Mixing Time of ASEP in a random environment

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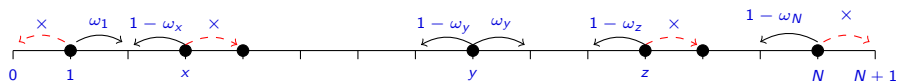
Seminário de probabilidade do IME-USP

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Setup

Environment: $\omega = (\omega_x)_x$ with values in $(0,1)$

Exclusion process with k particles in $[\![1, N]\!]$ with environment ω

(A) Each site is occupied by at most one particle (*the exclusion rule*).

(B) Each of the k particles independently performs a random walk such that a particle at site $x \in [\![1, N]\!]$

$$\begin{cases} \text{jumps to site } x+1 \text{ at rate } \omega_x & \text{for } x \leq N-1, \\ \text{jumps to site } x-1 \text{ at rate } 1-\omega_x & \text{for } x \geq 2, \end{cases}$$

if the target site is not occupied. (No particle is allowed to jump out of the segment.)

Setup

- State space (1: particle   0: empty site.)

$$\Omega_{N,k} := \left\{ \xi \in \{0,1\}^N : \sum_{x=1}^N \xi(x) = k \right\}.$$

- $\xi^{x,y}$: swap the contents of sites x, y of ξ

$$\forall z \in [\![1, N]\!], \quad \xi^{x,y}(z) = \xi(z)\mathbf{1}_{[\![1,N]\!]\setminus\{x,y\}} + \xi(x)\mathbf{1}_{\{y\}} + \xi(y)\mathbf{1}_{\{x\}}.$$

- Transition rates: (for $x \in [\![1, N-1]\!]$)

$$r^\omega(\xi, \xi^{x,x+1}) := \begin{cases} \omega_x & \text{if } \xi(x) = 1 \text{ and } \xi(x+1) = 0, \\ 1 - \omega_{x+1} & \text{if } \xi(x+1) = 1 \text{ and } \xi(x) = 0, \end{cases}$$

$$r^\omega(\xi, \xi') := 0 \quad \text{in all other cases.}$$

Setup

- Generator ($f : \Omega_{N,k} \to \mathbb{R}$)

$$(\mathcal{L}_{N,k}^{\omega}f)(\xi) := \sum_{x=1}^{N-1} r^{\omega}(\xi, \xi^{x,x+1})\big[f(\xi^{x,x+1}) - f(\xi)\big].$$

- Potential in the environment ω: $V^{\omega}(1) := 0$, and for $x \geq 2$

$$V^{\omega}(x) := \sum_{y=2}^{x} \log\left(\frac{1 - \omega_y}{\omega_{(y-1)}}\right).$$

  Potential for $\xi \in \Omega_{N,k}$:

$$V^{\omega}(\xi) := \sum_{x=1}^{N} V^{\omega}(x)\xi(x).$$

- Invariant probability measure

$$\pi_{N,k}^{\omega}(\xi) := \frac{1}{Z_{N,k}^{\omega}} e^{-V^{\omega}(\xi)} \quad \text{with} \quad Z_{N,k}^{\omega} := \sum_{\xi' \in \Omega_{N,k}} e^{-V^{\omega}(\xi')}.$$

  Detailed balance condition

$$\pi_{N,k}^{\omega}(\xi)r^{\omega}(\xi, \xi^{x,x+1}) = \pi_{N,k}^{\omega}(\xi^{x,x+1})r^{\omega}(\xi^{x,x+1}, \xi).$$

Assumptions

- $k = N^{\beta+o(1)} \leq \frac{1}{2}N$ particles where $\beta \in (0,1]$.

- $\omega = (\omega_x)_x$ is IID (law: $\mathbb{P}$, expectation: $\mathbb{E}$) satisfying:
  (1) Uniform ellipticity condition: $\exists \, \alpha \in (0,1/2)$ such that

$$\mathbb{P}\left(\omega_1 \in [\alpha, 1-\alpha]\right) = 1.$$

  (2)

$$\mathbb{E}\left[\log \frac{1-\omega_1}{\omega_1}\right] < 0,$$

  so that the random walk on $\mathbb{Z}$ is transient to the right.

