What we are doing here: the idea of adaptive correction

We have a system which can be cheracterised with the following set of equatins:

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k) \quad (1),$$

$$\mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k \quad (2),$$

Where (1) is state transition equation and (2) is measurement equation, $\mathbf{w}_k$ and $\mathbf{v}_k$ are process and measurement noise respectively, $\mathbf{u}_k$ is control vector.

We use the Extended Kalman Filter (EKF) as our main estimation algorithm and the H_∞ filter as backup algorithm to deal with Kalman filter divergence.

The EKF algorithm

Prediction step

$$\mathbf{x}_{k|k-1} = f(\mathbf{x}_{k-1|k-1}, \mathbf{u}_k, 0) \quad (3),$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^\top + \mathbf{B}_k \mathbf{Q}_k \mathbf{B}_k^\top \quad (4),$$

$$\mathbf{y}_k = \mathbf{z}_k - h(\mathbf{x}_{k|k-1}) \quad (5),$$

Correction step

The convenient form of correction step is:

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^\top + \mathbf{R}_k \quad (6),$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^\top \mathbf{S}_k^{-1} \quad (7),$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k \mathbf{y}_k \quad (8),$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\, \mathbf{P}_{k|k-1} (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^\top + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^\top \quad (9),$$

To compare the EKF with the H_∞ filter we need to rewrite the correction step in the following form:

$$\mathbf{P}_{k|k} = \left(\mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k\right)^{-1} \quad (10),$$

$$\mathbf{K}_k = \mathbf{P}_{k|k} \mathbf{H}_k^\top \mathbf{R}_k^{-1} \quad (11),$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k \mathbf{y}_k \quad (12),$$

Where:

$\mathbf{F}_k = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}_{k-1|k-1}, \mathbf{u}_k}, \quad \mathbf{B}_k = \left.\frac{\partial f}{\partial \mathbf{v}}\right|_{\mathbf{x}_{k-1|k-1}, \mathbf{u}_k}, \quad \mathbf{H}_k = \left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\mathbf{x}_{k|k-1}}, \quad \mathbf{Q}_k \geq 0$ is the

The H_∞ filter algorithm

The H_∞ filter algorithm is given in **[Banavar1992]** this algorithm can be written as follows:

Prediction step

$$\mathbf{x}_{k|k-1} = f(\mathbf{x}_{k-1|k-1}, \mathbf{u}_k, 0) \quad (13),$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^\top + \mathbf{B}_k \mathbf{Q}_k \mathbf{B}_k^\top \quad (14),$$

$$\mathbf{y}_k = \mathbf{z}_k - h(\mathbf{x}_{k|k-1}) \quad (15),$$

Correction step

$$\tilde{\mathbf{S}}_k = \mathbf{L}_b^\top \bar{\mathbf{S}}_k \mathbf{L}_k \quad (16),$$

$$\mathbf{P}_{k|k} = \left(\mathbf{P}_{k|k-1}^{-1} - \theta \cdot \tilde{\mathbf{S}}_k + \mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k\right)^{-1} \quad (17),$$

$$\mathbf{K}_k = \mathbf{P}_{k|k} \mathbf{H}_k^\top \mathbf{R}_k^{-1} \quad (18),$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{K}_k \mathbf{y}_k \quad (19),$$

Where:

$\bar{\mathbf{S}}_k > 0$ is user defined matrix, $\mathbf{L}_k$ is full rank user defined state weigthing matrix, $\theta$is user defined performance bounary value.

Filtering algorithms comparizon

If we compare equations (10) and (17) then we cen see that the least equation can be computed in two steps:

$$\mathbf{P}_b = \left(\mathbf{P}_{b-1}^{-1} - \theta \cdot \tilde{\mathbf{S}}_b\right)^{-1} = \mathbf{P}_{b-1} + \Delta\mathbf{Q}_b \quad (20),$$

$$\mathbf{P}_d = \left(\mathbf{P}_{d-1}^{-1} + \mathbf{H}_d^\top \mathbf{R}_d^{-1} \mathbf{H}_d\right)^{-1} \quad (21),$$

The order of these steps may vary, but the result should be ste same.

The step (21) is exactly the same as (10) which is used in the Kalman filter state covariance update.

The step (20) can be viewed as an additional process noise. Note that $\Delta\mathbf{Q}_b$ must be positive defnite for the existence of the filter.

