Spectral gap and cutoff of Simple Exclusion Process with IID conductances

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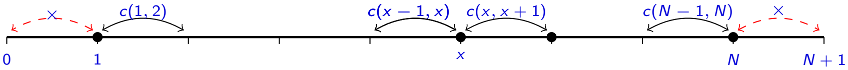
Setup: Simple Exclusion Process with inhomogeneous conductance

Setup: Simple Exclusion Process

Conductances: $(c(x, x+1))_{x\in\mathbb{N}}$ with values in $(0, \infty)$.
SEP with k particles in $[\![1, N]\!]$ with conductances $c(x, x+1)_{x\in\mathbb{N}}$.



(A) Each site is occupied by at most one particle (*the exclusion rule*).

(B) At each edge $\{x, x+1\}$ with $1 \leq x < N$, we place a Poisson clock with rate $c(x, x+1) > 0$. When a clock rings, we swap the contents of the two sites.

Setup

- State space (1: particle 0: empty site.)

$$\Omega_{N,k} := \left\{ \xi : [\![1, N]\!] \to \{0,1\} \;\middle|\; \sum_{i=1}^{N} \xi(i) = k \right\}.$$

- Generator: $(f : \Omega_{N,k} \mapsto \mathbb{R})$

$$(\mathcal{L}_{N,k} f)(\xi) := \sum_{i=1}^{N-1} c(i, i+1) \left[f(\xi \circ \tau_{i,i+1}) - f(\xi) \right],$$

where $\tau_{i,j}$ is the transposition of the two elements i and j.

- Uniform prob measure $\mu_{N,k}$ satisfies the detailed balance condition:

$$\mu(\xi)\mathcal{L}_{N,k}(\xi, \eta) = \mu(\eta)\mathcal{L}_{N,k}(\eta, \xi),$$

then μ is the invariant prob measure.

Setup

- Distance to equilibrium

$$d_{N,k}(t) \;:=\; \max_{\xi \in \Omega_{N,k}} \|P_t^{\xi} - \mu_{N,k}\|_{\mathrm{TV}}\,.$$

P_t^{ξ}: marginal distribution at instant t of the chain starting with ξ.

- ε-mixing time

$$t_{\mathrm{mix}}^{N,k}(\varepsilon) := \inf\left\{t \geq 0 : d_{N,k}(t) \leq \varepsilon\right\}\,.$$

- Cutoff: for all $\varepsilon \in (0,1)$,

$$\lim_{N\to\infty} \frac{t_{\mathrm{mix}}^{N,k}(\epsilon)}{t_{\mathrm{mix}}^{N,k}(1-\epsilon)} \;=\; 1\,.$$

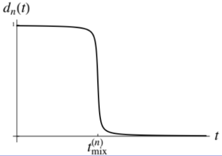


image from Levin and Peres

Setup

- : Spectral gap $\mathrm{gap}_{N,k}$: minimal nonzero eigenvalue of $-\mathcal{L}_{N,k}$

$$\mathrm{gap}_{N,k} \;:=\; \inf_{f\,:\,\mathrm{Var}_{\mu_{N,k}}(f)>0} \frac{-\langle f, \mathcal{L}_{N,k} f\rangle_{\mu_{N,k}}}{\mathrm{Var}_{\mu_{N,k}}(f)}$$

where $\mathrm{Var}_{\mu_{N,k}}(f) := \langle f, f\rangle_{\mu_{N,k}} - \langle f, \mathbf{1}\rangle^2_{\mu_{N,k}}$.

- Relation between spectral gap and mixing time/distance to equilibrium:

$$\frac{1}{\mathrm{gap}_{N,k}} \log \frac{1}{2\varepsilon} \;\leq\; t^{N,k}_{\mathrm{mix}}(\varepsilon) \;\leq\; \frac{1}{\mathrm{gap}_{N,k}} \log \frac{1}{2\varepsilon\mu_{\min}}$$

where $\mu_{\min} := \min_{\xi\in\Omega_{N,k}} \mu_{N,k}(\xi)$.

$$\lim_{t\to\infty} \frac{1}{t} \log d_{N,k}(t) \;=\; -\mathrm{gap}_{N,k}\,.$$

Question: How does the disordered setup (inhomogeneous conductances) affect the system in terms of spectral gap/mixing time?

