

Mixing Time of ASEP in a random environment

Shangjie Yang



TSIMF: Probability and Statistical Physics

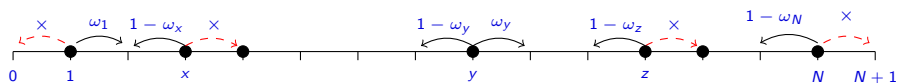
Jointed work with Hubert Lacoin (IMPA, Rio de Janeiro)

25/01/2024

Setup

Environment: $\omega = (\omega_x)_x$ with values in $(0, 1)$

Exclusion process with k particles in $\llbracket 1, N \rrbracket$ with environment ω



- (A) Each site is occupied by at most one particle (*the exclusion rule*).
- (B) Each of the k particles independently performs a random walk such that a particle at site $x \in \llbracket 1, N \rrbracket$

$$\begin{cases} \text{jumps to site } x+1 \text{ at rate } \omega_x & \text{for } x \leq N-1, \\ \text{jumps to site } x-1 \text{ at rate } 1-\omega_x & \text{for } x \geq 2, \end{cases}$$

if the target site is not occupied. (No particle is allowed to jump out of the segment.)

Setup

- State space (1: particle 0: empty site.)

$$\Omega_{N,k} := \left\{ \xi \in \{0,1\}^N : \sum_{x=1}^N \xi(x) = k \right\}.$$

- $\xi^{x,y}$: swap the contents of sites x, y of ξ

$$\forall z \in \llbracket 1, N \rrbracket, \quad \xi^{x,y}(z) = \xi(z) \mathbf{1}_{\llbracket 1, N \rrbracket \setminus \{x,y\}} + \xi(x) \mathbf{1}_{\{y\}} + \xi(y) \mathbf{1}_{\{x\}}.$$

- Transition rates: (for $x \in \llbracket 1, N-1 \rrbracket$)

$$r^\omega(\xi, \xi^{x,x+1}) := \begin{cases} \omega_x & \text{if } \xi(x) = 1 \text{ and } \xi(x+1) = 0, \\ 1 - \omega_{x+1} & \text{if } \xi(x+1) = 1 \text{ and } \xi(x) = 0, \\ r^\omega(\xi, \xi') := 0 & \text{in all other cases.} \end{cases}$$

Setup

- Generator ($f : \Omega_{N,k} \rightarrow \mathbb{R}$)

$$\mathcal{L}_{N,k}^\omega(f)(\xi) := \sum_{x=1}^{N-1} r^\omega(\xi, \xi^{x,x+1}) [f(\xi^{x,x+1}) - f(\xi)].$$

- Potential in the environment ω : $V^\omega(1) := 0$, and for $x \geq 2$

$$V^\omega(x) := \sum_{y=2}^x \log \left(\frac{1 - \omega_y}{\omega_{(y-1)}} \right).$$

Potential for $\xi \in \Omega_{N,k}$:

$$V^\omega(\xi) := \sum_{x=1}^N V^\omega(x) \xi(x).$$

- Invariant probability measure

$$\pi_{N,k}^\omega(\xi) := \frac{1}{Z_{N,k}^\omega} e^{-V^\omega(\xi)} \quad \text{with} \quad Z_{N,k}^\omega := \sum_{\xi' \in \Omega_{N,k}} e^{-V^\omega(\xi')}.$$

Detailed balance condition

$$\pi_{N,k}^\omega(\xi) r^\omega(\xi, \xi^{x,x+1}) = \pi_{N,k}^\omega(\xi^{x,x+1}) r^\omega(\xi^{x,x+1}, \xi).$$

Assumptions

- $\omega = (\omega_x)_x$ is IID (law: \mathbb{P} , expectation: \mathbb{E}).
- $k = N^{\beta+o(1)} \leq \frac{1}{2}N$ particles.
- Assume

$$\mathbb{E} \left[\log \frac{1 - \omega_1}{\omega_1} \right] < 0,$$

so that the random walk on \mathbb{Z} is transient to the right.

