Linear Response Boot Camp

Shigenobu Kimura <skimu@me.com>

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Abstract

An introduction to small signal analysis.

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1 Linear Response

In a linear system, the relation between the input and the output is linear, which means it satisfies the principle of superposition. Suppose that if such system is given an input of $x_1(t)$, it gives an output of $y_1(t)$, and that another input $x_2(t)$ gives another output $y_2(t)$, then the output will be $a\, y_1(t) + b\, y_2(t)$ if $a\, x_1(t) + b\, x_2(t)$ is given as the input. Let's say $F\{\cdot\}$ denotes relation between the input and the output of the system.

$$y_1(t) = F\left\{x_1(t)\right\}, \quad y_2(t) = F\left\{x_2(t)\right\}$$

When the system is linear

$$F\left\{a\, x_1(t) + b\, x_2(t)\right\} = F\left\{a\, x_1(t)\right\} + F\left\{b\, x_2(t)\right\} = a\, y_1(t) + b\, y_2(t),$$

where a and b are arbitrary constants.

1.1 Impulse response and frequency response

Let's decompose $x(t)$ into sum of impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau,$$

and put it into $y(t) = F\{x(t)\}$:

$$y(t) = F\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right\}.$$

Since the integral is nothing but sum of running parameter τ, we can bring F into the integral:

$$y(t) = \int_{-\infty}^{\infty} F\left\{x(\tau)\delta(t-\tau)\right\} d\tau,$$

and $x(\tau)$ is just a parameterized coefficient, it can be brought out from F:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, F\left\{\delta(t-\tau)\right\} d\tau,$$

Exchanging the order of multiplication and using $h(t)$ for impulse response $F\left\{\delta(t)\right\}$, we see $h(t-\tau)$ is contribution weight of the input at time τ to the output at time t:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau.$$

Now we would like to take a look at frequency response of the system. Output $y(t)$ for complex sine wave input, $x(t) = x_\omega \, e^{i\omega t}$ will be

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau) \, x_\omega \, e^{i\omega\tau} \, d\tau.$$

With a new variable $\tau' = t - \tau$, ($\tau = t - \tau'$, $d\tau = -d\tau'$), above becomes

$$y(t) = \int_{-\infty}^{\infty} h(\tau') \, x_\omega \, e^{i\omega(t-\tau')} \, d\tau'.$$

Bringing $x_\omega \, e^{i\omega t}$ out from the integral and dropping prime from τ yields

$$y(t) = x_\omega \, e^{i\omega t} \int_{-\infty}^{\infty} h(\tau) \, e^{-i\omega\tau} \, d\tau.$$

The integral is constant (does not depend on t), i.e., sine wave input gives sine wave output of the same frequency, which is anticipated result from a linear system. If we write $y(t) = y_\omega \, e^{i\omega t}$, we see frequency response is Fourier transform of impulse response.

$$H(i\omega) = \frac{y_\omega}{x_\omega} = \int_{-\infty}^{\infty} h(\tau) \, e^{-i\omega\tau} \, d\tau$$

Therefore inverse transform of frequency response will give impulse response:

$$h(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(i\omega) \, e^{i\omega\tau} \, d\omega.$$

1.2 Causal impulse response and Laplace transform

We have just learnt that frequency response is Fourier transform of impulse response:

$$H(i\omega) = \int_{-\infty}^{\infty} h(\tau) \, e^{-i\omega\tau} \, d\tau$$

When the system is causal, $h(t)$ is zero for $t < 0$, because the impulse is given at $t = 0$. Therefore above integral can start from $t = 0$,

$$H(i\omega) = \int_{0}^{\infty} h(\tau) \, e^{-i\omega\tau} \, d\tau$$

If we substitute $i\omega$ by a complex variable s, we get Laplace transform.

$$H(s) = \int_{0}^{\infty} h(\tau) \, e^{-s\tau} \, d\tau$$

The inverse transform will be

$$h(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(i\omega)\, e^{i\omega\tau}\, d\omega = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} H(s)\, e^{s\tau}\, ds.$$

When $h(t)$ is real, we see following relations between $H(s)$ and its complex conjugate.

$$H(s^*) = H(s)^* \quad \text{or} \quad H(-i\omega) = H(i\omega)^*$$

We use $\mathcal{L}\{\cdot\}$ and $\mathcal{L}^{-1}\{\cdot\}$ to denote Laplace transform and its inverse transform.

$$H(s) = \mathcal{L}\left\{h(t)\right\}, \qquad h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}$$

This is just a short hand for the integral form. The reason why we introduce a special symbol is that we rarely evaluate integrals in practical calculations, but just look up Appendix C instead.

Impedance, admittance and transfer function When $H(s)$ represents response of voltage to current, we call it impedance or impedance function. When $H(s)$ represents response of current to voltage, we call it admittance or admittance function. When $H(s)$ represents response of the same kind of quantity as input, we call it transfer function. For impedance we use $Z(s)$ instead of $H(s)$ to make dimensions of quantities clear. Similarly we use $Y(s)$ for admittance. To summarize, we use following names and symbols

Transfer function $H(s)$, impedance $Z(s)$, admittance $Y(s)$,

for response functions. Those are

$$\begin{aligned} H(s) &= \text{Laplace transform of impulse response, transfer function,} \\ Z(s) &= H(s) \text{ when it represents voltage response to current,} \\ Y(s) &= H(s) \text{ when it represents current response to voltage,} \end{aligned}$$

or

$$\begin{aligned} H(s) = \mathcal{L}\{h(t)\} &= \int_0^{\infty} h(\tau)\, e^{-s\tau}\, d\tau, \\ Z(s) = \mathcal{L}\{z(t)\} &= \int_0^{\infty} z(\tau)\, e^{-s\tau}\, d\tau, \\ Y(s) = \mathcal{L}\{y(t)\} &= \int_0^{\infty} y(\tau)\, e^{-s\tau}\, d\tau, \end{aligned}$$

where $h(t)$, $z(t)$ and $y(t)$ is impulse response:

$$x_{\text{out}}(t) = \int_{-\infty}^{t} h(t-\tau)\, x_{\text{in}}(\tau)\, d\tau,$$
$$v(t) = \int_{-\infty}^{t} z(t-\tau)\, i(\tau)\, d\tau,$$
$$i(t) = \int_{-\infty}^{t} y(t-\tau)\, v(\tau)\, d\tau.$$

Here I used x_{out} and x_{in} because we want to use $y(t)$ for impulse response of voltage to current.

From convolution theorem, we have

$$X_{\text{out}}(s) = H(s)\, X_{\text{in}}(s), \quad V(s) = Z(s)\, I(s), \quad I(s) = Y(s)\, V(s),$$

where

$$X_{\text{out}}(s) = \mathcal{L}\{x_{\text{out}}(t)\}, \quad V(s) = \mathcal{L}\{v(t)\}, \quad I(s) = \mathcal{L}\{i(t)\}.$$

$Z(s)$ has dimension of resistance, $Y(s)$ has dimension of conductance, and $H(s)$ is dimensionless. Substituting s by $i\omega$ in response functions gives frequency responses:

$$v_\omega = Z(i\omega)\, i_\omega, \qquad i_\omega = Y(i\omega)\, v_\omega,$$

where $v(t) = v_\omega\, e^{i\omega t}$ and $i(t) = i_\omega\, e^{i\omega t}$.

1.3 Laplace transform of basic functions

Let's take a look at Laplace transform of a few basic functions.

1.3.1 Impulse function

$$h(t) = \delta(t).$$

$$\mathcal{L}\left\{\delta(t)\right\} = \int_0^\infty \delta(\tau)\, e^{-s\tau}\, d\tau = e^{-s\tau}\Big|_{\tau=0} = 1.$$

1.3.2 Step function

$$h(t) = \theta(t) = \begin{cases} 0 & (t < 0), \\ 1 & (t \geq 0). \end{cases}$$

$$\mathcal{L}\left\{\theta(t)\right\} = \int_0^\infty e^{-s\tau}\, d\tau = \left[\frac{e^{-s\tau}}{-s}\right]_0^\infty = \frac{1}{s}.$$

1.3.3 Ramp function

$$h(t) = \begin{cases} 0 & (t < 0), \\ t & (t \geq 0). \end{cases}$$

$$\mathcal{L}\left\{t\right\} = \int_0^\infty \tau\, e^{-s\tau}\, d\tau = \left[\frac{\tau\, e^{-s\tau}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-s\tau}}{-s}\, d\tau = \frac{1}{s^2}$$

1.3.4 Exponential function

$$h(t) = \begin{cases} 0 & (t < 0), \\ e^{-t/\tau_1} & (t \geq 0). \end{cases}$$

$$\mathcal{L}\left\{e^{-t/\tau_1}\right\} = \int_0^\infty e^{-t/\tau_1} e^{-s\tau} d\tau = \left[-\frac{e^{-(1/\tau_1+s)\tau}}{1/\tau_1+s}\right]_0^\infty,$$

$$= \frac{1}{1/\tau_1+s} = \frac{\tau_1}{1+s\tau_1}.$$

In case $-1/\tau_1 = i\omega$,

$$\mathcal{L}\left\{e^{i\omega t}\right\} = \frac{1/(-i\omega)}{1+s/(-i\omega)} = \frac{1}{s-i\omega}.$$

To get real wave, we can add complex conjugate,

$$\mathcal{L}\left\{A\, e^{i\omega t} + A^* e^{-i\omega t}\right\} = \mathcal{L}\left\{2\,|A|\cos\left(\omega t+\varphi\right)\right\} = \frac{A}{s-i\omega} + \frac{A^*}{s+i\omega},$$

where $\varphi = \arg A$. Appendix C summarizes above results and some useful formulae.

1.4 Response functions of circuit elements

Consider a two terminal circuit element such as resistor R, capacitor C or inductor L. If we apply voltage $v(t)$ across the terminal, current will flow. Such current is regarded as response to input $v(t)$. Let's take a look at frequency response of these circuit elements to get corresponding response functions.

First, resistor. I-V relation of a resistor R is

$$i(t) = \frac{1}{R}\, v(t) \quad \text{or} \quad v(t) = R\, i(t).$$

It is trivial that frequency response function is

$$Y(i\omega) = 1/R, \qquad Z(i\omega) = R.$$

Next, capacitor. I-V relation of a capacitor C is

$$i(t) = C\, \frac{dv(t)}{dt} \quad \text{or} \quad v(t) = \frac{1}{C} \int^{t} i(t)\, dt.$$

Inserting $v(t) = v_\omega e^{i\omega t}$ and $i(t) = i_\omega e^{i\omega t}$ yields

$$Y(i\omega) = i_\omega / v_\omega = i\omega C, \qquad Z(i\omega) = v_\omega / i_\omega = \frac{1}{i\omega C}.$$

Similarly for inductor L,

$$Y(i\omega) = \frac{1}{i\omega L}, \qquad Z(i\omega) = i\omega L.$$

Therefore, by substituting $i\omega$ with s, response functions of resistor, capacitor and inductor is given as follows.

	$Z(s)$	$Y(s)$
Resistor	R	$\frac{1}{R}$
Capacitor	$\frac{1}{sC}$	sC
Inductor	sL	$\frac{1}{sL}$

1.5 Composition of impedance and admittance

When we have two impedances Z_1 and Z_2 in series, both share the same amount of current $i(t)$. Voltage across these two will be, in s-space,

$$V(s) = V_1(s) + V_2(s) = (Z_1(s) + Z_2(s))\, I(s) = Z(s)\, I(s),$$

where $V_1(s)$, $V_2(s)$ and $Z(s)$ is Laplace transform voltage across Z_1, Z_2 and the compound impedance function, respectively. Therefore

$$Z(s) = V(s)/I(s) = Z_1(s) + Z_2(s).$$

Similarly, when we have two admittances $Y_1(s)$ and $Y_2(s)$ in parallel, both share the same amount of voltage $v(t)$. Total current $I(s)$ will be sum of two currents

$$I(s) = I_1(s) + I_2(s) = (Y_1(s) + Y_2(s))\, V(s).$$

Therefore compound admittance

$$Y(s) = I(s)/V(s) = Y_1(s) + Y_2(s).$$

Example: Compound resistance and capacitance In case two resistors R_1 and R_2 connected in series, compound impedance R will be

$$R = R_1 + R_2,$$

and for parallel connection admittance will be

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Since impedance of resistor is inverse of its admittance, the compound impedance R for parallel connection,

$$R = \frac{1}{1/R_1 + 1/R_2}.$$

Similarly, for a series connection of two capacitors,

$$\frac{1}{sC} = \frac{1}{sC_1} + \frac{1}{sC_2}.$$

It behave as if it is a capacitor of capacitance of

$$C = \frac{1}{1/C_1 + 1/C_2}.$$

In case of two capacitors in parallel, it behaves as if its capacitance is the sum of two

$$C = C_1 + C_2.$$

These are consistent with the results obtained from electrostatics.

