

# Spectral gap and cutoff of Simple Exclusion Process with IID conductances

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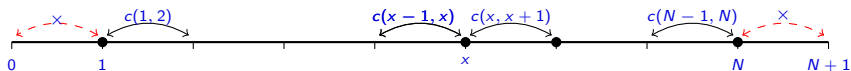
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# Setup: Simple Exclusion Process with inhomogeneous conductance

# Setup: Simple Exclusion Process (SEP)

Conductances:  $(c(x, x+1))_{x \in \mathbb{N}}$  with values in  $(0, \infty)$ .

SEP with  $k$  particles in  $\llbracket 1, N \rrbracket$  with conductances  $c(x, x+1)_{x \in \mathbb{N}}$ .



- (A) Each site is occupied by at most one particle (*the exclusion rule*).
- (B) At each edge  $\{x, x+1\}$  with  $1 \leq x < N$ , we place a Poisson clock with rate  $c(x, x+1) > 0$ . When a clock rings, we swap the contents of the two sites.

# Setup

- State space (1: particle      0: empty site.)

$$\Omega_{N,k} := \left\{ \xi : \llbracket 1, N \rrbracket \rightarrow \{0, 1\} \mid \sum_{i=1}^N \xi(i) = k \right\} .$$

- Generator: ( $f : \Omega_{N,k} \mapsto \mathbb{R}$ )

$$(\mathcal{L}_{N,k} f)(\xi) := \sum_{i=1}^{N-1} c(i, i+1) [f(\xi \circ \tau_{i,i+1}) - f(\xi)] ,$$

where  $\tau_{i,j}$  is the transposition of the two elements  $i$  and  $j$ .

- Uniform prob measure  $\mu_{N,k}$  satisfies the detailed balance condition:

$$\mu(\xi) \mathcal{L}_{N,k}(\xi, \eta) = \mu(\eta) \mathcal{L}_{N,k}(\eta, \xi),$$

then  $\mu$  is the invariant prob measure.

# Setup

- Distance to equilibrium

$$d_{N,k}(t) := \max_{\xi \in \Omega_{N,k}} \|P_t^\xi - \mu_{N,k}\|_{\text{TV}}.$$

$P_t^\xi$ : marginal distribution at instant  $t$  of the chain starting with  $\xi$ .

- $\varepsilon$ -mixing time

$$t_{\text{mix}}^{N,k}(\varepsilon) := \inf \{t \geq 0 : d_{N,k}(t) \leq \varepsilon\}.$$

- Cutoff: for all  $\varepsilon \in (0, 1)$ ,

$$\lim_{N \rightarrow \infty} \frac{t_{\text{mix}}^{N,k}(\varepsilon)}{t_{\text{mix}}^{N,k}(1-\varepsilon)} = 1.$$

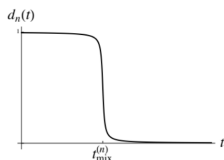


image from Levin and Peres

# Setup

- : Spectral gap  $\text{gap}_{N,k}$ : minimal nonzero eigenvalue of  $-\mathcal{L}_{N,k}$

$$\text{gap}_{N,k} := \inf_{f : \text{Var}_{\mu_{N,k}}(f) > 0} \frac{-\langle f, \mathcal{L}_{N,k} f \rangle_{\mu_{N,k}}}{\text{Var}_{\mu_{N,k}}(f)}$$

where  $\text{Var}_{\mu_{N,k}}(f) := \langle f, f \rangle_{\mu_{N,k}} - \langle f, \mathbf{1} \rangle_{\mu_{N,k}}^2$ .

- Relation between spectral gap and mixing time/distance to equilibrium:

$$\frac{1}{\text{gap}_{N,k}} \log \frac{1}{2\varepsilon} \leq t_{\text{mix}}^{N,k}(\varepsilon) \leq \frac{1}{\text{gap}_{N,k}} \log \frac{1}{2\varepsilon \mu_{\min}}$$

where  $\mu_{\min} := \min_{\xi \in \Omega_{N,k}} \mu_{N,k}(\xi)$ .

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log d_{N,k}(t) = -\text{gap}_{N,k}.$$

**Question:** How does the disordered setup (inhomogeneous conductances) affect the system in terms of spectral gap/mixing time?

