An Introduction to Noise

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Abstract

This document has two sections. The first is about mathematics tools to treat noise and the second is physics of thermal noise.

In Section 1, we will first review convenient mathematics concepts how to treat noise in electrical circuits, namely noise spectral density and autocorrelation. When we introduce noise spectral density, separate application of time average and ensemble average makes it easier to understand those concepts. Then build a few convenient formulae to treat transient behavior after a noise source turns on, which is common situation in discrete time systems. These formulae make it possible to estimate total noise of discrete time circuits (such as track and hold) by AC noise analysis.

In Section 2, we will go over some physics of thermal noise. We first derive Nyquist formula from two perspective. One is electrical circuit (macroscopic) point of view. The other is (classical) microscopic point of view. Then, we will learn relation between thermal noise and diffusion phenomena and behavior of a particle consisting gas, which gives some basis of mean free path model. Finally, we will see "shot noise" in semiconductor can also be understood as noise of diffusion current. In addition, we will briefly go over noise of electromagnetic wave, black-body radiation.

Contents

1 Noise Spectrum and Integrated Noise

In the linear response theory, our signals are either finite (impluse response) or periodic (frequency response). However noise is infinite (not periodic), we can not apply our linear response theory to noise as it is. But the good news is that noise is usually small and its mean square is finite. For small quantities, we expect the system respond lineary. We can use most of things we had learnt in linear response theory. In this section we will extend linear response theory to deal with infinite stream such as noise.

1.1 Measurement and average

Suppose that we sample an output (either voltage or current) of a system for N times during time period of T and assume that the system is in the same state (which characterizes fluctuation at the output) during that period. Let's say $v(t_n)$ is sampled value for n-th sample taken at t_n ($n = [1, N]$).[1] We take average of sampled value as the value of the output,

$$\langle v(t_n) \rangle_N = \frac{1}{N} \sum_{n=1}^{N} v(t_n),$$

and take difference between the value of output and sampled value as noise, i.e.,

$$v(t_n) = \langle v(t_n) \rangle_N + x(t_n),$$

where $x(t_n)$ is noise component of the output. Therefore, average of noise component is always zero by definition:

$$\langle x(t_n) \rangle_N = 0.$$

This is how we measure noise. However, following idealized averages (time average and ensemble average) are more convenient for theoretical study. The concept of two averages are very different, but it is almost always assumed the they give the same result. Because of this a lot of literature does not distinguish these averages but the author believes it makes much easier to understand the theory if we use them strictly when we construct the theory.

Time average Imagine that we could monitor the output over time period of T continuously. We can define noise component in similar way,

$$v(t) = \langle v(t) \rangle_T + x(t),$$

[1] Time points t_n are not necessarily equally spaced, they only needs to be almost uniformly distributed.

where

$$\langle v(t) \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} v(t)\, dt.$$

Therefore time average of noise component is zero by definition:

$$\langle x(t) \rangle_T = 0.$$

Ensemble average We can think of a lot of replicated system at the same time and take average over systems instead of repeated measurements. Average over a lot of replicated systems is called ensemble average. We denote ensemble average as $\langle x^2 \rangle$.

1.2 Noise spectral density

Imagine that we could monitor the output over time period of T continuously. In this case, we can write noise component $x(t)$ in Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} X_n\, e^{i\frac{2\pi n}{T}t}, \qquad X_n = \int_{-T/2}^{T/2} x(t)\, e^{-i\frac{2\pi n}{T}t}\, dt. \tag{1}$$

Since $x(t)$ is real number, changing sign of frequency gives complex conjugate:

$$X_{-n} = X_n^*.$$

X_0 is zero, since average of noise is zero:[2]

$$X_0 = T \langle x(t) \rangle_T = 0.$$

We are interested in mean square. Using above Fourier series,

$$\begin{aligned}
\left\langle x^2(t) \right\rangle_T &= \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t)\, dt, \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{X_n X_m}{T^2}\, e^{i\frac{2\pi(n+m)}{T}t}\, dt, \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{X_n^* X_m}{T^2}\, e^{i\frac{2\pi(m-n)}{T}t}\, dt. \qquad (n \to -n)
\end{aligned}$$

Recalling that

$$\delta_{n,m} = \frac{1}{T} \int_{-T/2}^{T/2} e^{i\frac{2\pi(m-n)}{T}t} dt,$$

[2] We need to be careful about this when we do $T \to \infty$.

we get

$$\langle x^2(t)\rangle_T \;=\; \sum_{n=-\infty}^{\infty} \frac{X_n^* X_n}{T} \cdot \frac{1}{T}.$$

Note that we do not need phase information of Fourier coefficients to calculate mean square ($X^*X = |X|^2$). Recalling that $X_0 = 0$ and that $X_{-n} = X_n^*$, above can be

$$\langle x^2(t)\rangle_T \;= 2\sum_{n=1}^{\infty} \frac{X_n^* X_n}{T} \cdot \frac{1}{T}.$$

Taking ensemble average yields

$$\langle x^2\rangle = 2\sum_{n=1}^{\infty} \left\langle \frac{X_n^* X_n}{T} \right\rangle \cdot \frac{1}{T}.$$

Mean square noise is decomposed into frequency components. The right hand side can be studied by frequency domain measurement (e.g. spectroscopy) and time period T corresponds to frequency resolution $1/T$. If there is a smooth function $S(f)$ which satisfies

$$\left\langle \frac{X_n^* X_n}{T} \right\rangle \;=\; S(n/T),$$

$\langle x^2\rangle$ is approximately written as,

$$\langle x^2\rangle \;\sim\; 2\int_{1/2T}^{\infty} S(f)\, df. \tag{2}$$

$S(f)$ ([V^2/Hz] or [A^2/Hz]) is called (double sided) noise spectral density. We most often use the term noise spectral density for single sided one, i.e., $2S(f)$ is the noise spectral density. Mean square noise calculated in the way expressed in Eq. (2) is called integrated noise. And when we just say noise, it likely refers noise spectral density.

Let's take a look at some known cases.

Example: RC low-pass filter $S(f)$ [V^2/Hz] for RC low-pass filter is known to be,[3]

$$2S(f) = \frac{4k\Theta R}{1 + (2\pi f\, RC)^2},$$

where Θ is absolute temperature and k is Boltzmann constant. Using

$$\int \frac{d\omega/2\pi}{1 + (\omega/\omega_0)^2} = \frac{\omega_0}{2\pi} \arctan\frac{\omega}{\omega_0},$$

[3] We will derive this formula later in Section 1.4.

mean square noise:

$$\left\langle x^2 \right\rangle = \int_{1/2T}^{\infty} 2S(f)\,df = \frac{k\Theta}{C}\left(1 - \frac{2}{\pi}\arctan 2\pi\frac{1}{2T}RC\right).$$

Recalling that $\arctan x \sim x$ for $x \ll 1$,

$$\left\langle x^2 \right\rangle \;\sim\; \frac{k\Theta}{C}\left(1 - \frac{2RC}{T}\right) \;\sim\; \frac{k\Theta}{C} \qquad (T \gg RC)\,.$$

This can also be written with $F = 1/T$ as,

$$\left\langle x^2 \right\rangle \sim \frac{k\Theta}{C} - 4k\Theta R\,F/2.$$

If we spend long time ($T/RC \gg 1$) for a measurement, mean square noise is insensitive to T. However if samples are concentrated within shorter time period we see smaller mean square noise (frequency component lower than $F/2$ falls into DC component). Figure 1 shows this situation.

Example: flicker noise Another example is flicker noise followed by noiseless low-pass filter of cut-off frequency f_0:

$$2S(f) = \frac{K}{f}\cdot\frac{1}{1+(f/f_0)^2},$$

where K is flicker noise coefficient (constant over f). Using

$$\int \frac{1}{f}\cdot\frac{1}{1+(f/f_0)^2}\;df = \frac{1}{2}\ln\frac{(f/f_0)^2}{1+(f/f_0)^2},$$

mean square noise:

$$\left\langle x^2 \right\rangle = \int_{F/2}^{\infty} \frac{K}{f}\cdot\frac{1}{1+(f/f_0)^2}\;df = \frac{K}{2}\ln\frac{1+\left(\frac{F/2}{f_0}\right)^2}{\left(\frac{F/2}{f_0}\right)^2},$$

where $F = 1/T$. Note that mean square diverges as $T \to \infty$, i.e., the longer we spend for a measurement the more we get noise. However this divergence is mild (logarithm).

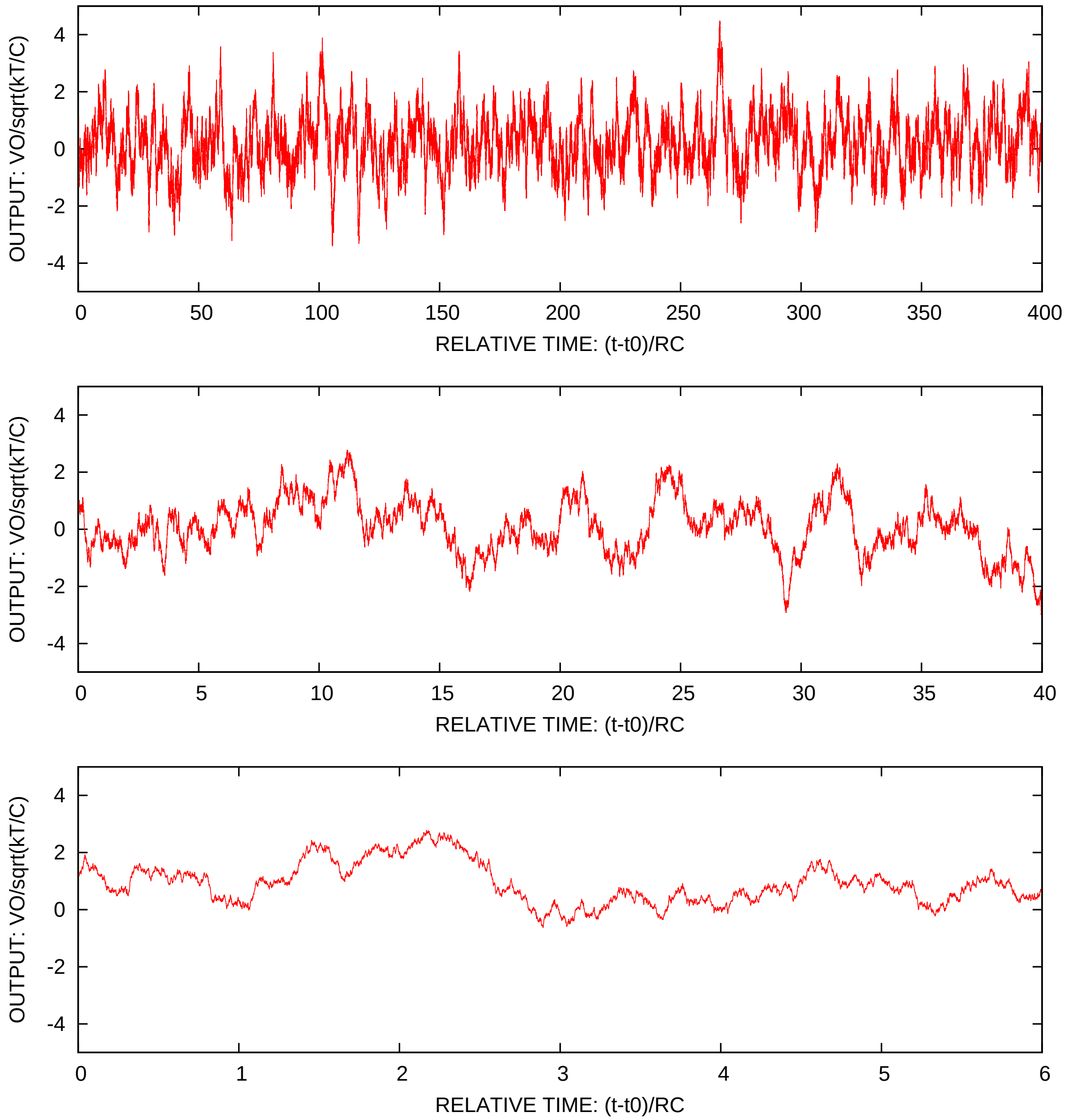


Figure 1: Simulated output waveform of RC low-pass filter for a few different time scale. For shorter period of time, we see less noise component (because high frequency component is filtered) and averaged value differs different time point of measurement. For longer time period, average gets closer to zero and mean square does not depend much on starting point of measurement.

1.3 Noise spectral density and autocorrelation

Eq. (2) becomes exact with $T \to \infty$. With this limit, Fourier series become Fourier integral:

$$x(t) = \lim_{T\to\infty} \int_{-\infty}^{\infty} X_T(f)\, e^{i\,2\pi f t}\, df, \qquad X_T(f) = \int_{-T/2}^{T/2} x(t)\, e^{-i\,2\pi f t}\, dt. \tag{3}$$

The reason why we have T explicitly is that $x(t)$ is in fact not square integrable ($\int_{-\infty}^{\infty} |x(t)|^2 dt$ diverges). Fourier transform of $x(t)$ is not well defined. However, what we want to calculate here is mean square, which is known to be finite. Convergence is established by taking limit $T \to \infty$ at very last. That's said, just as previous section, we see that mean square is decomposed into frequency components:

$$\begin{aligned}
\lim_{T\to\infty} \left\langle x^2(t) \right\rangle_T &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t)\, dt, \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_T(f)\, X_T(f')\, e^{i\,2\pi(f+f')t}\, df\, df'\, dt, \\
&= \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{X_T^*(f)\, X_T(f)}{T}\, df,
\end{aligned}$$

where we have used

$$\delta(f - f') = \int_{-\infty}^{\infty} e^{i\,2\pi(f-f')t}\, dt.$$

We define spectral density $S(f)$ by taking ensemble average of above as follows,

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} S(f)\, df, \qquad S(f) = \lim_{T\to\infty} \left\langle \frac{X_T^*(f)\, X_T(f)}{T} \right\rangle.$$

Note that $S(f) = S(-f)$ since $X_T(-f) = X_T^*(f)$. Therefore mean square noise can be calculated positive frequency only:

$$\left\langle x^2 \right\rangle = 2 \int_0^{\infty} S(f)\, df.$$

Let's take a look at following Fourier transform.

$$
\begin{aligned}
\lim_{T\to\infty}\int_{-\infty}^{\infty}&\frac{X_T^*(f)\,X_T(f)}{T}e^{i\,2\pi f\tau}df \\
&=\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}x(t)\,x(t')e^{i\,2\pi f(t-t'+\tau)}dtdt'df, \\
&=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}x(t)\,x(t')\delta(t-t'+\tau)dtdt', \\
&=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}x(t)\,x(t-\tau)dt, \\
&=\lim_{T\to\infty}\left\langle x(t)\,x(t-\tau)\right\rangle_T .
\end{aligned}
$$

This is autocorrelation. Therefore by taking ensemble average of above, it can be said that Fourier transform of spectral density is autocorrelation:

$$
\left\langle x(t)\,x(t-\tau)\right\rangle = \int_{-\infty}^{\infty} S(f)\,e^{i\,2\pi f\tau}\,df.
$$

Spectral density can be calculated from autocorrelation:

$$
S(f) = \int_{-\infty}^{\infty} \left\langle x(t)\,x(t-\tau)\right\rangle e^{i\,2\pi f\tau}\,d\tau.
$$

Autocorrelation can be found by analyzing time evolution of random process on the system in question.

We use $\phi(\tau)$ for autocorrelation not only for shorthand, but also to put emphasis on the fact that autocorrelation is a function of τ:

$$
\phi(\tau) = \left\langle x(t)\,x(t-\tau)\right\rangle .
$$

$\phi(\tau)$ is real and $\phi(0)$ is mean square by definition:

$$
\phi(0) = \left\langle x(t)\,x(t)\right\rangle = \left\langle x^2\right\rangle .
$$

It is unlikely that x at distant past affects present x, nor present x affects x at distant future, i.e.,

$$
\phi(\pm\infty) = \lim_{\tau\to\pm\infty}\left\langle x(t)\,x(t-\tau)\right\rangle = 0.
$$

$S(f)$ is positive real, and also even function, i.e.,

$$
S(f) > 0, \qquad S(-f) = S(f).
$$

We have just learned autocorrelation is Fourier transform of spectral density, vice versa:

$$\phi(\tau) = \int_{-\infty}^{\infty} S(f)\, e^{i\,2\pi f\tau}\, df, \qquad S(f) = \int_{-\infty}^{\infty} \phi(\tau)\, e^{-i\,2\pi f\tau}\, d\tau.$$

Therefore, $\phi(\tau)$ is even function because $S(f)$ is even function:

$$\phi(\tau) = \phi(-\tau).$$

Example 1: For a system of which autocorrelation is delta function:

$$\langle x(t)\, x(t-\tau)\rangle = K\,\delta(\tau).$$

Spectral density is flat:

$$S(f) = \int_{-\infty}^{\infty} K\delta(\tau)\, e^{i\,2\pi f\tau}\, d\tau = K.$$

Delta function autocorrelation represents memory less random process. Therefore if a system has flat noise spectrum, the noise is due to memory less random process.