1.6 AC power

Let's use a rule to convert complex sine wave to real sine wave like below

$$v(t) = \frac{1}{\sqrt{2}}\, v_\omega\, e^{i\omega t} + c.c., \qquad i(t) = \frac{1}{\sqrt{2}}\, i_\omega\, e^{i\omega t} + c.c.,$$

where $c.c.$ is complex conjugate of the first term. Suppose that these represent two terminal current and voltage, the average power over a period $T = 2\pi/\omega$ will be

$$P = \frac{1}{T}\int_0^T v(t)\, i(t)\, dt = \frac{1}{2}\,(v_\omega^*\, i_\omega + v_\omega\, i_\omega^*) = \mathrm{Re}(\, v_\omega^*\, i_\omega\,)\,.$$

Recalling that

$$V(s) = Z(s)\, I(s), \qquad I(s) = Y(s)\, V(s),$$

and that substituting s in a response function by $i\omega$ gives frequency response:

$$v_\omega = Z(i\omega)\, i_\omega, \qquad i_\omega = Y(i\omega)\, v_\omega,$$

power injected into the system is written with its impedance $Z(s)$ or admittance $Y(s)$, as follows.

$$P = \frac{1}{2}\,(Z^*(i\omega)\, i_\omega^*\, i_\omega + Z(i\omega)\, i_\omega\, i_\omega^*) = \mathrm{Re}\, Z(i\omega)\, |i_\omega|^2 \qquad \text{(by current source)}$$

$$P = \frac{1}{2}\,(v_\omega^*\, Y(i\omega)\, v_\omega + v_\omega Y^*(i\omega)\, v_\omega^*) = \mathrm{Re}\, Y(i\omega)\, |v_\omega|^2 \qquad \text{(by voltage source)}$$

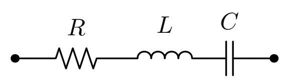
With this rule, $|v_\omega|$ and $|i_\omega|$ is rms value of $v(t)$ and $i(t)$, respectively.

$$\left\langle v^2 \right\rangle = |v_\omega|^2, \qquad \left\langle i^2 \right\rangle = |i_\omega|^2.$$

Note that proportionality coefficient is real part of frequency response function. From now on, we will use this rule unless otherwise noted.

1.7 Example: RLC resonator

Let's take a look at RLC resonator as an example of compound response function.



Impedance:

$$Z(s) = R + sL + \frac{1}{sC} = R \cdot \frac{1 + sRC + s^2LC}{sRC} = R \cdot \frac{(1 + s\,\tau_\oplus)(1 + s\,\tau_\ominus)}{s\,\tau_3},$$

where

$$\tau_\oplus + \tau_\ominus = RC, \quad \tau_\oplus\tau_\ominus = LC, \quad \tau_3 = RC,$$

$$1/\tau_{\oplus,\ominus} = \lambda \pm \sqrt{\lambda^2 - \omega_0^2}, \quad \lambda = R/2L, \quad \omega_0^2 = 1/LC.$$

1.7.1 Frequency response

Impedance:

$$Z(i\omega) = R + i\,\frac{\omega^2 LC - 1}{\omega C}.$$

Inductor and capacitor become transparent at $\omega = \omega_0$.
Admittance:

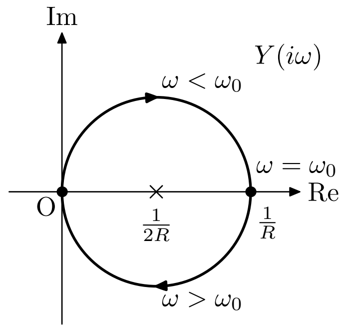
$$Y(i\omega) = \frac{1}{Z(i\omega)} = \frac{1}{R}\cdot\frac{i\dfrac{\omega}{\omega_0}\cdot\dfrac{2\lambda}{\omega_0}}{1-\left(\dfrac{\omega}{\omega_0}\right)^2 + i\dfrac{\omega}{\omega_0}\cdot\dfrac{2\lambda}{\omega_0}} = \frac{1}{R}\cdot\frac{iB}{A+iB} = Y' + i\,Y'',$$

where we have defined real quantities A, B, Y' and Y'' (Just for short hand). If we notice,

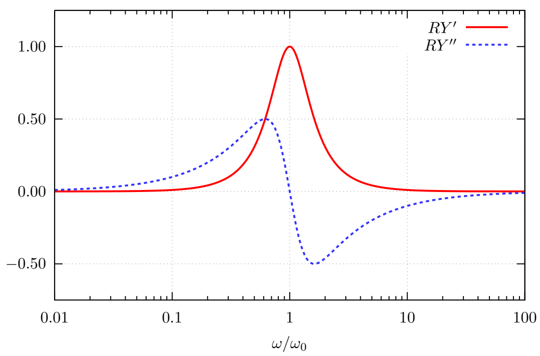
$$R\,Y(i\omega) = \frac{iB}{A+iB} = \frac{B^2 + iAB}{A^2+B^2} = x + i\,y,$$

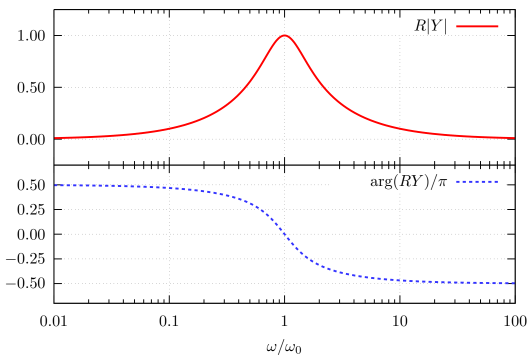
we see that $R\,Y(i\omega)$ is on a circle of diameter 1 in complex plane as shown in the right. Because x and y satisfy following equation.

$$(x-1/2)^2 + y^2 = 1/4$$



We can easily get some idea of relation between amplitude and phase from this circle. In case $\omega = -\infty$, $Y(i\omega)$ is at the origin. It moves clock wise as ω increases, and goes back to the origin at $\omega = \infty$. $\mathrm{Re}\,Y(i\omega)$ is maximum at $\omega = \omega_0$, i.e., if we place an AC voltage source (electromotive force) between two terminals of this resonator and make it a circuit, energy transfer from the voltage source to the resonator is maximum at ω_0. Below plots show real and imaginary part (left) and amplitude and phase (right).





We will see a few more thing about frequency response of RLC resonator in Section 1.11.4.

Q-factor Recalling that the mean square of current is the square of the absolute value of frequency component (complex current), average power being injected into the resonator P is written with complex current i_ω as

$$P = \mathrm{Re}\, Z(i\omega) \left\langle i^2 \right\rangle = R\, |i_\omega|^2.$$

In steady state, injected power is dissipated through the resistor. How about the inductor and the capacitor? What they do? They keep energy according to $LI^2/2$ and $CV^2/2$. Average energy $\bar{E}$ stored in L and C is sum of those:

$$\bar{E} = \frac{1}{2} L \left\langle i^2 \right\rangle + \frac{1}{2} C \left\langle v^2 \right\rangle,$$

where $\left\langle v^2 \right\rangle$ is mean square voltage across the capacitor. Recalling that $\left\langle v^2 \right\rangle = |v_\omega|^2$ and that

$$v_\omega = \frac{i_\omega}{i\omega C} \quad \rightarrow \quad |v_\omega|^2 = \frac{|i_\omega|^2}{\omega^2 C^2},$$

average energy stored in L and C:

$$\bar{E} \;=\; \frac{1}{2} L\, |i_\omega|^2 \;+\; \frac{1}{2} C\, \frac{|i_\omega|^2}{\omega^2 C^2} \;=\; \frac{1}{2} \left(\frac{\omega^2 LC + 1}{\omega^2 C} \right) |i_\omega|^2.$$

This amount of energy is injected at initial transient. (See Section 1.7.3) The ratio of the average energy stored and the average energy dissipated over a period of $1/\omega$ is called Q-factor:

$$Q(\omega) = \frac{\bar{E}}{P \cdot 1/\omega} = \frac{\omega^2 + \omega_0^2}{4\omega\lambda}.$$

Q_0 is defined at $\omega = \omega_0$:

$$Q_0 \;=\; Q(\omega_0) \;=\; \frac{\omega_0}{2\lambda} \qquad \text{or} \qquad \omega_0\, \frac{L}{R}.$$

Half-height width In case λ is small compared to ω_0, $|Y(i\omega)|$ has sharp peak at ω_0. $Y(i\omega)$ around ω_0 can be approximately written with $\epsilon = \omega - \omega_0$ as

$$Y(i\omega) = -\frac{1}{R} \cdot \frac{i\lambda}{\epsilon - i\lambda},$$

by dropping second order small quantities (coefficient of ϵ^2). Its square absolute value

$$|Y(i\omega)|^2 = \frac{1}{R^2} \cdot \frac{\lambda^2}{\lambda^2 + \epsilon^2}$$

gets half of its peak value at $\epsilon = \pm\lambda$. $2\lambda = \omega_0/Q_0$ is called half-height width or full width at half maximum (FWHM).

1.7.2 Step input

Suppose that DC voltage source V_0 is turned on at $t = 0$, i.e., $v(t)$ is a step function. Laplace transform of voltage across resonator will be V_0/s. Recalling that admittance

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{R} \cdot \frac{s\,\tau_3}{(1 + s\,\tau_\oplus)(1 + s\,\tau_\ominus)},$$

Laplace transform of current

$$\begin{aligned} I(s) = Y(s)\,V(s) &= \frac{1}{R} \cdot \frac{s\,\tau_3}{(1 + s\,\tau_\oplus)(1 + s\,\tau_\ominus)} \cdot \frac{V_0}{s}, \\ &= \frac{V_0\,\tau_3}{R} \cdot \frac{1}{(1 + s\,\tau_\oplus)(1 + s\,\tau_\ominus)}, \\ &\text{(By looking up Appendix C, we find)} \\ &= \frac{V_0\,\tau_3}{R} \cdot \frac{1}{\tau_\oplus - \tau_\ominus} \left(\frac{\tau_\oplus}{1 + s\,\tau_\oplus} - \frac{\tau_\ominus}{1 + s\,\tau_\ominus} \right). \end{aligned}$$

If we notice,

$$\frac{1}{\tau_\oplus - \tau_\ominus} = \frac{1/(\tau_\oplus\,\tau_\ominus)}{1/\tau_\ominus - 1/\tau_\oplus} = \frac{1}{LC} \cdot \frac{i}{2\,\omega_0 \sqrt{1 - (\lambda/\omega_0)^2}},$$

above become

$$I(s) = \frac{V_0}{L} \cdot \frac{i}{2\,\omega_0 \sqrt{1 - (\lambda/\omega_0)^2}} \left(\frac{\tau_\oplus}{1 + s\,\tau_\oplus} - \frac{\tau_\ominus}{1 + s\,\tau_\ominus} \right).$$

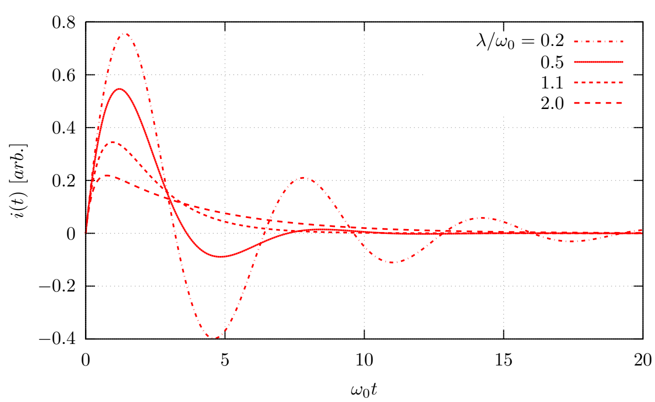
Here i is imaginary unit (not current). Taking inverse Laplace transform yields current $i(t)$ as a function of time:

$$i(t) = \mathcal{L}^{-1}\left\{I(s)\right\} = \frac{V_0}{L} \cdot \frac{i}{2\,\omega_0 \sqrt{1 - (\lambda/\omega_0)^2}} \left(e^{-t/\tau_\oplus} - e^{-t/\tau_\ominus} \right).$$

In case $\tau_{\oplus,\ominus}$ is real, i.e., $\lambda > \omega_0$, we see overshooting. Otherwise we see decaying oscillation of frequency ω' which is slightly lower than ω_0. Recalling that $1/\tau_{\oplus,\ominus} = \lambda \pm i\omega_0'$,

$$\begin{aligned} i(t) &= \frac{V_0}{\omega_0' L} \cdot e^{-\lambda t}\ \sin \omega_0' t, \\ \omega_0' &= \omega_0 \sqrt{1 - (\lambda/\omega_0)^2}. \end{aligned}$$

Wave forms are shown in the right.



1.7.3 Sine input

Let's apply sine wave input $v(t) = a\,e^{i\omega t}$ starting from $t = 0$. Laplace transform of such function is

$$V(s) = \mathcal{L}\{a\,e^{-t/\tau}\} = \frac{a\,\tau}{1+s\tau}, \quad \text{where} \quad -1/\tau = i\omega.$$

Current will be

$$\begin{aligned} I(s) &= Y(s)\,V(s), \\ &= \frac{1}{R}\frac{s\,\tau_3}{(1+s\,\tau_\oplus)(1+s\,\tau_\ominus)} \cdot \frac{a\,\tau}{1+s\,\tau}. \end{aligned}$$

In the end, we want inverse transform of $I(s)$. Our task is to rewrite $I(s)$ in a way we can find $\mathcal{L}^{-1}\{I(s)\}$ easily. (Just as we did in the previous section.) By looking up Appendix C, we find

$$\begin{aligned} I(s) = -\frac{a}{R} \cdot \frac{\tau_3}{\tau_\oplus - \tau_\ominus} \Bigg[& \frac{\tau}{\tau_\oplus - \tau}\left(\frac{\tau_\oplus}{1+s\,\tau_\oplus} - \frac{\tau}{1+s\,\tau}\right) \\ & - \frac{\tau}{\tau_\ominus - \tau}\left(\frac{\tau_\ominus}{1+s\,\tau_\ominus} - \frac{\tau}{1+s\,\tau}\right)\Bigg]. \end{aligned} \tag{1.1}$$

Let's focus on coefficient of $\mathcal{L}\{e^{i\omega t}\} = \tau/(1+s\tau)$, above becomes

$$\begin{aligned} I(s) = {} & \frac{a}{R} \cdot \frac{-\tau_3/\tau}{(1-\tau_\oplus/\tau)(1-\tau_\ominus/\tau)} \cdot \frac{\tau}{1+s\,\tau} \\ & + \frac{a}{R} \cdot \frac{\tau_3}{\tau_\oplus - \tau_\ominus}\left(\frac{1}{1-\tau_\oplus/\tau} \cdot \frac{\tau_\oplus}{1-s\,\tau_\oplus} - \frac{1}{1-\tau_\ominus/\tau} \cdot \frac{\tau_\ominus}{1-s\,\tau_\ominus}\right). \end{aligned}$$

Inserting $-1/\tau = i\omega$ and taking inverse transform yields

$$\begin{aligned} \mathcal{L}^{-1}\{I(s)\} = {} & \frac{a}{R} \cdot \frac{i\omega\,\tau_3}{(1+i\omega\,\tau_\oplus)(1+i\omega\,\tau_\ominus)} \cdot e^{i\omega t} \\ & + \frac{a}{R} \cdot \frac{\tau_3}{\tau_\oplus - \tau_\ominus}\left(\frac{1}{1+i\omega\,\tau_\oplus} \cdot e^{-t/\tau_\oplus} - \frac{1}{1+i\omega\,\tau_\ominus} \cdot e^{-t/\tau_\ominus}\right). \end{aligned} \tag{1.2}$$

Recalling that $1/\tau_{\oplus,\ominus} = \lambda \pm i\omega_0'$ and that $\lambda = R/2L > 0$, the last two terms vanish as t goes larger. Therefore we get anticipated result from frequency response.

$$\mathcal{L}^{-1}\{I(s)\} \sim Y(i\omega)\,a\,e^{i\omega t}. \qquad \text{(for large } t\text{)}$$

The vanishing two terms are used to charge energy into L and C.