- Distance to equilibrium & Mixing time:

$$d_{N,k}^\omega(t) := \max_{\xi \in \Omega_{N,k}} \|P_t^\xi - \pi_{N,k}^\omega\|_{\mathrm{TV}},$$

$$t_{\mathrm{mix}}^{N,k,\omega}(\varepsilon) := \inf\left\{t \geq 0: \, d_{N,k}^\omega(t) \leq \varepsilon\right\},$$
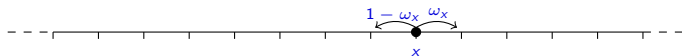
$$t_{\mathrm{mix}}^{N,k,\omega} := t_{\mathrm{mix}}^{N,k,\omega}(1/4).$$

Q: **How does the mixing time $t_{\mathrm{mix}}^{N,k,\omega}$ grow in terms of N and k for typical realization of ω?**

Outline

(1) Related results (RWRE, SEP with $\omega \equiv p$ or $\omega = (\omega_x)_x$ IID).

(2) Our result: $t_{\mathrm{mix}}^{N,k,\omega}$ grows like a power of N.

(4) Heuristic for the lower bounds: three mechanisms.

(4) Idea for the upper bound.

Related result: random walk in random environment on $\mathbb{Z}$

Given $(\omega_x)_x$, $(X_t)_{t \geq 0}$: a continuous-time random walk on $\mathbb{Z}$ starting at 0.
[Solomon '75 AOP] showed that

$$\begin{cases} \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] = 0 \Rightarrow (X_t)_{t \geq 0} \text{ is recurrent,} \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0 \Rightarrow \lim_{t \to \infty} X_t = \infty, \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] > 0 \Rightarrow \lim_{t \to \infty} X_t = -\infty. \end{cases}$$

Assuming $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$, set

$$\lambda = \lambda_{\mathbb{P}} := \inf \left\{ s > 0, \mathbb{E}\left[\left(\frac{1-\omega_1}{\omega_1} \right)^s \right] \geq 1 \right\} \in (0, \infty].$$

[Kesten, Kozlov, Spitzer '75 Compos. Math.] showed that
$$\begin{cases} \lim_{t \to \infty} \frac{X_t}{t} = \vartheta_{\mathbb{P}} > 0, & \text{if } \lambda > 1 \text{ (ballistic),} \\ \lim_{t \to \infty} \frac{\log(X_t)}{\log t} = \lambda, & \text{if } \lambda \in (0, 1] \text{ (subballistic) .} \end{cases}$$

Related results: many particles in homogenous environment

SSEP ($\omega \equiv \frac{1}{2}$): [Aldous '83 Lect. Notes Math.], [Wilson '04 AAP], [Lacoin '16 AOP]

$$t_{\mathrm{mix}}^{N,k_N} \asymp (N^2 \log k_N) = O(t_{\mathrm{mix}}^{N,1} \log N).$$

ASEP ($\omega \equiv p \neq \frac{1}{2}$): [Benjamini *et al.* '05 TAMS], [Labbé, Lacoin '19 AOP]

$$t_{\mathrm{mix}}^{N,k_N} \asymp N.$$

Related result: one particle in random environment

When $k = 1$, [Gantert, Kochler '18 ALEA] showed that if $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$,

$$\begin{cases} t_{\text{mix}}^{N,1,\omega}(\varepsilon) = [C(\mathbb{P}) + o(1)]N, & \text{if } \lambda_{\mathbb{P}} > 1, \\ \lim_{N \to \infty} \frac{\log t_{\text{mix}}^{N,1,\omega}}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}, & \text{if } \lambda_{\mathbb{P}} \in (0, 1]. \end{cases}$$

Potential $V^\omega : \mathbb{N} \to \mathbb{R}$

$$V^\omega(x) := \begin{cases} 0, & \text{for } x = 1, \\ \sum_{y=2}^{x} \log \left(\frac{1-\omega_y}{\omega_{y-1}} \right), & \text{for } x \geq 2. \end{cases}$$

The largest potential barrier: $\max_{1 \leq x < y \leq N} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N$.