## Previous Results

Homogeneous conductances  $c(x, x+1) \equiv 1$  for one particle ( $k = 1$ )

- Spectral gap

$$\text{gap}_{N,1} = 2(1 - \cos(\pi/N)) = (1 + o(1))\pi^2/N^2.$$

- Eigenfunctions

$$g_i^{(N)}(x) := \cos(i\pi(x - 1/2)/N), \quad 0 \leq i < N.$$

- Eigenvalues

$$-\lambda_i^{(N)} = -2(1 - \cos(i\pi/N)) \quad \mathcal{L}_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)}.$$

Homogeneous conductances  $c(x, x+1) \equiv 1$  for many particles

- [Aldous] [Wilson] [Lacoin] Assuming  $\liminf_{N \rightarrow \infty} \min(k, N - k) = \infty$ ,

$$t_{\min}^{N,k}(\varepsilon) = (1 + o(1)) \frac{N^2}{2\pi^2} \log k, \quad \text{gap}_{N,k} = \text{gap}_{N,1} = (1 + o(1)) \frac{\pi^2}{N^2}.$$

# Previous results

## Inhomogeneous conductance $c(x, x+1) > 0$

- Aldous' spectral gap conjecture (Proved by [Caputo, Liggett, Richthammer, JAMS '10]):

$$\text{gap}_{N,k} = \text{gap}_{N,1}, \quad \forall k \in \llbracket 1, N-1 \rrbracket.$$

- A function  $f : \llbracket 1, N \rrbracket \rightarrow \mathbb{R}$  for  $2 \leq b \leq c \leq N-1$   
Local maximum at  $\llbracket b, c \rrbracket$  if  $f$  is constant on  $\llbracket b, c \rrbracket$ ,  $f(b-1) < f(b)$  and  $f(c) > f(c+1)$ .  
Analogous definition holds for a local minimum.  
 $f$  is  $j$ -monotone if it displays exactly  $(j-1)$  distinct local extrema in  $\llbracket 2, N-1 \rrbracket$ .  
Nodal domains:

#connected components of  $\{x \in \llbracket 1, N \rrbracket, f(x) \neq 0\}$ .

- [Miclo]:  $L_{N,1} g_i^{(N)} = -\lambda_i^{(N)} g_i^{(N)}$  with  $0 = \lambda_0^{(N)} < \lambda_1^{(N)} < \dots < \lambda_{N-1}^{(N)}$   
 $g_i^{(N)}$  is  $i$ -monotone and has  $i+1$  nodal domains.



# Our results

## Proposition (Y. '24)

For any positive conductances  $(c(x, x+1))_{x \in \mathbb{N}}$ ,  $g_1^{(N)}$  is strictly monotone.

Write  $r(x, x+1) := 1/c(x, x+1)$  and  $r(n, m) := \sum_{x=n}^{m-1} r(x, x+1)$ .

Assume (LLN) condition

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sup_{2 \leq m \leq N} \left| (r^{(N)}(1, m) - (m-1)) \right| = 0. \quad (\text{LLN})$$

When  $(r^{(N)}(x-1, x))_{2 \leq x \leq N}$  is IID with expectation  $\mathbb{E}[r(x, x+1)] = 1$ , by the strong LLN we have

$$\mathbb{P} \left( \lim_{N \rightarrow \infty} \frac{1}{N} \max_{2 \leq m \leq N} |r(1, m) - (m-1)| = 0 \right) = 1.$$

# Our results

## Theorem (Y. '24)

*If the (LLN) condition on the resistances holds, we have*

$$\lim_{N \rightarrow \infty} \frac{N^2 \text{gap}_N}{\pi^2} = 1.$$

*Furthermore, concerning the shape and (weighted) derivative of the eigenfunction  $g_1$  with  $g_1(1) := 1$  corresponding to the spectral gap, i.e.  $\Delta^{(c)} g_1 = -\text{gap}_N \cdot g_1$  and setting*

$$h(x) := \cos\left(\frac{\pi(x - 1/2)}{N}\right), \quad \forall x \in \llbracket 1, N \rrbracket,$$

*we have  $((c\nabla f)(x) := c(x - 1, x)[f(x) - f(x - 1)])$*

$$\lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} |g_1(x) - h(x)| = 0,$$

$$\lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} |N(c\nabla g_1)(x) - N(\nabla h)(x)| = 0.$$

