Spectral gap and cutoff of Simple Exclusion Process with IID conductances

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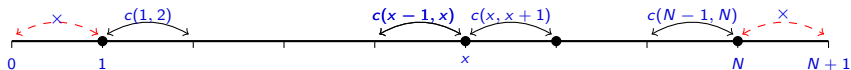Indian Institute of Technology Madras

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Setup: Simple Exclusion Process with inhomogeneous conductance

Setup: Simple Exclusion Process

Conductances: $(c(x, x+1))_{x \in \mathbb{N}}$ with values in $(0, \infty)$.

SEP with k particles in $[\![1, N]\!]$ with conductances $c(x, x+1)_{x \in \mathbb{N}}$.

(A) Each site is occupied by at most one particle (*the exclusion rule*).

(B) At each edge $\{x, x+1\}$ with $1 \leq x < N$, we place a Poisson clock with rate $c(x, x+1) > 0$. When a clock rings, we swap the contents of the two sites.

Setup

- State space (1: particle   0: empty site.)

$$\Omega_{N,k} := \left\{ \xi : [\![1, N]\!] \to \{0, 1\} \;\middle|\; \sum_{i=1}^{N} \xi(i) = k \right\}.$$

- Generator: $(f : \Omega_{N,k} \mapsto \mathbb{R})$

$$(\mathcal{L}_{N,k} f)(\xi) := \sum_{i=1}^{N-1} c(i, i+1) \left[f(\xi \circ \tau_{i,i+1}) - f(\xi) \right],$$

  where $\tau_{i,j}$ is the transposition of the two elements i and j.

- Uniform prob measure $\mu_{N,k}$ satisfies the detailed balance condition:

$$\mu(\xi) \mathcal{L}_{N,k}(\xi, \eta) = \mu(\eta) \mathcal{L}_{N,k}(\eta, \xi),$$

  then μ is the invariant prob measure.

Setup

- Distance to equilibrium

$$d_{N,k}(t) := \max_{\xi \in \Omega_{N,k}} \| P_t^\xi - \mu_{N,k} \|_{\mathrm{TV}} .$$

  P_t^ξ: marginal distribution at instant t of the chain starting with ξ.

- ε-mixing time

$$t_{\mathrm{mix}}^{N,k}(\varepsilon) := \inf \{ t \geq 0 : d_{N,k}(t) \leq \varepsilon \} .$$

- Cutoff: for all $\varepsilon \in (0,1)$,

$$\lim_{N \to \infty} \frac{t_{\mathrm{mix}}^{N,k}(\epsilon)}{t_{\mathrm{mix}}^{N,k}(1-\epsilon)} = 1 .$$
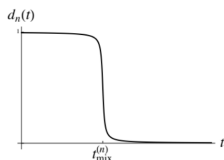

, image from Levin and Peres

Setup

- : Spectral gap $\mathrm{gap}_{N,k}$: minimal nonzero eigenvalue of $-\mathcal{L}_{N,k}$

$$\mathrm{gap}_{N,k} := \inf_{f \,:\, \mathrm{Var}_{\mu_{N,k}}(f) > 0} \frac{-\langle f, \mathcal{L}_{N,k} f \rangle_{\mu_{N,k}}}{\mathrm{Var}_{\mu_{N,k}}(f)}$$

  where $\mathrm{Var}_{\mu_{N,k}}(f) := \langle f, f \rangle_{\mu_{N,k}} - \langle f, \mathbf{1} \rangle_{\mu_{N,k}}^2$.

- Relation between spectral gap and mixing time/distance to equilibrium:

$$\frac{1}{\mathrm{gap}_{N,k}} \log \frac{1}{2\varepsilon} \;\le\; t_{\mathrm{mix}}^{N,k}(\varepsilon) \;\le\; \frac{1}{\mathrm{gap}_{N,k}} \log \frac{1}{2\varepsilon \mu_{\min}}$$

  where $\mu_{\min} := \min_{\xi \in \Omega_{N,k}} \mu_{N,k}(\xi)$.

$$\lim_{t \to \infty} \frac{1}{t} \log d_{N,k}(t) \;=\; -\mathrm{gap}_{N,k}\,.$$

Question: How does the disordered setup (inhomogeneous conductances) affect the system in terms of spectral gap/mixing time?

