\beta N - h|\phi^{-1}(0)|} \cdot \mathbb{P}_N^{\beta,h}(\phi) \sum_{n=0}^{|\phi^{-1}(0)|} \sum_{\substack{A \subseteq \phi^{-1}(0) \\ |A|=n}} \exp(-n(4\beta - h)) \\ & = e^{-4\beta N - h|\phi^{-1}(0)|} \left(1 + e^{-(4\beta - h)}\right)^{|\phi^{-1}(0)|} \mathbb{P}_N^{\beta,h}(\phi). \end{aligned}$$

- $A, A' \subseteq \phi^{-1}(0)$  with  $A \neq A'$ ,  $\Rightarrow U_A\phi \neq U_{A'}\phi$ .
- For  $\phi \neq \psi$ , if  $A \subseteq \phi^{-1}(0)$  and  $B \subseteq \psi^{-1}(0)$ ,  $\Rightarrow U_A\phi \neq U_B\psi$  (we can recover  $A$  from  $U_A\phi$  by zero-value sites, then proceed to recover  $\phi$ .)
- $\sum_{\phi \in \Omega_N} \sum_{A \subseteq \phi^{-1}(0)} \mathbb{P}_N^{\beta,h}(U_A\phi) \leq 1$ .

## Subcritical regime ( $h \in (0, h_w)$ ): Downward fluctuation

### Lemma

Let  $\beta \geq 1$ ,  $h \in [0, h_w)$  and  $\kappa > 0$ . Then for all  $m > \lceil \frac{1}{2\beta} \log(8\beta(1+\kappa)) \rceil$  and  $N \geq 1$  we have

$$\mathbb{P}_N^{\beta, h} \left( \left\{ |\phi^{-1}(0)| \leq \kappa N \right\} \cap \left\{ |\phi^{-1}([1, H-m])| \geq \frac{e^{-2\beta m}}{1 - e^{-2\beta}} N^2 \right\} \right) \leq \frac{1}{1 - e^{-\beta N}} e^{-\left(\frac{1}{2}e^{2\beta m} - 4\beta(1+\kappa)\right)N}.$$

Fix an integer  $\ell \in [1, H-m]$ . For  $A \subseteq \phi^{-1}(\ell)$ , define  $V_A \phi : \Lambda_N \mapsto \mathbb{Z}_+$

$$(V_A \phi)(x) := \begin{cases} 0, & \text{if } x \in \phi^{-1}(0), \\ 1, & \text{if } x \in A, \\ \phi(x) + 1, & \text{if } x \notin A \cup \phi^{-1}(0). \end{cases}$$

Observe that for  $x \in A$  and  $y \notin A \cup \phi^{-1}(0)$  with  $x \sim y$ ,

$$|(V_A\phi)(x) - (V_A\phi)(y)| = \phi(y) \leq |\ell - \phi(y)| + \ell,$$

$$\mathcal{H}_N(V_A\phi) \leq \mathcal{H}_N(\phi) + 4N + 4|\phi^{-1}(0)| + 4\ell|A|.$$

As  $|(V_A\phi)^{-1}(0)| = |\phi^{-1}(0)|$ ,

$$\mathbb{P}_N^{\beta,h}(V_A\phi) \geq \mathbb{P}_N^{\beta,h}(\phi) e^{-4\beta N - 4\beta|\phi^{-1}(0)| - 4\beta\ell|A|}.$$

Summing over all subsets

$$\begin{aligned} \sum_{A \subseteq \phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h}(V_A\phi) &\geq \mathbb{P}_N^{\beta,h}(\phi) \sum_{A \subseteq \phi^{-1}(\ell)} e^{-4\beta N - 4\beta|\phi^{-1}(0)| - 4\beta\ell|A|} \\ &\geq \mathbb{P}_N^{\beta,h}(\phi) \exp\left(-4\beta N - 4\beta|\phi^{-1}(0)| + \frac{1}{2}e^{-4\beta\ell}|\phi^{-1}(\ell)|\right). \end{aligned}$$

For  $A, A' \subseteq \phi^{-1}(\ell)$  with  $A \neq A'$ ,  $\Rightarrow V_A\phi \neq V_{A'}\phi$ .

