Typical height of the (2+1)-D Solid-on-Solid surface with pinning above a wall in the delocalized phase

Shangjie Yang

Bar-Ilan Probability Seminar

Joint work Naomi Feldheim

23/03/2023

Organization of the talk

1. Background and model

2. Our results

3. Ideas from the proof

Background and model

Background: qualitative approximation of Ising model

Three dimensional Ising model on a cube $[\![0, N+1]\!]^3$

- Spin value $\{1, -1\}$ on each site State space $\{-1, 1\}^{[\![0,N+1]\!]^3}$
- Boundary conditions: bottom face -1 and all the other face with $+1$
  $\sigma \in \{-1, 1\}^{[\![0,N+1]\!]^3}$

$$\sigma(x, y, z) = \begin{cases} -1 & \text{if } z = 0, \\ 1 & \text{if } z = N+1 \cup x \in \{0, N+1\} \cup y \in \{0, N+1\}. \end{cases}$$

- Ising measure ($\beta = \frac{1}{T} \gg 1$ inverse temperature)

$$\mathbb{P}_{N,\beta}(\sigma) = \frac{1}{Z_{N,\beta}} \exp\left(\beta \sum_{\substack{i,j \in [\![0,N+1]\!]^3 \\ i \sim j}} \sigma_i \sigma_j\right)$$

- **Q:** the "-1" component incident to the bottom face?

$$f : [\![1, N]\!]^2 \to [\![0, N]\!]$$

The solid-on-solid model: a crystal surface model

Introduced by [Burton, Cabrera, Frank '51] [Temperley '52]

$(d+1)$-D SOS model on $\mathbb{Z}^d$:

- Box $\Lambda_N := [\![1, N]\!]^d$ External boundary $\partial\Lambda_N$

$$\partial\Lambda_N := \left\{ x \in \mathbb{Z}^d \setminus \Lambda_N : \ \exists y \in \Lambda_N, x \sim y \right\}$$

- State space $\phi \in \widetilde{\Omega}_{\Lambda_N} := \mathbb{Z}^{\Lambda_N} = \{f : \Lambda_N \to \mathbb{Z}\}$ Hamiltonian (0 b.c.)

$$\mathcal{H}_N(\phi) := \sum_{\substack{\{x,y\} \subset \Lambda_N \\ x \sim y}} |\phi(x) - \phi(y)| + \sum_{\substack{x \in \Lambda_N, \ y \in \partial\Lambda_N \\ x \sim y}} |\phi(x)|.$$

- SOS probability measure ($\beta > 0$ inverse temperature)

$$\forall \phi \in \widetilde{\Omega}_N, \qquad \mathbf{P}_N^\beta(\phi) := \frac{1}{\widetilde{\mathcal{Z}}_N^\beta} e^{-\beta \mathcal{H}_N(\phi)}$$

$$\widetilde{\mathcal{Z}}_N^\beta := \sum_{\psi \in \widetilde{\Omega}_N} e^{-\beta \mathcal{H}_N(\psi)} \leq \left(\frac{1 + e^{-d\beta}}{1 - e^{-d\beta}} \right)^{|\Lambda_N|}$$

SOS: rigid/rough

- $d = 1$: rough (delocalized) [Temperley '52, '56] [Fisher '84]
  for all $\beta > 0$, the expectation of the absolute value of the height at the center diverges in the thermodynamic limit.

- $d \geq 3$: rigid (localized) [Bricmont, Fontaine, Lebowitz '82]
  for all $\beta > 0$, the expectation of the absolute value of the height at the center is uniformly bounded (by Peierls argument).

- $d = 2$ a phase transition between rough and rigid
  - rough: for small β ([Fröhlich, Spencer '81, '83])
  - rigid: for large β ([Brandenberger, Wayne '82], [Gallavotti, Martin-Löf, Miracle-Solé '73]).
  - Numerical critical point: $\beta_c \approx 0.806$

(2+1)-D SOS above a hard wall

- Above a hard wall

$$\forall \phi \in \Omega_N := \left\{ \phi \in \widetilde{\Omega}_N : \phi \geq 0 \right\}, \qquad \mathbb{P}_N^\beta(\phi) := \mathbf{P}_N^\beta(\phi) / \mathbf{P}_N^\beta(\Omega_N).$$

- [Bricmont, Mellouki, and Fröhlich '86]: for large β, the average height H of the surface satisfies

$$\frac{1}{C\beta} \log N \leq H \leq \frac{C}{\beta} \log N.$$

- [Caputo, Lubetzky, Martinelli, Sly, Toninelli '14] for $\beta \geq 1$, the typical height of the surface concentrates at

$$H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$$

with fluctuations of order $O(1)$.