Example 2: Spectral density [V^2/Hz] of RC low-pass filter is

$$2S(f) = \frac{4k\Theta R}{1+(2\pi f RC)^2}.$$

Autocorrelation will be

$$\phi(\tau) = \int_{-\infty}^{\infty} S(f)\, e^{i\,2\pi f\tau}\, df = \frac{k\Theta}{C}\, e^{-|\tau|/RC}.$$

The system would "forget its memory" after $t = RC$. If we look at the system in the time resolution much larger than its specific time (RC), the behavior should look as if it has delta function autocorrelation and indeed the spectral density is flat at $\omega \ll 1/RC$. $\phi(0)$ gives $k\Theta/C$.

If the spectrum is concentrated in lower frequency, autocorrelation becomes broader.

1.4 Composition of noise spectral density

Consider a system which has one noise source y of which spectral density $S_y(f)$ is known. Since noise is small, it is reasonable to assume that response of output x to noise source y is linear. First, we write noise source and its response in Fourier series:

$$x(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_T(n/T)\, e^{i\frac{2\pi n}{T} t}, \quad y(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} Y_T(n/T)\, e^{i\frac{2\pi n}{T} t}.$$

$Y_T(f)$ and $S_y(f)$ has following relation:

$$\lim_{T\to\infty} \left\langle \frac{Y_T^*(f)\, Y_T(f)}{T} \right\rangle = S_y(f).$$

Since $x(t)$ is linear response of $y(t)$,

$$X_T(f)/T = H(i\, 2\pi f)\, Y_T(f)/T \quad \rightarrow \quad X_T(f) = H(i\, 2\pi f)\, Y_T(f).$$

where $H(s)$ is transfer function, Laplace transform of impulse response. Therefore, noise spectral density at output:

$$\left\langle \frac{X_T^*\, X_T}{T} \right\rangle = \left\langle \frac{H^* Y_T^*\, H Y_T}{T} \right\rangle = |H|^2 \left\langle \frac{Y_T^*\, Y_T}{T} \right\rangle.$$

Here we take $T \to \infty$,

$$S_x(f) = |H(i\, 2\pi f)|^2\, S_y(f). \tag{4}$$

When we handle linear response, it is more convenient to use angular frequency $\omega = 2\pi f$ than ordinary frequency f. Here, we introduce following notation for single sided noise spectral density, so that we can easily go back and forth between those two kind of frequency.

$$\left\langle y^2 \right\rangle_f\, df = 2S_y(f)\, df, \qquad \left\langle y^2 \right\rangle_\omega\, d\omega = 2S_y(\omega/2\pi)\, \frac{d\omega}{2\pi}.$$

With this notation, Eq. (4) become

$$\left\langle x^2 \right\rangle_\omega = |H(i\omega)|^2\, \left\langle y^2 \right\rangle_\omega.$$

Similarly, if we have two noise sources y and z, Fourier coefficient of x is written with corresponding transfer function $F(s)$ and $G(s)$ and Fourier coefficient $Y_T(f)$ and $Z_T(f)$ as

$$X_T(f) = F(i\, 2\pi f)\, Y_T(f) + G(i\, 2\pi f)\, Z_T(f).$$

Assuming that there is no correlation between two noise sources i.e., they are independent each other:

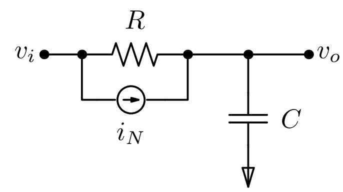
$$\langle Y_T^*(f)\, Z_T(f) \rangle = 0,$$

spectral density at output can be calculated as follows:

$$\left\langle x^2 \right\rangle_\omega = |F(i\omega)|^2\, \left\langle y^2 \right\rangle_\omega \;+\; |G(i\omega)|^2\, \left\langle z^2 \right\rangle_\omega.$$

Example: RC low-pass filter The only noise source is resistor R which is modeled as current source i_N in parallel. Noise spectral density is given by

$$\langle i_N^2 \rangle_\omega \, d\omega = \frac{4k\Theta}{R} \cdot \frac{d\omega}{2\pi}.$$



Equation for node v_o:

$$\frac{v_o - v_i}{R} + s\,C\,v_o - i_N = 0.$$

Therefore,

$$v_o = \frac{v_i}{1 + s\,RC} + \frac{R\,i_N}{1 + s\,RC} = F(s)\,v_i + G(s)\,i_N.$$

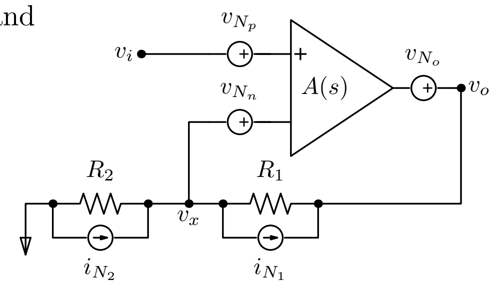
Spectral density on output due to resistor noise:

$$\langle v_o^2 \rangle_\omega \, d\omega = |G(i\omega)|^2 \langle i_N^2 \rangle_\omega \, d\omega = \frac{4k\Theta R}{1 + (\omega RC)^2} \cdot \frac{d\omega}{2\pi},$$

or

$$\langle v_o^2 \rangle_f \, df = \frac{4k\Theta R}{1 + (2\pi f RC)^2} \, df.$$

Example: Operational amplifier Opamp's noise is usually modeled as voltage sources on each terminal as shown v_{N_p}, v_{N_n} and v_{N_o} in the right. Opamp's transfer function:



$$A(s) = \frac{A_0}{1 + s\,\tau_A A_0} \sim \frac{1}{s\,\tau_A}.$$

Gain G is set by ratio of two resistors:

$$G = \frac{R_1 + R_2}{R_2} = 1 + \frac{R_1}{R_2}.$$

Opamp's equation and equation for node v_x is respectively,

$$v_o = v_{N_o} + A(s)\left((v_i + v_{N_p}) - (v_x + v_{N_n})\right), \quad \frac{v_x}{R_2} - i_{N_2} + \frac{v_x - v_o}{R_1} + i_{N_1} = 0.$$

Therefore,

$$v_o = \frac{G\left(v_i + v_{N_p} - v_{N_n}\right) + R_1\,(i_{N_1} - i_{N_2}) + s\,\tau_A G\,v_{N_o}}{1 + s\,\tau_A G},$$

or

$$v_o = E(s)\,v_i + E(s)\,v_{N_p} - E(s)\,v_{N_n} + F(s)\,i_{N_1} - F(s)\,i_{N_2} + H(s)\,v_{N_o},$$

where

$$E(s) = \frac{G}{1 + s\,\tau_A G}, \quad F(s) = \frac{R_1}{1 + s\,\tau_A G}, \quad H(s) = \frac{s\,\tau_A G}{1 + s\,\tau_A G}.$$

Recalling that noise current spectral density of a resistor $\left\langle i_N^2 \right\rangle_f$ is $4k\Theta/R$, contribution of feedback resistors:

$$\left\langle v_R^2 \right\rangle_\omega = |F(i\omega)|^2 \left(\left\langle i_{N_1}^2 \right\rangle_\omega + \left\langle i_{N_2}^2 \right\rangle_\omega \right) = \frac{4k\Theta R_1 G}{1 + (\omega\, \tau_A G)^2} \cdot \frac{1}{2\pi}.$$

Integrated noise:

$$\int_0^\infty \left\langle v_R^2 \right\rangle_\omega \, d\omega = \frac{k\Theta}{\tau_A / R_1}. \tag{5}$$

Input terminal's contribution:

$$\left\langle v_A^2 \right\rangle_\omega = |E(i\omega)|^2 \left(\langle v_{N_p}^2 \rangle_\omega + \langle v_{N_n}^2 \rangle_\omega \right) = \frac{G^2 (\langle v_{N_p}^2 \rangle_\omega + \langle v_{N_n}^2 \rangle_\omega)}{1 + (\omega\, \tau_A G)^2}.$$

Integrated noise:

$$\int_0^\infty \left\langle v_A^2 \right\rangle_\omega \, d\omega = \frac{G}{4\, \tau_A} \left(\langle v_{N_p}^2 \rangle_\omega + \langle v_{N_n}^2 \rangle_\omega \right). \tag{6}$$

$H(s)$ is high-pass filter, low frequency component of v_{N_o} is cut off. Contribution from v_{N_o} is usually very small.

In many cases, opamp A consists of transconductance amplifier followed by buffer amplifier. In such case, unity gain specific time and input terminal noise is given by

$$\tau_A \ \sim \ C_c / g_m, \qquad \langle v_{N_p}^2 \rangle_\omega = \langle v_{N_n}^2 \rangle_\omega \ \sim \ \frac{2k\Theta}{g_m} \cdot \frac{1}{2\pi},$$

where C_c and g_m is compensation capacitor and transconductance of the amplifier, respectively. Inserting these into Eq. (5) and (6) yields

$$\int_0^\infty \left\langle v_R^2 \right\rangle_\omega \, d\omega = \frac{k\Theta}{C_c} \cdot g_m R_1, \qquad \int_0^\infty \left\langle v_A^2 \right\rangle_\omega \, d\omega = \frac{k\Theta}{C_c} \cdot G.$$

Total input refereed integrated noise (excluding buffer noise) is sum of above divided by G^2:

$$\frac{1}{G^2} \int_0^\infty \left\langle v_o^2 \right\rangle_\omega \, d\omega = \frac{1}{G^2} \int_0^\infty \left(\left\langle v_R^2 \right\rangle_\omega + \left\langle v_A^2 \right\rangle_\omega \right) \, d\omega = \frac{k\Theta}{G C_c} \left(1 + \frac{g_m R_1}{G} \right).$$

As for inverting amplifier (set opamp's positive input to the ground and put signal at R_2), signal gain G_i is R_1/R_2. Therefore input referred integrated noise will be,

$$\frac{1}{G_i^2} \int_0^\infty \left\langle v_o^2 \right\rangle_\omega \, d\omega = \frac{k\Theta}{G_i C_c} \left(1 + \frac{1}{G_i} + g_m R_2 \right).$$

Noise floor from resistor, $4k\Theta R_1 G$, can be written with G_i as $4k\Theta G_i (R_1 + R_2)$.

In reality, we have excessive noise component such as flicker noise (MOSFET) or base resistance noise (BJT). Reference [3] and [2] treat such excessive noise component in detail.

1.5 Equivalent noise resistance and noise factor

In the last section, we have learnt that noise spectral density of RC low-pass filter is

$$\left\langle v_o^2 \right\rangle_f = \frac{4k\Theta R}{1+(2\pi f RC)^2}.$$

At low frequency it is $4k\Theta R$. $\left\langle v_o^2 \right\rangle_f$ can be obtained by simulation and $\left\langle v_o^2 \right\rangle_f$ at low frequencies (where it is flat) divided by $4k\Theta$ gives resistance R. Similarly, opamp's output noise spectral density is calculated using last section's example,

$$\left\langle v_o^2 \right\rangle_f = \left\langle v_R^2 \right\rangle_f + \left\langle v_A^2 \right\rangle_f = 4k\Theta R_1 G + 4k\Theta \cdot \frac{G^2}{g_m} = 4k\Theta \left(R_1 G + \frac{G^2}{g_m} \right).$$

Here we dropped frequency factor to focus on noise at low frequency. Again, $\left\langle v_o^2 \right\rangle_f / 4k\Theta$ gives a quantity in resistance which can easily be calculated from design parameters (R_1, R_2, g_m). We call $\left\langle v_o^2 \right\rangle_f / 4k\Theta$ as equivalent noise resistance. In this way, we can compare noise spectral densities obtained by simulation with the one calculated from design parameters. It will be more convenient to measure equivalent noise resistance in the unit of g_m because bandwidth of the amplifier is closely related with g_m:

$$\frac{\left\langle v_o^2 \right\rangle_f}{4k\Theta} = \left(R_1 G + \frac{G^2}{g_m} \right) = \frac{G^2}{g_m} \left(1 + \frac{g_m R_1}{G} \right).$$

Note that $g_m R_1$ stays constant if we scale everything wider, e.g., if we make everything twice as wide, g_m gets twice, R_1 become half. We call such quantity a scale invariant.

And input referred integrated noise is roughly baseline spectral density times 1/4 of its bandwidth. Recalling that bandwidth in this case is $1/\tau_A G$ and that $\tau_A = C_c/g_m$, input referred integrated noise will be

$$\frac{1}{G^2} \int_0^\infty \left\langle v_o^2 \right\rangle_f \, df = \frac{4k\Theta}{g_m} \left(1 + \frac{g_m R_1}{G} \right) \cdot \frac{1}{4\,\tau_A G} = \frac{k\Theta}{G C_c} \left(1 + \frac{g_m R_1}{G} \right).$$

Integrated noise divided by $k\Theta$ gives a quantity in (inverse of) capacitance which can easily be calculated from design parameters and we call it equivalent noise capacitance.

Noise factor It is more convenient measure equivalent noise capacitance in the unit of a capacitance specific to the system in question. In this case, it is GC_c. We call equivalent noise capacitance measured in the unit of such unit capacitance as noise factor. In this case, it is $(1 + g_m R_1/G)$. Note that noise factor is scale invariant, i.e., circuit it stays constant, if we scale everything wider or narrower while keeping current density the same.

1.6 Origin of white noise

Let us take a look at a series of impulse events which occurs randomly at fixed rate of P events per unit time. Let $x_T(t)$ be an observation of a such series of impulses for a time period T. $x_T(t)$ can be written as

$$x_T(t) = \sum_{j=1}^{N} \delta(t - t_j),$$

where N is number of events for a time period of T. If we observe this event long enough so that $PT \gg 1$, we can approximate N by PT. Fourier transform of $x_T(t)$:

$$X_T(f) = \int_{-T/2}^{T/2} x_T(t) e^{-i\,2\pi ft}\,dt = \sum_{j=1}^{PT} \int_{-T/2}^{T/2} \delta(t - t_j)\, e^{-i\,2\pi ft}\,dt = \sum_{j=1}^{PT} e^{-i\,2\pi ft_j}.$$

Spectral density:

$$\begin{aligned} S(f) &= \lim_{T\to\infty} \left\langle \frac{X_T^*(f) X_T(f)}{T} \right\rangle, \\ &= \lim_{T\to\infty} \sum_{j,k} \left\langle e^{i\,2\pi f(t_k - t_j)} \right\rangle \cdot \frac{1}{T}. \end{aligned}$$

Here time series $t_1,\ t_2,\ t_3, ...$ is randomly distributed for each observation, average of $e^{i\,2\pi f(t_k - t_j)}$ vanishes except for $j = k$ and the number of such term in the summation is PT. Therefore

$$S(f) = \lim_{T\to\infty} P\,T \cdot \frac{1}{T} = P,$$

P events per unit time means that event interval is $\tau = 1/P$ in average. In single sided notation:

$$\left\langle x^2 \right\rangle_f = 2S(f) = 2P = 2/\tau.$$

White noise source is mathematically modeled as a collection of consequences of these elementary impulses. If we know response of current, for example, against such elementary impulse, we can calculate noise spectral density from τ of that impulse. One example is current by itself. If x is the number of carrier passage at a given surface per unit time, current I will be q/τ and $q^2 \left\langle x^2 \right\rangle_f$ will be noise current spectral density. Therefore,

$$\left\langle i_N^2 \right\rangle = 2qI.$$

This is called shot noise.

We will study white noise in physical point of view later in Section 2.

1.7 Transient behavior

In discrete time systems such as data converters, we often find some of noise sources appear only one phase of the clock. For example MOS switch contributes noise only when it is closed. In this section we will study noise contribution of such noise sources as a function of time after it is turned on.

1.7.1 Frequency factor and effective bandwidth

We want to find mean square noise at output as a function of time t, in case noise source y is turned on at $t = 0$. We assume that y is white noise source. Suppose that $y(t)$ is noise voltage, its response $x(t)$ is written as follows:

$$x(t) = \int_0^t h(t-\tau)\, y(\tau)\, d\tau, \tag{7}$$

where $h(t)$ is impulse response of x to y. Ensemble average $\left\langle x^2(t)\right\rangle$ is the one we want.