Previous Results

Homogeneous conductances $c(x, x+1) \equiv 1$ for one particle ($k = 1$)

- Spectral gap
$$\mathrm{gap}_{N,1} = 2(1 - \cos(\pi/N)) = (1 + o(1))\pi^2/N^2 \,.$$

- Eigenfunctions
$$g_i^{(N)}(x) := \cos(i\pi(x - 1/2)/N) \,, \quad 0 \leq i < N \,.$$

- Eigenvalues
$$-\lambda_i^{(N)} = -2\left(1 - \cos(i\pi/N)\right) \quad \mathcal{L}_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)} \,.$$

Homogeneous conductances $c(x, x+1) \equiv 1$ for many particles

- [Aldous, Wilson, Lacoin] Assuming $\liminf_{N \to \infty} \min(k, N - k) = \infty$,

$$t_{\min}^{N,k}(\varepsilon) = (1 + o(1))\frac{N^2}{2\pi^2} \log k \,, \quad \mathrm{gap}_{N,k} = \mathrm{gap}_{N,1} = (1 + o(1))\frac{\pi^2}{N^2} \,.$$

Previous results

Inhomogeneous conductance $c(x, x+1) > 0$

- Aldous' spectral gap conjecture (Proved by [Caputo, Liggett, Richthammer, JAMS '10]):

$$\mathrm{gap}_{N,k} = \mathrm{gap}_{N,1}.$$

- A function $f : [\![1, N]\!] \to \mathbb{R}$ for $2 \leq b \leq c \leq N-1$
  Local maximum at $[\![b, c]\!]$ if f is constant on $[\![b, c]\!]$, $f(b-1) < f(b)$ and $f(c) > f(c+1)$.
  Analogous definition holds for a local minimum.
  f is j-monotone if it displays exactly $(j-1)$ distinct local extrema in $[\![2, N-1]\!]$.
  Nodal domains:

$$\#\text{connected components of } \{x \in [\![1, N]\!], f(x) \neq 0\}.$$

- [Miclo]: $L_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)}$ with $0 = \lambda_0^{(N)} < \lambda_1^{(N)} < \cdots \lambda_{N-1}^{(N)}$
  $g_i^{(N)}$ is i-monotone and has $i+1$ nodal domains.

Our results

Proposition (Y. '24)

For any positive conductances $(c(x, x+1))_{x \in \mathbb{N}}$, $g_1^{(N)}$ is strictly monotone.

Write $r(x, x+1) := 1/c(x, x+1)$ and $r(n, m) := \sum_{x=n}^{m-1} r(x, x+1)$.
Assume (LLN) condition

$$\limsup_{N \to \infty} \frac{1}{N} \sup_{1 \leq n < m \leq N} |(r(n, m) - (m - n)| = 0, \qquad \textbf{(LLN)}$$

which is equivalent to

$$\limsup_{N \to \infty} \frac{1}{N} \sup_{2 \leq m \leq N} \left|(r^{(N)}(1, m) - (m - 1)\right| = 0.$$

When $(r^{(N)}(x-1, x))_{2 \leq x \leq N}$ is IID with expectation $\mathbb{E}[r(x, x+1)] = 1$, by the strong LLN we have

$$\mathbb{P}\left(\lim_{N \to \infty} \frac{1}{N} \max_{2 \leq m \leq N} |r(1, m) - (m - 1)| = 0\right) = 1.$$

Our results

Theorem (Y. '24)

If the (LLN) condition on the resistances holds, we have

$$\lim_{N \to \infty} \frac{N^2 \mathrm{gap}_N}{\pi^2} = 1 \,.$$

Furthermore, concerning the shape and (weighted) derivative of the eigenfunction g_1 with $g_1(1) := 1$ corresponding to the spectral gap, i.e. $\Delta^{(c)} g_1 = -\mathrm{gap}_N \cdot g_1$ and setting

$$h(x) := \cos\left(\frac{\pi(x - 1/2)}{N}\right), \quad \forall x \in [\![1, N]\!] \,,$$

we have $((c\nabla f)(x) := c(x-1, x)[f(x) - f(x-1)])$

$$\lim_{N \to \infty} \sup_{x \in [\![1, N]\!]} |g_1(x) - h(x)| = 0 \,,$$

$$\lim_{N \to \infty} \sup_{x \in [\![1, N]\!]} |N(c\nabla g_1)(x) - N(\nabla h)(x)| = 0 \,.$$

