

# Metastability for expanding bubbles on a sticky substrate

Shangjie Yang



Harbin Institute of Technology

Jointed work with Hubert Lacoin (IMPA)

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# Education

- March 2017-April 2021      Instituto de Matemática Pura e Aplicada  
Ph.D. student      Adviser: Hubert Lacoin (2022 ICM speaker)
- March 2015-February 2017      Instituto de Matemática Pura e Aplicada  
Master student      Adviser: Vladas Sidoravicius (2014 ICM speaker)
- September 2013-December 2014      Nankai University  
Master student (dropout)      Adviser: Kainan Xiang
- September 2009-June 2013      Sun Yat-sen University  
Undergraduate student

# Publications and preprints

5. **N. Feldheim and S. Y.**, Typical height of the  $(2+1)$ -D Solid-on-Solid surface with pinning above a wall in the delocalized phase  
Stochastic Processes and their Applications. 165 (2023), 168-182.
4. **G. Amir and S. Yang**, The branching number of intermediate growth trees  
Arxiv: 2205.14238      submitted.
3. **H. Lacoïn and S.Y.**, Mixing time of the asymmetric simple exclusion process in a random environment      Arxiv: 2102.02606  
Accepted by Annals of Applied Probability
2. **H. Lacoïn and S.Y.**, Metastability for expanding bubbles on a sticky substrate  
Ann. Appl. Probab. 32(5): 3408-3449 (October 2022).
1. **S. Y.**, Cutoff for polymer pinning dynamics in the repulsive phase  
Ann. Inst. H. Poincaré Probab. Statist. 57(3): 1306-1335 (2021).

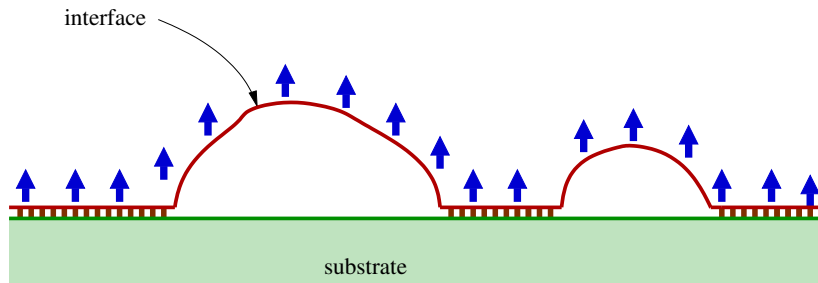
# The physical situation we are considering



= external force



= substrate/interface interaction



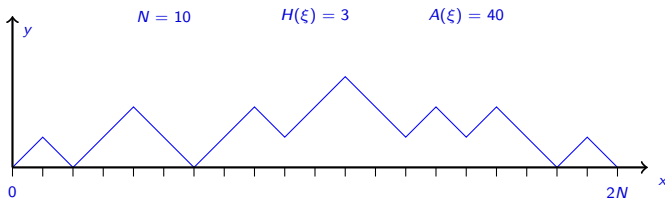
An interface is an element of

$$\Omega_N := \left\{ \xi \in \mathbb{Z}_+^{[0,2N]} : \xi(0) = \xi(2N) = 0 \text{ and } \forall x, |\xi(x) - \xi(x-1)| = 1 \right\}.$$

# The equilibrium measure

Given  $\xi \in \Omega_N$ ,

- $H(\xi) := \sum_{x=1}^{2N-1} \mathbf{1}_{\{\xi(x)=0\}}$  (# contacts with  $x$ -axis),
- $A(\xi) := \sum_{x=1}^{2N-1} \xi(x)$ : the area enclosed between  $\xi$  and the  $x$ -axis.



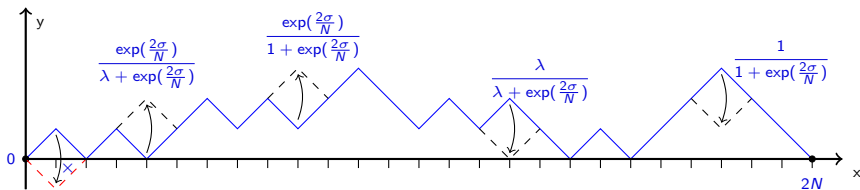
Given  $\lambda \geq 0$  and  $\sigma \geq 0$ , define  $\mu = \mu_N^{\lambda, \sigma}$  the probability on  $\Omega_N$ :

$$\mu(\xi) = \frac{2^{-2N} \lambda^{H(\xi)} e^{\frac{\sigma}{N} A(\xi)}}{Z_N(\lambda, \sigma)} \quad ; \quad Z_N(\lambda, \sigma) := 2^{-2N} \sum_{\xi \in \Omega_N} \lambda^{H(\xi)} e^{\frac{\sigma}{N} A(\xi)}.$$

## Corner-flip/Heat Bath dynamics $(\eta_t)_{t>0}$ on $\Omega_N$

Each coordinate is updated at rate one.