Now let us take a look at behavior for smaller t. Taking inverse transform of Eq. (1.1) yields

$$\mathcal{L}^{-1}\left\{I(s)\right\} = -\frac{a}{R}\cdot\frac{\tau_3}{\tau_\oplus - \tau_\ominus}\left[\frac{1/\tau_\oplus}{1/\tau_\oplus + i\omega}\left(e^{i\omega t} - e^{-t/\tau_\oplus}\right)\right.$$
$$\left. - \frac{1/\tau_\ominus}{1/\tau_\ominus + i\omega}\left(e^{i\omega t} - e^{-t/\tau_\ominus}\right)\right].$$

We see there's no current at $t = 0$.

Inserting $1/\tau_{\oplus,\ominus} = \lambda \pm i\omega_0'$, $\tau_3 = RC$, and bringing $e^{i\omega t}$ out of the bracket, we get

$$\mathcal{L}^{-1}\left\{I(s)\right\} = \frac{a\,e^{i\omega t}}{2i\omega_0' L}\left[\frac{\lambda + i\omega_0'}{\lambda + i(\omega + \omega_0')}\left(1 - e^{-(\lambda + i(\omega+\omega_0'))t}\right)\right.$$
$$\left. - \frac{\lambda - i\omega_0'}{\lambda + i(\omega - \omega_0')}\left(1 - e^{-(\lambda + i(\omega-\omega_0'))t}\right)\right].$$

Let us assume $\lambda \ll \omega_0$ where ω_0' is very close to ω_0 and current decays much slower than its ringing period of $1/\omega_0'$. In case when the input frequency ω is close to ω_0, the first term vibrates very rapidly but coefficient is small compared to the second term. Overall behavior is dominated by the second term, which is moving very slowly.

$$\mathcal{L}^{-1}\left\{I(s)\right\} = \frac{a\,e^{i\omega t}}{2i\omega_0 L}\cdot\frac{\lambda - i\omega_0}{\lambda + i(\omega - \omega_0)}\left(1 - e^{-(\lambda + i(\omega-\omega_0))t}\right) \tag{1.3}$$

Coefficient of $a\,e^{i\omega t}$ is the envelope of the wave.

If we take $\lambda \to 0$,

$$\mathcal{L}^{-1}\left\{I(s)\right\} = \frac{a\,e^{i\omega t}}{2i(\omega - \omega_0)L}\left(1 - e^{i(\omega_0 - \omega)t}\right).$$

This is beat. Peak amplitude grows as the input frequency ω getting closer to ω_0, and period of the envelope is getting longer.

For smaller t, where $\lambda + i(\omega - \omega_0)t \ll 1$, using $e^x \sim 1 + x$ for $x \ll 1$, we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{I(s)\right\} &= \frac{a\,e^{i\omega t}}{2L}\cdot\frac{i\omega_0 - \lambda}{i\omega_0}\cdot t,\\ &= \frac{t}{2L}\cdot a\,e^{i\omega t}. \qquad\qquad (\lambda \to 0)\end{aligned}$$

Envelope grows linearly at the beginning. Figure 1 shows a few example.

Here we got complex result because we put complex input. Real solution is obtained by adding complex conjugate, like shown in Section 1.6. Once real solution is obtained it may be interesting to see power transfer from voltage source to this resonator or energy stored in C and L as a function of t.

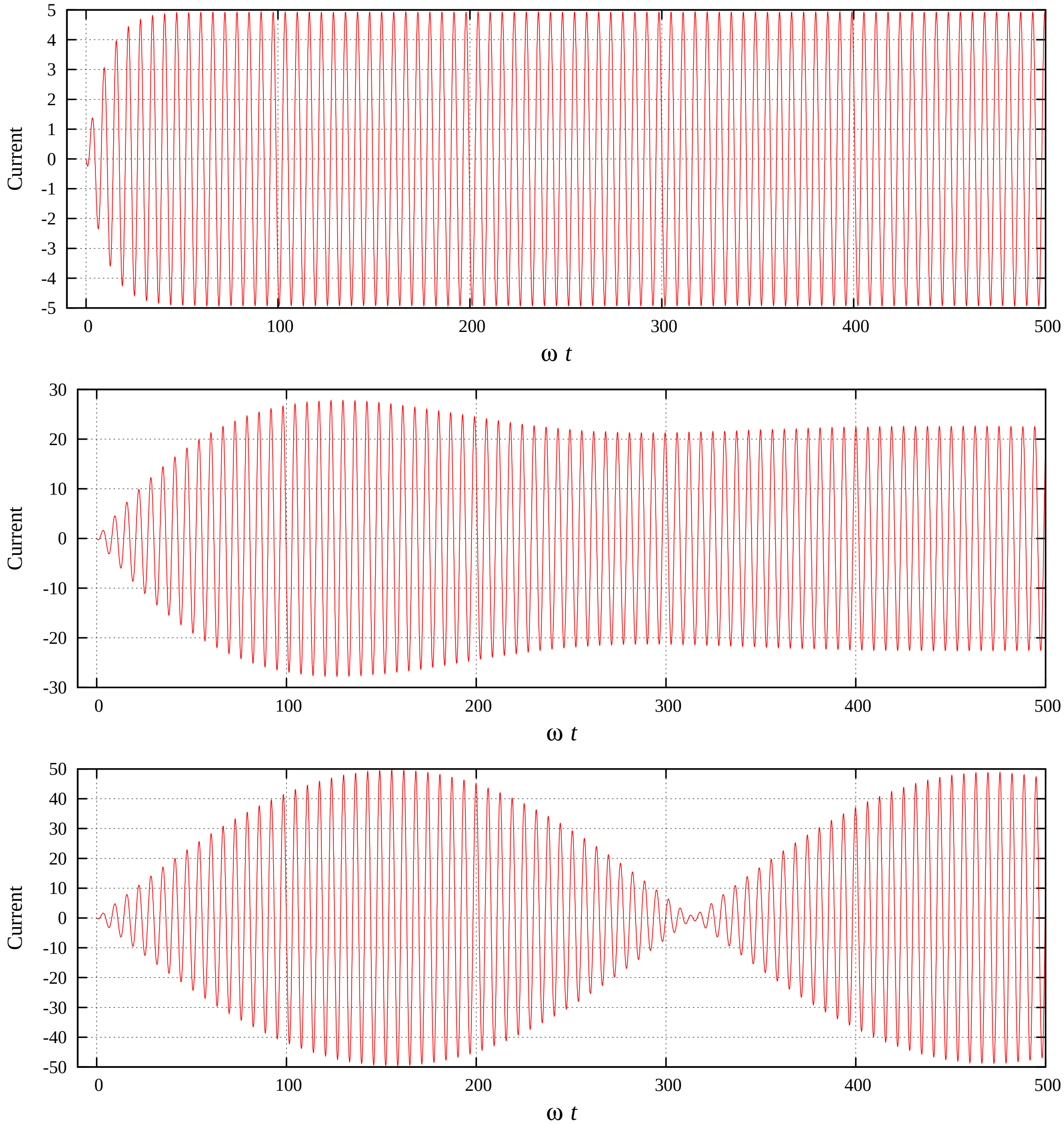


Figure 1: Current response of RLC driven by sine wave $ae^{i\omega t}$ (real part of Eq. (1.3)) with $L = 1.0$, $a = 1.0$, $\omega_0 = 0.98$, $\omega = 1.0$. (top) $\lambda = 0.1$, (mid) $\lambda = 0.01$, (bot) $\lambda = 0.0001$. Note in this plot L is fixed. This means R is reduced to make λ smaller resulting larger swing.

1.8 Circuit equations

(This section will be entirely rewritten)

1.8.1 Node equation

Sum of currents going out from a node is zero:

$$i_1(t) + i_2(t) + \ ... \ + i_N(t) = 0.$$

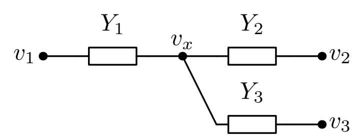
From the linearity of Laplace transform:

$$\mathcal{L}\{i_1(t) + i_2(t)\} = \mathcal{L}\{i_1(t)\} + \mathcal{L}\{i_2(t)\},$$

equation for $i_n(t)$ become, in s space,

$$I_1(s) + I_2(s) + \ ... \ + I_N(s) = 0.$$

where $I_n(s) = \mathcal{L}\{i_n(t)\}$. From now on, we are going to use lower case letters for both t space and s space functions to save number of symbols we deal with. For example, we are going to use i_n for both $I_n(s)$ and $i_n(t)$. This may sound confusing, you will be able to tell which is which easily from the context. Suppose a network like in below. Current going out from node v_x to each branch

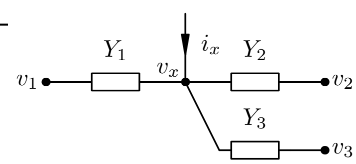


$$i_n = Y_n(s)\,(v_x - v_n)\,.$$

From the current conservation law, $\sum_n i_n = 0$, we get

$$\sum_n Y_n(s)\,(v_x - v_n) = 0$$

If we inject current i_x into node v_x, current conservation equation will be



$$\sum_n Y_n(s)\,(v_x - v_n) = i_x.$$

1.8.2 Circuit matrix

Consider a circuit network of N nodes and we inject current i_1 to node 1. Resulting node voltages V_i will be proportional to i_1,

$$v_i = z_{i1}\,i_1,$$

where z_{i1} is response function of v_i to i_1. Similarly, if we inject currents i_i to each node, resulting node voltages v_i will be linear combination of these:

$$v_i = \sum_{j=1}^{N} z_{ij}\, i_j.$$

We would like to find matrix (z) of which ij element is z_{ij}. Fortunately, inverse of (z) can be found by node equation as follows. Using $(y) = (z)^{-1}$, above equation become

$$i_i = \sum_{j=1}^{N} y_{ij}\, v_j, \qquad (y) = (z)^{-1}.$$

Let's say Y_{ij} is admittance between node i and j. If i and j is not connected, $Y_{ij} = 0$. Node equation will be

$$i_i = Y_{i0}\, v_i \;+\; \sum_{j \neq i} Y_{ij}\,(v_i - v_j),$$

where Y_{i0} is admittance to the ground. Therefore, diagonal and other elements of matrix (y) are, respectively,

$$y_{ii} = Y_{i0}\, v_i \;+\; \sum_{j \neq i} Y_{ij}, \qquad y_{ij} = -Y_{ij}.$$

Since matrix (Y) is symmetric ($Y_{ij} = Y_{ji}$), matrix (y) is also symmetric. Matrix (z) can be found by calculating inverse matrix of (y). Once we find (z), impedance at node 1 will be z_{11}, and $i_1 = v_1/z_{11}$. Transfer function from node 1 to node x will be

$$v_x \;=\; z_{x1}\, i_1 = \frac{z_{x1}}{z_{11}}\, v_1 \quad \rightarrow \quad v_x/v_1 = \frac{z_{x1}}{z_{11}}.$$

Recalling cramer's rule, $z_{ij} = \mathrm{adj}(y)_{ij}/\det(y)$, we find $v_x/v_1 = \mathrm{adj}(y)_{x1}/\mathrm{adj}(y)_{11}$.

Let's take a look at RC low pass filter shown in the next section and rename v_i to v_1 and v_o to v_2. Admittance between nodes will be

$$Y_{10} = 0, \quad Y_{12} = Y_{21} = 1/R, \quad Y_{20} = sC.$$

Therefore matrix elements

$$\begin{aligned} y_{11} &= Y_{10} + Y_{12} = 0 + 1/R = 1/R, \\ y_{12} &= y_{21} = -Y_{12} = -1/R, \\ y_{22} &= Y_{20} + Y_{21} = sC + 1/R, \end{aligned}$$

and matrix

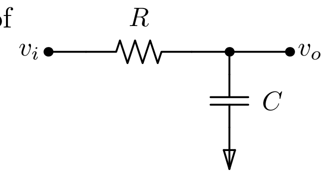
$$(y) = \begin{pmatrix} 1/R & -1/R \\ -1/R & 1/R + sC \end{pmatrix} = \frac{1}{R} \begin{pmatrix} 1 & -1 \\ -1 & 1 + sRC \end{pmatrix}.$$

Transfer function

$$v_o/v_i = v_2/v_1 = \frac{\mathrm{adj}(y)_{21}}{\mathrm{adj}(y)_{11}} = \frac{1}{1+sRC}.$$

This method is suitable when we want to study general properties of response functions or when you want to write a computer program to convert a netlist to circuit matrix (y) and find its solution (z). However, we would rather solve node equations than calculate adjugate, when we analyze a specific circuit. (Because it's usually easier, like shown in the next section.)