Our results

Remark

The method in the forementioned theorem also works for the other j-monotone eigenfunctions under the (LLN) assumption, i.e. with $K_0 \in \mathbb{N}$ being any prefixed constant, for all $1 \leq i \leq K_0$,

$$\lim_{N \to \infty} |\lambda_i N^2 / \pi^2 - i^2| = 0,$$

$$\lim_{N \to \infty} \sup_{x \in [\![1, N]\!]} \left| g_i(x) - \cos\left(\frac{i\pi(x - 1/2)}{N}\right) \right| = 0,$$

$$\lim_{N \to \infty} \sup_{x \in [\![1, N]\!]} |N(c\nabla g_i)(x) - N(\nabla h_i)(x)| = 0,$$

where $g_i(1) = 1$.

Our results: mixing time

Assumption

Exist constants $\upsilon \in (0,1)$ and $C_{\mathbb{P}} > 0$, a sequence of positive numbers $(\bar{\Upsilon}_N)_N > 0$ with $\lim_{N \to \infty} \bar{\Upsilon}_N = 0$ and $\lim_{N \to \infty} \bar{\Upsilon}_N \log N = \infty$ such that

$$\max_{1 \leq x < N} r(x, x+1) \leq C_{\mathbb{P}} \exp\left((\log N)^{\upsilon}\right),$$

$$\min_{1 \leq x < N} r(x, x+1) \geq \bar{\Upsilon}_N.$$

Exists $\varrho \in (0,1]$ and $c_{\varrho} > 0$ such that

$$c_{\varrho} N^{\varrho} \leq k_N \leq N/2.$$

Theorem (Y. '24)

Under (LLN) and the assumption above, for all $\varepsilon \in (0,1)$ we have

$$\lim_{N \to \infty} \frac{2\pi^2 t_{\mathrm{mix}}^{N,k}(\varepsilon)}{N^2 \log k_N} = 1.$$

Outline

- Idea for the $j-$monotonicity of eigenfunctions

- Idea for the spectral gap

- Idea for the shape & derivative of eigenfunction

- Idea for the lower bound on the mixing time

- Idea for the upper bound on the mixing time

Idea: $j-$monotonicity of eigenfunctions

Idea: j−monotonicity of eigenfunctions

$\mathcal{L}_{N,1}$ is a symmetric matrix. Then it is diagonalizable: $g_0 = \mathbf{1}$ and

$$\begin{cases} 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{N-1}, \\ \mathcal{L}_{N,1} g_i = -\lambda_i g_i \text{ and } g_i(1) = 1, & \forall\, i \in [\![0, N-1]\!], \\ \frac{1}{N} \sum_{x=1}^{N} g_i(x) g_j(x) = C_{i,j} \delta_{i,j}, & \forall\, i, j \in [\![0, N-1]\!]. \end{cases}$$

$\delta_{i,j}$: Kronecker delta $\quad (C_{i,i})_i$ are some positive constants. Observe:

$$(c \nabla g_i)(x+1) - (c \nabla g_i)(x) = -\lambda_i g_i(x) \Rightarrow$$
$$(c \nabla g_i)(x+1) = -\lambda_i \sum_{y=1}^{x} g_i(y).$$

Given $c(x, x+1)_x$, $g_i(1) = 1$ and λ_i together determine g_i, implying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}.$$

Assuming g_i is i-monotone, use the variational formula to show

$$g_1 \text{ is strictly monotone.}$$

$\mathrm{F}_i(\xi) := \sum_{x=1}^{N} g_i(x) \xi(x)$ is an eigenfunction of $\mathcal{L}_{N,k}$ with eigenvalue $-\lambda_i$. F_1 is monotone in the natural partial order $\Rightarrow \mathrm{gap}_{N,k} = \lambda_1$.

