# Cutoff for polymer pinning dynamics in the repulsive phase

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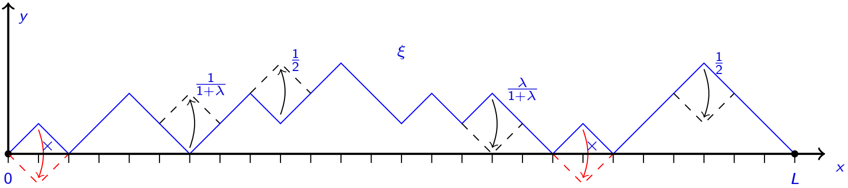
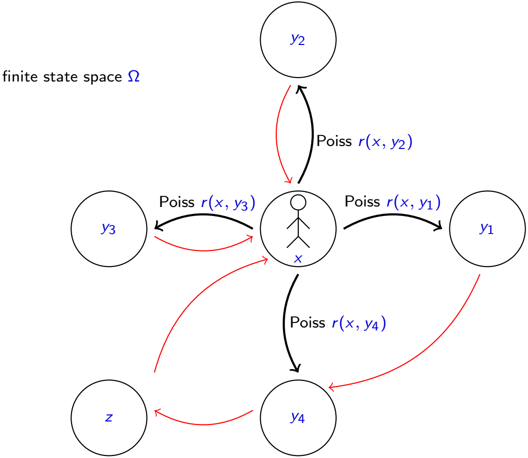
# Organization of the talk

1. Introduction to mixing for continuous-time Markov chains
   - Starting from 1980s
   - Aldous, Diaconis, etc.

2. Cutoff for polymer pinning dynamics in the repulsive phase



# Introduction to mixing for continuous-time Markov chains



# Setup

- Finite state space $\Omega$, elements $x, y, z \cdots$
- Generator: $\mathcal{L} = (r(x,y))_{x,y\in\Omega}$ is an $\Omega \times \Omega$ matrix:
  - Off diagonal elements are nonnegative;
  - Every row sum is equal to zero.

  Homeomorphism $\mathcal{L} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ (for $f \in \mathbb{R}^\Omega$)

$$(\mathcal{L}f)(x) := \sum_{y\in\Omega} r(x,y)\,(f(y) - f(x))\,.$$

- Markov semi-group $(P_t)_{t\geq 0}$:

$$P_t := e^{t\mathcal{L}} = \sum_{k=0}^{\infty} \frac{(t\mathcal{L})^k}{k!}\,,$$

$$P_t(x,y) \geq 0, \quad \sum_{y\in\Omega} P_t(x,y) = 1\,.$$

# Markov chain definition

The random process $(X_t)_{t\geq 0}$ is a continuous-time Markov chain with generator $\mathcal{L}$ and initial distribution $\nu$ if it is càdlàg and

-
$$\forall x \in \Omega, \quad \mathbb{P}\left[X_0 = x\right] = \nu(x)\,;$$

- Markov property: for $0 \leq t_1 < \cdots < t_n < s < s + t$,

$$\mathbb{P}\left[X_{s+t} = y | X_s = x; X_{t_k} = z_k, \forall k \leq n\right] = \mathbb{P}\left[X_{s+t} = y | X_s = x\right] = P_t(x, y)\,.$$

# Invariant probability measure

- $\mu$ is an invariant probability measure if

$$(\forall t \geq 0,\ \mu P_t \ = \ \mu) \quad \Leftrightarrow \quad \mu \mathcal{L} \ = \ 0\,.$$

- Irreducible: for all $x \neq y \in \Omega$, there exists a path $\Gamma_{xy} = (x, z_1, \cdots, z_{\ell-1}, y)$ with $r(z_{k-1}, z_k) > 0$ for all $1 \leq k \leq \ell(x,y)$.

**Theorem**

*If $(\Omega, \mathcal{L})$ is irreducible, there exists a unique invariant probability measure $\mu$, and the distribution $\mathbb{P}^\nu$ of $(X_t)_{t\geq 0}$ with initial distribution $\nu$ converges to $\mu$, i.e.*

$$\lim_{t\to\infty} \sum_{y\in\Omega} \left| \mathbb{P}^\nu \left[X_t = y\right] - \mu(y) \right| \ = \ 0\,.$$

# Distance to equilibrium

- The total variation distance: two probability measures $\alpha, \beta$ on $\Omega$,

$$\|\alpha - \beta\|_{\mathrm{TV}} \ := \ \sup_{A \subset \Omega} |\alpha(A) - \beta(A)| \ .$$

- The distance to equilibrium

$$d(t) \ := \ \max_{x \in \Omega} \|P_t(x, \cdot) - \mu\|_{\mathrm{TV}} \, .$$

- Given $\varepsilon \in (0,1)$, the $\varepsilon-$mixing time

$$t_{\mathrm{mix}}(\varepsilon) \ := \ \inf \{t \geq 0 : d(t) \leq \varepsilon\} \ .$$

Notation: $t_{\mathrm{mix}} \ := \ t_{\mathrm{mix}}(1/4) \, .$

# Markov chain sequence and cutoff

- A sequence of Markov chains $(\Omega_n, \mathcal{L}_n, \mu_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} |\Omega_n| = \infty$:

  $t_{\mathrm{mix}}^{(n)}(\varepsilon)$: the associated $\varepsilon-$mixing time.