The procedure to obtain $\left\langle x^2(t)\right\rangle$ is just as what we have done before. We first write $y(t)$ in Fourier series rather than integral, because $y(t)$ is not square integrable.

$$y(t) = \frac{1}{T}\sum_{\omega} Y_\omega\, e^{i\omega t}, \tag{8}$$

where sum is taken for all $\omega = 2\pi n$, $n = [-\infty \dots \infty]$. Then calculate $x^2(t)$ in terms of Y_ω and take ensemble average. Since y is white noise source, we expect that there is no correlation between different frequency components, i.e.,

$$\left\langle \frac{Y_\omega^*\, Y_{\omega'}}{T}\right\rangle = 0. \qquad \left(\omega' \neq \omega\right). \tag{9}$$

Finally bring $T \to \infty$, so that we can use

$$\lim_{T\to\infty} 2\left\langle \frac{Y_\omega^*\, Y_\omega}{T}\right\rangle = \lim_{T\to\infty} 2\left\langle \frac{|Y_\omega|^2}{T}\right\rangle = \left\langle y^2\right\rangle_\omega . \tag{10}$$

Now we execute the procedure. Inserting Eq. (8) into Eq. (7) yields

$$x(t) = \sum_{\omega} \frac{Y_\omega}{T}\int_0^t h(t-\tau)\, e^{i\omega\tau}\, d\tau.$$

Therefore

$$x^2(t) = \frac{1}{T}\sum_{\omega}\sum_{\omega'} \frac{Y_{\omega'}\, Y_\omega}{T}\int_0^t h(t-\tau)e^{i\omega\tau}\, d\tau \int_0^t h(t-\tau')\, e^{i\omega'\tau'}\, d\tau'.$$

Taking ensemble average and making substitution $\omega' \to -\omega'$ yields

$$\left\langle x^2(t)\right\rangle = \frac{1}{T}\sum_{\omega}\sum_{\omega'}\left\langle \frac{Y^*_{\omega'}\,Y_\omega}{T}\right\rangle \int_0^t h(t-\tau)\,e^{i\omega\tau}d\tau \int_0^t h(t-\tau')e^{-i\omega'\tau'}\,d\tau'.$$

Using Eq. (9) and noting that two integrals are complex conjugate each other if $\omega = \omega'$,

$$\left\langle x^2(t)\right\rangle = \frac{1}{T}\sum_{\omega}\left\langle \frac{Y^*_{\omega}\,Y_\omega}{T}\right\rangle \left|\int_0^t h(t-\tau)\,e^{i\omega\tau}\,d\tau\right|^2.$$

Taking limit $T \to \infty$, summation becomes integral:

$$\left\langle x^2(t)\right\rangle = \int_{-\infty}^{\infty}\frac{\left\langle y^2\right\rangle_\omega}{2}\left|\int_0^t h(t-\tau)\,e^{i\omega\tau}\,d\tau\right|^2 d\omega.$$

Since y is white $\left\langle y^2\right\rangle_\omega$ can be brought out from the integral. Since integrand of the outer integral is even function of ω, we take positive frequency only:

$$\begin{aligned}\left\langle x^2(t)\right\rangle &= \left\langle y^2\right\rangle_\omega \int_0^{\infty}\left|\int_0^t h(t-\tau)\,e^{i\omega\tau}\,d\tau\right|^2 d\omega,\\ &= \left\langle y^2\right\rangle_\omega \int_0^{\infty} f_t(\omega)\;d\omega = \left\langle y^2\right\rangle_\omega F(t).\end{aligned}$$

where we have defined frequency factor $f_t(\omega)$ and effective bandwidth $F(t)$:

$$f_t(\omega) = \left|\int_0^t h(t-\tau)\,e^{i\omega\tau}\,d\tau\right|^2, \qquad F(t) = \int_0^\infty f_t(\omega)\,d\omega = \frac{1}{2}\int_{-\infty}^{\infty} f_t(\omega)\,d\omega.$$

Frequency component of mean square noise at t can be defined as

$$\left\langle x^2(t)\right\rangle_\omega = \left\langle y^2\right\rangle_\omega f_t(\omega),$$

so that mean square noise can be written in integrated noise.

$$\left\langle x^2(t)\right\rangle = \int_0^\infty \left\langle x^2(t)\right\rangle_\omega d\omega.$$

Multiple source In case we have another independent (not correlated) white noise source z, its response is sum of these two,

$$x(t) = \int_0^t h(t-\tau)\,y(\tau)d\tau \;+\; \int_0^t g(t-\tau)\,z(\tau)d\tau.$$

We can repeat the same procedure. Using the fact y and z are not correlated, we get

$$\left\langle x^2(t)\right\rangle_\omega = \left\langle y^2\right\rangle_\omega \left|\int_0^t h(t-\tau)\,e^{i\omega\tau}\,d\tau\right|^2 \;+\; \left\langle z^2\right\rangle_\omega \left|\int_0^t g(t-\tau)\,e^{i\omega\tau}\,d\tau\right|^2.$$

Therefore we can calculate contribution from each noise source separately and then sum them up to obtain total amount.

Steady state limit Let us write transfer function in partial fraction expansion:

$$H(s) = \sum_k \frac{\tau_k}{1 + s\,\tau_k}\, a_k.$$

From the causality requirement, real part of τ_k is positive. In case $H(s)$ only has simple pole, impulse response function is

$$h(t) = \sum_k a_k\, e^{-t/\tau_k}.$$

Frequency factor $f_t(x)$:

$$f_t(\omega) = \left| \int_0^t h(t-\tau) e^{i\omega\tau}\, d\tau \right|^2 = \left| \sum_k \frac{a_k \tau_k}{1 + i\omega\,\tau_k} \left(e^{i\omega t} - e^{-t/\tau_k} \right) \right|^2.$$

For sufficiently large t, all e^{-t/τ_k} vanishes and we get steady state formula:

$$f_t(\omega) = \left| \sum_k \frac{a_k \tau_k}{1 + i\omega\,\tau_k}\, e^{i\omega t} \right|^2 = \left| H(i\omega)\, e^{i\omega t} \right|^2 = |H(i\omega)|^2\,, \qquad (t \to \infty)$$

therefore

$$\left\langle x^2(\infty) \right\rangle_\omega = |H(i\omega)|^2 \left\langle y^2 \right\rangle_\omega.$$

The same thing can be said for cases in which $H(s)$ has non-simple poles.

1.7.2 First order transfer function

$$H(s) = \frac{a_1 \tau_1}{1 + s\,\tau_1}, \qquad f_t(\omega) = \left| \frac{a_1 \tau_1}{1 + i\omega\,\tau_1} \left(e^{i\omega t} - e^{-t/\tau_1} \right) \right|^2.$$

Effective bandwidth:[4]

$$F(t) = \frac{1}{2} \int_{-\infty}^{\infty} f_t(\omega)\, d\omega = \frac{\pi}{2} \frac{(a_1 \tau_1)^2}{\tau_1} \left(1 - e^{-2t/\tau_1} \right). \tag{11}$$

For RC low-pass filter, $a_1 \tau_1 = 1$, $\tau_1 = RC$ and $\left\langle y^2 \right\rangle_\omega = 4k\Theta R / 2\pi$:

$$\begin{aligned} \left\langle x^2(t) \right\rangle &= \left\langle y^2 \right\rangle_\omega F(t), \\ &= \frac{4k\Theta R}{2\pi} \cdot \frac{\pi}{2RC} \left(1 - e^{-2t/RC} \right) = \frac{k\Theta}{C} \left(1 - e^{-2t/RC} \right). \end{aligned}$$

[4] Appendix D shows how to calculate this integral.

Example: ON resistance of MOS switch in track and hold Basic structure of track and hold circuit is RC low-pass filter in which R is replaced by MOS transistor. In hold mode, R is extremely high and i_N is extremely small. At transition to track mode ($t = 0$), i_N turns on. We sample output at T_s when the circuit goes back to hold mode. In usual operating conditions, however, T_s is at least a few times larger than its specific time $\tau = RC$, we do not get much noise reduction.

Example: Random walk Let us think about classic random walk problem[5]. When a particle is moving randomly, how far in average would it move in a time period of t?

The position x of the particle at t is simply integral of velocity over time:

$$x = \int_0^t v_N(t)\, dt.$$

Taking Laplace transform yields

$$X(s) = H(s)\, V_N(s), \qquad H(s) = \frac{1}{s}.$$

We take $V_N(s)$ as white noise source of which spectral density $\left\langle V_N^2 \right\rangle_\omega$. Note that dimension of $\left\langle V_N^2 \right\rangle_\omega$ is velocity squared per frequency, i.e., $[\mathrm{L}^2\mathrm{T}^{-2}(1/\mathrm{T})^{-1}] = [\mathrm{L}^2\mathrm{T}^{-1}]$. Using,

$$\frac{1}{s} = \lim_{\tau\to\infty} \frac{\tau}{1 + s\,\tau},$$

"effective bandwidth" is calculated from Eq. (11),

$$\begin{aligned} F(t) &= \lim_{\tau\to\infty} \frac{\pi}{2}\,\tau \left(1 - e^{-2t/\tau}\right), \\ &= \lim_{\tau\to\infty} \frac{\pi}{2}\,\tau\,(1 - 1 + 2t/\tau) = \pi\, t. \end{aligned}$$

Here we have used $e^\delta \sim 1 + \delta$ for small δ. Therefore square average distance of a random walk particle would make in a time period of t is

$$\left\langle x^2 \right\rangle = F(t) \left\langle V_N^2 \right\rangle_\omega = \pi \left\langle V_N^2 \right\rangle_\omega t.$$

A particle moving at random velocity does not stay in a definite region, even though its average velocity is zero ($\langle v_N(t) \rangle_T = 0$). If we think of an ensemble of particles concentrated at $x = 0$ at $t = 0$, square average of positions of particles would increase over time, which we observe as diffusion. We will come back this topic later in Section 2.

[5] Connection with random walk is pointed out by Manar El-Chammas.

1.7.3 Second order transfer function

$$H(s) = \frac{a_1\tau_1}{1+s\,\tau_1} + \frac{a_2\tau_2}{1+s\,\tau_2}.$$

Frequency factor is calculated as follows:

$$\begin{aligned} f_t(\omega) &= \left| \frac{a_1\tau_1\left(e^{i\omega\,t} - e^{-t/\tau_1}\right)}{1+i\omega\,\tau_1} + \frac{a_2\tau_2\left(e^{i\omega\,t} - e^{-t/\tau_2}\right)}{1+i\omega\,\tau_2} \right|^2, \\ &= \frac{(a_1\tau_1)^2\left(e^{i\omega\,t} - e^{-t/\tau_1}\right)\left(e^{-i\omega\,t} - e^{-t/\tau_1}\right)}{(1+i\omega\,\tau_1)\,(1-i\omega\,\tau_1)} \\ &\quad + \frac{(a_2\tau_2)^2\left(e^{i\omega\,t} - e^{-t/\tau_2}\right)\left(e^{-i\omega\,t} - e^{-t/\tau_2}\right)}{(1+i\omega\,\tau_2)\,(1-i\omega\,\tau_2)} \\ &\quad + \frac{a_1\tau_1\,a_2\tau_2\left(e^{i\omega\,t} - e^{-t/\tau_1}\right)\left(e^{-i\omega\,t} - e^{-t/\tau_2}\right)}{(1+i\omega\,\tau_1)\,(1-i\omega\,\tau_2)} + c.c.\ , \end{aligned}$$

where $c.c.$ is complex conjugate of the third term. Effective bandwidth:

$$\begin{aligned} F(t) = \frac{1}{2}\int_{-\infty}^{\infty} f_t(\omega)\,d\omega = \frac{\pi}{2}\Bigg[&\frac{(a_1\tau_1)^2}{\tau_1}\left(1-e^{-2t/\tau_1}\right) + \frac{(a_2\tau_2)^2}{\tau_2}\left(1-e^{-2t/\tau_2}\right) \\ &+ \frac{4a_1a_2}{1/\tau_1+1/\tau_2}\left(1-e^{-(1/\tau_1+1/\tau_2)t}\right)\Bigg]. \end{aligned} \tag{12}$$

Example: Base current noise of track and hold buffer In case track and hold circuit is followed by a bipolar transistor, its base current noise is integrated at the sampling capacitor and its spectral density diverges at low frequency.[2] Here output x is amount of charge at the next stage's sampling capacitor. Noise transfer function $H(s)$ is given by

$$H(s) = -\frac{\omega_0}{s} + \frac{a_1\tau_1}{1+s\,\tau_1},$$

where $1/\tau_1$ is overall bandwidth at the output. We can calculate effective bandwidth directly from this, however we can rewrite $H(s)$ as second order partial fraction with taking limit of $\tau_0 \to \infty$ as follows:

$$H(s) = -\frac{\omega_0}{s} + \frac{a_1\tau_1}{1+s\,\tau_1} = \lim_{\tau_0\to\infty}\left(-\frac{\omega_0\tau_0}{1+s\,\tau_0} + \frac{a_1\tau_1}{1+s\,\tau_1}\right).$$

Therefore effective bandwidth is obtained by substituting symbols and taking limit on Eq. (12). The result is

$$F(t) = \pi\,\omega_0^2\,t + \frac{\pi(a_1\tau_1)^2}{2\,\tau_1}\left(1-e^{-2t/\tau_1}\right) - 2\pi\,\omega_0\,a_1\tau_1\left(1-e^{-t/\tau_1}\right).$$

What we want is $F(T_s)$ where T_s is holding period. In usual operating conditions, T_s is at least a few times larger than τ_1, exponential terms vanish.

$$F(T_s) \;\sim\; \pi\,\omega_0^2\,T_s \,+\, \frac{\pi(a_1\tau_1)^2}{2\,\tau_1} \,-\, 2\pi\,\omega_0\,a_1\tau_1.$$

We see more noise at lower conversion rate. The first and the second term came from the first and the second term of $H(s)$, respectively. The third is cross product term and the effect can be seen more clearly by rewriting above as

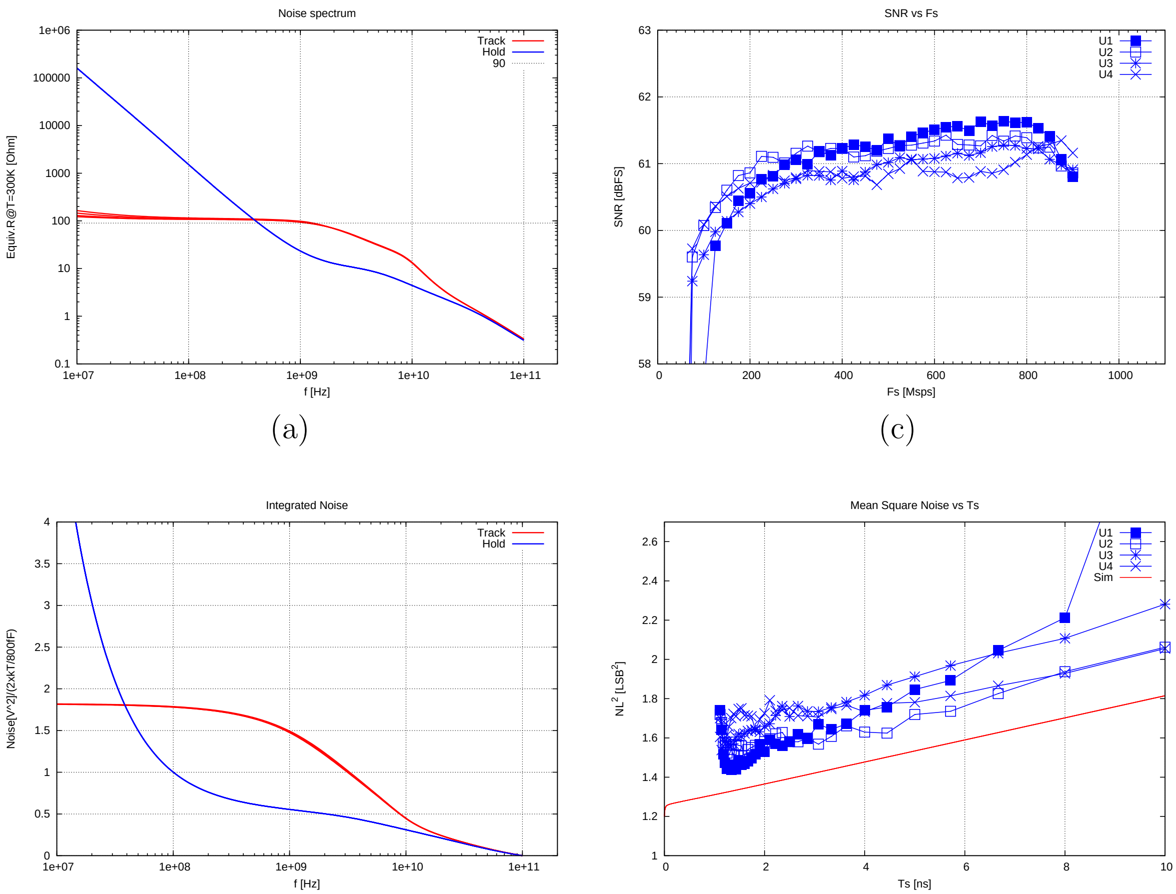
$$\frac{\pi(a_1\tau_1)^2}{2\,\tau_1} \,-\, 2\pi\,\omega_0\,a_1\tau_1 = \frac{\pi(a_1\tau_1)^2}{2\,\tau_1}\left(1 - \frac{4\,\omega_0}{a_1}\right).$$

In case the first term is dominating $F(T_s)$, it happens to be expressed as frequency domain integrated noise:

$$F(T_s) \;\sim\; \pi\,\omega_0^2\,T_s \;\sim\; \int_{\omega_s}^{\infty} |H(i\omega)|^2\,d\omega, \qquad \omega_s = \frac{1}{\pi\,T_s} = \frac{F_s}{\pi/2},$$

where F_s is sampling frequency $(2\,T_s = 1/F_s)$.

Figure 2 compares AC noise simulation and measurement data of a bipolar pipeline ADC. Please do not confuse this noise with the error caused by base current itself. Base current error is mainly common-mode and error voltage is proportional to time. Even if it shows up in differential mode causing gain error, noise power due to such error should be proportional to square of time.



