# Part X

# Data and Empirics

CHAPTER

# SIXTYTWO

# PANDAS FOR PANEL DATA

**Contents**



## 62.1 Overview

In an earlier lecture on pandas, we looked at working with simple data sets.

Econometricians often need to work with more complex data sets, such as panels.

Common tasks include

- Importing data, cleaning it and reshaping it across several axes.
- Selecting a time series or cross-section from a panel.
- Grouping and summarizing data.

`pandas` (derived from 'panel' and 'data') contains powerful and easy-to-use tools for solving exactly these kinds of problems.

In what follows, we will use a panel data set of real minimum wages from the OECD to create:

- summary statistics over multiple dimensions of our data
- a time series of the average minimum wage of countries in the dataset
- kernel density estimates of wages by continent

We will begin by reading in our long format panel data from a CSV file and reshaping the resulting `DataFrame` with `pivot_table` to build a `MultiIndex`.

Additional detail will be added to our `DataFrame` using pandas' `merge` function, and data will be summarized with the `groupby` function.

## 62.2 Slicing and Reshaping Data

We will read in a dataset from the OECD of real minimum wages in 32 countries and assign it to `realwage`.

The dataset can be accessed with the following link:

```
url1 = 'https://raw.githubusercontent.com/QuantEcon/lecture-python/master/source/_
↪static/lecture_specific/pandas_panel/realwage.csv'
```

```
import pandas as pd

# Display 6 columns for viewing purposes
pd.set_option('display.max_columns', 6)

# Reduce decimal points to 2
pd.options.display.float_format = '{:,.2f}'.format

realwage = pd.read_csv(url1)
```

Let's have a look at what we've got to work with

```
realwage.head()  # Show first 5 rows
```

```
   Unnamed: 0        Time  Country                                    Series  \
0           0  2006-01-01  Ireland  In 2015 constant prices at 2015 USD PPPs
1           1  2007-01-01  Ireland  In 2015 constant prices at 2015 USD PPPs
2           2  2008-01-01  Ireland  In 2015 constant prices at 2015 USD PPPs
3           3  2009-01-01  Ireland  In 2015 constant prices at 2015 USD PPPs
4           4  2010-01-01  Ireland  In 2015 constant prices at 2015 USD PPPs

  Pay period     value
0     Annual 17,132.44
1     Annual 18,100.92
2     Annual 17,747.41
3     Annual 18,580.14
4     Annual 18,755.83
```

The data is currently in long format, which is difficult to analyze when there are several dimensions to the data.

We will use `pivot_table` to create a wide format panel, with a `MultiIndex` to handle higher dimensional data.

`pivot_table` arguments should specify the data (values), the index, and the columns we want in our resulting dataframe.

By passing a list in columns, we can create a `MultiIndex` in our column axis

```
realwage = realwage.pivot_table(values='value',
                                index='Time',
                                columns=['Country', 'Series', 'Pay period'])
realwage.head()
```

```
Country                                      Australia         \
Series     In 2015 constant prices at 2015 USD PPPs
Pay period                                      Annual  Hourly
Time
2006-01-01                                   20,410.65  10.33
2007-01-01                                   21,087.57  10.67
2008-01-01                                   20,718.24  10.48
2009-01-01                                   20,984.77  10.62
2010-01-01                                   20,879.33  10.57

Country                                                           ...  \
Series     In 2015 constant prices at 2015 USD exchange rates    ...
Pay period                                             Annual    ...
Time                                                              ...
2006-01-01                                          23,826.64    ...
2007-01-01                                          24,616.84    ...
2008-01-01                                          24,185.70    ...
2009-01-01                                          24,496.84    ...
2010-01-01                                          24,373.76    ...

Country                                  United States  \
Series     In 2015 constant prices at 2015 USD PPPs
Pay period                                      Hourly
Time
2006-01-01                                        6.05
2007-01-01                                        6.24
2008-01-01                                        6.78
2009-01-01                                        7.58
2010-01-01                                        7.88

Country
Series     In 2015 constant prices at 2015 USD exchange rates
Pay period                                             Annual Hourly
Time
2006-01-01                                          12,594.40   6.05
2007-01-01                                          12,974.40   6.24
2008-01-01                                          14,097.56   6.78
2009-01-01                                          15,756.42   7.58
2010-01-01                                          16,391.31   7.88

[5 rows x 128 columns]
```

To more easily filter our time series data, later on, we will convert the index into a `DateTimeIndex`

```
realwage.index = pd.to_datetime(realwage.index)
type(realwage.index)
```

```
pandas.core.indexes.datetimes.DatetimeIndex
```

The columns contain multiple levels of indexing, known as a `MultiIndex`, with levels being ordered hierarchically (Country > Series > Pay period).

A `MultiIndex` is the simplest and most flexible way to manage panel data in pandas

```
type(realwage.columns)
```

```
pandas.core.indexes.multi.MultiIndex
```

```
realwage.columns.names
```

```
FrozenList(['Country', 'Series', 'Pay period'])
```

Like before, we can select the country (the top level of our `MultiIndex`)

```
realwage['United States'].head()
```

```
Series      In 2015 constant prices at 2015 USD PPPs         \
Pay period                                   Annual Hourly
Time
2006-01-01                                12,594.40   6.05
2007-01-01                                12,974.40   6.24
2008-01-01                                14,097.56   6.78
2009-01-01                                15,756.42   7.58
2010-01-01                                16,391.31   7.88

Series      In 2015 constant prices at 2015 USD exchange rates
Pay period                                             Annual Hourly
Time
2006-01-01                                          12,594.40   6.05
2007-01-01                                          12,974.40   6.24
2008-01-01                                          14,097.56   6.78
2009-01-01                                          15,756.42   7.58
2010-01-01                                          16,391.31   7.88
```

Stacking and unstacking levels of the `MultiIndex` will be used throughout this lecture to reshape our dataframe into a format we need.

`.stack()` rotates the lowest level of the column `MultiIndex` to the row index (`.unstack()` works in the opposite direction - try it out)

```
realwage.stack().head()
```

```
Country                                                    Australia  \
Series                 In 2015 constant prices at 2015 USD PPPs
Time       Pay period
2006-01-01 Annual                                          20,410.65
           Hourly                                              10.33
2007-01-01 Annual                                          21,087.57
           Hourly                                              10.67
2008-01-01 Annual                                          20,718.24

Country                                                                  \
Series                 In 2015 constant prices at 2015 USD exchange rates
Time       Pay period
2006-01-01 Annual                                                23,826.64
           Hourly                                                    12.06
2007-01-01 Annual                                                24,616.84
           Hourly                                                    12.46
2008-01-01 Annual                                                24,185.70

Country                                                      Belgium  ...  \
Series                 In 2015 constant prices at 2015 USD PPPs  ...
```

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```
Time       Pay period                                           ...
2006-01-01 Annual                                  21,042.28  ...
           Hourly                                      10.09  ...
2007-01-01 Annual                                  21,310.05  ...
           Hourly                                      10.22  ...
2008-01-01 Annual                                  21,416.96  ...

Country                                              United Kingdom  \
Series                In 2015 constant prices at 2015 USD exchange rates
Time       Pay period
2006-01-01 Annual                                         20,376.32
           Hourly                                              9.81
2007-01-01 Annual                                         20,954.13
           Hourly                                             10.07
2008-01-01 Annual                                         20,902.87

Country                                     United States  \
Series                In 2015 constant prices at 2015 USD PPPs
Time       Pay period
2006-01-01 Annual                                  12,594.40
           Hourly                                       6.05
2007-01-01 Annual                                  12,974.40
           Hourly                                       6.24
2008-01-01 Annual                                  14,097.56

Country
Series                In 2015 constant prices at 2015 USD exchange rates
Time       Pay period
2006-01-01 Annual                                         12,594.40
           Hourly                                              6.05
2007-01-01 Annual                                         12,974.40
           Hourly                                              6.24
2008-01-01 Annual                                         14,097.56

[5 rows x 64 columns]
```

We can also pass in an argument to select the level we would like to stack

```
realwage.stack(level='Country').head()
```

```
Series                In 2015 constant prices at 2015 USD PPPs         \
Pay period                                            Annual Hourly
Time       Country
2006-01-01 Australia                               20,410.65  10.33
           Belgium                                 21,042.28  10.09
           Brazil                                   3,310.51   1.41
           Canada                                  13,649.69   6.56
           Chile                                    5,201.65   2.22

Series                In 2015 constant prices at 2015 USD exchange rates
Pay period                                                      Annual Hourly
Time       Country
2006-01-01 Australia                                         23,826.64  12.06
           Belgium                                           20,228.74   9.70
           Brazil                                             2,032.87   0.87
           Canada                                            14,335.12   6.89
           Chile                                              3,333.76   1.42
```

Using a `DatetimeIndex` makes it easy to select a particular time period.

Selecting one year and stacking the two lower levels of the `MultiIndex` creates a cross-section of our panel data

```
realwage['2015'].stack(level=(1, 2)).transpose().head()
```

```
Time                                        2015-01-01         \
Series      In 2015 constant prices at 2015 USD PPPs
Pay period                                      Annual Hourly
Country
Australia                                    21,715.53  10.99
Belgium                                      21,588.12  10.35
Brazil                                        4,628.63   2.00
Canada                                       16,536.83   7.95
Chile                                         6,633.56   2.80

Time
Series      In 2015 constant prices at 2015 USD exchange rates
Pay period                                                Annual Hourly
Country
Australia                                              25,349.90  12.83
Belgium                                                20,753.48   9.95
Brazil                                                  2,842.28   1.21
Canada                                                 17,367.24   8.35
Chile                                                   4,251.49   1.81
```

For the rest of lecture, we will work with a dataframe of the hourly real minimum wages across countries and time, measured in 2015 US dollars.

To create our filtered dataframe (`realwage_f`), we can use the `xs` method to select values at lower levels in the multiindex, while keeping the higher levels (countries in this case)

```
realwage_f = realwage.xs(('Hourly', 'In 2015 constant prices at 2015 USD exchange␣
↪rates'),
                         level=('Pay period', 'Series'), axis=1)
realwage_f.head()
```

```
Country     Australia  Belgium  Brazil  ...  Turkey  United Kingdom  \
Time                                    ...
2006-01-01      12.06     9.70    0.87  ...    2.27            9.81
2007-01-01      12.46     9.82    0.92  ...    2.26           10.07
2008-01-01      12.24     9.87    0.96  ...    2.22           10.04
2009-01-01      12.40    10.21    1.03  ...    2.28           10.15
2010-01-01      12.34    10.05    1.08  ...    2.30            9.96

Country     United States
Time
2006-01-01           6.05
2007-01-01           6.24
2008-01-01           6.78
2009-01-01           7.58
2010-01-01           7.88

[5 rows x 32 columns]
```

## 62.3 Merging Dataframes and Filling NaNs

Similar to relational databases like SQL, pandas has built in methods to merge datasets together.

Using country information from WorldData.info, we'll add the continent of each country to `realwage_f` with the `merge` function.

The dataset can be accessed with the following link:

```
url2 = 'https://raw.githubusercontent.com/QuantEcon/lecture-python/master/source/_
↪static/lecture_specific/pandas_panel/countries.csv'
```

```
worlddata = pd.read_csv(url2, sep=';')
worlddata.head()
```

```
    Country (en) Country (de)        Country (local)  ... Deathrate  \
0    Afghanistan  Afghanistan  Afganistan/Afqanestan  ...     13.70
1          Egypt      Ägypten                   Misr  ...      4.70
2  Åland Islands  Ålandinseln                  Åland  ...      0.00
3        Albania      Albanien              Shqipëria  ...      6.70
4        Algeria      Algerien     Al-Jaza'ir/Algérie  ...      4.30

  Life expectancy                                                Url
0           51.30  https://www.laenderdaten.info/Asien/Afghanista...
1           72.70  https://www.laenderdaten.info/Afrika/Aegypten/...
2            0.00  https://www.laenderdaten.info/Europa/Aland/ind...
3           78.30  https://www.laenderdaten.info/Europa/Albanien/...
4           76.80  https://www.laenderdaten.info/Afrika/Algerien/...

[5 rows x 17 columns]
```

First, we'll select just the country and continent variables from `worlddata` and rename the column to 'Country'

```
worlddata = worlddata[['Country (en)', 'Continent']]
worlddata = worlddata.rename(columns={'Country (en)': 'Country'})
worlddata.head()
```

```
         Country Continent
0    Afghanistan      Asia
1          Egypt    Africa
2  Åland Islands    Europe
3        Albania    Europe
4        Algeria    Africa
```

We want to merge our new dataframe, `worlddata`, with `realwage_f`.

The pandas `merge` function allows dataframes to be joined together by rows.

Our dataframes will be merged using country names, requiring us to use the transpose of `realwage_f` so that rows correspond to country names in both dataframes

```
realwage_f.transpose().head()
```

```
Time       2006-01-01  2007-01-01  2008-01-01  ...  2014-01-01  2015-01-01  \
Country                                         ...
Australia       12.06       12.46       12.24  ...       12.67       12.83
Belgium          9.70        9.82        9.87  ...       10.01        9.95
```

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```
Brazil            0.87        0.92        0.96  ...       1.21        1.21
Canada            6.89        6.96        7.24  ...       8.22        8.35
Chile             1.42        1.45        1.44  ...       1.76        1.81

Time        2016-01-01
Country
Australia        12.98
Belgium           9.76
Brazil            1.24
Canada            8.48
Chile             1.91

[5 rows x 11 columns]
```

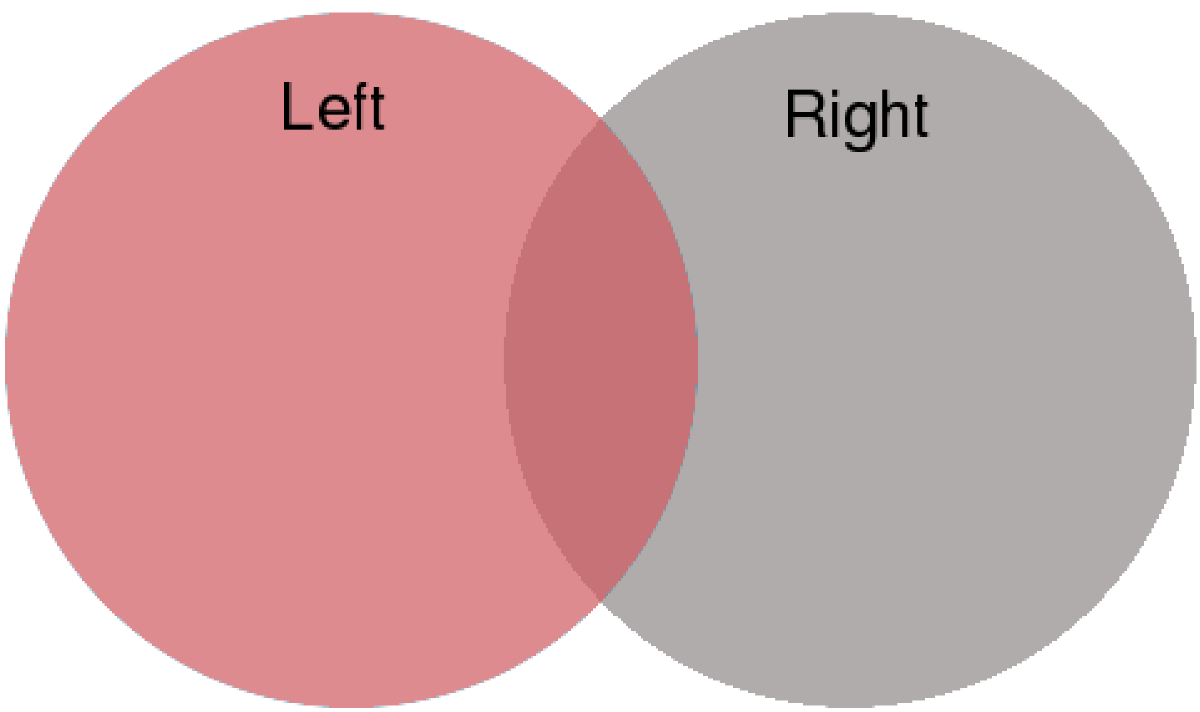
We can use either left, right, inner, or outer join to merge our datasets:

- left join includes only countries from the left dataset
- right join includes only countries from the right dataset
- outer join includes countries that are in either the left and right datasets
- inner join includes only countries common to both the left and right datasets

By default, `merge` will use an inner join.

Here we will pass `how='left'` to keep all countries in `realwage_f`, but discard countries in `worlddata` that do not have a corresponding data entry `realwage_f`.

This is illustrated by the red shading in the following diagram



We will also need to specify where the country name is located in each dataframe, which will be the `key` that is used to merge the dataframes 'on'.

Our 'left' dataframe (`realwage_f.transpose()`) contains countries in the index, so we set `left_index=True`.

Our 'right' dataframe (`worlddata`) contains countries in the 'Country' column, so we set `right_on='Country'`

```
merged = pd.merge(realwage_f.transpose(), worlddata,
                  how='left', left_index=True, right_on='Country')
merged.head()
```

```
        2006-01-01 00:00:00  2007-01-01 00:00:00  2008-01-01 00:00:00  ...  \
17.00                 12.06                12.46                12.24  ...
23.00                  9.70                 9.82                 9.87  ...
32.00                  0.87                 0.92                 0.96  ...
100.00                 6.89                 6.96                 7.24  ...
38.00                  1.42                 1.45                 1.44  ...

        2016-01-01 00:00:00    Country      Continent
17.00                 12.98  Australia      Australia
23.00                  9.76    Belgium         Europe
32.00                  1.24     Brazil  South America
100.00                 8.48     Canada  North America
38.00                  1.91      Chile  South America

[5 rows x 13 columns]
```

Countries that appeared in `realwage_f` but not in `worlddata` will have `NaN` in the Continent column.

To check whether this has occurred, we can use `.isnull()` on the continent column and filter the merged dataframe

```
merged[merged['Continent'].isnull()]
```

```
     2006-01-01 00:00:00  2007-01-01 00:00:00  2008-01-01 00:00:00  ...  \
nan                 3.42                 3.74                 3.87  ...
nan                 0.23                 0.45                 0.39  ...
nan                 1.50                 1.64                 1.71  ...

     2016-01-01 00:00:00             Country  Continent
nan                 5.28               Korea        NaN
nan                 0.55  Russian Federation        NaN
nan                 2.08     Slovak Republic        NaN

[3 rows x 13 columns]
```

We have three missing values!

One option to deal with NaN values is to create a dictionary containing these countries and their respective continents.

`.map()` will match countries in `merged['Country']` with their continent from the dictionary.

