Metastability for expanding bubbles on a sticky substrate

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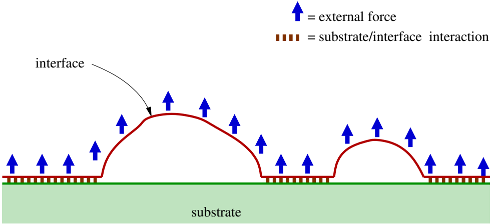
30/11/2023

Organization of the talk

1. Introduction to mixing for continuous-time Markov chains
   - Starting from 1980s
   - Aldous, Diaconis, etc.
2. Mixing time for an interface model



Setup

- Finite state space Ω, elements $x, y, z \cdots$
- Generator: $\mathcal{L} = (r(x,y))_{x,y\in\Omega}$ is an $\Omega \times \Omega$ matrix:
  - Off diagonal elements are nonnegative;
  - Every row sum is equal to zero.

  Homeomorphism $\mathcal{L} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ (for $f \in \mathbb{R}^\Omega$)

$$(\mathcal{L}f)(x) := \sum_{y\in\Omega} r(x,y)\left(f(y) - f(x)\right).$$

- Markov semi-group $(P_t)_{t\geq 0}$:

$$P_t := e^{t\mathcal{L}} = \sum_{k=0}^{\infty} \frac{(t\mathcal{L})^k}{k!},$$

$$P_t(x,y) \geq 0, \quad \sum_{y\in\Omega} P_t(x,y) = 1.$$

Markov chain definition

The random process $(X_t)_{t\geq 0}$ is a continuous-time Markov chain with generator $\mathcal{L}$ and initial distribution ν if it is càdlàg and

- $$\forall x \in \Omega, \quad \mathbb{P}\left[X_0 = x\right] = \nu(x);$$

- Markov property: for $0 \leq t_1 < \cdots < t_n < s < s + t$,

$$\mathbb{P}\left[X_{s+t} = y | X_s = x; X_{t_k} = z_k, \forall k \leq n\right] = \mathbb{P}\left[X_{s+t} = y | X_s = x\right] = P_t(x, y).$$

Invariant probability measure

- μ is an invariant probability measure if

$$(\forall t \geq 0, \mu P_t = \mu) \Leftrightarrow \mu \mathcal{L} = 0.$$

- Irreducible: for all $x \neq y \in \Omega$, there exists a path $\Gamma_{xy} = (x, z_1, \cdots, z_{\ell-1}, y)$ with $r(z_{k-1}, z_k) > 0$ for all $1 \leq k \leq \ell(x, y)$.

Theorem

If $(\Omega, \mathcal{L})$ is irreducible, there exists a unique invariant probability measure μ, and the distribution $\mathbb{P}^\nu$ of $(X_t)_{t\geq 0}$ with initial distribution ν converges to μ, i.e.

$$\lim_{t\to\infty} \sum_{y\in\Omega} \left| \mathbb{P}^\nu [X_t = y] - \mu(y) \right| = 0.$$

Distance to equilibrium

- The total variation distance: two probability measures α, β on Ω,

$$\|\alpha - \beta\|_{\mathrm{TV}} := \sup_{A \subset \Omega} |\alpha(A) - \beta(A)| .$$

- The distance to equilibrium

$$d(t) := \max_{x \in \Omega} \|P_t(x, \cdot) - \mu\|_{\mathrm{TV}}.$$

- Given $\varepsilon \in (0, 1)$, the $\varepsilon-$mixing time

$$t_{\mathrm{mix}}(\varepsilon) := \inf \{t \geq 0 : d(t) \leq \varepsilon\} .$$

Notation: $t_{\mathrm{mix}} := t_{\mathrm{mix}}(1/4)$.

Markov chain sequence and cutoff

- A sequence of Markov chains $(\Omega_n, \mathcal{L}_n, \mu_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} |\Omega_n| = \infty$:

  $t_{\mathrm{mix}}^{(n)}(\varepsilon)$: the associated $\varepsilon-$mixing time.