Related result: many particles in random environment

Assuming $\lambda > 1$ and $\lim_{N \to \infty} k_N/N = \theta \in (0,1)$, [Schmid '19 EJP] showed:

- When $\operatorname{ess\,inf} \omega_1 > 1/2$, $t_{\mathrm{mix}}^{N,k_N,\omega} \asymp N$ by comparison.

- When $\operatorname{ess\,inf} \omega_1 < 1/2$, $t_{\mathrm{mix}}^{N,k_N,\omega} \geq N^{1+\delta}$ for some $\delta > 0$.

- When $\operatorname{ess\,inf} \omega_1 = 1/2$, then

$$\liminf_{N \to \infty} t_{\mathrm{mix}}^{N,k_N,\omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\mathrm{mix}}^{N,k_N,\omega}(\varepsilon) \leq CN(\log N)^3,$$

  together with a quantitative lower bound if $\mathbb{P}[\omega_1 = 1/2] > 0$.

Q: **If $\operatorname{ess\,inf} \omega_1 < 1/2$, how does $t_{\mathrm{mix}}^{N,k_N,\omega}$ grow?**
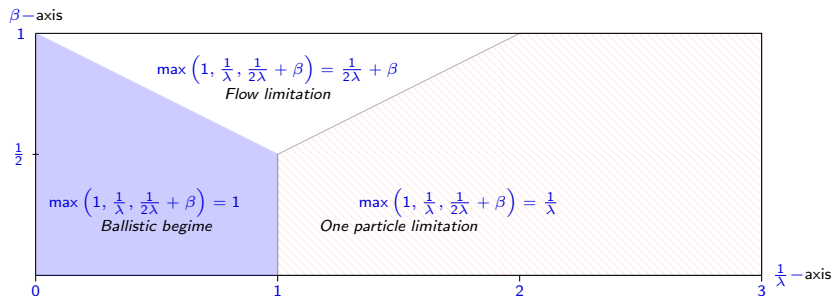
Our result

Theorem (Lacoin, Y.)

Assuming $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$, *ess* $\inf \omega_1 < \frac{1}{2}$, $k = N^{\beta+o(1)} \leq N/2$ *with* $\beta \in (0,1]$ *and the uniform ellipticity condition, with high probability we have*

$$c(\alpha, \mathbb{P}) N^{\max(1, \frac{1}{\lambda}, \beta+\frac{1}{2\lambda})+o(1)} \leq t_{\mathrm{mix}}^{N,k,\omega} \leq N^{C(\alpha,\mathbb{P})}.$$

Conjecture: Our lower bound is sharp.

Phase diagram (Conjecture)

The exponent of the mixing time with $k = N^{\beta + o(1)}$ particles

Lower bound on the mixing time: three mechanisms

- Mass transport cannot be faster than ballistic

- The leftmost particle is blocked by traps in the potential profile

- Potential barrier creates bottleneck for the particle flow

Typical configurations in equilibrium

$\begin{cases} \text{Every site of } [\![1, N-k-C]\!] \text{ is vacant,} \\ \text{Every site of } [\![N-k+C, N]\!] \text{ is occupied,} \end{cases}$ in equilibrium .
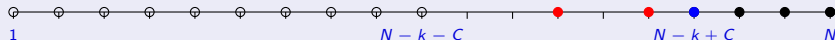
Figure: ○: empty sites   ●●●: particles

Figure: The minimal configuration $\xi_{\min}$

1° Mass transport cannot be faster than ballistic

The lower bound: $t_{\mathrm{mix}}^{N,k,\omega} = \Omega(N)$.