# Our results

## Remark

*The method in the forementioned theorem also works for the other  $j$ -monotone eigenfunctions under the (LLN) assumption, i.e. with  $K_0 \in \mathbb{N}$  being any prefixed constant, for all  $1 \leq i \leq K_0$ ,*

$$\begin{aligned}\lim_{N \rightarrow \infty} |\lambda_i N^2 / \pi^2 - i^2| &= 0, \\ \lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} \left| g_i(x) - \cos \left( \frac{i\pi(x - 1/2)}{N} \right) \right| &= 0, \\ \lim_{N \rightarrow \infty} \sup_{x \in \llbracket 1, N \rrbracket} |N(c\nabla g_i)(x) - N(\nabla h_i)(x)| &= 0,\end{aligned}$$

where  $g_i(1) = 1$ .

# Our results: mixing time

## Assumption

Exist constants  $\nu \in (0, 1)$  and  $C_{\mathbb{P}} > 0$ , a sequence of positive numbers  $(\bar{\Upsilon}_N)_N > 0$  with  $\lim_{N \rightarrow \infty} \bar{\Upsilon}_N = 0$  and  $\lim_{N \rightarrow \infty} \bar{\Upsilon}_N \log N = \infty$  such that

$$\begin{aligned} \max_{1 \leq x < N} r(x, x+1) &\leq C_{\mathbb{P}} \exp((\log N)^\nu), \\ \min_{1 \leq x < N} r(x, x+1) &\geq \bar{\Upsilon}_N. \end{aligned}$$

Exists  $\varrho \in (0, 1]$  and  $c_\varrho > 0$  such that

$$c_\varrho N^\varrho \leq k_N \leq N/2.$$

## Theorem (Y. '24)

Under (LLN) and the assumption above, for all  $\varepsilon \in (0, 1)$  we have

$$\lim_{N \rightarrow \infty} \frac{2\pi^2 t_{\text{mix}}^{N,k}(\varepsilon)}{N^2 \log k_N} = 1.$$

# Outline

- Idea for the  $j$ -monotonicity of eigenfunctions
- Idea for the spectral gap
- Idea for the shape & derivative of eigenfunction
- Idea for the lower bound on the mixing time
- Idea for the upper bound on the mixing time

Idea:  $j$ —monotonicity of eigenfunctions

## Idea: $j$ -monotonicity of eigenfunctions

$\mathcal{L}_{N,1}$  is a symmetric matrix. Then it is diagonalizable:  $g_0 = \mathbf{1}$  and

$$\begin{cases} 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}, \\ \mathcal{L}_{N,1} g_i = -\lambda_i g_i \text{ and } g_i(1) = 1, & \forall i \in \llbracket 0, N-1 \rrbracket, \\ \frac{1}{N} \sum_{x=1}^N g_i(x) g_j(x) = C_{i,j} \delta_{i,j}, & \forall i, j \in \llbracket 0, N-1 \rrbracket. \end{cases}$$

$\delta_{i,j}$ : Kronecker delta       $(C_{i,j})_i$  are some positive constants. Observe:

$$(c\nabla g_i)(x+1) - (c\nabla g_i)(x) = -\lambda_i g_i(x) \Rightarrow$$

$$(c\nabla g_i)(x+1) = -\lambda_i \sum_{y=1}^x g_i(y).$$

Given  $c(x, x+1)_x$ ,  $g_i(1) = 1$  and  $\lambda_i$  together determine  $g_i$ , implying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}.$$

Assuming  $g_i$  is  $i$ -monotone, use the variational formula to show

$g_1$  is strictly monotone.

$F_i(\xi) := \sum_{x=1}^N g_i(x) \xi(x)$  is an eigenfunction of  $\mathcal{L}_{N,k}$  with eigenvalue  $-\lambda_i$ .

$F_1$  is monotone in the natural partial order  $\Rightarrow \text{gap}_{N,k} = \lambda_1$ .

## Idea: $j$ -monotonicity of eigenfunctions

Setting  $c(N, N+1) = 1$ , for  $\lambda > 0$ , define  $f^\lambda : \llbracket 0, N+1 \rrbracket \mapsto \mathbb{R}$  by  $f^\lambda(0) = f^\lambda(1) = 1$  and for  $x \in \llbracket 1, N \rrbracket$ ,

$$f^\lambda(x+1) = f^\lambda(x) + \frac{1}{c(x, x+1)} \left[ (c\nabla f^\lambda)(x) - \lambda f^\lambda(x) \right].$$

Note that (the restriction to  $\llbracket 1, N \rrbracket$  of)  $f^\lambda$  is an eigenfunction of  $\mathcal{L}_{N,1}$  if and only if

$$f^\lambda(N+1) = f^\lambda(N).$$

There is no eigenfunction satisfying  $f^\lambda(1) = 0$  or  $f^\lambda(N) = 0$ .