Notice how countries not in our dictionary are mapped with `NaN`

```
missing_continents = {'Korea': 'Asia',
                      'Russian Federation': 'Europe',
                      'Slovak Republic': 'Europe'}

merged['Country'].map(missing_continents)
```

```
17.00         NaN
23.00         NaN
32.00         NaN
100.00        NaN
38.00         NaN
108.00        NaN
41.00         NaN
225.00        NaN
53.00         NaN
58.00         NaN
45.00         NaN
68.00         NaN
233.00        NaN
86.00         NaN
88.00         NaN
91.00         NaN
nan          Asia
117.00        NaN
122.00        NaN
123.00        NaN
138.00        NaN
153.00        NaN
151.00        NaN
174.00        NaN
175.00        NaN
nan        Europe
nan        Europe
198.00        NaN
200.00        NaN
227.00        NaN
241.00        NaN
240.00        NaN
Name: Country, dtype: object
```

We don't want to overwrite the entire series with this mapping.

`.fillna()` only fills in `NaN` values in `merged['Continent']` with the mapping, while leaving other values in the column unchanged

```
merged['Continent'] = merged['Continent'].fillna(merged['Country'].map(missing_
↪continents))

# Check for whether continents were correctly mapped

merged[merged['Country'] == 'Korea']
```

```
     2006-01-01 00:00:00  2007-01-01 00:00:00  2008-01-01 00:00:00  ...  \
nan                 3.42                 3.74                 3.87  ...

     2016-01-01 00:00:00  Country  Continent
nan                 5.28    Korea       Asia

[1 rows x 13 columns]
```

We will also combine the Americas into a single continent - this will make our visualization nicer later on.

To do this, we will use `.replace()` and loop through a list of the continent values we want to replace

```
replace = ['Central America', 'North America', 'South America']

for country in replace:
    merged['Continent'].replace(to_replace=country,
                                value='America',
                                inplace=True)
```

Now that we have all the data we want in a single `DataFrame`, we will reshape it back into panel form with a `MultiIndex`.

We should also ensure to sort the index using `.sort_index()` so that we can efficiently filter our dataframe later on.

By default, levels will be sorted top-down

```
merged = merged.set_index(['Continent', 'Country']).sort_index()
merged.head()
```

```
                      2006-01-01  2007-01-01  2008-01-01  ...  2014-01-01  \
Continent Country                                         ...
America   Brazil            0.87        0.92        0.96  ...        1.21
          Canada            6.89        6.96        7.24  ...        8.22
          Chile             1.42        1.45        1.44  ...        1.76
          Colombia          1.01        1.02        1.01  ...        1.13
          Costa Rica         nan         nan         nan  ...        2.41

                      2015-01-01  2016-01-01
Continent Country
America   Brazil            1.21        1.24
          Canada            8.35        8.48
          Chile             1.81        1.91
          Colombia          1.13        1.12
          Costa Rica        2.56        2.63

[5 rows x 11 columns]
```

While merging, we lost our `DatetimeIndex`, as we merged columns that were not in datetime format

```
merged.columns
```

```
Index([2006-01-01 00:00:00, 2007-01-01 00:00:00, 2008-01-01 00:00:00,
       2009-01-01 00:00:00, 2010-01-01 00:00:00, 2011-01-01 00:00:00,
       2012-01-01 00:00:00, 2013-01-01 00:00:00, 2014-01-01 00:00:00,
       2015-01-01 00:00:00, 2016-01-01 00:00:00],
      dtype='object')
```

Now that we have set the merged columns as the index, we can recreate a `DatetimeIndex` using `.to_datetime()`

```
merged.columns = pd.to_datetime(merged.columns)
merged.columns = merged.columns.rename('Time')
merged.columns
```

```
DatetimeIndex(['2006-01-01', '2007-01-01', '2008-01-01', '2009-01-01',
               '2010-01-01', '2011-01-01', '2012-01-01', '2013-01-01',
               '2014-01-01', '2015-01-01', '2016-01-01'],
              dtype='datetime64[ns]', name='Time', freq=None)
```

The `DatetimeIndex` tends to work more smoothly in the row axis, so we will go ahead and transpose `merged`

```
merged = merged.transpose()
merged.head()
```

```
Continent  America                  ...   Europe
Country     Brazil Canada Chile  ... Slovenia Spain United Kingdom
Time                                ...
2006-01-01    0.87   6.89  1.42  ...     3.92  3.99           9.81
2007-01-01    0.92   6.96  1.45  ...     3.88  4.10          10.07
2008-01-01    0.96   7.24  1.44  ...     3.96  4.14          10.04
2009-01-01    1.03   7.67  1.52  ...     4.08  4.32          10.15
2010-01-01    1.08   7.94  1.56  ...     4.81  4.30           9.96

[5 rows x 32 columns]
```

## 62.4 Grouping and Summarizing Data

Grouping and summarizing data can be particularly useful for understanding large panel datasets.

A simple way to summarize data is to call an aggregation method on the dataframe, such as `.mean()` or `.max()`.

For example, we can calculate the average real minimum wage for each country over the period 2006 to 2016 (the default is to aggregate over rows)

```
merged.mean().head(10)
```

```
Continent  Country
America    Brazil           1.09
           Canada           7.82
           Chile            1.62
           Colombia         1.07
           Costa Rica       2.53
           Mexico           0.53
           United States    7.15
Asia       Israel           5.95
           Japan            6.18
           Korea            4.22
dtype: float64
```

Using this series, we can plot the average real minimum wage over the past decade for each country in our data set

```
%matplotlib inline
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (11, 5)  #set default figure size
import matplotlib
matplotlib.style.use('seaborn')

merged.mean().sort_values(ascending=False).plot(kind='bar', title="Average real␣
↪minimum wage 2006 - 2016")

#Set country labels
country_labels = merged.mean().sort_values(ascending=False).index.get_level_values(
↪'Country').tolist()
plt.xticks(range(0, len(country_labels)), country_labels)
plt.xlabel('Country')
```

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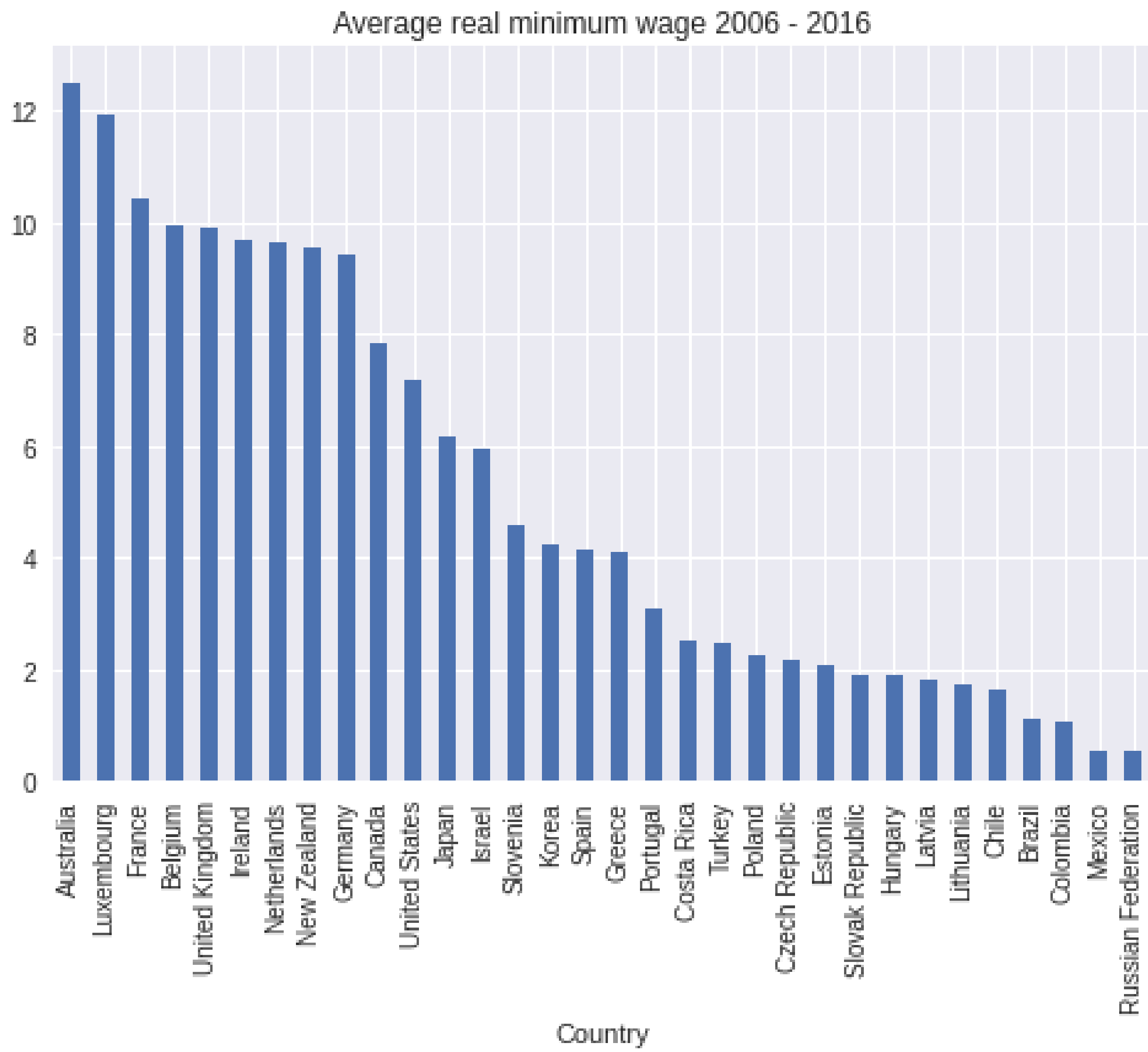
```
plt.show()
```




Passing in `axis=1` to `.mean()` will aggregate over columns (giving the average minimum wage for all countries over time)

```
merged.mean(axis=1).head()
```

```
Time
2006-01-01    4.69
2007-01-01    4.84
2008-01-01    4.90
2009-01-01    5.08
2010-01-01    5.11
dtype: float64
```

We can plot this time series as a line graph

```
merged.mean(axis=1).plot()
plt.title('Average real minimum wage 2006 - 2016')
plt.ylabel('2015 USD')
plt.xlabel('Year')
plt.show()
```

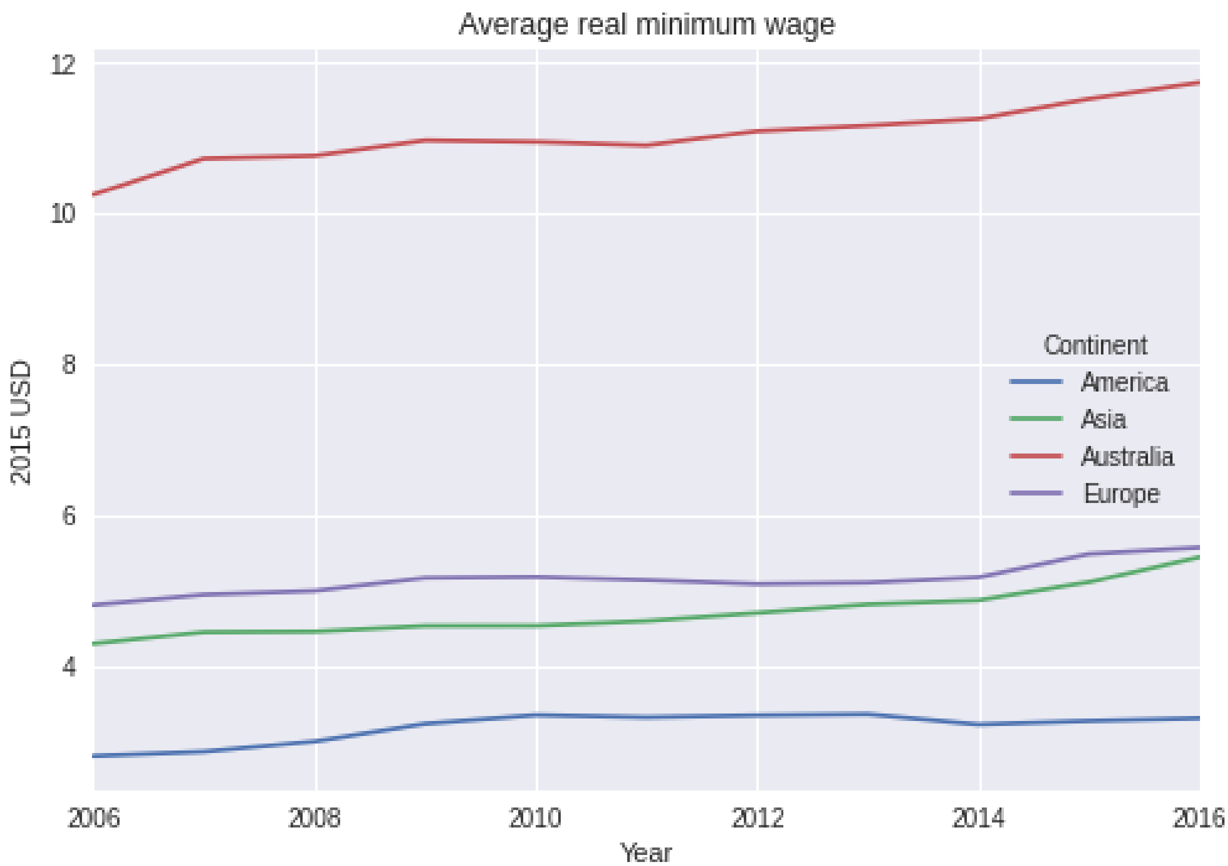

We can also specify a level of the `MultiIndex` (in the column axis) to aggregate over

```
merged.mean(level='Continent', axis=1).head()
```

```
Continent    America  Asia  Australia  Europe
Time
2006-01-01      2.80  4.29      10.25    4.80
2007-01-01      2.85  4.44      10.73    4.94
2008-01-01      2.99  4.45      10.76    4.99
2009-01-01      3.23  4.53      10.97    5.16
2010-01-01      3.34  4.53      10.95    5.17
```

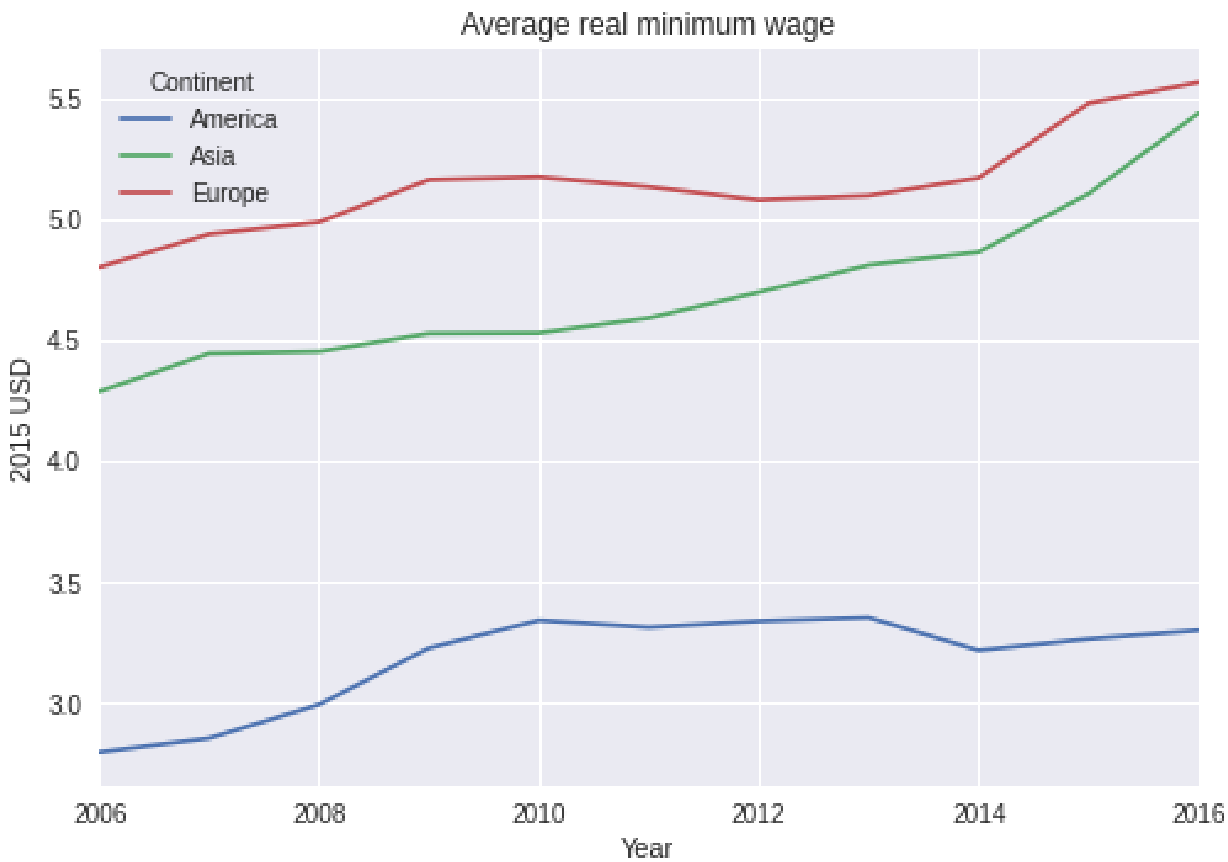
We can plot the average minimum wages in each continent as a time series

```
merged.mean(level='Continent', axis=1).plot()
plt.title('Average real minimum wage')
plt.ylabel('2015 USD')
plt.xlabel('Year')
plt.show()
```




We will drop Australia as a continent for plotting purposes

```
merged = merged.drop('Australia', level='Continent', axis=1)
merged.mean(level='Continent', axis=1).plot()
plt.title('Average real minimum wage')
plt.ylabel('2015 USD')
plt.xlabel('Year')
plt.show()
```




`.describe()` is useful for quickly retrieving a number of common summary statistics

```
merged.stack().describe()
```

```
Continent   America  Asia   Europe
count         69.00 44.00   200.00
mean           3.19  4.70     5.15
std            3.02  1.56     3.82
min            0.52  2.22     0.23
25%            1.03  3.37     2.02
50%            1.44  5.48     3.54
75%            6.96  5.95     9.70
max            8.48  6.65    12.39
```

This is a simplified way to use `groupby`.

Using `groupby` generally follows a 'split-apply-combine' process:

- split: data is grouped based on one or more keys
- apply: a function is called on each group independently
- combine: the results of the function calls are combined into a new data structure

The `groupby` method achieves the first step of this process, creating a new `DataFrameGroupBy` object with data split into groups.

Let's split `merged` by continent again, this time using the `groupby` function, and name the resulting object `grouped`

```
grouped = merged.groupby(level='Continent', axis=1)
grouped
```

```
<pandas.core.groupby.generic.DataFrameGroupBy object at 0x7fc17950dfa0>
```

Calling an aggregation method on the object applies the function to each group, the results of which are combined in a new data structure.