The time for $(\eta_t^{\min})_{t \geq 0}$ starting with $\xi_{\min} := \mathbf{1}_{\{1 \leq x \leq k\}}$ to reach equilibrium.
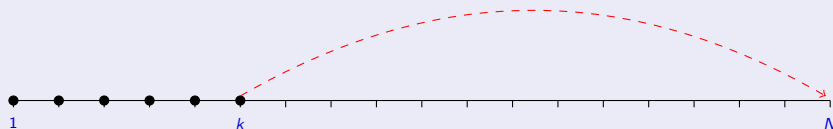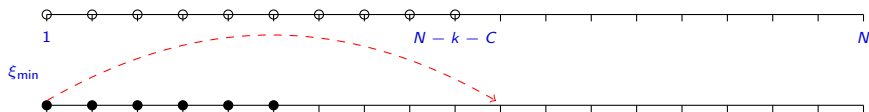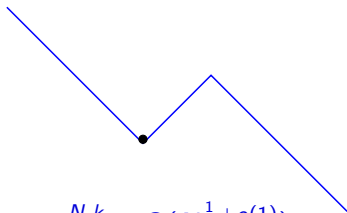

Figure: The configuration $\xi_{\min}$.

2° The leftmost particle is blocked by traps in the potential profile

Since $\operatorname{ess\,inf} \omega_1 < 1/2$, $V^\omega(x) = \sum_{y=2}^{x} \log\left(\frac{1-\omega_y}{\omega_{y-1}}\right)$ is non-monotone and
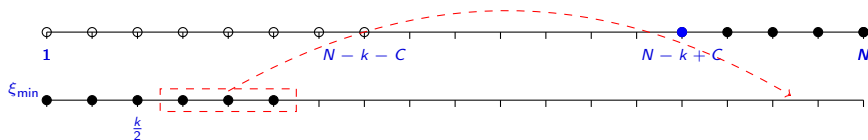
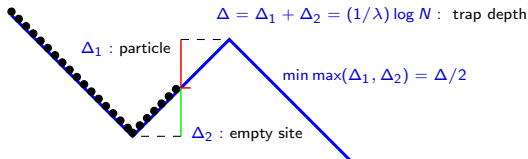$$\max_{1 \leq x < y \leq N/4} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N.$$

Then

$$t_{\mathrm{mix}}^{N,k} = \Omega(N^{\frac{1}{\lambda}+o(1)}).$$

3° Potential barrier creates bottleneck for the particle flow

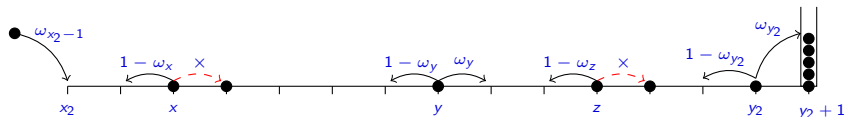The time for a particle to flow out of the trap is roughly $N^{\frac{1}{2\lambda}}$, and then

$$t_{\mathrm{mix}}^{N,k} = \Omega(N^{\beta + \frac{1}{2\lambda} + o(1)}).$$

$\Delta = \Delta_1 + \Delta_2 = (1/\lambda) \log N$: trap depth

Δ_1 : particle

$\min \max(\Delta_1, \Delta_2) = \Delta/2$

Δ_2 : empty site

Flow limitation: idea for the proof

- Deepest trap in $[\![N/2, 3N/4]\!]$: with $x_2(\omega) \leq y_2(\omega)$ as the two ends

$$V^\omega(y_2) - V^\omega(x_2) = \max_{N/2 \leq x \leq y \leq 3N/4} (V^\omega(y) - V^\omega(x)) \sim (1/\lambda) \log N.$$

- Partial order $\preccurlyeq$ on $\widetilde{\Omega}_{x_2, y_2}$:

$$\xi \preccurlyeq \xi' \quad \text{if} \quad \forall x \geq x_2, \quad \sum_{z=x}^{y_2+1} \xi(z) \leq \sum_{z=x}^{y_2+1} \xi'(z).$$

- Compare the dynamic in the interval $[s, s+t]$ with that starting from **0** at time s to obtain

$$\widetilde{\sigma}_{s+t}^{\mathbf{0}}(y_2 + 1) \geq \widetilde{\sigma}_s^{\mathbf{0}}(y_2 + 1) + (\vartheta_s \circ \widetilde{\sigma})_t^{\mathbf{0}}(y_2 + 1).$$