Idea: $j-$monotonicity of eigenfunctions

Setting $c(N, N+1) = 1$, for $\lambda > 0$, define $f^\lambda : [\![0, N+1]\!] \mapsto \mathbb{R}$ by $f^\lambda(0) = f^\lambda(1) = 1$ and for $x \in [\![1, N]\!]$,

$$f^\lambda(x+1) \ = \ f^\lambda(x) + \frac{1}{c(x, x+1)} \left[(c\nabla f^\lambda)(x) - \lambda f^\lambda(x) \right].$$

Note that (the restriction to $[\![1, N]\!]$ of) f^λ is an eigenfunction of $\mathcal{L}_{N,1}$ if and only if

$$f^\lambda(N+1) \ = \ f^\lambda(N).$$

There is no eigenfunction satisfying $f^\lambda(1) = 0$ or $f^\lambda(N) = 0$.

Idea: j−monotonicity of eigenfunctions

For $\lambda > 0$ and $x \in [\![1, N+1]\!]$, we set

$$b(\lambda, x) := -\frac{(c\nabla f^\lambda)(x)}{f^\lambda(x-1)}$$

convention: $b(\lambda, x) = \overline{\infty}$ if $f^\lambda(x-1) = 0$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\overline{\infty}\}$. We have

$$b(\lambda, x+1) = \frac{b(\lambda, x)}{1 - c(x-1, x)^{-1}b(\lambda, x)} + \lambda \,.$$

Given a fixed $c > 0$, define $\Xi^{(c)} : \mathbb{R} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ as

$$\Xi^{(c)}(\lambda, b) = \frac{b}{1 - c^{-1}b} + \lambda \,.$$

The function $b \mapsto \Xi^{(c)}(\lambda, b)$ may have zero, one or two fixed points depending on the values of λ and c, see the following figure.

Idea: j−monotonicity of eigenfunctions

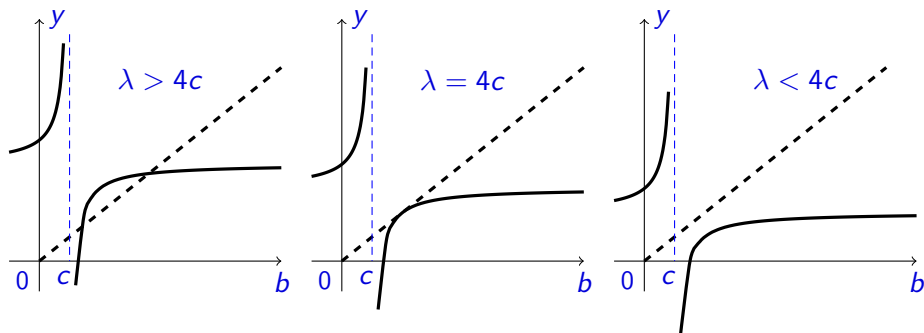$$\Xi^{(c)}(\lambda, b) = \frac{b}{1 - c^{-1}b} + \lambda.$$

Figure: Solid lines: $b \mapsto \Xi^{(c)}(b, \lambda)$ with $\lambda > 0$ fixed. Black dashed lines: $y = b$. Blue dashes lines: $b = c$.

If $b \mapsto \Xi^{(c)}(\lambda, b)$ has fixed points b_1 and b_2 (not necessarily distinct) such that $b_1 \leq b_2$, we define $I(\lambda, c) = [b_1, b_2]$. Otherwise, set $I(\lambda, c) = \varnothing$.

Idea: j−monotonicity of eigenfunctions

Define the "angle mapping" function

$$\varphi(c, \lambda, \theta) := \inf\{\theta' \geq \theta + \pi \mathbf{1}_{I(\lambda, c)}(\tan \theta) \ : \ \tan \theta' = \Xi^{(c)}(\lambda, \tan \theta)\}.$$

Recursively define an "angle" $\theta(\lambda, x)$: $\theta(\lambda, 1) = 0$ and for $x \in [\![1, N]\!]$,

$$\theta(\lambda, x + 1) := \varphi(c(x - 1, x), \lambda, \theta(\lambda, x))$$

with convention: $\tan(\pi/2 + k\pi) = \overline{\infty}$ for $k \in \mathbb{Z}$.

Lemma

For fixed $c, \lambda > 0$, the map $\theta \mapsto \varphi(c, \lambda, \theta)$ is continuous and strictly increasing.