When an update at  $x$  occurs at time  $t$ ,  $\eta_t$  is sampled according to the conditional equilibrium measure  $\mu_N^{\lambda, \sigma}(\cdot \mid \eta_{t-}(y), y \neq x)$ .



The measure  $\mu$  satisfies the detailed balance condition, *i.e.*

$$\mu(\xi)r(\xi, \xi^x) = \mu(\xi^x)r(\xi^x, \xi).$$

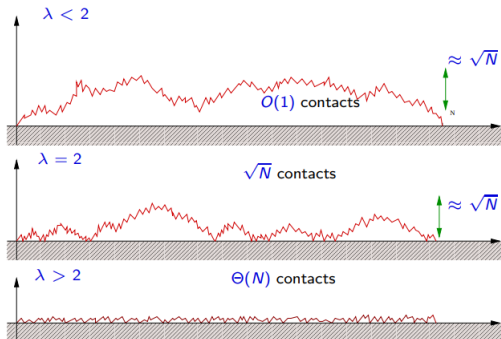
**$\mathbf{P}^\xi$ :** the distribution of the Markov chain  $(\eta_t)_{t \geq 0}$  starting from  $\xi$ .

$$T_N^{\lambda, \sigma} := \inf \left\{ t > 0 : \max_{\xi \in \Omega_N} \|\mathbf{P}_t^\xi - \mu\|_{\text{TV}} \leq 1/4 \right\}$$

# Presentation of our results for the interface model

- (1) Previous results about related models
- (2) Properties of the model at equilibrium
- (3) Slow/fast mixing and metastability ( $\sigma > 0$ )

# Equilibrium for $\sigma = 0$ [Fisher 1984 JSP]



If  $\sigma = 0$ , the system undergoes a transition at  $\lambda = 2$  between a pinned phase and an unpinned phase. This transition can be seen when looking at the free energy

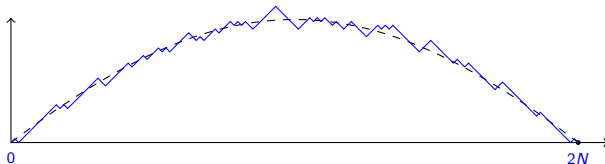
$$\lim_{N \rightarrow \infty} \frac{1}{2N} \log Z_N(\lambda, 0) = \log \left( \frac{\lambda}{2\sqrt{\lambda-1}} \right) \mathbf{1}_{\{\lambda > 2\}} =: F(\lambda).$$



## No wall constraint / WASEP interfaces [Labbé '18 Prob. Surv.]

If there is no wall constraint ( $\xi(x) < 0$  is allowed) and  $\lambda = 1$ , we have typically under the equilibrium measure ( $u \in [0, 2]$ )

$$\frac{\xi(\lceil uN \rceil)}{N} = \frac{1}{\sigma} \log \left( \frac{\cosh(\sigma)}{\cosh(\sigma(1-u))} \right) + o(1).$$



If  $\tilde{Z}_N(\sigma) := \frac{1}{2^{2N}} \sum_{\xi \in \tilde{\Omega}_N} e^{\frac{\sigma}{N} A(\xi)}$  denotes the corresponding partition function, we have

$$\lim_{N \rightarrow \infty} \frac{1}{2N} \log \tilde{Z}_N(\sigma) = G(\sigma) := \int_0^1 \log \cosh(\sigma(1-2u)) du.$$

# Equilibrium behavior

The two strategies to take benefit of the wall interaction and of the external force are different and cannot be combined.

Proposition (Lacoin, Y. '22)

*We have for any  $\lambda \in (0, \infty)$  and  $\sigma > 0$*

$$\lim_{N \rightarrow \infty} \frac{1}{2N} \log Z_N(\lambda, \sigma) = F(\lambda) \vee G(\sigma).$$