For example, we can return the number of countries in our dataset for each continent using `.size()`.

In this case, our new data structure is a `Series`

```
grouped.size()
```

```
Continent
America     7
Asia        4
Europe     19
dtype: int64
```

Calling `.get_group()` to return just the countries in a single group, we can create a kernel density estimate of the distribution of real minimum wages in 2016 for each continent.

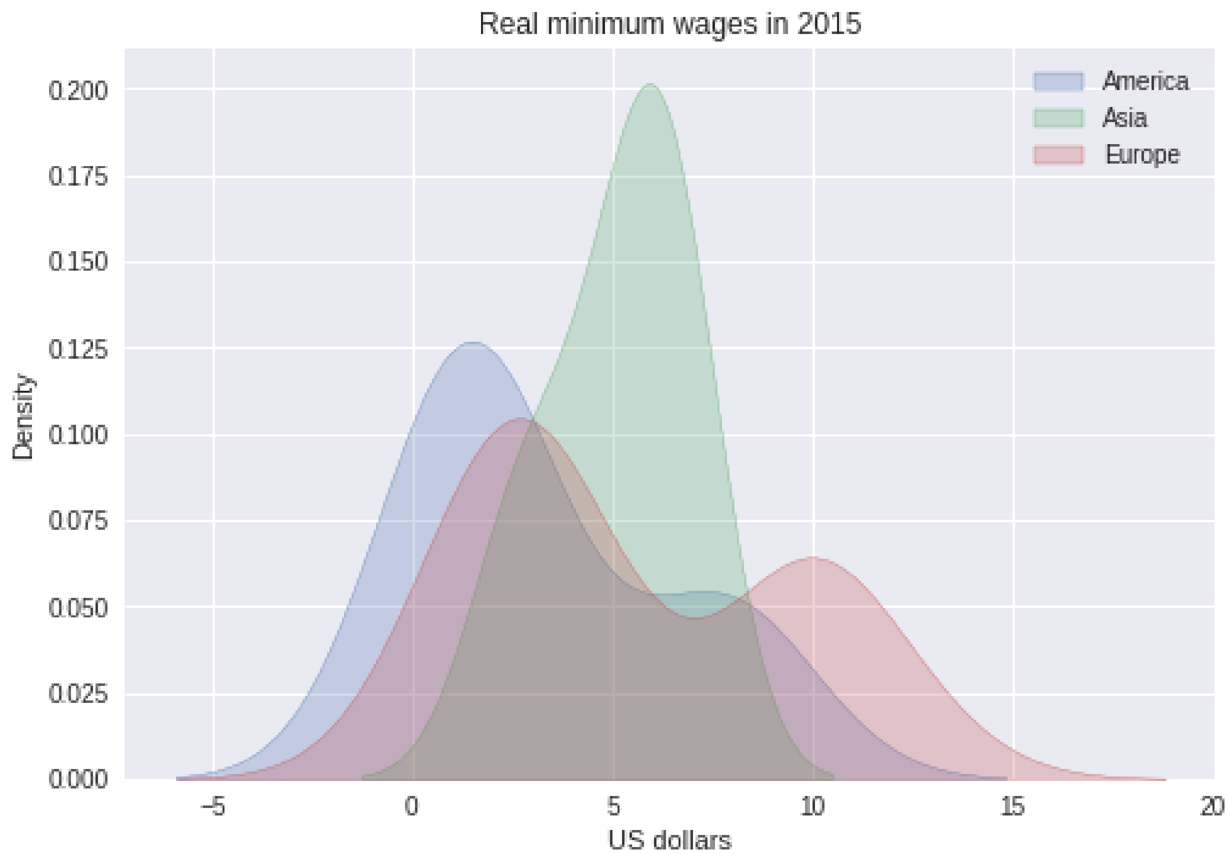
`grouped.groups.keys()` will return the keys from the `groupby` object

```
import seaborn as sns

continents = grouped.groups.keys()

for continent in continents:
    sns.kdeplot(grouped.get_group(continent)['2015'].unstack(), label=continent,␣
↪shade=True)

plt.title('Real minimum wages in 2015')
plt.xlabel('US dollars')
plt.legend()
plt.show()
```




## 62.5 Final Remarks

This lecture has provided an introduction to some of pandas' more advanced features, including multiindices, merging, grouping and plotting.

Other tools that may be useful in panel data analysis include xarray, a python package that extends pandas to N-dimensional data structures.

## 62.6 Exercises

### 62.6.1 Exercise 1

In these exercises, you'll work with a dataset of employment rates in Europe by age and sex from Eurostat.

The dataset can be accessed with the following link:

```
url3 = 'https://raw.githubusercontent.com/QuantEcon/lecture-python/master/source/_
↪static/lecture_specific/pandas_panel/employ.csv'
```

Reading in the CSV file returns a panel dataset in long format. Use `.pivot_table()` to construct a wide format dataframe with a `MultiIndex` in the columns.

Start off by exploring the dataframe and the variables available in the `MultiIndex` levels.

Write a program that quickly returns all values in the `MultiIndex`.

### 62.6.2 Exercise 2

Filter the above dataframe to only include employment as a percentage of 'active population'.

Create a grouped boxplot using `seaborn` of employment rates in 2015 by age group and sex.

**Hint:** `GEO` includes both areas and countries.

## 62.7 Solutions

### 62.7.1 Exercise 1

```
employ = pd.read_csv(url3)
employ = employ.pivot_table(values='Value',
                            index=['DATE'],
                            columns=['UNIT','AGE', 'SEX', 'INDIC_EM', 'GEO'])
employ.index = pd.to_datetime(employ.index) # ensure that dates are datetime format
employ.head()
```

```
UNIT       Percentage of total population                   ...  \
AGE                   From 15 to 24 years                   ...   
SEX                               Females                   ...   
INDIC_EM              Active population                     ...   
GEO                               Austria Belgium Bulgaria  ...   
DATE                                                        ...   
2007-01-01                          56.00   31.60    26.00  ...   
2008-01-01                          56.20   30.80    26.10  ...   
2009-01-01                          56.20   29.90    24.80  ...   
2010-01-01                          54.00   29.80    26.60  ...   
2011-01-01                          54.80   29.80    24.80  ...   

UNIT                                          Thousand persons           \
AGE                                        From 55 to 64 years            
SEX                                                      Total            
INDIC_EM   Total employment (resident population concept - LFS)           
GEO                                                Switzerland   Turkey   
DATE                                                                      
2007-01-01                                                 nan   1,282.00   
2008-01-01                                                 nan   1,354.00   
2009-01-01                                                 nan   1,449.00   
2010-01-01                                              640.00   1,583.00   
2011-01-01                                              661.00   1,760.00   

UNIT                        
AGE                         
SEX                         
INDIC_EM                    
GEO        United Kingdom   
DATE                        
2007-01-01       4,131.00   
2008-01-01       4,204.00   
2009-01-01       4,193.00   
```

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```
2010-01-01        4,186.00
2011-01-01        4,164.00

[5 rows x 1440 columns]
```

This is a large dataset so it is useful to explore the levels and variables available

```
employ.columns.names
```

```
FrozenList(['UNIT', 'AGE', 'SEX', 'INDIC_EM', 'GEO'])
```

Variables within levels can be quickly retrieved with a loop

```
for name in employ.columns.names:
    print(name, employ.columns.get_level_values(name).unique())
```

```
UNIT Index(['Percentage of total population', 'Thousand persons'], dtype='object',␣
↪name='UNIT')
AGE Index(['From 15 to 24 years', 'From 25 to 54 years', 'From 55 to 64 years'],␣
↪dtype='object', name='AGE')
SEX Index(['Females', 'Males', 'Total'], dtype='object', name='SEX')
INDIC_EM Index(['Active population', 'Total employment (resident population concept -␣
↪LFS)'], dtype='object', name='INDIC_EM')
GEO Index(['Austria', 'Belgium', 'Bulgaria', 'Croatia', 'Cyprus', 'Czech Republic',
       'Denmark', 'Estonia', 'Euro area (17 countries)',
       'Euro area (18 countries)', 'Euro area (19 countries)',
       'European Union (15 countries)', 'European Union (27 countries)',
       'European Union (28 countries)', 'Finland',
       'Former Yugoslav Republic of Macedonia, the', 'France',
       'France (metropolitan)',
       'Germany (until 1990 former territory of the FRG)', 'Greece', 'Hungary',
       'Iceland', 'Ireland', 'Italy', 'Latvia', 'Lithuania', 'Luxembourg',
       'Malta', 'Netherlands', 'Norway', 'Poland', 'Portugal', 'Romania',
       'Slovakia', 'Slovenia', 'Spain', 'Sweden', 'Switzerland', 'Turkey',
       'United Kingdom'],
      dtype='object', name='GEO')
```

### 62.7.2 Exercise 2

To easily filter by country, swap `GEO` to the top level and sort the `MultiIndex`

```
employ.columns = employ.columns.swaplevel(0,-1)
employ = employ.sort_index(axis=1)
```

We need to get rid of a few items in `GEO` which are not countries.

A fast way to get rid of the EU areas is to use a list comprehension to find the level values in `GEO` that begin with 'Euro'

```
geo_list = employ.columns.get_level_values('GEO').unique().tolist()
countries = [x for x in geo_list if not x.startswith('Euro')]
employ = employ[countries]
employ.columns.get_level_values('GEO').unique()
```

```
Index(['Austria', 'Belgium', 'Bulgaria', 'Croatia', 'Cyprus', 'Czech Republic',
       'Denmark', 'Estonia', 'Finland',
       'Former Yugoslav Republic of Macedonia, the', 'France',
       'France (metropolitan)',
       'Germany (until 1990 former territory of the FRG)', 'Greece', 'Hungary',
       'Iceland', 'Ireland', 'Italy', 'Latvia', 'Lithuania', 'Luxembourg',
       'Malta', 'Netherlands', 'Norway', 'Poland', 'Portugal', 'Romania',
       'Slovakia', 'Slovenia', 'Spain', 'Sweden', 'Switzerland', 'Turkey',
       'United Kingdom'],
      dtype='object', name='GEO')
```

Select only percentage employed in the active population from the dataframe

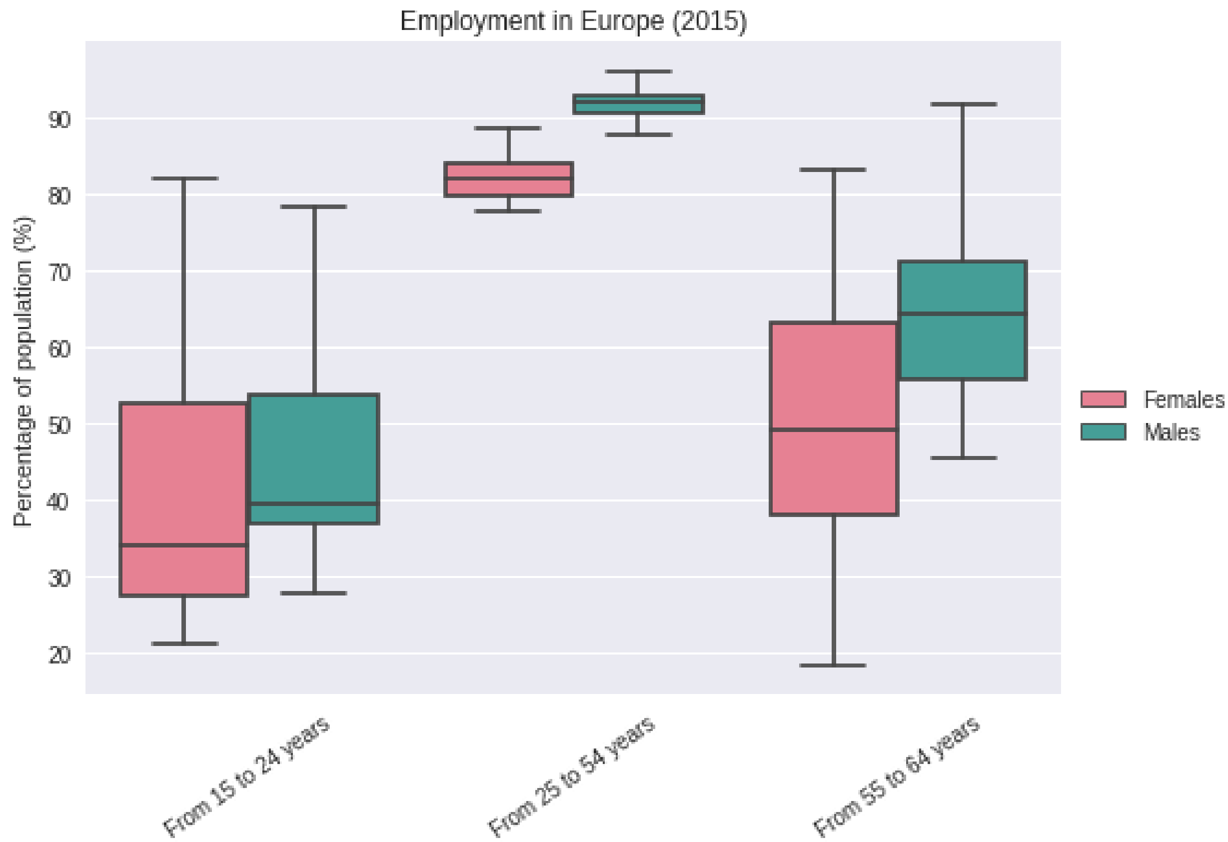
```
employ_f = employ.xs(('Percentage of total population', 'Active population'),
                     level=('UNIT', 'INDIC_EM'),
                     axis=1)
employ_f.head()
```

```
GEO                    Austria                ...      United Kingdom        \
AGE          From 15 to 24 years              ... From 55 to 64 years
SEX                    Females Males Total    ...             Females Males
DATE                                          ...
2007-01-01               56.00 62.90 59.40    ...               49.90 68.90
2008-01-01               56.20 62.90 59.50    ...               50.20 69.80
2009-01-01               56.20 62.90 59.50    ...               50.60 70.30
2010-01-01               54.00 62.60 58.30    ...               51.10 69.20
2011-01-01               54.80 63.60 59.20    ...               51.30 68.40

GEO
AGE
SEX          Total
DATE
2007-01-01 59.30
2008-01-01 59.80
2009-01-01 60.30
2010-01-01 60.00
2011-01-01 59.70

[5 rows x 306 columns]
```

Drop the 'Total' value before creating the grouped boxplot

```
employ_f = employ_f.drop('Total', level='SEX', axis=1)
```

```
box = employ_f['2015'].unstack().reset_index()
sns.boxplot(x="AGE", y=0, hue="SEX", data=box, palette=("husl"), showfliers=False)
plt.xlabel('')
plt.xticks(rotation=35)
plt.ylabel('Percentage of population (%)')
plt.title('Employment in Europe (2015)')
plt.legend(bbox_to_anchor=(1,0.5))
plt.show()
```


Employment in Europe (2015)
Percentage of population (%)
90
80
70
60
50
40
30
20
From 15 to 24 years
From 25 to 54 years
From 55 to 64 years
Females
Males

CHAPTER

SIXTYTHREE

# LINEAR REGRESSION IN PYTHON

Contents



In addition to what's in Anaconda, this lecture will need the following libraries:

```
!pip install linearmodels
```

## 63.1 Overview

Linear regression is a standard tool for analyzing the relationship between two or more variables.

In this lecture, we'll use the Python package `statsmodels` to estimate, interpret, and visualize linear regression models.

Along the way, we'll discuss a variety of topics, including

- simple and multivariate linear regression
- visualization
- endogeneity and omitted variable bias
- two-stage least squares

As an example, we will replicate results from Acemoglu, Johnson and Robinson's seminal paper [AJR01].

- You can download a copy here.

In the paper, the authors emphasize the importance of institutions in economic development.

The main contribution is the use of settler mortality rates as a source of *exogenous* variation in institutional differences.

Such variation is needed to determine whether it is institutions that give rise to greater economic growth, rather than the other way around.

Let's start with some imports:

```
%matplotlib inline
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (11, 5)  #set default figure size
import numpy as np
import pandas as pd
import statsmodels.api as sm
from statsmodels.iolib.summary2 import summary_col
from linearmodels.iv import IV2SLS
```

### 63.1.1 Prerequisites

This lecture assumes you are familiar with basic econometrics.

For an introductory text covering these topics, see, for example, [Woo15].

## 63.2 Simple Linear Regression

[AJR01] wish to determine whether or not differences in institutions can help to explain observed economic outcomes.

How do we measure *institutional differences* and *economic outcomes*?

In this paper,

- economic outcomes are proxied by log GDP per capita in 1995, adjusted for exchange rates.
- institutional differences are proxied by an index of protection against expropriation on average over 1985-95, constructed by the Political Risk Services Group.

These variables and other data used in the paper are available for download on Daron Acemoglu's webpage.

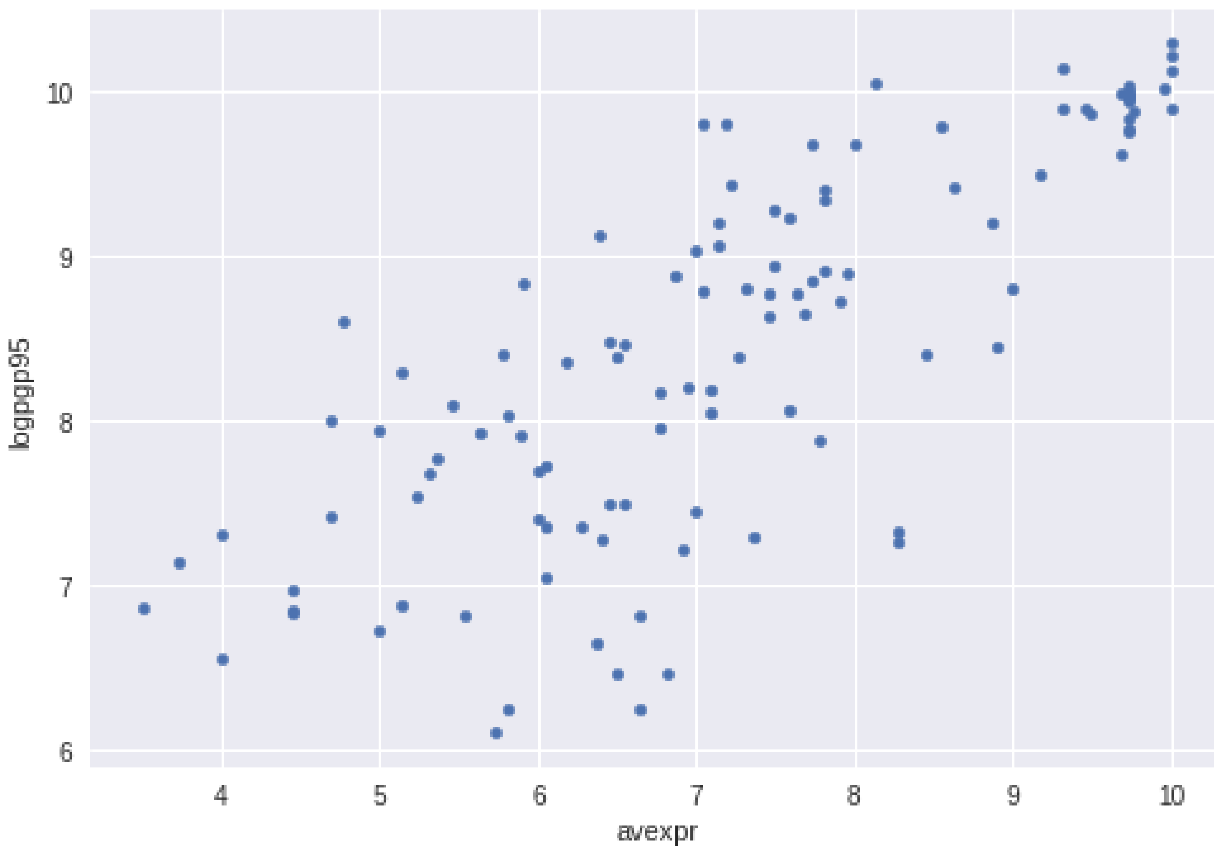
We will use pandas' `.read_stata()` function to read in data contained in the `.dta` files to dataframes