## Idea: $j$ -monotonicity of eigenfunctions

For  $\lambda > 0$  and  $x \in \llbracket 1, N+1 \rrbracket$ , we set

$$b(\lambda, x) := -\frac{(c\nabla f^\lambda)(x)}{f^\lambda(x-1)}$$

convention:  $b(\lambda, x) = \overline{\infty}$  if  $f^\lambda(x-1) = 0$ , and  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\overline{\infty}\}$ . We have

$$b(\lambda, x+1) = \frac{b(\lambda, x)}{1 - c(x-1, x)^{-1}b(\lambda, x)} + \lambda.$$

Given a fixed  $c > 0$ , define  $\Xi^{(c)} : \mathbb{R} \times \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$  as

$$\Xi^{(c)}(\lambda, b) = \frac{b}{1 - c^{-1}b} + \lambda.$$

The function  $b \mapsto \Xi^{(c)}(\lambda, b)$  may have zero, one or two fixed points depending on the values of  $\lambda$  and  $c$ , see the following figure.

# Idea: $j$ -monotonicity of eigenfunctions

$$\Xi^{(c)}(\lambda, b) = \frac{b}{1 - c^{-1}b} + \lambda.$$

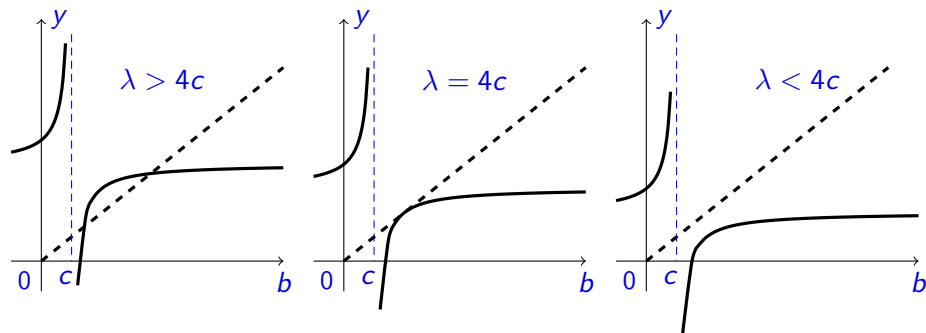


Figure: Solid lines:  $b \mapsto \Xi^{(c)}(b, \lambda)$  with  $\lambda > 0$  fixed. Black dashed lines:  $y = b$ . Blue dashes lines:  $b = c$ .

If  $b \mapsto \Xi^{(c)}(\lambda, b)$  has fixed points  $b_1$  and  $b_2$  (not necessarily distinct) such that  $b_1 \leq b_2$ , we define  $I(\lambda, c) = [b_1, b_2]$ . Otherwise, set  $I(\lambda, c) = \emptyset$ .

## Idea: $j$ –monotonicity of eigenfunctions

Define the “angle mapping” function

$$\varphi(c, \lambda, \theta) := \inf\{\theta' \geq \theta + \pi \mathbf{1}_{I(\lambda, c)}(\tan \theta) : \tan \theta' = \Xi^{(c)}(\lambda, \tan \theta)\}.$$

Recursively define an “angle”  $\theta(\lambda, x)$ :  $\theta(\lambda, 1) = 0$  and for  $x \in \llbracket 1, N \rrbracket$ ,

$$\theta(\lambda, x+1) := \varphi(c(x-1, x), \lambda, \theta(\lambda, x))$$

with convention:  $\tan(\pi/2 + k\pi) = \overline{\infty}$  for  $k \in \mathbb{Z}$ .

### Lemma

*For fixed  $c, \lambda > 0$ , the map  $\theta \mapsto \varphi(c, \lambda, \theta)$  is continuous and strictly increasing.*

### Lemma

*For fixed  $c, \theta > 0$ , the map  $\lambda \mapsto \varphi(c, \lambda, \theta)$  is strictly increasing and uniformly continuous in  $\theta$ .*

# Idea: $j$ -monotonicity of eigenfunctions

## Lemma

For fixed  $c > 0$ , the map  $(\lambda, \theta) \mapsto \varphi(c, \lambda, \theta)$  is jointly continuous.