(a) (c)

(b) (d)

Figure 2: (a) Noise spectral density of a bipolar track and hold circuit. In track phase it behaves like an RC low-pass filter. However in hold phase, it diverges as frequency goes low. This is because base current noise is integrated at the undriven capacitor. (b) Integrated noise $\int_f^\infty \langle v_N^2 \rangle_f \, df$. We can take value at $f = F_s/\pi^2$ as expected noise contribution of this block at sampling frequency F_s. (c) Measured SNR of pipeline ADC for a few device. Room Temp. Ain=–1dBFS. (d) Blue lines are SNR converted to mean square codes, using $\overline{\mathrm{NL}}^2 = (2^N/2\sqrt{2})^2/\mathrm{SNR}$. Red line is simulated noise of this ADC converter, calculated from AC noise spectral density by $\int_{F_s/\pi^2}^\infty \langle v_N^2 \rangle_f \, df$. The slope is originated low frequency divergence of base current noise. Distortion due to inter-stage gain error separates simulation and measurement.

1.7.4 Flicker noise

Let us consider following identity.

$$\frac{K}{\omega} = \frac{2K}{\pi} \int_0^\infty \frac{1}{1+\tau^2\omega^2}\, d\tau$$

The right hand side can be interpreted as infinite sum of low-pass filtered white noise source of which spectral density is $2K/\pi$, and the left hand side is spectral density of flicker noise. This is one of ways to model flicker noise. However, it is unlikely for a system to have infinite bandwidth ($\tau = 0$), there has to be lower limit for τ. Let's say this limit is τ_T. Spectral density of flicker noise $\left\langle x^2 \right\rangle_\omega$ can be written as follows:

$$\left\langle x^2 \right\rangle_\omega \;=\; \left\langle y_\tau^2 \right\rangle_\omega \int_{\tau_T}^\infty \frac{1}{1+\tau^2\omega^2}\, d\tau, \qquad \left\langle y_\tau^2 \right\rangle_\omega = \frac{2K}{\pi},$$

where y_τ is white noise source of which spectral density does not depend on τ. At low frequencies $\left\langle x^2 \right\rangle_\omega$ is proportional to $1/\omega$ and at high frequencies it is proportional to $1/\omega^2$, transition occurs around $\omega \sim 1/\tau_T$.

Effective bandwidth for flicker noise is the last expression of Eq. (11) with $a_1\tau_1 = 1$ integrated from τ_T to infinity,

$$F(t) \;=\; \frac{\pi}{2} \int_{\tau_T}^\infty \frac{1 - e^{-2t/\tau}}{\tau}\, d\tau.$$

Thanks to finite τ_T, this integral converges and there is a ξ which satisfies

$$\int_{\tau_T}^{2t\,\xi} \frac{1}{\tau}\, d\tau \;=\; \int_{\tau_T}^\infty \frac{1 - e^{-2t/\tau}}{\tau}\, d\tau,$$

because both integrand is positive and integrand of the right side is always smaller than that of the left. With such ξ, effective bandwidth is written as follows:

$$F(T_s) \;=\; \frac{\pi}{2} \int_{\tau_T}^{2T_s\,\xi} \frac{1}{\tau}\, d\tau \;=\; \frac{\pi}{2} \ln \frac{2T_s\,\xi}{\tau_T} = \pi^2 \int_{F_s/\xi}^{f_T} \frac{1}{f}\, df.$$

where $2T_s = 1/F_s$, $2\pi f_T = 1/\tau_T$. Again, it happens to be expressed as frequency domain integrated noise. ξ is a function of τ_T and we hope(!) $\xi \sim 1$ for $2T_s \gg \tau_T$.

2 A Few Topics on Thermal Noise

This section gives some physical basis on thermal noise formulae used in the previous sections. This section is not particularly useful in circuit design.

2.1 Macroscopic derivation of Nyquist formula

2.1.1 Statistical mechanics of tank circuit

Suppose that an LC tank is placed in the middle of empty space. The total energy of this system is the sum of energy stored in the inductor L and the capacitor C:

$$E = \frac{1}{2}LI^2 + \frac{1}{2}CV^2,$$

where I and V is current of the circuit and voltage across C, respectively. Recalling that $I = C\dot{V}$, this can be written as

$$E = \frac{1}{2}LC^2\dot{V}^2 + \frac{1}{2}CV^2.$$

This is identical to a harmonic oscillator

$$E = \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}m\,\omega_0^2\,x^2$$

with

$$m = LC^2, \qquad \omega_0^2 = 1/LC.$$

According to statistical mechanics, each harmonic oscillator has kT of energy in average at equilibrium.

$$\bar{E} = kT.$$

How can an isolated tank circuit acquire energy? In statistical mechanics, it is implicitly assumed that the system in question exchanges energy with the environment (or heat bath). The energy flow from the system to the environment is considered dissipation and the inverse can be considered thermal agitation. When net energy flow between the system and the environment is zero, the system is at equilibrium. In electric circuits, dissipation can be modeled as a resistor, and thermal agitation can be modeled as a voltage source. Therefore an LC tank circuit at equilibrium can be modeled as damped resonator. Circuit equation for such damped resonator is

$$L\,\dot{I} + RI + \frac{q}{C} = V, \qquad I = \dot{q}.$$

where q and V is charge stored in C, and electromotive force due to thermal agitation, respectively.

2.1.2 Damped oscillator

We first consider in case agitation force is absent to find rate of dissipation. Equation for q becomes

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q \;=\; 0,$$

with

$$\lambda = R/2L, \qquad \omega_0^2 = 1/LC.$$

Let's say $q = e^{rt}$. Inserting this to the equation yields

$$r^2 + 2\lambda r + \omega_0^2 \;=\; 0.$$

Therefore

$$r = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}.$$

In case $\lambda < \omega_0$ we get oscillating solution:

$$q = a\, e^{-\lambda} \cos(\omega t + \delta), \qquad \omega = \omega_0 \sqrt{1 - (\lambda/\omega_0)^2},$$

where a and δ is constant coming from initial condition. Furthermore, if we assume $\lambda \ll \omega_0$, shift of oscillation frequency from ω_0 become second order small quantity and we can neglect it.

$$\omega \;=\; \omega_0 \sqrt{1 - (\lambda/\omega_0)^2} \;\sim\; \omega_0 \left(1 - \frac{1}{2} (\lambda/\omega_0)^2 \right) \;\sim\; \omega_0.$$

Since $\lambda \ll \omega_0$, amplitude decays much slower than oscillation frequency, it can be treated as a constant for a cycle. Therefore average energy stored in L and C over a cycle is proportional to square of amplitude. Since amplitude decays at rate of λ, average energy decays at rate of 2λ:

$$\frac{d\bar{E}}{dt} = -2\lambda \bar{E}.$$

Agitation force V provides power to keep the system at equilibrium.

2.1.3 Damped resonator

Now we would like to turn back to damped resonator. Equation in s space:

$$\left(s\,L + R + \frac{1}{s\,C} \right) I = V.$$

Therefore frequency response:

$$\frac{I(\omega)}{V(\omega)} \;=\; \frac{i\omega\, C}{1 + i\omega\, RC + (i\omega)^2 LC}.$$

Note that here $V(\omega)$ is a quantity such that $|V(\omega)|^2$ becomes a power spectral density ([V^2/rad/s]). Similarly $I(\omega)$ is a quantity such that $|I(\omega)|^2$ becomes a power spectral density ([A^2/rad/s]). Using $\omega_0^2 = 1/LC$ and $\lambda = R/2L$,

$$I/V \;=\; \frac{1}{R} \cdot \frac{i\dfrac{\omega}{\omega_0} \cdot \dfrac{2\lambda}{\omega_0}}{1 - \left(\dfrac{\omega}{\omega_0}\right)^2 + i\dfrac{\omega}{\omega_0} \cdot \dfrac{2\lambda}{\omega_0}}.$$

Our focus is frequency near ω_0. Substituting ω with $\omega_0 + \epsilon$ and dropping second order small quantity yields

$$I/V \;=\; -\frac{1}{R} \cdot \frac{i\lambda}{\epsilon - i\lambda}.$$

Therefore power spectral density $p(\epsilon)$ and total power P is respectively

$$p(\epsilon) = \mathrm{Re}\,(V^* I) = \frac{\lambda^2}{\epsilon^2 + \lambda^2} \cdot \frac{|V(\omega)|^2}{R}, \qquad P = \int_{-\omega_0}^{\infty} p(\epsilon)\; d\epsilon.$$

Since $\lambda^2/(\epsilon^2 + \lambda^2)$ has sharp peak at ω_0, we can bring $|V|^2/R$ out from the integral using the value at ω_0. Also the integrand at $\epsilon \ll -\lambda$ does not contribute much to the integral, we can extend the integral from $-\infty$:

$$P \;\sim\; \frac{|V(\omega_0)|^2}{R} \int_{-\infty}^{\infty} \frac{\lambda^2}{\epsilon^2 + \lambda^2}\, d\epsilon \;=\; \frac{|V(\omega_0)|^2}{R} \cdot \pi\lambda.$$

To maintain equilibrium this has to be equal to the rate of dissipation $-d\bar{E}/dt$, i.e.,

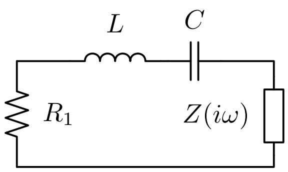
$$-\frac{d\bar{E}}{dt} = 2\lambda\bar{E} \;=\; P \sim \pi\lambda \frac{|V(\omega_0)|^2}{R}.$$

Recalling that $\bar{E} = kT$, we find

$$|V(\omega_0)|^2/R \;\sim\; 4kT/2\pi.$$

This is in fact exact, see Appendix E for details. Since the right hand side does not depend on ω_0, we can think of the environment as if it has voltage source with flat spectral density of $4kTR$ per Hz. Here R was barely a model for the environment. Can this be a real resistor? The answer is yes. In a real resistor, dissipation happens as a result of interaction between the carrier and everything else of the resistor, resulting resistance. "Everything else" is indeed a part of the environment. In usual circuit setups, dissipation due to resistance is much greater than dissipation due to other means such as radiation. We can regard resistor body by itself as the environment. Ideal L and C are used as a tool to show spectral density of agitation force is white. In fact, they are not essential to thermal agitation force. It is rate of dissipation (resistance) which balances with power of agitation force.

2.1.4 Equivalent circuit for impedance



Let us think about equilibrium of a resistor R_1 and an impedance Z connected through resonator like shown in the right. Here we suppose that two component are exchanging energy solely through the noise current. Because of resonator, energy is exchanged by frequency component of $\omega = 1/\sqrt{LC}$ of the current. At equilibrium energy transfer from R to Z and Z to R should be equal.

Suppose that v_1 and v_2 is noise electromotive force of R_1 and Z, respectively and that $v_1(\omega)$ and $v_2(\omega)$ is its frequency component, some quantity of which square magnitude $|v_1(\omega)|^2$ and $|v_2(\omega)|^2$ become spectral density. We know $|v_1(\omega)|^2$ is $4kTR_1/2\pi$ for any ω and we like to find formula for $|v_2(\omega)|^2$ from the condition that R_1 and Z are at thermal equilibrium, i.e., power transfer is balanced. Power transferred from R_1 to Z through frequency between ω and $\omega + d\omega$ is

$$\begin{aligned}
\mathrm{Re}(i_1(\omega)\cdot v_Z^*(\omega))\, d\omega &= \mathrm{Re}\left(\frac{v_1(\omega)}{R_1 + Z(i\omega)}\cdot\frac{Z^*(i\omega)\, v_1^*(\omega)}{R_1^* + Z^*(i\omega)}\right) d\omega, \\
&= \mathrm{Re}(Z^*(i\omega))\cdot\frac{|v_1(\omega)|^2}{|R_1 + Z(i\omega)|^2}\, d\omega, \\
&= \mathrm{Re}(Z^*(i\omega))\cdot\frac{4kTR_1/2\pi}{|R_1 + Z(i\omega)|^2}\, d\omega,
\end{aligned}$$

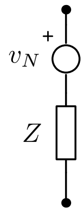
where i_1, v_Z is the circuit current due to v_1 and the voltage drop across Z, respectively. Power transferred from Z to R_1 through frequency between ω and $\omega + d\omega$ is

$$\begin{aligned}
\mathrm{Re}(i_2(\omega)\cdot v_R^*(\omega))\, d\omega &= \mathrm{Re}\left(\frac{v_2(\omega)}{R_1 + Z(i\omega)}\cdot\frac{R_1\, v_2^*(\omega)}{R_1 + Z^*(i\omega)}\right) d\omega, \\
&= \frac{R_1}{|R_1 + Z(i\omega)|^2}\cdot|v_2(\omega)|^2\, d\omega,
\end{aligned}$$

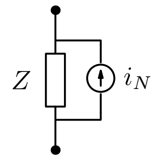
where i_2 and v_R is the circuit current due to v_2 and the voltage drop across R_1, respectively. At equilibrium power transfer between R_1 and Z is balanced. Therefore

$$|v_2(\omega)|^2 = \frac{4kT}{2\pi}\,\mathrm{Re}(Z(i\omega))\,. \tag{13}$$

We can use current source i_Z instead of electromotive force (voltage source). Power



$$\begin{aligned}\left\langle v_N^2 \right\rangle_\omega &= \frac{4kT}{2\pi}\,\mathrm{Re}(Z(i\omega)) \\ &= \frac{4kT}{2\pi}\,R(\omega) \\ Z(i\omega) &= R(\omega) + i\,X(\omega)\end{aligned}$$



$$\begin{aligned}\left\langle i_N^2 \right\rangle_\omega &= \frac{4kT}{2\pi}\,\mathrm{Re}\left(\frac{1}{Z(i\omega)}\right) \\ &= \frac{4kT}{2\pi}\,G(\omega) \\ 1/Z(i\omega) &= G(\omega) + i\,B(\omega)\end{aligned}$$

Figure 3: Equivalent circuit for (passive) impedance Z with noise source. (Left) With noise voltage source. (Right) With noise current source.

transferred from Z to R_1 through frequency between ω and $\omega + d\omega$ is

$$\begin{aligned}\mathrm{Re}(i_R(\omega) \cdot v_R(\omega)^*)\ d\omega &= \frac{|v_R(\omega)|^2}{R_1}\,d\omega, \\ &= \frac{1}{R_1}\left|\frac{i_Z}{1/R_1 + 1/Z(i\omega)}\right|^2 d\omega, \\ &= \frac{R_1\,|Z(i\omega)|^2}{|R_1 + Z(i\omega)|^2} \cdot |i_Z(\omega)|^2\,d\omega,\end{aligned}$$

where i_R and v_R is current and voltage at R_1 due to i_Z. This should be equal to power transfer from R to Z. Therefore

$$|i_Z(\omega)|^2 \;=\; \frac{4kT}{2\pi} \cdot \frac{\mathrm{Re}(Z(i\omega))}{|Z(i\omega)|^2} \;=\; \frac{4kT}{2\pi}\,\mathrm{Re}\left(\frac{1}{Z(i\omega)}\right). \tag{14}$$

If Z can be decomposed into real part and imaginary part like

$$Z(i\omega) = R(\omega) + i\,X(\omega),$$

noise can be modeled as a voltage source v_N in series with Z, of which power spectral density $\left\langle v_N^2 \right\rangle_f$ is equal to $4kTR(2\pi f)$.

If $1/Z$ can be decomposed into real part and imaginary part like

$$1/Z(i\omega) = G(\omega) + i\,B(\omega),$$

noise can be modeled as a current source i_N in parallel with Z, of which power spectral density $\left\langle i_N^2 \right\rangle_f$ is equal to $4kTG$.

Figure 3 summarizes these results.

2.2 Microscopic derivation of Nyquist formula

2.2.1 Macroscopic quantities in terms of microscopic quantities

Consider a conductor of length L, cross sectional area of A, resistance of R. When we apply voltage V across each end, current runs through this conductor will be

$$I = V/R.$$

Whereas relation between current density $j = I/A$ and electric filed $E = V/L$ is

$$j = \sigma E, \quad \text{or} \quad I/A = \sigma V/L,$$

where σ is called conductivity. Therefore relation between σ and R is,

$$R = \frac{L}{\sigma A}.$$

Suppose that current is running uniformly inside the conductor, i.e., carrier density (n) and average (drift) carrier velocity ($\bar{v}$) is also uniform:

$$j = n\,q\,\bar{v}, \qquad n = N/AL,$$

where q and N is charge of a carrier and total number of carriers, respectively. In this and following sections we use bar for average over carriers in a system and bracket for ensemble average. For example, $\bar{v}$ is average velocity of carries in a system and $\langle v_i \rangle$ is ensemble average of i-th carrier's velocity.