- Kingman's subbaddtive ergodic Theorem:
$$\mathbf{E}\left[\widetilde{\sigma}_t^{\mathbf{0}}(y_2 + 1)\right] \leq t\left[\lim_{s \to \infty} \frac{1}{s}\widetilde{\sigma}_s^{\mathbf{0}}(y_2 + 1)\right] = t\lim_{n \to \infty}\frac{n}{\mathcal{T}_n},$$
  where $\mathcal{T}_n$ is the instant of the nth particle jumps to the site $y_2 + 1$.
- When there is a particle jumping onto the site $y_2 + 1$, we move all the particles in the segment $[\![x_2, y_2]\!]$ to the site $y_2 + 1$. (The state becomes $\mathbf{0}$.)
$$\lim_{n \to \infty}\frac{n}{\mathcal{T}_n} \leq (y_2 - x_2 + 1)\frac{1}{\mathbf{E}[\mathcal{T}_1]}.$$

- Fill half of the trap and compare with a reversible M.C. with
$$\#\left\{x \in [\![x_2, y_2]\!] \ : \ V(x) \leq [V(y_2) + V(x_2)]/2 \ \text{particles.}\right.$$

- To wait for site x_2 to be empty or site y_2 to be occupied by a particle
$$\mathcal{T}_1 \geq \mathcal{T}' := \left\{t \geq 0 : \widetilde{\sigma}_t'(x_2) = 0 \ \text{or} \ \widetilde{\sigma}_t'(y_2) = 1\right\}.$$

Upper bound on the mixing time

- Coupling, Peres-Winkler inequality

- Flow method: a lower bound on the spectral gap

- Guide particles to the right

Idea for the upper bound

- A coupling: if $\xi \leq \xi'$ (i.e. $\forall x, \sum_{i=x}^{N} \xi(i) \leq \sum_{i=x}^{N} \xi'(i)$), then
$$\forall t \geq 0, \quad \eta_t^\xi \leq \eta_t^{\xi'},$$
$$d_{N,k}^\omega(t) \leq \max_{\xi,\xi'} \|P_t^\xi - P_t^{\xi'}\|_{\mathrm{TV}} \leq \max_{\xi,\xi'} \mathbf{P}\left[\eta_t^\xi \neq \eta_t^{\xi'}\right].$$

- $\xi_{\max} := \mathbf{1}_{\{N-k+1 \leq x \leq N\}}$ $\quad (\eta_t^{\min})_{t \geq 0}$ starts from $\xi_{\min} = \mathbf{1}_{\{1 \leq x \leq k\}}$,
$$\tau := \inf\left\{t \geq 0 : \eta_t^{\min} = \xi_{\max}\right\},$$
$$\forall t \geq \tau, \forall \xi, \xi' \quad \Rightarrow \quad \eta_t^\xi = \eta_t^{\xi'}.$$

  By Markov property and stochastic domination,
$$d_{N,k}^\omega(nt) \leq \mathbb{P}[\tau > nt] \leq \mathbb{P}[\tau > t]^n \leq \left(1 - \mathbb{P}\left[\eta_t^{\min} = \xi_{\max}\right]\right)^n.$$

- Censoring scheme $\mathcal{C} : [0, \infty) \mapsto \mathcal{P}(E)$ (E: edges in $[\![1, N]\!]$)
$(P_t^\mathcal{C})_{t \geq 0}$: a transition is performed if it doesn't cross an edge in $\mathcal{C}(t)$.
Peres-Winkler inequality: for any $\xi \in \Omega_{N,k}$ and any $\mathcal{C}$
$$P_t(\xi, \xi_{\max}) \geq P_t(\xi_{\min}, \xi_{\max}) \geq P_t^\mathcal{C}(\xi_{\min}, \xi_{\max}).$$

