

# Mixing Time of ASEP in a random environment

Shangjie Yang



Seminário de probabilidade do IME-USP

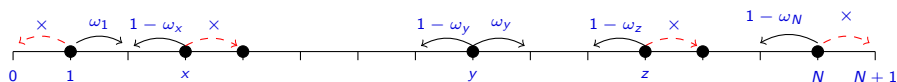
Jointed work with Hubert Lacoin (IMPA, Rio de Janeiro)

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# Setup

Environment:  $\omega = (\omega_x)_x$  with values in  $(0, 1)$

Exclusion process with  $k$  particles in  $\llbracket 1, N \rrbracket$  with environment  $\omega$



- (A) Each site is occupied by at most one particle (*the exclusion rule*).
- (B) Each of the  $k$  particles independently performs a random walk such that a particle at site  $x \in \llbracket 1, N \rrbracket$

$$\begin{cases} \text{jumps to site } x+1 \text{ at rate } \omega_x & \text{for } x \leq N-1, \\ \text{jumps to site } x-1 \text{ at rate } 1-\omega_x & \text{for } x \geq 2, \end{cases}$$

if the target site is not occupied. (No particle is allowed to jump out of the segment.)

# Setup

- State space (1: particle      0: empty site.)

$$\Omega_{N,k} := \left\{ \xi \in \{0,1\}^N : \sum_{x=1}^N \xi(x) = k \right\}.$$

- $\xi^{x,y}$ : swap the contents of sites  $x, y$  of  $\xi$

$$\forall z \in \llbracket 1, N \rrbracket, \quad \xi^{x,y}(z) = \xi(z) \mathbf{1}_{\llbracket 1, N \rrbracket \setminus \{x,y\}} + \xi(x) \mathbf{1}_{\{y\}} + \xi(y) \mathbf{1}_{\{x\}}.$$

- Transition rates: (for  $x \in \llbracket 1, N-1 \rrbracket$ )

$$r^\omega(\xi, \xi^{x,x+1}) := \begin{cases} \omega_x & \text{if } \xi(x) = 1 \text{ and } \xi(x+1) = 0, \\ 1 - \omega_{x+1} & \text{if } \xi(x+1) = 1 \text{ and } \xi(x) = 0, \\ r^\omega(\xi, \xi') := 0 & \text{in all other cases.} \end{cases}$$

# Setup

- Generator ( $f : \Omega_{N,k} \rightarrow \mathbb{R}$ )

$$(\mathcal{L}_{N,k}^\omega f)(\xi) := \sum_{x=1}^{N-1} r^\omega(\xi, \xi^{x,x+1}) [f(\xi^{x,x+1}) - f(\xi)].$$

- Potential in the environment  $\omega$ :  $V^\omega(1) := 0$ , and for  $x \geq 2$

$$V^\omega(x) := \sum_{y=2}^x \log \left( \frac{1 - \omega_y}{\omega_{(y-1)}} \right).$$

Potential for  $\xi \in \Omega_{N,k}$ :

$$V^\omega(\xi) := \sum_{x=1}^N V^\omega(x) \xi(x).$$

- Invariant probability measure

$$\pi_{N,k}^\omega(\xi) := \frac{1}{Z_{N,k}^\omega} e^{-V^\omega(\xi)} \quad \text{with} \quad Z_{N,k}^\omega := \sum_{\xi' \in \Omega_{N,k}} e^{-V^\omega(\xi')}.$$

Detailed balance condition

$$\pi_{N,k}^\omega(\xi) r^\omega(\xi, \xi^{x,x+1}) = \pi_{N,k}^\omega(\xi^{x,x+1}) r^\omega(\xi^{x,x+1}, \xi).$$

# Assumptions

- $k = N^{\beta+o(1)} \leq \frac{1}{2}N$  particles where  $\beta \in (0, 1]$ .
- $\omega = (\omega_x)_x$  is IID ( law:  $\mathbb{P}$ , expectation:  $\mathbb{E}$ ) satisfying:  
(1) Uniform ellipticity condition:  $\exists \alpha \in (0, 1/2)$  such that

$$\mathbb{P}(\omega_1 \in [\alpha, 1 - \alpha]) = 1.$$

(2)

$$\mathbb{E} \left[ \log \frac{1 - \omega_1}{\omega_1} \right] < 0,$$

so that the random walk on  $\mathbb{Z}$  is transient to the right.