1.9 Example: RC low-pass filter



Let's take a look at RC low-pass filter as an example of circuit equation. Equation for node v_o is

$$\frac{1}{R}(v_o - v_i) \;+\; sC\,v_o = 0.$$

Therefore transfer function

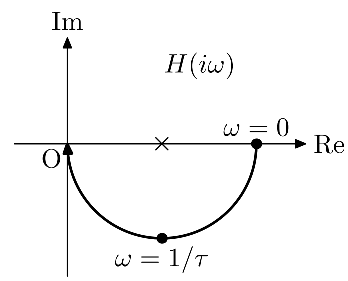
$$H(s) = v_o/v_i = \frac{1}{1+s\,\tau}, \qquad \tau = RC.$$

1.9.1 Frequency response

If we decompose frequency response function $H(i\omega)$ into real and imaginary part:

$$H(i\omega) = \frac{1}{1+i\omega\,\tau} = \frac{1}{1+\omega^2\tau^2} \;+\; i\frac{-\omega\,\tau}{1+\omega^2\tau^2} = x + i\,y$$

we see that $H(i\omega)$ is on a circle of



$$(x-1/2)^2 \;+\; y^2 = 1/4,$$

just like admittance of RLC resonator. But this time, $H(i\omega)$ makes only half circle like shown in the right. If we write $H(i\omega)$ in polar form like this:

$$H(i\omega) = |H(i\omega)|\,e^{i\phi}$$

magnitude and phase is, respectively,

$$|H(i\omega)| = \sqrt{x^2+y^2} = \frac{1}{\sqrt{1+\omega^2\tau^2}} \quad \text{and} \quad \phi = \tan^{-1} y/x = -\tan^{-1}\omega\tau.$$

In case $\omega \ll 1/\tau$,

$$H(i\omega) = \frac{1}{\sqrt{1+\omega^2\tau^2}} \;\sim\; 1 - \frac{1}{2}\cdot(\omega\tau)^2 \;\sim\; 1, \qquad \phi \;\sim -\omega\tau,$$

($(\omega\tau)^2$ is second order small quantity) therefore response to sine wave,

$$H(i\omega)\,e^{i\omega t} = |H(i\omega)|\,e^{i\phi}\cdot e^{i\omega t} \;\sim\; e^{i\omega(t-\tau)}.$$

Output follows input with time delay of τ.

In case $\omega \gg 1/\tau$, $\phi \sim \pi/2$, and in case $\omega = 1/\tau$, $\phi = \pi/4$ and $|H(i\omega)| = 1/\sqrt{2}$. Figure 2 shows waveform of these cases.

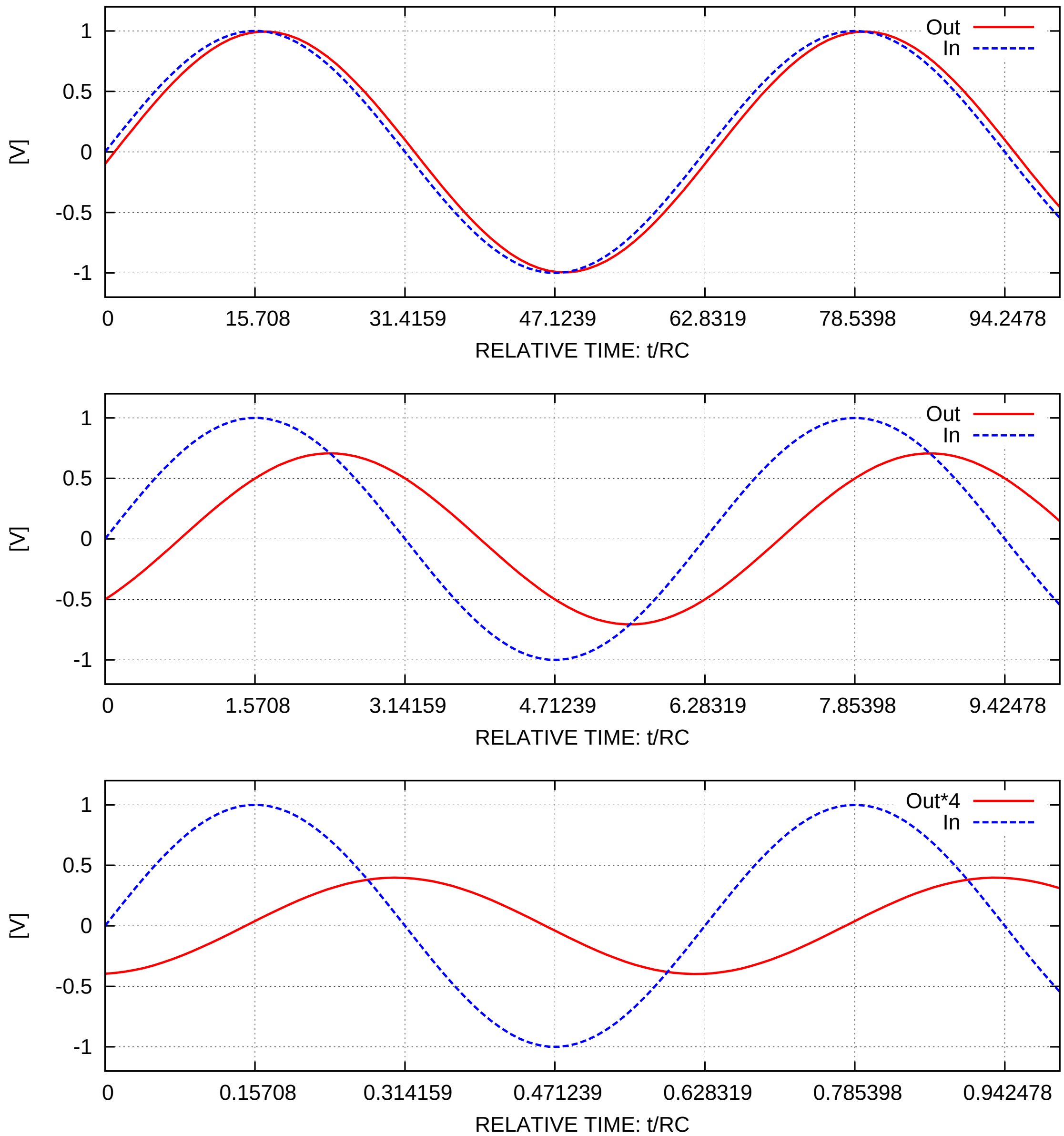


Figure 2: Input and output waveform of RC low-pass filter for a few input frequency. In $= 1.0\text{V} \times \sin \omega t$. top) $\omega = 0.1/\tau$, mid) $\omega = 1.0/\tau$, bot) $\omega = 10/\tau$, where $\tau = RC$. For low input frequencies where $\omega \ll 1/\tau$, output follows input with constant delay of τ in time. For high frequencies where $\omega \gg 1/\tau$, phase delay is $\pi/2$ or $90°$, not sensitive to the frequency. The output waveform (red) of the bottom is scaled up by 4, to make phase relation clearer.

1.9.2 Sine input

Recalling that

$$\mathcal{L}\{e^{-t/\tau}\} = \frac{\tau}{1+s\tau},$$

Laplace transform of complex sine wave $a\,e^{i\omega t}$ will be

$$v_i = \mathcal{L}\{a\,e^{i\omega t}\} = \frac{a/(-i\omega)}{1+s/(-i\omega)}.$$

Therefore response of output v_o to sine wave input v_i:

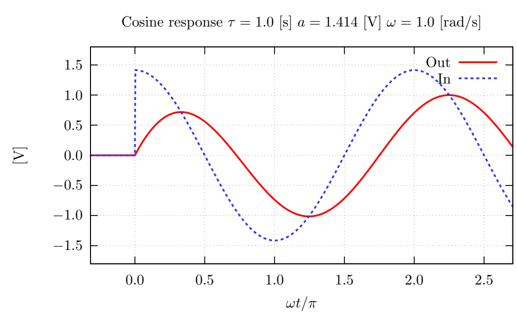
$$\begin{aligned}
v_o = H(s)\cdot v_i &= \frac{1}{1+s\tau}\cdot\frac{a/(-i\omega)}{1+s/(-i\omega)},\\
&= \frac{a}{-i\omega}\cdot\frac{1}{1+s\tau}\cdot\frac{1}{1+s/(-i\omega)},\\
&= \frac{a}{-i\omega}\cdot\frac{1}{\tau-1/(-i\omega)}\left(\frac{\tau}{1+s\tau}-\frac{1/(-i\omega)}{1+s/(-i\omega)}\right),\\
&= \frac{a}{1+i\omega\,\tau}\left(\frac{1/(-i\omega)}{1+s/(-i\omega)}-\frac{\tau}{1+s\tau}\right).
\end{aligned}$$

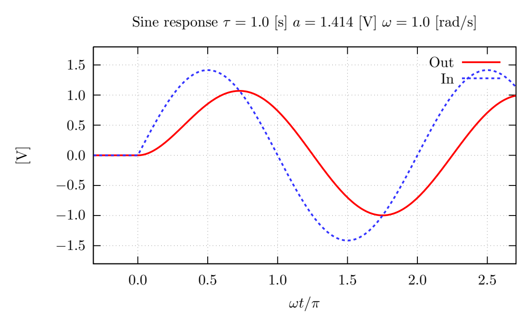
In t space, this is

$$v_o(t) = \mathcal{L}^{-1}\left\{H(s)\cdot v_i\right\} = \frac{a}{1+i\omega\tau}\left(e^{i\omega t}-e^{-t/\tau}\right).$$

The second term represents transient of input from static zero to sine wave at $t=0$. For $t \gg \tau$, the second term vanishes and we get something anticipated from the frequency response:

$$\frac{1}{1+i\omega\tau}\cdot a\,e^{i\omega t}.$$



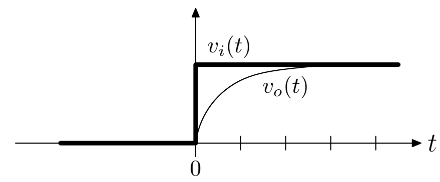


These are input and output wave forms for cosine and sine input, for example. Discontinuity of cosine input at $t=0$ results discontinuity of derivative of the output.

1.9.3 Step input

Let's take a look at response to step input $v_i(t)$:

$$v_i(t) = \begin{cases} 0, & (t < 0) \\ V_0. & (t \geq 0) \end{cases}$$



By looking up Appendix C, we find the Laplace transform of $v_i(t)$ is V_0/s, and

$$v_o \;=\; H(s) \cdot \frac{V_0}{s} \;=\; \frac{V_0}{s} \cdot \frac{1}{1+s\tau} \;=\; V_0 \left(\frac{1}{s} - \frac{\tau}{1+s\tau} \right).$$

Taking inverse transform yields

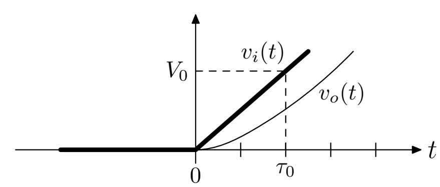
$$v_o(t) \;=\; V_0 \left(1 - e^{-t/\tau} \right).$$

The output v_o approaches V_0 at rate of τ.

1.9.4 Ramp input

Ramp input $v_i(t)$:

$$v_i(t) = \begin{cases} 0, & (t < 0) \\ V_0 \cdot t/\tau_0. & (t \geq 0) \end{cases}$$



By looking up Appendix C, we find the Laplace transform of $v_i(t)$ is $V_0/s^2\tau_0$, and

$$v_o \;=\; H(s) \cdot \frac{V_0}{s^2\,\tau_0} \;=\; \frac{V_0}{\tau_0} \cdot \frac{1}{s^2\,(1+s\tau)} \;=\; V_0\,\frac{\tau}{\tau_0} \left(\frac{\tau}{1+s\tau} + \frac{1}{s^2\,\tau} - \frac{1}{s} \right).$$

Taking inverse transform yields

$$v_o(t) \;=\; V_0\,\frac{\tau}{\tau_0} \left(e^{-t/\tau} + t/\tau - 1 \right).$$

Note that $v_o(t)$ and its derivative at $t = 0$ is zero:

$$v_o(0) = 0, \qquad \left.\frac{dv_o}{dt}\right|_{t=0} \;=\; V_0\,\frac{\tau}{\tau_0} \left.\left(-\frac{1}{\tau} e^{-t/\tau} + 1/\tau \right)\right|_{t=0} \;=\; 0.$$

The output (v_o) follows the input (v_i) with delay of τ, after a while ($t \gg \tau$):

$$v_o(t) \;\sim\; V_0\,(t-\tau)\,/\tau_0 \;=\; v_i(t-\tau). \qquad (t \gg \tau)$$

1.9.5 Output impedance

Let's say i_o is the current coming in from the outside through the terminal v_o. Current conservation equation for node v_o becomes

$$\frac{1}{R}\,(v_o - v_i) + sC\,v_o = i_o.$$

Therefore v_o is linear combination of input voltage v_i and current i_o like this:

$$v_o = \frac{v_i}{1+sRC} \;+\; \frac{R\,i_o}{1+sRC}.$$

We sometimes use a notation, v_o/v_i, for response function, to remind ourselves the meaning of the function and to save the number of symbols involved. Note that v_o/v_i is not always v_o divided by v_i, but it is always response of v_o to v_i. With this notation we rewrite above formula like below:

$$v_o = v_o/v_i \cdot v_i + v_o/i_o \cdot i_o,$$

where v_o/v_i is the transfer function $H(s)$:

$$H(s) = v_o/v_i = \frac{1}{1+sRC},$$

and v_o/i_o is the output impedance $Z_o(s)$:

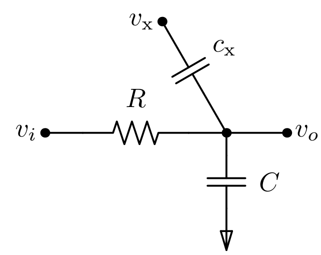
$$Z_o(s) = v_o/i_o = \frac{R}{1+sRC}.$$

Therefore,

$$v_o = H(s)\,v_i + Z_o(s)\,i_o.$$

Note: i_o is injected current at node v_o.