Adaptive correction derivation

Divergence criterion

Kalman filter divergence is detected when the folowing criterion is met:

$$\mathbf{y}_b^\top \mathbf{S}_b^{-1} \mathbf{y}_b > \beta_n \quad (22),$$

where β_n is threshold $\chi^2_{\alpha,n}$ value.

Correction: Initial derivation

After adaptive correction step case we get:

$$\mathbf{y}_b^\top \mathbf{S}_{corr}^{-1} \mathbf{y}_b = \beta_n \quad (23),$$

By multimlication of the equation (23) by $\mathbf{y}_b$ on the left and by $\mathbf{y}_b^\top$ on the rigth sides we get:

$$\mathbf{y}_b \mathbf{y}_b^\top \mathbf{S}_{corr}^{-1} \mathbf{y}_b \mathbf{y}_b^\top = \beta_n \cdot \mathbf{y}_b \mathbf{y}_b^\top \quad (24),$$

By compating left and igth sides of the equation (24) we get:

$$\mathbf{y}_b y_b^\top \mathbf{S}_{corr}^{-1} = \beta_n \cdot \mathbf{I},$$

or

$$\mathbf{y}_b \mathbf{y}_b^\top = \beta_n \cdot \mathbf{S}_{corr},$$

or

$$\mathbf{S}_{corr} = \frac{\mathbf{y}_b \mathbf{y}_b^\top}{\beta_n} \quad (25),$$

Correction: No limits approach

By substitution of (20) rigth side to (16) we get:

$$\mathbf{S}_{corr} = S_b + \mathbf{H}_b \Delta\mathbf{Q}_b \mathbf{H}_b^\top \quad (26),$$

and so substitution (26) to (25) gives us:

$$S_b + \mathbf{H}_b \Delta\mathbf{Q}_b \mathbf{H}_b^\top = \frac{\mathbf{y}_b \mathbf{y}_b^\top}{\beta_n} \quad (27),$$

or

$$A_b = \mathbf{H}_b \Delta \mathbf{Q}_b \mathbf{H}_b^\top = \frac{\mathbf{y}_b \mathbf{y}_b^\top}{\beta_n} - S_b \quad (28),$$

on the other hand (20) gives us:

$$\mathbf{P}_b = \left(\mathbf{P}_{b-1}^{-1} - \theta \cdot \tilde{\mathbf{S}}_b\right)^{-1} = \mathbf{P}_{b-1} + \Delta \mathbf{Q}_b$$

or

$$\mathbf{P}_b = \left(\mathbf{P}_{b-1}^{-1} - \theta \cdot \mathbf{L}_b^\top \bar{\mathbf{S}}_b \mathbf{L}_b\right)^{-1} = \left(\mathbf{P}_{b-1}^{-1} - \mathbf{L}_b^\top \mathbf{M}_b^{-1} \mathbf{L}_b\right)^{-1}$$, where $\mathbf{M}_b^{-1} = \theta \cdot \bar{\mathbf{S}}_b$

or

$$\mathbf{P}_b = \mathbf{P}_{b-1} + \mathbf{P}_{b-1} \mathbf{L}_b^\top \left(\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top\right)^{-1} \mathbf{L}_b \mathbf{P}_{b-1}$$

which means that:

$$\Delta \mathbf{Q}_b = \mathbf{P}_{b-1} \mathbf{L}_b^\top \left(\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top\right)^{-1} \mathbf{L}_b \mathbf{P}_{b-1} \quad (29),$$

from (28) we get:

$$\mathbf{A}_b = \mathbf{H}_b \Delta \mathbf{Q}_b \mathbf{H}_b^\top = \mathbf{H}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top \left(\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top\right)^{-1} \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{H}_b^\top = \mathbf{C}_b \Big(\mathbf{M}_b -$$

or

$$\mathbf{A}_b = \mathbf{C}_b \left(\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top\right)^{-1} \mathbf{C}_b^\top$$

or

$$\mathbf{C}_b^+ \mathbf{A}_b = \left(\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top\right)^{-1} \mathbf{C}_b^\top$$

or

$$\left(\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top\right) \mathbf{C}_b^+ = \mathbf{C}_b^\top \mathbf{A}_b^{-1}$$

or

$$\mathbf{M}_b - \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top = \mathbf{C}_b^\top \mathbf{A}_b^{-1} \mathbf{C}_b$$

or

$$\mathbf{M}_b = \mathbf{C}_b^\top \mathbf{A}_b^{-1} \mathbf{C}_b + \mathbf{L}_b \mathbf{P}_{b-1} \mathbf{L}_b^\top \quad (30),$$

where:

$$\mathbf{C}_b = \mathbf{L}_b\mathbf{P}_{b-1}\mathbf{H}_b^\top \quad (31),$$

so for $\Delta\mathbf{Q}_b$ we have:

$$\Delta\mathbf{Q}_b = \mathbf{P}_{b-1}\mathbf{L}_b^\top\left(\mathbf{C}_b^\top\mathbf{A}_b^{-1}\mathbf{C}_b + \mathbf{L}_b\mathbf{P}_{b-1}\mathbf{L}_b^\top - \mathbf{L}_b\mathbf{P}_{b-1}\mathbf{L}_b^\top\right)^{-1}\mathbf{L}_b\mathbf{P}_{b-1} = \mathbf{P}_{b-1}\mathbf{L}_b^\top\Big(($$

or

$$\Delta\mathbf{Q}_b = \mathbf{P}_{b-1}\mathbf{L}_b^\top\mathbf{C}_b^+\mathbf{A}_b\mathbf{C}_b^{+\top}\mathbf{L}_b\mathbf{P}_{b-1} \quad (32),$$

The Kalman filter with H_∞ correction (32) exists when:

$$\Delta\mathbf{Q}_b = \mathbf{P}_{b-1}\mathbf{L}_b^\top\mathbf{C}_b^+\mathbf{A}_b\mathbf{C}_b^{+\top}\mathbf{L}_b\mathbf{P}_{b-1} > 0,$$

According to **[Horn et al]** the above citeria is met when:

- Maxtix $\mathbf{Z}_b = \mathbf{P}_{b-1}\mathbf{L}_b^\top\mathbf{C}_b^+$ has full rank and
- $\mathbf{A}_b > 0$

Let us consider $\mathbf{Z}_b = \mathbf{P}_{b-1}\mathbf{L}_b^\top\mathbf{C}_b^+$:

- as $\mathbf{P}_{b-1} > 0$ and $\mathbf{L}_b$ has full rank then $\mathbf{P}_{b-1}\mathbf{L}_b$ also has full rank;
- as both $\mathbf{H}_b$ and $\mathbf{L}_b$ have full rank, and $\mathbf{P}_{b-1} > 0$, then $\mathbf{C}_{b-1}$ also has full rank, consequently: the product $\mathbf{C}_b^\top\mathbf{C}_b > 0$ (see **[Horn et al]**), and so, $\mathbf{C}_b^+ = \left(\mathbf{C}_b^\top\mathbf{C}_b\right)^{-1}\mathbf{C}_b^\top$ also have full rank;
- as both product $\mathbf{P}_{b-1}\mathbf{L}_b$ and the matrix $\mathbf{C}_b^+$ have full rank, then $\mathbf{Z}_b$ also has full rank.

Let us consider the matrix $\mathbf{A}_b$:

- $\mathbf{S}_b \geq \mathbf{H}_b\Delta\mathbf{P}_{b-1}\mathbf{H}_b^\top > 0$
- now we must consider the product $\mathbf{x}^\top\left(\mathbf{y}_b\mathbf{y}_b^\top\right)\mathbf{x}$ for $\mathbf{x} \neq 0$:
  - if the dimensionality of x is greater than one, then $\mathbf{x}^\top\left(\mathbf{y}_b\mathbf{y}_b^\top\right)\mathbf{x} = \left(\mathbf{x}^\top\mathbf{y}_b\right)\left(\mathbf{y}_b^\top\mathbf{x}\right) \geq 0$, consequently, regarding to positive definiteness of S b , we can not guarante the existense of the filter;
  - if the dimensionality of x is one (when we implement sequential filter), then $\mathbf{x}^\top\left(\mathbf{y}_b\mathbf{y}_b^\top\right)\mathbf{x} = y_b^2 x^(2) > 0$,
    - in such case the equation (28) reduces to $a_b = \dfrac{y_b^2}{\beta_n} - s_b$
    - the criterion (22) is met when $\dfrac{y_b^2}{\beta_1} > s_b$ or $\dfrac{y_b^2}{\beta_n} - s_b > 0$
    - so when thecriterion (22) is met we have $\mathbf{A}_b = a_b > 0,$

thus trere is at least one sequential filter implementation.

So we can select $\mathbf{L}_b$, $\bar{\mathbf{S}}_b$, θ parameters of H_∞ filter so that we can use operation (20) to correct the Kalman filter divergence. Our selection is reduced to definition of $\mathbf{L}_b$ and y_b to be used in certain filtering algorithm.