Typical height of (2+1)-D SOS above a wall

Theorem (Caputo, Lubetzky, Martinelli, Sly, Toninelli '14)

There exist two universal constants $C, K > 0$ such that for all $\beta \geq 1$ and all integer $k \geq K$, we have for all N,

$$\mathbb{P}_N^{\beta}\left(|\{x \in \Lambda_N : \ \phi(x) \geq H + k\}| > e^{-2\beta k} N^2\right) \leq e^{-Ce^{-2\beta k} N\left(1 \wedge e^{-2\beta k} N \log^{-8} N\right)}$$

and

$$\mathbb{P}_N^{\beta}\left(|\{x \in \Lambda_N : \ \phi(x) \leq H - k\}| > e^{-2\beta k} N^2\right) \leq e^{-e^{\beta k} N}.$$

Entropic repulsion: In the large β regime, the presence of an impenetrable wall pushes the surface up to the height of order $\frac{1}{4\beta} \log N$, instead of remaining uniformly bounded when no wall is present.

The $(2+1)$-D SOS surface with pinning above a wall

- State space $\Omega_N = \mathbb{Z}_+^{\Lambda_N}$
- Inverse temperature $\beta > 0$, pinning parameter $h \geq 0$
- Probability measure $\mathbb{P}_N^{\beta,h}$: above a wall, with 0 b.c., pinning reward h,

$$\mathbb{P}_N^{\beta,h}(\phi) := \frac{1}{\mathcal{Z}_N^{\beta,h}} e^{-\beta\mathcal{H}_N(\phi)+h|\{x\in\Lambda_N:\ \phi(x)=0\}|},$$

$$\mathcal{Z}_N^{\beta,h} := \sum_{\phi\in\Omega_N} e^{-\beta\mathcal{H}_N(\phi)+h|\{x\in\Lambda:\ \phi(x)=0\}|} \leq e^{h|\Lambda_N|}\left(\frac{1+e^{-2\beta}}{1-e^{-2\beta}}\right)^{|\Lambda_N|}.$$

- Free energy ($log\mathcal{Z}_\Lambda^{\beta,h}$ is superadditive for disjoint boxes $\Rightarrow \exists$ limit)

$$\mathrm{F}(\beta,h) := \lim_{N\to\infty}\frac{1}{N^2}\log\mathcal{Z}_N^{\beta,h}$$

$\mathrm{F}(\beta,h)$ is increasing and convex in h by Hölder's inequality: $\theta\in[0,1]$

$$\mathcal{Z}_N^{\beta,\theta h_1+(1-\theta)h_2} \leq \left(\mathcal{Z}_N^{\beta,h_1}\right)^{\theta}\cdot\left(\mathcal{Z}_N^{\beta,h_2}\right)^{1-\theta}.$$

The $(2+1)$-D SOS surface with pinning above a wall

- When $\mathrm{F}(\beta, h)$ is differentiable in h, the convexity allows us to exchange the order of limit and derivative to obtain the asymptotic contact fraction

$$\partial_h \mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N^2} \mathbb{E}_N^{\beta,h}\left[|\phi^{-1}(0)|\right].$$

- [Chalker '82]: Existence of criticality

$$h_w(\beta) := \sup\left\{h \in \mathbb{R}_+ : \mathrm{F}(\beta, h) = \mathrm{F}(\beta, 0)\right\} > 0 \quad \text{for all } \beta > 0$$

  separates the delocalized phase $(\partial_h \mathrm{F}(\beta, h) = 0)$ from the localized phase $(\partial_h \mathrm{F}(\beta, h) > 0)$. Furthermore, for all $\beta > 0$

$$\log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right) \le h_w(\beta) \le \log\left(\frac{16(e^{4\beta}+1)}{e^{4\beta}-1}\right).$$

- [Alexander, Dunlop, Miracle-Solé, '11]: the lower bound above is asymptotically sharp, and when h decreases to h_w the system undergoes a sequence of layering transitions.