Parasitic coupling on output Suppose that there is coupling between the output v_o and an external node v_x, like shown in the right. Equation for node v_o is



$$\frac{1}{R}(v_o - v_i) + sC\,v_o + sc_\mathrm{x}\,(v_o - v_\mathrm{x}) = 0.$$

Therefore

$$v_o = \frac{1}{1+s\,RC'}\,v_i \;+\; \frac{sR\,c_\mathrm{x}}{1+sRC'}\,v_\mathrm{x}, \qquad C' = C + c_\mathrm{x}.$$

Step response to v_x (v_i stay unchanged):

$$v_o = \frac{sRc_\mathrm{x}}{1+sRC'} \cdot \frac{V_\mathrm{x}}{s} = \frac{c_\mathrm{x}V_\mathrm{x}}{C'} \cdot \frac{RC'}{1+sRC'}. \tag{1.4}$$

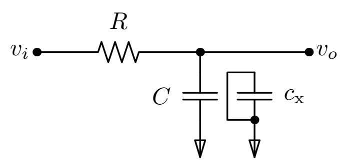
In t space (taking inverse transform):

$$v_o(t) = V_\mathrm{x} \cdot \frac{c_\mathrm{x}}{C'} \cdot e^{-t/RC'}.$$

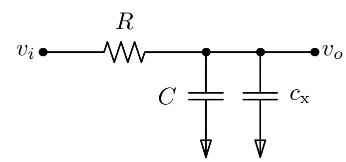
If c_x is relatively small (like parasitic coupling) $C' \sim C$. Eq. (1.4) may be rewritten as

$$v_o = \frac{R}{1+s\,RC} \cdot Q_\mathrm{x} = Z_o\,Q_\mathrm{x}, \qquad Q_\mathrm{x} = c_\mathrm{x}V_\mathrm{x}. \tag{1.5}$$

Recalling that inverse Laplace transform of constant is delta function, we see that v_o can also be interpreted as response to impulse current $Q_\mathrm{x}\,\delta(t)$ injected into v_o. Suppose that C is initially charged to V_o (v_i is tied to DC level of V_o) and uncharged, relatively small capacitor c_x is connected in parallel with C at $t = 0$:



$\Longrightarrow$

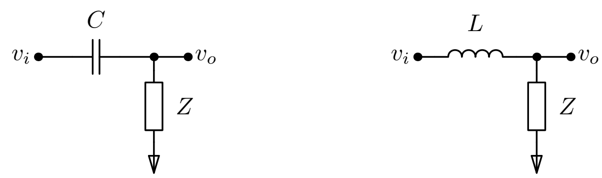


This is effectively giving impulse current of $-c_\mathrm{x}V_o\,\delta(t)$ to node v_o. By taking inverse transform of Eq. (1.5) we get

$$v_o(t) = -V_o \cdot \frac{c_\mathrm{x}}{C} \cdot e^{-t/RC}.$$

Output voltage drops by a factor of c_x/C, and goes back to the original value at rate of RC.

AC coupling and decoupling Let's take a look at other popular AC coupling and decoupling techniques in terms of frequency response, as yet another example of circuit equation. Here Z represents impedance at a node, say v_o, of your circuit and we are interested in coupling to an external node, say v_i. We connect v_i and v_o through a capacitor or an inductor.



For capacitor, equation for node v_x is

$$\frac{v_o}{Z} + sC(v_o - v_i) = 0 \quad \to \quad v_o/v_i = \frac{sC\,Z}{1 + sC\,Z}.$$

Therefore, frequency response

$$v_o/v_i = \frac{i\omega C\,Z}{1 + i\omega C\,Z}$$

This becomes unity as ω gets higher and becomes zero for very low frequency. Meaning that a capacitor behaves as short circuit for high frequency and open circuit for low frequency.

Similarly for inductor,

$$\frac{v_o}{Z} + \frac{v_o - v_i}{sL} = 0 \quad \to \quad v_o/v_i = \frac{Z}{Z + sL}.$$

Frequency response

$$v_o/v_i = \frac{Z}{Z + i\omega L}$$

This becomes unity as ω gets lower and becomes zero for very high frequency. Meaning that an inductor behaves as short circuit for low frequency and open circuit for high frequency.

1.10 Partial fraction expansion and impulse response

Recalling that our response function z_{ij} is written with circuit matrix y_{ij} as follows,

$$(z) = (y)^{-1} \quad \rightarrow \quad z_{ij} = \frac{\mathrm{adj}(y)_{ij}}{\det(y)},$$

and that

$$y_{ii} = Y_{i0}\, v_i \; + \; \sum_{j \neq i} Y_{ij}, \qquad y_{ij} = -Y_{ij}.$$

Since each Y_{ij} is either $1/R$, sC, or $1/sL$, trans impedance z_{ij} should be rational function of s, and so as transfer functions z_{ji}/z_{ii}, i.e., all the response functions we care has a form of

$$H(s) = H_0 \cdot \frac{1 + s\,u_1 + s^2\,u_2 + ... + s^n\,u_n}{1 + s\,t_1 + s^2\,t_2 + ... + s^d\,t_d},$$

where d and n is the order denominator and numerator, respectively. In case the denominator does not have root at the origin, we can factorize the denominator like

$$1 + s\,t_1 + s^2\,t_2 + ... + s^d\,t_d, = (1 + s\,\tau_1)(1 + s\,\tau_2)...(1 + s\,\tau_d),$$

where $-1/\tau_i$ are roots of the denominator. And in case $n < d$, $H(s)$ can be written in partial fraction expansion:

$$H(s) = \sum_k \frac{\tau_k}{1 + s\,\tau_k}\,a_k. \tag{1.6}$$

It will turn out all the response function we want to know falls into this case.[1] In case the denominator has root at the origin, we can bring one of τ_k to infinity. Recalling that

$$h(\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} H(s)\; e^{s\tau}\, ds,$$

with response functions we care (rational with $n < d$, i.e., $|H(s)| \rightarrow 0$ as $|s| \rightarrow \infty$), we can use residue theorem to calculate impulse response from frequency response. From Jordan's lemma, above integral is either the sum of residues in the right half of s plane (for $\tau < 0$) or that in the left half of s plane (for $\tau > 0$). Therefore $H(s)$ should not

[1] Well, there's an exception. It is an ideal high-pass filter:

$$H(s) = \frac{s\,\tau_c}{1 + s\,\tau_c}.$$

This does not become zero with $|s| \rightarrow \infty$. However, we can calculate step response and sine wave response since response as a whole ($H(s)\frac{1}{s}$, or $H(s)\frac{\tau}{1+s\tau}$) can be written in a form of Eq. (1.6). And they give reasonable results.

have pole in the right half of s plane to meet causality condition, $h(\tau) = 0$ for $\tau < 0$. In other words real part of all τ_k should be positive. In case $H(s)$ has poles on imaginary axis, we can still get meaningful transient behavior, by shifting poles slightly to the left by λ and then take $\lambda \to 0$ at the end, like we did in Section 1.7.3. This operation is equivalent to bringing corresponding τ_k to infinity. Inverse transform of Eq. (1.6) is

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \sum_k a_k \, e^{-t/\tau_k}, \quad \text{for} \quad (t > 0).$$

Since real part of τ_k is positive, $h(t)$ decays as time goes by. In a system which does not respond before input, will also forget what it received, or it comes back to the original state by itself after disturbance, the system is stable. Wait a minute, there must be cases where the system is causal and unstable, like latch? Appendix A provides some note on this. In short, frequency response is not well defined in such system.

Useful facts

- If a system consists of R and C only, all poles are located on real axis in left half plane. (No ringing)
- If a system consists of R and C and L only, like those we have looked at so far, real part of all poles are negative. (Always stable, but may ring).
- Above coming from the fact that all component parameters (resistance, capacitance, and inductance) are positive.

Useful formula Since we are going to find response functions by hand, (muscle aided design) we do not want to deal with high order polynomials. And for lower order polynomials there's a few useful formula in Appendix C (only two pages). We have been using these formula already and that's all we need to know.

Kramers-Kronig and Bode Causality condition, which cuts freedom of impulse response function by half, leads to a fact that real part and imaginary part of frequency response function $H(i\omega)$ can not be independent. If you know one, you can calculate the other from it. The relation is known as Kramers-Kronig relations (Appendix B). We do not use this relation because our objective is to find response functions of known circuit network. We can always find full response functions (both real and imaginary part). However it is useful when you want to know the response function of an unknown system by measurement. It will be difficult to measure exact I-V relation, however power-amplitude relation is much easier to measure. Since power is related with amplitude by real part of the response function, we can get real part of response function from such measurement. Then we can calculate imaginary part from the real part to get complete response function. Similarly there's a relation between phase and amplitude of $H(i\omega)$, which is known as Bode's theorem.

1.11 Bode plot for frequency response

We have just learnt that response function is rational function of s:

$$H(s) = H_0 \cdot \frac{1 + s\,u_1 + s^2\,u_2 + ... + s^n\,u_n}{1 + s\,t_1 + s^2\,t_2 + ... + s^d\,t_d},$$

and that partial fraction expansion is convenient way to see impulse response. In this section, we learn convenient way to look at frequency response and find a way to get some idea of impulse response from frequency response.

1.11.1 General formulation

Because $H(s)$ is rational function, it can also be written in factorized form:

$$H(s) = H_0 \cdot \frac{(1 + s\,\tau_{n1})(1 + s\,\tau_{n2})...(1 + s\,\tau_{nn})}{(1 + s\,\tau_{d1})(1 + s\,\tau_{d2})...(1 + s\,\tau_{dd})},$$

where $-1/\tau_{n1}, -1/\tau_{n2}, ..., -1/\tau_{nn}$ and $-1/\tau_{d1}, -1/\tau_{d2}, ..., -1/\tau_{dd}$ are roots of numerator and denominator, respectively. In case one of polynomials has root at the origin, we can take it by bringing one of τ to infinity. Taking log of absolute value yields

$$\begin{aligned}\log|H(i\omega)| = \log|H_0| \;&+\; \log|1 + i\omega\,\tau_{n1}| \;+\; \log|1 + i\omega\,\tau_{n2}| \;+\; ... \;+\; \log|1 + i\omega\,\tau_{nn}| \\ &-\; \log|1 + i\omega\,\tau_{d1}| \;-\; \log|1 + i\omega\,\tau_{d2}| \;-\; ... \;-\; \log|1 + i\omega\,\tau_{dd}|\,.\end{aligned}$$

Let's sort τ_{ni}, τ_{di} with respect to real part of corresponding root of polynomials:

$$\log|H(i\omega)| = \log|H_0| \;+\; \sum_i \sigma_i \log|1 + i\omega\,\tau_i|\,, \quad \mathrm{Re}(1/\tau_{i+1}) \geq \mathrm{Re}(1/\tau_i),$$

where

$$\sigma_i = \begin{cases} +1 & \text{if } -1/\tau_i \text{ is one of roots of numerator } (-1/\tau_{nj}), \\ -1 & \text{if } -1/\tau_i \text{ is one of roots of denominator } (-1/\tau_{dj}). \end{cases}$$

As for phase,

$$\arg\left(H\left(i\omega\right)\right) = \arg\left(H_0\right) \;+\; \sum_i \sigma_i \arg\left(1 + i\omega\,\tau_i\right),$$

Note that $(1 + i\omega\,\tau)\,/\tau = (i\omega + 1/\tau)$ is vector in complex plane from point $-1/\tau$ to point $i\omega$, i.e., $|i\omega + 1/\tau|$ is distance in complex plane between point $i\omega$ and point $-1/\tau$ and $\arg(i\omega + 1/\tau)$ is angle to the real axis.

1.11.2 Real roots

For real τ, each term can be estimated as follows:

$$\log|1+i\omega\,\tau| \;=\; \begin{cases} \sim 0 & (\omega\,\tau \ll 1), \\ \log\sqrt{2} & (\omega\,\tau = 1), \\ \sim \log|\omega\,\tau| = \log|\omega| + const. & (\omega\,\tau \gg 1), \end{cases}$$

and

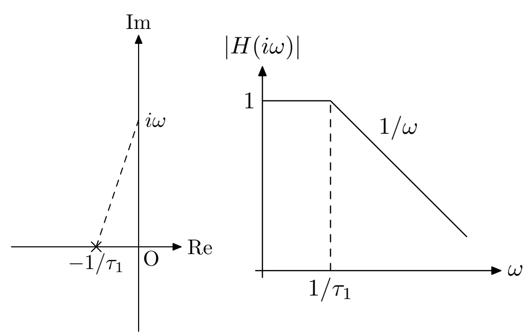
$$\arg(1+i\omega\,\tau) \;=\; \begin{cases} \sim 0 & (\omega \ll 1/\tau), \\ \pi/4 & (\omega = 1/\tau), \\ \sim \pi/2 & (\omega \gg 1/\tau). \end{cases}$$

Therefore, in case all roots are real, the following can be said when we sweep ω from zero to infinity:

- Whenever ω crosses one of $1/\tau_i$, slope of $\log|H(i\omega)|$ in log scale increases (or decreases) by 1 and $\arg(H(i\omega))$ increases (or decreases) by $\pi/2$ if $1/\tau_i$ is one of τ_{nj} (or τ_{dj}).