  Q: **How does $t_{\mathrm{mix}}^{(n)}(\varepsilon)$ grow in terms of $n$ and $\varepsilon$ ?**

- Precutoff:

$$\sup_{\varepsilon\in(0,\frac{1}{2})} \limsup_{n\to\infty} \frac{t_{\mathrm{mix}}^{(n)}(\varepsilon)}{t_{\mathrm{mix}}^{(n)}(1-\varepsilon)} < \infty \,.$$

- Cutoff: for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} \frac{t_{\mathrm{mix}}^{(n)}(\epsilon)}{t_{\mathrm{mix}}^{(n)}(1-\epsilon)} = 1\,. \quad \Leftrightarrow \quad \lim_{n\to\infty} d_n\left(ct_{\mathrm{mix}}^{(n)}\right) = \begin{cases} 1 & \text{if } c < 1\,, \\ 0 & \text{if } c > 1\,. \end{cases}$$

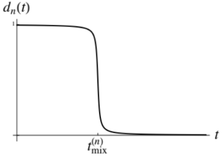



image from Levin and Peres

# Spectral gap of reversible chain

- The detailed balance condition: if for all $x, y \in \Omega$

$$\mu(x)r(x,y) = \mu(y)r(y,x)\,. \qquad \text{Then } \mu\mathcal{L} = 0\,.$$

- Spectral gap: minimal nonzero eigenvalue of $-\mathcal{L}$

$$\langle f, g\rangle_\mu \;:=\; \sum_{x\in\Omega} \mu(x)f(x)g(x)\,, \qquad \mathrm{Var}_\mu(f) \;:=\; \langle f, f\rangle_\mu - \langle f, \mathbf{1}\rangle_\mu^2\,,$$

$$\mathrm{gap} \;:=\; \inf_{\mathrm{Var}_\mu(f)>0} \frac{-\langle f, \mathcal{L}f\rangle_\mu}{\mathrm{Var}_\mu(f)}\,.$$

- Relaxation time: $t_{\mathrm{rel}} := \frac{1}{\mathrm{gap}}$.

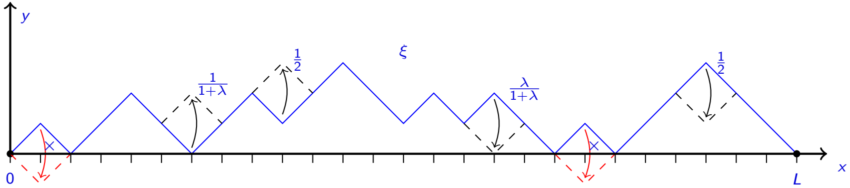
Letting $\mu_{\min} := \min_{x\in\Omega} \mu(x)$, for $\varepsilon \in (0,1)$ we have

$$t_{\mathrm{rel}} \log \frac{1}{2\varepsilon} \;\le\; t_{\mathrm{mix}}(\varepsilon) \;\le\; t_{\mathrm{rel}} \log \frac{1}{2\varepsilon\mu_{\min}}\,,$$

$$\lim_{t\to\infty} \frac{1}{t} \log d(t) \;=\; -\mathrm{gap}\,.$$

# Cutoff for polymer pinning dynamics in the repulsive phase

# The physical situation we are considering



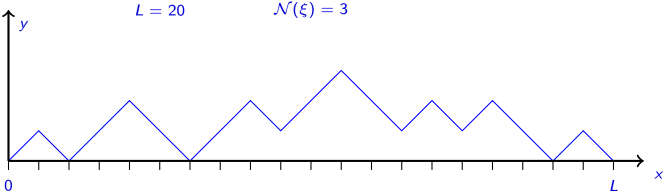

An interface is an element of

$$\Omega_L := \left\{ \xi \in \mathbb{Z}_+^{[\![0,L]\!]} : \ \xi(0) = \xi(L) = 0 \text{ and } \forall\, x,\ |\xi(x) - \xi(x-1)| = 1 \right\} .$$

# The equilibrium measure

Given $\xi \in \Omega_L$,

- $\mathcal{N}(\xi) := \sum_{x=1}^{L-1} \mathbf{1}_{\{\xi(x)=0\}}$ (# contacts with $x-$axis).

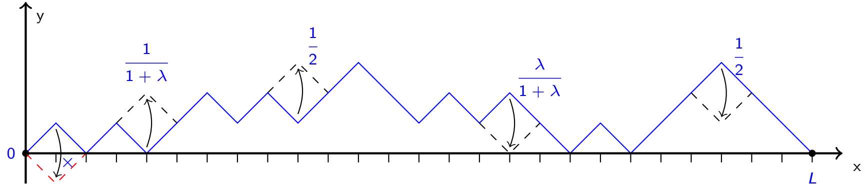



Given $\lambda \geq 0$, define $\mu = \mu_L^\lambda$ the probability on $\Omega_L$:

$$\mu(\xi) = \frac{\lambda^{\mathcal{N}(\xi)}}{Z_L(\lambda)}, \qquad Z_L(\lambda) := \sum_{\xi' \in \Omega_L} \lambda^{\mathcal{N}(\xi')}.$$

# Corner-flip/Heat Bath dynamics $(\eta_t)_{t\geq 0}$ on $\Omega_L$

Each coordinate is updated at rate one.
When an update at $x$ occurs at time $t$, $\eta_t$ is sampled according to the conditional equilibrium measure $\mu_L^\lambda(\cdot \mid \eta_{t-}(y), y \neq x)$.