The $(2+1)$-D SOS surface with pinning above a wall

- [Lacoin '18]: for $\beta > \beta_c \in (\log 2, \log 3)$

$$h_w(\beta) = \log\left(\frac{e^{4\beta}}{e^{4\beta}-1}\right)$$

  and there exists a constant C_β such that

$$\forall u \in (0,1], \quad C^{-1}u^3 \leq \mathrm{F}(\beta, u + h_w(\beta)) - \mathrm{F}(\beta, h_w(\beta)) \leq Cu^3.$$

- [Lacoin '21]: when $h > h_w$, a complete picture of the typical height, the Gibbs states and regularity of the free energy.

- **Q:** When $0 \leq h \leq h_w$, how does the surface look like?

Our results: typical height in delocalized phase

Our result (Subcritical regime: $h \in (0, h_w)$

Pinning does not change the typical height ($h = 0$).

Theorem (N. Feldheim, Y. '23)

Fix $\beta \geq 1$, $h \in (0, h_w)$ *and* $N \geq 1$. *Let* $H = \left\lfloor \frac{1}{4\beta} \log N \right\rfloor$.

1. *There exist universal constants* $C, K > 0$ *s.t. for all integer* $m \geq K$,
$$\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([H+m,\infty))\right| > e^{-2\beta m}N^2\right) \leq e^{-Ce^{-2\beta m}N\left(1\wedge e^{-2\beta m}N\log^{-8}N\right)}.$$
2. *For* $\delta > 0$ *and* $m \in \mathbb{N}$ *we have*
$$\mathbb{P}_N^{\beta,h}\left(\left|\phi^{-1}([0,H-m])\right| > 2e^{-2\beta m}N^2\right) \leq 3e^{-\min\left(\frac{1}{2}e^{2\beta m}-4\beta(1+\kappa),\,\delta\right)N}.$$

where (for $h \in (0, h_w)$, $e^{-h} + e^{-4\beta} > 1$)

$$\kappa(\beta, h, \delta) := \frac{4\beta + \delta}{\log\left(e^{-h} + e^{-4\beta}\right)}.$$

At criticality: conjecture and result

At $h = h_w$, Lacoin conjectured: the surface height concentrates around

$$H_w := \left\lfloor \frac{1}{6\beta} \log N \right\rfloor,$$

with fluctuations similar to the subcritical regime.

Isolated and non-isolated zeros

$$q_1(\phi) := \{x \in \Lambda_N : \ \phi(x) = 0, \forall y \in \Lambda_N, y \sim x, \phi(y) \geq 1\},$$
$$q_{2+}(\phi) := \{x \in \Lambda_N : \ \phi(x) = 0, \exists y \in \Lambda_N, y \sim x, \phi(y) = 0\}.$$

Theorem (N. Feldheim, Y. '23)

For $\beta \geq 1$ and $h = h_w$, we have for all $N \in \mathbb{N}$ and $C > 0$:

$$\mathbb{P}_N^{\beta, h_w} \left(\phi \in \Omega_N : \ |q_{2+}(\phi)| \geq CN\right) \leq e^{-N\left(\frac{C}{20} e^{-6\beta} - 4\beta\right)}.$$

At criticality: lower bound on the height

Proposition

For all $\beta \geq 1$, $C > 0$, $h = h_w$, $N \in \mathbb{N}$ *and* $m \in \mathbb{N}$, *letting* $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$ *we have*

$$\mathbb{P}_N^{\beta,h_w}\left(\left\{|\phi^{-1}(0)| \leq CN^{\frac{4}{3}}\right\} \bigcap \left\{\left|\phi^{-1}([1, H_w - m])\right| \geq 2e^{-2\beta m}N^2\right\}\right) \leq 2\exp\left(4\beta N + 4\beta CN^{\frac{4}{3}} - \frac{1}{2}e^{2\beta m}N^{\frac{4}{3}}\right).$$

It suffices to prove that for large enough $C > 0$, we have

$$\mathbb{P}_N^{\beta,h_w}\left(|q_1(\phi)| > CN^{4/3}\right) = o(1)$$

in order to obtain a lower bound on the typical height of the surface at criticality, matching the conjectured height H_w.