- Uniform ellipticity condition: $\exists \alpha \in (0, 1/2)$ such that
- Mixing time $\mathbb{P}(\omega_1 \in [\alpha, 1 - \alpha]) = 1$.

$$d_{N,k}^\omega(t) := \max_{\xi \in \Omega_{N,k}} \|P_t^\xi - \pi_{N,k}^\omega\|_{\text{TV}},$$

$$t_{\text{mix}}^{N,k,\omega}(\varepsilon) := \inf \{ t \geq 0 : d_{N,k}^\omega(t) \leq \varepsilon \},$$

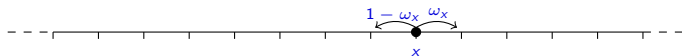
$$t_{\text{mix}}^{N,k,\omega} := t_{\text{mix}}^{N,k,\omega}(1/4).$$

Q: How does the mixing time $t_{\text{mix}}^{N,k,\omega}$ grow in terms of N and k for typical realization of ω ?

Outline

- (1) Related results (RWRE, SEP with $\omega \equiv p$ or $\omega = (\omega_x)_x$ IID).
- (2) Our result: $t_{\text{mix}}^{N,k,\omega}$ grows like a power of N .
- (4) Heuristic for the lower bounds: three mechanisms.
- (4) Idea for the upper bound.

Related result: random walk in random environment on \mathbb{Z}



Given $(\omega_x)_x, (X_t)_{t \geq 0}$: a continuous-time random walk on \mathbb{Z} starting at 0. [Solomon '75 AOP] showed that

$$\begin{cases} \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] = 0 \Rightarrow (X_t)_{t \geq 0} \text{ is recurrent,} \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0 \Rightarrow \lim_{t \rightarrow \infty} X_t = \infty, \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] > 0 \Rightarrow \lim_{t \rightarrow \infty} X_t = -\infty. \end{cases}$$

Assuming $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$, set

$$\lambda = \lambda_{\mathbb{P}} := \inf \left\{ s > 0, \mathbb{E} \left[\left(\frac{1-\omega_1}{\omega_1} \right)^s \right] \geq 1 \right\} \in (0, \infty].$$

[Kesten, Kozlov, Spitzer '75 Compos. Math.] showed that

$$\begin{cases} \lim_{t \rightarrow \infty} \frac{X_t}{t} = \vartheta_{\mathbb{P}} > 0, & \text{if } \lambda > 1 \text{ (ballistic),} \\ \lim_{t \rightarrow \infty} \frac{\log(X_t)}{\log t} = \lambda, & \text{if } \lambda \in (0, 1] \text{ (subballistic).} \end{cases}$$

Related results: many particles in homogenous environment

SSEP ($\omega \equiv \frac{1}{2}$): [Aldous '83 Lect. Notes Math.], [Wilson '04 AAP], [Lacoin '16 AOP]

$$t_{\text{mix}}^{N,k_N} \asymp (N^2 \log k_N) = O(t_{\text{mix}}^{N,1} \log N).$$

ASEP ($\omega \equiv p \neq \frac{1}{2}$): [Benjamini *et al.* '05 TAMS], [Labbé, Lacoin '19 AOP]

$$t_{\text{mix}}^{N,k_N} \asymp N.$$

Related result: one particle in random environment

When $k = 1$, [Gantert, Kochler '18 ALEA] showed that if $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$,

$$\begin{cases} t_{\text{mix}}^{N,1,\omega}(\varepsilon) = [C(\mathbb{P}) + o(1)]N, & \text{if } \lambda_{\mathbb{P}} > 1, \\ \lim_{N \rightarrow \infty} \frac{\log t_{\text{mix}}^{N,1,\omega}}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}, & \text{if } \lambda_{\mathbb{P}} \in (0, 1]. \end{cases}$$

Potential $V^\omega : \mathbb{N} \rightarrow \mathbb{R}$

$$V^\omega(x) := \begin{cases} 0, & \text{for } x = 1, \\ \sum_{y=2}^x \log \left(\frac{1-\omega_y}{\omega_{y-1}} \right), & \text{for } x \geq 2. \end{cases}$$

The largest potential barrier: $\max_{1 \leq x < y \leq N} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N$.