Flow method [gap : minimal nonzero eigenvalue of $-\mathcal{L}_{N,k}^\omega$]
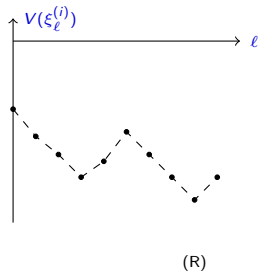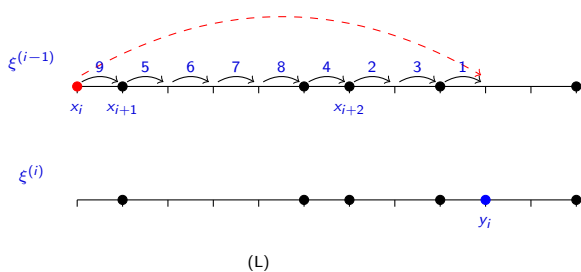
$$\mathrm{gap} \geq \exp\left(-C(\alpha)N\right),$$

$$t_{\mathrm{mix}}^{N,k,\omega}(\varepsilon) \leq \mathrm{gap}^{-1}\log\frac{1}{2\varepsilon\mu_{\min}}.$$

Ground state $\xi^*: \pi_{N,k}^\omega(\xi^*) = \max_{\xi\in\Omega_{N,k}}\pi_{N,k}^\omega(\xi)$

Connect ξ with ξ^*: if $\xi(x_i) = 1 - \xi^*(x_i) = 1$ and $\xi(y_i) = 1 - \xi^*(y_i) = 0$. We move the particle at site x_i to y_i as follows:

(L)   (R)

In this process, the potential V^ω can increase at most $c(\alpha)N$.

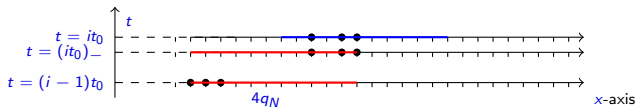# Guide particles to the right

- case $k \leq q_N = C(\mathbb{P}) \log N$:

  Run k particles in a segment of length $4q_N$.

  In equilibrium, all the k particles are in the right half $2q_N$ sites w.h.p.

  Let $\bar{\xi}(1)$ be the position of the leftmost particle. We have

$$\pi^{\omega}_{[x+1,x+4q_N],k} \left[\bar{\xi}(1) \leq x + 2q_N \right] \leq N^{-3}, \text{ for } \xi \in \Omega_{[x+1,x+4q_N],k}.$$

$t_0 = t^{4q_N,k,\omega}_{\mathrm{mix}} \left(1/N^3 \right) \leq \exp(C(\alpha)4q_N)$.

intersection between the red segment and the blue segment $= 2q_N$.

$$\lim_{\varepsilon \to 0} \inf_{\substack{N \geq 1 \\ k \in [\![1, N/2]\!]}} \mathbb{P} \left[\pi^{\omega}_{N,k}(\xi_{\max}) > \varepsilon \right] = 1.$$

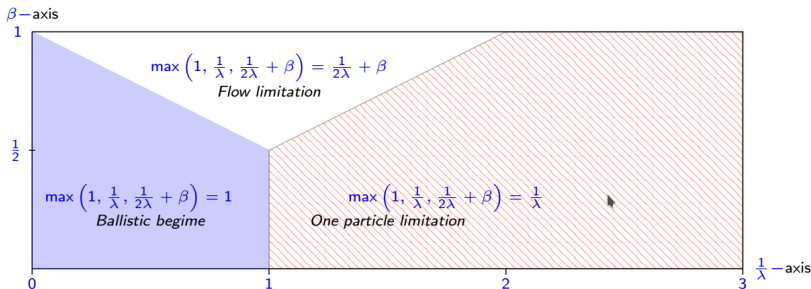# Guide particles to the right

- Case $k > q_N$: send the rightmost $k - q_N$ particles to the rightmost $k - q_N$ sites one by one.
  We repeat the first step for the leftmost q_N particles.

$$\pi^{\omega}_{[x+1, x+4q_N], k} \left[\xi(x + 4q_N) = 1 \right] \geq 1 - N^{-3}.$$
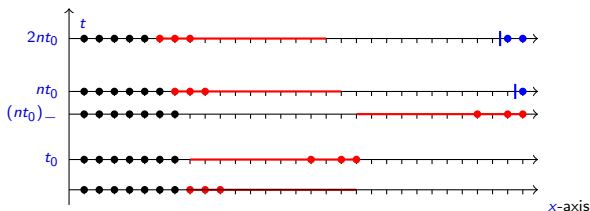
The exponent of the mixing time with $k = N^{\beta + o(1)}$ particles

Thank you for your attention