Ideas from the proof

Subcritical regime ($h \in (0, h_w)$): Upward fluctuation

- Partial order "$\leq$" on $\Omega_N \times \Omega_N$:

$$\phi \leq \psi \quad \Leftrightarrow \quad \forall x \in \Lambda_N,\ \phi(x) \leq \psi(x).$$

- Function $f : \Omega_N \mapsto \mathbb{R}$ is increasing if $\phi \leq \psi \quad \Rightarrow \quad f(\phi) \leq f(\psi)$.
- Event $\mathcal{A} \subset \Omega_N$ is increasing if $\mathbf{1}_{\mathcal{A}}$ is increasing.
- (μ_1, μ_2 on Ω_N) μ_2 dominates μ_1 ($\mu_1 \preceq \mu_2$) if for any bounded increasing function $f : \Omega_N \mapsto \mathbb{R}$,

$$\mu_1(f) \leq \mu_2(f).$$

Lemma

For all $\beta > 0$ and $0 \leq h_1 \leq h_2$, we have

$$\mathbb{P}_N^{\beta,h_2} \preceq \mathbb{P}_N^{\beta,h_1}.$$

Proof: Verify Holley's condition $\mathbb{P}^{h_1}(\phi \vee \psi)\mathbb{P}^{h_2}(\phi \wedge \psi) \geq \mathbb{P}^{h_1}(\phi)\mathbb{P}^{h_2}(\psi)$.

Subcritical regime ($h \in (0, h_w)$): Upward fluctuation

- The following event is increasing

$$\left\{\phi \in \Omega_N : \ |\{x \in \Lambda_N : \ \phi(x) \geq H + m\}| > e^{-2\beta m} N^2 \right\}$$

- [Caputo, Lubetzky, Martinelli, Sly, Toninelli '14] There exist two universal constants $C, K > 0$ such that for all $\beta \geq 1$ and all integer $k \geq K$, we have for all N,

$$\mathbb{P}_N^{\beta,0}\left(|\{x \in \Lambda_N : \ \phi(x) \geq H + k\}| > e^{-2\beta k} N^2\right) \leq e^{-Ce^{-2\beta k} N \left(1 \wedge e^{-2\beta k} N \log^{-8} N\right)}$$

- $\mathbb{P}_N^{\beta,h} \preceq \mathbb{P}_N^{\beta,0}$

Combining the three items, we obtain the desired upper bound on the upward fluctuation.

Subcritical regime ($h \in (0, h_w)$) : Downward fluctuation

Lemma

For all $\beta \geq 1$, $h \in [0, h_w)$, $\delta > 0$ *and* $N \geq 1$, *we have*

$$\mathbb{P}_N^{\beta,h}\left(|\phi^{-1}(0)| \geq \kappa N\right) \leq e^{-\delta N},$$

where $\kappa = \kappa(\beta, h, \delta) = \frac{4\beta+\delta}{\log\left(e^{-h}+e^{-4\beta}\right)}$.

Idea: Lift the surface up by one

For $\phi \in \Omega_N$ and each $A \subseteq \phi^{-1}(0)$, we define $U_A\phi : \Lambda_N \mapsto \mathbb{Z}_+$ as follows

$$(U_A\phi)(x) := \begin{cases} \phi(x) + 1, & \text{if } x \notin A, \\ 0, & \text{if } x \in A. \end{cases}$$

The action U_A increases the height of each site in $\Lambda_N \setminus A$ by one, we have

$$\mathcal{H}_N(U_A\phi) \leq \mathcal{H}_N(\phi) + 4|A| + 4N,$$
$$|\phi^{-1}(0)| - |(U_A\phi)^{-1}(0)| = |\phi^{-1}(0) \setminus A|.$$

$$\mathbb{P}_N^{\beta,h}(U_A\phi) \geq \mathbb{P}_N^{\beta,h}(\phi)\cdot\exp\left(-h|\phi^{-1}(0)\setminus A| - 4\beta|A| - 4\beta N\right),$$

$$\begin{aligned}
&\sum_{A\subseteq\phi^{-1}(0)} \mathbb{P}_N^{\beta,h}(U_A\phi)\\
&\geq e^{-4\beta N}\cdot\mathbb{P}_N^{\beta,h}(\phi)\sum_{A\subseteq\phi^{-1}(0)}\exp\left(-h|\phi^{-1}(0)\setminus A| - 4\beta|A|\right)\\
&= e^{-4\beta N - h|\phi^{-1}(0)|}\cdot\mathbb{P}_N^{\beta,h}(\phi)\sum_{n=0}^{|\phi^{-1}(0)|}\sum_{\substack{A\subseteq\phi^{-1}(0)\\|A|=n}}\exp\left(-n(4\beta-h)\right)\\
&= e^{-4\beta N - h|\phi^{-1}(0)|}\left(1+e^{-(4\beta-h)}\right)^{|\phi^{-1}(0)|}\mathbb{P}_N^{\beta,h}(\phi).
\end{aligned}$$