Previous Results

Homogeneous conductances $c(x, x+1) \equiv 1$ for one particle ($k = 1$)

- Spectral gap

$$\mathrm{gap}_{N,1} = 2(1 - \cos(\pi/N)) = (1 + o(1))\pi^2/N^2 \,.$$

- Eigenfunctions

$$g_i^{(N)}(x) := \cos(i\pi(x - 1/2)/N) \,, \quad 0 \le i < N \,.$$

- Eigenvalues

$$-\lambda_i^{(N)} = -2\,(1 - \cos(i\pi/N)) \quad \mathcal{L}_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)} \,.$$

Homogeneous conductances $c(x, x+1) \equiv 1$ for many particles

- [Aldous, Wilson, Lacoin] Assuming $\liminf_{N\to\infty} \min(k, N-k) = \infty$,

$$t_{\min}^{N,k}(\varepsilon) = (1 + o(1))\frac{N^2}{2\pi^2}\log k \,, \quad \mathrm{gap}_{N,k} = \mathrm{gap}_{N,1} = (1 + o(1))\frac{\pi^2}{N^2} \,.$$

Previous results

Inhomogeneous conductance $c(x, x+1) > 0$

- Aldous' spectral gap conjecture (Proved by [Caputo, Liggett, Richthammer, JAMS '10]):

$$\mathrm{gap}_{N,k} = \mathrm{gap}_{N,1}\,.$$

- A function $f : [\![1, N]\!] \to \mathbb{R}$ for $2 \leq b \leq c \leq N-1$
  Local maximum at $[\![b, c]\!]$ if f is constant on $[\![b, c]\!]$, $f(b-1) < f(b)$ and $f(c) > f(c+1)$.
  Analogous definition holds for a local minimum.
  f is j-monotone if it displays exactly $(j-1)$ distinct local extrema in $[\![2, N-1]\!]$.
  Nodal domains:

$$\#\text{connected components of } \{x \in [\![1, N]\!], f(x) \neq 0\}\,.$$

- [Miclo]: $L_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)}$ with $0 = \lambda_0^{(N)} < \lambda_1^{(N)} < \cdots \lambda_{N-1}^{(N)}$
  $g_i^{(N)}$ is i-monotone and has $i+1$ nodal domains.

Our results

Proposition (Y. '24)

For any positive conductances $(c(x,x+1))_{x\in\mathbb{N}}$, $g_1^{(N)}$ *is strictly monotone.*

Write $r(x,x+1) := 1/c(x,x+1)$ and $r(n,m) := \sum_{x=n}^{m-1} r(x,x+1)$.
Assume (LLN) condition

$$\limsup_{N\to\infty} \frac{1}{N} \sup_{1\le n<m\le N} |(r(n,m)-(m-n)| \;=\; 0\,, \qquad \textbf{(LLN)}$$

which is equivalent to

$$\limsup_{N\to\infty} \frac{1}{N} \sup_{2\le m\le N} \left|(r^{(N)}(1,m)-(m-1)\right| \;=\; 0\,.$$

When $(r^{(N)}(x-1,x))_{2\le x\le N}$ is IID with expectation $\mathbb{E}[r(x,x+1)] = 1$, by the strong LLN we have

$$\mathbb{P}\left(\lim_{N\to\infty} \frac{1}{N} \max_{2\le m\le N} |r(1,m)-(m-1)| = 0\right) \;=\; 1\,.$$

Our results

Theorem (Y. '24)

If the (LLN) condition on the resistances holds, we have

$$\lim_{N\to\infty} \frac{N^2 \mathrm{gap}_N}{\pi^2} \;=\; 1\,.$$

Furthermore, concerning the shape and (weighted) derivative of the eigenfunction g_1 *with* $g_1(1) := 1$ *corresponding to the spectral gap, i.e.* $\Delta^{(c)} g_1 = -\mathrm{gap}_N \cdot g_1$ *and setting*

$$h(x) := \cos\left(\frac{\pi(x-1/2)}{N}\right), \quad \forall\, x \in [\![1, N]\!]\,,$$

we have $((c\nabla f)(x) := c(x-1,x)[f(x) - f(x-1)])$

$$\lim_{N\to\infty} \sup_{x\in[\![1,N]\!]} |g_1(x) - h(x)| \;=\; 0\,,$$

$$\lim_{N\to\infty} \sup_{x\in[\![1,N]\!]} |N(c\nabla g_1)(x) - N(\nabla h)(x)| \;=\; 0\,.$$

Our results

Remark

The method in the forementioned theorem also works for the other j*-monotone eigenfunctions under the (LLN) assumption, i.e. with* $K_0 \in \mathbb{N}$ *being any prefixed constant, for all* $1 \le i \le K_0$,

$$\lim_{N\to\infty} |\lambda_i N^2/\pi^2 - i^2| \;=\; 0\,,$$

$$\lim_{N\to\infty} \sup_{x\in[\![1,N]\!]} \left| g_i(x) - \cos\left(\frac{i\pi(x-1/2)}{N}\right)\right| \;=\; 0\,,$$

$$\lim_{N\to\infty} \sup_{x\in[\![1,N]\!]} |N(c\nabla g_i)(x) - N(\nabla h_i)(x)| \;=\; 0\,,$$

where $g_i(1) = 1$.

