Mixing Time of ASEP in a random environment

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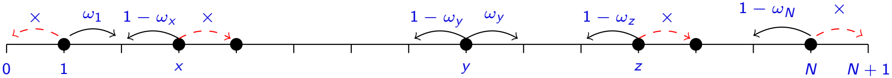
Setup

Environment: $\omega = (\omega_x)_x$ with values in $(0,1)$
Exclusion process with k particles in $[\![1,N]\!]$ with environment ω



(A) Each site is occupied by at most one particle (*the exclusion rule*).

(B) Each of the k particles independently performs a random walk such that a particle at site $x \in [\![1,N]\!]$

$$\begin{cases} \text{jumps to site } x+1 \text{ at rate } \omega_x & \text{for } x \le N-1, \\ \text{jumps to site } x-1 \text{ at rate } 1-\omega_x & \text{for } x \ge 2, \end{cases}$$

if the target site is not occupied. (No particle is allowed to jump out of the segment.)

Setup

- State space (1: particle 0: empty site.)

$$\Omega_{N,k} := \left\{ \xi \in \{0,1\}^N : \sum_{x=1}^{N} \xi(x) = k \right\}.$$

- $\xi^{x,y}$: swap the contents of sites x, y of ξ

$$\forall z \in [\![1,N]\!], \quad \xi^{x,y}(z) = \xi(z)\mathbf{1}_{[\![1,N]\!]\setminus\{x,y\}} + \xi(x)\mathbf{1}_{\{y\}} + \xi(y)\mathbf{1}_{\{x\}}.$$

- Transition rates: (for $x \in [\![1, N-1]\!]$)

$$r^{\omega}(\xi, \xi^{x,x+1}) := \begin{cases} \omega_x & \text{if } \xi(x) = 1 \text{ and } \xi(x+1) = 0, \\ 1 - \omega_{x+1} & \text{if } \xi(x+1) = 1 \text{ and } \xi(x) = 0, \end{cases}$$

$$r^{\omega}(\xi, \xi') := 0 \quad \text{in all other cases.}$$

Setup

- Generator ($f : \Omega_{N,k} \to \mathbb{R}$)

$$(\mathcal{L}^{\omega}_{N,k} f)(\xi) := \sum_{x=1}^{N-1} r^{\omega}(\xi, \xi^{x,x+1})\big[f(\xi^{x,x+1}) - f(\xi)\big].$$

- Potential in the environment ω: $V^{\omega}(1) := 0$, and for $x \geq 2$

$$V^{\omega}(x) := \sum_{y=2}^{x} \log\left(\frac{1-\omega_y}{\omega_{(y-1)}}\right).$$

Potential for $\xi \in \Omega_{N,k}$:

$$V^{\omega}(\xi) := \sum_{x=1}^{N} V^{\omega}(x)\xi(x).$$

- Invariant probability measure

$$\pi^{\omega}_{N,k}(\xi) := \frac{1}{Z^{\omega}_{N,k}} e^{-V^{\omega}(\xi)} \quad \text{with} \quad Z^{\omega}_{N,k} := \sum_{\xi' \in \Omega_{N,k}} e^{-V^{\omega}(\xi')}.$$

Detailed balance condition

$$\pi^{\omega}_{N,k}(\xi) r^{\omega}(\xi, \xi^{x,x+1}) \;=\; \pi^{\omega}_{N,k}(\xi^{x,x+1}) r^{\omega}(\xi^{x,x+1}, \xi)\,.$$

Assumptions

- $k = N^{\beta+o(1)} \leq \frac{1}{2}N$ particles where $\beta \in (0,1]$.
- $\omega = (\omega_x)_x$ is IID (law: $\mathbb{P}$, expectation: $\mathbb{E}$) satisfying:
  (1) Uniform ellipticity condition: $\exists\, \alpha \in (0, 1/2)$ such that
  $$\mathbb{P}\left(\omega_1 \in [\alpha, 1-\alpha]\right) = 1.$$
  (2)
  $$\mathbb{E}\left[\log \frac{1-\omega_1}{\omega_1}\right] < 0,$$
  so that the random walk on $\mathbb{Z}$ is transient to the right.
- Distance to equilibrium & Mixing time:

$$d^{\omega}_{N,k}(t) := \max_{\xi \in \Omega_{N,k}} \|P^{\xi}_t - \pi^{\omega}_{N,k}\|_{\mathrm{TV}}\,,$$
$$t^{N,k,\omega}_{\mathrm{mix}}(\varepsilon) \;:=\; \inf\left\{t \geq 0 :\; d^{\omega}_{N,k}(t) \;\leq\; \varepsilon\right\}\,,$$
$$t^{N,k,\omega}_{\mathrm{mix}} := t^{N,k,\omega}_{\mathrm{mix}}(1/4)\,.$$

Q: **How does the mixing time $t^{N,k,\omega}_{\mathrm{mix}}$ grow in terms of N and k for typical realization of ω?**

Outline

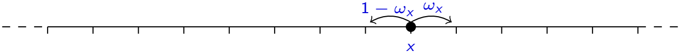
(1) Related results (RWRE, SEP with $\omega \equiv p$ or $\omega = (\omega_x)_x$ IID).

(2) Our result: $t_{\mathrm{mix}}^{N,k,\omega}$ grows like a power of N.

(4) Heuristic for the lower bounds: three mechanisms.

(4) Idea for the upper bound.

Related result: random walk in random environment on $\mathbb{Z}$



Given $(\omega_x)_x$, $(X_t)_{t\geq 0}$: a continuous-time random walk on $\mathbb{Z}$ starting at 0.
[Solomon '75 AOP] showed that

$$\begin{cases} \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] = 0 \Rightarrow (X_t)_{t\geq 0} \text{ is recurrent,} \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0 \Rightarrow \lim_{t\to\infty} X_t = \infty, \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] > 0 \Rightarrow \lim_{t\to\infty} X_t = -\infty. \end{cases}$$

Assuming $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$, set

$$\lambda = \lambda_{\mathbb{P}} := \inf \left\{ s > 0, \mathbb{E}\left[\left(\frac{1-\omega_1}{\omega_1} \right)^s \right] \geq 1 \right\} \in (0, \infty].$$

[Kesten, Kozlov, Spitzer '75 Compos. Math.] showed that

$$\begin{cases} \lim_{t\to\infty} \frac{X_t}{t} = \vartheta_{\mathbb{P}} > 0, & \text{if } \lambda > 1 \text{ (ballistic),} \\ \lim_{t\to\infty} \frac{\log(X_t)}{\log t} = \lambda, & \text{if } \lambda \in (0,1] \text{ (subballistic)} \end{cases}.$$

Related results: many particles in homogenous environment

SSEP ($\omega \equiv \frac{1}{2}$): [Aldous '83 Lect. Notes Math.], [Wilson '04 AAP], [Lacoin '16 AOP]

$$t_{\text{mix}}^{N,k_N} \asymp (N^2 \log k_N) = O(t_{\text{mix}}^{N,1} \log N).$$

ASEP ($\omega \equiv p \neq \frac{1}{2}$): [Benjamini *et al.* '05 TAMS], [Labbé, Lacoin '19 AOP]

$$t_{\text{mix}}^{N,k_N} \asymp N.$$

Related result: one particle in random environment

When $k = 1$, [Gantert, Kochler '18 ALEA] showed that if $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$,

$$\begin{cases} t_{\mathrm{mix}}^{N,1,\omega}(\varepsilon) = [C(\mathbb{P}) + o(1)]N, & \text{if } \lambda_{\mathbb{P}} > 1, \\ \lim_{N\to\infty} \frac{\log t_{\mathrm{mix}}^{N,1,\omega}}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}, & \text{if } \lambda_{\mathbb{P}} \in (0,1]. \end{cases}$$

Potential $V^\omega : \mathbb{N} \to \mathbb{R}$

$$V^\omega(x) := \begin{cases} 0, & \text{for } x = 1, \\ \sum_{y=2}^{x} \log\left(\frac{1-\omega_y}{\omega_{y-1}}\right), & \text{for } x \geq 2. \end{cases}$$

The largest potential barrier: $\max_{1\leq x<y\leq N} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N$.