- $A, A' \subseteq \phi^{-1}(0)$ with $A \neq A'$, $\Rightarrow U_A\phi \neq U_{A'}\phi$.
- For $\phi \neq \psi$, if $A \subseteq \phi^{-1}(0)$ and $B \subseteq \psi^{-1}(0)$, $\Rightarrow U_A\phi \neq U_B\psi$ (we can recover A from $U_A\phi$ by zero-value sites, then proceed to recover ϕ.)
- $\sum_{\phi\in\Omega_N}\sum_{A\subset\phi^{-1}(0)}\mathbb{P}_N^{\beta,h}(U_A\phi) \leq 1$.

Subcritical regime ($h \in (0, h_w)$): Downward fluctuation

Lemma

Let $\beta \geq 1$, $h \in [0, h_w)$ and $\kappa > 0$. Then for all $m > \lceil \frac{1}{2\beta} \log(8\beta(1+\kappa)) \rceil$ and $N \geq 1$ we have

$$\mathbb{P}_N^{\beta,h}\left(\left\{|\phi^{-1}(0)| \leq \kappa N\right\} \bigcap \left\{\left|\phi^{-1}([1, H-m])\right| \geq \frac{e^{-2\beta m}}{1-e^{-2\beta}} N^2\right\}\right)$$
$$\leq \frac{1}{1-e^{-\beta N}} e^{-\left(\frac{1}{2}e^{2\beta m} - 4\beta(1+\kappa)\right)N}.$$

Fix an integer $\ell \in [1, H-m]$. For $A \subseteq \phi^{-1}(\ell)$, define $V_A\phi : \Lambda_N \mapsto \mathbb{Z}_+$

$$(V_A\phi)(x) := \begin{cases} 0, & \text{if } x \in \phi^{-1}(0), \\ 1, & \text{if } x \in A, \\ \phi(x)+1, & \text{if } x \notin A \cup \phi^{-1}(0). \end{cases}$$

Observe that for $x \in A$ and $y \notin A \cup \phi^{-1}(0)$ with $x \sim y$,

$$|(V_A\phi)(x) - (V_A\phi)(y)| = \phi(y) \le |\ell - \phi(y)| + \ell,$$

$$\mathcal{H}_N(V_A\phi) \le \mathcal{H}_N(\phi) + 4N + 4|\phi^{-1}(0)| + 4\ell|A|.$$

As $|(V_A\phi)^{-1}(0)| = |\phi^{-1}(0)|$,

$$\mathbb{P}_N^{\beta,h}(V_A\phi) \ge \mathbb{P}_N^{\beta,h}(\phi)\, e^{-4\beta N - 4\beta|\phi^{-1}(0)| - 4\beta\ell|A|}.$$

Summing over all subsets

$$\sum_{A \subseteq \phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h}(V_A\phi) \ge \mathbb{P}_N^{\beta,h}(\phi) \sum_{A \subseteq \phi^{-1}(\ell)} e^{-4\beta N - 4\beta|\phi^{-1}(0)| - 4\beta\ell|A|}$$

$$\ge \mathbb{P}_N^{\beta,h}(\phi) \exp\left(-4\beta N - 4\beta|\phi^{-1}(0)| + \tfrac{1}{2} e^{-4\beta\ell}|\phi^{-1}(\ell)|\right).$$

For $A, A' \subseteq \phi^{-1}(\ell)$ with $A \ne A'$, $\Rightarrow V_A\phi \ne V_{A'}\phi$.
For $\phi \ne \psi \in \Omega_N$, $A \subset \phi^{-1}(\ell)$ and $B \subset \psi^{-1}(\ell)$, $\Rightarrow V_A\phi \ne V_B\psi$. (we can recover A by $1-$valued sites of $V_A\phi$ and then proceed to recover ϕ)

$$1 \ge \sum_{\substack{\phi:\ |\phi^{-1}(\ell)| \ge e^{-2\beta j} N^2 \\ |\phi^{-1}(0)| \le \kappa N}} \sum_{A \subset \phi^{-1}(\ell)} \mathbb{P}_N^{\beta,h}(V_A\phi) \quad (j = H - \ell)$$