$f^\lambda$  is an eigenfunction if and only if  $\theta(\lambda, N+1)$  is a multiple of  $\pi$ .

$$f^\lambda \text{ is an eigenfunction} \quad \Leftrightarrow \quad \theta(\lambda, N+1) = k\pi \text{ for } k \in \llbracket 0, N-1 \rrbracket.$$

Let  $\lambda_k > 0$  denote the unique number satisfying  $\theta(\lambda_k, N+1) = k\pi$  and set  $f_k := f^{\lambda_k}$ . Let  $x_i \in \llbracket 1, N \rrbracket$  such that  $\theta(\lambda_k, x_i) \leq i\pi < \theta(\lambda_k, x_i + 1)$  for  $i \in \llbracket 1, k-1 \rrbracket$ .

## Lemma

For  $\lambda_k$  mentioned above and the associated sequence  $(x_i)_i$ , we have that  $\# \{(x_i)_i\} = k-1$ ,  $1 < x_i < N$  are the local extrema of  $f_k$  (or the pair  $\{x_i - 1, x_i\}$  when  $\theta(\lambda_k, x_i) = i\pi$ ) and no any other local extrema.

Idea: the spectral gap

## Idea for the spectral gap

Setting  $B^{(N)}(x) := b(\lambda, x)N$  and  $\lambda := \alpha/N^2$ , we have

$$B^{(N)}(x+1) = \frac{B^{(N)}(x)}{1 - N^{-1}r^{(N)}(x-1, x)B^{(N)}(x)} + \frac{\alpha}{N},$$

which starts from  $B^{(N)}(1) := 0$ .

$$N[B(x+1) - B(x)] = \frac{r(x-1, x)B(x)^2}{1 - B(x)r(x-1, x)N^{-1}} + \alpha$$

Intuition: the asymptotic ODE

$$\begin{cases} \frac{dy}{dx} = y^2 + \alpha, & x \in (0, 1) \\ y(0) = 0. \end{cases}$$

Its unique solution:

$$y(x) = \sqrt{\alpha} \tan(\sqrt{\alpha} \cdot x).$$

Therefore  $\alpha = i^2\pi^2$ .

Idea: shape/derivative of the  
eigenfunction

## Idea for shape of the eigenfunction

Eigenfunction corresponding to the spectral gap when  $r(j-1, j) \equiv 1$

$$h(x) = h_N(x) = \cos\left(\frac{\pi(x-1/2)}{N}\right).$$

The spectral gap is

$$\bar{\lambda} := 2 \left[ 1 - \cos\left(\frac{\pi}{N}\right) \right] = \frac{\pi^2}{N^2} + O\left(\frac{1}{N^4}\right).$$

By  $b(\lambda, x) = -\frac{(c\nabla g)(x)}{g(x-1)}$  and  $b(\lambda, x) = B(x)/N$ , for  $x \geq 2$  we have

$$g(x) = [1 - r(x-1, x)N^{-1}B(x)] g(x-1).$$

Writing  $u(x) := h(x) - g(x)$ , we have

$$\begin{aligned} u(x) &= u(x-1) \left[ 1 - \frac{r(x-1, x)B(x)}{N} \right] \\ &\quad + \frac{h(x-1)}{N} [r(x-1, x)B(x) - \bar{B}(x)]. \end{aligned}$$

Iterate the equation above to conclude the proof for  $x \leq N/3$ .



## Idea for the derivative of the eigenfunction

For  $x \leq N/3$ : by

$$(c\nabla g_i)(x+1) = -\lambda_i \sum_{y=1}^x g_i(y),$$

we have

$$\begin{aligned} & |N(c\nabla g)(x) - N(\nabla h)(x)| \\ &= \left| -N\lambda_1 \sum_{k=1}^{x-1} g(k) + N\bar{\lambda} \sum_{k=1}^{x-1} h(k) \right| \\ &\leq \left| -N\lambda_1 \sum_{k=1}^{x-1} [g(k) - h(k)] \right| + \left| N(\bar{\lambda} - \lambda_1) \sum_{k=1}^{x-1} h(k) \right| \\ &\leq N\lambda_1 \sum_{k=1}^{x-1} |[g(k) - h(k)]| + N|\bar{\lambda} - \lambda_1|(x-1). \end{aligned}$$