  Q: **How does $t_{\mathrm{mix}}^{(n)}(\varepsilon)$ grow in terms of n and ε?**

- Precutoff:

$$\sup_{\varepsilon\in(0,\frac{1}{2})} \limsup_{n\to\infty} \frac{t_{\mathrm{mix}}^{(n)}(\varepsilon)}{t_{\mathrm{mix}}^{(n)}(1-\varepsilon)} < \infty.$$

- Cutoff: for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} \frac{t_{\mathrm{mix}}^{(n)}(\epsilon)}{t_{\mathrm{mix}}^{(n)}(1-\epsilon)} = 1. \quad \Leftrightarrow \quad \lim_{n\to\infty} d_n\left(ct_{\mathrm{mix}}^{(n)}\right) = \begin{cases} 1 & \text{if } c < 1, \\ 0 & \text{if } c > 1. \end{cases}$$

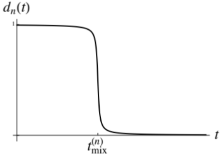


image from Levin and Peres

Spectral gap of reversible chain

- The detailed balance condition: if for all $x, y \in \Omega$

$$\mu(x)r(x,y) = \mu(y)r(y,x) \qquad (\Rightarrow \quad \mu\mathcal{L} = 0).$$

- Spectral gap: minimal nonzero eigenvalue of $-\mathcal{L}$

$$\langle f, g\rangle_\mu := \sum_{x\in\Omega} \mu(x)f(x)g(x), \qquad \mathrm{Var}_\mu(f) := \langle f, f\rangle_\mu - \langle f, \mathbf{1}\rangle_\mu^2,$$

$$\mathrm{gap} := \inf_{\mathrm{Var}_\mu(f)>0} \frac{-\langle f, \mathcal{L}f\rangle_\mu}{\mathrm{Var}_\mu(f)}.$$

- Relaxation time: $t_{\mathrm{rel}} := \frac{1}{\mathrm{gap}}$.

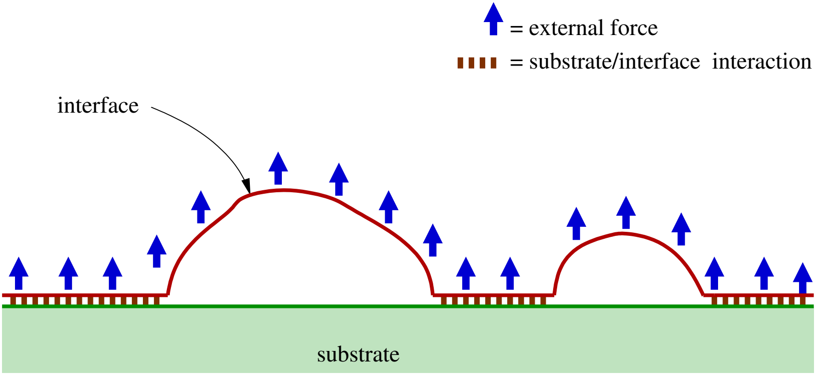
Letting $\mu_{\min} := \min_{x\in\Omega} \mu(x)$, for $\varepsilon \in (0,1)$ we have

$$t_{\mathrm{rel}} \log \frac{1}{2\varepsilon} \le t_{\mathrm{mix}}(\varepsilon) \le t_{\mathrm{rel}} \log \frac{1}{2\varepsilon\mu_{\min}},$$

$$\lim_{t\to\infty} \frac{1}{t} \log d(t) = -\mathrm{gap}.$$

Mixing time for an interface model

The physical situation we are considering



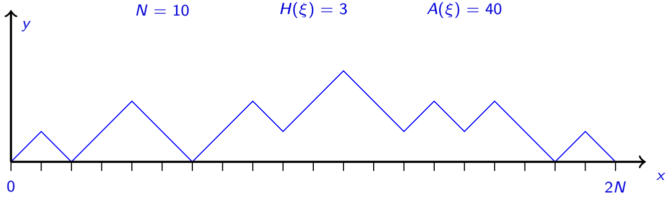
An interface is an element of

$$\Omega_N := \left\{ \xi \in \mathbb{Z}_+^{[\![0,2N]\!]} : \ \xi(0) = \xi(2N) = 0 \text{ and } \forall x\,,\ |\xi(x) - \xi(x-1)| = 1 \right\}.$$