At criticality

Observation: for $x_1, x_2, x_3, x_4 \in \mathbb{Z}_+$,

$$\sum_{k=-\infty}^{0} \exp\left(-\beta \sum_{i=1}^{4} |x_i - k|\right) = \exp\left(h_w - \beta \sum_{i=1}^{4} x_i\right).$$

A new state space (only allows isolated negative value sites.)

$$\Omega_N^* := \{\psi : \Lambda_N \to \mathbb{Z} \mid \text{if } \psi(x) \le -1, \forall y \in \Lambda_N, y \sim x, \psi(y) \ge 1\}.$$

Note: if $\psi \in \Omega_N^*$, then $\max(\psi, 0) \in \Omega_N$. we have

$$\mathcal{Z}_N^{\beta,h_w} = \sum_{\psi \in \Omega_N^*} \exp\left(-\beta \mathcal{H}_N(\psi) + h_w |q_{2+}(\psi)|\right).$$

Define a new probability measure $\widetilde{\mathbb{P}}_N$ on Ω_N^* as follows:

$$\forall \psi \in \Omega_N^*, \quad \widetilde{\mathbb{P}}_N(\psi) := \frac{1}{\mathcal{Z}_N^{\beta,h_w}} \exp\left(-\beta \mathcal{H}_N(\psi) + h_w |q_{2+}(\psi)|\right).$$

Relation between $\widetilde{\mathbb{P}}_N$ and $\mathbb{P}_N^{\beta,h_w}$: for any $\phi \in \Omega_N$,

$$\mathbb{P}_N^{\beta,h_w}(\phi) = \widetilde{\mathbb{P}}_N\left(\{\psi \in \Omega_N^* : \max(\psi, 0) = \phi\}\right).$$

At criticality

Since for any $\psi \in \Omega_N^*$, we have $q_{2+}(\max(\psi, 0)) = q_{2+}(\psi)$, then

$$\mathbb{P}_N^{\beta,h_w}\left(\{\phi \in \Omega_N : \ |q_{2+}(\phi)| \geq CN\}\right) = \widetilde{\mathbb{P}}_N\left(\{\psi \in \Omega_N^* : \ |q_{2+}(\psi)| \geq CN\}\right).$$

For any subset $A \subseteq q_{2+}(\psi)$, $\mathcal{N}(A)$: the edge boundary of A

$$\mathcal{N}(A) := \left\{\{x, y\} \in E(\mathbb{Z}^2) : \ x \in A, y \in A^{\complement}\right\}.$$

Define $U_A\psi \in \Omega_N^*$ as

$$(U_A\psi)(x) := \begin{cases} \psi(x) + 1 & \text{if } x \notin A, \\ 0 & \text{if } x \in A. \end{cases}$$

Notation: for fixed $\psi \in \Omega_N^*$, write $q_{2+}(A) := q_{2+}(U_A\psi)$.
Observe $\mathcal{H}_N(U_A\psi) \leq \mathcal{H}_N(\psi) + 4\beta N + \beta|\mathcal{N}(A)|$, then

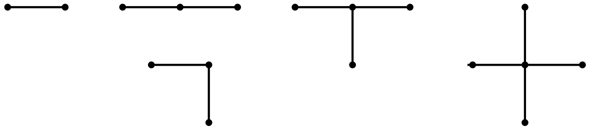
$$\widetilde{\mathbb{P}}_N(U_A\psi) \geq \widetilde{\mathbb{P}}_N(\psi)\exp\left(-4\beta N - \beta|\mathcal{N}(A)| - h_w\left(|q_{2+}(\psi)| - |q_{2+}(A)|\right)\right).$$