Our results: mixing time

Assumption

Exist constants $\upsilon \in (0,1)$ *and* $C_{\mathbb{P}} > 0$, *a sequence of positive numbers* $(\bar{\Upsilon}_N)_N > 0$ *with* $\lim_{N\to\infty} \bar{\Upsilon}_N = 0$ *and* $\lim_{N\to\infty} \bar{\Upsilon}_N \log N = \infty$ *such that*

$$\max_{1\le x<N} r(x,x+1) \ \le \ C_{\mathbb{P}} \exp\left((\log N)^{\upsilon}\right) ,$$

$$\min_{1\le x<N} r(x,x+1) \ \ge \ \bar{\Upsilon}_N .$$

Exists $\varrho \in (0,1]$ *and* $c_\varrho > 0$ *such that*

$$c_\varrho N^\varrho \ \le \ k_N \ \le \ N/2 .$$

Theorem (Y. '24)

Under (LLN) and the assumption above, for all $\varepsilon \in (0,1)$ *we have*

$$\lim_{N\to\infty} \frac{2\pi^2 t_{\mathrm{mix}}^{N,k}(\varepsilon)}{N^2 \log k_N} \ = \ 1 .$$

Outline

- Idea for the $j-$monotonicity of eigenfunctions
- Idea for the spectral gap
- Idea for the shape & derivative of eigenfunction
- Idea for the lower bound on the mixing time
- Idea for the upper bound on the mixing time

Idea: $j-$monotonicity of eigenfunctions

Idea: $j-$monotonicity of eigenfunctions

$\mathcal{L}_{N,1}$ is a symmetric matrix. Then it is diagonalizable: $g_0 = \mathbf{1}$ and

$$\begin{cases} 0 \ = \ \lambda_0 \ < \ \lambda_1 \ \le \ \lambda_2 \ \le \ \cdots \ \le \ \lambda_{N-1}\,, & \\ \mathcal{L}_{N,1} g_i \ = \ -\lambda_i g_i \text{ and } g_i(1) \ = \ 1\,, & \forall\, i \in [\![0, N-1]\!]\,, \\ \frac{1}{N}\sum_{x=1}^{N} g_i(x) g_j(x) = C_{i,j}\delta_{i,j}\,, & \forall\, i,j \in [\![0, N-1]\!]\,. \end{cases}$$

$\delta_{i,j}$: Kronecker delta $\quad (C_{i,i})_i$ are some positive constants. Observe:

$$(c\nabla g_i)(x+1) - (c\nabla g_i)(x) = -\lambda_i g_i(x) \Rightarrow$$

$$(c\nabla g_i)(x+1) \ = \ -\lambda_i \sum_{y=1}^{x} g_i(y)\,.$$

Given $c(x,x+1)_x$, $g_i(1) = 1$ and λ_i together determine g_i, implying

$$0 \ = \ \lambda_0 \ < \ \lambda_1 \ < \ \lambda_2 \ < \ \cdots \ < \ \lambda_{N-1}\,.$$

Assuming g_i is i-monotone, use the variational formula to show

g_1 is strictly monotone.

$\mathrm{F}_i(\xi) := \sum_{x=1}^{N} g_i(x)\xi(x)$ is an eigenfunction of $\mathcal{L}_{N,k}$ with eigenvalue $-\lambda_i$.
F_1 is monotone in the natural partial order $\Rightarrow \mathrm{gap}_{N,k} = \lambda_1$.

Idea: $j-$monotonicity of eigenfunctions

Setting $c(N, N+1) = 1$, for $\lambda > 0$, define $f^\lambda : [\![0, N+1]\!] \mapsto \mathbb{R}$ by $f^\lambda(0) = f^\lambda(1) = 1$ and for $x \in [\![1, N]\!]$,

$$f^\lambda(x+1) \ = \ f^\lambda(x) + \frac{1}{c(x, x+1)} \left[(c\nabla f^\lambda)(x) - \lambda f^\lambda(x) \right] .$$

Note that (the restriction to $[\![1, N]\!]$ of) f^λ is an eigenfunction of $\mathcal{L}_{N,1}$ if and only if

$$f^\lambda(N+1) \ = \ f^\lambda(N) .$$

There is no eigenfunction satisfying $f^\lambda(1) = 0$ or $f^\lambda(N) = 0$.