```
df1 = pd.read_stata('https://github.com/QuantEcon/lecture-python/blob/master/source/_
↪static/lecture_specific/ols/maketable1.dta?raw=true')
df1.head()
```

```
  shortnam   euro1900  excolony    avexpr  logpgp95  cons1  cons90  democ00a  \
0      AFG   0.000000       1.0       NaN       NaN    1.0     2.0       1.0
1      AGO   8.000000       1.0  5.363636  7.770645    3.0     3.0       0.0
2      ARE   0.000000       1.0  7.181818  9.804219    NaN     NaN       NaN
3      ARG  60.000004       1.0  6.386364  9.133459    1.0     6.0       3.0
4      ARM   0.000000       0.0       NaN  7.682482    NaN     NaN       NaN

   cons00a    extmort4    logem4  loghjypl  baseco
0      1.0   93.699997  4.540098       NaN     NaN
1      1.0  280.000000  5.634789 -3.411248     1.0
2      NaN         NaN       NaN       NaN     NaN
3      3.0   68.900002  4.232656 -0.872274     1.0
4      NaN         NaN       NaN       NaN     NaN
```

Let's use a scatterplot to see whether any obvious relationship exists between GDP per capita and the protection against expropriation index

```
plt.style.use('seaborn')

df1.plot(x='avexpr', y='logpgp95', kind='scatter')
plt.show()
```




The plot shows a fairly strong positive relationship between protection against expropriation and log GDP per capita.

Specifically, if higher protection against expropriation is a measure of institutional quality, then better institutions appear to be positively correlated with better economic outcomes (higher GDP per capita).

Given the plot, choosing a linear model to describe this relationship seems like a reasonable assumption.

We can write our model as

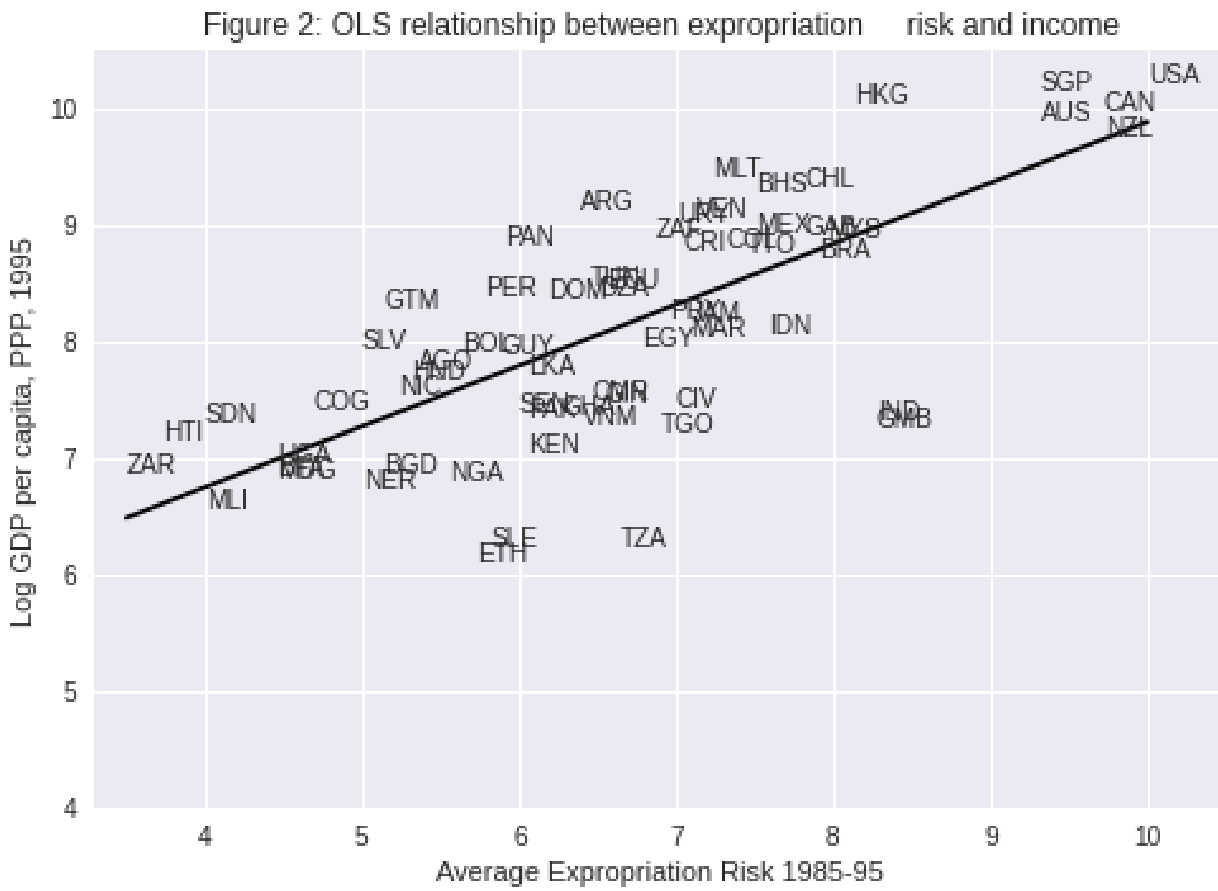
$$logpgp95_i = \beta_0 + \beta_1 avexpr_i + u_i$$

where:

- $\beta_0$ is the intercept of the linear trend line on the y-axis
- $\beta_1$ is the slope of the linear trend line, representing the *marginal effect* of protection against risk on log GDP per capita
- $u_i$ is a random error term (deviations of observations from the linear trend due to factors not included in the model)

Visually, this linear model involves choosing a straight line that best fits the data, as in the following plot (Figure 2 in [AJR01])

```
# Dropping NA's is required to use numpy's polyfit
df1_subset = df1.dropna(subset=['logpgp95', 'avexpr'])

# Use only 'base sample' for plotting purposes
df1_subset = df1_subset[df1_subset['baseco'] == 1]

X = df1_subset['avexpr']
y = df1_subset['logpgp95']
labels = df1_subset['shortnam']

# Replace markers with country labels
fig, ax = plt.subplots()
ax.scatter(X, y, marker='')

for i, label in enumerate(labels):
    ax.annotate(label, (X.iloc[i], y.iloc[i]))

# Fit a linear trend line
ax.plot(np.unique(X),
         np.poly1d(np.polyfit(X, y, 1))(np.unique(X)),
         color='black')

ax.set_xlim([3.3,10.5])
ax.set_ylim([4,10.5])
ax.set_xlabel('Average Expropriation Risk 1985-95')
ax.set_ylabel('Log GDP per capita, PPP, 1995')
ax.set_title('Figure 2: OLS relationship between expropriation \
    risk and income')
plt.show()
```




The most common technique to estimate the parameters ($\beta$'s) of the linear model is Ordinary Least Squares (OLS).

As the name implies, an OLS model is solved by finding the parameters that minimize *the sum of squared residuals*, i.e.

$$\min_{\hat{\beta}} \sum_{i=1}^{N} \hat{u}_i^2$$

where $\hat{u}_i$ is the difference between the observation and the predicted value of the dependent variable.

To estimate the constant term $\beta_0$, we need to add a column of 1's to our dataset (consider the equation if $\beta_0$ was replaced with $\beta_0 x_i$ and $x_i = 1$)

```
df1['const'] = 1
```

Now we can construct our model in `statsmodels` using the OLS function.

We will use `pandas` dataframes with `statsmodels`, however standard arrays can also be used as arguments

```
reg1 = sm.OLS(endog=df1['logpgp95'], exog=df1[['const', 'avexpr']], \
    missing='drop')
type(reg1)
```

```
statsmodels.regression.linear_model.OLS
```

So far we have simply constructed our model.

We need to use `.fit()` to obtain parameter estimates $\hat{\beta}_0$ and $\hat{\beta}_1$

```python
results = reg1.fit()
type(results)
```

```
statsmodels.regression.linear_model.RegressionResultsWrapper
```

We now have the fitted regression model stored in `results`.

To view the OLS regression results, we can call the `.summary()` method.

Note that an observation was mistakenly dropped from the results in the original paper (see the note located in maketable2.do from Acemoglu's webpage), and thus the coefficients differ slightly.

```python
print(results.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:               logpgp95   R-squared:                       0.611
Model:                            OLS   Adj. R-squared:                  0.608
Method:                 Least Squares   F-statistic:                     171.4
Date:                Thu, 07 Oct 2021   Prob (F-statistic):           4.16e-24
Time:                        21:22:23   Log-Likelihood:                -119.71
No. Observations:                 111   AIC:                             243.4
Df Residuals:                     109   BIC:                             248.8
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          4.6261      0.301     15.391      0.000       4.030       5.222
avexpr         0.5319      0.041     13.093      0.000       0.451       0.612
==============================================================================
Omnibus:                        9.251   Durbin-Watson:                   1.689
Prob(Omnibus):                  0.010   Jarque-Bera (JB):                9.170
Skew:                          -0.680   Prob(JB):                       0.0102
Kurtosis:                       3.362   Cond. No.                         33.2
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly␣
↪specified.
```

From our results, we see that

- The intercept $\hat{\beta}_0 = 4.63$.
- The slope $\hat{\beta}_1 = 0.53$.
- The positive $\hat{\beta}_1$ parameter estimate implies that. institutional quality has a positive effect on economic outcomes, as we saw in the figure.
- The p-value of 0.000 for $\hat{\beta}_1$ implies that the effect of institutions on GDP is statistically significant (using $p < 0.05$ as a rejection rule).
- The R-squared value of 0.611 indicates that around 61% of variation in log GDP per capita is explained by protection against expropriation.

Using our parameter estimates, we can now write our estimated relationship as

$$\widehat{logpgp95}_i = 4.63 + 0.53 \ avexpr_i$$

This equation describes the line that best fits our data, as shown in Figure 2.

We can use this equation to predict the level of log GDP per capita for a value of the index of expropriation protection.

For example, for a country with an index value of 7.07 (the average for the dataset), we find that their predicted level of log GDP per capita in 1995 is 8.38.

```
mean_expr = np.mean(df1_subset['avexpr'])
mean_expr
```

```
6.515625
```

```
predicted_logpdp95 = 4.63 + 0.53 * 7.07
predicted_logpdp95
```

```
8.3771
```

An easier (and more accurate) way to obtain this result is to use `.predict()` and set $constant = 1$ and $avexpr_i = mean\_expr$

```
results.predict(exog=[1, mean_expr])
```

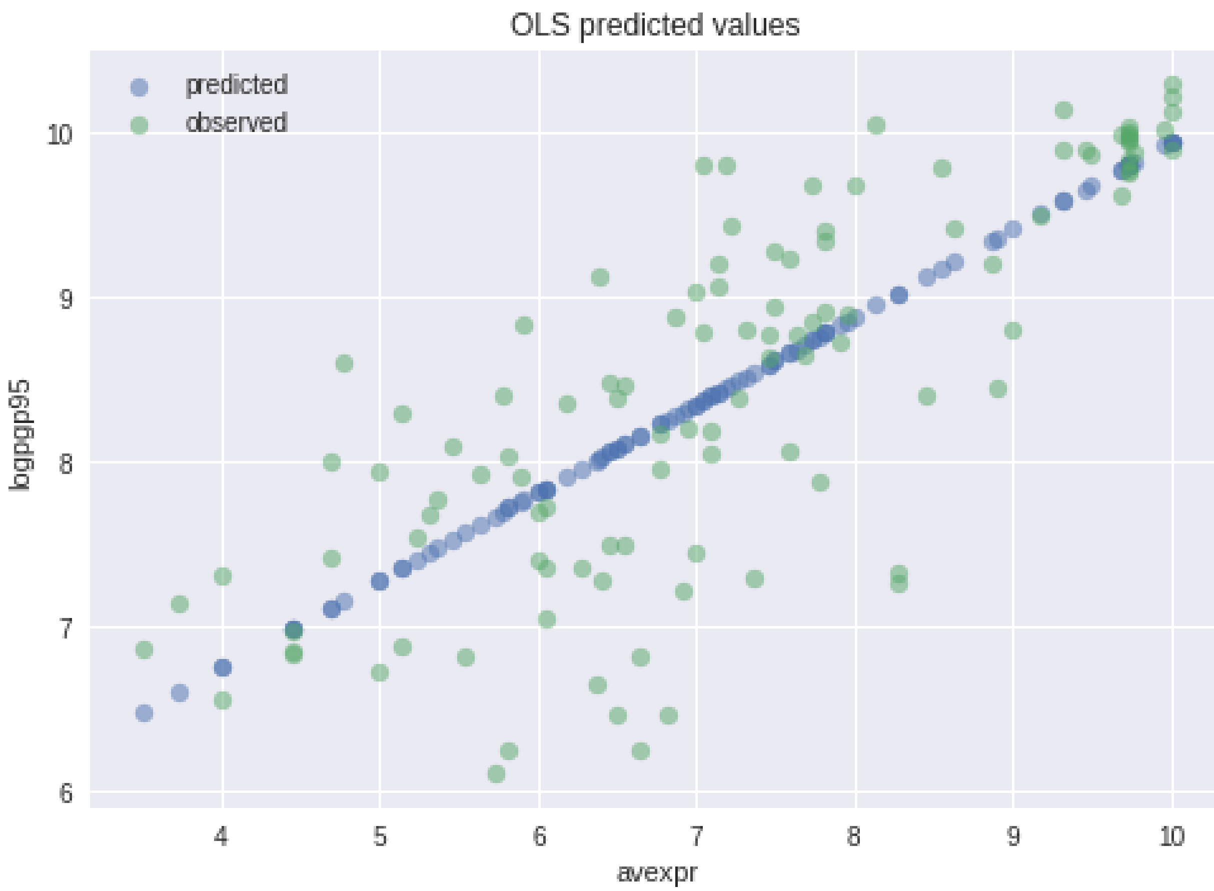
```
array([8.09156367])
```

We can obtain an array of predicted $logpgp95_i$ for every value of $avexpr_i$ in our dataset by calling `.predict()` on our results.

Plotting the predicted values against $avexpr_i$ shows that the predicted values lie along the linear line that we fitted above.

The observed values of $logpgp95_i$ are also plotted for comparison purposes

```
# Drop missing observations from whole sample

df1_plot = df1.dropna(subset=['logpgp95', 'avexpr'])

# Plot predicted values

fix, ax = plt.subplots()
ax.scatter(df1_plot['avexpr'], results.predict(), alpha=0.5,
        label='predicted')

# Plot observed values

ax.scatter(df1_plot['avexpr'], df1_plot['logpgp95'], alpha=0.5,
        label='observed')

ax.legend()
ax.set_title('OLS predicted values')
ax.set_xlabel('avexpr')
ax.set_ylabel('logpgp95')
plt.show()
```




## 63.3 Extending the Linear Regression Model

So far we have only accounted for institutions affecting economic performance - almost certainly there are numerous other factors affecting GDP that are not included in our model.

Leaving out variables that affect $logpgp95_i$ will result in **omitted variable bias**, yielding biased and inconsistent parameter estimates.

We can extend our bivariate regression model to a **multivariate regression model** by adding in other factors that may affect $logpgp95_i$.

[AJR01] consider other factors such as:

- the effect of climate on economic outcomes; latitude is used to proxy this
- differences that affect both economic performance and institutions, eg. cultural, historical, etc.; controlled for with the use of continent dummies

Let's estimate some of the extended models considered in the paper (Table 2) using data from `maketable2.dta`

```
df2 = pd.read_stata('https://github.com/QuantEcon/lecture-python/blob/master/source/_
↪static/lecture_specific/ols/maketable2.dta?raw=true')

# Add constant term to dataset
df2['const'] = 1
```

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```python
# Create lists of variables to be used in each regression
X1 = ['const', 'avexpr']
X2 = ['const', 'avexpr', 'lat_abst']
X3 = ['const', 'avexpr', 'lat_abst', 'asia', 'africa', 'other']

# Estimate an OLS regression for each set of variables
reg1 = sm.OLS(df2['logpgp95'], df2[X1], missing='drop').fit()
reg2 = sm.OLS(df2['logpgp95'], df2[X2], missing='drop').fit()
reg3 = sm.OLS(df2['logpgp95'], df2[X3], missing='drop').fit()
```

Now that we have fitted our model, we will use `summary_col` to display the results in a single table (model numbers correspond to those in the paper)

```python
info_dict={'R-squared' : lambda x: f"{x.rsquared:.2f}",
           'No. observations' : lambda x: f"{int(x.nobs):d}"}

results_table = summary_col(results=[reg1,reg2,reg3],
                            float_format='%0.2f',
                            stars = True,
                            model_names=['Model 1',
                                         'Model 3',
                                         'Model 4'],
                            info_dict=info_dict,
                            regressor_order=['const',
                                             'avexpr',
                                             'lat_abst',
                                             'asia',
                                             'africa'])

results_table.add_title('Table 2 - OLS Regressions')

print(results_table)
```

```
        Table 2 - OLS Regressions
=========================================
                 Model 1 Model 3 Model 4
-----------------------------------------
const            4.63*** 4.87*** 5.85***
                 (0.30)  (0.33)  (0.34)
avexpr           0.53*** 0.46*** 0.39***
                 (0.04)  (0.06)  (0.05)
lat_abst                 0.87*   0.33
                         (0.49)  (0.45)
asia                             -0.15
                                 (0.15)
africa                           -0.92***
                                 (0.17)
other                            0.30
                                 (0.37)
R-squared        0.61    0.62    0.72
R-squared Adj.   0.61    0.62    0.70
R-squared        0.61    0.62    0.72
No. observations 111     111     111
=========================================
Standard errors in parentheses.
```

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```
* p<.1, ** p<.05, ***p<.01
```

## 63.4 Endogeneity

As [AJR01] discuss, the OLS models likely suffer from **endogeneity** issues, resulting in biased and inconsistent model estimates.

Namely, there is likely a two-way relationship between institutions and economic outcomes:

- richer countries may be able to afford or prefer better institutions
- variables that affect income may also be correlated with institutional differences
- the construction of the index may be biased; analysts may be biased towards seeing countries with higher income having better institutions

To deal with endogeneity, we can use **two-stage least squares (2SLS) regression**, which is an extension of OLS regression.

This method requires replacing the endogenous variable $avexpr_i$ with a variable that is:

1. correlated with $avexpr_i$
2. not correlated with the error term (ie. it should not directly affect the dependent variable, otherwise it would be correlated with $u_i$ due to omitted variable bias)

The new set of regressors is called an **instrument**, which aims to remove endogeneity in our proxy of institutional differences.

The main contribution of [AJR01] is the use of settler mortality rates to instrument for institutional differences.

They hypothesize that higher mortality rates of colonizers led to the establishment of institutions that were more extractive in nature (less protection against expropriation), and these institutions still persist today.

Using a scatterplot (Figure 3 in [AJR01]), we can see protection against expropriation is negatively correlated with settler mortality rates, coinciding with the authors' hypothesis and satisfying the first condition of a valid instrument.

