Part II

Linear Programming

CHAPTER

THIRTEEN

LINEAR PROGRAMMING

13.1 Overview

Linear programming problems either maximize or minimize a linear objective function subject to a set of linear equality and/or inequality constraints.

Linear programs come in pairs:

- an original **primal** problem, and
- an associated **dual** problem.

If a primal problem involves **maximization**, the dual problem involves **minimization**.

If a primal problem involves **minimization**, the dual problem involves **maximization**.

We provide a standard form of a linear program and methods to transform other forms of linear programming problems into a standard form.

We tell how to solve a linear programming prolem using **Scipy**.

We describe the important concept of complementary slackness and how it relates to the dual problem.

Let's start with some standard imports.

```
import numpy as np
from scipy.optimize import linprog
import matplotlib.pyplot as plt
from matplotlib.patches import Polygon
%matplotlib inline
```

13.2 Objective Function and Constraints

We want to minimize a **cost function** $c'x = \sum_{i=1}^{n} c_i x_i$ over feasible values of $x = (x_1, x_2, \ldots, x_n)'$.

Here

- $c = (c_1, c_2, \ldots, c_n)'$ is a **unit cost vector**, and
- $x = (x_1, x_2, \ldots, x_n)'$ is a vector of **decision variables**

Decision variables are restricted to satisfy a set of linear equality and/or inequality constraints.

We describe the constraints with the following collections of n-dimensional vectors a_i and scalars b_i and associated sets indexing the equality and inequality constraints:

- a_i for $i \in M_i$, where M_1, M_2, M_3 are each sets of indexes

and a collection of scalers

- b_i for $i \in N_i$, where N_1, N_2, N_3 are each sets of indexes.

A linear programming can be stated as [Ber97]:

$$
\begin{aligned}
\min_x \; & c'x \\
\text{subject to } \; & a_i'x \geq b_i, && i \in M_1 \\
& a_i'x \leq b_i, && i \in M_2 \\
& a_i'x = b_i, && i \in M_3 \\
& x_j \geq 0, && j \in N_1 \\
& x_j \leq 0, && j \in N_2 \\
& x_j \text{ unrestricted}, && j \in N_3
\end{aligned}
\tag{1}
$$

A vector x that satisfies all of the constraints is called a **feasible solution**.

A collection of all feasible solutions is called a **feasible set**.

A feasible solution x that minimizes the cost function is called an **optimal solution**.

The corresponding value of cost function $c'x$ is called the **optimal value**.

If the feasible set is empty, we say that solving the linear programming problem is **infeasible**.

If, for any $K \in \mathbb{R}$, there exists a feasible solution x such that $c'x < K$, we say that the problem is **unbounded** or equivalently that the optimal value is $-\infty$.

13.3 Example 1: Production Problem

This example was created by [Ber97]

Suppose that a factory can produce two goods called Product 1 and Product 2.

To produce each product requires both material and labor.

Selling each product generates revenue.

Required per unit material and labor inputs and revenues are shown in table below:

	Product 1	Product 2
Material	2	5
Labor	4	2
Revenue	3	4

30 units of material and 20 units of labor available.

A firm's problem is to construct a production plan that uses its 30 units of materials and 20 unites of labor to maximize its revenue.

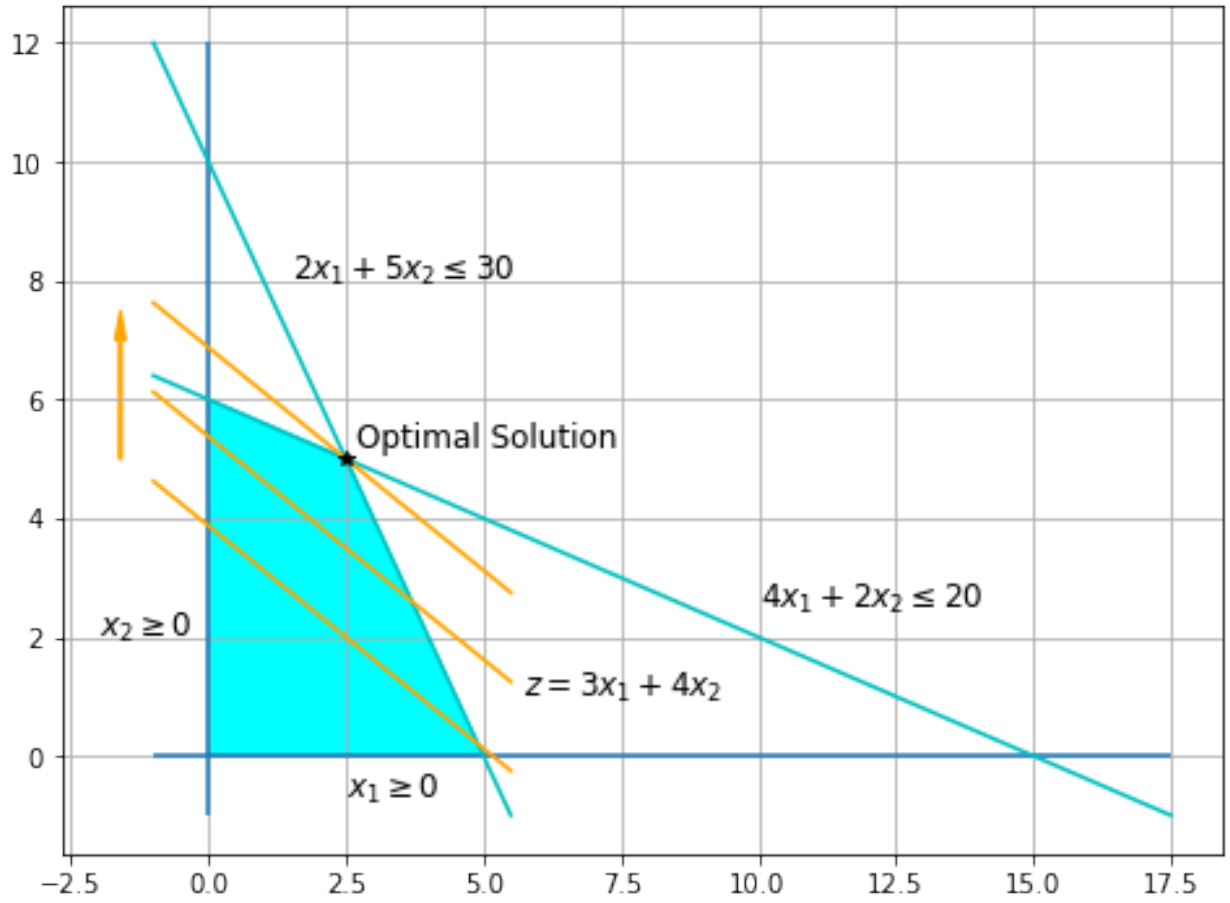
Let x_i denote the quantity of Product i that the firm produces.