- Distance to equilibrium & Mixing time:

$$d_{N,k}^\omega(t) := \max_{\xi \in \Omega_{N,k}} \|P_t^\xi - \pi_{N,k}^\omega\|_{\text{TV}},$$

$$t_{\text{mix}}^{N,k,\omega}(\varepsilon) := \inf \{ t \geq 0 : d_{N,k}^\omega(t) \leq \varepsilon \},$$

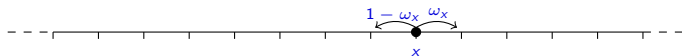
$$t_{\text{mix}}^{N,k,\omega} := t_{\text{mix}}^{N,k,\omega}(1/4).$$

**Q:** How does the mixing time  $t_{\text{mix}}^{N,k,\omega}$  grow in terms of  $N$  and  $k$  for typical realization of  $\omega$ ?

# Outline

- (1) Related results (RWRE, SEP with  $\omega \equiv p$  or  $\omega = (\omega_x)_x$  IID).
- (2) Our result:  $t_{\text{mix}}^{N,k,\omega}$  grows like a power of  $N$ .
- (4) Heuristic for the lower bounds: three mechanisms.
- (4) Idea for the upper bound.

## Related result: random walk in random environment on $\mathbb{Z}$



Given  $(\omega_x)_x, (X_t)_{t \geq 0}$ : a continuous-time random walk on  $\mathbb{Z}$  starting at 0. [Solomon '75 AOP] showed that

$$\begin{cases} \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] = 0 \Rightarrow (X_t)_{t \geq 0} \text{ is recurrent,} \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0 \Rightarrow \lim_{t \rightarrow \infty} X_t = \infty, \\ \mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] > 0 \Rightarrow \lim_{t \rightarrow \infty} X_t = -\infty. \end{cases}$$

Assuming  $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$ , set

$$\lambda = \lambda_{\mathbb{P}} := \inf \left\{ s > 0, \mathbb{E} \left[ \left( \frac{1-\omega_1}{\omega_1} \right)^s \right] \geq 1 \right\} \in (0, \infty].$$

[Kesten, Kozlov, Spitzer '75 Compos. Math.] showed that

$$\begin{cases} \lim_{t \rightarrow \infty} \frac{X_t}{t} = \vartheta_{\mathbb{P}} > 0, & \text{if } \lambda > 1 \text{ (ballistic),} \\ \lim_{t \rightarrow \infty} \frac{\log(X_t)}{\log t} = \lambda, & \text{if } \lambda \in (0, 1] \text{ (subballistic).} \end{cases}$$

## Related results: many particles in homogenous environment

SSEP ( $\omega \equiv \frac{1}{2}$ ): [Aldous '83 Lect. Notes Math.], [Wilson '04 AAP], [Lacoin '16 AOP]

$$t_{\text{mix}}^{N,k_N} \asymp (N^2 \log k_N) = O(t_{\text{mix}}^{N,1} \log N).$$

ASEP ( $\omega \equiv p \neq \frac{1}{2}$ ): [Benamini *et al.* '05 TAMS], [Labbé, Lacoin '19 AOP]

$$t_{\text{mix}}^{N,k_N} \asymp N.$$



## Related result: one particle in random environment

When  $k = 1$ , [Gantert, Kochler '18 ALEA] showed that if  $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$ ,

$$\begin{cases} t_{\text{mix}}^{N,1,\omega}(\varepsilon) = [C(\mathbb{P}) + o(1)]N, & \text{if } \lambda_{\mathbb{P}} > 1, \\ \lim_{N \rightarrow \infty} \frac{\log t_{\text{mix}}^{N,1,\omega}}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}, & \text{if } \lambda_{\mathbb{P}} \in (0, 1]. \end{cases}$$

Potential  $V^\omega : \mathbb{N} \rightarrow \mathbb{R}$

$$V^\omega(x) := \begin{cases} 0, & \text{for } x = 1, \\ \sum_{y=2}^x \log \left( \frac{1-\omega_y}{\omega_{y-1}} \right), & \text{for } x \geq 2. \end{cases}$$

The largest potential barrier:  $\max_{1 \leq x < y \leq N} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N$ .