The measure $\mu$ satisfies the detailed balance condition, *i.e.*

$$\mu(\xi) r(\xi, \xi^x) \;=\; \mu(\xi^x) r(\xi^x, \xi)\,.$$

$\mathbf{P}^\xi$: the distribution of the Markov chain $(\eta_t^\xi)_{t\geq 0}$ starting from $\xi$.
$T_{\mathrm{mix}}^{L,\lambda}(\varepsilon)$: associated $\varepsilon-$mixing time.

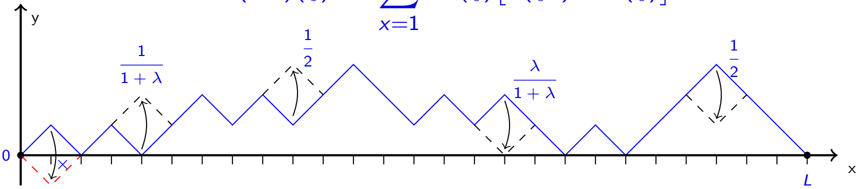
# Generator

- $\xi^x$: obtained by flipping the corner at coordinate $x$ of the path $\xi$ provided there is a corner at $x$ and $\xi^x \in \Omega_L$.
- jump rate

$$R_x(\xi) := \begin{cases} \frac{1}{2} & \text{if } \xi(x-1) = \xi(x+1) > 1\,, \\ \frac{\lambda}{1+\lambda} & \text{if } (\xi(x-1), \xi(x), \xi(x+1)) = (1,2,1)\,, \\ \frac{1}{1+\lambda} & \text{if } (\xi(x-1), \xi(x), \xi(x+1)) = (1,0,1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

- Generator: for $f : \Omega_L \to \mathbb{R}$

$$(\mathcal{L}f)(\xi) := \sum_{x=1}^{L-1} R_x(\xi)\big[f(\xi^x) - f(\xi)\big].$$



# Presentation of our results for the polymer pinning model

(1) Properties of the model at equilibrium

(2) Previous results for the polymer pinning dynamics

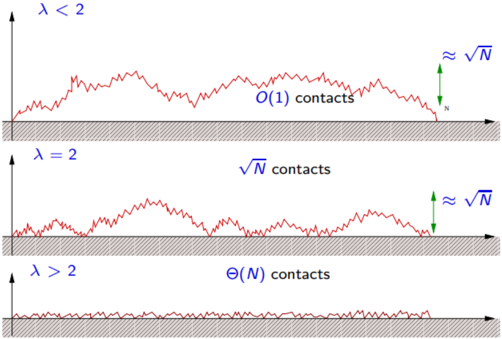
(3) Our results for the polymer pinning dynamics in diffusive regime

(4) Idea: Lower bound on the mixing time for $\lambda \in [0, 2)$

(5) Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

(6) Idea: Upper bound on the mixing time for $\lambda \in (1, 2)$ concerning the extremal initial conditions

# Equilibrium Properties [Fisher 1984]




A transition at $\lambda = 2$ between a pinned phase and an unpinned phase. This transition can be seen when looking at the free energy

$$\lim_{L\to\infty} \frac{1}{L} \log 2^{-L} Z_L(\lambda) = \log\left(\frac{\lambda}{2\sqrt{\lambda-1}}\right)\mathbf{1}_{\{\lambda>2\}} =: F(\lambda).$$

# Details about the partition function

- Asymptotic of the partition function

$$2^{-L} Z_L(\lambda) \sim C(\lambda) \times \begin{cases} L^{-3/2} & \text{if } \lambda \in [0,2), \\ L^{-1/2} & \text{if } \lambda = 2, \\ e^{LF(\lambda)} & \text{if } \lambda \in (2,\infty). \end{cases}$$

- How to calculate it? No positive constraint state space:

$$\widetilde{\Omega}_L := \left\{ \xi \in \mathbb{Z}^{[\![0,\,L]\!]} : \ \xi(0) = \xi(L) = 0 \text{ and } \forall x \ , \ |\xi(x) - \xi(x-1)| = 1 \right\}.$$

$$\widetilde{Z}_L(\lambda) := \sum_{\xi \in \widetilde{\Omega}_L} \lambda^{\mathcal{N}(\xi)}, \qquad \widetilde{Z}_L(\lambda) = 2 Z_L(2\lambda),$$

- Renewal process viewpoint ($\mathbf{P}$ : SRW $\quad \widetilde{\mathbf{P}}$ : renewal law)

$$K(n) := \mathbf{P}\left(S_1 \neq 0, S_2 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0\right), \quad \forall\, n \geq 1.$$

$$\widetilde{K}(n) := \lambda e^{-2n\widetilde{F}(\lambda)} K(n).$$

$$e^{-L\widetilde{F}(\lambda)} 2^{-L} Z_L(\lambda) = e^{-L\widetilde{F}(\lambda)} \sum_{k=1}^{L/2} \sum_{\substack{(n_1,\dots,n_k) \\ \sum_{i=1}^k n_i = L/2}} \prod_{i=1}^k K(n_i)\lambda$$

$$= \sum_{k=1}^{L/2} \sum_{\substack{(n_1,\dots,n_k) \\ \sum_{i=1}^k n_i = L/2}} \prod_{i=1}^k \widetilde{K}(n_i) = \widetilde{\mathbf{P}}\left(L \in \tau\right).$$