This problem can be formulated as:

$$
\begin{aligned}
\max_{x_1, x_2} \; & z = 3x_1 + 4x_2 \\
\text{subject to } \; & 2x_1 + 5x_2 \leq 30 \\
& 4x_1 + 2x_2 \leq 20 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The following graph illustrates the firm's constraints and iso-revenue lines.

```
fig, ax = plt.subplots(figsize=(8, 6))
ax.grid()

# Draw constraint lines
ax.hlines(0, -1, 17.5)
ax.vlines(0, -1, 12)
ax.plot(np.linspace(-1, 17.5, 100), 6-0.4*np.linspace(-1, 17.5, 100), color="c")
ax.plot(np.linspace(-1, 5.5, 100), 10-2*np.linspace(-1, 5.5, 100), color="c")
ax.text(1.5, 8, "$2x_1 + 5x_2 \leq 30$", size=12)
ax.text(10, 2.5, "$4x_1 + 2x_2 \leq 20$", size=12)
ax.text(-2, 2, "$x_2 \geq 0$", size=12)
ax.text(2.5, -0.7, "$x_1 \geq 0$", size=12)

# Draw the feasible region
feasible_set = Polygon(np.array([[0, 0],
                                 [0, 6],
                                 [2.5, 5],
                                 [5, 0]]),
                       color="cyan")
ax.add_patch(feasible_set)

# Draw the objective function
ax.plot(np.linspace(-1, 5.5, 100), 3.875-0.75*np.linspace(-1, 5.5, 100), color="orange
 ↪")
ax.plot(np.linspace(-1, 5.5, 100), 5.375-0.75*np.linspace(-1, 5.5, 100), color="orange
 ↪")
ax.plot(np.linspace(-1, 5.5, 100), 6.875-0.75*np.linspace(-1, 5.5, 100), color="orange
 ↪")
ax.arrow(-1.6, 5, 0, 2, width = 0.05, head_width=0.2, head_length=0.5, color="orange")
ax.text(5.7, 1, "$z = 3x_1 + 4x_2$", size=12)

# Draw the optimal solution
ax.plot(2.5, 5, "*", color="black")
ax.text(2.7, 5.2, "Optimal Solution", size=12)

plt.show()
```



The blue region is the feasible set within which all constraints are satisfied.

Parallel orange lines are iso-revenue lines.

The firm's objective is to find the parallel orange lines to the upper boundary of the feasible set.

The intersection of the feasible set and the highest orange line delineates the optimal set.

In this example, the optimal set is the point $(2.5, 5)$.

13.4 Example 2: Investment Problem

We now consider a problem posed and solved by [Hu18].

A mutual fund has $\$100,000$ to be invested over a three year horizon.

Three investment options are available:

1. **Annuity:** the fund can pay a same amount of new capital at the beginning of each of three years and receive a payoff of 130% of **total capital** invested at the end of the third year. Once the mutual fund decides to invest in this annuity, it has to keep investing in all subsequent years in the three year horizon.
2. **Bank account:** the fund can deposit any amount into a bank at the beginning of each year and receive its capital plus 6% interest at the end of that year. In addition, the mutual fund is permitted to borrow no more than $20,000 at the beginning of each year and is asked to pay back the amount borrowed plus 6% interest at the end of the year. The mutual fund can choose whether to deposit or borrow at the beginning of each year.

3. **Corporate bond:** At the beginning of the second year, a corporate bond becomes available. The fund can buy an amount that is no more than \$50,000 of this bond at the beginning of the second year and at the end of the third year receive a payout of 130% of the amount invested in the bond.

The mutual fund's objective is to maximize total payout that it owns at the end of the third year.

We can formulate this as a linear programming problem.

Let x_1 be the amount of put in the annuity, x_2, x_3, x_4 be bank deposit balances at the beginning of the three years, and x_5 be the amount invested in the corporate bond.

When x_2, x_3, x_4 are negative, it means that the mutual fund has borrowed from bank.

The table below shows the mutual fund's decision variables together with the timing protocol described above:

	Year 1	Year 2	Year 3
Annuity	x_1	x_1	x_1
Bank account	x_2	x_3	x_4
Corporate bond	0	x_5	0

The mutual fund's decision making proceeds according to the following timing protocol:

1. At the beginning of the first year, the mutual fund decides how much to invest in the annuity and how much to deposit in the bank. This decision is subject to the constraint:

$$x_1 + x_2 = 100,000$$

2. At the beginning of the second year, the mutual fund has a bank balance of $1.06x_2$. It must keep x_1 in the annuity. It can choose to put x_5 into the corporate bond, and put x_3 in the bank. These decisions are restricted by

$$x_1 + x_5 = 1.06x_2 - x_3$$

3. At the beginning of the third year, the mutual fund has a bank account balance equal to $1.06x_3$. It must again invest x_1 in the annuity, leaving it with a bank account balance equal to x_4. This situation is summarized by the restriction:

$$x_1 = 1.06x_3 - x_4$$

The mutual fund's objective function, i.e., its wealth at the end of the third year is:

$$1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5$$

Thus, the mutual fund confronts the linear program:

$$\begin{aligned}
\max_x \quad & 1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5 \\
\text{subject to} \quad & x_1 + x_2 = 100,000 \\
& x_1 - 1.06x_2 + x_3 + x_5 = 0 \\
& x_1 - 1.06x_3 + x_4 = 0 \\
& x_2 \geq -20,000 \\
& x_3 \geq -20,000 \\
& x_4 \geq -20,000 \\
& x_5 \leq 50,000 \\
& x_j \geq 0, \quad j = 1,5 \\
& x_j \text{ unrestricted}, \quad j = 2,3,4
\end{aligned}$$

13.5 Standard Form

For purposes of

- unifying linear programs that are initially stated in superficially different forms, and
- having a form that is convenient to put into black-box software packages,

it is useful to devote some effort to describe a **standard form**.

Our standard form is:

$$\begin{aligned} \min_x \; & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \text{subject to } \; & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ & \quad \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The standard form LP problem can be expressed concisely as:

$$\begin{aligned} \min_x \; & c' x \\ \text{subject to } \; & Ax = b \\ & x >= 0 \end{aligned} \tag{2}$$

Here, $Ax = b$ means that the i-th entry of Ax equals the i-th entry of b for every i.

Similarly, $x >= 0$ means that x_j is greater than 0 for every j.

13.5.1 Useful Transformations

It is useful to know how to transform a problem that initially is not stated in the standard form into one that is.

By deploying the following steps, any linear programming problem can be transformed into an equivalent standard form linear programming problem.