Related result: many particles in random environment

Assuming $\lambda > 1$ and $\lim_{N \rightarrow \infty} k_N/N = \theta \in (0, 1)$, [Schmid '19 EJP] showed:

- When $\text{ess inf } \omega_1 > 1/2$, $t_{\text{mix}}^{N, k_N, \omega} \asymp N$ by comparison.
- When $\text{ess inf } \omega_1 < 1/2$, $t_{\text{mix}}^{N, k_N, \omega} \geq N^{1+\delta}$ for some $\delta > 0$.
- When $\text{ess inf } \omega_1 = 1/2$, then

$$\liminf_{N \rightarrow \infty} t_{\text{mix}}^{N, k_N, \omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\text{mix}}^{N, k_N, \omega}(\varepsilon) \leq CN(\log N)^3,$$

together with a quantitative lower bound if $\mathbb{P}[\omega_1 = 1/2] > 0$.

Q: If $\text{ess inf } \omega_1 < 1/2$, how does $t_{\text{mix}}^{N, k_N, \omega}$ grow?

Our result

Theorem (Lacoin, Y.)

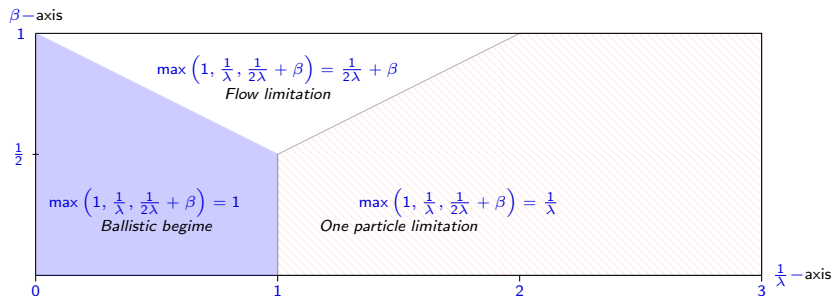
Assuming $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$, $\text{ess inf } \omega_1 < \frac{1}{2}$, $k = N^{\beta+o(1)} \leq N/2$ with $\beta \in (0, 1]$ and the uniform ellipticity condition, with high probability we have

$$c(\alpha, \mathbb{P}) N^{\max(1, \frac{1}{\lambda}, \beta + \frac{1}{2\lambda}) + o(1)} \leq t_{\text{mix}}^{N, k, \omega} \leq N^{C(\alpha, \mathbb{P})}.$$

Conjecture: Our lower bound is sharp.

Phase diagram (Conjecture)

The exponent of the mixing time with $k = N^{\beta+o(1)}$ particles



Lower bound on the mixing time: three mechanisms

- Mass transport cannot be faster than ballistic
- The leftmost particle is blocked by traps in the potential profile
- Potential barrier creates bottleneck for the particle flow

Typical configurations in equilibrium

$\left\{ \begin{array}{l} \text{Every site of } [1, N - k - C] \text{ is vacant,} \\ \text{Every site of } [N - k + C, N] \text{ is occupied,} \end{array} \right.$
 in equilibrium .

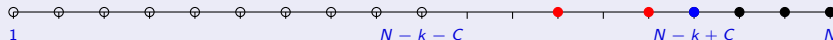


Figure: ○: empty sites ●●●: particles



Figure: The minimal configuration ξ_{\min}

1° Mass transport cannot be faster than ballistic

The lower bound: $t_{\text{mix}}^{N,k,\omega} = \Omega(N)$.

The time for $(\eta_t^{\min})_{t \geq 0}$ starting with $\xi_{\min} := \mathbf{1}_{\{1 \leq x \leq k\}}$ to reach equilibrium.