# Previous results: Polymer pinning dynamics

[Caputo, Martinelli, Toninelli '08]:

- When $\lambda \in [0,2)$, $\mathrm{gap} \asymp L^{-2}$ and there is a precutoff, i.e.

$$\frac{1+o(1)}{2\pi^2} L^2 \log L \ \le\ T_{\mathrm{mix}}^{L,\lambda}(\epsilon) \ \le\ \frac{6+o(1)}{\pi^2} L^2 \log L\,.$$

- When $\lambda = 2$, $\mathrm{gap} \asymp L^{-2}$

$$cL^2 \le T_{\mathrm{mix}}^{L,\lambda}(1/4) \le \frac{6+o(1)}{\pi^2} L^2 \log L\,.$$

- When $\lambda > 2$, $\mathrm{gap} \le cL^{-1}$

$$T_{\mathrm{mix}}^{L,\lambda}(1/4) \ge cL^2\,,$$

  where $c$ is independent of $\lambda$.
- When $\lambda = \infty$,

$$T_{\mathrm{mix}}^{L,\lambda}(1/4) \le L^2\,.$$

[Lacoin '14] identified the constant in the mixing (hitting) time when $\lambda = \infty$ for smooth initial profile.

# Our main result: cutoff

Understand the pattern of relaxation to equilibrium, and in particular identify the mixing time.

$$T^{L,\lambda}_{\mathrm{mix}}(\varepsilon) := \inf\left\{ t \ : \ \forall \xi \in \Omega_N, \ \left\| \mathbf{P}^{\xi}_t - \mu \right\|_{\mathrm{TV}} \leq \varepsilon \right\}.$$

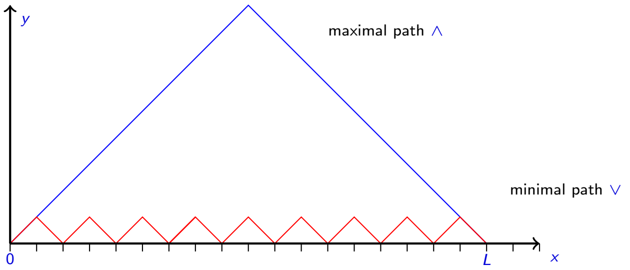
**Theorem (Y, '21 (cutoff))**

*When* $\lambda \in [0,1]$, *for all* $\epsilon \in (0,1)$ *we have*

$$\lim_{L\to\infty} \frac{\pi^2 T^{L,\lambda}_{\mathrm{mix}}(\varepsilon)}{L^2 \log L} = 1.$$

# Our main result: partial cutoff

$$\breve{T}^{L,\lambda}_{\mathrm{mix}}(\varepsilon) := \inf\left\{t \ : \max\left(\left\|\mathbf{P}^{\wedge}_t - \mu\right\|_{\mathrm{TV}}, \left\|\mathbf{P}^{\vee}_t - \mu\right\|_{\mathrm{TV}}\right) \leq \varepsilon\right\}.$$




**Theorem ( Y, '21 (Partial cutoff))**

*If $\lambda \in (1,2)$, for all $\epsilon \in (0,1)$ we have*

$$\lim_{L\to\infty} \frac{\pi^2\, \breve{T}^{L,\lambda}_{\mathrm{mix}}(\varepsilon)}{L^2 \log L} = 1.$$

# Polymer pinning dynamics

Idea: Lower bound on the mixing time for $\lambda \in [0, 2)$

# Idea for the lower bound when $\lambda \in [0,2)$

A weighted area function $\Phi\colon \Omega_L \to \mathbb{R}$ [introduced by Wilson '04]:

$$\Phi(\xi) := \sum_{x=1}^{L-1} \xi(x) \sin\left(\frac{\pi x}{L}\right).$$

- Under equilibrium $\mu$, $\Phi$ is at most of order $L^{3/2}$ since

$$\sup_{\lambda \geq 0,\, L \in 2\mathbb{N}} \sup_{x \in [\![1, L-1]\!]} \mu_L^\lambda \left( \frac{(\xi(x))^k}{L^{k/2}} \right) < \infty.$$

- For the dynamics $(\eta_t^\wedge)_{t \geq 0}$ starting from the highest path $\wedge$, $\Phi$ is initially of order $L^2$;
- To show the time required by $\Phi(\eta_t^\wedge)$ to become of order $L^{3/2}$ is at least $(1-o(1))\frac{1}{\pi^2}L^2 \log L$, we estimate the mean $\mathbf{E}\left[\Phi(\eta_t^\wedge)\right]$ and its fluctuation by building a Dynkin's martingale and controlling the martingale bracket.

# Idea for Lower bound when $\lambda \in [0,2)$ $[\kappa_L := 1 - \cos(\pi/L)]$

- [Caputo, Martinelli, Toninelli]

$$\mathbb{E}[\Phi(\eta^{\wedge}_{t_0})] \;\geq\; \Phi(\eta^{\wedge}_0)e^{-\kappa_L t_0} - c(\lambda)L^{3/2} \geq 2C_\varepsilon L^{3/2}\,.$$

Notation: $t_0 := \frac{1}{\pi^2}L^2\log L - C_\varepsilon L^2$ ($C_\varepsilon \gg 1$.)