1. **Objective Function:** If a problem is originally a constrained **maximization** problem, we can construct a new objective function that is the additive inverse of the original objective function. The transformed problem is then a **minimization** problem.
2. **Decision Variables:** Given a variable x_j satisfying $x_j \leq 0$, we can introduce a new variable $x'_j = -x_j$ and subsitute it into original problem. Given a free variable x_i with no restriction on its sign, we can introduce two new variables x_j^+ and x_j^- satisfying $x_j^+, x_j^- \geq 0$ and replace x_j by $x_j^+ - x_j^-$.
3. **Inequality constraints:** Given an inequality constraint $\sum_{j=1}^n a_{ij} x_j \leq 0$, we can introduce a new variable s_i, called a **slack variable** that satisfies $s_i \geq 0$ and replace the original constraint by $\sum_{j=1}^n a_{ij} x_j + s_i = 0$.

Let's apply the above steps to the two examples described above.

13.5.2 Example 1: Production Problem

The original problem is:

$$\begin{aligned} \max_{x_1, x_2} \ & 3x_1 + 4x_2 \\ \text{subject to } \ & 2x_1 + 5x_2 \leq 30 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

This problem is equivalent to the following problem with a standard form:

$$\begin{aligned} \min_{x_1, x_2} \ & -(3x_1 + 4x_2) \\ \text{subject to } \ & 2x_1 + 5x_2 + s_1 = 30 \\ & 4x_1 + 2x_2 + s_2 = 20 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

13.5.3 Example 2: Investment Problem

The original problem is:

$$\begin{aligned} \max_{x} \ & 1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5 \\ \text{subject to } \ & x_1 + x_2 = 100,000 \\ & x_1 - 1.06x_2 + x_3 + x_5 = 0 \\ & x_1 - 1.06x_3 + x_4 = 0 \\ & x_2 \geq -20,000 \\ & x_3 \geq -20,000 \\ & x_4 \geq -20,000 \\ & x_5 \leq 50,000 \\ & x_j \geq 0, \quad j = 1, 5 \\ & x_j \text{ unrestricted}, \quad j = 2, 3, 4 \end{aligned}$$

This problem is equivalent to the following problem with a standard form:

$$\begin{aligned} \min_{x} \ & -(1.30 \cdot 3x_1 + 1.06x_4^+ - 1.06x_4^- + 1.30x_5) \\ \text{subject to } \ & x_1 + x_2^+ - x_2^- = 100,000 \\ & x_1 - 1.06(x_2^+ - x_2^-) + x_3^+ - x_3^- + x_5 = 0 \\ & x_1 - 1.06(x_3^+ - x_3^-) + x_4^+ - x_4^- = 0 \\ & x_2^- - x_2^+ + s_1 = 20,000 \\ & x_3^- - x_3^+ + s_2 = 20,000 \\ & x_4^- - x_4^+ + s_3 = 20,000 \\ & x_5 + s_4 = 50,000 \\ & x_j \geq 0, \quad j = 1, 5 \\ & x_j^+, x_j^- \geq 0, \quad j = 2, 3, 4 \\ & s_j \geq 0, \quad j = 1, 2, 3, 4 \end{aligned}$$

13.6 Computations

The package *scipy.optimize* provides a function ***linprog*** to solve linear programming problems with a form below:

$$\begin{aligned} \min_x \ & c'x \\ \text{subject to } & A_{ub}x \leq b_{ub} \\ & A_{eq}x = b_{eq} \\ & l \leq x \leq u \end{aligned}$$

Note: By default $l = 0$ and $u =$ None unless explicitly specified with the argument 'bounds'.

Let's apply this great Python tool to solve our two example problems.

13.6.1 Example 1: Production Problem

The problem is:

$$\begin{aligned} \max_{x_1, x_2} \ & 3x_1 + 4x_2 \\ \text{subject to } & 2x_1 + 5x_2 \leq 30 \\ & 4x_1 + 2x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

```
# Construct parameters
c_ex1 = np.array([3, 4])

# Inequality constraints
A_ex1 = np.array([[2, 5],
                  [4, 2]])
b_ex1 = np.array([30,20])

# Solve the problem
# we put a negative sign on the objective as linprog does minimization
res_ex1 = linprog(-c_ex1, A_ub=A_ex1, b_ub=b_ex1, method='revised simplex')

res_ex1
```

```
     con: array([], dtype=float64)
     fun: -27.5
 message: 'Optimization terminated successfully.'
     nit: 2
   slack: array([0., 0.])
  status: 0
 success: True
       x: array([2.5, 5. ])
```

The optimal plan tells the factory to produce 2.5 units of Product 1 and 5 units of Product 2; that generates a maximizing value of revenue of 27.5.

We are using the *linprog* function as a **black box**.

Inside it, Python first transforms the problem into standard form.

To do that, for each inequality constraint it generates one slack variable.

Here the vector of slack variables is a two-dimensional numpy array that equals $b_{ub} - A_{ub}x$.

See official documentation for more details.

Note: This problem is to maximize the objective, so that we need to put a minus sign in front of parameter vector c.

13.6.2 Example 2: Investment Problem

The problem is:

$$
\begin{aligned}
\max_x \; & 1.30 \cdot 3x_1 + 1.06x_4 + 1.30x_5 \\
\text{subject to } \; & x_1 + x_2 = 100,000 \\
& x_1 - 1.06x_2 + x_3 + x_5 = 0 \\
& x_1 - 1.06x_3 + x_4 = 0 \\
& x_2 \geq -20,000 \\
& x_3 \geq -20,000 \\
& x_4 \geq -20,000 \\
& x_5 \leq 50,000 \\
& x_j \geq 0, \quad j = 1,5 \\
& x_j \text{ unrestricted}, \quad j = 2,3,4
\end{aligned}
$$

Let's solve this problem using *linprog*.

```
# Construct parameters
rate = 1.06

# Objective function parameters
c_ex2 = np.array([1.30*3, 0, 0, 1.06, 1.30])

# Inequality constraints
A_ex2 = np.array([[1,  1,  0,  0,  0],
                  [1, -rate, 1, 0, 1],
                  [1, 0, -rate, 1, 0]])
b_ex2 = np.array([100000, 0, 0])

# Bounds on decision variables
bounds_ex2 = [(  0,    None),
              (-20000, None),
              (-20000, None),
              (-20000, None),
              (  0,   50000)]

# Solve the problem
res_ex2 = linprog(-c_ex2, A_eq=A_ex2, b_eq=b_ex2,
                  bounds=bounds_ex2, method='revised simplex')

res_ex2
```

```
     con: array([ 1.45519152e-11, -2.18278728e-11,  0.00000000e+00])
     fun: -141018.24349792692
```