Let's take a look at some simple cases. The first is RC low-pass filter:

$$H(s) = \frac{1}{1+s\,\tau_1}.$$



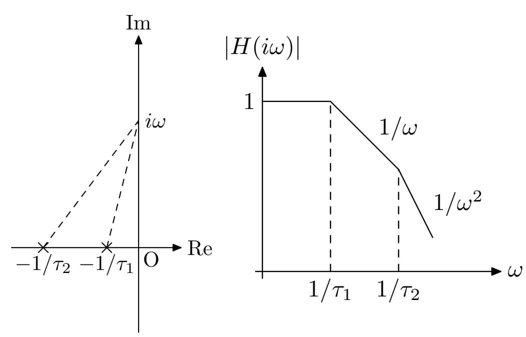
This one has a pole at $-1/\tau_1$. Log of absolute value of frequency response:

$$\log|H(i\omega)| = -\log|1+i\omega\tau_1|.$$

$|H(i\omega)|$ is unity for $\omega \ll 1/\tau_1$, and as ω crosses $1/\tau_1$ slope become -1 in log scale (proportional to ω^{-1}). At the same time, phase changes from 0 to $\pi/2$.

Next example will be two pole response function with both poles are at real axis.

$$H(s) = \frac{1}{(1+s\,\tau_1)(1+s\,\tau_2)}.$$



Log of its absolute value:

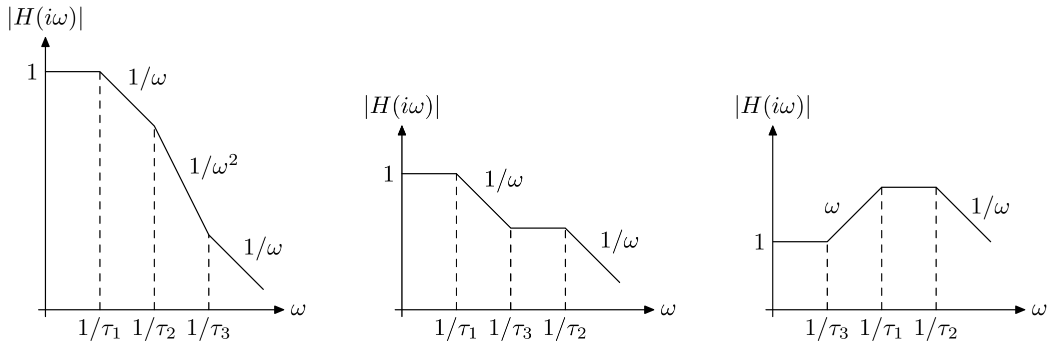
$$\begin{aligned}\log|H(i\omega)| = &-\log|1+i\omega\tau_1| \\ &-\log|1+i\omega\tau_2|.\end{aligned}$$

In case $1/\tau_2 \gg 1/\tau_1$, $\log|1+i\omega\tau_2|$ is still close to zero for $\omega \ll 1/\tau_2$, we will see sloe of -1 in between $1/\tau_1$ and $1/\tau_2$. And slope becomes -2 at $\omega \gg 1/\tau_2$.

The last example for real roots is

$$H(s) = \frac{(1 + s\,\tau_3)}{(1 + s\,\tau_1)(1 + s\,\tau_2)},$$
$$\log|H(i\omega)| = +\log|1 + i\omega\tau_3| - \log|1 + i\omega\tau_1| - \log|1 + i\omega\tau_2|.$$

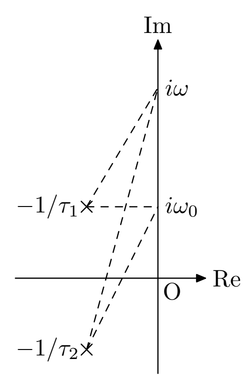
The shape of $|H(i\omega)$ depends on the relation between τ_3 and $\tau_{1,2}$. Below plots are for $1/\tau_1 < 1/\tau_2 < 1/\tau_3$, $1/\tau_1 < 1/\tau_3 < 1/\tau_2$, $1/\tau_3 < 1/\tau_1 < 1/\tau_2$, from left to right.



1.11.3 Complex roots

Now let us think about cases polynomials have complex roots, where impulse response has ringing components. Let's say $-1/\tau_1$ is one of complex roots of a polynomial. Since all of component parameters are real, coefficients of s^n in the polynomial are all real. If such polynomial has a complex root, its complex conjugate is also one of roots. Therefore complex conjugate of $-1/\tau_1$ is also a root of the polynomial.

Let's take a look at a simple case



$$H(s) \;=\; \frac{1}{(1 + s\,\tau_1)(1 + s\,\tau_2)},$$

Here τ_1 is complex, $1/\tau_2$ must be complex conjugate of $1/\tau_1$. Let's say λ and $i\omega_0$ is real part and imaginary part of $-1/\tau_1$:

$$-1/\tau_1 = -\lambda + i\omega_0, \quad -1/\tau_2 = -\lambda - i\omega_0.$$

Log of absolute value of frequency response is

$$\log|H(i\omega)| \;=\; -\log|i\omega + 1/\tau_1| - \log|i\omega + 1/\tau_2| + const.$$

If we sweep ω from 0 to infinity, the second term decreases monotonically as ω getting larger, however the first term has peak at $\omega = \omega_0$ where $-1/\tau_1$ and $i\omega$ is closest. Similarly, if numerator has a complex root, we should see a notch. For ω much greater than $1/\tau_1$ or $1/\tau_2$ both terms gives $1/\omega$, resulting slope of -2 (proportional to $1/\omega^2$).

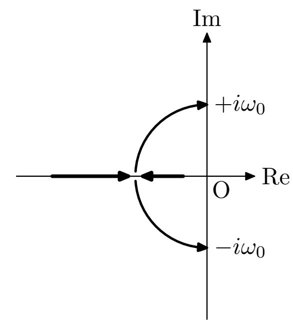
1.11.4 Example: Transfer function of RLC

Let's take a look at RLC resonator again. This time, we like to look at response of voltage across the capacitor. Suppose that resonator is driven by v_i and that the voltage across the capacitor is v_o. Transfer function $H(s) = v_o/v_i$ will be

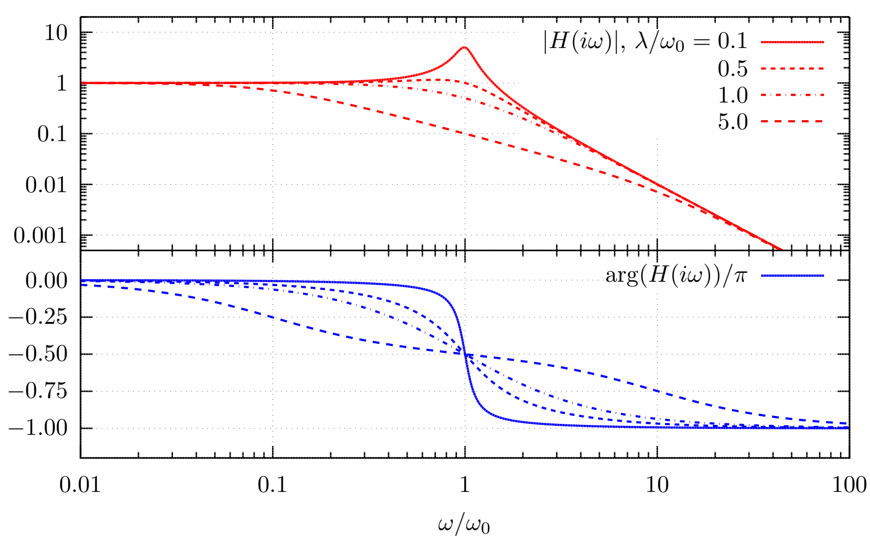
$$H(s) = \frac{1}{(1 + s\,\tau_\oplus)(1 + s\tau_\ominus)},$$

where

$$1/\tau_\oplus = \lambda + \sqrt{\lambda^2 - \omega_0^2}, \qquad 1/\tau_\ominus = \lambda - \sqrt{\lambda^2 - \omega_0^2}.$$



In case $\lambda > \omega_0$, both poles are on the real axis. As we decrease λ, two poles are getting closer, and they are on top of each other when $\lambda = \omega_0$. Then as we go further, poles become complex and reaches at imaginary axis with $\lambda = 0$, where we see sustained oscillation. Recalling that $\lambda = R/2L$ which has to be positive, we cannot go any further. However, for some reason, if R becomes negative, the system becomes unstable. This pole movement with decreasing R while keeping L an C constant, is shown in the right.
Below shows frequency response $|H(i\omega)|$ and $\arg H(i\omega)$ with a few λ. For $\lambda/\omega_0 = 5.0$, we have two separate corner with $1/\omega$ slope in between. For $\lambda/\omega_0 = 0.1$, we see sharp peak at $\omega = \omega_0$, as the result of complex poles. In any cases, we see slope of -2 at $\omega \gg \omega_0$. Figure 3 shows real part and imaginary part of $H(i\omega)$ and $|H(i\omega)|$ in linear scale. We saw step response of current in Section 1.7.2.



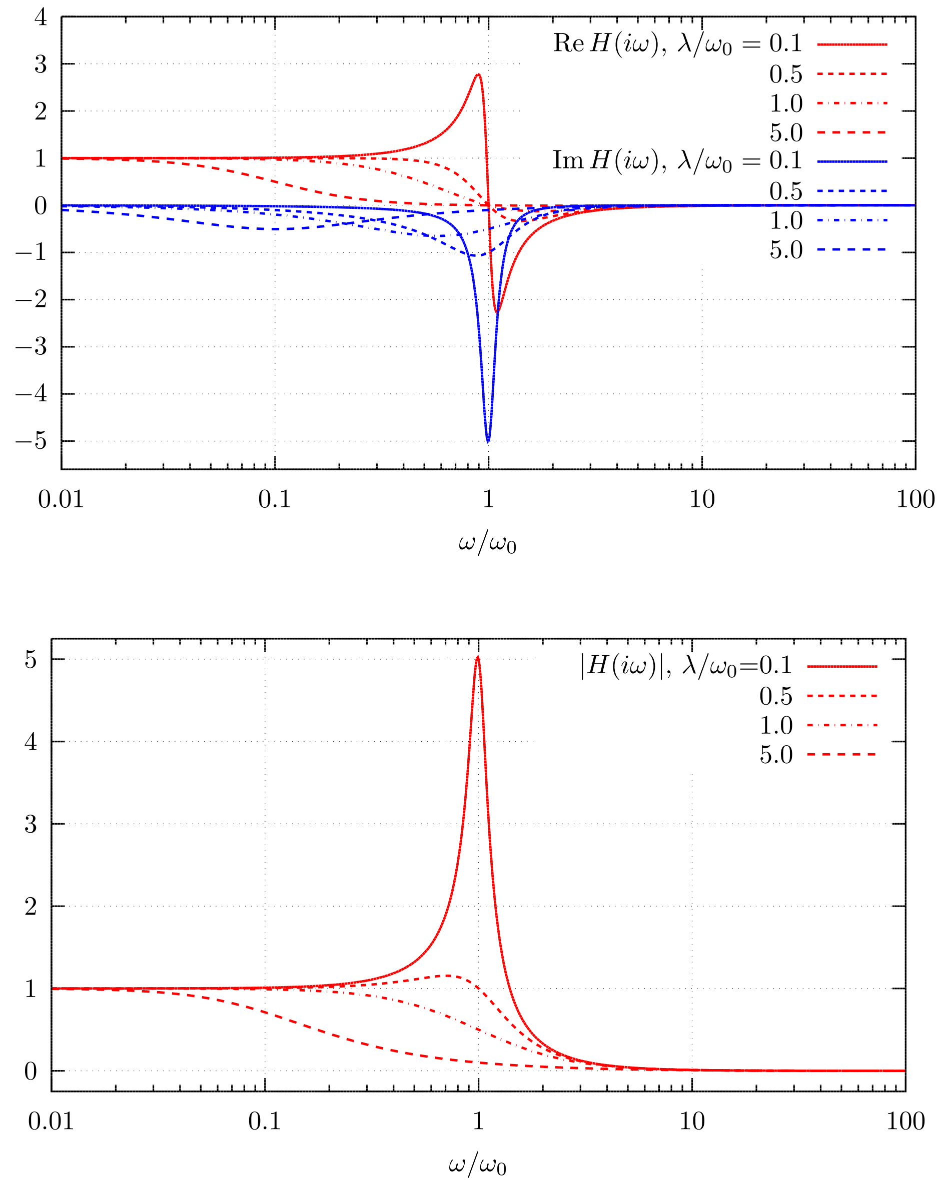


Figure 3: (top) Real and imaginary part of $H(i\omega)$, where $H(i\omega)$ is transfer function of RLC resonator from driving voltage v_i to voltage across the capacitor v_o, i.e., $H(s) = v_o/v_i$. (bot) $|H(i\omega)|$ in linear scale.