- Build a Dynkin's martingale: $F(t,\xi) = \exp(\kappa_L(t-t_0))\Phi(\xi)$

$$M_t := F(t,\eta^{\wedge}_t) - F(0,\eta^{\wedge}_0) - \int_0^t (\partial_s + \mathcal{L})F(s,\eta^{\wedge}_s)\mathrm{d}s\,.$$

$$\Psi(\xi) := \sum_{x=1}^{L-1}\sin\left(\frac{\pi x}{L}\right)\left[\mathbf{1}_{\{\xi(x-1)=\xi(x-1)=0\}} - \left(\frac{\lambda-1}{\lambda+1}\right)\mathbf{1}_{\{\xi(x-1)=\xi(x+1)=1\}}\right].$$

$$(\partial_t + \mathcal{L})F(t,\eta^{\wedge}_t) = e^{\kappa_L(t-t_0)}\Psi(\eta^{\wedge}_t)\,.$$

Each transition can change $M_t$ in absolute value by at most $2e^{\kappa_L(t-t_0)}$

$$\partial_t\langle M.\rangle_t \leq \sum_{x=1}^{L-1}4e^{2\kappa_L(t-t_0)} \leq 4Le^{2\kappa_L(t-t_0)}.$$

# Idea for Lower bound when $\lambda \in [0, 2)$

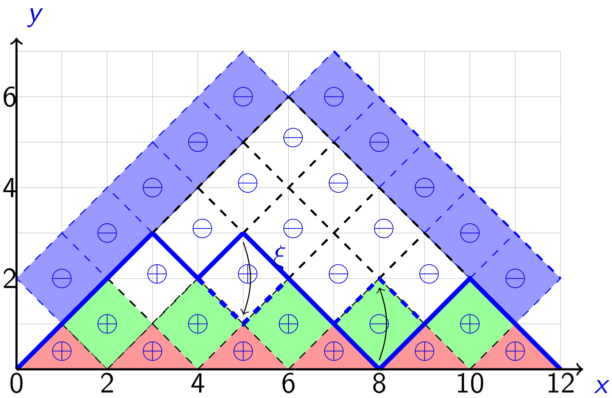
- $$\mathbb{E}[M_{t_0}^2] = \mathbb{E}[\langle M.\rangle_{t_0}] \le \int_0^{t_0} 4Le^{2\kappa_L(t-t_0)}dt \le \frac{8L^3}{\pi^2}.$$
- $\Psi(\xi) = \sum_{x=1}^{L-1} \sin\left(\frac{\pi x}{L}\right) \left[\mathbf{1}_{\{\xi_{x-1}=\xi_{x-1}=0\}} - \left(\frac{\lambda-1}{\lambda+1}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}\right].$
$$B(t) := \int_0^t e^{\kappa_L(s-t_0)}\Psi(\eta_s^\wedge)\mathrm{d}s.$$
$$\mathbb{E}[|B(t_0)|] \le \mathbb{E}\left[\int_0^{t_0} e^{\kappa_L(t-t_0)}|\Psi(\eta_t^\wedge)|\mathrm{d}t\right]$$
$$\le C(\lambda)\kappa_L^{-1}\sum_{x=1}^{L-1}\sin\left(\frac{\pi x}{L}\right)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}} \le C(\lambda)L^{3/2}.$$
- $$\mathbb{P}\left[\left|\Phi(\sigma_{t_0}^\wedge) - \mathbb{E}[\Phi(\sigma_{t_0}^\wedge)]\right| \ge C_\varepsilon L^{3/2}\right] = \mathbb{P}\left[\left|M_{t_0} + B(t_0) - \mathbb{E}[B(t_0)]\right| \ge C_\varepsilon L^{3/2}\right]$$
$$\le \mathbb{P}\left[\left|M_{t_0}\right| \ge \frac{1}{3}C_\varepsilon L^{3/2}\right] + \mathbb{P}\left[\left|B(t_0)\right| \ge \frac{1}{3}C_\varepsilon L^{3/2}\right]$$
$$\le \frac{9\mathbb{E}[M_{t_0}^2]}{C_\varepsilon^2 L^3} + \frac{3\mathbb{E}[|B(t_0)|]}{C_\varepsilon L^{3/2}} \le \varepsilon \text{ if } C_\varepsilon \text{ is sufficiently large.}$$

# Polymer pinning dynamics

Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

# Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- A graphical construction preserves the monotonicity.




- Partial order " $\leq$ " on $\Omega_L$: $(\xi \leq \xi') \Leftrightarrow (\forall x \in [\![0, L]\!],\ \xi(x) \leq \xi'(x))$ .
- The graphical construction preserves monotonicity

$$\xi \leq \xi' \Rightarrow \eta_t^{\xi} \leq \eta_t^{\xi'}, \quad \forall\, t \geq 0\,.$$

$$\eta_t^{\vee} \leq \eta_t^{\xi} \leq \eta_t^{\wedge}, \quad \forall\, t \geq 0\,, \forall\, \xi \in \Omega_L\,.$$

# Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- Reduce the problem to estimate the coalescing time

$$\|\mathbf{P}_t^\xi - \mathbf{P}_t^\mu\|_{\mathrm{TV}} \leq \sum_{\xi' \in \Omega_L} \mu(\xi') \|\mathbf{P}_t^\xi - \mathbf{P}_t^{\xi'}\|_{\mathrm{TV}} \leq \max_{\xi' \in \Omega_L} \|\mathbf{P}_t^\xi - \mathbf{P}_t^{\xi'}\|_{\mathrm{TV}} .$$

$$\|\mathbf{P}_t^\xi - \mathbf{P}_t^{\xi'}\|_{\mathrm{TV}} \leq \mathbb{P}[\eta_t^\xi \neq \eta_t^{\xi'}] \leq \mathbb{P}[\eta_t^\wedge \neq \eta_t^\vee] .$$

Coalescing times

$$\begin{aligned}
\widetilde{\tau} &:= \inf\left\{t > 0 \ : \ \eta_t^\wedge = \eta_t^\vee\right\} , \\
\tau' &:= \inf\left\{t > 0 \ : \ \eta_t^\vee = \eta_t^\mu\right\} , \\
\tau &:= \inf\left\{t > 0 \ : \ \eta_t^\wedge = \eta_t^\mu\right\} , \\
\widetilde{\tau} &= \max\left(\tau, \tau'\right) .
\end{aligned}$$

It is more practical to deal with $\tau'$, $\tau$ than $\widetilde{\tau}$.

# Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

- An area function $\overline{\Phi}\colon \Omega_L \to [0, \infty)$ given by
$$\overline{\Phi}(\xi) := \sum_{x=1}^{L-1} \xi(x)\overline{\cos}_\beta(x) \quad \overline{\cos}_\beta(x) := \cos\left(\beta(x - L/2)/L\right)$$
where $\beta < \pi$ and $\beta$ is chosen sufficiently close to $\pi$.
$$\xi \le \xi' \quad \Rightarrow \quad \overline{\Phi}(\xi) \le \overline{\Phi}(\xi'),$$
$$\delta_{\min} := \min_{\xi \le \xi', \xi \neq \xi'} \left(\overline{\Phi}(\xi') - \overline{\Phi}(\xi)\right) = 2\cos\left(\frac{\beta(L/2 - 1)}{L}\right) \ge \frac{1}{2}(\pi - \beta).$$

- The area function between the paths $A_t := \delta_{\min}^{-1}\left[\overline{\Phi}(\eta^\wedge) - \overline{\Phi}(\eta^\mu)\right]$
$A_t - A_0 - \int_0^t \mathcal{L}A_s \mathrm{d}s$ is a Dynkin martingale
$A_t$ is a supermartingale when $\lambda \in [0, 1]$.

- $$\mathcal{L}\xi_x = (\Delta\xi)_x + \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} + \left(\frac{1-\lambda}{1+\lambda}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}.$$

  For $\lambda \in [0,1]$, if $\xi \le \xi'$,

  $$\mathcal{L}\xi_x - (\Delta\xi)_x \ge \mathcal{L}\xi'_x - (\Delta\xi')_x\,, \qquad \forall\, x \in [\![1, L-1]\!]\,.$$

$$\sum_{x=1}^{L-1} \cos(x)\left((\Delta\xi')_x - (\Delta\xi)_x\right) = -\left(1-\cos\left(\beta/L\right)\right)\sum_{x=1}^{L-1}\cos(x)(\xi'_x - \xi_x).$$

$$\begin{aligned}
(\mathcal{L}\overline{\Phi})(\xi') - (\mathcal{L}\overline{\Phi})(\xi) &= \sum_{x=1}^{L-1} \overline{\cos}(x)((\Delta\xi')_x - (\Delta\xi)_x + \mathcal{L}\xi'_x - (\Delta\xi')_x - (\mathcal{L}\xi_x - (\Delta\xi)_x)) \\
&\le \sum_{x=1}^{L-1} \overline{\cos}(x)\Big((\Delta\xi)'_x - (\Delta\xi)_x\Big) \\
&= -\left(1-\cos\left(\beta/L\right)\right)\sum_{x=1}^{L-1}\overline{\cos}(x)(\xi'_x - \xi_x)\,, \\
&\Rightarrow A_t \text{ is a supermartingale}\,.
\end{aligned}$$

# Idea: Upper bound on the mixing time for $\lambda \in [0, 1]$

(1) The decay rate of $\mathbb{E}[A_t]$ is at least $1 - \cos(\beta/L) \Rightarrow$

$$t_{\delta/2} := \frac{1+\delta/2}{\pi^2} L^2 \log L\,, \quad A_{t_{\delta/2}} \ll L^{3/2}\,.$$

(2) For $t \geq t_{\delta/2}$, applying the supermartingale approach [Labbé, Lacoin, '20] to show: it only takes an extra amount of time of order $L^2$ for $A_t$ to shrink from $L^{3/2}$ to zero. Idea:

$\eta > 0$: sufficiently small, $K := \lceil 1/(2\eta) \rceil > 1/(2\eta)$. Define $(\mathcal{T}_i)_{i=2}^{K}$ by

$$\mathcal{T}_2 := \inf\left\{ t \geq t_{\delta/2} : A_t \leq L^{\frac{3}{2}-2\eta} \right\}\,,$$

$$\mathcal{T}_i := \inf\left\{ t \geq \mathcal{T}_{i-1} : A_t \leq L^{\frac{3}{2}-i\eta} \right\}\,, \text{ for } i \in [\![3, k]\!]\,,$$

$$\mathcal{T}_\infty := \max\left(\tau_1, t_{\delta/2}\right)\,.$$

To show: ($\Delta\mathcal{T}_i := \mathcal{T}_i - \mathcal{T}_{i-1}$ for $3 \leq i \leq K$)

$$\lim_{L\to\infty} \mathbb{P}\left[ \{\mathcal{T}_2 = t_{\delta/2}\} \cap \left( \bigcap_{i=3}^{K} \{\Delta\mathcal{T}_i \leq 2^{-i}L^2\} \right) \cap \{\mathcal{T}_\infty - \mathcal{T}_K \leq L^2\} \right] = 1\,.$$