For  $N/3 \leq x \leq 2N/3$ : use  $A(x) = 1/B(x)$

$$A(x) \left[ 1 - \frac{g(x)}{g(x-1)} \right] = r(x-1, x)N^{-1}.$$

Idea: the lower bound on the mixing time

## Idea for the lower bound on the mixing time

$$F(\xi) = F_1(\xi) = \sum_{1 \leq x \leq N} \xi(x) g_1(x)$$

is an eigenfunction satisfying  $\mathcal{L}_{N,k} F = -\text{gap}_N \cdot F$ . For  $t_0 > 0$ , define

$$F(t, \xi) := e^{\lambda_1(t-t_0)} F(\xi), \quad \forall \xi \in \Omega_{N,k},$$

and study the Dynkin martingale

$$M_t := F(t, \eta_t^\nu) - F(0, \eta_0^\nu) - \int_0^t (\partial_s + \mathcal{L}_{N,k}) F(s, \eta_s^\nu) ds.$$

$$\mathbf{E}[F(\eta_{t_0}^\nu)] = \mathbf{E}[F(t_0, \eta_{t_0}^\nu)] = \mathbf{E}[F(0, \eta_0^\nu)] = e^{-\lambda_1 t_0} \mathbf{E}[F(\eta_0^\nu)].$$

$$\mathbf{E}[M_{t_0}^2] = \mathbf{E}\left[\int_0^{t_0} \partial_s \langle M \rangle_s ds\right]$$

$$\bar{\eta}_t^\nu(x, x+1) := \eta_t^\nu(x)(1 - \eta_t^\nu(x+1)) + \eta_t^\nu(x+1)(1 - \eta_t^\nu(x))$$

$$\partial_t \langle M \cdot \rangle_t = e^{2\lambda_1(t-t_0)} \sum_{x=1}^{N-1} \bar{\eta}_t^\nu(x, x+1) r(x, x+1) [c(x, x+1)(g(x) - g(x+1))]^2.$$

## Idea for the lower bound on the mixing time

At equilibrium

$$\mathbf{E} [F(\eta_{t_0}^\mu)] = \mu_{N,k}(F) = \frac{k}{N} \sum_{1 \leq x \leq N} g_1(x) = 0, \quad \text{Var}_\mu(F) \asymp k.$$

- ① If  $\nu$  concentrates at one configuration, then

$$\mathbf{E} [F(\eta_{t_0}^\nu) - \mathbf{E} [F(\eta_{t_0}^\nu)]]^2 = \mathbf{E} [F(t_0, \eta_{t_0}^\nu) - F(0, \eta_0^\nu)]^2 = \mathbf{E} [M_{t_0}^2].$$

- ② If  $\nu$  is non-degenerated, we have

$$\begin{aligned} & \mathbf{E} [F(\eta_{t_0}^\nu) - \mathbf{E} [F(\eta_{t_0}^\nu)]]^2 \\ &= \mathbf{E} [F(t_0, \eta_{t_0}^\nu) - F(0, \eta_0^\nu) + F(0, \eta_0^\nu) - \mathbf{E} [F(0, \eta_0^\nu)]]^2 \\ &\leq 2\mathbf{E} [M_{t_0}^2] + 2\mathbf{E} [F(0, \eta_0^\nu) - \mathbf{E} [F(0, \eta_0^\nu)]]^2. \end{aligned}$$

If  $N/64 \leq k \leq N/2$ , take  $\nu = \delta_\wedge$ .

If  $(\log N)^{1+\gamma} \leq k < N/64$ , take  $\nu$  as follows: sample a configuration according to  $\mu_{N,2k}$ , keep the first  $k$  particles and project the rest to be empty sites.