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```
 message: 'Optimization terminated successfully.'
     nit: 4
   slack: array([], dtype=float64)
  status: 0
 success: True
       x: array([ 24927.75474306,  75072.24525694,   4648.8252293 , -20000.        ,
        50000.        ])
```

Python tells us that the best investment strategy is:

1. At the beginning of the first year, the mutual fund should buy $\$24,927.75$ of the annuity. Its bank account balance should be $\$75,072.25$.
2. At the beginning of the second year, the mutual fund should buy $\$50,000$ of the corporate bond and keep invest in the annuity. Its bank account balance should be $\$4,648.83$.
3. At the beginning of the third year, the mutual fund should borrow $\$20,000$ from the bank and invest in the annuity.
4. At the end of the third year, the mutual fund will get payouts from the annuity and corporate bond and repay its loan from the bank. At the end it will own \$141018.24, so that it's total net rate of return over the three periods is 41.02%.

13.7 Duality

Associated with a linear programming of form (1) with m constraints and n decision variables, there is an **dual** linear programming problem that takes the form (please see [Ber97])

$$
\begin{aligned}
\max_{p} \; & b'p \\
\text{subject to } \; & p_i \geq 0, && i \in M_1 \\
& p_i \leq 0, && i \in M_2 \\
& p_i \text{ unrestricted}, && i \in M_3 \\
& A_j'p \leq c_j, && j \in N_1 \\
& A_j'p \geq c_j, && j \in N_2 \\
& A_j'p = c_j, && j \in N_3
\end{aligned}
$$

Where A_j is j-th column of the m by n matrix A.

Note: In what follows, we shall use a_i' to denote the i-th row of A and A_j to denote the j-th column of A.

$$
A = \begin{bmatrix} a_1' \\ a_2' \\ \\ a_m' \end{bmatrix}.
$$

To construct the dual of linear programming problem (1), we proceed as follows:

1. For every constraint $a_i'x \geq (\leq or =) b_i$, $j = 1, 2, ..., m$, in the primal problem, we construct a corresponding dual variable p_i. p_i is restricted to be positive if $a_i'x \geq b_i$ or negative if $a_i'x \leq b_i$ or unrestricted if $a_i'x = b_i$. We construct the m-dimensional vector p with entries p_i.

2. For every variable x_j, $j = 1, 2, ..., n$, we construct a corresponding dual constraint $A'_j p \geq (\leq \ or \ =) c_j$. The constraint is $A'_j p \geq c_j$ if $x_j \leq 0$, $A'_j p \leq c_j$ if $x_j \geq 0$ or $A'_j p = c_j$ if x_j is unrestricted.
3. The dual problem is to **maximize** objective function $b'p$.

For a **maximization** problem, we can first transform it to an equivalent minimization problem and then follow the above steps above to construct the dual **minimization** problem.

We can easily verify that **the dual of a dual problem is the primal problem**.

The following table summarizes relationships between objects in primal and dual problems.

Objective: Min	Objective: Max
m constraints	m variables
constraint $\geq$	variable ≥ 0
constraint $\leq$	variable ≤ 0
constraint $=$	variable free
n variables	n constraints
variable ≥ 0	constraint $\leq$
variable ≤ 0	constraint $\geq$
variable free	constraint $=$

As an example, the dual problem of the standard form (2) is:

$$\begin{aligned} \max_p \ & b'p \\ \text{subject to } \ & A'p \leq c \end{aligned}$$

As another example, consider a linear programming problem with form:

$$\begin{aligned} \max_x \ & c'x \\ \text{subject to } \ & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{3}$$

Its dual problem is:

$$\begin{aligned} \min_p \ & b'p \\ \text{subject to } \ & A'p \geq c \\ & p \geq 0 \end{aligned}$$

13.8 Duality Theorems

Primal and dual problems are linked by powerful **duality theorems** that have **weak** and **strong** forms.

The duality theorems provide the foundations of enlightening economic interpretations of linear programming problems.

Weak duality: For linear programming problem (1), if x and p are feasible solutions to the primal and the dual problems, respectively, then

$$b'p \leq c'x$$

Strong duality: For linear programming problem (1), if the primal problem has an optimal solution x, then the dual problem also has an optimal solution. Denote an optimal solution of the dual problem as p. Then

$$b'p = c'x$$

According to strong duality, we can find the optimal value for the primal problem by solving the dual problem.

But the dual problem tells us even more as we shall see next.

13.8.1 Complementary Slackness

Let x and p be feasible solutions to the primal problem (1) and its dual problem, respectively.

Then x and p are also optimal solutions of the primal and dual problems if and only if:

$$p_i(a_i'x - b_i) = 0, \quad \forall i,$$
$$x_j(A_j'p - c_j) = 0, \quad \forall j.$$

This means that $p_i = 0$ if $a_i'x - b_i \neq 0$ and $x_j = 0$ if $A_j'p - c_j \neq 0$.

These are the celebrated **complementary slackness** conditions.

Let's interpret them.

13.8.2 Interpretations

Let's take a version of problem (3) as a production problem and consider its associated dual problem.

A factory produce n products with m types of resources.

Where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, let

- x_j denote quantities of product j to be produced
- a_{ij} denote required amount of resource i to make one unit of product j,
- b_i denotes the avaliable amount of resource i
- c_j denotes the revenue generated by producing one unit of product j.

Dual variables: By strong duality, we have

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = b_1p_1 + b_2p_2 + \cdots + b_mp_m.$$

Evidently, a one unit change of b_i results in p_i units change of revenue.

Thus, a dual variable can be interpreted as the **value** of one unit of resource i.

This is why it is often called the **shadow price** of resource i.

For feasible but not optimal primal and dual solutions x and p, by weak duality, we have

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n < b_1p_1 + b_2p_2 + \cdots + b_mp_m.$$

Note: Here, the expression is opposite to the statement above since primal problem is a minimization problem.

When a strict inequality holds, the solution is not optimal because it doesn't fully utilize all valuable resources.

Evidently,

- if a shadow price p_i is larger than the market price for Resource i, the factory should buy more Resource i and expand its scale to generate more revenue;
- if a shadow price p_i is less than the market price for Resource i, the factory should sell its Resource i.

Complementary slackness: If there exists i such that $a_i' x - b_i < 0$ for some i, then $p_i = 0$ by complementary slackness. $a_i' x - b_i < 0$ means that to achieve its optimal production, the factory doesn't require as much Resource i as it has. It is reasonable that the shadow price of Resource i is 0: some of its resource i is redundant.

If there exists j such that $A_j' p - c_j > 0$, then $x_j = 0$ by complementary slackness. $A_j' p - c_j > 0$ means that the value of all resources used when producing one unit of product j is greater than its cost.

This means that producing another product that can more efficiently utilize these resources is a better choice than producing product j

Since producing product j is not optimal, x_j should equal 0.

13.8.3 Example 1: Production Problem

This problem is one specific instance of the problem (3), whose economic meaning is interpreted above.

Its dual problem is:

$$\begin{aligned} \min_{x_1, x_2} \ & 30p_1 + 20p_2 \\ \text{subject to } \ & 2p_1 + 4p_2 \geq 3 \\ & 5p_1 + 2p_2 \geq 4 \\ & p_1, p_2 \geq 0 \end{aligned}$$

We then solve this dual problem by function *linprog*. Since parameters used here are defined before when solving the primal problem, we don't need to define them here.