Related result: many particles in random environment

Assuming $\lambda > 1$ and $\lim_{N\to\infty} k_N/N = \theta \in (0,1)$, [Schmid '19 EJP] showed:

- When $\operatorname{ess\,inf} \omega_1 > 1/2$, $t_{\mathrm{mix}}^{N,k_N,\omega} \asymp N$ by comparison.
- When $\operatorname{ess\,inf} \omega_1 < 1/2$, $t_{\mathrm{mix}}^{N,k_N,\omega} \geq N^{1+\delta}$ for some $\delta > 0$.
- When $\operatorname{ess\,inf} \omega_1 = 1/2$, then

$$\liminf_{N\to\infty} t_{\mathrm{mix}}^{N,k_N,\omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\mathrm{mix}}^{N,k_N,\omega}(\varepsilon) \leq CN(\log N)^3,$$

  together with a quantitative lower bound if $\mathbb{P}[\omega_1 = 1/2] > 0$.

Q: **If $\operatorname{ess\,inf} \omega_1 < 1/2$, how does $t_{\mathrm{mix}}^{N,k_N,\omega}$ grow?**

Our result

Theorem (Lacoin, Y.)

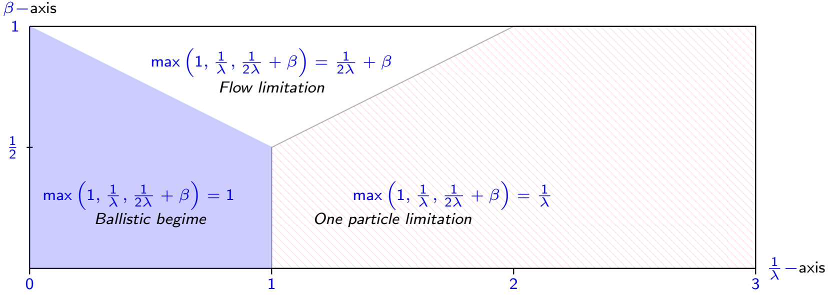
Assuming $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$, $\operatorname{ess\,inf} \omega_1 < \frac{1}{2}$, $k = N^{\beta + o(1)} \leq N/2$ *with* $\beta \in (0, 1]$ *and the uniform ellipticity condition, with high probability we have*

$$c(\alpha, \mathbb{P}) N^{\max(1, \frac{1}{\lambda}, \beta + \frac{1}{2\lambda}) + o(1)} \leq t_{\mathrm{mix}}^{N,k,\omega} \leq N^{C(\alpha, \mathbb{P})}.$$

Conjecture: Our lower bound is sharp.

Phase diagram (Conjecture)

The exponent of the mixing time with $k = N^{\beta+o(1)}$ particles



Lower bound on the mixing time: three mechanisms

- Mass transport cannot be faster than ballistic
- The leftmost particle is blocked by traps in the potential profile
- Potential barrier creates bottleneck for the particle flow

Typical configurations in equilibrium

$$\begin{cases} \text{Every site of } [\![1, N-k-C]\!] \text{ is vacant,} \\ \text{Every site of } [\![N-k+C, N]\!] \text{ is occupied,} \end{cases} \quad \text{in equilibrium}\ .$$

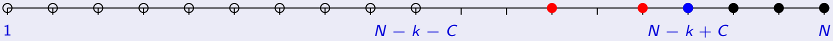


Figure: ○: empty sites ●●●: particles

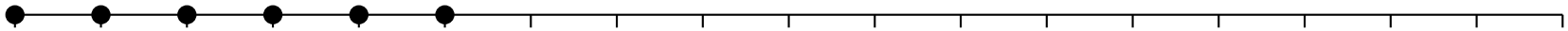


Figure: The minimal configuration $\xi_{\min}$

1° Mass transport cannot be faster than ballistic

The lower bound: $t_{\mathrm{mix}}^{N,k,\omega} = \Omega(N)$.

The time for $(\eta_t^{\mathsf{min}})_{t\geq 0}$ starting with $\xi_{\mathsf{min}} := \mathbf{1}_{\{1\leq x\leq k\}}$ to reach equilibrium.