The equilibrium measure

Given $\xi \in \Omega_N$,

- $H(\xi) := \sum_{x=1}^{2N-1} \mathbf{1}_{\{\xi(x)=0\}}$ (# contacts with $x-$axis),
- $A(\xi) := \sum_{x=1}^{2N-1} \xi(x)$: the area enclosed between ξ and the x-axis.

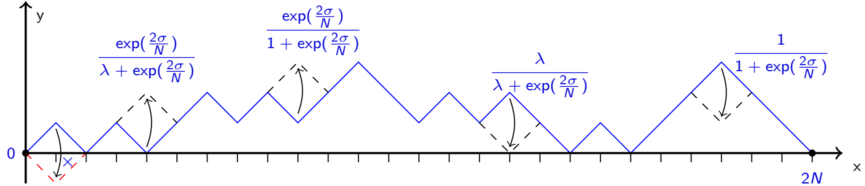


Given $\lambda \geq 0$ and $\sigma \geq 0$, define $\mu = \mu_N^{\lambda,\sigma}$ the probability on Ω_N:

$$\mu(\xi) = \frac{2^{-2N}\lambda^{H(\xi)}e^{\frac{\sigma}{N}A(\xi)}}{Z_N(\lambda,\sigma)} \quad ; \quad Z_N(\lambda,\sigma) := 2^{-2N}\sum_{\xi\in\Omega_N}\lambda^{H(\xi)}e^{\frac{\sigma}{N}A(\xi)}.$$

Corner-flip/Heat Bath dynamics $(\eta_t)_{t\geq 0}$ on Ω_N

Each coordinate is updated at rate one.
When an update at x occurs at time t, η_t is sampled according to the conditional equilibrium measure $\mu_N^{\lambda,\sigma}(\cdot \mid \eta_{t-}(y), y \neq x)$.



The measure μ satisfies the detailed balance condition, *i.e.*

$$\mu(\xi)r(\xi,\xi^x) = \mu(\xi^x)r(\xi^x,\xi).$$

$\mathbf{P}^{\xi}$: the distribution of the Markov chain $(\eta_t)_{t\geq 0}$ starting from ξ.
$T_N^{\lambda,\sigma}(\varepsilon)$: associated $\varepsilon-$mixing time.

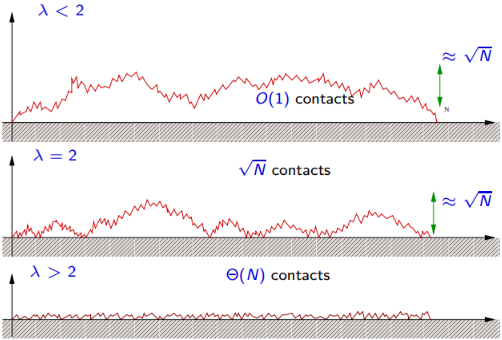
Presentation of our results for the interface model

(1) Previous results about related models

(2) Properties of the model at equilibrium

(3) Slow/fast mixing and metastability ($\sigma > 0$)

Equilibrium for $\sigma = 0$ [Fisher 1984 JSP]



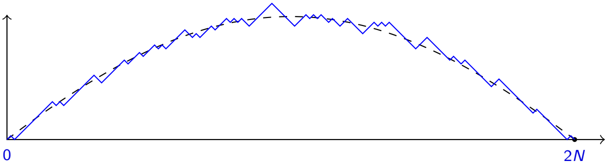
If $\sigma = 0$, the system undergoes a transition at $\lambda = 2$ between a pinned phase and an unpinned phase. This transition can be seen when looking at the free energy

$$\lim_{N\to\infty} \frac{1}{2N} \log Z_N(\lambda, 0) = \log\left(\frac{\lambda}{2\sqrt{\lambda - 1}}\right)\mathbf{1}_{\{\lambda>2\}} =: F(\lambda).$$

No wall constraint / WASEP interfaces [Labbé '18 Prob. Surv.]