```
# Solve the dual problem
res_ex1_dual = linprog(b_ex1, A_ub=-A_ex1.T, b_ub=-c_ex1, method='revised simplex')

res_ex1_dual
```

```
     con: array([], dtype=float64)
     fun: 27.5
 message: 'Optimization terminated successfully.'
     nit: 2
   slack: array([0., 0.])
  status: 0
 success: True
       x: array([0.625 , 0.4375])
```

The optimal of the dual problem is 27.5, which is the same as the primal problem's. This matches the strong duality. The shadow prices for materials and labor are 0.625 and 0.4375, respectively.

13.8.4 Example 2: Investment Problem

The dual problem is:

$$
\begin{aligned}
\min_{p} \; & 100,000p_1 - 20,000p_4 - 20,000p_5 - 20,000p_6 + 50,000p_7 \\
\text{subject to } & p_1 + p_2 + p_3 \geq 1.30 \cdot 3 \\
& p_1 - 1.06p_2 + p_4 = 0 \\
& p_2 - 1.06p_3 + p_5 = 0 \\
& p_3 + p_6 = 1.06 \\
& p_2 + p_7 \geq 1.30 \\
& p_i \text{ unrestricted}, \quad i = 1, 2, 3 \\
& p_i \leq 0, \quad i = 4, 5, 6 \\
& p_7 \geq 0
\end{aligned}
$$

We then solve this dual problem by function *linprog*.

```
# Objective function parameters
c_ex2_dual = np.array([100000, 0, 0, -20000, -20000, -20000, 50000])

# Equality constraints
A_eq_ex2_dual = np.array([[1, -1.06,     0,  1,  0,  0,  0],
                          [0,     1, -1.06,  0,  1,  0,  0],
                          [0,     0,     1,  0,  0,  1,  0]])
b_eq_ex2_dual = np.array([0, 0, 1.06])

# Inequality constraints
A_ub_ex2_dual = - np.array([[1, 1, 1, 0, 0, 0, 0],
                           [0, 1, 0, 0, 0, 0, 1]])
b_ub_ex2_dual = - np.array([1.30*3, 1.30])

# Bounds on decision variables
bounds_ex2_dual = [(None, None),
                   (None, None),
                   (None, None),
                   (None,    0),
                   (None,    0),
                   (None,    0),
                   (   0, None)]

# Solve the dual problem
res_ex2_dual = linprog(c_ex2_dual, A_eq=A_eq_ex2_dual, b_eq=b_eq_ex2_dual,
                       A_ub=A_ub_ex2_dual, b_ub=b_ub_ex2_dual, bounds=bounds_ex2_dual,
                       method='revised simplex')

res_ex2_dual
```

```
     con: array([-2.22044605e-16,  0.00000000e+00,  0.00000000e+00])
     fun: 141018.2434979269
 message: 'Optimization terminated successfully.'
     nit: 4
   slack: array([0., 0.])
  status: 0
 success: True
       x: array([ 1.37644176,  1.29852997,  1.22502827,  0.        ,  0.        ,
       -0.16502827,  0.00147003])
```

The optimal value for the dual problem is 141018.24, which is the same as the primal problem's.

Now, let's interpret the dual variables.

By strong duality and also our numerical results, we have that optimal value is:

$$100,000p_1 - 20,000p_4 - 20,000p_5 - 20,000p_6 + 50,000p_7.$$

We know if b_i changes one dollor, then the optimal payoff in the end of the third year will change p_i dollars.

For $i = 1$, this means if the initial capital changes by one dollar, then the optimal payoff in the end of the third year will change p_1 dollars.

Thus, p_1 is the potential value of one more unit of initial capital, or the shadow price for initial capital.

We can also interpret p_1 as the prospective value in the end of the third year coming from having one more dollar to invest at the beginning of the first year.

If the mutual fund can raise money at a cost lower than $p_1 - 1$, then it should raise more money to increase its revenue.

But if it bears a cost of funds higher than $p_1 - 1$, it shouldn't do that.

For $i = 4, 5, 6$, this means, if the amount of capital that the fund is permitted to borrow from the bank changes by one dollar, the optimal pay out at the end of the third year will change p_i dollars.

Thus, for $i = 4, 5, 6$, $|p_i|$ reflects the value of one dollar that the mutual fund can borrow from the bank at the beginning of the $i - 3$-th year.

$|p_i|$ is the shadow price for the loan amount. (We use absolute value here since $p_i \leq 0$.)

If the interest rate is lower than $|p_i|$, then the mutual fund should borrow to increase its optimal payoff; if the interest rate is higher, it is better to not do this.

For $i = 7$, this means that if the amount of the corporate bond the mutual fund can buy changes one dollar, then the optimal payoff will change p_7 dollars at the end of the third year. Again, p_7 is the shadow price for the amount of the corporate bond the mutual fund can buy.

As for numerical results,

1. $p_1 = 1.38$, which means one dollar of initial capital is worth $\$1.38$ at the end of the third year.
2. $p_4 = p_5 = 0$, which means the loan amounts at the beginning of the first and second year are worth nothing. Recall that the optimal solution to the primal problem, $x_2, x_3 > 0$, which means at the beginning of the first and second year, the mutual fund has a postive bank account and borrows no capital from the bank. Thus, it is reasonable that the loan amounts at the beginning of the first and second year are valueless. This is what the complementary slackness conditions mean in this setting.
3. $p_6 = -0.16$, which means one dollar of the loan amount at the beginning of the third year is worth $\$0.16$. Since $|p_6|$ is higher than the interest rate 6%, the mutual fund should borrow as much as possible at the beginning of the third year. Recall that the optimal solution to the primal problem is $x_4 = -20,000$ which means the mutual fund borrows money from the bank as much as it can.
4. $p_7 = 0.0015$, which means one dollar of the amount of the corporate bond that the mutual fund can buy is worth $\$0.0015$.

CHAPTER

FOURTEEN

OPTIMAL TRANSPORT

14.1 Overview

The **transportation** or **optimal transport** problem is interesting both because of its many applications and its important role in the history of economic theory.

In this lecture, we describe the problem, tell how **linear programming** is a key tool for solving it, then provide some examples.

We will provide other applications in followup lectures.

The optimal transport problem was studied in early work about linear programming, as summarized for example by [DSS58]. A modern reference about applications in economics is [Gal16].

We shall solve our problems first by using the scipy function *linprog* and then the quantecon program *linprog_simplex*.

```
!pip install --upgrade quantecon
```

Let's start with some imports.