Lemma

For fixed $c, \theta > 0$, the map $\lambda \mapsto \varphi(c, \lambda, \theta)$ is strictly increasing and uniformly continuous in θ.

Idea: j−monotonicity of eigenfunctions

Lemma

For fixed $c > 0$, the map $(\lambda, \theta) \mapsto \varphi(c, \lambda, \theta)$ is jointly continuous.

f^λ is an eigenfunction if and only if $\theta(\lambda, N + 1)$ is a multiple of π.

$$f^\lambda \text{ is an eigenfunction} \quad \Leftrightarrow \quad \theta(\lambda, N + 1) = k\pi \text{ for } k \in [\![0, N - 1]\!].$$

Let $\lambda_k > 0$ denote the unique number satisfying $\theta(\lambda_k, N + 1) = k\pi$ and set $f_k := f^{\lambda_k}$. Let $x_i \in [\![1, N]\!]$ such that $\theta(\lambda_k, x_i) \le i\pi < \theta(\lambda_k, x_i + 1)$ for $i \in [\![1, k - 1]\!]$.

Lemma

For λ_k mentioned above and the associated sequence $(x_i)_i$, we have that $\# \{(x_i)_i\} = k - 1$, $1 < x_i < N$ are the local extrema of f_k (or the pair $\{x_i - 1, x_i\}$ when $\theta(\lambda_k, x_i) = i\pi$) and no any other local extrema.

Idea: the spectral gap

Idea for the spectral gap

Setting $B^{(N)}(x) := b(\lambda, x)N$ and $\lambda := \alpha/N^2$, we have

$$B^{(N)}(x+1) = \frac{B^{(N)}(x)}{1 - N^{-1}r^{(N)}(x-1,x)B^{(N)}(x)} + \frac{\alpha}{N},$$

which starts from $B^{(N)}(1) := 0$.

$$N[B(x+1) - B(x)] = \frac{r(x-1,x)B(x)^2}{1 - B(x)r(x-1,x)N^{-1}} + \alpha$$

Intuition: the asymptotic ODE

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + \alpha, & x \in (0,1) \\ y(0) = 0. \end{cases}$$

Its unique solution:

$$y(x) = \sqrt{\alpha}\tan\left(\sqrt{\alpha} \cdot x\right).$$

Therefore $\alpha = i^2\pi^2$.

Idea: shape/derivative of the eigenfunction

Idea for shape of the eigenfunction

Eigenfunction corresponding to the spectral gap when $r(j-1, j) \equiv 1$

$$h(x) = h_N(x) = \cos\left(\frac{\pi(x - 1/2)}{N}\right).$$

The spectral gap is

$$\overline{\lambda} := 2\left[1 - \cos\left(\frac{\pi}{N}\right)\right] = \frac{\pi^2}{N^2} + O\left(\frac{1}{N^4}\right).$$

By $b(\lambda, x) = -\frac{(c\nabla g)(x)}{g(x-1)}$ and $b(\lambda, x) = B(x)/N$, for $x \geq 2$ we have

$$g(x) = \left[1 - r(x-1, x)N^{-1}B(x)\right]g(x-1).$$

Writing $u(x) := h(x) - g(x)$, we have

$$u(x) = u(x-1)\left[1 - \frac{r(x-1, x)B(x)}{N}\right]$$
$$+ \frac{h(x-1)}{N}\left[r(x-1, x)B(x) - \overline{B}(x)\right].$$

Iterate the equation above to conclude the proof for $x \leq N/3$.

Idea for the derivative of the eigenfunction

For $x \leq N/3$: by

$$(c\nabla g_i)(x+1) = -\lambda_i \sum_{y=1}^{x} g_i(y),$$

we have

$$|N(c\nabla g)(x) - N(\nabla h)(x)|$$

$$= \left| -N\lambda_1 \sum_{k=1}^{x-1} g(k) + N\overline{\lambda} \sum_{k=1}^{x-1} h(k) \right|$$

$$\leq \left| -N\lambda_1 \sum_{k=1}^{x-1} [g(k) - h(k)] \right| + \left| N(\overline{\lambda} - \lambda_1) \sum_{k=1}^{x-1} h(k) \right|$$

$$\leq N\lambda_1 \sum_{k=1}^{x-1} |[g(k) - h(k)]| + N|\overline{\lambda} - \lambda_1|(x-1).$$