If there is no wall constraint ($\xi(x) < 0$ is allowed) and $\lambda = 1$, we have typically under the equilibrium measure ($u \in [0, 2]$)

$$\frac{\xi(\lceil uN \rceil)}{N} = \frac{1}{\sigma} \log\left(\frac{\cosh(\sigma)}{\cosh(\sigma(1-u))}\right) + o(1).$$



If $\widetilde{Z}_N(\sigma) := \frac{1}{2^{2N}} \sum_{\xi \in \widetilde{\Omega}_N} e^{\frac{\sigma}{N} A(\xi)}$ denotes the corresponding partition function, we have

$$\lim_{N\to\infty} \frac{1}{2N} \log \widetilde{Z}_N(\sigma) = G(\sigma) := \int_0^1 \log\cosh(\sigma(1-2u))\mathrm{d}u.$$

Equilibrium behavior

The two strategies to take benefit of the wall interaction and of the external force are different and cannot be combined.

Proposition (Lacoin, Y. '22)

We have for any $\lambda \in (0, \infty)$ *and* $\sigma > 0$

$$\lim_{N \to \infty} \frac{1}{2N} \log Z_N(\lambda, \sigma) = F(\lambda) \vee G(\sigma).$$

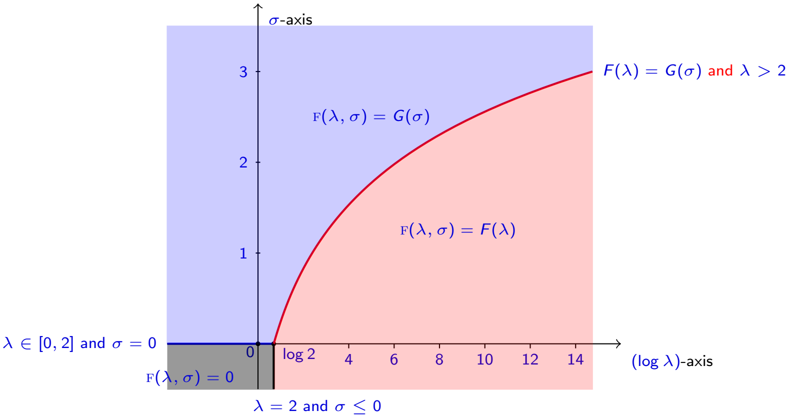
(A) *If* $G(\sigma) > F(\lambda)$*, then* $Z_N(\lambda, \sigma) \asymp \frac{1}{\sqrt{N}} e^{2NG(\sigma)}$.

(B) *If* $F(\lambda) \geq G(\sigma)$*, then* $Z_N(\lambda, \sigma) \asymp e^{2NF(\lambda)}$.

From this result we derive the detailed behavior of the paths.

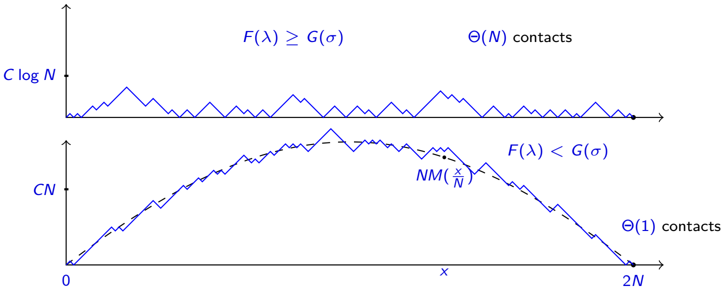
Free energy

$$\mathrm{F}(\lambda, \sigma) := \lim_{N \to \infty} \frac{1}{2N} \log Z_N(\lambda, \sigma).$$



Theorem: macroscopic shape

$$M_\sigma(u) = \frac{1}{\sigma}\log\left(\frac{\cosh(\sigma)}{\cosh(\sigma(1-u))}\right).$$



Dynamical polymer pinning model/WASEP

The problem of mixing time for interface with pinning or WASEP has been studied in previous works.