Idea: $j-$monotonicity of eigenfunctions

For $\lambda > 0$ and $x \in [\![1, N+1]\!]$, we set

$$b(\lambda, x) \; := \; -\frac{(c\nabla f^{\lambda})(x)}{f^{\lambda}(x-1)}$$

convention: $b(\lambda, x) = \overline{\infty}$ if $f^{\lambda}(x-1) = 0$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\overline{\infty}\}$. We have

$$b(\lambda, x+1) \; = \; \frac{b(\lambda, x)}{1 - c(x-1, x)^{-1} b(\lambda, x)} + \lambda \, .$$

Given a fixed $c > 0$, define $\Xi^{(c)} : \mathbb{R} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ as

$$\Xi^{(c)}(\lambda, b) \; = \; \frac{b}{1 - c^{-1} b} + \lambda \, .$$

The function $b \mapsto \Xi^{(c)}(\lambda, b)$ may have zero, one or two fixed points depending on the values of λ and c, see the following figure.

Idea: $j-$monotonicity of eigenfunctions

$$\Xi^{(c)}(\lambda, b) \; = \; \frac{b}{1 - c^{-1}b} + \lambda \, .$$

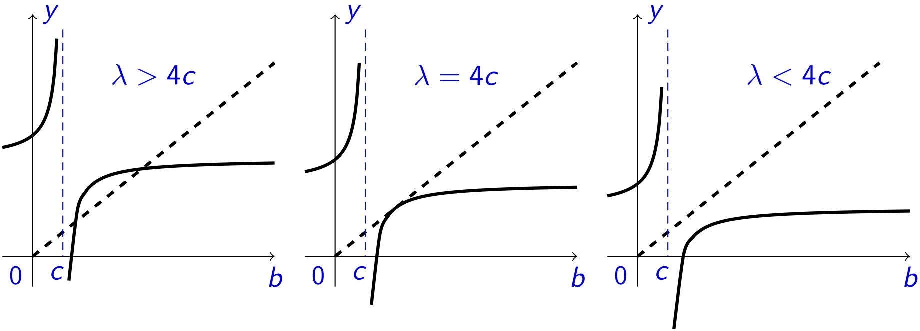


Figure: Solid lines: $b \mapsto \Xi^{(c)}(b, \lambda)$ with $\lambda > 0$ fixed. Black dashed lines: $y = b$. Blue dashes lines: $b = c$.

If $b \mapsto \Xi^{(c)}(\lambda, b)$ has fixed points b_1 and b_2 (not necessarily distinct) such that $b_1 \leq b_2$, we define $I(\lambda, c) = [b_1, b_2]$. Otherwise, set $I(\lambda, c) = \varnothing$.

Idea: $j-$monotonicity of eigenfunctions

Define the "angle mapping" function

$$\varphi(c, \lambda, \theta) \; := \; \inf\{\theta' \geq \theta + \pi \mathbf{1}_{I(\lambda,c)}(\tan\theta) \; : \; \tan\theta' = \Xi^{(c)}(\lambda, \tan\theta)\}\,.$$

Recursively define an "angle" $\theta(\lambda, x)$: $\theta(\lambda, 1) = 0$ and for $x \in [\![1, N]\!]$,

$$\theta(\lambda, x+1) \; := \; \varphi(c(x-1, x), \lambda, \theta(\lambda, x))$$

with convention: $\tan(\pi/2 + k\pi) = \overline{\infty}$ for $k \in \mathbb{Z}$.

Lemma

For fixed $c, \lambda > 0$, the map $\theta \mapsto \varphi(c, \lambda, \theta)$ is continuous and strictly increasing.

Lemma

For fixed $c, \theta > 0$, the map $\lambda \mapsto \varphi(c, \lambda, \theta)$ is strictly increasing and uniformly continuous in θ.

Idea: $j-$monotonicity of eigenfunctions

Lemma

For fixed $c > 0$, the map $(\lambda, \theta) \mapsto \varphi(c, \lambda, \theta)$ is jointly continuous.

f^λ is an eigenfunction if and only if $\theta(\lambda, N+1)$ is a multiple of π.

$$f^\lambda \text{ is an eigenfunction} \quad \Leftrightarrow \quad \theta(\lambda, N+1) = k\pi \ \text{ for } k \in [\![0, N-1]\!] .$$

Let $\lambda_k > 0$ denote the unique number satisfying $\theta(\lambda_k, N+1) = k\pi$ and set $f_k := f^{\lambda_k}$. Let $x_i \in [\![1, N]\!]$ such that $\theta(\lambda_k, x_i) \leq i\pi < \theta(\lambda_k, x_i + 1)$ for $i \in [\![1, k-1]\!]$.

Lemma

For λ_k mentioned above and the associated sequence $(x_i)_i$, we have that $\#\{(x_i)_i\} = k - 1$, $1 < x_i < N$ are the local extrema of f_k (or the pair $\{x_i - 1, x_i\}$ when $\theta(\lambda_k, x_i) = i\pi$) and no any other local extrema.