# Idea: Upper bound on the mixing time for $\lambda \in [0,1]$

- During the time interval $[\mathcal{T}_{i-1}, \mathcal{T}_i]$ for $3 \le i \le K$, apply the surpermartingale approach ([Labbé, Lacoin '20]) to show w.h.p.

$$\langle A_\cdot \rangle_{\mathcal{T}_i} - \langle A_\cdot \rangle_{\mathcal{T}_{i-1}} \le L^{3-2(i-1)\eta + \frac{1}{2}\eta},$$
$$\langle A_\cdot \rangle_{\mathcal{T}_\infty} - \langle A_\cdot \rangle_{\mathcal{T}_K} \le L^2.$$

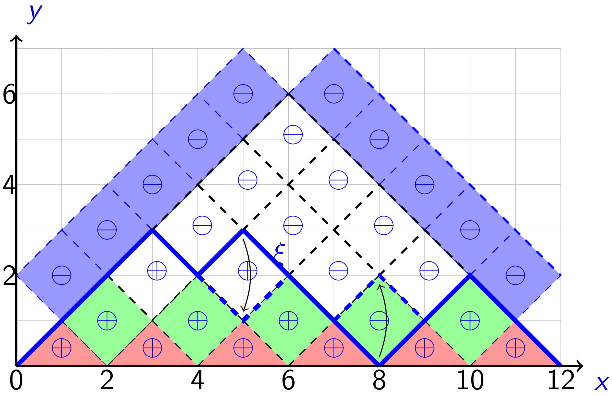
- Compare $\mathcal{T}_\infty - \mathcal{T}_K$ with $\langle A_\cdot \rangle_{\mathcal{T}_\infty} - \langle A_\cdot \rangle_{\mathcal{T}_K}$. As $\partial_t \langle A_\cdot \rangle \ge 1$ for all $t < \mathcal{T}_\infty$, we have

$$\mathcal{T}_\infty - \mathcal{T}_K \le \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} \partial_t \langle A_\cdot \rangle \mathrm{d}t = \langle A_\cdot \rangle_{\mathcal{T}_\infty} - \langle A_\cdot \rangle_{\mathcal{T}_K}.$$

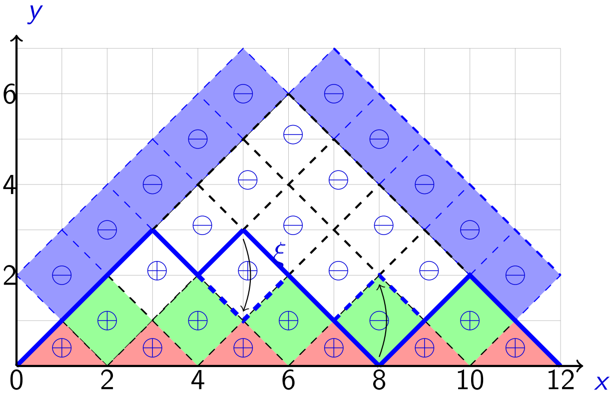
- For $3 \le i \le K$, to compare $\langle A_\cdot \rangle_{\mathcal{T}_i} - \langle A_\cdot \rangle_{\mathcal{T}_{i-1}}$ with $\mathcal{T}_i - \mathcal{T}_{i-1}$, we provide a better lower bound on $\partial_t \langle A_\cdot \rangle$ in terms of the highest point of $\eta_t^\wedge$ and the maximal length of a monotone segment of $\eta_t^\mu$. We use induction method to show that $\mathcal{T}_i - \mathcal{T}_{i-1} \le 2^{-i} L^2$ for all $i \in [\![3, K]\!]$, arguing by contradiction.

# Upper bound on the mixing time for $\lambda \in (1,2)$ concerning extremal initial conditions

- When $\lambda \in (1,2)$, $A_t$ is not a super-martingale due to the entropic repulsion of the hard wall.




- Censoring: canceling any prescribed updates in any given spin positions and any chosen time intervals.

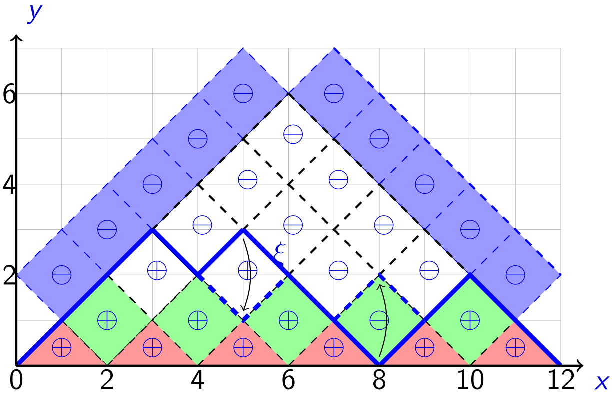



- Peres-Winkler inequality: for monotone spin systems, censoring delays mixing for dynamics starting with extremal initial condition.
  For any prescribed censoring scheme $\mathcal{C}$, for all $\lambda \in [0, \infty)$, all $t \geq 0$ and $\xi \in \{\wedge, \vee\}$, we have

$$\|\mathbf{P}_t^{\xi} - \mu\|_{\mathrm{TV}} \leq \|\mathbf{P}_t^{\xi,\mathcal{C}} - \mu\|_{\mathrm{TV}}\,.$$

# Idea for the upper bound concerning the maximal initial condition when $\lambda \in (1, 2)$

- We censor updates in those spin positions colored green for $t < t_{\delta/2}$, and don't censor any update for $t \geq t_{\delta/2}$.