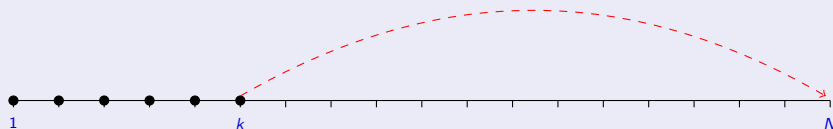
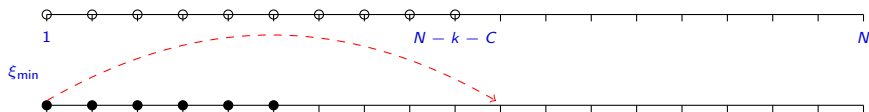


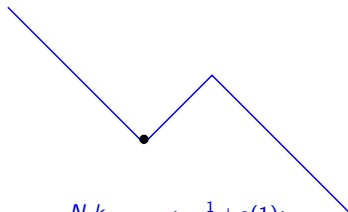
Figure: The configuration ξ_{\min} .

2° The leftmost particle is blocked by traps in the potential profile



Since $\text{ess inf } \omega_1 < 1/2$, $V^\omega(x) = \sum_{y=2}^x \log \left(\frac{1-\omega_y}{\omega_{y-1}} \right)$ is non-monotone and

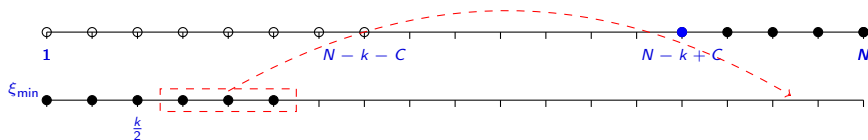
$$\max_{1 \leq x < y \leq N/4} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N.$$



Then

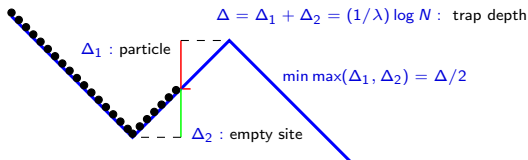
$$t_{\text{mix}}^{N,k} = \Omega(N^{\frac{1}{\lambda} + o(1)}).$$

3° Potential barrier creates bottleneck for the particle flow



The time for a particle to flow out of the trap is roughly $N^{\frac{1}{2\lambda}}$, and then

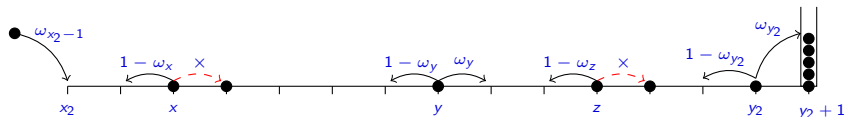
$$t_{\text{mix}}^{N,k} = \Omega(N^{\beta + \frac{1}{2\lambda} + o(1)}).$$



Flow limitation: idea for the proof

- Deepest trap in $\llbracket N/2, 3N/4 \rrbracket$: with $x_2(\omega) \leq y_2(\omega)$ as the two ends

$$V^\omega(y_2) - V^\omega(x_2) = \max_{N/2 \leq x \leq y \leq 3N/4} (V^\omega(y) - V^\omega(x)) \sim (1/\lambda) \log N.$$



- Partial order \preceq on $\tilde{\Omega}_{x_2, y_2}$:

$$\xi \preceq \xi' \quad \text{if} \quad \forall x \geq x_2, \quad \sum_{z=x}^{y_2+1} \xi(z) \leq \sum_{z=x}^{y_2+1} \xi'(z).$$

- Compare the dynamic in the interval $[s, s+t]$ with that starting from $\mathbf{0}$ at time s to obtain

$$\tilde{\sigma}_{s+t}^{\mathbf{0}}(y_2 + 1) \geq \tilde{\sigma}_s^{\mathbf{0}}(y_2 + 1) + (\vartheta_s \circ \tilde{\sigma})_t^{\mathbf{0}}(y_2 + 1).$$

- Kingman's subadditive ergodic Theorem:

$$\mathbf{E} \left[\tilde{\sigma}_t^0(y_2 + 1) \right] \leq t \left[\lim_{s \rightarrow \infty} \frac{1}{s} \tilde{\sigma}_s^0(y_2 + 1) \right] = t \lim_{n \rightarrow \infty} \frac{n}{\mathcal{T}_n},$$

where \mathcal{T}_n is the instant of the n^{th} particle jumps to the site $y_2 + 1$.