```python
# Dropping NA's is required to use numpy's polyfit
df1_subset2 = df1.dropna(subset=['logem4', 'avexpr'])

X = df1_subset2['logem4']
y = df1_subset2['avexpr']
labels = df1_subset2['shortnam']

# Replace markers with country labels
fig, ax = plt.subplots()
ax.scatter(X, y, marker='')

for i, label in enumerate(labels):
    ax.annotate(label, (X.iloc[i], y.iloc[i]))

# Fit a linear trend line
ax.plot(np.unique(X),
         np.poly1d(np.polyfit(X, y, 1))(np.unique(X)),
         color='black')

ax.set_xlim([1.8,8.4])
ax.set_ylim([3.3,10.4])
```

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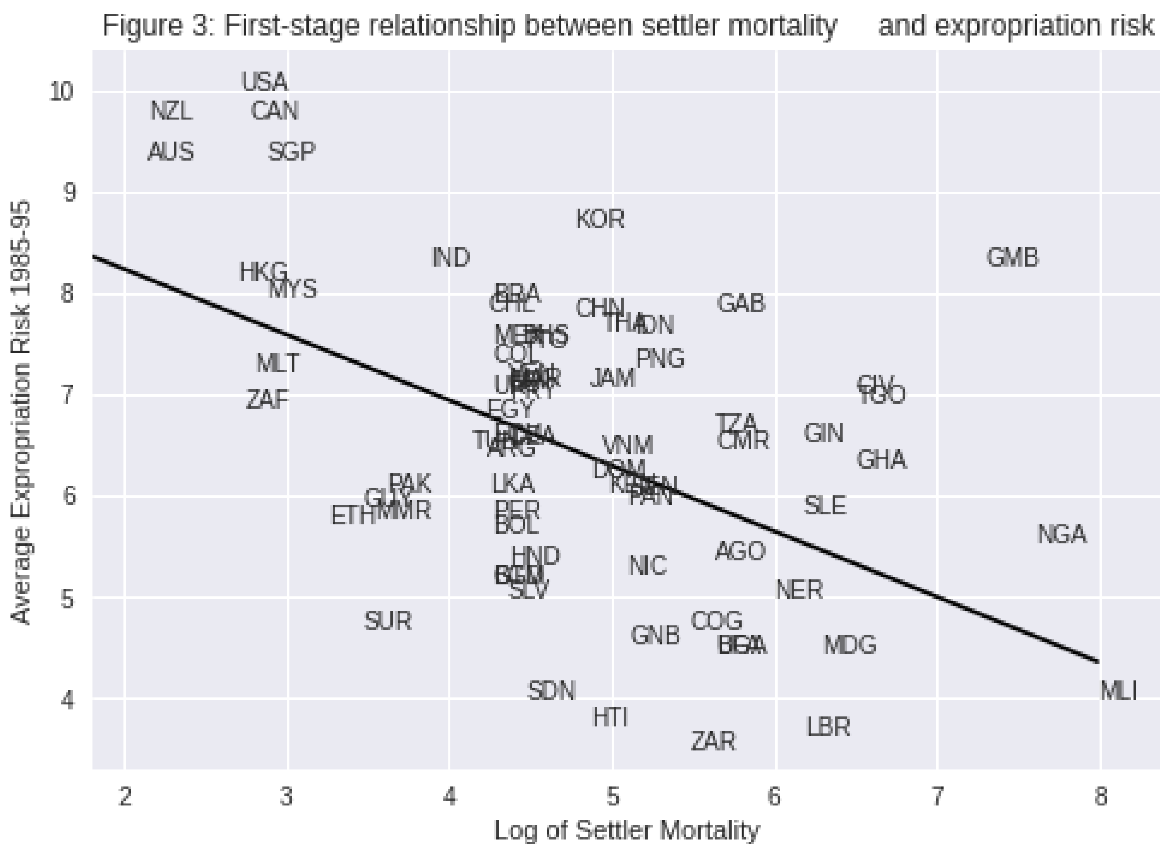
```
ax.set_xlabel('Log of Settler Mortality')
ax.set_ylabel('Average Expropriation Risk 1985-95')
ax.set_title('Figure 3: First-stage relationship between settler mortality \
    and expropriation risk')
plt.show()
```




The second condition may not be satisfied if settler mortality rates in the 17th to 19th centuries have a direct effect on current GDP (in addition to their indirect effect through institutions).

For example, settler mortality rates may be related to the current disease environment in a country, which could affect current economic performance.

[AJR01] argue this is unlikely because:

- The majority of settler deaths were due to malaria and yellow fever and had a limited effect on local people.
- The disease burden on local people in Africa or India, for example, did not appear to be higher than average, supported by relatively high population densities in these areas before colonization.

As we appear to have a valid instrument, we can use 2SLS regression to obtain consistent and unbiased parameter estimates.

**First stage**

The first stage involves regressing the endogenous variable ($avexpr_i$) on the instrument.

The instrument is the set of all exogenous variables in our model (and not just the variable we have replaced).

Using model 1 as an example, our instrument is simply a constant and settler mortality rates $logem4_i$.

Therefore, we will estimate the first-stage regression as

$$avexpr_i = \delta_0 + \delta_1 logem4_i + v_i$$

The data we need to estimate this equation is located in `maketable4.dta` (only complete data, indicated by `baseco = 1`, is used for estimation)

```
# Import and select the data
df4 = pd.read_stata('https://github.com/QuantEcon/lecture-python/blob/master/source/_
↪static/lecture_specific/ols/maketable4.dta?raw=true')
df4 = df4[df4['baseco'] == 1]

# Add a constant variable
df4['const'] = 1

# Fit the first stage regression and print summary
results_fs = sm.OLS(df4['avexpr'],
                    df4[['const', 'logem4']],
                    missing='drop').fit()
print(results_fs.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                 avexpr   R-squared:                       0.270
Model:                            OLS   Adj. R-squared:                  0.258
Method:                 Least Squares   F-statistic:                     22.95
Date:                Thu, 07 Oct 2021   Prob (F-statistic):           1.08e-05
Time:                        21:22:25   Log-Likelihood:                -104.83
No. Observations:                  64   AIC:                             213.7
Df Residuals:                      62   BIC:                             218.0
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          9.3414      0.611     15.296      0.000       8.121      10.562
logem4        -0.6068      0.127     -4.790      0.000      -0.860      -0.354
==============================================================================
Omnibus:                        0.035   Durbin-Watson:                   2.003
Prob(Omnibus):                  0.983   Jarque-Bera (JB):                0.172
Skew:                           0.045   Prob(JB):                        0.918
Kurtosis:                       2.763   Cond. No.                         19.4
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
↪specified.
```

**Second stage**

We need to retrieve the predicted values of $avexpr_i$ using `.predict()`.

We then replace the endogenous variable $avexpr_i$ with the predicted values $\widehat{avexpr}_i$ in the original linear model.

Our second stage regression is thus

$$logpgp95_i = \beta_0 + \beta_1 \widehat{avexpr}_i + u_i$$

```
df4['predicted_avexpr'] = results_fs.predict()

results_ss = sm.OLS(df4['logpgp95'],
                    df4[['const', 'predicted_avexpr']]).fit()
print(results_ss.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:               logpgp95   R-squared:                       0.477
Model:                            OLS   Adj. R-squared:                  0.469
Method:                 Least Squares   F-statistic:                     56.60
Date:                Thu, 07 Oct 2021   Prob (F-statistic):           2.66e-10
Time:                        21:22:25   Log-Likelihood:                -72.268
No. Observations:                  64   AIC:                             148.5
Df Residuals:                      62   BIC:                             152.9
Df Model:                           1
Covariance Type:            nonrobust
====================================================================================
                       coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------------
const                1.9097      0.823      2.320      0.024       0.264       3.555
predicted_avexpr     0.9443      0.126      7.523      0.000       0.693       1.195
==============================================================================
Omnibus:                       10.547   Durbin-Watson:                   2.137
Prob(Omnibus):                  0.005   Jarque-Bera (JB):               11.010
Skew:                          -0.790   Prob(JB):                      0.00407
Kurtosis:                       4.277   Cond. No.                         58.1
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
 specified.
```

The second-stage regression results give us an unbiased and consistent estimate of the effect of institutions on economic outcomes.

The result suggests a stronger positive relationship than what the OLS results indicated.

Note that while our parameter estimates are correct, our standard errors are not and for this reason, computing 2SLS 'manually' (in stages with OLS) is not recommended.

We can correctly estimate a 2SLS regression in one step using the linearmodels package, an extension of `statsmodels`

Note that when using `IV2SLS`, the exogenous and instrument variables are split up in the function arguments (whereas before the instrument included exogenous variables)

```
iv = IV2SLS(dependent=df4['logpgp95'],
            exog=df4['const'],
            endog=df4['avexpr'],
            instruments=df4['logem4']).fit(cov_type='unadjusted')

print(iv.summary)
```

```
                          IV-2SLS Estimation Summary
==============================================================================
Dep. Variable:               logpgp95   R-squared:                      0.1870
Estimator:                    IV-2SLS   Adj. R-squared:                 0.1739
No. Observations:                  64   F-statistic:                    37.568
```

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```
Date:                  Thu, Oct 07 2021   P-value (F-stat)                0.0000
Time:                          21:22:25   Distribution:                  chi2(1)
Cov. Estimator:              unadjusted

                             Parameter Estimates
==============================================================================
            Parameter  Std. Err.     T-stat    P-value    Lower CI    Upper CI
------------------------------------------------------------------------------
const          1.9097     1.0106     1.8897     0.0588     -0.0710      3.8903
avexpr         0.9443     0.1541     6.1293     0.0000      0.6423      1.2462
==============================================================================

Endogenous: avexpr
Instruments: logem4
Unadjusted Covariance (Homoskedastic)
Debiased: False
```

Given that we now have consistent and unbiased estimates, we can infer from the model we have estimated that institutional differences (stemming from institutions set up during colonization) can help to explain differences in income levels across countries today.

[AJR01] use a marginal effect of 0.94 to calculate that the difference in the index between Chile and Nigeria (ie. institutional quality) implies up to a 7-fold difference in income, emphasizing the significance of institutions in economic development.

## 63.5 Summary

We have demonstrated basic OLS and 2SLS regression in `statsmodels` and `linearmodels`.

If you are familiar with R, you may want to use the formula interface to `statsmodels`, or consider using r2py to call R from within Python.

## 63.6 Exercises

### 63.6.1 Exercise 1

In the lecture, we think the original model suffers from endogeneity bias due to the likely effect income has on institutional development.

Although endogeneity is often best identified by thinking about the data and model, we can formally test for endogeneity using the **Hausman test**.

We want to test for correlation between the endogenous variable, $avexpr_i$, and the errors, $u_i$

$$H_0 : Cov(avexpr_i, u_i) = 0 \quad (no\ endogeneity)$$
$$H_1 : Cov(avexpr_i, u_i) \neq 0 \quad (endogeneity)$$

This test is running in two stages.

First, we regress $avexpr_i$ on the instrument, $logem4_i$

$$avexpr_i = \pi_0 + \pi_1 logem4_i + \upsilon_i$$

Second, we retrieve the residuals $\hat{\upsilon}_i$ and include them in the original equation

$$logpgp95_i = \beta_0 + \beta_1 avexpr_i + \alpha\hat{\upsilon}_i + u_i$$

If $\alpha$ is statistically significant (with a p-value < 0.05), then we reject the null hypothesis and conclude that $avexpr_i$ is endogenous.

Using the above information, estimate a Hausman test and interpret your results.

### 63.6.2 Exercise 2

The OLS parameter $\beta$ can also be estimated using matrix algebra and `numpy` (you may need to review the numpy lecture to complete this exercise).

The linear equation we want to estimate is (written in matrix form)

$$y = X\beta + u$$

To solve for the unknown parameter $\beta$, we want to minimize the sum of squared residuals

$$\min_{\hat{\beta}} \hat{u}'\hat{u}$$

Rearranging the first equation and substituting into the second equation, we can write

$$\min_{\hat{\beta}} (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

Solving this optimization problem gives the solution for the $\hat{\beta}$ coefficients

$$\hat{\beta} = (X'X)^{-1}X'y$$

Using the above information, compute $\hat{\beta}$ from model 1 using `numpy` - your results should be the same as those in the `statsmodels` output from earlier in the lecture.

## 63.7 Solutions

### 63.7.1 Exercise 1

```
# Load in data
df4 = pd.read_stata('https://github.com/QuantEcon/lecture-python/blob/master/source/_
↪static/lecture_specific/ols/maketable4.dta?raw=true')

# Add a constant term
df4['const'] = 1

# Estimate the first stage regression
reg1 = sm.OLS(endog=df4['avexpr'],
              exog=df4[['const', 'logem4']],
              missing='drop').fit()

# Retrieve the residuals
df4['resid'] = reg1.resid
```

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```
# Estimate the second stage residuals
reg2 = sm.OLS(endog=df4['logpgp95'],
              exog=df4[['const', 'avexpr', 'resid']],
              missing='drop').fit()

print(reg2.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:               logpgp95   R-squared:                       0.689
Model:                            OLS   Adj. R-squared:                  0.679
Method:                 Least Squares   F-statistic:                     74.05
Date:                Thu, 07 Oct 2021   Prob (F-statistic):           1.07e-17
Time:                        21:22:25   Log-Likelihood:                -62.031
No. Observations:                  70   AIC:                             130.1
Df Residuals:                      67   BIC:                             136.8
Df Model:                           2
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          2.4782      0.547      4.530      0.000       1.386       3.570
avexpr         0.8564      0.082     10.406      0.000       0.692       1.021
resid         -0.4951      0.099     -5.017      0.000      -0.692      -0.298
==============================================================================
Omnibus:                       17.597   Durbin-Watson:                   2.086
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               23.194
Skew:                          -1.054   Prob(JB):                     9.19e-06
Kurtosis:                       4.873   Cond. No.                         53.8
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly␣
↪specified.
```

The output shows that the coefficient on the residuals is statistically significant, indicating $avexpr_i$ is endogenous.

### 63.7.2 Exercise 2

```
# Load in data
df1 = pd.read_stata('https://github.com/QuantEcon/lecture-python/blob/master/source/_
↪static/lecture_specific/ols/maketable1.dta?raw=true')
df1 = df1.dropna(subset=['logpgp95', 'avexpr'])

# Add a constant term
df1['const'] = 1

# Define the X and y variables
y = np.asarray(df1['logpgp95'])
X = np.asarray(df1[['const', 'avexpr']])

# Compute β_hat
β_hat = np.linalg.solve(X.T @ X, X.T @ y)
```

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```
# Print out the results from the 2 x 1 vector β_hat
print(f'β_0 = {β_hat[0]:.2}')
print(f'β_1 = {β_hat[1]:.2}')
```

```
β_0 = 4.6
β_1 = 0.53
```

It is also possible to use `np.linalg.inv(X.T @ X) @ X.T @ y` to solve for $\beta$, however `.solve()` is preferred as it involves fewer computations.

CHAPTER

SIXTYFOUR

# MAXIMUM LIKELIHOOD ESTIMATION

**Contents**



## 64.1 Overview

In a *previous lecture*, we estimated the relationship between dependent and explanatory variables using linear regression.

But what if a linear relationship is not an appropriate assumption for our model?

One widely used alternative is maximum likelihood estimation, which involves specifying a class of distributions, indexed by unknown parameters, and then using the data to pin down these parameter values.

The benefit relative to linear regression is that it allows more flexibility in the probabilistic relationships between variables.

Here we illustrate maximum likelihood by replicating Daniel Treisman's (2016) paper, Russia's Billionaires, which connects the number of billionaires in a country to its economic characteristics.

The paper concludes that Russia has a higher number of billionaires than economic factors such as market size and tax rate predict.