- When $\sigma = 0$, the mixing time is at most of order $N^2 \log N$ [Caputo, Martinelli, Toninelli '08 EJP]:

$$e.g. \quad T_N^{\lambda,0} \asymp N^2 \log N, \text{ and } \mathrm{gap} \asymp N^{-2} \text{ for } \lambda \in [0,2).$$

- Without wall and pinning, [Levin, Peres '16 JSP] [Labbé, Lacoin '20 AAP]

$$\forall \varepsilon \in (0,1), \quad T_N^{\sigma}(\varepsilon) \asymp N^2 \log N.$$

Our main result: $\lambda > 2$ and $\sigma \geq 0$

Theorem (Lacoin, Y. '22)

When $\lambda > 2$ and $\sigma \geq 0$, then there exists $\sigma_c(\lambda) > 0$ such that

$$\begin{cases} T_N^{\lambda,\sigma} \leq N^C & \text{if } \sigma \leq \sigma_c(\lambda), \\ T_N^{\lambda,\sigma} = e^{2NE(\lambda,\sigma)} N^{O(1)} & \text{if } \sigma > \sigma_c(\lambda), \end{cases}$$

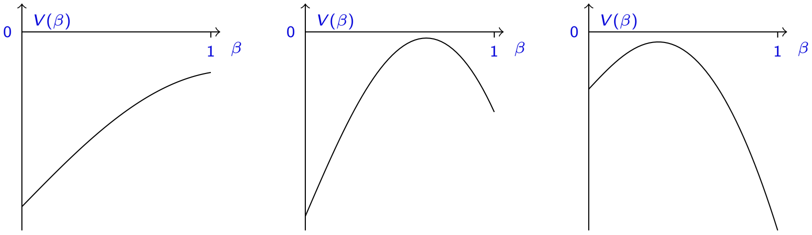
where $\sigma_c(\lambda)$ and $E(\lambda,\sigma) > 0$ are explicit.

We believe when $\lambda \in [0, 2]$ and $\sigma \geq 0$, there exists some constant C

$$T_N^{\lambda,\sigma} \leq N^C.$$

Heuristic for $\lambda > 2$ and $\sigma \geq 0$

β: fraction of the largest excursion $\qquad V(\beta) := -(1-\beta)F(\lambda) - \beta G(\beta\sigma)$
(paths with only one large excursion of size $2\beta N$: $e^{-2NV(\beta)}$.)



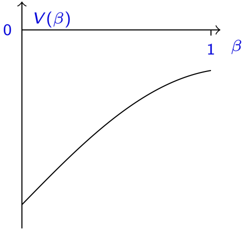
(A) (B) (C)

(A) If $G(\sigma) + \sigma G'(\sigma) \leq F(\lambda)$, then the pinned region can grow without obstruction and the system should mix in polynomial time.

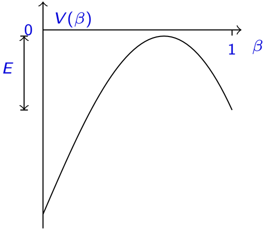
(B) If $G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma)$, then the system starting from the fully unpinned state takes a long time to reach the fully pinned equilibrium state.

(C) If $F(\lambda) < G(\sigma)$, then the system starting from the fully pinned state takes a long time to reach the fully unpinned equilibrium state.