For $N/3 \leq x \leq 2N/3$: use $A(x) = 1/B(x)$

$$A(x) \left[1 - \frac{g(x)}{g(x-1)} \right] = r(x-1, x)N^{-1}.$$

Idea: the lower bound on the mixing time

Idea for the lower bound on the mixing time

$$\mathrm{F}(\xi) = \mathrm{F}_1(\xi) = \sum_{1 \leq x \leq N} \xi(x) g_1(x)$$

is an eigenfunction satisfying $\mathcal{L}_{N,k} \mathrm{F} = -\mathrm{gap}_N \cdot \mathrm{F}$. For $t_0 > 0$, define

$$F(t, \xi) := e^{\lambda_1(t-t_0)} \mathrm{F}(\xi), \quad \forall \, \xi \in \Omega_{N,k},$$

and study the Dynkin martingale

$$M_t := F(t, \eta_t^\nu) - F(0, \eta_0^\nu) - \int_0^t (\partial_s + \mathcal{L}_{N,k}) F(s, \eta_s^\nu) \, \mathrm{d}s.$$

$$\mathbf{E}\left[\mathrm{F}\left(\eta_{t_0}^\nu\right)\right] = \mathbf{E}\left[F\left(t_0, \eta_{t_0}^\nu\right)\right] = \mathbf{E}\left[F(0, \eta_0^\nu)\right] = e^{-\lambda_1 t_0} \mathbf{E}\left[\mathrm{F}\left(\eta_0^\nu\right)\right].$$

$$\mathbf{E}[M_{t_0}^2] = \mathbf{E}[\int_0^{t_0} \partial_s \langle M \rangle_s \mathrm{d}s]$$

$$\overline{\eta}_t^\nu(x, x+1) := \eta_t^\nu(x)\left(1 - \eta_t^\nu(x+1)\right) + \eta_t^\nu(x+1)\left(1 - \eta_t^\nu(x)\right)$$

$$\partial_t \langle M. \rangle_t = e^{2\lambda_1(t-t_0)} \sum_{x=1}^{N-1} \overline{\eta}_t^\nu(x, x+1) r(x, x+1) \left[c(x, x+1)(g(x) - g(x+1))\right]^2.$$

Idea for the lower bound on the mixing time

At equilibrium

$$\mathbf{E}\left[\mathrm{F}(\eta_{t_0}^{\mu})\right] = \mu_{N,k}(\mathrm{F}) = \frac{k}{N} \sum_{1 \le x \le N} g_1(x) = 0, \qquad \mathrm{Var}_{\mu}(\mathrm{F}) \asymp k.$$

1. If ν concentrates at one configuration, then
$$\mathbf{E}\left[\mathrm{F}(\eta_{t_0}^{\nu}) - \mathbf{E}\left[\mathrm{F}(\eta_{t_0}^{\nu})\right]\right]^2 = \mathbf{E}\left[F(t_0, \eta_{t_0}^{\nu}) - F(0, \eta_0^{\nu})\right]^2 = \mathbf{E}\left[M_{t_0}^2\right].$$

2. If ν is non-degenerated, we have
$$\mathbf{E}\left[\mathrm{F}(\eta_{t_0}^{\nu}) - \mathbf{E}\left[\mathrm{F}(\eta_{t_0}^{\nu})\right]\right]^2$$
$$= \mathbf{E}\left[F(t_0, \eta_{t_0}^{\nu}) - F(0, \eta_0^{\nu}) + F(0, \eta_0^{\nu}) - \mathbf{E}\left[F(0, \eta_0^{\nu})\right]\right]^2$$
$$\le 2\mathbf{E}\left[M_{t_0}^2\right] + 2\mathbf{E}\left[F(0, \eta_0^{\nu}) - \mathbf{E}\left[F(0, \eta_0^{\nu})\right]\right]^2.$$

If $N/64 \le k \le N/2$, take $\nu = \delta_{\wedge}$.
If $(\log N)^{1+\gamma} \le k < N/64$, take ν as follows: sample a configuration according to $\mu_{N,2k}$, keep the first k particles and project the rest to be empty sites.