## Related result: many particles in random environment

Assuming  $\lambda > 1$  and  $\lim_{N \rightarrow \infty} k_N/N = \theta \in (0, 1)$ , [Schmid '19 EJP] showed:

- When  $\text{ess inf } \omega_1 > 1/2$ ,  $t_{\text{mix}}^{N, k_N, \omega} \asymp N$  by comparison.
- When  $\text{ess inf } \omega_1 < 1/2$ ,  $t_{\text{mix}}^{N, k_N, \omega} \geq N^{1+\delta}$  for some  $\delta > 0$ .
- When  $\text{ess inf } \omega_1 = 1/2$ , then

$$\liminf_{N \rightarrow \infty} t_{\text{mix}}^{N, k_N, \omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\text{mix}}^{N, k_N, \omega}(\varepsilon) \leq CN(\log N)^3,$$

together with a quantitative lower bound if  $\mathbb{P}[\omega_1 = 1/2] > 0$ .

**Q:** If  $\text{ess inf } \omega_1 < 1/2$ , how does  $t_{\text{mix}}^{N, k_N, \omega}$  grow?

# Our result

## Theorem (Lacoin, Y.)

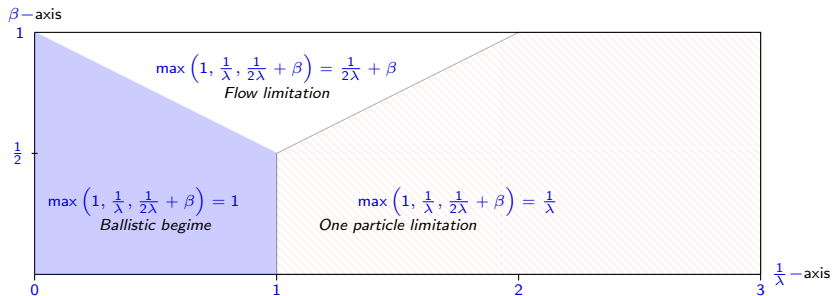
*Assuming  $\mathbb{E}[\log \frac{1-\omega_1}{\omega_1}] < 0$ ,  $\text{ess inf } \omega_1 < \frac{1}{2}$ ,  $k = N^{\beta+o(1)} \leq N/2$  with  $\beta \in (0, 1]$  and the uniform ellipticity condition, with high probability we have*

$$c(\alpha, \mathbb{P}) N^{\max(1, \frac{1}{\lambda}, \beta + \frac{1}{2\lambda}) + o(1)} \leq t_{\text{mix}}^{N, k, \omega} \leq N^{C(\alpha, \mathbb{P})}.$$

Conjecture: Our lower bound is sharp.

# Phase diagram (Conjecture)

The exponent of the mixing time with  $k = N^{\beta+o(1)}$  particles



## Lower bound on the mixing time: three mechanisms

- Mass transport cannot be faster than ballistic
- The leftmost particle is blocked by traps in the potential profile
- Potential barrier creates bottleneck for the particle flow

# Typical configurations in equilibrium

$\left\{ \begin{array}{l} \text{Every site of } [1, N - k - C] \text{ is vacant,} \\ \text{Every site of } [N - k + C, N] \text{ is occupied,} \end{array} \right.$ 
 in equilibrium .

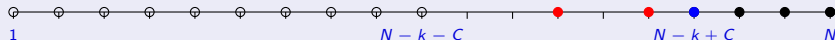


Figure: ○: empty sites      ●●●: particles



Figure: The minimal configuration  $\xi_{\min}$

# 1° Mass transport cannot be faster than ballistic

The lower bound:  $t_{\text{mix}}^{N,k,\omega} = \Omega(N)$ .

The time for  $(\eta_t^{\min})_{t \geq 0}$  starting with  $\xi_{\min} := \mathbf{1}_{\{1 \leq x \leq k\}}$  to reach equilibrium.