For  $\phi \neq \psi \in \Omega_N$ ,  $A \subset \phi^{-1}(\ell)$  and  $B \subset \psi^{-1}(\ell)$ ,  $\Rightarrow V_A\phi \neq V_B\psi$ . (we can recover  $A$  by 1-valued sites of  $V_A\phi$  and then proceed to recover  $\phi$ )

$$1 \geq \sum_{\substack{\phi: |\phi^{-1}(\ell)| \geq e^{-2\beta j} N^2 \\ |\phi^{-1}(0)| \leq \kappa N}} \sum_{A \subset \phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h}(V_A\phi) \quad (j = H - \ell)$$

## At criticality

Observation: for  $x_1, x_2, x_3, x_4 \in \mathbb{Z}_+$ ,

$$\sum_{k=-\infty}^0 \exp \left( -\beta \sum_{i=1}^4 |x_i - k| \right) = \exp \left( h_w - \beta \sum_{i=1}^4 x_i \right).$$

A new state space (only allows isolated negative value sites.)

$$\Omega_N^* := \{ \psi : \Lambda_N \rightarrow \mathbb{Z} \mid \text{if } \psi(x) \leq -1, \forall y \in \Lambda_N, y \sim x, \psi(y) \geq 1 \}.$$

Note: if  $\psi \in \Omega_N^*$ , then  $\max(\psi, 0) \in \Omega_N$ . we have

$$\mathcal{Z}_N^{\beta, h_w} = \sum_{\psi \in \Omega_N^*} \exp(-\beta \mathcal{H}_N(\psi) + h_w |q_{2+}(\psi)|).$$

Define a new probability measure  $\tilde{\mathbb{P}}_N$  on  $\Omega_N^*$  as follows:

$$\forall \psi \in \Omega_N^*, \quad \tilde{\mathbb{P}}_N(\psi) := \frac{1}{\mathcal{Z}_N^{\beta, h_w}} \exp(-\beta \mathcal{H}_N(\psi) + h_w |q_{2+}(\psi)|).$$

Relation between  $\tilde{\mathbb{P}}_N$  and  $\mathbb{P}_N^{\beta, h_w}$ : for any  $\phi \in \Omega_N$ ,

$$\mathbb{P}_N^{\beta, h_w}(\phi) = \tilde{\mathbb{P}}_N(\{ \psi \in \Omega_N^* : \max(\psi, 0) = \phi \}).$$

## At criticality

Since for any  $\psi \in \Omega_N^*$ , we have  $q_{2+}(\max(\psi, 0)) = q_{2+}(\psi)$ , then

$$\mathbb{P}_N^{\beta, h_w}(\{\phi \in \Omega_N : |q_{2+}(\phi)| \geq CN\}) = \tilde{\mathbb{P}}_N(\{\psi \in \Omega_N^* : |q_{2+}(\psi)| \geq CN\}).$$

For any subset  $A \subseteq q_{2+}(\psi)$ ,  $\mathcal{N}(A)$ : the edge boundary of  $A$

$$\mathcal{N}(A) := \left\{ \{x, y\} \in E(\mathbb{Z}^2) : x \in A, y \in A^c \right\}.$$

Define  $U_A \psi \in \Omega_N^*$  as

$$(U_A \psi)(x) := \begin{cases} \psi(x) + 1 & \text{if } x \notin A, \\ 0 & \text{if } x \in A. \end{cases}$$

Notation: for fixed  $\psi \in \Omega_N^*$ , write  $q_{2+}(A) := q_{2+}(U_A \psi)$ .

Observe  $\mathcal{H}_N(U_A \psi) \leq \mathcal{H}_N(\psi) + 4\beta N + \beta|\mathcal{N}(A)|$ , then

$$\tilde{\mathbb{P}}_N(U_A \psi) \geq \tilde{\mathbb{P}}_N(\psi) \exp(-4\beta N - \beta|\mathcal{N}(A)| - h_w(|q_{2+}(\psi)| - |q_{2+}(A)|)).$$



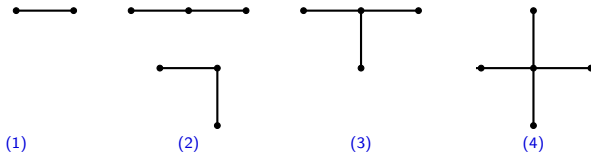
$V_1, V_2, \dots, V_k$ : connected components of  $q_{2+}(\psi)$ , write  $A_i = A \cap V_i$ .  
Sum over all subsets  $A \subseteq q_{2+}(\psi)$  to obtain