Now equation of motion for each carrier:

$$m\dot{v}_i \;=\; -\frac{m}{\tau_c}\,v_i \;+\; q\bar{E},$$

where m is mass of carrier, τ_c represents average "drag force" from all sorts of interaction between carrier and lattice vibration, impurity etc., and $\bar{E}$ is V/L. The reason why we have a bar on top of E is that, in microscopic point of view, each carrier should see local electric field from surrounding atoms. However, since conductor as a whole is electrically neutral those local modulation vanishes in average. After sufficient time carrier particles catch up steady speed where filed force and drag force are balanced:

$$\frac{m}{\tau_c}\,\bar{v} = q\bar{E} \quad \rightarrow \quad \bar{v} = \frac{q\,\tau_c}{m}\,\bar{E} = \mu\bar{E},$$

where $\mu = q\,\tau_c/m$ is called carrier mobility. Therefore conductivity is written with q, m and τ_c as follows:

$$j = n\,q\,\bar{v} = nq\mu\bar{E} = \frac{n\,q^2\bar{v}}{m}\,\bar{E} \quad \rightarrow \quad \sigma = nq\mu = \frac{n\,q^2\tau_c}{m}.$$

2.2.2 Random agitation force

Now let us take a close look at each carrier when $V = 0$. Average field $\bar{E}$ is of course zero, however carrier receives force from surroundings as a result of thermal agitation. Let $\mathcal{E}_i$ represent such thermal agitation force and we assume it is white noise. In such case equation of motion for each carrier is

$$m\dot{v}_i \;=\; -\frac{m}{\tau_c}\,v_i \;+\; q\,\mathcal{E}_i,$$

From Laplace transform of the equation

$$s\,mv_i(s) \;=\; -\frac{m}{\tau_c}\,v_i(s) + q\mathcal{E}_i \quad\rightarrow\quad v_i(s)/\mathcal{E}_i \;=\; \frac{q\,\tau_c}{m}\cdot\frac{1}{1+s\,\tau_c},$$

we get relation between spectral densities

$$\left\langle v_i^2\right\rangle_\omega = \frac{(q\,\tau_c)^2}{m^2}\cdot\frac{1}{1+\omega^2\tau_c^2}\;\left\langle \mathcal{E}_i^2\right\rangle_\omega, \tag{15}$$

and mean square velocity

$$\left\langle v_i^2\right\rangle = \int_0^\infty \left\langle v_i^2\right\rangle_\omega d\omega = \frac{\pi}{2}\cdot\frac{q^2\tau_c}{m^2}\left\langle \mathcal{E}_i^2\right\rangle_\omega.$$

At equilibrium this should be equal to the thermal velocity kT/m, therefore spectral density of agitation force is found to be

$$\left\langle v_i^2\right\rangle = \frac{\pi}{2}\cdot\frac{q^2\tau_c}{m^2}\left\langle \mathcal{E}_i^2\right\rangle_\omega = \frac{kT}{m} \quad\rightarrow\quad \left\langle \mathcal{E}_i^2\right\rangle_\omega = \frac{4kT}{2\pi}\cdot\frac{1}{q\mu}. \tag{16}$$

2.2.3 Electromotive force

For arbitrary cross section, there is nA carriers per unit length and average agitation force per unit length over a cross section $\bar{\mathcal{E}}$ is $\sum \mathcal{E}_i/nA$ where sum is taken over nA carriers. Therefore ensemble mean square

$$\left\langle \bar{\mathcal{E}}^2\right\rangle \;=\; \frac{1}{(nA)^2}\cdot nA\left\langle \mathcal{E}_i^2\right\rangle \;=\; \frac{1}{nA}\left\langle \mathcal{E}_i^2\right\rangle.$$

Mean square electromotive force across the conductor can be summed up for its length:

$$\left\langle V^2\right\rangle \;=\; L\left\langle \bar{\mathcal{E}}^2\right\rangle \;=\; \frac{L}{nA}\left\langle \mathcal{E}_i^2\right\rangle.$$

Same thing can be said for each frequency component:

$$\left\langle V^2\right\rangle_\omega \;=\; \frac{L}{nA}\left\langle \mathcal{E}_i^2\right\rangle_\omega.$$

Therefore, using $\sigma = nq\mu$, $R = L/A\sigma$,

$$\left\langle V^2\right\rangle_\omega \;=\; \frac{L}{nA}\left\langle \mathcal{E}_i^2\right\rangle_\omega \;=\; 4kTR\,/\,2\pi.$$

2.2.4 Noise current

Recalling that current is expressed by number of carriers inside the conductor N as

$$I = An\,q\,\bar{v} = ALn\,q\,\bar{v}/L = N\bar{v}\,q/L,$$

and that $\bar{v}$ is average of each carrier's velocity:

$$\bar{v} = \frac{1}{N}\sum_i v_i \quad \rightarrow \quad \left\langle \bar{v}^2 \right\rangle = \frac{1}{N^2}\,N\left\langle v_i^2 \right\rangle = \frac{1}{N}\left\langle v_i^2 \right\rangle,$$

we find noise current is expressed by mean square of carrier velocity:

$$\left\langle i_N^2 \right\rangle = \left\langle I^2 \right\rangle = N^2 q^2 \left\langle \bar{v}^2 \right\rangle / L^2 = N q^2 \left\langle v_i^2 \right\rangle / L^2.$$

Same thing can be said for frequency component:

$$\left\langle i_N^2 \right\rangle_\omega = N q^2 \left\langle v_i^2 \right\rangle_\omega / L^2$$

Inserting Eq. (15) and Eq. (16), we find spectral density of noise current

$$\left\langle i_N^2 \right\rangle_\omega = \frac{N}{L^2}\,q^2\left\langle v_i^2 \right\rangle_\omega = \frac{Nq\mu}{L^2}\cdot\frac{4kT}{2\pi}\cdot\frac{1}{1+\omega^2\tau_c^2}. \tag{17}$$

Recalling that $\sigma = nq\mu$, $R = L/A\sigma$, this reduces to the thermal noise formula for $\omega \ll 1/\tau_c$,

$$\left\langle i_N^2 \right\rangle_f = 2\pi\left\langle i_N^2 \right\rangle_\omega = 4kT/R$$

Note that Eq. (17) corresponds to Eq. (14), i.e., noise current power spectral density is proportional to conductance.

2.2.5 Drift current

Now we apply voltage V to the conductor and let net current flow. As we will see later in Section 2.2.8, τ_c can be interpreted as average time that particles "fly" freely before impacted by thermal agitation force. Thermal velocity v_T times τ_c is called mean free path $l = v_T\,\tau_c$. Thermal velocity v_T can be written with l as

$$v_T^2 = \frac{kT}{m}, \quad v_T = \frac{l}{\tau_c} \quad \rightarrow \quad v_T = \frac{kT}{m}\cdot\frac{\tau_c}{l} \;=\; \mu\,\frac{kT/q}{l}.$$

Recalling that drift velocity v_D is $\mu\bar{E} = \mu V/L$, ratio between drift velocity and thermal velocity is

$$\frac{v_D}{v_T} = \frac{V}{kT/q}\cdot\frac{l}{L}.$$

In usual circuit setups, V and kT/q is about the same order. Mean free path is in the order of nano meters, whereas L is in the order of micro meters, drift current hardly affects each carrier's activity, hence noise:

$$\left\langle v_i^2 \right\rangle \;=\; v_D^2 + v_T^2 \;\sim\; v_T^2.$$

2.2.6 Diffusion phenomena and Nernst-Einstein relation

Now we would like to see how carrier particle travel by thermal agitation force. We place x-axis along with the conductor and set $x = 0$ somewhere in the middle of the conductor. Equation of motion for a carrier at the origin with $\dot{x} = 0$ at $t = 0$ is

$$m\ddot{x}_i \;=\; -\frac{m}{\tau_c}\,\dot{x}_i + q\mathcal{E}_i \quad \text{or} \quad x_i/\mathcal{E}_i \;=\; \frac{q\,\tau_c/m}{s\,(1+s\,\tau_c)} \;=\; \mu\left(\frac{1}{s} - \frac{\tau_c}{1+s\,\tau_c}\right).$$

Using result obtained in Example of Section 1.7.3,

$$\begin{aligned}\left\langle x_i(t)^2\right\rangle \;&=\; \pi\mu^2\left(t + \frac{\tau_c}{2}\left(1 - e^{-2t/\tau_c}\right) - 2\tau_c\left(1 - e^{-t/\tau_c}\right)\right)\left\langle\mathcal{E}_i^2\right\rangle_\omega,\\ &\sim\; \pi\mu^2 t\cdot\left\langle\mathcal{E}_i^2\right\rangle_\omega. \qquad\qquad (t \gg \tau_c)\end{aligned}$$

Inserting Eq. (16) to this, we get

$$\left\langle x_i(t)^2\right\rangle \;=\; 2\mu\,kT/q\cdot t. \tag{18}$$

Carrier particle does not stay where it used be but travels indefinitely even though average of its driving force ($\bar{\mathcal{E}}_i$) is zero. We observe this as diffusion phenomena.[6]

For diffusion phenomena, there is an empirical macroscopic law:

$$j \;=\; -D\,\frac{\partial n}{\partial x}.$$

Combining this and law of continuity,

$$\frac{\partial n}{\partial t} \;+\; \frac{\partial j}{\partial x} \;=\; 0,$$

yields diffusion equation:

$$\frac{\partial n}{\partial t} \;=\; D\,\frac{\partial^2 n}{\partial x^2}.$$

One solution is

$$n(x,t) = \frac{N}{\sqrt{4\pi Dt}}\;e^{-x^2/4Dt} \quad \text{for} \quad n(x,0) = N\delta(x).$$

Therefore mean square of a carrier position in case it is at $x = 0$ at $t = 0$:

$$\left\langle x(t)^2\right\rangle \;=\; \int_{-\infty}^{\infty} x^2\,n(x,t)\,dx = 2D\,t. \tag{19}$$

By comparing with Eq. (18), we get Nernst-Einstein[7] relation:

$$D/\mu = kT/q.$$

[6] See Appendix A.

[7] According to Wannier, this formula was used by Nernst in 1884 (before Einstein), to determine the magnitude of charge carried by an individual ion in a solution. [9]

2.2.7 Diffusion constant and carrier velocity

We have just learnt average displacement of carrier $\langle x(t)^2 \rangle$ is expressed using diffusion constant D as

$$\langle x(t)^2 \rangle \;=\; 2D\,t.$$

How about velocity? By combining Nernst-Einstein relation $qD/\mu = kT$ and thermal velocity $\langle v_i^2 \rangle = kT/m$ and mobility $\mu = q\tau_c/m$, we find

$$\langle v_i^2 \rangle \;=\; D/\tau_c.$$

Recalling that autocorrelation is Fourier transform of spectral density, we find autocorrelation of velocity $\phi(t) = \langle v_i(0)v_i(t) \rangle$ using Eq. (15) and $\phi(0) = \langle v_i^2 \rangle = D/\tau_c$,

$$\phi(t) = \langle v_i(0)v_i(t) \rangle = \frac{1}{2}\int_{-\infty}^{\infty} \langle v_i^2 \rangle_\omega \, e^{i\omega t} d\omega = \frac{D}{\tau_c} e^{-|t|/\tau_c}.$$

Note that factor 1/2 is came from the fact that $\langle v_i \rangle_\omega$ is single sided spectral density. Therefore

$$\langle v_i^2 \rangle_\omega \;=\; \frac{4D}{2\pi} \cdot \frac{1}{1+\omega^2\tau_c^2}.$$

From Eq. (17) noise current spectral density

$$\langle i_N^2 \rangle_\omega \;=\; \frac{Nq^2}{L^2} \cdot \frac{4D}{2\pi} \cdot \frac{1}{1+\omega^2\tau_c^2} = \frac{4q}{2\pi} \cdot \frac{A\,qD\,n}{L} \cdot \frac{1}{1+\omega^2\tau_c^2}.$$

2.2.8 Mean free time

The formula we derived in Section 1.7 is valid for zero initial conditions, hence Eq. (18) is for such condition. The average displacement is insensitive to the initial condition after a certain period of time, however the solution with initial condition gives an interesting idea for τ_c. Namely when a particle is moving at v_0 at $t = 0$, average displacement is given by

$$\langle x^2(t) \rangle = v_0^2\, t^2 \quad \text{for} \quad t \ll \tau_c,$$

meaning that the particle "flies" freely for a short period of time before it is impacted by thermal agitation force, and τ_c can be interpreted as mean free time (before "collision"). Here we would like to derive above formula according to Uhlenbeck and Ornstein.[7]

With $A(t) = q\mathcal{E}_i/m$, the equation for $v = \dot{x}$ can be written as

$$\dot{v} + \frac{1}{\tau_c} v \;=\; A(t).$$

If we notice that

$$\frac{d}{dt}\left(e^{t/\tau_c}\,v\right) \;=\; e^{t/\tau_c}\,\dot{v}+\frac{1}{\tau_c}e^{t/\tau_c}\,v \;=\; e^{t/\tau_c}A(t),$$

$$\rightarrow \qquad e^{t/\tau_c}\,v \;=\; v_0+\int_0^t e^{\xi/\tau_c}A(\xi)\,d\xi,$$

v can be written as follows:

$$v \;=\; v_0\,e^{-t/\tau_c}+e^{-t/\tau_c}\int_0^t e^{\xi/\tau_c}A(\xi)\,d\xi. \tag{20}$$

Therefore v^2:

$$v^2 \;=\; v_0^2\,e^{-2t/\tau_c} \;+\; 2\,v_0\,e^{-2t/\tau_c}\int_0^t e^{\xi/\tau_c}A(\xi)\,d\xi$$
$$+\; e^{-2t/\tau_c}\int_0^t\int_0^t e^{(\xi+\eta)/\tau_c}A(\xi)A(\eta)\,d\xi d\eta.$$

If we take ensemble average, the second term vanishes since $\langle A(t)\rangle = 0$.

$$\left\langle v^2(t)\right\rangle \;=\; v_0^2\,e^{-2t/\tau_c} \;+\; e^{-2t/\tau_c}\int_0^t\int_0^t e^{(\xi+\eta)/\tau_c}\left\langle A(\xi)A(\eta)\right\rangle\,d\xi d\eta \tag{21}$$

We may want to use the fact that $A(t)$ is white noise, its autocorrelation function is proportional to delta function. However we would like to proceed a bit more for later convenience. By taking $r = \xi + \eta$ and $s = \xi - \eta$ as new variables the integral become

$$\int_0^t\int_0^t e^{(\xi+\eta)/\tau_c}\left\langle A(\xi)A(\eta)\right\rangle\,d\xi d\eta = \frac{1}{2}\int_0^{2t} e^{r/\tau_c}\int_{-t}^{t}\left\langle A\left((r+s)/2\right)A\left((r-s)/2\right)\right\rangle\,ds\,dr,$$

where factor 1/2 is Jacobian $\partial(\xi,\eta)/\partial(r,s)$. Recalling that autocorrelation depends only on difference of time, inner integral does not depend on r:

$$\int_{-t}^{t}\left\langle A\left((r+s)/2\right)A\left((r-s)/2\right)\right\rangle\,ds \;=\; \int_{-t}^{t}\left\langle A(0)A(t)\right\rangle\,dt.$$

Therefore, we can execute integral for r, then Eq. (21) becomes

$$\left\langle v^2(t)\right\rangle \;=\; v_0^2\,e^{-2t/\tau_c} \;+\; \frac{\tau_c}{2}\left(1-e^{-2t/\tau_c}\right)\int_{-t}^{t}\left\langle A(0)A(t)\right\rangle\,dt. \tag{22}$$

Now we want to use $A(t)$ is white noise. Let's say $\langle A(0)A(0)\rangle = C\delta(t)$, and using the fact that $\left\langle v^2\right\rangle$ should be thermal velocity kT/m for $t\rightarrow\infty$, we find

$$\left\langle v^2(t)\right\rangle \;=\; v_0^2\,e^{-2t/\tau_c}+\frac{kT}{m}\left(1-e^{-2t/\tau_c}\right), \qquad C=\frac{2kT}{m\tau_c}.$$

Integrating Eq. (20) yields

$$x - x_0 \; = \; v_0 \, \tau_c \left(1 - e^{-t/\tau_c} \right) + \int_0^t e^{-\eta/\tau_c} \int_0^\eta e^{\xi/\tau_c} A(\xi) \, d\xi d\eta.$$

By squaring, averaging and calculating the integral in the same way as before, we get

$$\left\langle (x - x_0)^2 \right\rangle = v_0 \, \tau_c^2 \left(1 - e^{-t/\tau_c} \right) + \frac{2\mu kT}{q} \left(t + \frac{\tau_c}{2} \left(1 - e^{-2t/\tau_c} \right) - 2\tau_c \left(1 - e^{-t/\tau_c} \right) \right).$$

This reduces to the same result we get in the previous section for $t \gg \tau_c$, and for $t \ll \tau_c$ it becomes the formula we want.

$$\left\langle (x - x_0)^2 \right\rangle \; = \; v_0^2 \, t^2$$

As for velocity, it approaches thermal velocity very quickly at rate of $\tau_c/2$. Therefore we can draw a naive picture of particles flying at thermal velocity v_T and changing its direction at rate of $1/\tau_c$, or after $l = v_T \, \tau_c$ of flight.

Non white case Even in case $A(t)$ is not white, we can still get some interesting result as long as $\langle A(t) \rangle = 0$. Eq. (22) becomes for $t \to \infty$,

$$\left\langle v^2 \right\rangle \; = \; \lim_{t\to\infty} \left\langle v^2(t) \right\rangle \; = \; \frac{\tau_c}{2} \int_{-\infty}^{\infty} \langle A(0) A(t) \rangle \; dt.$$

Recalling that $A(t) = q\mathcal{E}_i/m$ and that $\left\langle v^2 \right\rangle = kT/m$, we get relation between mean free time and thermal agitation force:

$$\frac{1}{\tau_c} \; = \; \frac{q^2}{2 \, m \, kT} \int_{-\infty}^{\infty} \langle \mathcal{E}_i(0) \mathcal{E}_i(t) \rangle \; dt.$$

The right hand side may be calculated by considering details how carriers are scattered. Recalling that conductivity σ is $n \, q^2 \tau_c / m$, this can be seen as a formula which gives resistance (τ_c) from noise ($\mathcal{E}_i$), while Nyquist formula gives noise from resistance.