We'll require the following imports:

```
%matplotlib inline
import matplotlib.pyplot as plt
plt.rcParams["figure.figsize"] = (11, 5)  #set default figure size
import numpy as np
```

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```
from numpy import exp
from scipy.special import factorial
import pandas as pd
from mpl_toolkits.mplot3d import Axes3D
import statsmodels.api as sm
from statsmodels.api import Poisson
from scipy import stats
from scipy.stats import norm
from statsmodels.iolib.summary2 import summary_col
```

### 64.1.1 Prerequisites

We assume familiarity with basic probability and multivariate calculus.

## 64.2 Set Up and Assumptions

Let's consider the steps we need to go through in maximum likelihood estimation and how they pertain to this study.

### 64.2.1 Flow of Ideas

The first step with maximum likelihood estimation is to choose the probability distribution believed to be generating the data.

More precisely, we need to make an assumption as to which *parametric class* of distributions is generating the data.

- e.g., the class of all normal distributions, or the class of all gamma distributions.

Each such class is a family of distributions indexed by a finite number of parameters.

- e.g., the class of normal distributions is a family of distributions indexed by its mean $\mu \in (-\infty, \infty)$ and standard deviation $\sigma \in (0, \infty)$.

We'll let the data pick out a particular element of the class by pinning down the parameters.

The parameter estimates so produced will be called **maximum likelihood estimates**.

### 64.2.2 Counting Billionaires

Treisman [Tre16] is interested in estimating the number of billionaires in different countries.

The number of billionaires is integer-valued.

Hence we consider distributions that take values only in the nonnegative integers.

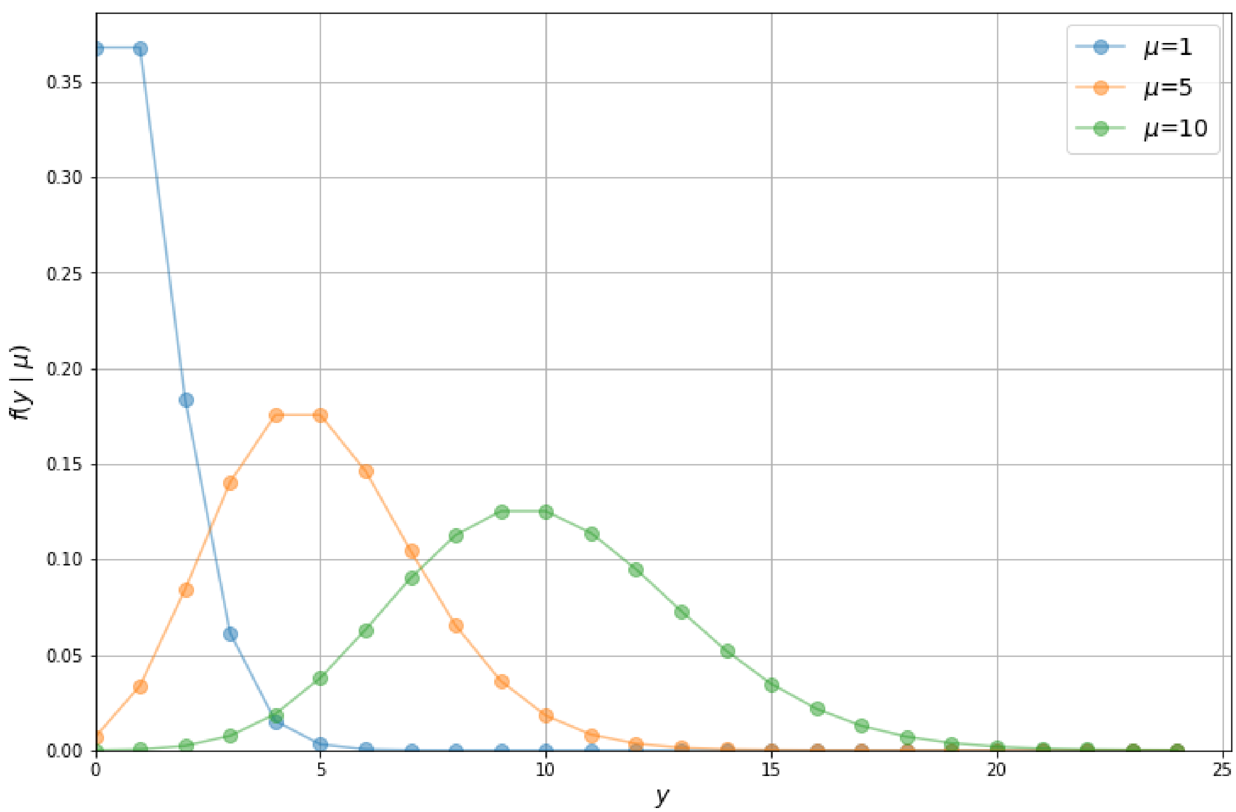
(This is one reason least squares regression is not the best tool for the present problem, since the dependent variable in linear regression is not restricted to integer values)

One integer distribution is the Poisson distribution, the probability mass function (pmf) of which is

$$f(y) = \frac{\mu^y}{y!} e^{-\mu}, \qquad y = 0, 1, 2, \ldots, \infty$$

We can plot the Poisson distribution over $y$ for different values of $\mu$ as follows

```
poisson_pmf = lambda y, μ: μ**y / factorial(y) * exp(-μ)
y_values = range(0, 25)

fig, ax = plt.subplots(figsize=(12, 8))

for μ in [1, 5, 10]:
    distribution = []
    for y_i in y_values:
        distribution.append(poisson_pmf(y_i, μ))
    ax.plot(y_values,
            distribution,
            label=f'$\mu$={μ}',
            alpha=0.5,
            marker='o',
            markersize=8)

ax.grid()
ax.set_xlabel('$y$', fontsize=14)
ax.set_ylabel('$f(y \mid \mu)$', fontsize=14)
ax.axis(xmin=0, ymin=0)
ax.legend(fontsize=14)

plt.show()
```




Notice that the Poisson distribution begins to resemble a normal distribution as the mean of $y$ increases.

Let's have a look at the distribution of the data we'll be working with in this lecture.

Treisman's main source of data is *Forbes'* annual rankings of billionaires and their estimated net worth.

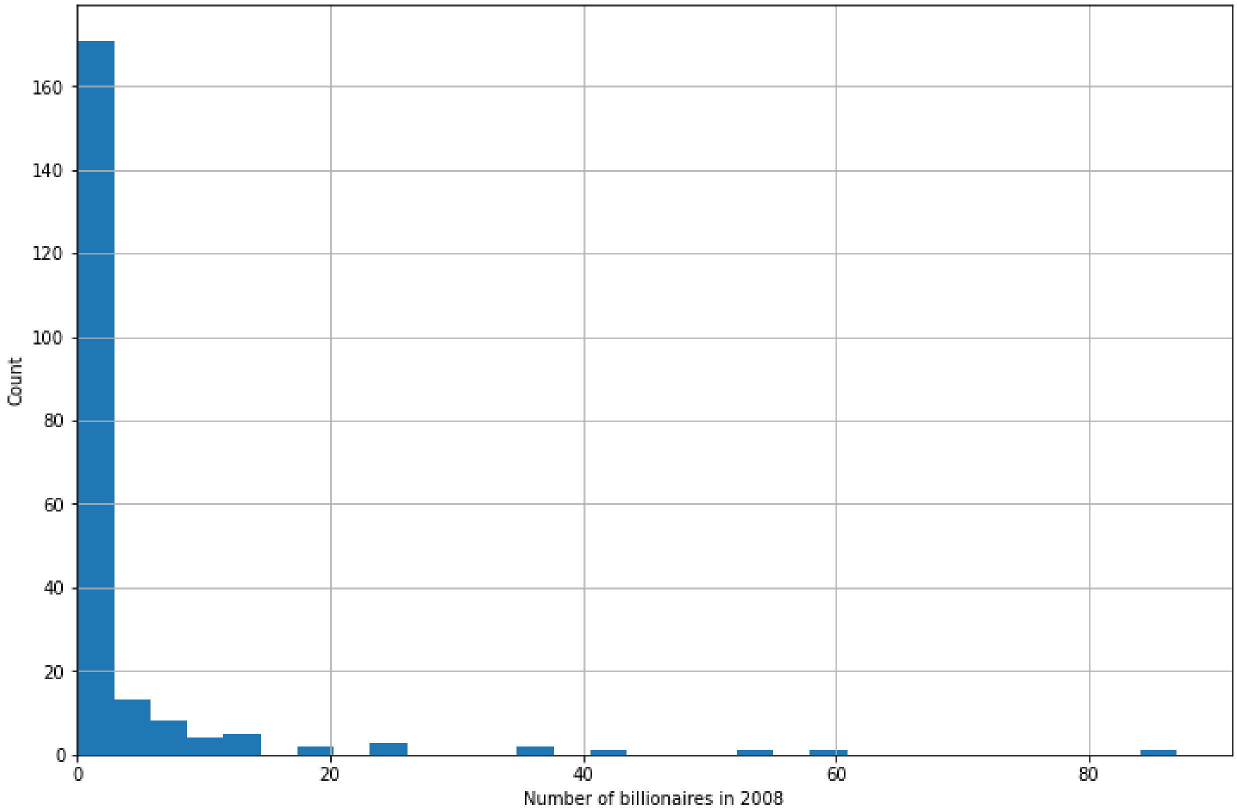
The dataset `mle/fp.dta` can be downloaded from here or its AER page.

```
pd.options.display.max_columns = 10

# Load in data and view
df = pd.read_stata('https://github.com/QuantEcon/lecture-python/blob/master/source/_
↪static/lecture_specific/mle/fp.dta?raw=true')
df.head()
```

```
         country  ccode    year    cyear  numbil  ...   topint08     rintr  \
0  United States    2.0  1990.0  21990.0     NaN  ...  39.799999  4.988405
1  United States    2.0  1991.0  21991.0     NaN  ...  39.799999  4.988405
2  United States    2.0  1992.0  21992.0     NaN  ...  39.799999  4.988405
3  United States    2.0  1993.0  21993.0     NaN  ...  39.799999  4.988405
4  United States    2.0  1994.0  21994.0     NaN  ...  39.799999  4.988405

   noyrs  roflaw  nrrents
0   20.0    1.61      NaN
1   20.0    1.61      NaN
2   20.0    1.61      NaN
3   20.0    1.61      NaN
4   20.0    1.61      NaN

[5 rows x 36 columns]
```

Using a histogram, we can view the distribution of the number of billionaires per country, `numbil0`, in 2008 (the United States is dropped for plotting purposes)

```
numbil0_2008 = df[(df['year'] == 2008) & (
    df['country'] != 'United States')].loc[:, 'numbil0']

plt.subplots(figsize=(12, 8))
plt.hist(numbil0_2008, bins=30)
plt.xlim(left=0)
plt.grid()
plt.xlabel('Number of billionaires in 2008')
plt.ylabel('Count')
plt.show()
```




From the histogram, it appears that the Poisson assumption is not unreasonable (albeit with a very low $\mu$ and some outliers).

## 64.3 Conditional Distributions

In Treisman's paper, the dependent variable — the number of billionaires $y_i$ in country $i$ — is modeled as a function of GDP per capita, population size, and years membership in GATT and WTO.

Hence, the distribution of $y_i$ needs to be conditioned on the vector of explanatory variables $\mathbf{x}_i$.

The standard formulation — the so-called *poisson regression* model — is as follows:

$$f(y_i \mid \mathbf{x}_i) = \frac{\mu_i^{y_i}}{y_i!} e^{-\mu_i}; \qquad y_i = 0, 1, 2, \ldots, \infty. \tag{1}$$

$$\text{where } \mu_i = \exp(\mathbf{x}_i'\beta) = \exp(\beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik})$$

To illustrate the idea that the distribution of $y_i$ depends on $\mathbf{x}_i$ let's run a simple simulation.

We use our `poisson_pmf` function from above and arbitrary values for $\beta$ and $\mathbf{x}_i$

```
y_values = range(0, 20)

# Define a parameter vector with estimates
β = np.array([0.26, 0.18, 0.25, -0.1, -0.22])

# Create some observations X
datasets = [np.array([0, 1, 1, 1, 2]),
```

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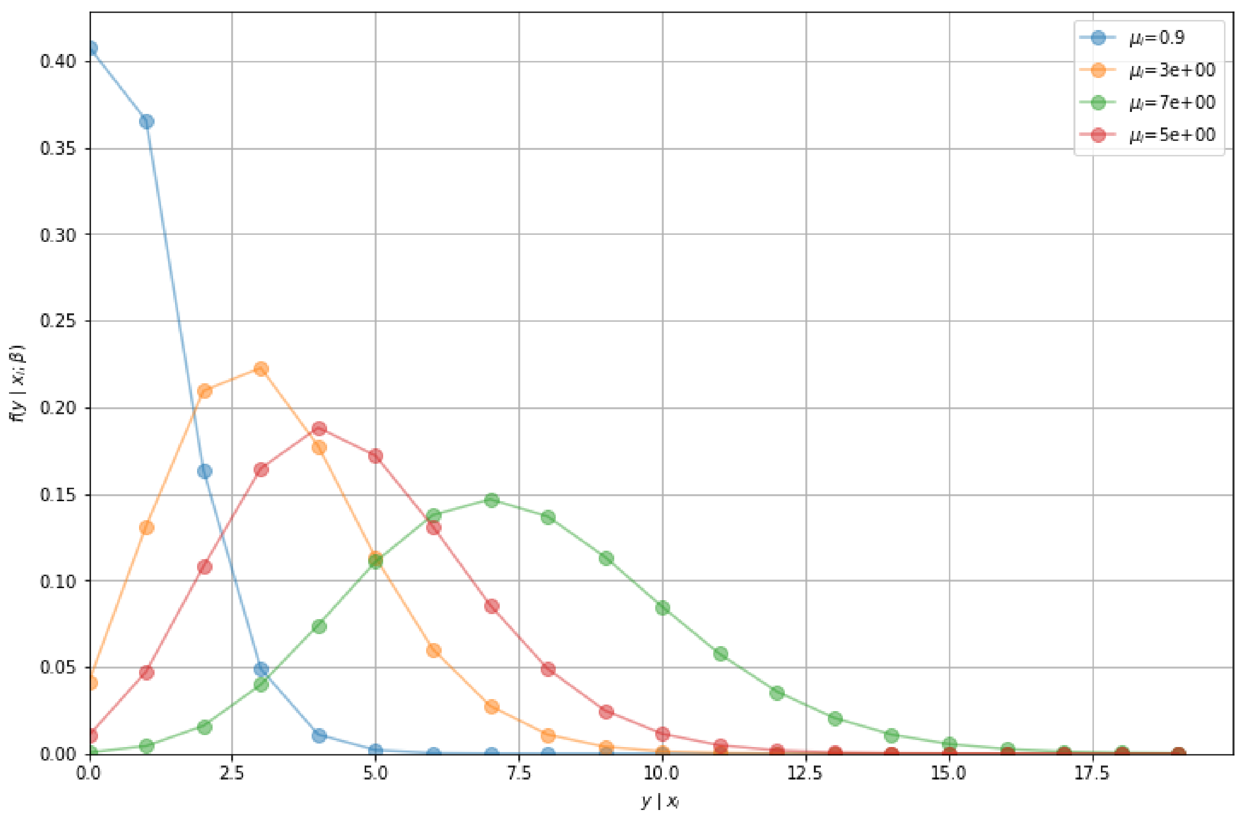
```
            np.array([2, 3, 2, 4, 0]),
            np.array([3, 4, 5, 3, 2]),
            np.array([6, 5, 4, 4, 7])]


fig, ax = plt.subplots(figsize=(12, 8))

for X in datasets:
    μ = exp(X @ β)
    distribution = []
    for y_i in y_values:
        distribution.append(poisson_pmf(y_i, μ))
    ax.plot(y_values,
            distribution,
            label=f'$\mu_i$={μ:.1}',
            marker='o',
            markersize=8,
            alpha=0.5)

ax.grid()
ax.legend()
ax.set_xlabel('$y \mid x_i$')
ax.set_ylabel(r'$f(y \mid x_i; \beta )$')
ax.axis(xmin=0, ymin=0)
plt.show()
```




We can see that the distribution of $y_i$ is conditional on $\mathbf{x}_i$ ($\mu_i$ is no longer constant).

## 64.4 Maximum Likelihood Estimation

In our model for number of billionaires, the conditional distribution contains 4 ($k = 4$) parameters that we need to estimate.

We will label our entire parameter vector as $\beta$ where

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

To estimate the model using MLE, we want to maximize the likelihood that our estimate $\hat{\beta}$ is the true parameter $\beta$.

Intuitively, we want to find the $\hat{\beta}$ that best fits our data.

First, we need to construct the likelihood function $\mathcal{L}(\beta)$, which is similar to a joint probability density function.

Assume we have some data $y_i = \{y_1, y_2\}$ and $y_i \sim f(y_i)$.

If $y_1$ and $y_2$ are independent, the joint pmf of these data is $f(y_1, y_2) = f(y_1) \cdot f(y_2)$.

If $y_i$ follows a Poisson distribution with $\lambda = 7$, we can visualize the joint pmf like so

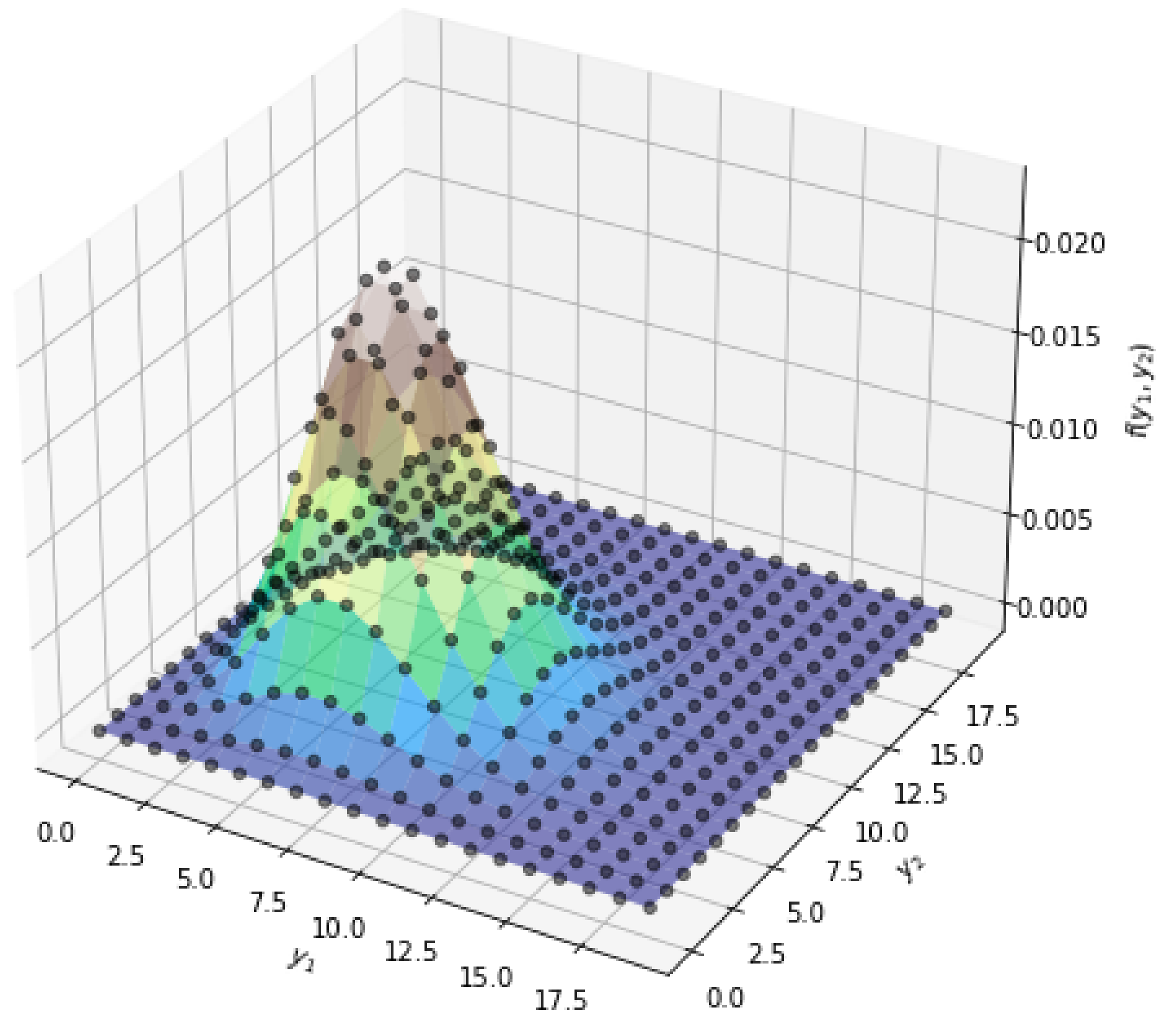
```
def plot_joint_poisson(μ=7, y_n=20):
    yi_values = np.arange(0, y_n, 1)