- When there is a particle jumping onto the site $y_2 + 1$, we move all the particles in the segment $[[x_2, y_2]]$ to the site $y_2 + 1$. (The state becomes **0**.)

$$\lim_{n \rightarrow \infty} \frac{n}{\mathcal{T}_n} \leq (y_2 - x_2 + 1) \frac{1}{\mathbf{E}[\mathcal{T}_1]}.$$

- Fill half of the trap and compare with a reversible M.C. with

$$\# \{x \in [[x_2, y_2]] : V(x) \leq [V(y_2) + V(x_2)]/2\} \text{ particles.}$$

- To wait for site x_2 to be empty or site y_2 to be occupied by a particle

$$\mathcal{T}_1 \geq \mathcal{T}' := \{t \geq 0 : \tilde{\sigma}'_t(x_2) = 0 \text{ or } \tilde{\sigma}'_t(y_2) = 1\}.$$

Upper bound on the mixing time

- Coupling, Peres-Winkler inequality
- Flow method: a lower bound on the spectral gap
- Guide particles to the right

Idea for the upper bound

- A coupling: if $\xi \leq \xi'$ (i.e. $\forall x, \sum_{i=x}^N \xi(i) \leq \sum_{i=x}^N \xi'(i)$), then

$$\forall t \geq 0, \quad \eta_t^\xi \leq \eta_t^{\xi'},$$

$$d_{N,k}^\omega(t) \leq \max_{\xi, \xi'} \|P_t^\xi - P_t^{\xi'}\|_{\text{TV}} \leq \max_{\xi, \xi'} \mathbf{P} \left[\eta_t^\xi \neq \eta_t^{\xi'} \right].$$

- $\xi_{\max} := \mathbf{1}_{\{N-k+1 \leq x \leq N\}}$ (η_t^{\min}) $_{t \geq 0}$ starts from $\xi_{\min} = \mathbf{1}_{\{1 \leq x \leq k\}}$,

$$\tau := \inf \{ t \geq 0 : \eta_t^{\min} = \xi_{\max} \},$$

$$\forall t \geq \tau, \forall \xi, \xi' \Rightarrow \eta_t^\xi = \eta_t^{\xi'}.$$

By Markov property and stochastic domination,

$$d_{N,k}^\omega(nt) \leq \mathbb{P} [\tau > nt] \leq \mathbb{P} [\tau > t]^n \leq (1 - \mathbb{P} [\eta_t^{\min} = \xi_{\max}])^n.$$

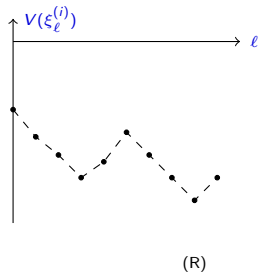
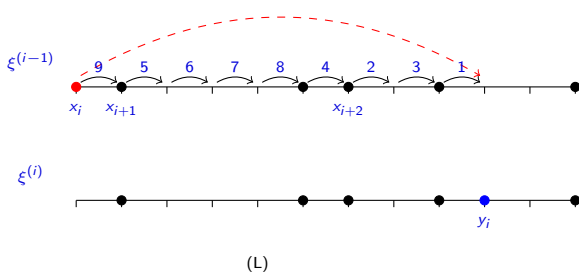
- Censoring scheme $\mathcal{C} : [0, \infty) \mapsto \mathcal{P}(E)$ (E : edges in $\llbracket 1, N \rrbracket$)
 $(P_t^{\mathcal{C}})_{t \geq 0}$: a transition is performed if it doesn't cross an edge in $\mathcal{C}(t)$.
 Peres-Winkler inequality: for any $\xi \in \Omega_{N,k}$ and any \mathcal{C}

$$P_t(\xi, \xi_{\max}) \geq P_t(\xi_{\min}, \xi_{\max}) \geq P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}).$$