An explanation for the lower bound  $t_{\text{mix}} \geq \frac{1}{2\lambda_1} \log k - \frac{C}{\lambda_1}$ :

$$ke^{-\lambda_1 \cdot (\frac{1}{2\lambda_1} \log k - \frac{C}{\lambda_1})} = e^C \sqrt{k}.$$

Idea: the upper bound on the mixing time

## Idea: the upper bound on the mixing time

Height function:

$$\xi \in \Omega_{N,k} \quad \rightarrow \quad h^\xi(x) := \sum_{y=1}^x \xi(y) - \frac{k}{N}x.$$

A partial order:

$$(\xi \leq \xi') \quad \Leftrightarrow \quad \left( h^\xi(x) \leq h^{\xi'}(x), \forall x \in \llbracket 1, N \rrbracket \right).$$

Attractive:

$$\left( h^\xi \leq h^{\xi'} \right) \quad \Rightarrow \quad \left( \forall t \geq 0, h_t^\xi \leq h_t^{\xi'} \right).$$

Coalescing time:

$$\begin{aligned} T_1 &:= \inf \left\{ t \geq 0 : h_t^\wedge = h_t^\mu \right\}, \\ T_2 &:= \inf \left\{ t \geq 0 : h_t^\vee = h_t^\mu \right\}. \end{aligned}$$

## Idea: the upper bound on the mixing time

Construct a supermartingale: inspired by [Wilson, '04], embed the segment  $\llbracket 1, N \rrbracket$  in  $\llbracket -\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor \rrbracket$  and place conductance  $(c(x, x+1) = 1)_{x \notin [1, N-1]}$ . The principle eigenfunction satisfies:

$$\lim_{N \rightarrow \infty} \sup_{x \in \llbracket -\lfloor \delta N \rfloor, N + \lfloor \delta N \rfloor \rrbracket} \left| G(x) - \cos \left( \frac{\pi(x + \lfloor \delta N \rfloor + 1/2)}{\bar{N}} \right) \right| = 0.$$

Define  $\bar{G}(x) := G(x) - G(x+1) > 0$  and

$$\mathbf{F}(\xi) := \sum_{x=1}^{N-1} h^\xi(x) \bar{G}(x).$$

For  $\xi, \xi' \in \Omega_{N,k}$  with  $\xi \leq \xi'$ , since  $h^\xi(x) \leq h^{\xi'}(x)$  and  $\bar{G}(x) > 0$ ,

$$\mathbf{F}(\xi) \leq \mathbf{F}(\xi').$$

Furthermore, if  $\xi \leq \xi'$  with  $\xi \neq \xi'$ , we have  $\mathbf{F}(\xi) < \mathbf{F}(\xi')$ .

## Idea: upper bound on the mixing time

Using  $h^\xi(0) = h^\xi(N) = 0$  and for  $x \in \llbracket 1, N-1 \rrbracket$

$$\begin{aligned}(\mathcal{L}_{N,k} h^\xi)(x) &= c(x, x+1) [\xi(x+1) - \xi(x)] \\ &= c(x, x+1) \left[ \left( h^\xi(x+1) - h^\xi(x) \right) - \left( h^\xi(x) - h^\xi(x-1) \right) \right]\end{aligned}$$

we obtain

$$\begin{aligned}(\mathcal{L}_{N,k} \mathbf{F})(\xi) &= \sum_{x=1}^{N-1} \bar{G}(x) (\mathcal{L}_{N,k} h^\xi)(x) \\ &= -\bar{\lambda}_1 \mathbf{F}(\xi) - h^\xi(1) c(0, 1) \bar{G}(0) - h^\xi(N-1) c(N, N+1) \bar{G}(N)\end{aligned}$$

where  $\bar{\lambda}_1$  is the spectral gap of the system in the longer line segment.



## Idea: upper bound on the mixing time

For  $\xi \leq \xi'$ ,

$$\begin{aligned} (\mathcal{L}_{N,k} \mathbf{F})(\xi') - (\mathcal{L}_{N,k} \mathbf{F})(\xi) &= -\bar{\lambda}_1 [\mathbf{F}(\xi') - \mathbf{F}(\xi)] \\ &\quad - \left[ h^{\xi'}(1) - h^{\xi}(1) \right] c(0,1) \bar{G}(0) - \left[ h^{\xi'}(N-1) - h^{\xi}(N-1) \right] c(N,N+1) \bar{G}(N) \\ &\leq -\bar{\lambda}_1 [\mathbf{F}(\xi') - \mathbf{F}(\xi)] . \end{aligned}$$

Then  $(\mathbf{F}(h_t^\wedge) - \mathbf{F}(h_t^\mu))_{t \geq 0}$  is a supermartingale with decay rate  $\bar{\lambda}_1$ .

Combine the approaches in [Lacoin AOP'16] and [Labbé, Lacoin AAP'20] to adapt to the disordered setup to conclude the proof.