1.12 Typical waveforms of impulse response

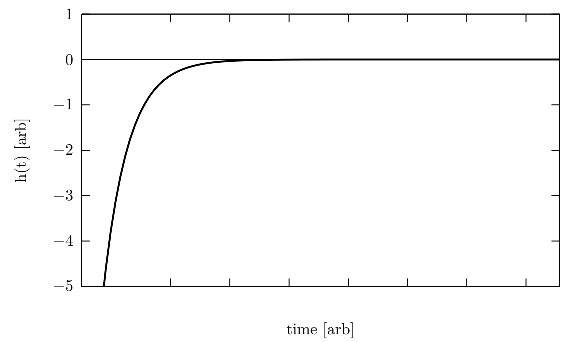
We have just learnt that transfer function can written as rational function like follows,

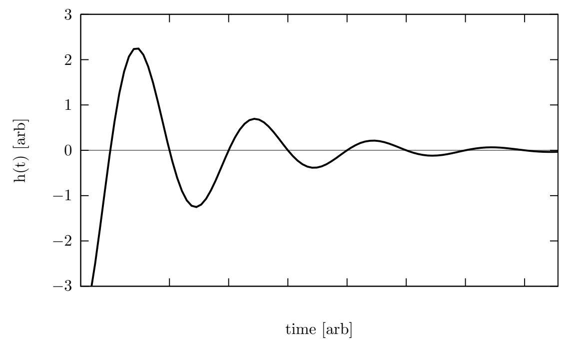
$$H(s) = H_0 \cdot \frac{(1 + s\,\tau_{n1})(1 + s\,\tau_{n2})...(1 + s\,\tau_{nn})}{(1 + s\,\tau_1)(1 + s\,\tau_2)...(1 + s\,\tau_d)} = \sum_k \frac{\tau_k}{1 + s\,\tau_k}\,a_k,$$

where $-1/\tau_{n1}, -1/\tau_{n2}, ..., -1/\tau_{nn}$ and $-1/\tau_1, -1/\tau_2, ..., -1/\tau_d$ are called zeros and poles, respectively. Let's take a looks at some typical waveforms. The simplest and most desired transfer function for many cases would be one real pole, and its impulse response is exponential settling.

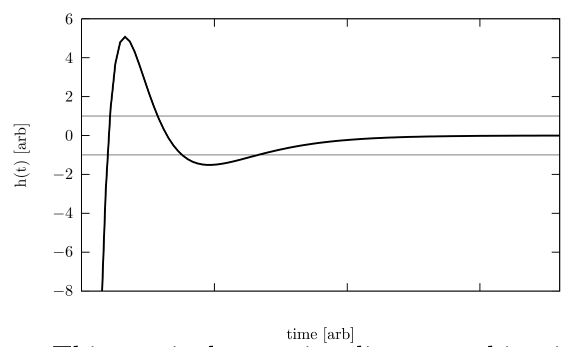
$$H(s) = H_0 \cdot \frac{1}{1 + s\tau}, \qquad h(t) = \frac{H_0}{\tau} \cdot e^{-t/\tau}.$$

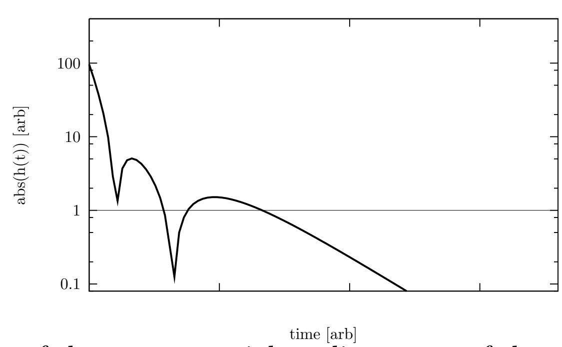
The waveform would look like below left. It gets about one third (1/e) over every τ.





If τ gets imaginary part, waveform would show ringing behavior like shown above right. Note that denominator has to be at least second order to have complex pole. Another response function we often observe is overshooting behavior like below left.





This particular one is a linear combination of three exponential settling terms of three different time constant. It is important to distinguish between ringing and overshooting. In ringing, zero-crossing points has specific time period, whereas overshooting does not. Above right is the semi-log plot of overshooting behavior, we see three segments. Each corresponds to dominant settling term.

1.13 Small signal

Now we would like to think about how we want to treat highly non-linear semiconductor devices in terms of linear response. The first thing I would like to point out is that any smooth curve looks straight if you look at it close enough. In other words, suppose $y = f(x)$ is such a curve, if we look at it around $x = x_0$,

$$f(x) = f(x_0) + \frac{\partial f}{\partial x} \cdot (x - x_0) + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x} \cdot (x - x_0)^2 + \ldots ,$$

When $(x - x_0)$ is small, $(x - x_0)^2$ is even smaller. Therefore, within sufficiently small $\Delta x = x - x_0$, $\Delta f = f(x) - f(x_0)$ is proportional to Δx:

$$\Delta f = \frac{\partial f}{\partial x} \cdot \Delta x.$$

For semiconductor devices, we usually set a DC bias current and put signal as small displacement from the DC operating point.

1.13.1 p-n junction diode

From Shockley formula, conductance is calculated as follows:

$$I = I_s\, e^{V/vt} \qquad \rightarrow \qquad g = \frac{\Delta I}{\Delta V} = \left.\frac{\partial I_s e^{V/vt}}{\partial V}\right|_{I=I_{\mathrm{bias}}} = \frac{I_{\mathrm{bias}}}{v_t}.$$

It is known that p-n junction has capacitance which consists of junction capacitance (c_j) and diffusion capacitance (c_d) which can be put in parallel with g. Therefore admittance

$$Y(s) = g + sc_j + sc_d = g(1 + sc/g), \qquad c = c_j + c_d.$$

Process technology and size of the device determines I_s and c as a function of operating condition (bias and temperature). However g_D is function of bias current and $v_t = kT/q$ only, it is less sensitive to process technology and size.

1.13.2 BJT

Let us recall large signal BJT equations for active or weak saturation region:

$$I_{\mathrm{B}} = I_s\, e^{V_{\mathrm{BE}}/v_t}\left(1 + \frac{V_{\mathrm{CE}}}{V_A}\right), \quad I_{\mathrm{C}} = \beta_0\, I_{\mathrm{B}},$$

where I_{B} and I_{C} is base and collector current, respectively. Note that early voltage V_A and β_0 are function of operating condition. We usually set up circuit so that device has desired collector bias current I_C. Therefore we want expressions for coefficients in terms of I_C. First transconductance

$$g_m = \frac{\partial I_\mathrm{C}}{\partial V_\mathrm{BE}} = \frac{I_C}{v_t}.$$

Note that this relation is independent of process technology. As we will see later, g_m plays central role in transistor circuits, we'd like use g_m as much as possible for other coefficients. Next conductance between base and emitter

$$g_{be} = \frac{\partial I_\mathrm{B}}{\partial V_\mathrm{BE}} = \frac{I_\mathrm{B}}{v_t} = \frac{I_\mathrm{C}}{\beta_0 v_t} = \frac{g_m}{\beta_0}.$$

Since base-emitter is made of p-n junction, we have a capacitor in parallel. We customary call complex resistance between base and emitter as r_π:

$$\frac{1}{r_\pi} = g_{be} + sc_{be} = \frac{g_m}{\beta_0}\left(1 + s\, c_{be}/g_m \cdot \beta_0\right),$$

or using complex $\beta(s)$

$$r_\pi = \frac{\beta(s)}{g_m}, \qquad \beta(s) = \frac{\beta_0}{1 + s\,\tau_T \beta_0}, \qquad \tau_T = \frac{c_{be}}{g_m}.$$

Conductance between collector and emitter

$$g_{ce} = \frac{\partial I_\mathrm{C}}{\partial V_\mathrm{CE}} = \beta_0 I_s e^{V_\mathrm{BE}/v_t} \cdot \frac{1}{V_A} = \frac{I_\mathrm{C}}{1 + V_\mathrm{CE}/V_A} \cdot \frac{1}{V_A} = \frac{I_\mathrm{C}}{V_A + V_\mathrm{CE}}.$$

Since V_A is usually much greater than V_CE,

$$g_{ce} \;\sim\; \frac{I_\mathrm{C}}{V_A} = g_m \cdot \frac{v_t}{V_A}.$$

We customary call inverse of g_{ce} as r_o.

$$r_o = \frac{1}{g_{ce}} = \frac{1}{g_m} \cdot \frac{V_A}{v_t}$$

Since g_m plays central role in transistor circuits, we like to measure r_π and r_o in the unit of $1/g_m$,

$$g_m r_\pi = \beta(s), \qquad g_m r_o = V_A/v_t.$$

Note that if we keep current density the same, these quantities do not depend on device size. Finally, since we likely want to use transistor at fairly high current density, series resistance of p-n junction is not negligible. We place r_b between base terminal and the junction, it appears $g_m r_b$ is also a function of current density. Figure 4 shows model circuit, with collector base capacitance c_μ.

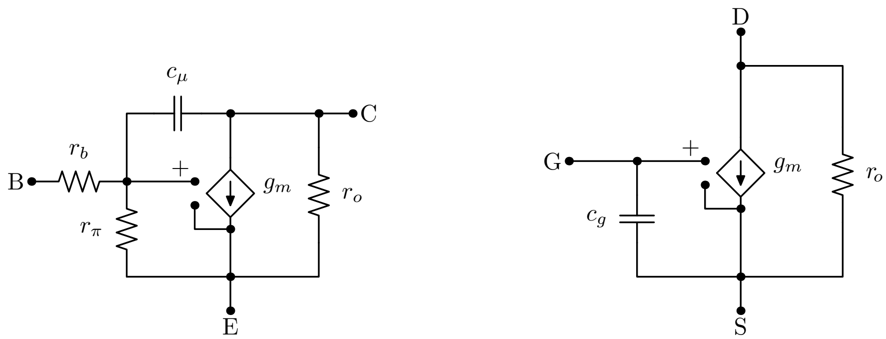


Figure 4: Naive small signal model for BJT (left) and MOSFET (right).

1.13.3 MOSFET

Small signal model for MOSFET is shown in Figure 4. For MOSFETs, transconductance of long channel device in active saturation region

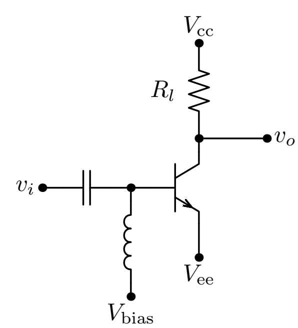
$$g_m = \frac{I_\mathrm{D}}{m\, v_t}, \qquad m = \frac{V_\mathrm{GS} - V_\mathrm{th}}{2\, v_t} = \frac{1}{v_t}\sqrt{\frac{J_\mathrm{D}}{2\mu_n C_\mathrm{ox}(1/L)}}$$

and in (deep) subthreshold active region

$$g_m = \frac{I_\mathrm{D}}{n\, v_t}, \qquad n = 1 + \frac{C_j}{C_\mathrm{ox}}.$$

In active saturation region, transconductance is proportional to the square root of the drain current density and temperature dependence is coming through mobility with constant drain current density. Factor m indicates "efficiency" of transconductance relative to BJT. Factor n called subthreshold slope and it is usually about 1.3 for planar transistors.

1.13.4 Circuit analysis



Let us take a look at an RF amplifier shown in the right. V_cc and V_ee are supply voltage and we usually take V_ee as reference level, i.e., $V_\mathrm{ee} = 0$. V_bias sets up V_BE so that we get desired collector bias current I_C. R_l represents load. R_l may not be linear in a large signal sense. The input v_i is driven by external circuit and it enters through a capacitor. DC level of v_o is determined by R_l and I_C, which is, in case R_l is linear,

$$V_o = V_\mathrm{cc} - R_l I_\mathrm{C}.$$

We like to keep V_o high enough so that not to squeeze V_{CE} of the transistor. We usually set up V_o higher than V_{bias} to keep base-collector junction from heavily forward biased.

$$V_{\mathrm{CB}} = V_o - V_{\mathrm{bias}} \ > \ 0$$

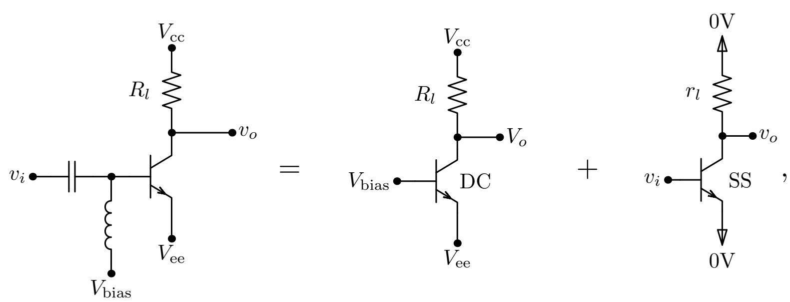
Now we want to put small RF signal on v_i. The DC voltage at base terminal of the transistor is determined by the capacitor and the inductor and the transistor (base-emitter junction). Since v_i is high frequency, the capacitor acts a short circuit and the inductor acts as an open circuit. Therefore v_i goes directly the base terminal:

$$V_{\mathrm{B}} = V_{\mathrm{bias}} + v_i.$$

The input voltage v_i modulates base current by small amount i_b, i.e., total base current is $I_{\mathrm{B}} + i_b$. The same thing can be said for collector current. The voltage at the output terminal is also modulated by small amount v_o which is equal to the impedance of R_l multiplied by collector current i_c:

$$V_o + v_o = V_o - r_l \, i_c,$$

where r_l is the impedance of R_l operating at DC bias current of I_{C}. Therefore the circuit can be decomposed into DC component and small-signal component as follows.



where transistor labeled as DC obeys the large signal equation, the first equation of Section 1.13.2. Response of base and collector current, i_b and i_c, to the input v_i is described by the small signal model shown in the previous section. Therefore transistor labeled as SS can be replaced by the small signal model shown in the left of Figure 4.