Idea: the spectral gap

Idea for the spectral gap

Setting $B^{(N)}(x) := b(\lambda, x)N$ and $\lambda := \alpha/N^2$, we have

$$B^{(N)}(x+1) = \frac{B^{(N)}(x)}{1 - N^{-1} r^{(N)}(x-1,x) B^{(N)}(x)} + \frac{\alpha}{N},$$

which starts from $B^{(N)}(1) := 0$.

$$N\left[B(x+1) - B(x)\right] = \frac{r(x-1,x)B(x)^2}{1 - B(x)\, r(x-1,x) N^{-1}} + \alpha$$

Intuition: the asymptotic ODE

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + \alpha, & x \in (0,1) \\ y(0) = 0. \end{cases}$$

Its unique solution:

$$y(x) = \sqrt{\alpha} \tan\left(\sqrt{\alpha} \cdot x\right).$$

Therefore $\alpha = i^2 \pi^2$.

Idea for the spectral gap

Proposition

If ((**LLN**)) *holds, then for any* $\varepsilon \in (0, \pi/2)$ *we have*

$$\limsup_{N\to\infty} \max_{u\in\left[0,(\frac{\pi}{2}-\varepsilon)\alpha^{-1/2}\right]} \left| B^{(N)}(\lceil uN\rceil) - \sqrt{\alpha}\tan(\sqrt{\alpha}u)\right| = 0 .$$

$$\delta_N^{(1)} := \max_{x\in[\![2,N]\!]} \frac{r(x-1,x)}{N} \le \frac{1}{N} + \frac{1}{N}\sup_{1\le n<m\le N} |(r(n,m)-(m-n)| \to 0 ,$$

$$1+z \le \frac{1}{1-z} \le 1+(1+3(\delta_N^{(1)})^{1/2})z , \text{ for } z \in \left[0, (\delta_N^{(1)})^{1/2}\right] ,$$

$$(r')^{(N)}(x-1,x) := (1+3(\delta_N^{(1)})^{1/2})r^{(N)}(x-1,x)$$

$$\begin{cases} \widehat{B}(1) = \widetilde{B}(1) = B(1) = 0, \\ \widehat{B}\,(x+1) \;=\; \widehat{B}(x) + N^{-1}r(x-1,x)\left(\widehat{B}(x)\right)^2 + \frac{\alpha}{N} , & \text{for } x \ge 1 , \\ \widetilde{B}\,(x+1) \;=\; \widetilde{B}(x) + N^{-1}r'(x-1,x)\left(\widetilde{B}(x)\right)^2 + \frac{\alpha}{N} , & \text{for } x \ge 1 . \end{cases}$$

Idea for the spectral gap

As long as

$$\max_{y \in [\![0, x-1]\!]} \widetilde{B}(y) \leq (\delta_N^{(1)})^{-1/2}$$

we have

$$\widehat{B}(x) \ \leq \ B(x) \ \leq \ \widetilde{B}(x)\,.$$

Sufficient to prove the proposition concerning $\widehat{B}$ and $\widetilde{B}$.
First deal with those coordinates x which are small.

Lemma

If (**(LLN)**) *holds, then for any* $\varepsilon > 0$ *we have*

$$\limsup_{N \to \infty} \max_{x : \frac{\sqrt{\alpha} r(1, x-1)}{N} \leq \frac{\pi}{2} - \varepsilon} \left| \widehat{B}(x) - \sqrt{\alpha} \tan\left(\frac{\sqrt{\alpha} r(1, x-1)}{N} \right) \right| \ = \ 0\,.$$

Idea for the spectral gap

Setting

$$\begin{cases} Y(1) := 0\,, \\ Y(x) \ := \ \sqrt{\alpha}\tan\left(\frac{\sqrt{\alpha}r(1,x-1)}{N}\right)\,,\ \forall\, x \geq 2\,, \end{cases}$$

and using the formula for the tangent of the difference of two angles,

$$Y(x+1) \ = \ Y(x) + N^{-1}r(x-1,x)Y^2(x) + \frac{\alpha}{N}r(x-1,x) + q_N(x)\,,$$

where $|q_N(x)| \leq C(\alpha,\varepsilon)r(x-1,x)^2N^{-2}$ for all x.
Set $w_N(1) = \gamma(1) = 0$, and for $x \geq 2$,

$$w_N(x) \ := \ \frac{\alpha}{N}[x-1-r(1,x-1)] - \sum_{y=1}^{x-1} q_N(y)\,,$$

$$\gamma(x) \ := \ \widehat{B}(x) - Y(x) - w_N(x)\,.$$

Idea for the spectral gap

$$
\begin{aligned}
&\gamma(x+1)-\gamma(x)\\
&= N^{-1}r(x-1,x)\left[\widehat{B}(x)^2 - Y(x)^2\right]\\
&= N^{-1}r(x-1,x)\cdot[2Y(x)+\gamma(x)+w_N(x)]\cdot[\gamma(x)+w_N(x)]\,.
\end{aligned}
$$

