Finite Element Analysis II - Home Task No. 2

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Heat Flow Problem

Differential equation:

$$\dot{u} - k\frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad 0 < x < L$$

Where $k = \frac{\kappa}{\rho c} = const.$

Boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

$$u(L) = 0$$

Initial conditions:

$$u(t=0) = u_0 = const.$$

Analytical solution:

$$u(x,t) = \sum_{n=0} \frac{2sin(n+0.5)\pi}{(n+0.5)\pi} e^{-(n+0.5)^2\pi^2 t} cos\left((n+0.5)\pi x\right)$$

Semi discrete problem (matrix):

$$\boldsymbol{M}\dot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{F}$$

$$\boldsymbol{d}(0) = \boldsymbol{d_0}$$

6th row and column are discarded due to the BC:

$$\boldsymbol{M} = \frac{\rho c h}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{\kappa}{h}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Generalized Trapezoidal Methods

$$\boldsymbol{M}\boldsymbol{v}_{\boldsymbol{n+1}} + \boldsymbol{K}\boldsymbol{d}_{\boldsymbol{n+1}} = \boldsymbol{F}_{\boldsymbol{n+1}}$$

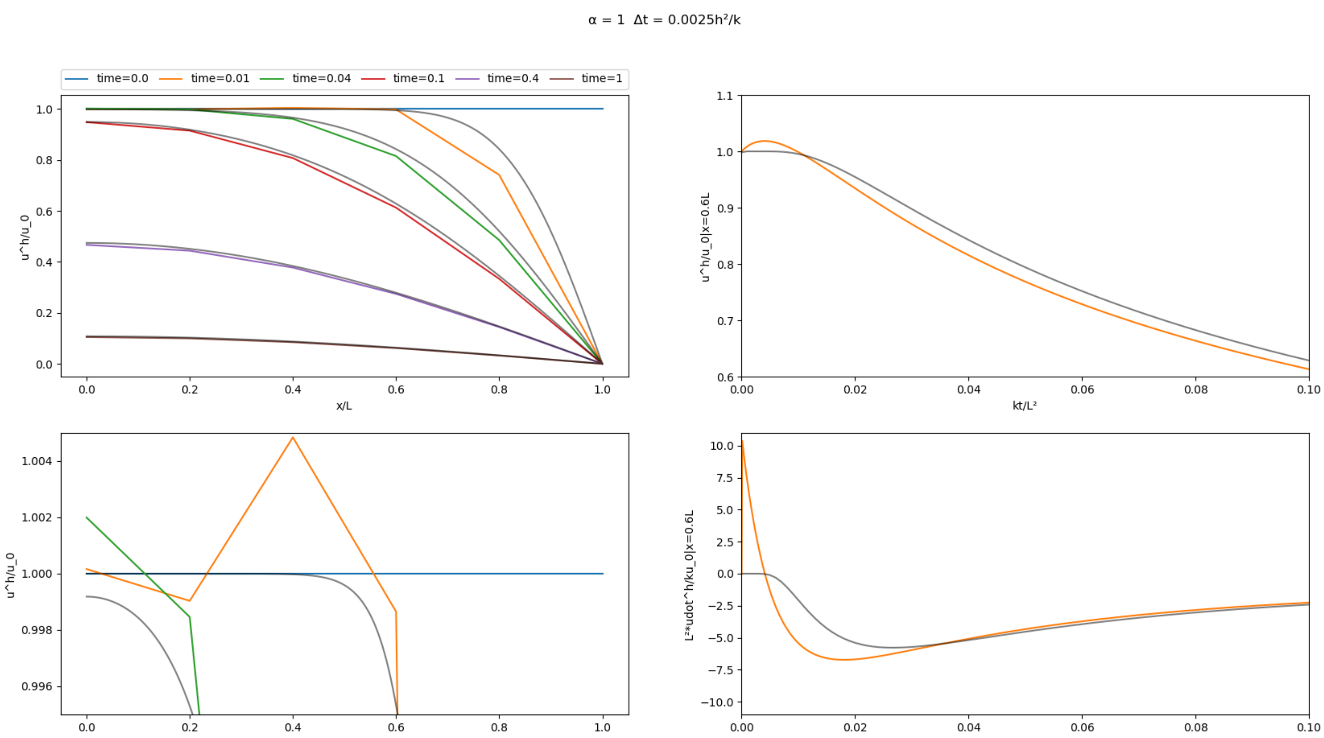
$$\boldsymbol{d}_{\boldsymbol{n+1}} = \boldsymbol{d}_{\boldsymbol{n}} + \Delta t[(\boldsymbol{1}-\boldsymbol{\alpha})\boldsymbol{v}_{\boldsymbol{n}} + \boldsymbol{\alpha}\boldsymbol{v}_{\boldsymbol{n+1}}]$$