    # Create coordinate points of X and Y
    X, Y = np.meshgrid(yi_values, yi_values)

    # Multiply distributions together
    Z = poisson_pmf(X, μ) * poisson_pmf(Y, μ)

    fig = plt.figure(figsize=(12, 8))
    ax = fig.add_subplot(111, projection='3d')
    ax.plot_surface(X, Y, Z.T, cmap='terrain', alpha=0.6)
    ax.scatter(X, Y, Z.T, color='black', alpha=0.5, linewidths=1)
    ax.set(xlabel='$y_1$', ylabel='$y_2$')
    ax.set_zlabel('$f(y_1, y_2)$', labelpad=10)
    plt.show()

plot_joint_poisson(μ=7, y_n=20)
```




Similarly, the joint pmf of our data (which is distributed as a conditional Poisson distribution) can be written as

$$f(y_1, y_2, \ldots, y_n \mid \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; \beta) = \prod_{i=1}^{n} \frac{\mu_i^{y_i}}{y_i!} e^{-\mu_i}$$

$y_i$ is conditional on both the values of $\mathbf{x}_i$ and the parameters $\beta$.

The likelihood function is the same as the joint pmf, but treats the parameter $\beta$ as a random variable and takes the observations $(y_i, \mathbf{x}_i)$ as given

$$\begin{aligned}\mathcal{L}(\beta \mid y_1, y_2, \ldots, y_n \; ; \; \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = & \prod_{i=1}^{n} \frac{\mu_i^{y_i}}{y_i!} e^{-\mu_i} \\ = & f(y_1, y_2, \ldots, y_n \mid \; \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; \beta)\end{aligned}$$

Now that we have our likelihood function, we want to find the $\hat{\beta}$ that yields the maximum likelihood value

$$\max_{\beta} \mathcal{L}(\beta)$$

In doing so it is generally easier to maximize the log-likelihood (consider differentiating $f(x) = x\exp(x)$ vs. $f(x) = \log(x) + x$).

Given that taking a logarithm is a monotone increasing transformation, a maximizer of the likelihood function will also be a maximizer of the log-likelihood function.

In our case the log-likelihood is

$$\begin{aligned}
\log \mathcal{L}(\beta) = & \log \Big( f(y_1; \beta) \cdot f(y_2; \beta) \cdot \ldots \cdot f(y_n; \beta) \Big) \\
&= \sum_{i=1}^{n} \log f(y_i; \beta) \\
&= \sum_{i=1}^{n} \log \Big( \frac{\mu_i^{y_i}}{y_i!} e^{-\mu_i} \Big) \\
&= \sum_{i=1}^{n} y_i \log \mu_i - \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \log y!
\end{aligned}$$

The MLE of the Poisson to the Poisson for $\hat{\beta}$ can be obtained by solving

$$\max_{\beta} \Big( \sum_{i=1}^{n} y_i \log \mu_i - \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \log y! \Big)$$

However, no analytical solution exists to the above problem – to find the MLE we need to use numerical methods.

## 64.5 MLE with Numerical Methods

Many distributions do not have nice, analytical solutions and therefore require numerical methods to solve for parameter estimates.

One such numerical method is the Newton-Raphson algorithm.

Our goal is to find the maximum likelihood estimate $\hat{\beta}$.

At $\hat{\beta}$, the first derivative of the log-likelihood function will be equal to 0.

Let's illustrate this by supposing

$$\log \mathcal{L}(\beta) = -(\beta - 10)^2 - 10$$

```
β = np.linspace(1, 20)
logL = -(β - 10) ** 2 - 10
dlogL = -2 * β + 20

fig, (ax1, ax2) = plt.subplots(2, sharex=True, figsize=(12, 8))

ax1.plot(β, logL, lw=2)
ax2.plot(β, dlogL, lw=2)

ax1.set_ylabel(r'$log \mathcal{L(\beta)}$',
               rotation=0,
               labelpad=35,
               fontsize=15)
ax2.set_ylabel(r'$\frac{dlog \mathcal{L(\beta)}}{d \beta}$ ',
               rotation=0,
```

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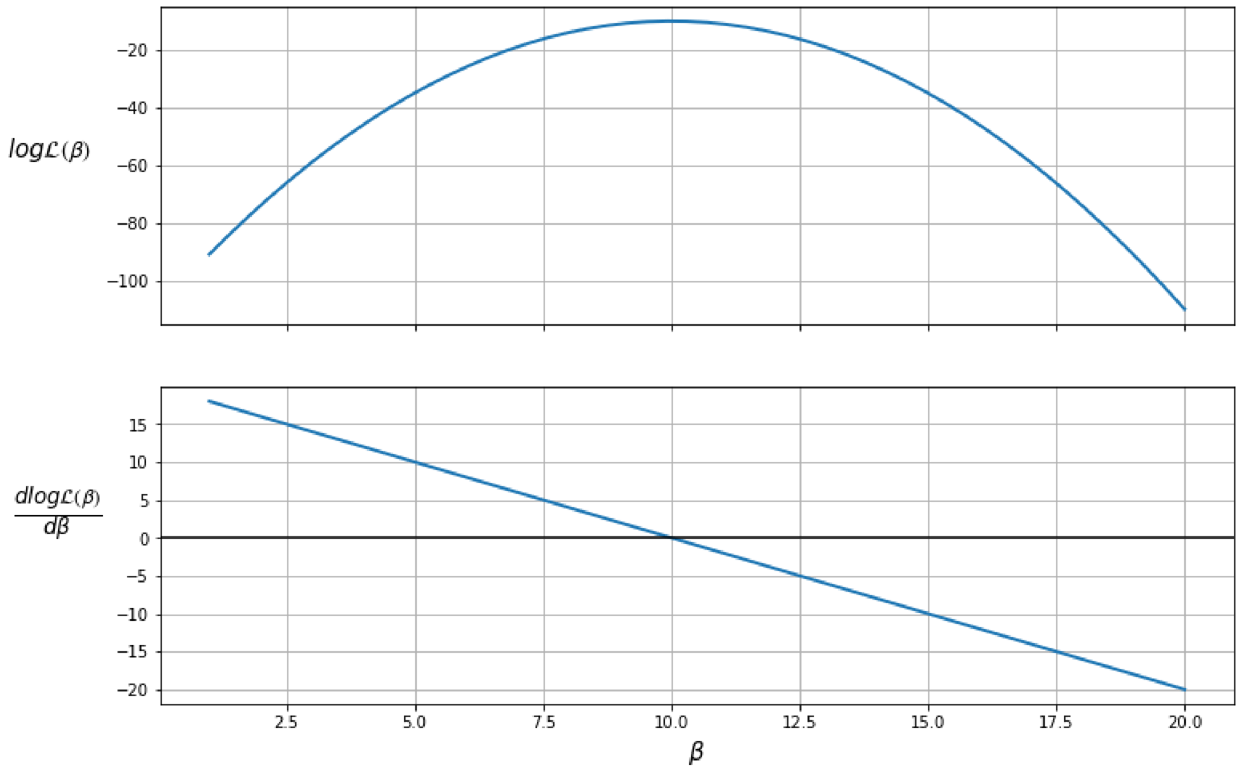
```
                labelpad=35,
                fontsize=19)
ax2.set_xlabel(r'$\beta$', fontsize=15)
ax1.grid(), ax2.grid()
plt.axhline(c='black')
plt.show()
```




The plot shows that the maximum likelihood value (the top plot) occurs when $\frac{d \log \mathcal{L}(\beta)}{d\beta} = 0$ (the bottom plot).

Therefore, the likelihood is maximized when $\beta = 10$.

We can also ensure that this value is a *maximum* (as opposed to a minimum) by checking that the second derivative (slope of the bottom plot) is negative.

The Newton-Raphson algorithm finds a point where the first derivative is 0.

To use the algorithm, we take an initial guess at the maximum value, $\beta_0$ (the OLS parameter estimates might be a reasonable guess), then

1. Use the updating rule to iterate the algorithm

$$\beta_{(k+1)} = \beta_{(k)} - H^{-1}(\beta_{(k)})G(\beta_{(k)})$$

   where:

$$G(\beta_{(k)}) = \frac{d \log \mathcal{L}(\beta_{(k)})}{d\beta_{(k)}}$$

$$H(\beta_{(k)}) = \frac{d^2 \log \mathcal{L}(\beta_{(k)})}{d\beta_{(k)} d\beta'_{(k)}}$$

2. Check whether $\beta_{(k+1)} - \beta_{(k)} < tol$

- If true, then stop iterating and set $\hat{\beta} = \beta_{(k+1)}$
- If false, then update $\beta_{(k+1)}$

As can be seen from the updating equation, $\beta_{(k+1)} = \beta_{(k)}$ only when $G(\beta_{(k)}) = 0$ ie. where the first derivative is equal to 0.

(In practice, we stop iterating when the difference is below a small tolerance threshold)

Let's have a go at implementing the Newton-Raphson algorithm.

First, we'll create a class called `PoissonRegression` so we can easily recompute the values of the log likelihood, gradient and Hessian for every iteration

```
class PoissonRegression:

    def __init__(self, y, X, β):
        self.X = X
        self.n, self.k = X.shape
        # Reshape y as a n_by_1 column vector
        self.y = y.reshape(self.n,1)
        # Reshape β as a k_by_1 column vector
        self.β = β.reshape(self.k,1)

    def μ(self):
        return np.exp(self.X @ self.β)

    def logL(self):
        y = self.y
        μ = self.μ()
        return np.sum(y * np.log(μ) - μ - np.log(factorial(y)))

    def G(self):
        y = self.y
        μ = self.μ()
        return X.T @ (y - μ)

    def H(self):
        X = self.X
        μ = self.μ()
        return -(X.T @ (μ * X))
```

Our function `newton_raphson` will take a `PoissonRegression` object that has an initial guess of the parameter vector $\beta_0$.

The algorithm will update the parameter vector according to the updating rule, and recalculate the gradient and Hessian matrices at the new parameter estimates.

Iteration will end when either:

- The difference between the parameter and the updated parameter is below a tolerance level.
- The maximum number of iterations has been achieved (meaning convergence is not achieved).

So we can get an idea of what's going on while the algorithm is running, an option `display=True` is added to print out values at each iteration.

```
def newton_raphson(model, tol=1e-3, max_iter=1000, display=True):

    i = 0
    error = 100  # Initial error value
```

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```
    # Print header of output
    if display:
        header = f'{"Iteration_k":<13}{"Log-likelihood":<16}{"θ":<60}'
        print(header)
        print("-" * len(header))

    # While loop runs while any value in error is greater
    # than the tolerance until max iterations are reached
    while np.any(error > tol) and i < max_iter:
        H, G = model.H(), model.G()
        β_new = model.β - (np.linalg.inv(H) @ G)
        error = β_new - model.β
        model.β = β_new

        # Print iterations
        if display:
            β_list = [f'{t:.3}' for t in list(model.β.flatten())]
            update = f'{i:<13}{model.logL():<16.8}{β_list}'
            print(update)

        i += 1

    print(f'Number of iterations: {i}')
    print(f'β_hat = {model.β.flatten()}')

    # Return a flat array for β (instead of a k_by_1 column vector)
    return model.β.flatten()
```

Let's try out our algorithm with a small dataset of 5 observations and 3 variables in $\mathbf{X}$.

```
X = np.array([[1, 2, 5],
              [1, 1, 3],
              [1, 4, 2],
              [1, 5, 2],
              [1, 3, 1]])

y = np.array([1, 0, 1, 1, 0])

# Take a guess at initial βs
init_β = np.array([0.1, 0.1, 0.1])

# Create an object with Poisson model values
poi = PoissonRegression(y, X, β=init_β)

# Use newton_raphson to find the MLE
β_hat = newton_raphson(poi, display=True)
```

```
Iteration_k  Log-likelihood  θ
-----------------------------------------------------------------------------
↪---
0            -4.3447622      ['-1.49', '0.265', '0.244']
1            -3.5742413      ['-3.38', '0.528', '0.474']
2            -3.3999526      ['-5.06', '0.782', '0.702']
3            -3.3788646      ['-5.92', '0.909', '0.82']
4            -3.3783559      ['-6.07', '0.933', '0.843']
```

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```
5           -3.3783555      ['-6.08', '0.933', '0.843']
Number of iterations: 6
β_hat = [-6.07848205  0.93340226  0.84329625]
```

As this was a simple model with few observations, the algorithm achieved convergence in only 6 iterations.

You can see that with each iteration, the log-likelihood value increased.

Remember, our objective was to maximize the log-likelihood function, which the algorithm has worked to achieve.

Also, note that the increase in $\log \mathcal{L}(\beta_{(k)})$ becomes smaller with each iteration.

This is because the gradient is approaching 0 as we reach the maximum, and therefore the numerator in our updating equation is becoming smaller.

The gradient vector should be close to 0 at $\hat{\beta}$

```
poi.G()
```

```
array([[-3.95169226e-07],
       [-1.00114804e-06],
       [-7.73114559e-07]])
```

The iterative process can be visualized in the following diagram, where the maximum is found at $\beta = 10$

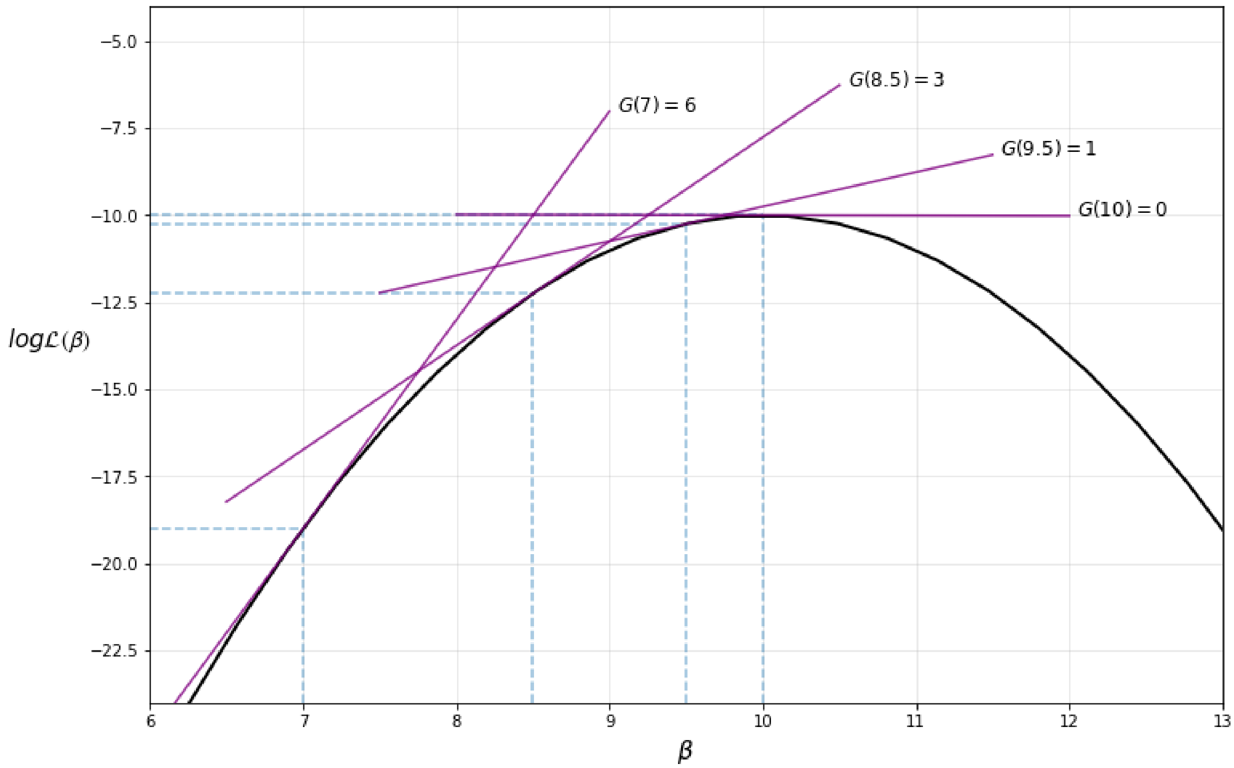
```
logL = lambda x: -(x - 10) ** 2 - 10

def find_tangent(β, a=0.01):
    y1 = logL(β)
    y2 = logL(β+a)
    x = np.array([[β, 1], [β+a, 1]])
    m, c = np.linalg.lstsq(x, np.array([y1, y2]), rcond=None)[0]
    return m, c

β = np.linspace(2, 18)
fig, ax = plt.subplots(figsize=(12, 8))
ax.plot(β, logL(β), lw=2, c='black')

for β in [7, 8.5, 9.5, 10]:
    β_line = np.linspace(β-2, β+2)
    m, c = find_tangent(β)
    y = m * β_line + c
    ax.plot(β_line, y, '-', c='purple', alpha=0.8)
    ax.text(β+2.05, y[-1], f'$G({β}) = {abs(m):.0f}$', fontsize=12)
    ax.vlines(β, -24, logL(β), linestyles='--', alpha=0.5)
    ax.hlines(logL(β), 6, β, linestyles='--', alpha=0.5)

ax.set(ylim=(-24, -4), xlim=(6, 13))
ax.set_xlabel(r'$\beta$', fontsize=15)
ax.set_ylabel(r'$log \mathcal{L(\beta)}$',
              rotation=0,
              labelpad=25,
              fontsize=15)
ax.grid(alpha=0.3)
plt.show()
```




Note that our implementation of the Newton-Raphson algorithm is rather basic — for more robust implementations see, for example, scipy.optimize.

## 64.6 Maximum Likelihood Estimation with `statsmodels`

Now that we know what's going on under the hood, we can apply MLE to an interesting application.

We'll use the Poisson regression model in `statsmodels` to obtain a richer output with standard errors, test values, and more.

`statsmodels` uses the same algorithm as above to find the maximum likelihood estimates.

Before we begin, let's re-estimate our simple model with `statsmodels` to confirm we obtain the same coefficients and log-likelihood value.