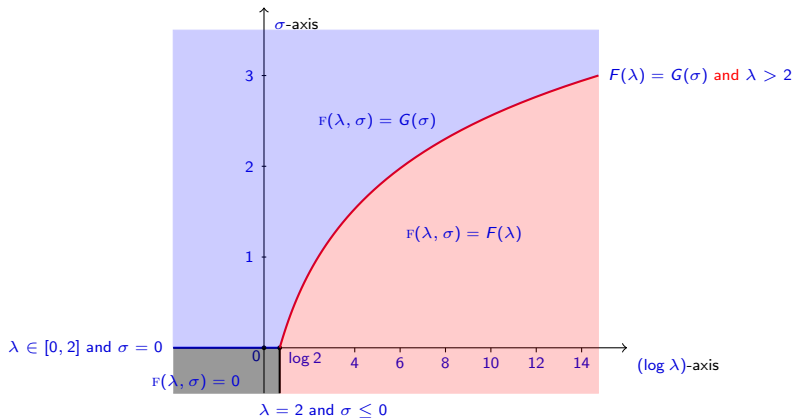
(A) *If  $G(\sigma) > F(\lambda)$ , then  $Z_N(\lambda, \sigma) \asymp \frac{1}{\sqrt{N}} e^{2NG(\sigma)}$ .*

(B) *If  $F(\lambda) \geq G(\sigma)$ , then  $Z_N(\lambda, \sigma) \asymp e^{2NF(\lambda)}$ .*

From this result we derive the detailed behavior of the paths.

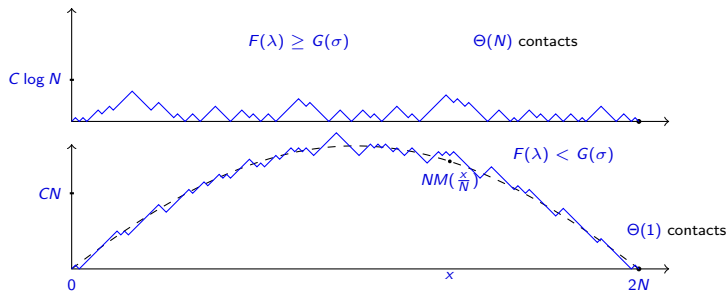
# Free energy

$$F(\lambda, \sigma) := \lim_{N \rightarrow \infty} \frac{1}{2N} \log Z_N(\lambda, \sigma).$$



# Theorem: macroscopic shape

$$M_\sigma(u) = \frac{1}{\sigma} \log \left( \frac{\cosh(\sigma)}{\cosh(\sigma(1-u))} \right).$$



# Dynamical polymer pinning model/WASEP

The problem of mixing time for interface with pinning or WASEP has been studied in previous works.

- When  $\sigma = 0$ , the mixing time is at most of order  $N^2 \log N$  [Caputo, Martinelli, Toninelli '08 EJP]:

$$\text{e.g. } T_N^{\lambda,0} \asymp N^2 \log N, \text{ and gap } \asymp N^{-2} \text{ for } \lambda \in [0, 2).$$

- Without wall and pinning, [Levin, Peres '16 JSP] [Labbé, Lacoïn '20 AAP]

$$T_N^\sigma \asymp N^2 \log N.$$

Our main result:  $\lambda > 2$  and  $\sigma \geq 0$

Theorem (Lacoin, Y. '22)

When  $\lambda > 2$  and  $\sigma \geq 0$ , then there exists  $\sigma_c(\lambda) > 0$  such that

$$\begin{cases} T_N^{\lambda, \sigma} \leq N^C & \text{if } \sigma \leq \sigma_c(\lambda), \\ T_N^{\lambda, \sigma} = e^{2NE(\lambda, \sigma)} N^{O(1)} & \text{if } \sigma > \sigma_c(\lambda), \end{cases}$$

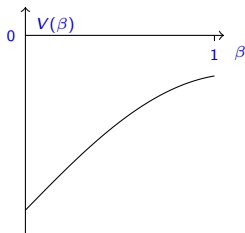
where  $\sigma_c(\lambda)$  and  $E(\lambda, \sigma) > 0$  are explicit.

We believe when  $\lambda \in [0, 2]$  and  $\sigma \geq 0$ , there exists some constant  $C$

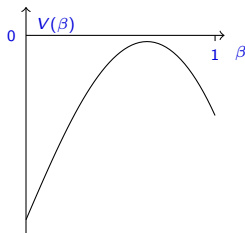
$$T_N^{\lambda, \sigma} \leq N^C.$$

## Heuristic for $\lambda > 2$ and $\sigma \geq 0$

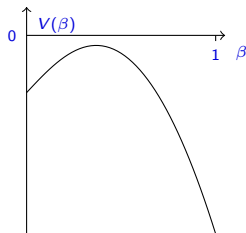
$\beta$ : fraction of the largest excursion       $V(\beta) := -(1 - \beta)F(\lambda) - \beta G(\beta\sigma)$   
(paths with only one large excursion of size  $2\beta N$ :  $e^{-2NV(\beta)}$ .)