Now may be the time to set up circuit equation and study behavior of this circuit. However we like to stop here. Curious reader may refer Reference [1, 2], which demonstrates small signal and noise analysis of various transistor circuits.[2] As for noise analysis Reference [3] provides an introduction.

[2] These are basically my design notes. It is a bit messy and not quite reader friendly with a lot of errors.

Acknowledgments

Section 1.1 borrows ideas from Hidetoshi Takahashi's book [4]. The author thanks following people for reviewing manuscript and valuable comments: Manar El-Chammas, and Tanwei Yan.

Appendix A Causality and Stability

In Section 1.10, we saw as if causality and stability are identical. The reason is that we presumed that all impulse response has corresponding frequency response when we construct our linear response theory. In other words, we take the path of integral

$$h(\tau) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} H(s)\, e^{s\tau}\, ds,$$

on the imaginary axis and divide s plane into left half plane and right half plane. With this path $H(i\omega)$ is given its meaning as frequency response function.

However, even if we find poles in right half of the s plane, we can still get useful and causal impulse response function by shifting the path (which is on the imaginary axis) to the right to include all the poles. If you use such path, $h(\tau)$ is still zero at $\tau < 0$, which means causal and for $\tau > 0$, $h(\tau)$ will grow exponentially. There are cases when it is useful to find time constant of growth or $h(+0)$, but it is very rare and the system is likely meta-stable and things like $H(i\omega)$, $Z(i\omega)$ and $Y(i\omega)$ lose their meaning as frequency response function.

Appendix B Kramers Kronig relations

Let's review properties of impulse response function $h(t)$ and its Laplace transform $H(s) = \mathcal{L}\{h(t)\}$. First, the response is causal.

$$h(t) = 0 \quad \text{for} \quad t < 0 \tag{B.1}$$

Second, input in the remote past does not influence the present.

$$h(t) \to 0 \quad \text{as} \quad t \to \infty \tag{B.2}$$

Third, $h(t)$ is real.

$$H(s)^* = H(s^*). \tag{B.3}$$

Eqs. (B.1) and (B.2) ensures that $h(t)$ has meaningful Fourier transform.

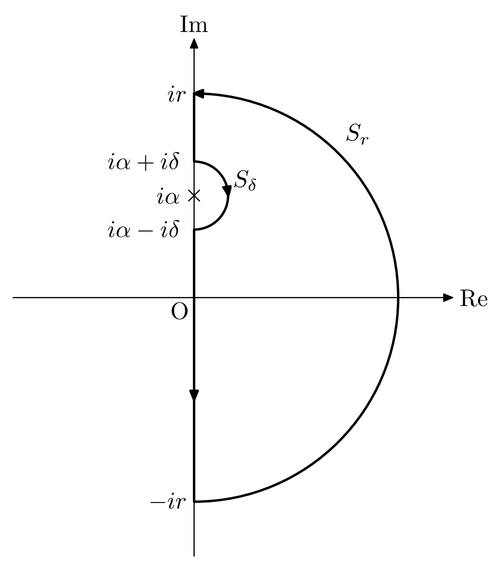
$$H(i\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt \tag{B.4}$$

And therefore,

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} H(s)\, e^{st}\, ds.$$

To meet Eq. (B.1), $H(s)$ has to be analytic in the right half plane (RHP).

Now, let's derive Kramers Kronig relation. We start from following contour integral.



$$\oint \frac{H(s)}{s - i\alpha}\, ds$$

The path is shown in the right. Since $H(s)$ is analytic in the right half plane, this integral is zero. Therefore,

$$\oint = 0 = \int_{ir}^{i(\alpha+\delta)} + \int_{S_\delta} + \int_{i(\alpha-\delta)}^{-ir} + \int_{S_r}$$

Because $H(i\omega)$ has Fourier transform, $|H(s)| \to 0$ as $|s| \to \infty$. Therefore, the integral over S_r vanishes as $r \to \infty$.

$$\lim_{r\to\infty} \int_{S_r} \frac{H(s)}{s - i\alpha}\, ds = 0$$

As for S_δ, let us add and subtract $H(i\alpha)$ term like below,

$$\int_{S_\delta} \frac{H(s)}{s - i\alpha}\, ds = \int_{S_\delta} \frac{H(s) - H(i\alpha)}{s - i\alpha}\, ds \ + \ H(i\alpha)\int_{S_\delta} \frac{1}{s - i\alpha}\, ds$$

Since $H(s)$ is continuous at α, the first term vanishes as $\delta \to 0$. With $s - i\alpha = \delta\, e^{i\theta}$ and therefor $ds = i\delta\, e^{i\theta} d\theta$, the second term is calculated as follows.

$$H(i\alpha) \int_{S_\delta} \frac{1}{s - i\alpha}\, ds = H(i\alpha) \int_{\pi/2}^{-\pi/2} \frac{1}{\delta\, e^{i\theta}}\, i\delta\, e^{i\theta} d\theta = -i\pi H(i\alpha).$$

Therefore, with $s = i\omega$,

$$\begin{aligned}
\lim_{\substack{r\to\infty \\ \delta\to 0}} \oint \frac{H(s)}{s - i\alpha}\, ds = 0 &= \lim_{\delta\to 0} \left[\int_{\infty}^{\alpha+\delta} \frac{H(i\omega)}{\omega - \alpha}\, d\omega \;+\; \int_{\alpha-\delta}^{-\infty} \frac{H(i\omega)}{\omega - \alpha}\, d\omega \right] \;-\; i\pi H(i\alpha), \\
&= \lim_{\delta\to 0} \left[-\int_{\alpha+\delta}^{\infty} \frac{H(i\omega)}{\omega - \alpha}\, d\omega \;-\; \int_{-\infty}^{\alpha-\delta} \frac{H(i\omega)}{\omega - \alpha}\, d\omega \right] \;-\; i\pi H(i\alpha).
\end{aligned}$$

Finally,

$$H(i\alpha) \;=\; -\frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{H(i\omega)}{\omega - \alpha}\, d\omega,$$

where P indicates principle value integral.

$$P \int_{-\infty}^{\infty} \;=\; \lim_{\delta\to 0} \left[\int_{-\infty}^{\alpha-\delta} \;+\; \int_{\alpha+\delta}^{\infty} \right]$$

Some reader might have noticed that the sign is different from other literature. That's because our definition of Fourier transform, Eq. (B.4), is different from others, resulting sign of frequency is flipped. Their negative frequency is our positive frequency, vice versa. To accommodate, let's use negative sign for the imaginary part. That is

$$H(i\omega) = H'(\omega) - iH''(\omega),$$

where both H' and H'' are real function of ω. With this, we find following Hilbert transform pair,

$$\begin{aligned}
H'(\alpha) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{H''(\omega)}{\omega - \alpha}\, d\omega, \\
H''(\alpha) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{H'(\omega)}{\omega - \alpha}\, d\omega
\end{aligned}$$

Using Eq. (B.3), i.e., $H''(-\omega) = H'(\omega)$ and $H''(-\omega) = -H''(\omega)$, we can convert these integrals using positive frequency only,

$$\begin{aligned}
H'(\alpha) &= \frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega H''(\omega)}{\omega^2 - \alpha^2}\, d\omega, \\
H''(\alpha) &= -\frac{2\alpha}{\pi} P \int_{0}^{\infty} \frac{H'(\omega)}{\omega^2 - \alpha^2}\, d\omega.
\end{aligned}$$

These are Kramers Kronig relations.

Appendix C Useful Formula

Parallel impedance operator

$$(r_1//r_2) = \frac{1}{1/r_1 + 1/r_2} = \frac{r_1 r_2}{r_1 + r_2}, \quad (r_1//r_2//r_3) = \frac{1}{1/r_1 + 1/r_2 + 1/r_3} = \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}$$

$$(r_1//r_2) = (r_2//r_1), \quad \frac{1}{(r_1//r_2)} + \frac{1}{r_3} = \frac{1}{(r_1//r_2//r_3)}, \quad (r_1//c_1) = \frac{r_1}{1 + s\, r_1 c_1}$$

Minimum value

$$\min\left(\frac{A}{x} + Bx\right) = 2\sqrt{AB}, \qquad x_{\min} = \sqrt{\frac{A}{B}}.$$

Integral

$$\int_0^\infty \frac{d\omega/2\pi}{1 + (\omega/\omega_0)^2} = \frac{\omega_0}{2\pi} \tan^{-1}\left(\frac{\omega}{\omega_0}\right)\Big|_0^\infty = \frac{1}{4}\omega_0$$

$$\int_{\omega_s}^\infty \frac{d\omega}{\omega(1 + (\omega/\omega_0)^2)} = \frac{1}{2} \ln \frac{(\omega/\omega_0)^2}{1 + (\omega/\omega_0)^2}\Big|_{\omega_s}^\infty = \frac{1}{2} \ln \frac{1 + (\omega_s/\omega_0)^2}{(\omega_s/\omega_0)^2}$$

Two pole response function and its impulse response

$$v_o/v_i = \frac{1}{1 + s\, b + s^2\, a} = \frac{1}{(1 + s\, \tau_\oplus)(1 + s\, \tau_\ominus)} \qquad (a > 0,\ b > 0)$$

$$1/\tau_{\oplus,\ominus} = \frac{b \pm \sqrt{b^2 - 4a}}{2a} = \frac{b}{2a}\left(1 \pm \sqrt{1 - 4a/b^2}\right)$$

Discriminant $4a/b^2$:

$$\begin{aligned} 4a/b^2 < 1 \rightarrow &\quad \text{Exponential settling (overshooting)} \\ = 1 \rightarrow &\quad \text{Critical damping} \\ > 1 \rightarrow &\quad \text{Ringing} \end{aligned}$$

If $4a/b^2 \ll 1$,

$$1/\tau_\oplus = b/a - 1/b, \qquad 1/\tau_\ominus = 1/b$$

Canonical form of two pole amplifier

$$A(s) = \frac{N}{Q + s\, B + s^2\, A} = \frac{A_0}{(1 + s\, \tau_A A_0)(1 + s\, \tau_\oplus)}$$

If $4AQ/B^2 \ll 1$:

$$A_0 = N/Q, \quad \tau_A = B/N, \quad 1/\tau_\oplus = B/A - Q/B$$

Laplace transform

$$\mathcal{L}\{\delta(t)\} = 1, \qquad \mathcal{L}\{1\} = \frac{1}{s}, \qquad \mathcal{L}\{t\} = \frac{1}{s^2}, \qquad \mathcal{L}\{e^{-t/\tau_1}\} = \frac{\tau_1}{1+s\,\tau_1}$$

$$\mathcal{L}\{t/\tau_1\, e^{-t/\tau_1}\} = \frac{\tau_1}{(1+s\,\tau_1)^2}$$

$$\frac{1}{(1+s\,\tau_1)(1+s\,\tau_2)} = \frac{1}{\tau_1-\tau_2}\left(\frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_2}{1+s\,\tau_2}\right)$$

$$\frac{s}{(1+s\,\tau_1)(1+s\,\tau_2)} = -\frac{1}{\tau_1-\tau_2}\left(\frac{1}{\tau_1}\cdot\frac{\tau_1}{1+s\,\tau_1} - \frac{1}{\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}\right)$$

$$\frac{1+s\,\tau_3}{(1+s\,\tau_1)(1+s\,\tau_2)} = \frac{1}{\tau_1-\tau_2}\left(\frac{\tau_1-\tau_3}{\tau_1}\cdot\frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_2-\tau_3}{\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}\right)$$

$$\frac{s}{(1+s\,\tau_1)^2} = \frac{1}{\tau_1^2}\left(\frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_1}{(1+s\,\tau_1)^2}\right)$$

$$\frac{1}{s\,(1+s\,\tau_1)} = \frac{1}{s} - \frac{\tau_1}{1+s\,\tau_1}$$

$$\frac{1}{s^2\,(1+s\,\tau_1)} = \frac{1}{s^2} - \frac{\tau_1}{s} + \frac{\tau_1^2}{1+s\,\tau_1}$$

$$\frac{1}{s\,(1+s\,\tau_1)(1+s\,\tau_2)} = \frac{1}{s} - \frac{\tau_1}{\tau_1-\tau_2}\cdot\frac{\tau_1}{1+s\,\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}$$

$$\frac{1}{s\,(1+s\,\tau_1)^2} = \frac{1}{s} - \frac{\tau_1}{1+s\,\tau_1} - \frac{\tau_1}{(1+s\,\tau_1)^2}$$

$$\frac{1+s\,\tau_3}{s\,(1+s\,\tau_1)(1+s\,\tau_2)} = \frac{1}{s} - \frac{\tau_1-\tau_3}{\tau_1-\tau_2}\cdot\frac{\tau_1}{1+s\,\tau_1} + \frac{\tau_2-\tau_3}{\tau_1-\tau_2}\cdot\frac{\tau_2}{1+s\,\tau_2}$$

Approximation If $\tau_1 \gg \tau_2$,

$$\frac{s}{(1+s\,\tau_1)(1+s\,\tau_2)} \sim \frac{1}{\tau_1\tau_2}\left(\frac{\tau_2}{1+s\,\tau_2} - \frac{\tau_2}{\tau_1}\cdot\frac{\tau_1}{1+s\,\tau_1}\right)$$

$$\frac{1}{s\,(1+s\,\tau_1)(1+s\,\tau_2)} \sim \frac{1}{s} - \frac{\tau_1}{1+s\,\tau_1} + \frac{\tau_2}{\tau_1}\cdot\frac{\tau_2}{1+s\,\tau_2}$$

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