In our range of x, $Y(x)$ is uniformly bounded and

$$|w_N(x)| \le \delta_N := \alpha\delta_N^{(0)} + 6C(\alpha,\varepsilon)(\delta_N^{(0)})^{1/2} \to 0\,.$$

We argue by induction that $|\gamma(x)| \le 1$ for all x in the range.
It holds for $x = 1, 2$ (for all N big enough).
Suppose it holds for all $k \le x$, then

$$|2Y(x)+\gamma(x)+w_N(x)| \le C = C(\alpha,\varepsilon)\,,$$

and

$$|\gamma(k+1)| \;\le\; \left(1 + CN^{-1}r(k-1,k)\right)|\gamma(k)| + CN^{-1}r(k-1,k)\delta_N\,.$$

Idea for the spectral gap

Iterating this inequality from $k = x$ backward to $k = 2$, we obtain

$$
\begin{aligned}
|\gamma(x+1)| &\le |\gamma(2)| \prod_{j=2}^{x} \left(1 + \frac{Cr(j-1,j)}{N}\right) \\
&\quad + \sum_{j=2}^{x} \frac{Cr(j-1,j)}{N} \delta_N \prod_{i=j+1}^{x} \left(1 + \frac{Cr(i-1,i)}{N}\right) \\
&\le |\gamma(2)| \exp\left(\frac{Cr(1,x)}{N}\right) + \frac{Cr(1,x)}{N} \delta_N \exp\left(\frac{Cr(1,x)}{N}\right) \\
&\le C' \delta_N \,.
\end{aligned}
$$

The assumption $|\gamma(x+1)| \le 1$ is verified, and we can conclude:

$$
\begin{aligned}
&\limsup_{N\to\infty} \max_{x:\frac{\sqrt{\alpha} r(1,x-1)}{N} \le \pi/2-\varepsilon} |\widehat{B}(x) - Y(x)| \\
\le\; &\limsup_{N\to\infty} \max_{x:\frac{\sqrt{\alpha} r(1,x-1)}{N} \le \pi/2-\varepsilon} |\gamma(x)| + |w_N(x)| \;=\; 0\,.
\end{aligned}
$$

Idea for the spectral gap

For the second segment:

$$\frac{\pi}{4} \leq \frac{\sqrt{\alpha} r(1, x-1)}{N} \leq \frac{3\pi}{4} \,.$$

Consider:

$$A^{(N)}(x) \;:=\; \frac{1}{B^{(N)}(x)} \,.$$

Motivation: avoid dealing with the explosion of the tangent fun at the neighbor of $\pi/2$. we have

$$N\left[A\left(x+1\right)-A\left(x\right)\right] \;=\; \frac{\alpha r(x-1,x)A\left(x\right)N^{-1}-r(x-1,x)-\alpha A\left(x\right)^{2}}{1+\alpha A\left(x\right)N^{-1}-\alpha r(x-1,x)N^{-2}} \,.$$

For the last segment: $\frac{\sqrt{\alpha} r(1,x-1)}{N} > \frac{3\pi}{4}$, consider $B(x)$.

Idea: shape/derivative of the eigenfunction

Idea for shape/derivative of the eigenfunction

Eigenfunction corresponding to the spectral gap when $r(j-1,j) \equiv 1$

$$h(x) \;=\; h_N(x) \;=\; \cos\left(\frac{\pi(x-1/2)}{N}\right).$$

The spectral gap is

$$\overline{\lambda} \;:=\; 2\left[1-\cos\left(\frac{\pi}{N}\right)\right] \;=\; \frac{\pi^2}{N^2} + O\left(\frac{1}{N^4}\right).$$

By $b(\lambda, x) \;=\; -\frac{(c\nabla g)(x)}{g(x-1)}$ and $b(\lambda, x) = B(x)/N$, for $x \geq 2$ we have

$$g(x) \;=\; \left[1 - r(x-1,x)N^{-1}B\left(x\right)\right] g(x-1).$$

Writing $u(x) := h(x) - g(x)$, we have

$$\begin{aligned} u(x) \;=\; & u(x-1)\left[1 - \frac{r(x-1,x)B\left(x\right)}{N}\right] \\ & + \frac{h(x-1)}{N}\left[r(x-1,x)B\left(x\right) - \overline{B}\left(x\right)\right]. \end{aligned}$$

Iterate the equation above to conclude the proof for the first segment.