(A)



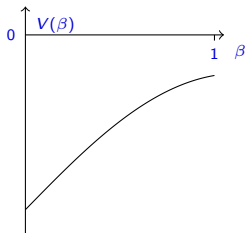
(B)



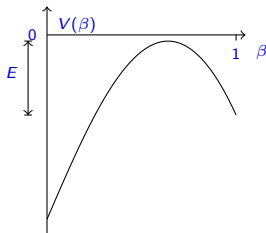
(C)

- (A) If  $G(\sigma) + \sigma G'(\sigma) \leq F(\lambda)$ , then the pinned region can grow without obstruction and the system should mix in polynomial time.
- (B) If  $G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma)$ , then the system starting from the fully unpinned state takes a long time to reach the fully pinned equilibrium state.
- (C) If  $F(\lambda) < G(\sigma)$ , then the system starting from the fully pinned state takes a long time to reach the fully unpinned equilibrium state.

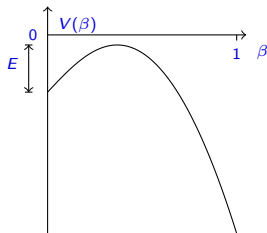
$$V(\beta) = -(1 - \beta)F(\lambda) - \beta G(\beta\sigma)$$



(A)



(B)



(C)

## Activation Energy

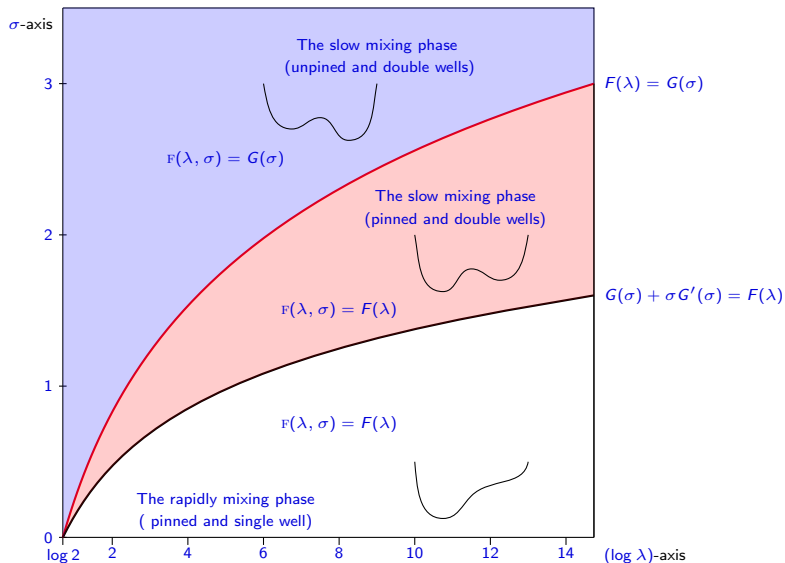
The size of the effective potential barrier to be overcome in case (B) and (C) is equal to

$$E(\lambda, \sigma) := F(\lambda) \wedge G(\sigma) - [(1 - \beta^*)F(\lambda) + \beta^* G(\beta^* \sigma)]$$

with  $\beta^*$  such that  $V(\beta^*) = \max_{\beta \in [0,1]} V(\beta)$ .



# Our result: phase diagram (for $\lambda > 2$ and $\sigma \geq 0$ )



# Metastability

Assuming  $E(\lambda, \sigma) > 0$ , let  $\mathcal{H}_N$  denote the domain of attraction of the unstable local equilibrium of the dynamics:

$$\mathcal{H}_N := \begin{cases} \{\xi \in \Omega_N : L_{\max}(\xi) > \beta^* N\} & \text{if } G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma), \\ \{\xi \in \Omega_N : L_{\max}(\xi) \leq \beta^* N\} & \text{if } F(\lambda) < G(\sigma), \end{cases}$$

where

$$L_{\max}(\xi) := \max\{y - x : \xi_{2x} = 0, \xi_{2y} = 0, \forall z \in \llbracket x, y \rrbracket, \xi_{2z} > 0\}.$$

Theorem (Lacoin, Y. '22)

We have

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N(\cdot | \mathcal{H}_N)} \left( \eta_{t T_{\text{rel}}^N(\lambda, \sigma)} \in \mathcal{H}_N \right) = \exp(-t),$$

where  $T_{\text{rel}}^N(\lambda, \sigma) = e^{2NE(\lambda, \sigma)} N^{O(1)}$  is the relaxation time of the system.

# Proof ingredients

- Lower bound on mixing time follows directly from the heuristics using bottleneck arguments.
- For the upper bound, the hard part is to show that the system always mixes fast within  $\mathcal{H}_N$  and  $\mathcal{H}_N^c$ . The proof is intricate and relies on chain decomposition argument [Jerrum *et al.* '04 AAP].
- Once fast mixing in each potential well is proved, the metastability statement follows from a general meta-theorem [Beltran and Landim '15 PTRF].

Thank you for your attention