```
import numpy as np
from scipy.optimize import linprog
from quantecon.optimize import linprog_simplex
```

14.2 The Linear Programming Problem

Suppose that m factories produce goods that must be sent to n locations.

Let

- x_{ij} denote the quantity shipped from factory i to location j
- c_{ij} denote the cost of shipping one unit from factory i to location j
- p_i denote the capacity of factory i and q_j denote the amount required at location j.
- $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

A planner wants to minimize total transportation costs subject to the following constraints:

- The amount shipped **from** each factory must equal its capacity.
- The amount shipped **to** each location must equal the quantity required there.

The planner's problem can be expressed as the following constrained minimization problem:

$$\begin{aligned} \min_{x_{ij}} \ & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to } & \sum_{j=1}^{n} x_{ij} = p_i, \quad i = 1, 2, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} = q_j, \quad j = 1, 2, \ldots, n \\ & x_{ij} \geq 0 \end{aligned} \tag{1}$$

This is an **optimal transport problem** with

- mn decision variables, namely, the entries x_{ij} and
- $m + n$ constraints.

Summing the q_j's across all j's and the p_i's across all i's indicates that the total capacity of all the factories equals total requirements at all locations:

$$\sum_{j=1}^{n} q_j = \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} p_i \tag{2}$$

The presence of the restrictions in (2) will be the source of one redundancy in the complete set of restrictions that we describe below.

More about this later.

14.2.1 Vectorizing a Matrix of Decision Variables

A **matrix** of decision variables x_{ij} appears in problem (1).

The Scipy function *linprog* expects to see a **vector** of decision variables.

This situation impels us to want to rewrite our problem in terms of a **vector** of decision variables.

Let

- X, C be $m \times n$ matrices with entries x_{ij}, c_{ij},
- p be m-dimensional vector with entries p_i,
- q be n-dimensional vector with entries q_j.

Where $\mathbf{1}_n$ denotes n-dimensional column vector $(1, 1, \ldots, 1)'$, our problem can now be expressed compactly as:

$$\begin{aligned} \min_{X} \ & \text{tr}(C'X) \\ \text{subject to } & X\,\mathbf{1}_n = p \\ & X'\,\mathbf{1}_m = q \\ & X \geq 0 \end{aligned}$$

We can convert the matrix X into a vector by stacking all of its columns into a column vector.

Doing this is called **vectorization**, an operation that we denote $\text{vec}(X)$.

Similarly, we convert the matrix C into an mn-dimensional vector $\text{vec}(C)$.

The objective function can be expressed as the inner product between $\text{vec}(C)$ and $\text{vec}(X)$:

$$\text{vec}(C)' \cdot \text{vec}(X).$$

To express the constraints in terms of $\text{vec}(X)$, we use a **Kronecker product** denoted by $\otimes$ and defined as follows.

Suppose A is an $m \times s$ matrix with entries (a_{ij}) and that B is an $n \times t$ matrix.

A **Kronecker product** of A and B is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1s}B \\ a_{21}B & a_{22}B & \dots & a_{2s}B \\ & & \vdots & \\ a_{m1}B & a_{m2}B & \dots & a_{ms}B \end{bmatrix}.$$

$A \otimes B$ is an $mn \times st$ matrix.

It has the property that for any $m \times n$ matrix X

$$\text{vec}(A'XB) = (B' \otimes A')\,\text{vec}(X). \tag{3}$$

We can now express our constraints in terms of $\text{vec}(X)$.

Let $A = \mathbf{I}'_m, B = \mathbf{1}_n$.

By equation (3)

$$X\,\mathbf{1}_n = \text{vec}(X\,\mathbf{1}_n) = \text{vec}(\mathbf{I}_m X\,\mathbf{1}_n) = (\mathbf{1}'_n \otimes \mathbf{I}_m)\,\text{vec}(X).$$

where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix.

Constraint $X\,\mathbf{1}_n = p$ can now be written as:

$$(\mathbf{1}'_n \otimes \mathbf{I}_m)\,\text{vec}(X) = p.$$

Similarly, the constraint $X'\,\mathbf{1}_m = q$ can be rewriten as:

$$(\mathbf{I}_n \otimes \mathbf{1}'_m)\,\text{vec}(X) = q.$$

Our problem can now be expressed in terms of an mn-dimensional vector of decision variables:

$$\begin{aligned} \min_z \ & \text{vec}(C)'z \\ \text{subject to } \ & Az = b \\ & z \geq 0 \end{aligned} \tag{4}$$

where

$$A = \begin{bmatrix} \mathbf{1}'_n \otimes \mathbf{I}_m \\ \mathbf{I}_n \otimes \mathbf{1}'_m \end{bmatrix}, b = \begin{bmatrix} p \\ q \end{bmatrix}$$

where $z = \text{vec}(X)$.

Example:

We now provide an example that takes the form (4) that we'll solve by deploying the function *linprog*.

The table below provides numbers for the requirements vector q, the capacity vector p, and entries c_{ij} of the cost-of-shipping matrix C.

The numbers in the above table tell us to construct the following objects:

$$\begin{aligned} m = 3, n = 5, \\ p = (50, 100, 150)', q = (25, 115, 60, 30, 70)', \\ C = \begin{bmatrix} 10 & 15 & 20 & 20 & 40 \\ 20 & 40 & 15 & 30 & 30 \\ 30 & 35 & 40 & 55 & 25 \end{bmatrix}. \end{aligned}$$

Let's write Python code that sets up the problem and solves it.

```
# Define parameters
m = 3
n = 5

p = np.array([50, 100, 150])
q = np.array([25, 115, 60, 30, 70])

C = np.array([[10, 15, 20, 20, 40],
              [20, 40, 15, 30, 30],
              [30, 35, 40, 55, 25]])

# Vectorize matrix C
C_vec = C.reshape((m*n, 1), order='F')

# Construct matrix A by Kronecker product
A1 = np.kron(np.ones((1, n)), np.identity(m))
A2 = np.kron(np.identity(n), np.ones((1, m)))
A = np.vstack([A1, A2])

# Construct vector b
b = np.hstack([p, q])

# Solve the primal problem
res = linprog(C_vec, A_eq=A, b_eq=b, method='Revised simplex')

# Print results
print("message:", res.message)
print("nit:", res.nit)
print("fun:", res.fun)
print("z:", res.x)
print("X:", res.x.reshape((m,n), order='F'))
```

```
message: Optimization terminated successfully.
nit: 12
fun: 7225.0
z: [15. 10.  0. 35.  0. 80.  0. 60.  0.  0. 30.  0.  0.  0. 70.]
X: [[15. 35.  0.  0.  0.]
 [10.  0. 60. 30.  0.]
 [ 0. 80.  0.  0. 70.]]
```

```
<ipython-input-3-ee1aac536182>:24: OptimizeWarning: A_eq does not appear to be of␣
 ↪full row rank. To improve performance, check the problem formulation for redundant␣
 ↪equality constraints.
  res = linprog(C_vec, A_eq=A, b_eq=b, method='Revised simplex')
```