$V_1, V_2, \cdots, V_k$: connected components of $q_{2+}(\psi)$, write $A_i = A \cap V_i$.
Sum over all subsets $A \subseteq q_{2+}(\psi)$ to obtain

$$
\begin{aligned}
&\sum_{A \subseteq q_{2+}(\psi)} \widetilde{\mathbb{P}}_N(U_A \psi) \\
&\geq \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N - h_w |q_{2+}(\psi)|\right) \sum_{A \subseteq q_{2+}(\psi)} \exp\left(-\beta |\mathcal{N}(A)| + h_w |q_{2+}(A)|\right) \\
&= \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N - h_w |q_{2+}(\psi)|\right) \sum_{A_1, \cdots, A_k} \prod_{i=1}^{k} \exp\left(-\beta |\mathcal{N}(A_i)| + h_w |q_{2+}(A_i)|\right) \\
&= \widetilde{\mathbb{P}}_N(\psi) \exp\left(-4\beta N\right) \prod_{i=1}^{k} \exp\left(-h_w |V_i|\right) \sum_{A_i \subseteq V_i} \exp\left(-\beta |\mathcal{N}(A_i)| + h_w |q_{2+}(A_i)|\right)
\end{aligned}
$$

$q_{2+}(\psi) = V_1 \cup V_2 \cdots \cup V_k$ disjoint union.
Focus on each connected component V_i

For each connected graph (V_i, E_i), after deleting some edges, it can be covered by the following four types of patterns.



(1) (2) (3) (4)

Lemma

If $\beta \geq 1$ and V is the vertex set of one of the patterns shown above, then

$$\exp\left(-h_w|V|\right) \sum_{B \subseteq V} \exp\left(-\beta|\mathcal{N}(B)| + h_w|q_{2+}(B)|\right) \geq 1 + \frac{1}{2}e^{-6\beta}.$$

This lemma implies

$$\sum_{A \subseteq q_{2+}(\psi)} \widetilde{\mathbb{P}}_N(U_A\psi) \geq e^{-4\beta N}\widetilde{\mathbb{P}}_N(\psi)\left(1 + \frac{1}{2}e^{-6\beta}\right)^{|q_{2+}(\psi)|/5}.$$

- For $A \neq B \subset q_{2+}(\psi)$, we have $U_A\psi \neq U_B\psi$ since $(U_A\psi)|_{A\setminus B} = 0$ and $(U_B\psi)|_{A\setminus B} = 1$.
- For $\psi, \psi' \in \Omega_N^*$ and $A \subset q_{2+}(\psi), A' \subset q_{2+}(\psi')$, we have

  $$U_A\psi \neq U_{A'}\psi'.$$

  To see this, note that

  $$A = \{x \in \Lambda_N : \ (U_A\psi)(x) = 0, \ \exists y \in \Lambda_N, \ y \sim x, \ (U_A\psi)(y) \in \{0,1\}\}.$$

  Thus, given $U_A\psi$, we can first recover the set A and then proceed to recover ψ.
- $$\begin{aligned} 1 &\geq \sum_{\psi\in\Omega_N^*:\ |q_{2+}(\psi)|\geq CN} \ \sum_{A\subset q_{2+}(\psi)} \widetilde{\mathbb{P}}_N\left(U_A\psi\right) \\ &\geq \sum_{\psi\in\Omega_N^*:\ |q_{2+}(\psi)|\geq CN} e^{-4\beta N}\, \widetilde{\mathbb{P}}_N(\psi) \left(1 + \tfrac{1}{2}e^{-6\beta}\right)^{|q_{2+}(\psi)|/5} \end{aligned}$$

Proposition

For all $\beta \geq 1$, $C > 0$, $h = h_w$, $N \in \mathbb{N}$ and $m \in \mathbb{N}$, letting $H_w = \lfloor \frac{1}{6\beta} \log N \rfloor$ we have

$$\mathbb{P}_N^{\beta, h_w} \left(\left\{ |\phi^{-1}(0)| \leq CN^{\frac{4}{3}} \right\} \bigcap \left\{ \left|\phi^{-1}([1, H_w - m])\right| \geq 2e^{-2\beta m} N^2 \right\} \right) \\ \leq 2 \exp\left(4\beta N + 4\beta C N^{\frac{4}{3}} - \frac{1}{2} e^{2\beta m} N^{\frac{4}{3}} \right).$$

Idea: Fix an integer $\ell \in [1, H_w - m]$. For $A \subseteq \phi^{-1}(\ell)$, define $V_A\phi : \Lambda_N \mapsto \mathbb{Z}_+$

$$(V_A\phi)(x) := \begin{cases} 0, & \text{if } x \in \phi^{-1}(0), \\ 1, & \text{if } x \in A, \\ \phi(x) + 1, & \text{if } x \notin A \cup \phi^{-1}(0). \end{cases}$$

Thank you for your attention!