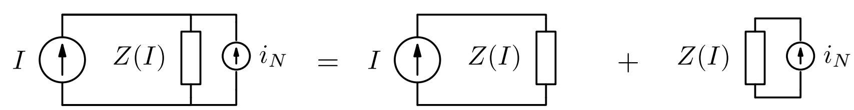
A few words on Brownian motion Brownian motion is motion of a small but macroscopic scale particle (pollen) caused by bombardment of surrounding molecules (water). Its bombardment of molecules is a bunch of small impulses, the particle is considered to be driven by white noise. Equation of motion for this particle is the same as the case of a carrier driven by thermal agitation force, the only difference is mass of the particle is huge. Carrier particles change their direction completely randomly by a single impulse, while it is less significant for a particle in Brownian motion. Furthermore time scale of observation is much longer than the rate of impulse, which makes it difficult to measure exact velocity (or kinetic energy) at any instant, thermal velocity tends to be underestimated. However, displacement is relatively easy to measure and agreement with observation and theory (Eq. (18)) was one of the evidence of molecular nature of matter.

2.3 Nyquist's derivation of Nyquist formula

The author recommends everyone to read Nyquist's original paper, "*Thermal agitation of electric charge in conductors*", Phys. Rev. **32**, 110 (1928). This is a great example of how we want to explain things. In some sense Section 2.1 is an inferior elaboration of this classic paper. However, I imagine, he must have analyzed some specific model like what we did in Section 2.2 and examine the result he'd got carefully before he came up with his explanation. Classical carrier particle is not fully represent the reality but it was a good starting point and what I demonstrated there was an example of how to formulate and analyze a very specific problem. By carefully examining the result or the process to the result, you may be able to find what is the essential point and may also be able to set up more generic problem. (In this case, it was the rate of energy transfer between a system to the other.)

2.4 Net current

When a resistor has net current, i.e., there is net energy flow from the resistor to the environment, we can't use statistical mechanics, can we? The total current is superposition of net current and noise current and we can think net current as a part of the environment. We can apply statistical mechanics for the noise component as long as temperature of the resistor is uniform.



In case resistance is a function of net current I, noise spectral density

$$\left\langle i_N^2 \right\rangle_f \;=\; 4kT\,\mathrm{Re}\left(\frac{1}{Z(I)}\right).$$

Wrong example Recalling that conductance g of p-n junction diode is I_D/v_t, noise spectral density for p-n junction diode would be,

$$\left\langle i_N^2 \right\rangle_f \;=\; 4kTg = 4kTI_D/v_t = 4qI_D. \qquad \text{(wrong!)}$$

Therefore p-n junction diode is noiseless when net current is zero.... Wait a minute, there's something wrong.

2.5 Semiconductor noise

2.5.1 Shot noise

To get the correct answer we need to start from Shockley formula,

$$I_D = I_s \left(e^{V_D/v_t} - 1 \right).$$

At equilibrium where net current is zero, conductance

$$g(I_D)|_{I_D=0} \;=\; \left.\frac{\partial I_D}{\partial V_D}\right|_{I_D=0} = \frac{I_s}{v_t}.$$

Therefore we get finite noise spectral density:

$$\left\langle i_N^2(0) \right\rangle_f \;=\; 4kTg(0) = 4kTI_s/v_t = 4qI_s.$$

With net current of I_D, semiconductor noise is usually explained by "shot noise". Shockley formula can be interpreted as sum of forward going and reverse going current:

$$I_D \;=\; I_F - I_R,$$

where

$$I_F \;=\; I_s e^{V_D/v_t}, \qquad I_R = I_s.$$

"Shot noise" explanation states that each current has independent noise component of $\left\langle i_F^2 \right\rangle_f = 2qI_F$ and $\left\langle i_R^2 \right\rangle_f = 2qI_R$. Here's an observation.

$$\left\langle i_N^2(I_D) \right\rangle_f \;=\; \left\langle i_F^2 \right\rangle_f + \left\langle i_R^2 \right\rangle_f \;=\; 2qI_F + 2qI_R.$$

At equilibrium where $I_D = 0$, this reduces to above thermal noise formula:

$$\left\langle i_N^2(0) \right\rangle_f = 2qI_s + 2qI_s = 4qI_s.$$

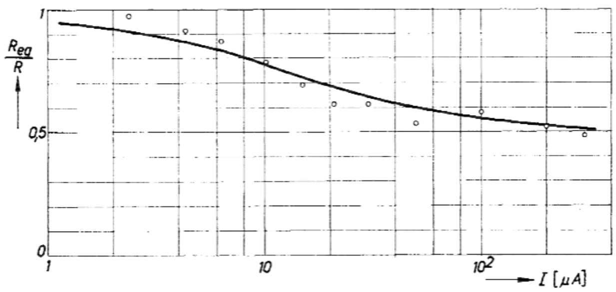
However at $I_D \gg I_s$, where saturation current is negligible ($I_D \sim I_F$), we get only half of the "wrong" conclusion shown in the previous page:

$$\left\langle i_N^2(I_D) \right\rangle_f \;=\; 2qI_D = 2kTg(I_D).$$

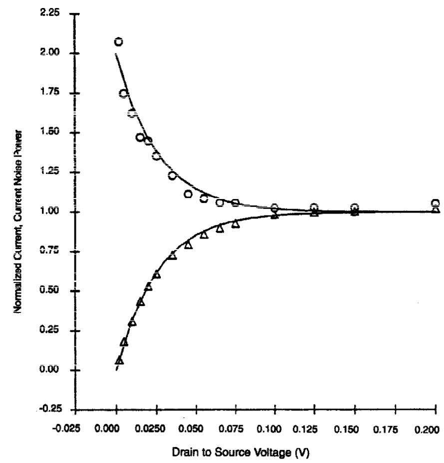
Figure 4 (a) shows measurement data. Same thing can be said for subthreshold MOSFET and measurement data is shown in Figure 4 (b). Note that noise magnitude is still proportional to conductance and temperature, only it becomes half in semiconductor. As we will see in the next section, "shot noise" can be understood as thermal noise of diffusion current. As for MOSFET in active region, where drift current plays more role, noise suppression happens only partially. We use γ to represent this partial suppression:

$$\left\langle i_N^2 \right\rangle_f = \gamma 4kTg_m,$$

where $\gamma = 2/3$ for long channel device. [8]



(a)



(b)

Figure 4: (a) Equivalent noise resistance (according to Nyquist equation) over differential resistance of diode junction (vertical) as dependent on diode forward current (horizontal) of junction diode 1N91. Curve calculated, taking account of diode resistance of 20-ohm outside junction. Points measured. [1] (b) Measured current and noise characteristics of a subthreshold MOS transistor. The lower curve is the current normalized by its saturation value I_{sat} so that it is 1 in saturation and zero when V_{ds} is 0. The upper curve is the noise spectral density ΔI^2 normalized by dividing it by $2qI_{sat}\Delta f$, where Δf is the bandwidth and q is the charge on the electron. As the transistor moves from the linear region to saturation, the noise spectral density decreases by a factor of two. The lines are fits to theory using the measured value of the saturation current and the value for the charge on the electron $q = 1.6 \times 10^{-19}$C. [5]

2.5.2 Diffusion current noise

Suppose that some external force makes non-uniform carrier density, there must be diffusion current

$$j = -qD\,\frac{\partial n(x)}{\partial x}.$$

In case total current is dominated by this diffusion current, gradient of $n(x)$ is constant because current at any cross section must be the same:

$$\frac{\partial n(x)}{\partial x} = \frac{n(L) - n(0)}{L} = \frac{n(0)}{L}\left(\frac{n(L)}{n(0)} - 1\right), \qquad \text{(constant)}$$

Therefore total current

$$I = Aj = \frac{A\,qD\,n(0)}{L}\left(1 - \frac{n(L)}{n(0)}\right) = I_s\left(1 - \frac{n(L)}{n(0)}\right),$$

where

$$I_s = \frac{A\,qD\,n(0)}{L}.$$

Depletion region of p-n junction makes it possible to create such situation described here (carrier density gradient, absence of drift current). In subthreshold MOSFET, for example, $n(0) \propto \exp(-qV_{gs}/kT)$ and $n(L)/n(0) = \exp(qV_{ds}/kT)$.[6] Therefore

$$I_d \;=\; I_s\left(1 - e^{qV_{ds}/kT}\right), \qquad I_s \;\propto\; \frac{A}{L}\,e^{-qV_{gs}/kT}.$$

This is current-voltage relation MOSFET in subthreshold region. Note that q is negative for NCH transistors.

As for noise, since diffusion current is nothing but current driven by thermal agitation force, we can use Eq. (17). Inserting $\bar{N} = AL(n(0) + n(L))/2$ into Eq. (17) yields

$$\left\langle i_N^2 \right\rangle_\omega = \frac{A}{L} \cdot \frac{n(0) + n(L)}{2} \cdot \frac{4kTq\mu}{2\pi}.$$

Using Nernst-Einstein relation $D = \mu kT/q$, we get "shot noise",

$$\left\langle i_N^2 \right\rangle_\omega = \frac{2q}{2\pi} \cdot \frac{A\,qD\,n(0)}{L}\left(1 + \frac{n(L)}{n(0)}\right) = \frac{2q}{2\pi} I_s\left(1 + \frac{n(L)}{n(0)}\right).$$

For example, noting that device current I is very close to I_s for $n(0) \gg n(L)$,

$$\left\langle i_N^2 \right\rangle_\omega = 2qI/2\pi. \quad (n(0) \gg n(L)), \quad \left\langle i_N^2 \right\rangle_\omega = 4qI_s/2\pi \quad (n(L) = n(0))$$

"Shot noise" is explained in terms of thermal noise.

2.6 Electromagnetic wave

In Section 2.1, we discussed statistical mechanics of an LC tank circuit which exchanges energy with the environment. For an isolated LC tank circuit, energy is exchanged by electromagnetic wave (radiation). Here we would like to focus on statistical mechanics of electromagnetic wave in empty space at equilibrium with the environment. Such electromagnetic wave is known as black-body radiation. There is a lot of excellent literature on black-body radiation available, we just go over briefly.

As shown in Appendix C, electromagnetic wave is also considered as ensemble of harmonic oscillators. Due to the fact that electromagnetic wave is transverse, there are two oscillators for a wave vector $\boldsymbol{k}$. The number of possible wave vectors in a volume element $d^3\boldsymbol{k}$ is $Vd^3\boldsymbol{k}/(2\pi)^3$. Using dispersion relation $\omega = ck$ and polar coordinate, this can be written as

$$\frac{Vd^3\boldsymbol{k}}{(2\pi)^3} = \frac{Vk^2 \sin\theta dk d\theta d\varphi}{(2\pi)^3} = \frac{V\omega^2 \sin\theta d\omega d\theta d\varphi}{(2\pi\, c)^3}.$$

Therefore, by integrating angular part and multiplying by 2 for that fact that each $\boldsymbol{k}$ has two harmonic oscillators, the number of oscillators with frequencies between ω and $\omega + d\omega$ is found to be

$$\frac{V\omega^2 d\omega}{\pi^2 c^3}.$$

At equilibrium, each harmonic oscillator has energy of kT, therefore energy of electromagnetic wave with frequencies between ω and $\omega + d\omega$ is

$$dU_\omega = \frac{VkT\omega^2 d\omega}{\pi^2 c^3}.$$

Spectral density of electromagnetic wave has frequency dependence while that of current doesn't. This is the consequence of the fact that electromagnetic wave is three dimensional, while current is one dimensional. However, this Rayleigh-Jeans law does not match with observation at high-frequencies. Furthermore total energy or "integrated noise", $\int_0^\infty dU_\omega$, diverges. To get correct answer, we need to go to quantum mechanics. In fact, this problem of black-body radiation led discovery of quantum nature of elementary interactions of matter. According to quantum statistics, each harmonic oscillator of electromagnetic field has energy of

$$\frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \frac{\hbar\omega}{2} \;=\; \hbar\omega \coth\frac{\hbar\omega}{2kT} \tag{23}$$

instead of kT, which gives famous Planck's formula instead of Rayleigh-Jeans law:

$$du_\omega = dU_\omega/V \;=\; \frac{\hbar}{\pi^2 c^3} \cdot \frac{\omega^3 d\omega}{e^{\hbar/kT} - 1}.$$

Here we have omitted zero-point energy term ($\hbar\omega/2$) in Eq. (23), we will come back to this later. Planck's formula has peak at $\hbar\omega/kT \sim 2.8$, which enable us measure temperature of the matter by spectroscopy of electromagnetic wave. Total energy is obtained by integrating Planck's formula:

$$u_\omega = \int_0^\infty \frac{\hbar}{\pi^2 c^3} \cdot \frac{\omega^3\, d\omega}{e^{\hbar\omega/kT} - 1} = \frac{\pi^2 (kT)^4}{15 c^3 \hbar^3}$$

Total energy is proportional to T^4, which is called Stephan-Boltzmann law. Eq. (23) gives kT at low frequencies where $\hbar\omega \ll kT$. Therefore quantum effect arises at high frequencies. At high frequencies where $\hbar\omega \gg kT$, Planck's formula reduces to Wien's formula:

$$du_\omega = \frac{\hbar}{\pi^2 c^3}\, \omega^3 e^{-\hbar\omega/kT} d\omega. \tag{24}$$

Both Wien and Boltzmann derived their formulae (aside from exact proportionality coefficient) from classical thermodynamics. Appendix B provides supplemental material from the classical point of view.

For electromagnetic wave, we had to go quantum mechanics to get correct spectral density throughout all frequency range. How about carrier? Should we use Eq (23) instead of kT which seemingly gives better results? I doubt it. For normal room temperature, $kT/\hbar$ is about 6THz, which is far above frequencies of our interest and $\hbar/kT$ is 25fs, which is about the same order or smaller than τ_c of metals. From Eq. (17) high frequency noise component is cut-off at $\omega \sim 1/\tau_c$ anyway. It is unlikely that quantum mechanical effect shows up in usual electronic circuit setups.

Zero-point energy Zero point energy gives infinite contribution to the total energy, however it is constant. Existence of such term does not affect thermodynamic relations. (In general physics laws, change of energy is important and we can always alter reference point of energy.) Therefore we can omit this term for total energy calculation. However, in case electromagnetic wave is confined in a very narrow volume, the number of possible wave vector reduces significantly and contribution of zero-point energy term to the total energy also reduces, resulting total energy contribution of this term become a function of volume. This can be observed as attracting force between to metal plates placed a few nano meter apart. (Casimir effect)

Appendix A Random Walk and Diffusion

In section 2.2.6, we introduced diffusion equation rather precipitously and we came up with an idea that particles driven by white noise fly freely for some distance l in Section 2.2.8. Here we will derive diffusion equation from this free flying model, which can directly be translated into random walk problem.

Consider a particle hopping for length l either to the left $(-l)$ or to the right $(+l)$. For each time step $t_n = n\tau$, that particle makes decision whether it is going to stay or move to the left or move to the right. The position of a particle after n step, $x(t_n)$, can be written as follows:

$$x(t_n) = e_1 + e_2 + \ldots + e_n,$$

where e_i is the displacement the particle made for each step, i.e., either $+l$ (moved to the right) or $-l$ (left) or 0 (stay). Let's say probability for a particle to make move is p, expectation value for e_i and e_i^2 is respectively,

$$\begin{aligned}\langle e_i \rangle &= p \cdot (+l) \ + \ p \cdot (-l) \ + \ (1-2p) \cdot (0) = 0, \\ \left\langle e_i^2 \right\rangle &= p \cdot (+l)^2 \ + \ p \cdot (-l)^2 \ + \ (1-2p) \cdot (0)^2 = 2\,p\,l^2.\end{aligned}$$

And we assume each step is independent, i.e., $\langle e_i e_j \rangle = \langle e_i \rangle \langle e_j \rangle = 0$ for $i \neq j$. Therefore expectation value for $x(t_n)$ and $x^2(t_n)$ is respectively,

$$\langle x(t_n) \rangle = \sum_{i=1}^{N} \langle e_i \rangle = 0, \quad \left\langle x^2(t_n) \right\rangle = \sum_{i,j} \langle e_i e_j \rangle = \sum_{i} \left\langle e_i^2 \right\rangle = 2Np\,l^2 = 2 \cdot \frac{p\,l^2}{\tau} \cdot t_n.$$

We get Eq. (19) with $D = p\,l^2/\tau$.