```
C.reshape((m*n, 1), order='F')
```

```
array([[10],
       [20],
       [30],
       [15],
       [40],
       [35],
       [20],
       [15],
       [40],
```

(continues on next page)

(continued from previous page)

```
       [20],
       [30],
       [55],
       [40],
       [30],
       [25]])
```

```
C.reshape((m*n, 1), order='C')
```

```
array([[10],
       [15],
       [20],
       [20],
       [40],
       [20],
       [40],
       [15],
       [30],
       [30],
       [30],
       [35],
       [40],
       [55],
       [25]])
```

```
C.reshape((m*n, 1), order='A')
```

```
array([[10],
       [15],
       [20],
       [20],
       [40],
       [20],
       [40],
       [15],
       [30],
       [30],
       [30],
       [35],
       [40],
       [55],
       [25]])
```

Interpreting the warning:

The above warning message from scipy pointing out that A is not full rank.

This indicates that the problem has been set up to include one or more redundant constraints.

Here, the source of the redundancy is that the set of restrictions (2).

Let's explore this further by printing out A and staring at it.

```
A
```

```
array([[1., 0., 0., 1., 0., 0., 1., 0., 0., 1., 0., 0., 1., 0., 0.],
       [0., 1., 0., 0., 1., 0., 0., 1., 0., 0., 1., 0., 0., 1., 0.],
       [0., 0., 1., 0., 0., 1., 0., 0., 1., 0., 0., 1., 0., 0., 1.],
       [1., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 1., 1., 1., 0., 0., 0., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 1., 1., 1., 0., 0., 0., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 1., 0., 0., 0.],
       [0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 1., 1., 1.]])
```

The singularity of A reflects that the first three constraints and the last five constraints both require that "total requirements equal total capacities" expressed in (2).

One equality constraint here is redundant.

Below we drop one of the equality constraints, and use only 7 of them.

After doing this, we attain the same minimized cost.

However, we find a different transportation plan.

Though it is a different plan, it attains the same cost!

```
linprog(C_vec, A_eq=A[:-1], b_eq=b[:-1], method='Revised simplex')
```

```
     con: array([0., 0., 0., 0., 0., 0., 0.])
     fun: 7225.0
 message: 'Optimization terminated successfully.'
     nit: 13
   slack: array([], dtype=float64)
  status: 0
 success: True
       x: array([ 0., 25.,  0., 35.,  0., 80.,  0., 60.,  0., 15., 15.,  0.,  0.,
        0., 70.])
```

```
%timeit linprog(C_vec, A_eq=A[:-1], b_eq=b[:-1], method='Revised simplex')
```

```
3.47 ms ± 9 µs per loop (mean ± std. dev. of 7 runs, 100 loops each)
```

```
%timeit linprog(C_vec, A_eq=A, b_eq=b, method='Revised simplex')
```

```
<magic-timeit>:1: OptimizeWarning: A_eq does not appear to be of full row rank. To␣
↪improve performance, check the problem formulation for redundant equality␣
↪constraints.
```

```
3.94 ms ± 177 µs per loop (mean ± std. dev. of 7 runs, 100 loops each)
```

Evidently, it is slightly quicker to work with the system that removed a redundant constraint.

Let's drill down and do some more calculations to help us understand whether or not our finding **two** different optimal transport plans reflects our having dropped a redundant equality constraint.

Hint

It will turn out that dropping a redundant equality isn't really what mattered.

To verify our hint, we shall simply use **all** of the original equality constraints (including a redundant one), but we'll just shuffle the order of the constraints.

```
arr = np.arange(m+n)
```

```
sol_found = []
cost = []

# simulate 1000 times
for i in range(1000):

    np.random.shuffle(arr)
    res_shuffle = linprog(C_vec, A_eq=A[arr], b_eq=b[arr], method='Revised simplex')

    # if find a new solution
    sol = tuple(res_shuffle.x)
    if sol not in sol_found:
        sol_found.append(sol)
        cost.append(res_shuffle.fun)
```

```
<ipython-input-12-4ae183291d9d>:8: OptimizeWarning: A_eq does not appear to be of␣
↪full row rank. To improve performance, check the problem formulation for redundant␣
↪equality constraints.
  res_shuffle = linprog(C_vec, A_eq=A[arr], b_eq=b[arr], method='Revised simplex')
```

```
for i in range(len(sol_found)):
    print(f"transportation plan {i}: ", sol_found[i])
    print(f"     minimized cost {i}: ", cost[i])
```

```
transportation plan 0:  (15.0, 10.0, 0.0, 35.0, 0.0, 80.0, 0.0, 60.0, 0.0, 0.0, 30.0,␣
↪0.0, 0.0, 0.0, 70.0)
     minimized cost 0:  7225.0
transportation plan 1:  (0.0, 25.0, 0.0, 35.0, 0.0, 80.0, 0.0, 60.0, 0.0, 15.0, 15.0,␣
↪0.0, 0.0, 0.0, 70.0)
     minimized cost 1:  7225.0
```

Ah hah! As you can see, putting constraints in different orders in this case uncovers two optimal transportation plans that achieve the same minimized cost.

These are the same two plans computed early.

Next, we show that leaving out the first constraint "accidentally" leads to the initial plan that we computed.

```
linprog(C_vec, A_eq=A[1:], b_eq=b[1:], method='Revised simplex')
```

```
     con: array([0., 0., 0., 0., 0., 0., 0.])
     fun: 7225.0
 message: 'Optimization terminated successfully.'
     nit: 12
   slack: array([], dtype=float64)
  status: 0
 success: True
       x: array([15., 10.,  0., 35.,  0., 80.,  0., 60.,  0.,  0., 30.,  0.,  0.,
         0., 70.])
```

Let's compare this transport plan with

```
res.x
```

```
array([15., 10.,  0., 35.,  0., 80.,  0., 60.,  0.,  0., 30.,  0.,  0.,
        0., 70.])
```

Here the matrix X contains entries x_{ij} that tell amounts shipped **from** factor $i = 1, 2, 3$ **to** location $j = 1, 2, ..., 5$.

The vector z evidently equals vec(X).

The minimized cost from the optimal transport plan is given by the fun variable.

We can also solve an optimal transportation problem using a powerful tool from `quantecon`, namely,`quantecon.optimize.linprog_simplex`.

It uses the same simplex algorithm as `scipy.optimize.linprog`, but the program is accelerated by using `numba`.

As you will see very soon, by using `scipy.optimize.linprog` the time required to solve an optimal transportation problem can be reduced significantly.

```
# construct matrices/vectors for linprog_simplex
c = C.flatten()

# Equality constraints
A_eq = np.zeros((m+n, m*n))
for i in range(m):
    for j in range(n):
        A_eq[i, i*n+j] = 1
        A_eq[m+j, i*n+j] = 1

b_eq = np.hstack([p, q])
```

Since `quantecon.optimize.linprog_simplex` does maximization instead of minimization, we need to put a negative sign before vector `c`.