$$\boldsymbol{M}^{*}\boldsymbol{v}_{\boldsymbol{n+1}} = \boldsymbol{F}^{*}_{\boldsymbol{n+1}}$$

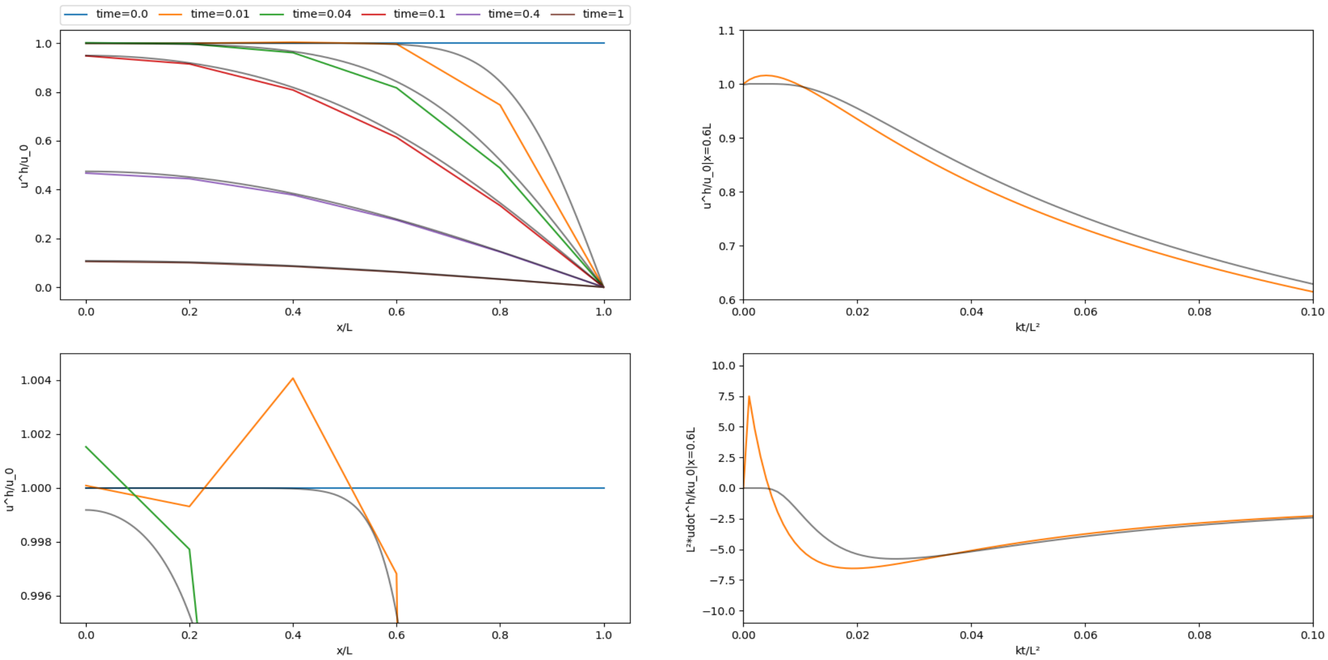
$$\boldsymbol{where}\ \boldsymbol{M}^{*} = \boldsymbol{M} + \boldsymbol{\alpha\Delta tK} \quad \boldsymbol{and}\ \ \boldsymbol{F}^{*} = \boldsymbol{F}(\boldsymbol{t}_{\boldsymbol{n+1}}) - \boldsymbol{K}\widetilde{\boldsymbol{d}}$$