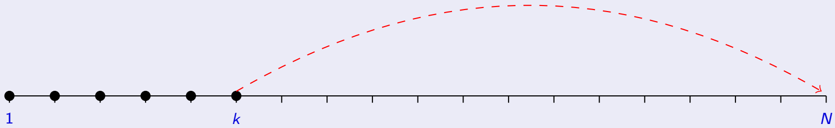
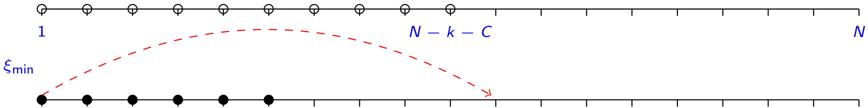


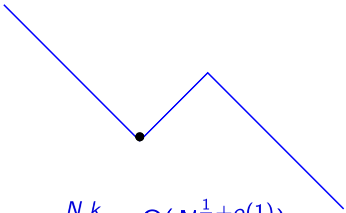
Figure: The configuration ξ_{min}.

2° The leftmost particle is blocked by traps in the potential profile



Since $\operatorname{ess\,inf}\omega_1 < 1/2$, $V^\omega(x) = \sum_{y=2}^{x} \log\left(\frac{1-\omega_y}{\omega_{y-1}}\right)$ is non-monotone and

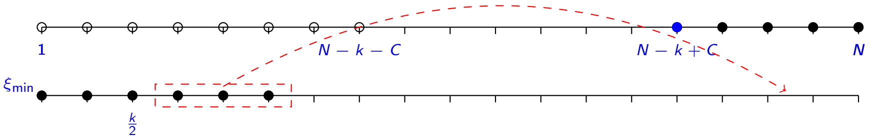
$$\max_{1\le x<y\le N/4} V^\omega(y) - V^\omega(x) \sim (1/\lambda)\log N.$$



Then

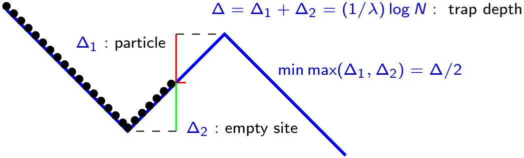
$$t_{\mathrm{mix}}^{N,k} = \Omega(N^{\frac{1}{\lambda}+o(1)}).$$

3° Potential barrier creates bottleneck for the particle flow



The time for a particle to flow out of the trap is roughly $N^{\frac{1}{2\lambda}}$, and then

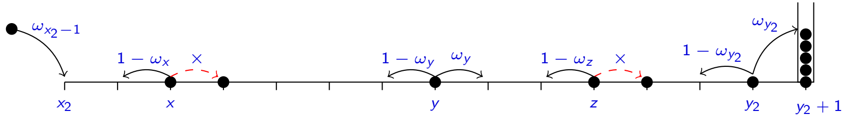
$$t_{\mathrm{mix}}^{N,k} = \Omega(N^{\beta+\frac{1}{2\lambda}+o(1)}).$$



Flow limitation: idea for the proof

- Deepest trap in $[\![N/2, 3N/4]\!]$: with $x_2(\omega) \leq y_2(\omega)$ as the two ends

$$V^\omega(y_2) - V^\omega(x_2) = \max_{N/2 \leq x \leq y \leq 3N/4} (V^\omega(y) - V^\omega(x)) \sim (1/\lambda) \log N.$$



- Partial order $\preccurlyeq$ on $\widetilde{\Omega}_{x_2,y_2}$:

$$\xi \preccurlyeq \xi' \quad \text{if} \quad \forall x \geq x_2, \quad \sum_{z=x}^{y_2+1} \xi(z) \leq \sum_{z=x}^{y_2+1} \xi'(z).$$

- Compare the dynamic in the interval $[s, s+t]$ with that starting from $\mathbf{0}$ at time s to obtain

$$\widetilde{\sigma}^{\mathbf{0}}_{s+t}(y_2+1) \geq \widetilde{\sigma}^{\mathbf{0}}_{s}(y_2+1) + (\vartheta_s \circ \widetilde{\sigma})^{\mathbf{0}}_t(y_2+1).$$