Idea for the shape/derivative of the eigenfunction

For the first segment: by

$$(c\nabla g_i)(x+1) \;=\; -\lambda_i \sum_{y=1}^{x} g_i(y)\,,$$

we have

$$\begin{aligned}
& |N(c\nabla g)(x) - N(\nabla h)(x)| \\
= & \left| -N\lambda_1 \sum_{k=1}^{x-1} g(k) + N\overline{\lambda} \sum_{k=1}^{x-1} h(k) \right| \\
\le & \left| -N\lambda_1 \sum_{k=1}^{x-1} [g(k) - h(k)] \right| + \left| N\left(\overline{\lambda} - \lambda_1\right) \sum_{k=1}^{x-1} h(k) \right| \\
\le & \, N\lambda_1 \sum_{k=1}^{x-1} |[g(k) - h(k)]| + N|\overline{\lambda} - \lambda_1|(x-1)\,.
\end{aligned}$$

For the second segment: use

$$A(x)\left[1 - \frac{g(x)}{g(x-1)}\right] \;=\; r(x-1,x)N^{-1}\,.$$

Idea: the lower bound on the mixing time

Idea for the lower bound on the mixing time

$$\mathrm{F}(\xi) = \mathrm{F}_1(\xi) = \sum_{1 \le x \le N} \xi(x) g_1(x)$$

is an eigenfunction satisfying $\mathcal{L}_{N,k}\mathrm{F} = -\mathrm{gap}_N \cdot \mathrm{F}$. For $t_0 > 0$, define

$$F(t,\xi) := e^{\lambda_1(t-t_0)}\mathrm{F}(\xi)\,, \quad \forall\, \xi \in \Omega_{N,k}\,,$$

and study the Dynkin martingale

$$M_t \;:=\; F(t,\eta_t^\nu) - F(0,\eta_0^\nu) - \int_0^t \left(\partial_s + \mathcal{L}_{N,k}\right) F(s,\eta_s^\nu)\,\mathrm{d}s\,.$$

$$\mathbf{E}\left[\mathrm{F}\left(\eta_{t_0}^\nu\right)\right] \;=\; \mathbf{E}\left[F\left(t_0,\eta_{t_0}^\nu\right)\right] \;=\; \mathbf{E}\left[F(0,\eta_0^\nu)\right] \;=\; e^{-\lambda_1 t_0}\mathbf{E}\left[\mathrm{F}\left(\eta_0^\nu\right)\right]\,.$$

$$\mathbf{E}[M_{t_0}^2] = \mathbf{E}[\int_0^{t_0} \partial_s \langle M\rangle_s \mathrm{d}s]$$

$$\overline{\eta}_t^\nu(x,x+1) := \eta_t^\nu(x)\left(1-\eta_t^\nu(x+1)\right) + \eta_t^\nu(x+1)\left(1-\eta_t^\nu(x)\right)$$

$$\partial_t \langle M.\rangle_t \;=\; e^{2\lambda_1(t-t_0)} \sum_{x=1}^{N-1} \overline{\eta}_t^\nu(x,x+1) r(x,x+1)\left[c(x,x+1)(g(x)-g(x+1))\right]^2\,.$$

Idea for the lower bound on the mixing time

At equilibrium

$$\mathbf{E}\left[\mathrm{F}(\eta^{\mu}_{t_0})\right] = \mu_{N,k}(\mathrm{F}) = \frac{k}{N}\sum_{1\le x\le N} g_1(x) = 0, \qquad \mathrm{Var}_{\mu}(\mathrm{F}) \asymp k.$$

1. If ν concentrates at one configuration, then
$$\mathbf{E}\left[\mathrm{F}(\eta^{\nu}_{t_0}) - \mathbf{E}\left[\mathrm{F}(\eta^{\nu}_{t_0})\right]\right]^2 = \mathbf{E}\left[F(t_0,\eta^{\nu}_{t_0}) - F(0,\eta^{\nu}_0)\right]^2 = \mathbf{E}\left[M^2_{t_0}\right].$$
2. If ν is non-degenerated, we have
$$\begin{aligned}&\mathbf{E}\left[\mathrm{F}(\eta^{\nu}_{t_0}) - \mathbf{E}\left[\mathrm{F}(\eta^{\nu}_{t_0})\right]\right]^2\\ =\ &\mathbf{E}\left[F(t_0,\eta^{\nu}_{t_0}) - F(0,\eta^{\nu}_0) + F(0,\eta^{\nu}_0) - \mathbf{E}\left[F(0,\eta^{\nu}_0)\right]\right]^2\\ \le\ &2\mathbf{E}\left[M^2_{t_0}\right] + 2\mathbf{E}\left[F(0,\eta^{\nu}_0) - \mathbf{E}\left[F(0,\eta^{\nu}_0)\right]\right]^2.\end{aligned}$$

If $N/64 \le k \le N/2$, take $\nu = \delta_{\wedge}$.
If $(\log N)^{1+\gamma} \le k < N/64$, take ν as follows: sample a configuration according to $\mu_{N,2k}$, keep the first k particles and project the rest to be empty sites.