```
res_qe = linprog_simplex(-c, A_eq=A_eq, b_eq=b_eq)
```

Since the two LP solvers use the same simplex algorithm, we expect to get exactly the same solutions

```
res_qe.x.reshape((m, n), order='C')
```

```
array([[15., 35.,  0.,  0.,  0.],
       [10.,  0., 60., 30.,  0.],
       [ 0., 80.,  0.,  0., 70.]])
```

```
res.x.reshape((m, n), order='F')
```

```
array([[15., 35.,  0.,  0.,  0.],
       [10.,  0., 60., 30.,  0.],
       [ 0., 80.,  0.,  0., 70.]])
```

Let's do a speed comparison between `scipy.optimize.linprog` and `quantecon.optimize.linprog_simplex`.

```
# scipy.optimize.linprog
%timeit res = linprog(C_vec, A_eq=A[:-1, :], b_eq=b[:-1], method='Revised simplex')
```

```
3.44 ms ± 21 µs per loop (mean ± std. dev. of 7 runs, 100 loops each)
```

```
# quantecon.optimize.linprog_simplex
%timeit out = linprog_simplex(-c, A_eq=A_eq, b_eq=b_eq)
```

```
22.9 µs ± 134 ns per loop (mean ± std. dev. of 7 runs, 10000 loops each)
```

As you can see, the `quantecon.optimize.linprog_simplex` is almost 200 times faster.

14.3 The Dual Problem

Let u, v denotes vectors of dual decision variables with entries $(u_i), (v_j)$.

The **dual** to **minimization** problem (1) is the **maximization** problem:

$$\begin{aligned} &\max_{u_i, v_j} \sum_{i=1}^{m} p_i u_i + \sum_{j=1}^{n} q_j v_j \\ &\text{subject to } \; u_i + v_j \leq c_{ij}, \; i = 1, 2, \dots, m; \; j = 1, 2, \dots, n \end{aligned} \tag{5}$$

The dual problem is also a linear programming problem.

It has $m + n$ dual variables and mn constraints.

Vectors u and v of **values** are attached to the first and the second sets of primal constraits, respectively.

Thus, u is attached to the constraints

- $(\mathbf{1}'_n \otimes \mathbf{I}_m) \operatorname{vec}(X) = p$

and v is attached to constraints

- $(\mathbf{I}_n \otimes \mathbf{1}'_m) \operatorname{vec}(X) = q$.

Components of the vectors u and v of **values** are **shadow prices** of the quantities appearing on the right sides of those constraints.

We can write the dual problem as

$$\begin{aligned} &\max_{u_i, v_j} pu + qv \\ &\text{subject to } \; A' \begin{bmatrix} u \\ v \end{bmatrix} = \operatorname{vec}(C) \end{aligned} \tag{6}$$

For the same numerical example described above, let's solve the dual problem.

```
# Solve the dual problem
res_dual = linprog(-b, A_ub=A.T, b_ub=C_vec,
                   bounds=[(None, None)]*(m+n), method='Revised simplex')

#Print results
print("message:", res_dual.message)
print("nit:", res_dual.nit)
print("fun:", res_dual.fun)
print("u:", res_dual.x[:m])
print("v:", res_dual.x[-n:])
```

```
message: Optimization terminated successfully.
nit: 7
fun: -7225.0
u: [ 5. 15. 25.]
v: [ 5. 10.  0. 15.  0.]
```

We can also solve the dual problem using quantecon.optimize.linprog_simplex.

```
res_dual_qe = linprog_simplex(b_eq, A_ub=A_eq.T, b_ub=c)
```

And the shadow prices computed by the two programs are identical.

```
res_dual_qe.x
```

```
array([ 5., 15., 25.,  5., 10.,  0., 15.,  0.])
```

```
res_dual.x
```

```
array([ 5., 15., 25.,  5., 10.,  0., 15.,  0.])
```

We can compare computational times from using our two tools.

```
%timeit linprog(-b, A_ub=A.T, b_ub=C_vec, bounds=[(None, None)]*(m+n), method=
↪'Revised simplex')
```

```
2.64 ms ± 35.8 µs per loop (mean ± std. dev. of 7 runs, 100 loops each)
```

```
%timeit linprog_simplex(b_eq, A_ub=A_eq.T, b_ub=c)
```

```
168 µs ± 591 ns per loop (mean ± std. dev. of 7 runs, 10000 loops each)
```

`quantecon.optimize.linprog_simplex` solves the dual problem 10 times faster.

Just for completeness, let's solve the dual problems with nonsingular A matrices that we create by dropping a redundant equality constraint.

Try first leaving out the first constraint:

```
linprog(-b[1:], A_ub=A[1:].T, b_ub=C_vec,
        bounds=[(None, None)]*(m+n-1), method='Revised simplex')
```

```
     con: array([], dtype=float64)
     fun: -7225.0
 message: 'Optimization terminated successfully.'
     nit: 7
   slack: array([ 0.,  0.,  0.,  0., 15.,  0., 15.,  0., 15.,  0.,  0., 15., 35.,
       15.,  0.])
  status: 0
 success: True
       x: array([10., 20., 10., 15.,  5., 20.,  5.])
```

Not let's instead leave out the last constraint:

```
linprog(-b[:-1], A_ub=A[:-1].T, b_ub=C_vec,
        bounds=[(None, None)]*(m+n-1), method='Revised simplex')
```

```
     con: array([], dtype=float64)
     fun: -7225.0
 message: 'Optimization terminated successfully.'
     nit: 11
   slack: array([ 0.,  0.,  0.,  0., 15.,  0., 15.,  0., 15.,  0.,  0., 15., 35.,
        15.,  0.])
  status: 0
 success: True
       x: array([ 5., 15., 25.,  5., 10.,  0., 15.])
```

14.3.1 Interpretation of dual problem

By **strong duality**, we know that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} = \sum_{i=1}^{m} p_i u_i + \sum_{j=1}^{n} q_j v_j$$

One unit more capacity in factory i, i.e. p_i, results in u_i more transportation costs.

Thus, u_i describes the cost of shipping one unit **from** factory i.

Call this the ship-out cost of one unit shipped from factory i.

Similarly, v_j is the cost of shipping one unit **to** location j.

Call this the ship-in cost of one unit to location j.

Strong duality implies that total transprotation costs equals total ship-out costs **plus** total ship-in costs.

It is reasonable that, for one unit of a product, ship-out cost u_i **plus** ship-in cost v_j should equal transportation cost c_{ij}.

This equality is assured by **complementary slackness** conditions that state that whenever $x_{ij} > 0$, meaning that there are positive shipments from factory i to location j, it must be true that $u_i + v_j = c_{ij}$.