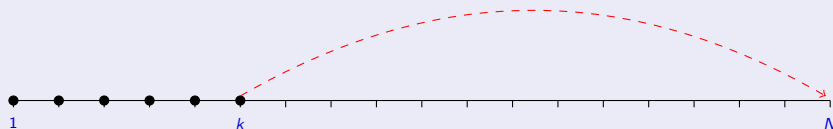
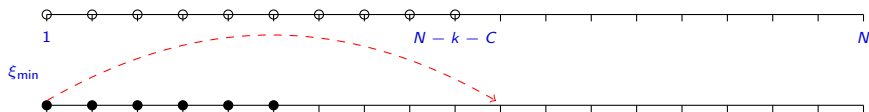


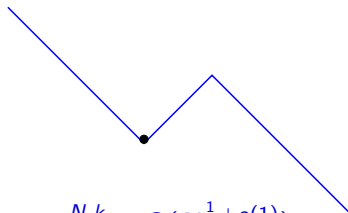
Figure: The configuration  $\xi_{\min}$ .

2° The leftmost particle is blocked by traps in the potential profile



Since  $\text{ess inf } \omega_1 < 1/2$ ,  $V^\omega(x) = \sum_{y=2}^x \log \left( \frac{1-\omega_y}{\omega_{y-1}} \right)$  is non-monotone and

$$\max_{1 \leq x < y \leq N/4} V^\omega(y) - V^\omega(x) \sim (1/\lambda) \log N.$$

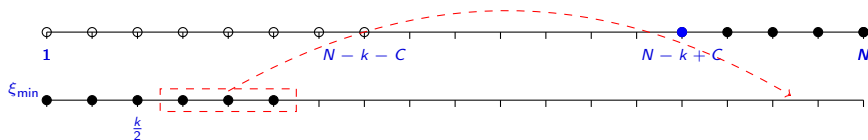


Then

$$t_{\text{mix}}^{N,k} = \Omega(N^{\frac{1}{\lambda} + o(1)}).$$

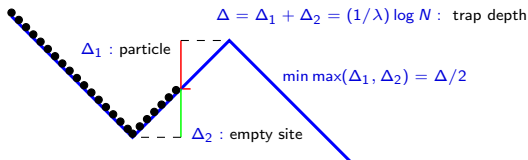


### 3° Potential barrier creates bottleneck for the particle flow



The time for a particle to flow out of the trap is roughly  $N^{\frac{1}{2\lambda}}$ , and then

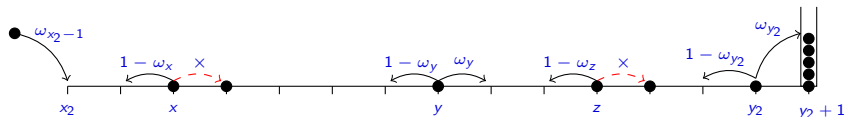
$$t_{\text{mix}}^{N,k} = \Omega(N^{\beta + \frac{1}{2\lambda} + o(1)}).$$



## Flow limitation: idea for the proof

- Deepest trap in  $\llbracket N/2, 3N/4 \rrbracket$ : with  $x_2(\omega) \leq y_2(\omega)$  as the two ends

$$V^\omega(y_2) - V^\omega(x_2) = \max_{N/2 \leq x \leq y \leq 3N/4} (V^\omega(y) - V^\omega(x)) \sim (1/\lambda) \log N.$$



- Partial order  $\preceq$  on  $\tilde{\Omega}_{x_2, y_2}$ :

$$\xi \preceq \xi' \quad \text{if} \quad \forall x \geq x_2, \quad \sum_{z=x}^{y_2+1} \xi(z) \leq \sum_{z=x}^{y_2+1} \xi'(z).$$

- Compare the dynamic in the interval  $[s, s + t]$  with that starting from  $\mathbf{0}$  at time  $s$  to obtain

$$\tilde{\sigma}_{s+t}^{\mathbf{0}}(y_2 + 1) \geq \tilde{\sigma}_s^{\mathbf{0}}(y_2 + 1) + (\vartheta_s \circ \tilde{\sigma})_t^{\mathbf{0}}(y_2 + 1).$$

- Kingman's subadditive ergodic Theorem:

$$\mathbf{E} \left[ \tilde{\sigma}_t^0(y_2 + 1) \right] \leq t \left[ \lim_{s \rightarrow \infty} \frac{1}{s} \tilde{\sigma}_s^0(y_2 + 1) \right] = t \lim_{n \rightarrow \infty} \frac{n}{\mathcal{T}_n},$$

where  $\mathcal{T}_n$  is the instant of the  $n^{\text{th}}$  particle jumps to the site  $y_2 + 1$ .