```
X = np.array([[1, 2, 5],
              [1, 1, 3],
              [1, 4, 2],
              [1, 5, 2],
              [1, 3, 1]])

y = np.array([1, 0, 1, 1, 0])

stats_poisson = Poisson(y, X).fit()
print(stats_poisson.summary())
```

```
Optimization terminated successfully.
         Current function value: 0.675671
         Iterations 7
```

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```
                          Poisson Regression Results
==============================================================================
Dep. Variable:                      y   No. Observations:                    5
Model:                        Poisson   Df Residuals:                        2
Method:                           MLE   Df Model:                            2
Date:                Thu, 07 Oct 2021   Pseudo R-squ.:                  0.2546
Time:                        21:25:01   Log-Likelihood:                -3.3784
converged:                       True   LL-Null:                       -4.5325
Covariance Type:            nonrobust   LLR p-value:                    0.3153
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -6.0785      5.279     -1.151      0.250     -16.425       4.268
x1             0.9334      0.829      1.126      0.260      -0.691       2.558
x2             0.8433      0.798      1.057      0.291      -0.720       2.407
==============================================================================
```

Now let's replicate results from Daniel Treisman's paper, Russia's Billionaires, mentioned earlier in the lecture.

Treisman starts by estimating equation (1), where:

- $y_i$ is $number\ of\ billionaires_i$
- $x_{i1}$ is $\log GDP\ per\ capita_i$
- $x_{i2}$ is $\log population_i$
- $x_{i3}$ is $years\ in\ GATT_i$ – years membership in GATT and WTO (to proxy access to international markets)

The paper only considers the year 2008 for estimation.

We will set up our variables for estimation like so (you should have the data assigned to `df` from earlier in the lecture)

```python
# Keep only year 2008
df = df[df['year'] == 2008]

# Add a constant
df['const'] = 1

# Variable sets
reg1 = ['const', 'lngdppc', 'lnpop', 'gattwto08']
reg2 = ['const', 'lngdppc', 'lnpop',
        'gattwto08', 'lnmcap08', 'rintr', 'topint08']
reg3 = ['const', 'lngdppc', 'lnpop', 'gattwto08', 'lnmcap08',
        'rintr', 'topint08', 'nrrents', 'roflaw']
```

Then we can use the `Poisson` function from `statsmodels` to fit the model.

We'll use robust standard errors as in the author's paper

```python
# Specify model
poisson_reg = sm.Poisson(df[['numbil0']], df[reg1],
                         missing='drop').fit(cov_type='HC0')
print(poisson_reg.summary())
```

```
Optimization terminated successfully.
         Current function value: 2.226090
         Iterations 9
                          Poisson Regression Results
```

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```
==============================================================================
Dep. Variable:                numbil0   No. Observations:                  197
Model:                        Poisson   Df Residuals:                      193
Method:                           MLE   Df Model:                            3
Date:                Thu, 07 Oct 2021   Pseudo R-squ.:                  0.8574
Time:                        21:25:01   Log-Likelihood:                -438.54
converged:                       True   LL-Null:                       -3074.7
Covariance Type:                  HC0   LLR p-value:                     0.000
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const        -29.0495      2.578    -11.268      0.000     -34.103     -23.997
lngdppc        1.0839      0.138      7.834      0.000       0.813       1.355
lnpop          1.1714      0.097     12.024      0.000       0.980       1.362
gattwto08      0.0060      0.007      0.868      0.386      -0.008       0.019
==============================================================================
```

Success! The algorithm was able to achieve convergence in 9 iterations.

Our output indicates that GDP per capita, population, and years of membership in the General Agreement on Tariffs and Trade (GATT) are positively related to the number of billionaires a country has, as expected.

Let's also estimate the author's more full-featured models and display them in a single table

```
regs = [reg1, reg2, reg3]
reg_names = ['Model 1', 'Model 2', 'Model 3']
info_dict = {'Pseudo R-squared': lambda x: f"{x.prsquared:.2f}",
             'No. observations': lambda x: f"{int(x.nobs):d}"}
regressor_order = ['const',
                   'lngdppc',
                   'lnpop',
                   'gattwto08',
                   'lnmcap08',
                   'rintr',
                   'topint08',
                   'nrrents',
                   'roflaw']
results = []

for reg in regs:
    result = sm.Poisson(df[['numbil0']], df[reg],
                        missing='drop').fit(cov_type='HC0',
                                            maxiter=100, disp=0)
    results.append(result)

results_table = summary_col(results=results,
                            float_format='%0.3f',
                            stars=True,
                            model_names=reg_names,
                            info_dict=info_dict,
                            regressor_order=regressor_order)
results_table.add_title('Table 1 - Explaining the Number of Billionaires \
                        in 2008')
print(results_table)
```

```
Table 1 - Explaining the Number of Billionaires                         in 2008
=================================================
```

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```
                    Model 1    Model 2    Model 3
-------------------------------------------------
const              -29.050*** -19.444*** -20.858***
                   (2.578)    (4.820)    (4.255)
lngdppc            1.084***   0.717***   0.737***
                   (0.138)    (0.244)    (0.233)
lnpop              1.171***   0.806***   0.929***
                   (0.097)    (0.213)    (0.195)
gattwto08          0.006      0.007      0.004
                   (0.007)    (0.006)    (0.006)
lnmcap08                      0.399**    0.286*
                              (0.172)    (0.167)
rintr                         -0.010     -0.009
                              (0.010)    (0.010)
topint08                      -0.051***  -0.058***
                              (0.011)    (0.012)
nrrents                                  -0.005
                                         (0.010)
roflaw                                   0.203
                                         (0.372)
Pseudo R-squared 0.86         0.90       0.90
No. observations 197          131        131
=================================================
Standard errors in parentheses.
* p<.1, ** p<.05, ***p<.01
```

The output suggests that the frequency of billionaires is positively correlated with GDP per capita, population size, stock market capitalization, and negatively correlated with top marginal income tax rate.

To analyze our results by country, we can plot the difference between the predicted an actual values, then sort from highest to lowest and plot the first 15

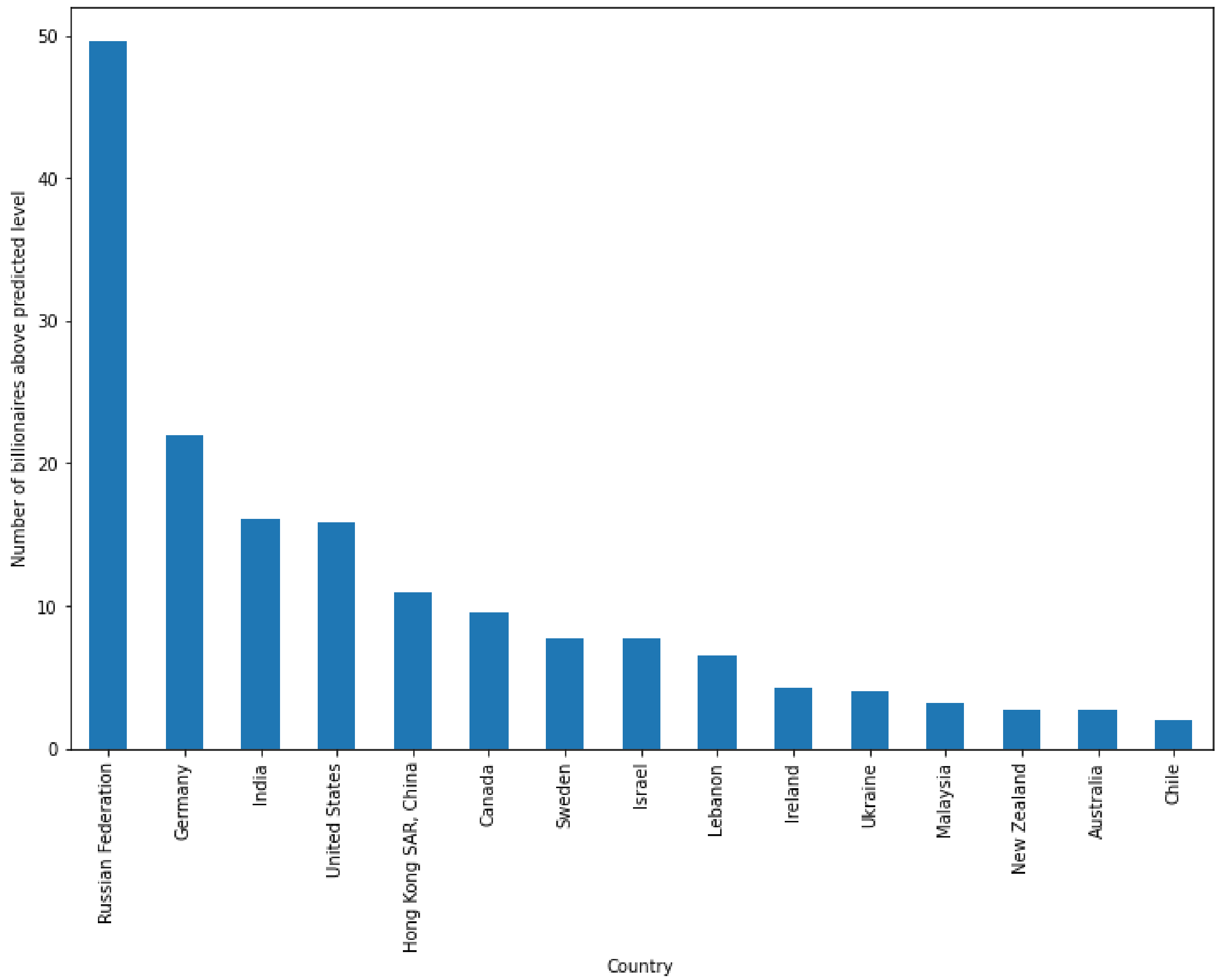
```
data = ['const', 'lngdppc', 'lnpop', 'gattwto08', 'lnmcap08', 'rintr',
        'topint08', 'nrrents', 'roflaw', 'numbil0', 'country']
results_df = df[data].dropna()

# Use last model (model 3)
results_df['prediction'] = results[-1].predict()

# Calculate difference
results_df['difference'] = results_df['numbil0'] - results_df['prediction']

# Sort in descending order
results_df.sort_values('difference', ascending=False, inplace=True)

# Plot the first 15 data points
results_df[:15].plot('country', 'difference', kind='bar',
                    figsize=(12,8), legend=False)
plt.ylabel('Number of billionaires above predicted level')
plt.xlabel('Country')
plt.show()
```




As we can see, Russia has by far the highest number of billionaires in excess of what is predicted by the model (around 50 more than expected).

Treisman uses this empirical result to discuss possible reasons for Russia's excess of billionaires, including the origination of wealth in Russia, the political climate, and the history of privatization in the years after the USSR.

## 64.7 Summary

In this lecture, we used Maximum Likelihood Estimation to estimate the parameters of a Poisson model.

`statsmodels` contains other built-in likelihood models such as Probit and Logit.

For further flexibility, `statsmodels` provides a way to specify the distribution manually using the `GenericLikelihoodModel` class - an example notebook can be found here.

## 64.8 Exercises

### 64.8.1 Exercise 1

Suppose we wanted to estimate the probability of an event $y_i$ occurring, given some observations.

We could use a probit regression model, where the pmf of $y_i$ is

$$
\begin{aligned}
f(y_i; \beta) = \mu_i^{y_i} (1-\mu_i)^{1-y_i}, \quad y_i = 0, 1 \\
\text{where} \quad \mu_i = \Phi(\mathbf{x}_i' \beta)
\end{aligned}
$$

$\Phi$ represents the *cumulative normal distribution* and constrains the predicted $y_i$ to be between 0 and 1 (as required for a probability).

$\beta$ is a vector of coefficients.

Following the example in the lecture, write a class to represent the Probit model.

To begin, find the log-likelihood function and derive the gradient and Hessian.

The `scipy` module `stats.norm` contains the functions needed to compute the cmf and pmf of the normal distribution.

### 64.8.2 Exercise 2

Use the following dataset and initial values of $\beta$ to estimate the MLE with the Newton-Raphson algorithm developed earlier in the lecture

$$
\mathbf{X} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 5 & 6 \\ 1 & 3 & 5 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \beta_{(0)} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}
$$

Verify your results with `statsmodels` - you can import the Probit function with the following import statement

```python
from statsmodels.discrete.discrete_model import Probit
```

Note that the simple Newton-Raphson algorithm developed in this lecture is very sensitive to initial values, and therefore you may fail to achieve convergence with different starting values.

## 64.9 Solutions

### 64.9.1 Exercise 1

The log-likelihood can be written as

$$
\log \mathcal{L} = \sum_{i=1}^{n} \left[ y_i \log \Phi(\mathbf{x}_i' \beta) + (1 - y_i) \log(1 - \Phi(\mathbf{x}_i' \beta)) \right]
$$

Using the **fundamental theorem of calculus**, the derivative of a cumulative probability distribution is its marginal distribution

$$
\frac{\partial}{\partial s} \Phi(s) = \phi(s)
$$

where $\phi$ is the marginal normal distribution.

The gradient vector of the Probit model is

$$\frac{\partial \log \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \left[ y_i \frac{\phi(\mathbf{x}_i'\beta)}{\Phi(\mathbf{x}_i'\beta)} - (1 - y_i) \frac{\phi(\mathbf{x}_i'\beta)}{1 - \Phi(\mathbf{x}_i'\beta)} \right] \mathbf{x}_i$$

The Hessian of the Probit model is

$$\frac{\partial^2 \log \mathcal{L}}{\partial \beta \partial \beta'} = -\sum_{i=1}^{n} \phi(\mathbf{x}_i'\beta) \left[ y_i \frac{\phi(\mathbf{x}_i'\beta) + \mathbf{x}_i'\beta \Phi(\mathbf{x}_i'\beta)}{[\Phi(\mathbf{x}_i'\beta)]^2} + (1 - y_i) \frac{\phi(\mathbf{x}_i'\beta) - \mathbf{x}_i'\beta(1 - \Phi(\mathbf{x}_i'\beta))}{[1 - \Phi(\mathbf{x}_i'\beta)]^2} \right] \mathbf{x}_i \mathbf{x}_i'$$

Using these results, we can write a class for the Probit model as follows

```python
class ProbitRegression:

    def __init__(self, y, X, β):
        self.X, self.y, self.β = X, y, β
        self.n, self.k = X.shape

    def μ(self):
        return norm.cdf(self.X @ self.β.T)

    def ϕ(self):
        return norm.pdf(self.X @ self.β.T)

    def logL(self):
        μ = self.μ()
        return np.sum(y * np.log(μ) + (1 - y) * np.log(1 - μ))

    def G(self):
        μ = self.μ()
        ϕ = self.ϕ()
        return np.sum((X.T * y * ϕ / μ - X.T * (1 - y) * ϕ / (1 - μ)),
                      axis=1)

    def H(self):
        X = self.X
        β = self.β
        μ = self.μ()
        ϕ = self.ϕ()
        a = (ϕ + (X @ β.T) * μ) / μ**2
        b = (ϕ - (X @ β.T) * (1 - μ)) / (1 - μ)**2
        return -(ϕ * (y * a + (1 - y) * b) * X.T) @ X
```

## 64.9.2 Exercise 2

```python
X = np.array([[1, 2, 4],
              [1, 1, 1],
              [1, 4, 3],
              [1, 5, 6],
              [1, 3, 5]])

y = np.array([1, 0, 1, 1, 0])

# Take a guess at initial βs
```

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```
β = np.array([0.1, 0.1, 0.1])

# Create instance of Probit regression class
prob = ProbitRegression(y, X, β)

# Run Newton-Raphson algorithm
newton_raphson(prob)
```

```
Iteration_k  Log-likelihood  θ
-----------------------------------------------------------------------------------------
↪---
0            -2.3796884      ['-1.34', '0.775', '-0.157']
1            -2.3687526      ['-1.53', '0.775', '-0.0981']
2            -2.3687294      ['-1.55', '0.778', '-0.0971']
3            -2.3687294      ['-1.55', '0.778', '-0.0971']
Number of iterations: 4
β_hat = [-1.54625858  0.77778952 -0.09709757]
```

```
array([-1.54625858,  0.77778952, -0.09709757])
```

```
# Use statsmodels to verify results

print(Probit(y, X).fit().summary())
```

```
Optimization terminated successfully.
         Current function value: 0.473746
         Iterations 6
                          Probit Regression Results
==============================================================================
Dep. Variable:                      y   No. Observations:                    5
Model:                         Probit   Df Residuals:                        2
Method:                           MLE   Df Model:                            2
Date:                Thu, 07 Oct 2021   Pseudo R-squ.:                  0.2961
Time:                        21:25:01   Log-Likelihood:                -2.3687
converged:                       True   LL-Null:                       -3.3651
Covariance Type:            nonrobust   LLR p-value:                    0.3692
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const         -1.5463      1.866     -0.829      0.407      -5.204       2.111
x1             0.7778      0.788      0.986      0.324      -0.768       2.323
x2            -0.0971      0.590     -0.165      0.869      -1.254       1.060
==============================================================================
```

# Part XI

# Other

CHAPTER

SIXTYFIVE

# TROUBLESHOOTING

**Contents**



This page is for readers experiencing errors when running the code from the lectures.

## 65.1 Fixing Your Local Environment

The basic assumption of the lectures is that code in a lecture should execute whenever

1. it is executed in a Jupyter notebook and
2. the notebook is running on a machine with the latest version of Anaconda Python.

You have installed Anaconda, haven't you, following the instructions in this lecture?

Assuming that you have, the most common source of problems for our readers is that their Anaconda distribution is not up to date.

Here's a useful article on how to update Anaconda.

Another option is to simply remove Anaconda and reinstall.

You also need to keep the external code libraries, such as QuantEcon.py up to date.

For this task you can either

- use conda install -y quantecon on the command line, or
- execute !conda install -y quantecon within a Jupyter notebook.

If your local environment is still not working you can do two things.

First, you can use a remote machine instead, by clicking on the Launch Notebook icon available for each lecture

Second, you can report an issue, so we can try to fix your local set up.

We like getting feedback on the lectures so please don't hesitate to get in touch.

## 65.2 Reporting an Issue

One way to give feedback is to raise an issue through our issue tracker.

Please be as specific as possible. Tell us where the problem is and as much detail about your local set up as you can provide.

Another feedback option is to use our discourse forum.

Finally, you can provide direct feedback to contact@quantecon.org