$$\begin{aligned}
& \sum_{A \subseteq q_{2+}(\psi)} \tilde{\mathbb{P}}_N(U_A \psi) \\
& \geq \tilde{\mathbb{P}}_N(\psi) \exp(-4\beta N - h_w |q_{2+}(\psi)|) \sum_{A \subseteq q_{2+}(\psi)} \exp(-\beta |\mathcal{N}(A)| + h_w |q_{2+}(A)|) \\
& = \tilde{\mathbb{P}}_N(\psi) \exp(-4\beta N - h_w |q_{2+}(\psi)|) \sum_{A_1, \dots, A_k} \prod_{i=1}^k \exp(-\beta |\mathcal{N}(A_i)| + h_w |q_{2+}(A_i)|) \\
& = \tilde{\mathbb{P}}_N(\psi) \exp(-4\beta N) \prod_{i=1}^k \exp(-h_w |V_i|) \sum_{A_i \subseteq V_i} \exp(-\beta |\mathcal{N}(A_i)| + h_w |q_{2+}(A_i)|)
\end{aligned}$$

$q_{2+}(\psi) = V_1 \cup V_2 \cdots \cup V_k$  disjoint union.

Focus on each connected component  $V_i$

For each connected graph  $(V_i, E_i)$ , after deleting some edges, it can be covered by the following four types of patterns.



### Lemma

If  $\beta \geq 1$  and  $V$  is the vertex set of one of the patterns shown above, then

$$\exp(-h_w|V|) \sum_{B \subseteq V} \exp(-\beta|\mathcal{N}(B)| + h_w|q_{2+}(B)|) \geq 1 + \frac{1}{2}e^{-6\beta}.$$

This lemma implies

$$\sum_{A \subseteq q_{2+}(\psi)} \tilde{\mathbb{P}}_N(U_A \psi) \geq e^{-4\beta N} \tilde{\mathbb{P}}_N(\psi) \left(1 + \frac{1}{2}e^{-6\beta}\right)^{|q_{2+}(\psi)|/5}.$$

- For  $A \neq B \subset q_{2+}(\psi)$ , we have  $U_A\psi \neq U_B\psi$  since  $(U_A\psi)|_{A \setminus B} = 0$  and  $(U_B\psi)|_{A \setminus B} = 1$ .
- For  $\psi, \psi' \in \Omega_N^*$  and  $A \subset q_{2+}(\psi), A' \subset q_{2+}(\psi')$ , we have

$$U_A\psi \neq U_{A'}\psi'.$$

To see this, note that

$$A = \{x \in \Lambda_N : (U_A\psi)(x) = 0, \exists y \in \Lambda_N, y \sim x, (U_A\psi)(y) \in \{0, 1\}\}.$$

Thus, given  $U_A\psi$ , we can first recover the set  $A$  and then proceed to recover  $\psi$ .

•

$$\begin{aligned}
1 &\geq \sum_{\psi \in \Omega_N^* : |q_{2+}(\psi)| \geq CN} \sum_{A \subset q_{2+}(\psi)} \tilde{\mathbb{P}}_N(U_A\psi) \\
&\geq \sum_{\psi \in \Omega_N^* : |q_{2+}(\psi)| \geq CN} e^{-4\beta N} \tilde{\mathbb{P}}_N(\psi) \left(1 + \frac{1}{2}e^{-6\beta}\right)^{|q_{2+}(\psi)|/5}
\end{aligned}$$

## Proposition

For all  $\beta \geq 1$ ,  $C > 0$ ,  $h = h_w$ ,  $N \in \mathbb{N}$  and  $m \in \mathbb{N}$ , letting  $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$  we have

$$\begin{aligned} \mathbb{P}_N^{\beta, h_w} \left( \left\{ |\phi^{-1}(0)| \leq CN^{\frac{4}{3}} \right\} \cap \left\{ |\phi^{-1}([1, H_w - m])| \geq 2e^{-2\beta m} N^2 \right\} \right) \\ \leq 2 \exp \left( 4\beta N + 4\beta CN^{\frac{4}{3}} - \frac{1}{2} e^{2\beta m} N^{\frac{4}{3}} \right). \end{aligned}$$

Idea: Fix an integer  $\ell \in [1, H_w - m]$ . For  $A \subseteq \phi^{-1}(\ell)$ , define  $V_A \phi : \Lambda_N \mapsto \mathbb{Z}_+$

$$(V_A \phi)(x) := \begin{cases} 0, & \text{if } x \in \phi^{-1}(0), \\ 1, & \text{if } x \in A, \\ \phi(x) + 1, & \text{if } x \notin A \cup \phi^{-1}(0). \end{cases}$$

Thank you for your attention!