- Kingman's subbadditive ergodic Theorem:

$$\mathbf{E}\left[\widetilde{\sigma}_t^{\mathbf{0}}(y_2+1)\right] \leq t\left[\lim_{s\to\infty}\frac{1}{s}\widetilde{\sigma}_s^{\mathbf{0}}(y_2+1)\right] = t\lim_{n\to\infty}\frac{n}{\mathcal{T}_n},$$

where $\mathcal{T}_n$ is the instant of the n^{th} particle jumps to the site y_2+1.
- When there is a particle jumping onto the site y_2+1, we move all the particles in the segment $[\![x_2, y_2]\!]$ to the site y_2+1. (The state becomes $\mathbf{0}$.)

$$\lim_{n\to\infty}\frac{n}{\mathcal{T}_n} \leq (y_2-x_2+1)\frac{1}{\mathbf{E}[\mathcal{T}_1]}.$$

- Fill half of the trap and compare with a reversible M.C. with

$$\#\left\{x\in[\![x_2,y_2]\!] \ : \ V(x)\leq[V(y_2)+V(x_2)]/2\right\} \text{ particles.}$$

- To wait for site x_2 to be empty or site y_2 to be occupied by a particle

$$\mathcal{T}_1 \geq \mathcal{T}' := \left\{t\geq 0 : \widetilde{\sigma}'_t(x_2)=0 \ \text{ or } \ \widetilde{\sigma}'_t(y_2)=1\right\}.$$

Upper bound on the mixing time

- Coupling, Peres-Winkler inequality
- Flow method: a lower bound on the spectral gap
- Guide particles to the right

Idea for the upper bound

- A coupling: if $\xi \le \xi'$ (*i.e.* $\forall x$, $\sum_{i=x}^N \xi(i) \le \sum_{i=x}^N \xi'(i)$), then
$$\forall t \ge 0, \quad \eta_t^{\xi} \le \eta_t^{\xi'},$$
$$d_{N,k}^{\omega}(t) \le \max_{\xi,\xi'} \|P_t^{\xi} - P_t^{\xi'}\|_{\mathrm{TV}} \le \max_{\xi,\xi'} \mathbf{P}\left[\eta_t^{\xi} \ne \eta_t^{\xi'}\right].$$
- $\xi_{\max} := \mathbf{1}_{\{N-k+1 \le x \le N\}}$ $\quad (\eta_t^{\min})_{t\ge 0}$ starts from $\xi_{\min} = \mathbf{1}_{\{1\le x\le k\}}$,
$$\tau := \inf\left\{t \ge 0 : \eta_t^{\min} = \xi_{\max}\right\},$$
$$\forall t \ge \tau, \forall \xi, \xi' \quad \Rightarrow \quad \eta_t^{\xi} = \eta_t^{\xi'}.$$
By Markov property and stochastic domination,
$$d_{N,k}^{\omega}(nt) \le \mathbb{P}[\tau > nt] \le \mathbb{P}[\tau > t]^n \le \left(1 - \mathbb{P}\left[\eta_t^{\min} = \xi_{\max}\right]\right)^n.$$
- Censoring scheme $\mathcal{C} : [0,\infty) \mapsto \mathcal{P}(E)$ (E: edges in $[\![1, N]\!]$)
$(P_t^{\mathcal{C}})_{t\ge 0}$: a transition is performed if it doesn't cross an edge in $\mathcal{C}(t)$.
Peres-Winkler inequality: for any $\xi \in \Omega_{N,k}$ and any $\mathcal{C}$
$$P_t(\xi, \xi_{\max}) \ge P_t(\xi_{\min}, \xi_{\max}) \ge P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}).$$