Now we want to consider particle distribution. Suppose that there is $n(x,t)$ particle at position x at time t, $n(x, t+\tau)$ will be sum of number of particles that 1) stay, 2) come from the left, 3) come from the right:

$$n(x, t+\tau) = (1-2p) \cdot n(x,t) + p \cdot n(x-l, t) + p \cdot n(x+l, t). \tag{25}$$

We would like to observe smooth $n(x,t)$ in a scale much greater than l and τ, we take Taylor expansion of $n(x,t)$:

$$\begin{aligned} n(x, t+\tau) \ &= \ n(x,t) \ + \ \tau \cdot \frac{\partial\, n(x,t)}{\partial t}, \\ n(x \pm l, t) \ &= \ n(x,t) \ \pm \ l \cdot \frac{\partial\, n(x,t)}{\partial x} \ + \ \frac{l^2}{2} \cdot \frac{\partial^2\, n(x,t)}{\partial x^2}. \end{aligned}$$

Inserting these into Eq. (25) yields diffusion equation:

$$\frac{\partial\, n(x,t)}{\partial t} \ = \ \frac{p\,l^2}{\tau} \cdot \frac{\partial^2\, n(x,t)}{\partial^2 x}.$$

(The first order term in Taylor expansion with respect to l cancels out.)

Appendix B Black-body Radiation

In section 2.6, we have discussed black body radiation using statistical mechanics (equipartition theorem) since it follows discussion of Nyquist formula which also uses equipartition theorem. In this appendix, we will derive Stephan-Boltzmann law and Wien's formula from pure classical point of view (electromagnetism and thermodynamics) to show how far we can go without quantum mechanics.

B.1 Gas equation of electromagnetic wave

Consider a cubic box of L made of perfect conductor. If there is a electromagnetic wave inside the box, it will be reflect back without losing energy, resulting persistent standing wave. In case electromagnetic wave is isotropic, one can show[8] that pressure p is equal to one third of electromagnetic energy per unit volume u,

$$p = \frac{1}{3}u,$$

by calculating force acting on a wall. Therefore

$$pV = \frac{1}{3}U$$

can be considered to be gas equation for isotropic electromagnetic wave, where $U = Vu$ is internal energy of electromagnetic "gas". Note that for the ideal gas, corresponding equation is $pV = \frac{2}{3}U$ and U does not depend on V.

B.2 Stephan-Boltzmann Law

Once we know gas equation, we can apply thermodynamics to it. From $U = TS + F$

$$\begin{aligned} u(T) = \left(\frac{\partial U}{\partial V}\right)_T &= T\left(\frac{\partial S}{\partial V}\right)_T + \left(\frac{\partial F}{\partial V}\right)_T, \\ &= T\left(\frac{\partial p}{\partial T}\right)_V - p, \end{aligned}$$

where we used thermodynamic relation $p = -(\partial F/\partial V)_T$ and $(\partial S/\partial V)_T = (\partial p/\partial T)_V$. Inserting $p = u/3$, we get

$$u(T) = \frac{T}{3}\frac{du(T)}{dT} - \frac{u(T)}{3} \quad \rightarrow \quad T\frac{du(T)}{dT} = 4u(T).$$

Therefore u is proportional to T^4. We use a for proportionality constant.

$$u(T) = aT^4.$$

[8] See Appendix C.

B.3 Wien's law

We'd like to decompose energy into frequency components, or spectral density u_ω:

$$u(T) = \int_0^\infty u_\omega \, d\omega,$$

and find what we can say about u_ω. Wien showed that u_ω can be written with a function f as

$$u_\omega = \omega^3 f(\omega/T) \tag{26}$$

by considering adiabatic expansion of electromagnetic wave confined in perfect conductor and Doppler shift associated with the expansion. Above formula is called Wien's law. Once we know u_ω for one particular temperature, we can find u_ω for any temperature using this formula. Although we can not say any further about f, we can show that if u_ω has a peak, the location has to be proportional to T by solving $du_\omega/d\omega = 0$. Wien speculated his formula Eq. (24) from Eq. (26) and observation data, with $\hbar$ in the exponent as one of fitting parameters. It is interesting to see both Rayleigh-Jeans law and Planck's formula has the form of Eq. (26).

Here I would like to derive Eq. (26) by using dimension analysis rather than demonstrating Wien's original derivation which can be found elsewhere. In general, dimension analysis can not give you a proof, but it is quick and sometimes very useful like this case.

What we want is energy spectral density per unit volume u_ω. First of all, we know that temperature is average energy of the system and that it always shows up in the form of kT in laws of physics, where k is Boltzmann constant. Since we want to find spectral density of electromagnetic wave, let us use speed of light c as well as ω and kT and there is no other physical quantity to specify this system. Rayleigh-Jeans law is obtained by making energy spectral density from these three quantities:

$$u_\omega = A\frac{kT}{c^3}\omega^2, \tag{27}$$

where A is dimensionless constant. u_ω has to satisfy Stephan-Boltzmann law:

$$u(T) = \int_0^\infty u_\omega d\omega = aT^4 \tag{28}$$

Suppose that this dimensionless quantity is in fact a function of ω and T, we can write such function as $A(\omega T^r)$.[9] Note that at this moment we do not know how to

[9] In case A's dependency is $\omega^s T^r$, we can always rewrite A as $A((\omega T^{r/s})^s)$, and replace A and r by $A'(x) = A(x^s)$ and $r' = r/s$.

make dimensionless quantity just from ω and T. We will come back this issue shortly. Inserting Eq. (27) into Eq. (28) yields

$$u(T) = \frac{kT}{c^3}\int_0^\infty \omega^2 A(\omega T^r)d\omega.$$

We want to take T out of the integral, by introducing $y = \omega T^r$,

$$u(T) = \frac{kT}{c^3}\int_0^\infty \frac{y^2}{T^{2r}}A(y)\frac{1}{T^r}dy = \frac{kT^{(1-3r)}}{c^3}\int_0^\infty y^2 A(y)dy.$$

Thus, since the integral is constant, r has to be -1 to meet $u(T) \propto T^4$. Finally u_ω can be written as follows:

$$u_\omega \propto A(\omega/T)\cdot\frac{kT}{c^3}\omega^2 = \omega^3\cdot\frac{A(\omega/T)}{c^3}kT/\omega.$$

This is Wien's law. Now we have to come back to the issue mentioned earlier – how to construct dimensionless quantity from ω and T. This issue is now reduced to how to construct dimensionless quantity from ω/T. We know Boltzmann constant which relates temperature to energy. Therefore it suggests a universal constant which relates frequency to energy. Planck's constant is indeed such a constant.

B.4 Black-body

Why we only care about electromagnetic wave in a section titled "black-body radiation" which seemingly about radiation from some kind of material?

In 1860, Kirchhoff showed, by thermodynamic argument, that ratio of absorption coefficient of light and radiation intensity does not depend on the material.[10] Let's say, for a material X at temperature T, $A_{\mathrm{X}}(\omega, T)$ is absorption coefficient of frequency ω, i.e., material X reflects back $(1 - A_{\mathrm{X}}(\omega, T))$ of light arrives and absorbs $A_{\mathrm{X}}(\omega, T)$ of it. Suppose that $J_{\mathrm{X}}(\omega, T)$ is radiation intensity of X, Kirchhoff showed that $J_{\mathrm{X}}(\omega, T)/A_{\mathrm{X}}(\omega, T)$ is a universal function which depends only on ω and T. Therefore if we measure radiation of a material which absorbs light perfectly, i.e., absorption coefficient is unity, we can calculate radiation intensity from its absorption coefficient. Such material, because it absorbs light perfectly, is called black-body. A small hole to a cavity is a close approximation of such black-body, since light can go inside freely and has to be reflected by wall many times before coming out from the hole, likely vanishes by that time. Light comes out from the hole is mostly leakage of electromagnetic wave inside. Therefore black-body radiation is in fact electromagnetic wave in an empty space at equilibrium with surrounding material.

[10] Just like Nyquist did for conductors, considering two material at thermal equilibrium through an ideal optical filter. Nyquist used perfectly matching transmission line (no reflection). It is not difficult to show. By the way, thermal noise depends only on resistance and temperature and not on the kind of material conductor is made of.

Appendix C Isotropic Electromagnetic wave

Our goal is to find pressure of electromagnetic wave confined in walls of a cubic box made of perfect conductor. There's no field outside the box. However, we can think of a periodic system by placing the same box one next to the other infinitely, so that we can use Fourier transform to represent periodic function.

C.1 Periodic boundary condition

Let's say the length of one side of the box is L and place x, y, z axis along with the box as usual. A function f of position $\boldsymbol{r}$ inside the box can be decomposed into frequency components:

$$f(\boldsymbol{r}) = \sum_{\boldsymbol{k}} F_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \qquad k_i = \frac{2\pi n_i}{L}, \qquad i = \{x, y, z\},$$

where n_x, n_y, n_z is integer. There is orthogonality of Fourier basis:

$$\delta_{\boldsymbol{k},\boldsymbol{k}'} = \frac{1}{V}\int_V e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\; d^3\boldsymbol{r}$$

Differentiating k_x with respect to n_x yields

$$\Delta k_x = \frac{2\pi}{L}\Delta n_x.$$

In another words, the number of possible k_x within k_x and $k_x + \Delta k_x$ is

$$\Delta n_x = \frac{L}{2\pi}\Delta k_x.$$

Similarly, the number of possible $\boldsymbol{k}$ within volume element $\Delta k_x \Delta k_y \Delta k_z$ at $\boldsymbol{k}$:

$$\frac{L}{2\pi}\Delta k_x \cdot \frac{L}{2\pi}\Delta k_y \cdot \frac{L}{2\pi}\Delta k_x \;= \frac{L^3}{(2\pi)^3}\Delta k_x \Delta k_y \Delta k_z = \frac{V}{(2\pi)^3}\Delta^3\boldsymbol{k},$$

where $V = L^3$ is the volume of the box. $\Delta^3\boldsymbol{k}$ is just a short hand for volume element $\Delta k_x \Delta k_y \Delta k_z$. We have used this in Section 2.6.

C.2 Plane wave solution for electromagnetic field

Now we want to find electromagnetic field in empty space. There is no charge or current inside the box. In such case Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$) with zero scalar potential ($\phi(\boldsymbol{r}) = 0$) is convenient choice. Maxwell's equation for empty space:

$$\nabla\times\boldsymbol{E} + \frac{1}{c}\frac{\partial c\boldsymbol{B}}{\partial t} = 0, \qquad \nabla\times c\boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = 0,$$
$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}.$$

Note that $\nabla\cdot\boldsymbol{E} = 0$ and $\nabla\cdot\boldsymbol{B} = 0$ is automatically satisfied with above equations. Here we treat $c\boldsymbol{B}$ (speed of light times $\boldsymbol{B}$ field) as one symbol so that $c\boldsymbol{B}$ has the same dimension as $\boldsymbol{E}$. ($c\partial t$ is length, ∇ is inverse of length).

Inserting the second line of the equations into the second of the first line yields

$$\nabla\times\nabla\times\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial^2 t} = 0 \quad\rightarrow\quad \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial^2 t} = 0,$$

where we used $\nabla\times\nabla\times\boldsymbol{A} = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$. Since $\nabla\cdot\boldsymbol{A} = 0$, we get wave equation:

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial^2 t} = 0.$$

To find solution we first apply periodic boundary condition:

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

There will be current on the surface of walls (because it is made of conductor) which sets another boundary condition other than periodicity such as $\boldsymbol{E}$ has to be normal to a wall etc. However we would like to leave it for later and take a look at just one plane wave first.

Plane wave, noting that $\boldsymbol{A}(\boldsymbol{r})$ is real:

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{a}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + c.c.,$$

where $c.c.$ is complex conjugate of the first term, namely $\boldsymbol{a}^*_{\boldsymbol{k}} \exp(-i\boldsymbol{k}\cdot\boldsymbol{r})$. Inserting this into the wave equation above yields

$$\ddot{\boldsymbol{a}}_{\boldsymbol{k}} \;=\; -(ck)^2\, \boldsymbol{a}_{\boldsymbol{k}}, \tag{29}$$

where k is norm of $\boldsymbol{k}$, i.e., $k = |\boldsymbol{k}|$. Therefore $\boldsymbol{a}_{\boldsymbol{k}}$ can have time dependence of $\exp(\pm ickt)$, but we take $\exp(-ickt)$ for traveling wave in $\boldsymbol{k}$ direction. From $\nabla\cdot\boldsymbol{A} = 0$,

$$\nabla\cdot\boldsymbol{A} = i\boldsymbol{k}\cdot\left(\boldsymbol{a}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \boldsymbol{a}^*_{\boldsymbol{k}}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right) = 0. \tag{30}$$

$\boldsymbol{E}$ field, noting that $\boldsymbol{a}_{\boldsymbol{k}} \propto \exp(-ickt)$:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} = ick\left(\boldsymbol{a}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \boldsymbol{a}^*_{\boldsymbol{k}}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right). \tag{31}$$

Because of Eq. (30), $\boldsymbol{E}$ is perpendicular to the wave vector $\boldsymbol{k}$. $\boldsymbol{B}$ field:

$$c\boldsymbol{B} = c\,\nabla\times\boldsymbol{A} = \frac{\boldsymbol{k}}{k}\times ick\left(\boldsymbol{a}_{\boldsymbol{k}}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \boldsymbol{a}^*_{\boldsymbol{k}}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right) = \frac{\boldsymbol{k}}{k}\times\boldsymbol{E}.$$

Therefore $E = cB$, and $\boldsymbol{B}$ is perpendicular to both $\boldsymbol{k}$ and $\boldsymbol{E}$. Energy density for a plane wave of wave vector $\boldsymbol{k}$:

$$u_{\boldsymbol{k}} = \frac{\varepsilon_0}{2}\left(E^2 + c^2B^2\right) = \varepsilon_0 E^2 = \varepsilon_0\,\boldsymbol{E}\cdot\boldsymbol{E}.$$

Poynting vector, using $\boldsymbol{A}\times\boldsymbol{B}\times\boldsymbol{C} = (\boldsymbol{A}\cdot\boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C}$ and $\boldsymbol{k}\cdot\boldsymbol{E} = 0$:

$$\begin{aligned}\boldsymbol{s}_{\boldsymbol{k}} &= c\,\varepsilon_0\boldsymbol{E}\times c\boldsymbol{B} = c\,\varepsilon_0\boldsymbol{E}\times\frac{\boldsymbol{k}}{k}\times\boldsymbol{E} = c\,\varepsilon_0\left(E^2\,\frac{\boldsymbol{k}}{k} - \left(\boldsymbol{E}\cdot\frac{\boldsymbol{k}}{k}\right)\boldsymbol{E}\right),\\ &= \frac{\boldsymbol{k}}{k}\,c\,\varepsilon_0 E^2 = \frac{\boldsymbol{k}}{k}\,c\,u_{\boldsymbol{k}}.\end{aligned}$$

Energy flows $\boldsymbol{k}$ direction at speed of c. Recalling that momentum is energy flux divided by c squared, relation between momentum density $\boldsymbol{g}_{\boldsymbol{k}}$ and energy density will be

$$\boldsymbol{g}_{\boldsymbol{k}} = \frac{\boldsymbol{s}_{\boldsymbol{k}}}{c^2} = \frac{\boldsymbol{k}}{ck}\,u_{\boldsymbol{k}}.$$

These quantities are vibrating over space and time. To obtain pressure which is flux of average momentum density, let us average $u_{\boldsymbol{k}}$ over the volume.

$$\bar{u}_{\boldsymbol{k}} = \frac{1}{V}\int u_{\boldsymbol{k}}\;d^3\boldsymbol{r} = \frac{1}{V}\int \varepsilon_0\,\boldsymbol{E}\cdot\boldsymbol{E}\;d^3\boldsymbol{r}$$

Inserting Eq. (31) and noting that vibrating term vanishes with integration, we get

$$\bar{u}_{\boldsymbol{k}} = 2\,\varepsilon_0(ck)^2\,\boldsymbol{a}_{\boldsymbol{k}}\cdot\boldsymbol{a}^*_{\boldsymbol{k}}\,.$$

Since $\boldsymbol{a}_{\boldsymbol{k}} \propto \exp(-ickt)$, $\bar{u}_{\boldsymbol{k}}$ does not have time dependence. Similarly, Poynting vector:

$$\bar{\boldsymbol{s}}_{\boldsymbol{k}} = \frac{\boldsymbol{k}}{k}\,c\,\bar{u}_{\boldsymbol{k}}.$$

As we will see in the next section, pressure is nothing but flow of momentum density, when a plane wave is reflected by a wall normal to $\boldsymbol{k}$ the wall receives pressure $p_{\boldsymbol{k}}$ of

$$p_{\boldsymbol{k}} = \text{momentum density}\times\text{speed} = c\,\bar{g}_{\boldsymbol{k}} = \bar{s}_{\boldsymbol{k}}/c = \bar{u}_{\boldsymbol{k}}.$$

For isotropic wave, it is superposition of plane waves with total momentum is zero, i.e., $\boldsymbol{k}$ is uniformly distributed over x, y and z directions. Total pressure p will be one third of $\bar{u}$, total energy density averaged over the volume. From orthogonality of plane waves, $\bar{u}$ is simply sum of $\bar{u}_{\boldsymbol{k}}$. Therefore,

$$p = \frac{1}{3}\bar{u}, \qquad \bar{u} = \sum_{\boldsymbol{k}} \bar{u}_{\boldsymbol{k}}\,.$$

We have obtained what we want without applying the final boundary condition of walls made of perfect conductor. In fact, for black-body radiation, this is enough and preferable. At thermal equilibrium, walls radiate back the same amount of energy (momentum) it absorbs, which gives the same amount reaction as reflection by perfect conductors in average.