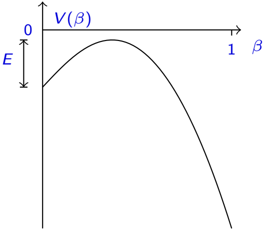
$V(\beta) = -(1-\beta)F(\lambda) - \beta G(\beta\sigma)$



(A)



(B)



(C)

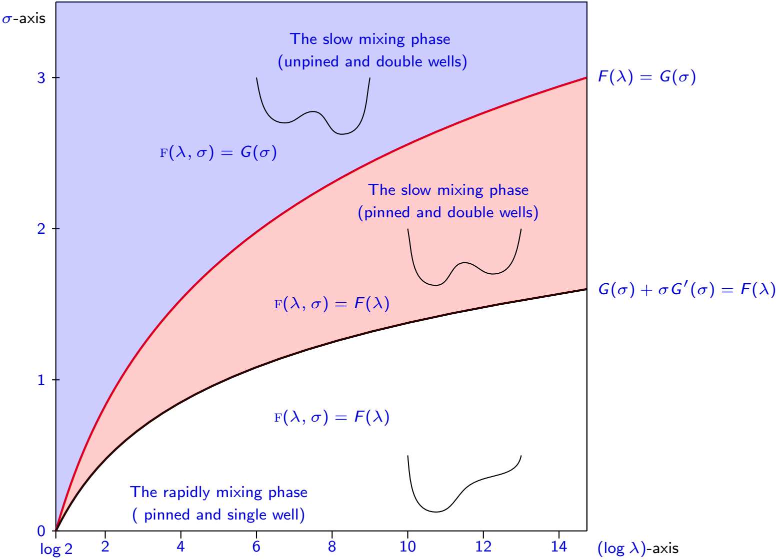
Activation Energy

The size of the effective potential barrier to be overcome in case (B) and (C) is equal to

$$E(\lambda,\sigma) := F(\lambda) \wedge G(\sigma) - [(1-\beta^*)F(\lambda) + \beta^* G(\beta^*\sigma)]$$

with β^* such that $V(\beta^*) = \max_{\beta\in[0,1]} V(\beta)$.

Our result: phase diagram (for $\lambda > 2$ and $\sigma \geq 0$)



Metastability

Assuming $E(\lambda, \sigma) > 0$, let $\mathcal{H}_N$ denote the domain of attraction of the unstable local equilibrium of the dynamics:

$$\mathcal{H}_N := \begin{cases} \{\xi \in \Omega_N : L_{\max}(\xi) > \beta^* N\} & \text{if } G(\sigma) \leq F(\lambda) < G(\sigma) + \sigma G'(\sigma), \\ \{\xi \in \Omega_N : L_{\max}(\xi) \leq \beta^* N\} & \text{if } F(\lambda) < G(\sigma), \end{cases}$$

where

$$L_{\max}(\xi) := \max\{y - x \ : \ \xi_{2x} = 0, \xi_{2y} = 0, \forall z \in [\![x, y]\!], \xi_{2z} > 0\}.$$

Theorem (Lacoin, Y. '22)

We have

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N(\cdot|\mathcal{H}_N)} \left(\eta_{tT^N_{\mathrm{rel}}(\lambda,\sigma)} \in \mathcal{H}_N \right) = \exp(-t),$$

where $T^N_{\mathrm{rel}}(\lambda, \sigma) = e^{2NE(\lambda,\sigma)} N^{O(1)}$ *is the relaxation time of the system.*

Proof ingredients

- Lower bound on mixing time follows directly from the heuristics using bottleneck arguments.
- For the upper bound, the hard part is to show that the system always mixes fast within $\mathcal{H}_N$ and $\mathcal{H}_N^\complement$. The proof is intricate and relies on chain decomposition argument [Jerrum *et al.* '04 AAP].
- Once fast mixing in each potential well is proved, the metastability statement follows from a general meta-theorem [Beltran and Landim '15 PTRF].

Thank you for your attention