- When there is a particle jumping onto the site  $y_2 + 1$ , we move all the particles in the segment  $[[x_2, y_2]]$  to the site  $y_2 + 1$ . (The state becomes **0**.)

$$\lim_{n \rightarrow \infty} \frac{n}{\mathcal{T}_n} \leq (y_2 - x_2 + 1) \frac{1}{\mathbf{E}[\mathcal{T}_1]}.$$

- Fill half of the trap and compare with a reversible M.C. with

$$\# \{x \in [[x_2, y_2]] : V(x) \leq [V(y_2) + V(x_2)]/2\} \text{ particles.}$$

- To wait for site  $x_2$  to be empty or site  $y_2$  to be occupied by a particle

$$\mathcal{T}_1 \geq \mathcal{T}' := \{t \geq 0 : \tilde{\sigma}'_t(x_2) = 0 \text{ or } \tilde{\sigma}'_t(y_2) = 1\}.$$

# Upper bound on the mixing time

- Coupling, Peres-Winkler inequality
- Flow method: a lower bound on the spectral gap
- Guide particles to the right

## Idea for the upper bound

- A coupling: if  $\xi \leq \xi'$  ( i.e.  $\forall x, \sum_{i=x}^N \xi(i) \leq \sum_{i=x}^N \xi'(i)$ ), then

$$\forall t \geq 0, \quad \eta_t^\xi \leq \eta_t^{\xi'},$$

$$d_{N,k}^\omega(t) \leq \max_{\xi, \xi'} \|P_t^\xi - P_t^{\xi'}\|_{\text{TV}} \leq \max_{\xi, \xi'} \mathbf{P} \left[ \eta_t^\xi \neq \eta_t^{\xi'} \right].$$

- $\xi_{\max} := \mathbf{1}_{\{N-k+1 \leq x \leq N\}}$  ( $\eta_t^{\min}$ ) $_{t \geq 0}$  starts from  $\xi_{\min} = \mathbf{1}_{\{1 \leq x \leq k\}}$ ,

$$\tau := \inf \{ t \geq 0 : \eta_t^{\min} = \xi_{\max} \},$$

$$\forall t \geq \tau, \forall \xi, \xi' \Rightarrow \eta_t^\xi = \eta_t^{\xi'}.$$

By Markov property and stochastic domination,

$$d_{N,k}^\omega(nt) \leq \mathbb{P} [\tau > nt] \leq \mathbb{P} [\tau > t]^n \leq (1 - \mathbb{P} [\eta_t^{\min} = \xi_{\max}])^n.$$

- Censoring scheme  $\mathcal{C} : [0, \infty) \mapsto \mathcal{P}(E)$  ( $E$ : edges in  $\llbracket 1, N \rrbracket$ )  
 $(P_t^{\mathcal{C}})_{t \geq 0}$ : a transition is performed if it doesn't cross an edge in  $\mathcal{C}(t)$ .  
 Peres-Winkler inequality: for any  $\xi \in \Omega_{N,k}$  and any  $\mathcal{C}$

$$P_t(\xi, \xi_{\max}) \geq P_t(\xi_{\min}, \xi_{\max}) \geq P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}).$$

Flow method [gap : minimal nonzero eigenvalue of  $-\mathcal{L}_{N,k}^\omega$ ]