Correction: An approach with limitations on H_∞ filter parameters

I order to correct Kalman filter divergence we will use H_∞ filter with th following parameters:

$$\mathbf{L}_k = \mathbf{I},$$

$$\bar{\mathbf{S}}_k = \frac{\alpha_k}{1+\alpha_k}\mathbf{P}_{k-1}^{-1},$$

$$\theta = 1.$$

So the step (20) will reduce to:

$$\mathbf{P}_b = \left(\mathbf{P}_{b-1}^{-1} - \frac{\alpha_b}{1+\alpha_b}\mathbf{P}_{b-1}^{-1}\right)^{-1} = \mathbf{P}_{b-1} + \alpha_b \mathbf{P}_{b-1},$$

so

$$\Delta\mathbf{Q}_b = \alpha_b \mathbf{P}_{b-1} \quad (33)$$

Let's substitude the eq. (33) to the eq. (28):

$$\alpha_b \mathbf{H}_b \mathbf{P}_{b-1} \mathbf{H}_b^\top = \frac{\mathbf{y}_b \mathbf{y}_b^\top}{\beta_n} - S_b,$$

or

$$\alpha_b \mathbf{D}_b = \frac{\mathbf{y}_b \mathbf{y}_b^\top}{\beta_n} - S_b,$$

or

$$\frac{1}{\alpha_b}\mathbf{D}_b^{-1} = \left(\frac{\mathbf{y}_b \mathbf{y}_b^\top}{\beta_n} - S_b\right)^{-1} = \left(-S_b + \mathbf{y}_b \frac{1}{\beta_n} \mathbf{y}_b^\top\right)^{-1},$$

or

$$\frac{1}{\alpha_b}\mathbf{D}_b^{-1} = -S_b^{-1} + S_b^{-1}\mathbf{y}_b\left(\mathbf{y}_b^\top \mathbf{S}_b^{-1}\mathbf{y}_b - \beta_n\right)^{-1}\mathbf{y}_b^\top S_b^{-1} \quad (34),$$

where:

$$\mathbf{D}_b = \mathbf{H}_b \mathbf{P}_{b-1} \mathbf{H}_b^\top$$

Let's multiply the eq. (34) with $\mathbf{y}_b^\top$ on the left and with $\mathbf{y}_b$ on the rigth:

$$\frac{1}{\alpha_b}\mathbf{y}_b^\top \mathbf{D}_b^{-1}\mathbf{y}_b = -\mathbf{y}_b^\top S_b^{-1}\mathbf{y}_b + \mathbf{y}_b^\top S_b^{-1}\mathbf{y}_b\left(\mathbf{y}_b^\top \mathbf{S}_b^{-1}\mathbf{y}_b - \beta_n\right)^{-1}\mathbf{y}_b^\top S_b^{-1}\mathbf{y}_b = \mathbf{y}_b^\top S_b^{-1}\mathbf{y}$$

,

or

$$\frac{\alpha_b}{\mathbf{y}_b^\top \mathbf{D}_b^{-1}\mathbf{y}_b} = \frac{1}{\beta_n} - \frac{1}{\mathbf{y}_b^\top S_b^{-1}\mathbf{y}_b}$$

or

$$\alpha_b = \mathbf{y}_b^\top \mathbf{D}_b^{-1}\mathbf{y}_b\left(\frac{1}{\beta_n} - \frac{1}{\mathbf{y}_b^\top S_b^{-1}\mathbf{y}_b}\right) \qquad (35)$$

The H_∞ filter exists as long as $\alpha_b \geq 0$.

- Since $\mathbf{H}_b$ has full rank and $\mathbf{P}_{b-1} > 0$ the matrix $\mathbf{D}_b$ is positive definite and

  $\mathbf{y}_b^\top \mathbf{D}_b^{-1}\mathbf{y}_b > 0.$

- According to the expression (22):

  $$\left(\frac{1}{\beta_n} - \frac{1}{\mathbf{y}_b^\top S_b^{-1}\mathbf{y}_b}\right) > 0$$

So $\alpha_b > 0$ for all cases when the correction is necessary i.e. the existence of H_∞ filter is proven.

References

[Banavar1992] R. Banavar, “A game theoretic approach to linear dynamic estimation”, Doctoral Dissertation, University of Texas at Austin, May 1992.

[Horn et al] R.A. Horn, C.R Johnson, Charles R. “Matrix Analysis (2nd ed.).”, Cambridge University Press, 2013, ISBN 978-0-521-38632-6.