Therefore, the dynamics $(\eta_t^{\wedge,\mathcal{C}})_{0 \leq t < t_{\delta/2}}$ does not touch the $x-$axis except at the two coordinates $x = 0,\ L$.

- By the cutoff theorem for $\lambda = 1$, the distribution of $\eta^{\wedge,\mathcal{C}}_{t_{\delta/2}}$ is close to $\mu^0_L$ in total variation distance.
- 1) The Radon-Nikodym derivative of $\mu^0_L$ with respect to $\mu^\lambda_L$ is bounded by a constant, by

$$2^{-L} Z_L(\lambda) \;\sim\; C(\lambda) \times \begin{cases} L^{-3/2} & \text{if } \lambda \in [0,2), \\ L^{-1/2} & \text{if } \lambda = 2, \\ e^{LF(\lambda)} & \text{if } \lambda \in (2,\infty). \end{cases}$$

  2)

$$\mathrm{gap}_{L,\lambda} \geq \kappa_L \;=\; 1 - \cos\left(\frac{\pi}{L}\right).$$

  3) Combining Cauchy-Schwarz inequality and the reversibility of the Markov chain, for any probability distribution $\nu$ on $\Omega_L$, [Caputo, Lacoin, Martinelli, Simenhaus, Toninelli '12] proves that

$$\|\nu P_t - \mu\|_{\mathrm{TV}} \;\leq\; \frac{1}{2} e^{-t\cdot \mathrm{gap}_{L,\lambda}} \sqrt{\mathrm{Var}_\mu(\rho)}\,,$$

  where $\rho := \frac{d\nu}{d\mu}$ and $\mathrm{Var}_\mu(\rho) := \mu(\rho^2) - \mu(\rho)^2$.

Therefore, the distribution of $\eta^{\wedge}_{t_{\delta/2}+C_\varepsilon L^2}$ is close to $\mu$.

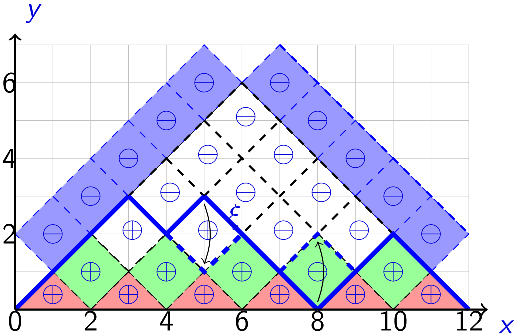
# Idea for the upper bound concerning the minimal initial condition when $\lambda \in (1,2)$

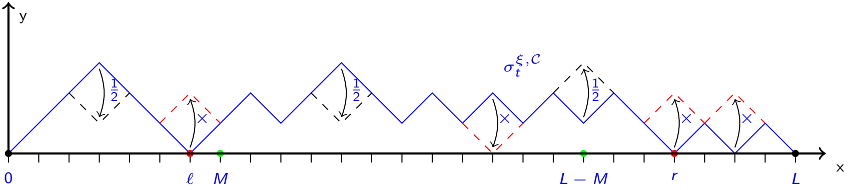
(i) Run the dynamics $(\eta_t^{\vee})_{0 \le t < S_0(L)}$ without censoring, where

$$s_0(L) := 10L^{16/9} \log L \ll L^2 \log L\,.$$

W.h.p. $\eta^{\vee}_{s_0(L)}$ does not touch the $x-$axis in the interval $[\![M, L-M]\!]$ for some $M$ sufficiently large.

(ii) In the time interval $[s_0(L), s_0(L) + t_{\delta/2})$, censor updates in those spin positions colored green.






- By the cutoff theorem for $\lambda = 1$, roughly speaking, the distribution of $\eta^{\vee,\mathcal{C}}_{s_0(L)+t_{\delta/2}}$ is close to $\mu_L^0$ in total variation distance.
- Then we run the dynamics without censoring with an extra amount of time of order $L^2$, and apply

**[CLMST '12]**

$$\|\nu P_t - \mu\|_{\mathrm{TV}} \le \frac{1}{2} e^{-t\cdot \mathrm{gap}_{L,\lambda}} \sqrt{\mathrm{Var}_\mu(\rho)},$$

where $\rho := \frac{d\nu}{d\mu}$ and $\mathrm{Var}_\mu(\rho) := \mu(\rho^2) - \mu(\rho)^2$.

to conclude the proof.

# Open question

To understand the effect of entropic repulsion on the dynamics. In particular, to prove

For $\lambda \in (1,2)$, we have

$$T_{\mathrm{mix}}^{L,\lambda}(\varepsilon) \;\leq\; \frac{1+o(1)}{\pi^2} L^2 \log L \,.$$

Thank you for your attention!