Flow method [gap : minimal nonzero eigenvalue of $-\mathcal{L}_{N,k}^\omega$]
 $\text{gap} \geq \exp(-C(\alpha)N)$ and $t_{\text{mix}}^{N,k,\omega}(N^{-3}) \leq \text{gap}^{-1} \log \frac{N^3}{\mu_{\min}} \leq \exp(C'(\alpha)N)$.

Ground state $\xi^* : \pi_{N,k}^\omega(\xi^*) = \max_{\xi \in \Omega_{N,k}} \pi_{N,k}^\omega(\xi)$

Connect ξ with ξ^* : if $\xi(x_i) = 1 - \xi^*(x_i) = 1$ and $\xi(y_i) = 1 - \xi^*(y_i) = 0$, we move the particle at site x_i to y_i as follows:



In this process, the potential V^ω can increase at most $c(\alpha)N$.

Guide particles to the right

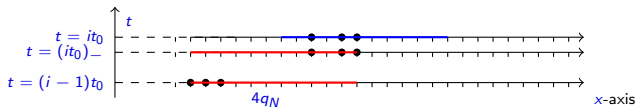
- case $k \leq q_N = C(\mathbb{P}) \log N$:

Run k particles in a segment of length $4q_N$.

In equilibrium, all the k particles are in the right half $2q_N$ sites w.h.p.

Let $\bar{\xi}(1)$ be the position of the leftmost particle. We have

$$\pi_{[x+1, x+4q_N], k}^\omega [\bar{\xi}(1) \leq x + 2q_N] \leq N^{-3}, \text{ for } \xi \in \Omega_{[x+1, x+4q_N], k}.$$



$$t_0 = t_{\text{mix}}^{4q_N, k, \omega} \leq \exp(C(\alpha)4q_N).$$

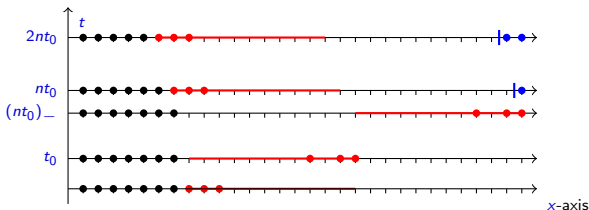
intersection between the red segment and the blue segment $= 2q_N$.

$$\lim_{\varepsilon \rightarrow 0} \inf_{\substack{N \geq 1 \\ k \in [1, N/2]}} \mathbb{P} [\pi_{N, k}^\omega (\xi_{\max}) > \varepsilon] = 1.$$

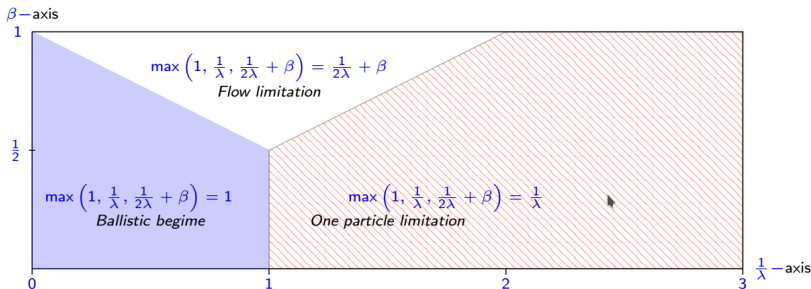
Guide particles to the right

- case $k > q_N$: send the rightmost $k - q_N$ particles to the rightmost $k - q_N$ sites one by one
we repeat the first step for the leftmost q_N particles.

$$\pi_{[x+1, x+4q_N], k}^\omega [\xi(x + 4q_N) = 1] \geq 1 - N^{-3}.$$



The exponent of the mixing time with $k = N^{\beta+o(1)}$ particles



Thank you for your attention