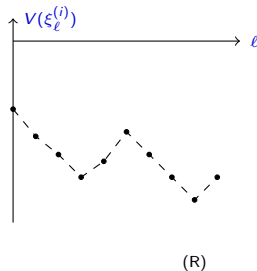
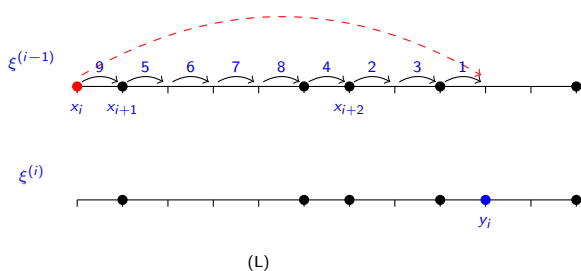
$$\text{gap} \geq \exp(-C(\alpha)N),$$

$$t_{\text{mix}}^{N,k,\omega}(\varepsilon) \leq \text{gap}^{-1} \log \frac{1}{2\varepsilon\mu_{\min}}.$$

Ground state  $\xi^* : \pi_{N,k}^\omega(\xi^*) = \max_{\xi \in \Omega_{N,k}} \pi_{N,k}^\omega(\xi)$

Connect  $\xi$  with  $\xi^*$ : if  $\xi(x_i) = 1 - \xi^*(x_i) = 1$  and  $\xi(y_i) = 1 - \xi^*(y_i) = 0$ .

We move the particle at site  $x_i$  to  $y_i$  as follows:



In this process, the potential  $V^\omega$  can increase at most  $c(\alpha)N$ .

# Guide particles to the right

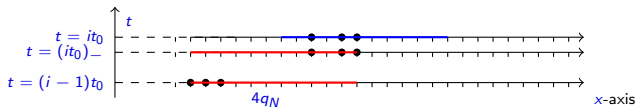
- case  $k \leq q_N = C(\mathbb{P}) \log N$ :

Run  $k$  particles in a segment of length  $4q_N$ .

In equilibrium, all the  $k$  particles are in the right half  $2q_N$  sites w.h.p.

Let  $\bar{\xi}(1)$  be the position of the leftmost particle. We have

$$\pi_{[x+1, x+4q_N], k}^\omega [\bar{\xi}(1) \leq x + 2q_N] \leq N^{-3}, \text{ for } \xi \in \Omega_{[x+1, x+4q_N], k}.$$



$$t_0 = t_{\text{mix}}^{4q_N, k, \omega} (1/N^3) \leq \exp(C(\alpha)4q_N).$$

# intersection between the red segment and the blue segment  $= 2q_N$ .

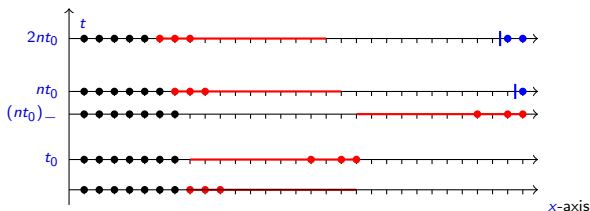
$$\lim_{\varepsilon \rightarrow 0} \inf_{\substack{N \geq 1 \\ k \in [1, N/2]}} \mathbb{P} [\pi_{N, k}^\omega (\xi_{\max}) > \varepsilon] = 1.$$

# Guide particles to the right

- Case  $k > q_N$ : send the rightmost  $k - q_N$  particles to the rightmost  $k - q_N$  sites one by one.

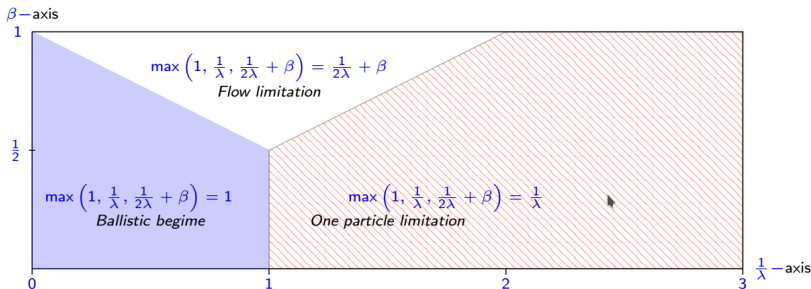
We repeat the first step for the leftmost  $q_N$  particles.

$$\pi_{[x+1, x+4q_N], k}^\omega [\xi(x + 4q_N) = 1] \geq 1 - N^{-3}.$$





The exponent of the mixing time with  $k = N^{\beta+o(1)}$  particles



Thank you for your attention