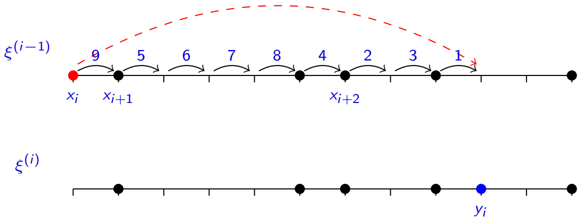
Flow method [gap : minimal nonzero eigenvalue of $-\mathcal{L}^{\omega}_{N,k}$]

$$\mathrm{gap} \geq \exp\left(-C(\alpha)N\right),$$

$$t_{\mathrm{mix}}^{N,k,\omega}(\varepsilon) \leq \mathrm{gap}^{-1} \log \frac{1}{2\varepsilon\mu_{\min}}.$$

Ground state ξ^* : $\pi^{\omega}_{N,k}(\xi^*) = \max_{\xi\in\Omega_{N,k}} \pi^{\omega}_{N,k}(\xi)$

Connect ξ with ξ^*: if $\xi(x_i) = 1 - \xi^*(x_i) = 1$ and $\xi(y_i) = 1 - \xi^*(y_i) = 0$. We move the particle at site x_i to y_i as follows:



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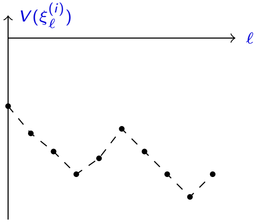

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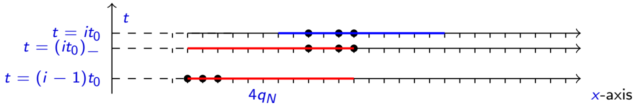
In this process, the potential V^{ω} can increase at most $c(\alpha)N$.

Guide particles to the right

- case $k \leq q_N = C(\mathbb{P}) \log N$:
  Run k particles in a segment of length $4q_N$.
  In equilibrium, all the k particles are in the right half $2q_N$ sites w.h.p.
  Let $\bar{\xi}(1)$ be the position of the leftmost particle. We have

$$\pi^{\omega}_{[x+1,x+4q_N],k}\left[\bar{\xi}(1) \leq x + 2q_N\right] \leq N^{-3}, \text{ for } \xi \in \Omega_{[x+1,x+4q_N],k}.$$



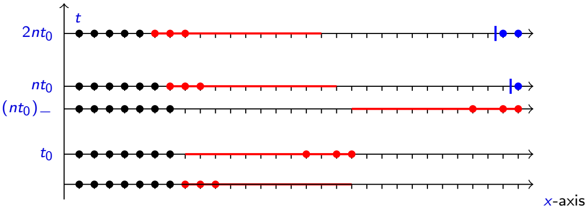
$t_0 = t^{4q_N,k,\omega}_{\mathrm{mix}}\left(1/N^3\right) \leq \exp(C(\alpha)4q_N)$.
$\#$ intersection between the red segment and the blue segment $= 2q_N$.

$$\lim_{\varepsilon \to 0} \inf_{\substack{N \geq 1 \\ k \in [\![1, N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}(\xi_{\max}) > \varepsilon\right] = 1.$$

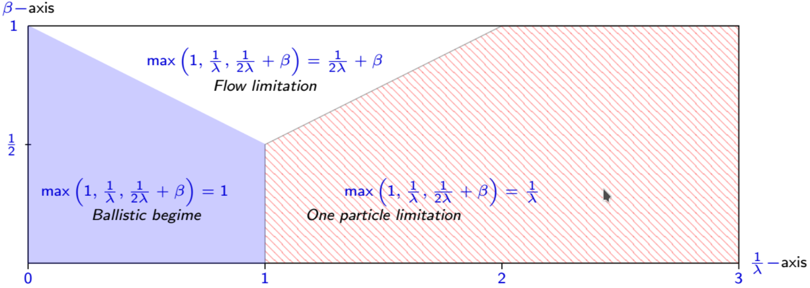
Guide particles to the right

- Case $k > q_N$: send the rightmost $k - q_N$ particles to the rightmost $k - q_N$ sites one by one.
  We repeat the first step for the leftmost q_N particles.

$$\pi^{\omega}_{[x+1, x+4q_N], k}\left[\xi(x + 4q_N) = 1\right] \geq 1 - N^{-3}.$$



The exponent of the mixing time with $k = N^{\beta+o(1)}$ particles



Thank you for your attention