Eq. (29) indicates that $\boldsymbol{a_k}$ is a vector harmonic oscillator, however $\boldsymbol{a_k}$ has to be in a plane normal to $\boldsymbol{k}$, the degree of freedom (number of independent oscillators) is 2 (not 3). In another words, each $\boldsymbol{k}$ has two harmonic oscillators.

C.3 Energy/Momentum flow and pressure

Consider a particle in a cubic box of $V{=}L^3$, moving at speed of v_x in x direction. When the particle bounced back at a wall of the box, momentum change is $2mv_x$ and it will travel $2L$ before come back to the same wall. Therefore average force on the wall

$$\begin{aligned} F &= \textit{momentum change per impact} \times \textit{number of impact per unit time},\\ &= 2mv_x \cdot \frac{v_x}{2L} = v_x \cdot \frac{mv_x}{L}. \end{aligned}$$

Pressure p is flow of momentum density mv_x/V, or twice of energy density $mv_x^2/2V$:

$$p = F/L^2 = v_x \cdot \frac{mv_x}{V} = 2 \cdot \frac{1}{2}mv_x^2 \Big/ V \tag{32}$$

When we have N particles in the box consisting uniform gas, momentum flow is distributed over x, y and z direction uniformly, pV is two thirds of the total energy U:

$$p = \frac{2}{3} \cdot \frac{U}{V}, \qquad U = \sum_{i=1}^{N} \frac{1}{2}mv_i^2,$$

where $\boldsymbol{v}_i$ is velocity of i-th particle.

In the special theory of relativity, energy u and momentum $\boldsymbol{g}$ of mass particle m is

$$u = \frac{mc^2}{\sqrt{1-(v/c)^2}}, \qquad \boldsymbol{g} = \frac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}}.$$

Therefore, relation between energy and momentum, $\boldsymbol{g} = \boldsymbol{v}u/c^2$ holds for both mass particle and electromagnetic wave, i.e., momentum is equal to energy flow divided by c^2. In case $v \ll c$, kinetic energy is second order small quantity: $u \sim mc^2$, and momentum is energy flow divided by c^2, i.e., $g \sim v \cdot mc^2/c^2 = mv$, resulting $u = g^2/2m$ and $pV = \frac{2}{3}U$. In case $m = 0$, $v = c$, we got $u = cg$ and $pV = \frac{1}{3}U$.

C.4 Statistical Mechanics

We have so far derived gas equation of electromagnetic wave from kinetics. The same can be obtained in terms of statistical mechanics. According to the (classical) statistical mechanics, the system partition function Z for N non-interacting particles is

$$Z = \frac{1}{N!} \cdot (f)^N,$$

where f is one-particle partition function

$$f = \frac{1}{(2\pi\hbar)^3} \iint e^{-\varepsilon(\boldsymbol{p},\boldsymbol{r})/kT} \, d^3\boldsymbol{p} \, d^3\boldsymbol{r}.$$

Here we use $\varepsilon(\boldsymbol{p},\boldsymbol{r})$ and $\boldsymbol{p}$ for energy and momentum of a particle to align with notation found in many text books. Since we are thinking of free particles in an empty space (no external potential), energy does not depend on $\boldsymbol{r}$, it only depends on magnitude of momentum $p = |\boldsymbol{p}|$. Therefore integral over $\boldsymbol{r}$ is equal to the volume V and integral over $\boldsymbol{p}$ can be separated into polar coordinate, with angular part is simply 4π:

$$f = \frac{4\pi V}{(2\pi\hbar)^3} \int_0^\infty p^2 \, e^{-\varepsilon(p)/kT} \, dp.$$

Since $\varepsilon(p)$ is a function of p only, integral is a function of kT only, therefore using constant a and γ, f can be written as follows.

$$f = aVT^\gamma = aV\beta^{-\gamma}, \qquad \beta = 1/kT.$$

Let us recall that internal energy U and pressure P can be calculated from partition function as follows.

$$U = -\frac{\partial \ln Z}{\partial \beta}, \qquad P = kT\frac{\partial \ln Z}{\partial V}.$$

Internal energy

$$U = -\frac{\partial}{\partial \beta} \ln(f)^N = -N\frac{\partial}{\partial \beta} \ln aVT^\gamma = \gamma N\beta{-}1 = \gamma NkT,$$

and pressure

$$P = NkT\frac{\partial}{\partial V} \ln aVT^\gamma = \frac{NkT}{V}.$$

Therefore

$$PV = NkT = U/\gamma.$$

Now, let us evaluate the integral to find γ. For non-relativistic particles, $\varepsilon = p^2/2m$. Therefore, by looking up gauss integral formula, we find

$$\int_0^\infty p^2 \exp\left(-\frac{p^2}{2mkT}\right) dp \quad \propto \quad (kT)^{3/2}.$$

As for electromagnetic wave, $\varepsilon = cp$, performing "integrate by part" two times we find

$$\int_0^\infty p^2 \exp\left(-\frac{cp}{kT}\right) dp \quad \propto \quad (kT)^3.$$

Therefore

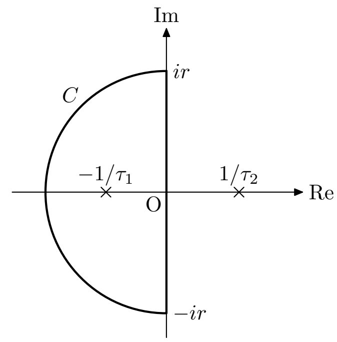
$$\begin{aligned} PV &= \frac{2}{3}U, \qquad \text{(for non-relativistic particle)} \\ PV &= \frac{1}{3}U. \qquad \text{(for electromagnetic wave)} \end{aligned}$$

$PV = NkT$ is not valid for quantum mechanical gas but it can be shown that above relations between P, V and U still hold for quantum mechanical case for both non-relativistic particle and, of course, photon. Curious reader may refer, for example, §56 of Landau & Lifshitz [4].

Appendix D Details of Eq. (11)

We want to find following integral:

$$I = \int_{-\infty}^{\infty} f(i\omega)\, e^{i\omega t}\, d\omega/2\pi,$$



where

$$f(i\omega) = \frac{1}{(1+i\omega\tau_1)(1-i\omega\tau_2)}, = \frac{1}{\tau_1\tau_2}\cdot\frac{1}{(1/\tau_1+i\omega)(1/\tau_2-i\omega)}.$$

We extend domain to complex plane and take path C shown. With $z = i\omega = re^{i\theta}$,

$$\oint_C f(z)\, e^{zt} dz/2\pi i = \int_{C_1} f(z)\, e^{zt} dz/2\pi i + \int_{C_2} f(z)\, e^{zt} dz/2\pi i,$$

where path C_1 and path C_2 is line from $-ir$ to ir and semi-circle from ir to $-ir$, respectively. I can be written as

$$I = \lim_{r\to\infty}\int_{C_1} f(z)\, e^{zt} dz/2\pi i = \lim_{r\to\infty}\left(\oint_C f(z)\, e^{zt} dz/2\pi i \; - \int_{C_2} f(z)\, e^{zt} dz/2\pi i\right).$$

Since $f(z) \to 0$ as $r \to \infty$, integral on C_2 vanishes if $t > 0$ (Jordan's lemma). The first term can be calculated using residue theorem:

$$\oint_C f(z)\, e^{zt} dz/2\pi i = \lim_{z\to -1/\tau_1} (z + 1/\tau_1)\, f(z) e^{zt} = \frac{e^{-t/\tau_1}}{\tau_1+\tau_2}. \qquad (t > 0)$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(1+i\omega\tau_1)(1+i\omega\tau_2)}\cdot\frac{d\omega}{2\pi} \;=\; \frac{e^{-t/\tau_1}}{\tau_1+\tau_2}. \qquad (t > 0)$$

For $t < 0$, we can use semi-circle on the other side. Result is

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(1+i\omega\tau_1)(1+i\omega\tau_2)}\cdot\frac{d\omega}{2\pi} \;=\; \frac{e^{-t/\tau_2}}{\tau_1+\tau_2}. \qquad (t < 0)$$

In case $\tau_1 = \tau_2 = \tau$, we get same answer for $t < 0$ and $t > 0$:

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1+\omega^2\tau^2}\cdot\frac{d\omega}{2\pi} \;=\; \frac{1}{2\tau}\, e^{-|t|/\tau}.$$

Appendix E Noise of RLC resonator

In Section 2.1.3, we used approximation using $\epsilon = \omega - \omega_0$ to evaluate integral of absolute square of admittance,

$$i/v \;\sim\; -\frac{1}{R} \cdot \frac{i\lambda}{\epsilon - i\lambda}, \tag{33}$$

resulting Nyquist formula,

$$\left\langle v^2 \right\rangle_\omega \;\sim\; 4kTR/2\pi.$$

In fact, the integral can obtained exactly using the same method as Appendix D and above "∼" can be replaced by "=". We could demonstrate it, however, here we like to calculate autocorrelation of current instead. We start from admittance $Y(i\omega)$,

$$Y(i\omega) = \frac{1}{R} \frac{i\omega\tau_3}{(1 + i\omega\tau_\oplus)(1 + i\omega\tau_\ominus)},$$

where

$$1/\tau_{\oplus,\ominus} = \lambda \pm i\omega_0', \quad \omega_0' = \sqrt{1 - (\lambda/\omega_0)^2}, \quad \tau_3 = RC, \quad \omega_0^2 = 1/LC, \quad \lambda = R/2L.$$

Recalling that autocorrelation $\phi(\tau)$ is Fourier transform of spectral density:

$$\phi(\tau) = \langle i(0)i(\tau) \rangle = \frac{1}{2} \int_{-\infty}^{\infty} \left\langle i^2 \right\rangle_\omega e^{-i\omega\tau} d\omega,$$

where factor 1/2 is came from the fact that $\left\langle i^2 \right\rangle_\omega$ is single sided. Using

$$\left\langle i^2 \right\rangle_\omega = |Y(i\omega)|^2 \left\langle v^2 \right\rangle_\omega, \quad \left\langle v^2 \right\rangle_\omega = 4kTR/2\pi,$$

we get with $z = i\omega$,

$$\phi(\tau) = \frac{2kTR}{L^2} \int_{-i\infty}^{i\infty} \frac{\omega^2 e^{-z\tau}}{(z + 1/\tau_\oplus)(z + 1/\tau_\ominus)(z - 1/\tau_\oplus)(z - 1/\tau_\ominus)} \cdot \frac{dz}{2\pi i}$$

Using the same method as Appendix D, we find

$$\phi(\tau) = \frac{kTR}{L^2} \frac{1}{2\lambda \cdot 2i\omega_0'} \left(\frac{1}{\tau_\oplus} e^{-|\tau|/\tau_\oplus} - \frac{1}{\tau_\ominus} e^{-|\tau|/\tau_\ominus} \right).$$

Note that absolute sign of $|\tau|$ came from the fact that proper semicircle is different for $\tau > 0$ and $\tau < 0$. For $\tau = 0$, $\phi(0) = \left\langle i^2 \right\rangle = kT/L$, which matches result using Eq. (33). For $\omega_0 < \lambda$, $i\omega_0'$, $1/\tau_{\oplus,\ominus}$ are real, resulting overshooting. For $\omega_0 > \lambda$, ω_0' is real, resulting damped oscillation:

$$\phi(\tau) = \frac{kTR}{2L^2} e^{-\lambda\tau} \left(\frac{\cos\omega_0'\tau}{\lambda} - \frac{\sin\omega_0'\tau}{\omega_0} \right).$$

Appendix F Useful Formula

Parallel impedance operator

$$(r_1//r_2) = \frac{1}{1/r_1 + 1/r_2} = \frac{r_1 r_2}{r_1 + r_2}, \quad (r_1//r_2//r_3) = \frac{1}{1/r_1 + 1/r_2 + 1/r_3} = \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}$$

$$(r_1//r_2) = (r_2//r_1), \quad \frac{1}{(r_1//r_2)} + \frac{1}{r_3} = \frac{1}{(r_1//r_2//r_3)}, \quad (r_1//c_1) = \frac{r_1}{1 + s\, r_1 c_1}$$

Minimum value

$$\min\left(\frac{A}{x} + Bx\right) = \sqrt{AB}, \qquad x_{\min} = \sqrt{\frac{A}{B}}.$$

Integral

$$\int_0^\infty \frac{d\omega/2\pi}{1 + (\omega/\omega_0)^2} = \frac{\omega_0}{2\pi} \tan^{-1}\left(\frac{\omega}{\omega_0}\right)\bigg|_0^\infty = \frac{1}{4}\omega_0$$

$$\int_{\omega_s}^\infty \frac{d\omega}{\omega(1 + (\omega/\omega_0)^2)} = \frac{1}{2} \ln \frac{(\omega/\omega_0)^2}{1 + (\omega/\omega_0)^2}\bigg|_{\omega_s}^\infty = \frac{1}{2} \ln \frac{1 + (\omega_s/\omega_0)^2}{(\omega_s/\omega_0)^2}$$

Two pole response function and its impulse response

$$v_o/v_i = \frac{1}{1 + s\,b + s^2\,a} = \frac{1}{(1 + s\,\tau_\oplus)(1 + s\,\tau_\ominus)} \qquad (a > 0,\ b > 0)$$

$$1/\tau_{\oplus,\ominus} = \frac{b \pm \sqrt{b^2 - 4a}}{2a} = \frac{b}{2a}\left(1 \pm \sqrt{1 - 4a/b^2}\right)$$

Discriminant $4a/b^2$:

$$\begin{aligned} 4a/b^2 < 1 \rightarrow &\quad \text{Exponential settling (overshooting)} \\ = 1 \rightarrow &\quad \text{Critical damping} \\ > 1 \rightarrow &\quad \text{Ringing} \end{aligned}$$

If $4a/b^2 \ll 1$,

$$1/\tau_\oplus = b/a - 1/b, \qquad 1/\tau_\ominus = 1/b$$

Canonical form of two pole amplifier

$$A(s) = \frac{N}{Q + s\,B + s^2\,A} = \frac{A_0}{(1 + s\,\tau_A A_0)(1 + s\,\tau_\oplus)}$$

If $4AQ/B^2 \ll 1$:

$$A_0 = N/Q, \quad \tau_A = B/N, \quad 1/\tau_\oplus = B/A - Q/B$$

Laplace transform

$$\mathcal{L}\{\delta(t)\} = 1, \quad \mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{e^{-t/\tau_1}\} = \frac{\tau_1}{1+s\,\tau_1}, \quad \mathcal{L}\{t/\tau_1\, e^{-t/\tau_1}\} = \frac{\tau_1}{(1+s\,\tau_1)^2}.$$

$$\begin{aligned}
\frac{1}{(1+s\,\tau_1)(1+s\,\tau_2)} &= \frac{1}{\tau_1-\tau_2}\left(\frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_2}{1+s\,\tau_2}\right)\\
\frac{s}{(1+s\,\tau_1)(1+s\,\tau_2)} &= -\frac{1}{\tau_1-\tau_2}\left(\frac{1}{\tau_1}\cdot\frac{\tau_1}{1+s\,\tau_1} - \frac{1}{\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}\right)\\
\frac{1+s\,\tau_3}{(1+s\,\tau_1)(1+s\,\tau_2)} &= \frac{1}{\tau_1-\tau_2}\left(\frac{\tau_1-\tau_3}{\tau_1}\cdot\frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_2-\tau_3}{\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}\right)\\
\frac{s}{(1+s\,\tau_1)^2} &= \frac{1}{\tau_1^2}\left(\frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_1}{(1+s\,\tau_1)^2}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s\,(1+s\,\tau_1)} &= \frac{1}{s} - \frac{\tau_1}{1+s\,\tau_1}\\
\frac{1}{s\,(1+s\,\tau_1)(1+s\,\tau_2)} &= \frac{1}{s} - \frac{\tau_1}{\tau_1-\tau_2}\cdot\frac{\tau_1}{1+s\,\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}\\
\frac{1}{s\,(1+s\,\tau_1)^2} &= \frac{1}{s} - \frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_1}{(1+s\,\tau_1)^2}\\
\frac{1+s\,\tau_3}{s\,(1+s\,\tau_1)(1+s\,\tau_2)} &= \frac{1}{s} - \frac{\tau_1-\tau_3}{\tau_1-\tau_2}\cdot\frac{\tau_1}{1+s\,\tau_1} + \frac{\tau_2-\tau_3}{\tau_1-\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}
\end{aligned}$$

Approximation If $\tau_1 \gg \tau_2$,

$$\begin{aligned}
\frac{s}{(1+s\,\tau_1)(1+s\,\tau_2)} &\sim \frac{1}{\tau_1\tau_2}\left(\frac{\tau_2}{1+s\,\tau_2} - \frac{\tau_2}{\tau_1}\cdot\frac{\tau_1}{1+s\,\tau_1}\right)\\
\frac{1}{s\,(1+s\,\tau_1)(1+s\,\tau_2)} &\sim \frac{1}{s} - \frac{\tau_1}{1+s\,\tau_1} + \frac{\tau_2}{\tau_1}\cdot\frac{\tau_2}{1+s\,\tau_2}
\end{aligned}$$

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