Idea: the upper bound on the mixing time

Idea: the upper bound on the mixing time

Height function:

$$\xi \in \Omega_{N,k} \quad \to \quad h^{\xi}(x) \; := \; \sum_{y=1}^{x} \xi(y) - \frac{k}{N} x \,.$$

A partial order:

$$\left(\xi \;\leq\; \xi'\right) \quad \Leftrightarrow \quad \left(h^{\xi}(x) \;\leq\; h^{\xi'}(x) \,, \forall\, x \in [\![1, N]\!]\right) \,.$$

Attractive:

$$\left(h^{\xi} \leq h^{\xi'}\right) \quad \Rightarrow \quad \left(\forall\, t \geq 0,\; h^{\xi}_t \leq h^{\xi'}_t\right) \,.$$

Coalescing time:

$$T_1 \;:=\; \inf\left\{t \geq 0 :\; h^{\wedge}_t = h^{\mu}_t\right\} \,,$$
$$T_2 \;:=\; \inf\left\{t \geq 0 :\; h^{\vee}_t = h^{\mu}_t\right\} \,.$$

Idea: the upper bound on the mixing time

Construct a supermartingale: inspired by [Wilson, '04], embed the segment $[\![1, N]\!]$ in $[\![-\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor]\!]$ and place conductance $(c(x, x+1) = 1)_{x \notin [1, N-1]}$. The principle eigenfunction satisfies:

$$\lim_{N \to \infty} \sup_{x \in [\![-\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor]\!]} \left| G(x) - \cos\left(\frac{\pi(x + \lfloor \delta N \rfloor + 1/2)}{\bar{N}} \right) \right| = 0 .$$

Define

$$\mathbf{F}(\xi) := \sum_{x=1}^{N-1} h^{\xi}(x) \bar{G}(x) .$$

For $\xi, \xi' \in \Omega_{N,k}$ with $\xi \leq \xi'$, since $h^{\xi}(x) \leq h^{\xi'}(x)$ and $\bar{G}(x) > 0$,

$$\mathbf{F}(\xi) \leq \mathbf{F}(\xi') .$$

Furthermore, if $\xi \leq \xi'$ with $\xi \neq \xi'$, we have $\mathbf{F}(\xi) < \mathbf{F}(\xi')$.

Idea: upper bound on the mixing time

Using $h^{\xi}(0) = h^{\xi}(N) = 0$ and for $x \in [\![1, N-1]\!]$

$$\begin{aligned}\left(\mathcal{L}_{N,k} h^{\xi}\right)(x) &= c(x, x+1)\left[\xi(x+1) - \xi(x)\right] \\ &= c(x, x+1)\left[\left(h^{\xi}(x+1) - h^{\xi}(x)\right) - \left(h^{\xi}(x) - h^{\xi}(x-1)\right)\right]\end{aligned}$$

we obtain

$$\begin{aligned}\left(\mathcal{L}_{N,k}\mathbf{F}\right)(\xi) &= \sum_{x=1}^{N-1} \bar{G}(x)\left(\mathcal{L}_{N,k} h^{\xi}\right)(x) \\ &= -\bar{\lambda}_1 \mathbf{F}(\xi) - h^{\xi}(1)c(0,1)\bar{G}(0) - h^{\xi}(N-1)c(N, N+1)\bar{G}(N)\end{aligned}$$

where $\bar{\lambda}_1$ is the spectral gap of the system in the longer line segment.

Idea: upper bound on the mixing time

For $\xi \leq \xi'$,

$$
\begin{aligned}
&(\mathcal{L}_{N,k}\mathbf{F})(\xi') - (\mathcal{L}_{N,k}\mathbf{F})(\xi) \;=\; -\bar{\lambda}_1 \left[\mathbf{F}(\xi') - \mathbf{F}(\xi)\right] \\
&- \left[h^{\xi'}(1) - h^{\xi}(1)\right] c(0,1)\bar{G}(0) - \left[h^{\xi'}(N-1) - h^{\xi}(N-1)\right] c(N,N+1)\bar{G}(N) \\
&\leq\; -\bar{\lambda}_1 \left[\mathbf{F}(\xi') - \mathbf{F}(\xi)\right] .
\end{aligned}
$$

Then $(\mathbf{F}(h_t^{\wedge}) - \mathbf{F}(h_t^{\mu}))_{t\geq 0}$ is a supermartingale with decay rate $\bar{\lambda}_1$.

Combine the approachs in [Lacoin AOP'16] and [Labbé, Lacoin AAP'20] to adapt to the disordered setup to conclude the proof.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