Idea: the upper bound on the mixing time

Idea: the upper bound on the mixing time

Height function:

$$\xi \in \Omega_{N,k} \quad \rightarrow \quad h^\xi(x) := \sum_{y=1}^x \xi(y) - \frac{k}{N}x.$$

A partial order:

$$(\xi \leq \xi') \quad \Leftrightarrow \quad \left(h^\xi(x) \leq h^{\xi'}(x), \forall x \in [\![1, N]\!] \right).$$

Attractive:

$$\left(h^\xi \leq h^{\xi'} \right) \quad \Rightarrow \quad \left(\forall\, t \geq 0,\; h_t^\xi \leq h_t^{\xi'} \right).$$

Coalescing time:

$$T_1 := \inf \left\{ t \geq 0 : h_t^\wedge = h_t^\mu \right\},$$
$$T_2 := \inf \left\{ t \geq 0 : h_t^\vee = h_t^\mu \right\}.$$

Idea: the upper bound on the mixing time

Construct a supermartingale: inspired by [Wilson, '04], embed the segment $[\![1, N]\!]$ in $[\![-\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor]\!]$ and place conductance $(c(x, x+1) = 1)_{x \notin [1, N-1]}$. The principle eigenfunction satisfies:

$$\lim_{N \to \infty} \sup_{x \in [\![-\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor]\!]} \left| G(x) - \cos\left(\frac{\pi(x + \lfloor \delta N \rfloor + 1/2)}{\bar{N}}\right) \right| = 0.$$

Define $\bar{G}(x) := G(x) - G(x+1) > 0$ and

$$\mathbf{F}(\xi) := \sum_{x=1}^{N-1} h^\xi(x) \bar{G}(x).$$

For $\xi, \xi' \in \Omega_{N,k}$ with $\xi \leq \xi'$, since $h^\xi(x) \leq h^{\xi'}(x)$ and $\bar{G}(x) > 0$,

$$\mathbf{F}(\xi) \leq \mathbf{F}(\xi').$$

Furthermore, if $\xi \leq \xi'$ with $\xi \neq \xi'$, we have $\mathbf{F}(\xi) < \mathbf{F}(\xi')$.

Idea: upper bound on the mixing time

Using $h^\xi(0) = h^\xi(N) = 0$ and for $x \in [\![1, N-1]\!]$

$$
\left(\mathcal{L}_{N,k} h^\xi\right)(x) = c(x, x+1)\left[\xi(x+1) - \xi(x)\right]
$$
$$
= c(x, x+1)\left[\left(h^\xi(x+1) - h^\xi(x)\right) - \left(h^\xi(x) - h^\xi(x-1)\right)\right]
$$

we obtain

$$
\left(\mathcal{L}_{N,k}\mathbf{F}\right)(\xi) = \sum_{x=1}^{N-1} \bar{G}(x)\left(\mathcal{L}_{N,k}h^\xi\right)(x)
$$
$$
= -\bar{\lambda}_1\mathbf{F}(\xi) - h^\xi(1)c(0,1)\bar{G}(0) - h^\xi(N-1)c(N, N+1)\bar{G}(N)
$$

where $\bar{\lambda}_1$ is the spectral gap of the system in the longer line segment.

Idea: upper bound on the mixing time

For $\xi \leq \xi'$,

$$\left(\mathcal{L}_{N,k}\mathbf{F}\right)(\xi') - \left(\mathcal{L}_{N,k}\mathbf{F}\right)(\xi) = -\bar{\lambda}_1\left[\mathbf{F}(\xi') - \mathbf{F}(\xi)\right]$$
$$-\left[h^{\xi'}(1) - h^{\xi}(1)\right] c(0,1)\bar{G}(0) - \left[h^{\xi'}(N-1) - h^{\xi}(N-1)\right] c(N, N+1)\bar{G}(N)$$
$$\leq -\bar{\lambda}_1\left[\mathbf{F}(\xi') - \mathbf{F}(\xi)\right] .$$

Then $(\mathbf{F}(h_t^\wedge) - \mathbf{F}(h_t^\mu))_{t\geq 0}$ is a supermartingale with decay rate $\bar{\lambda}_1$.

Combine the approaches in [Lacoin AOP'16] and [Labbé, Lacoin AAP'20] to adapt to the disordered setup to conclude the proof.