In this task, the problem was solved with Forward Euler ($\alpha = 0$), Crank-Nicolson ($\alpha = \frac{1}{2}$) and Backward Euler ($\alpha = 1$). The mass was lumped for the Forward Euler method and thus the problem was solved explicitly.

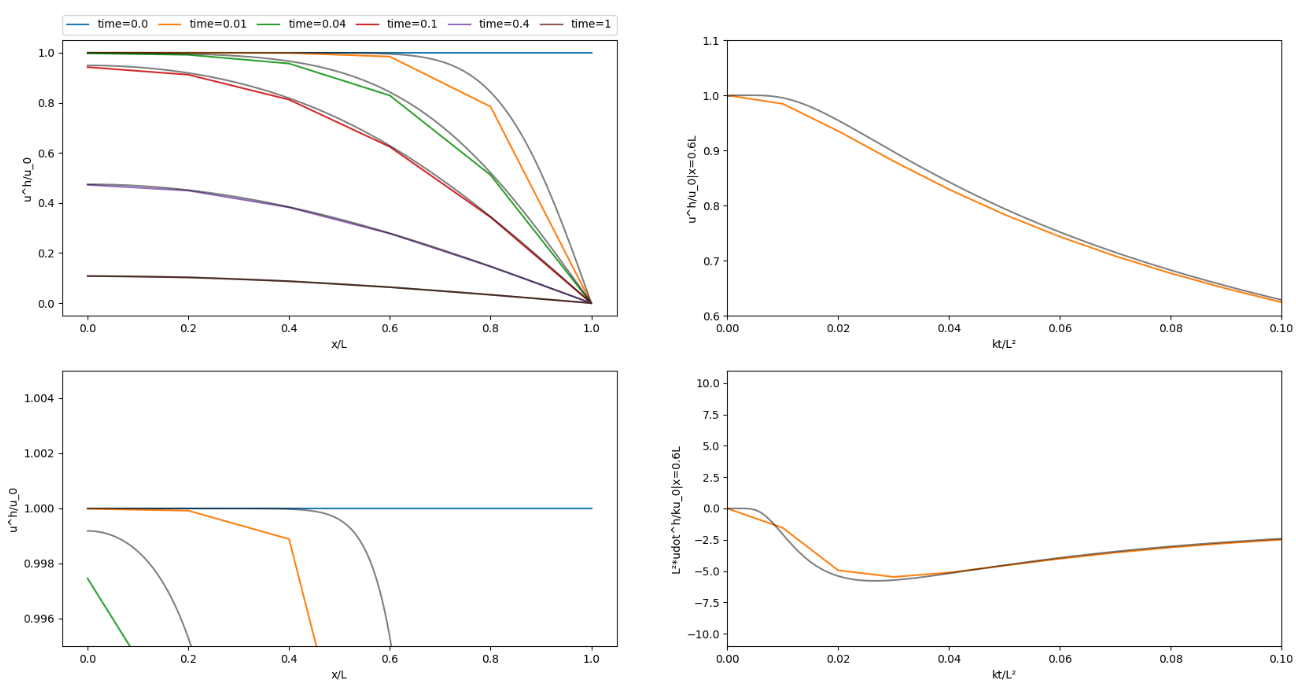
1) Reproduction of the Backward Euler results:



$\alpha = 1 \;\; \Delta t = 0.025h^2/k$

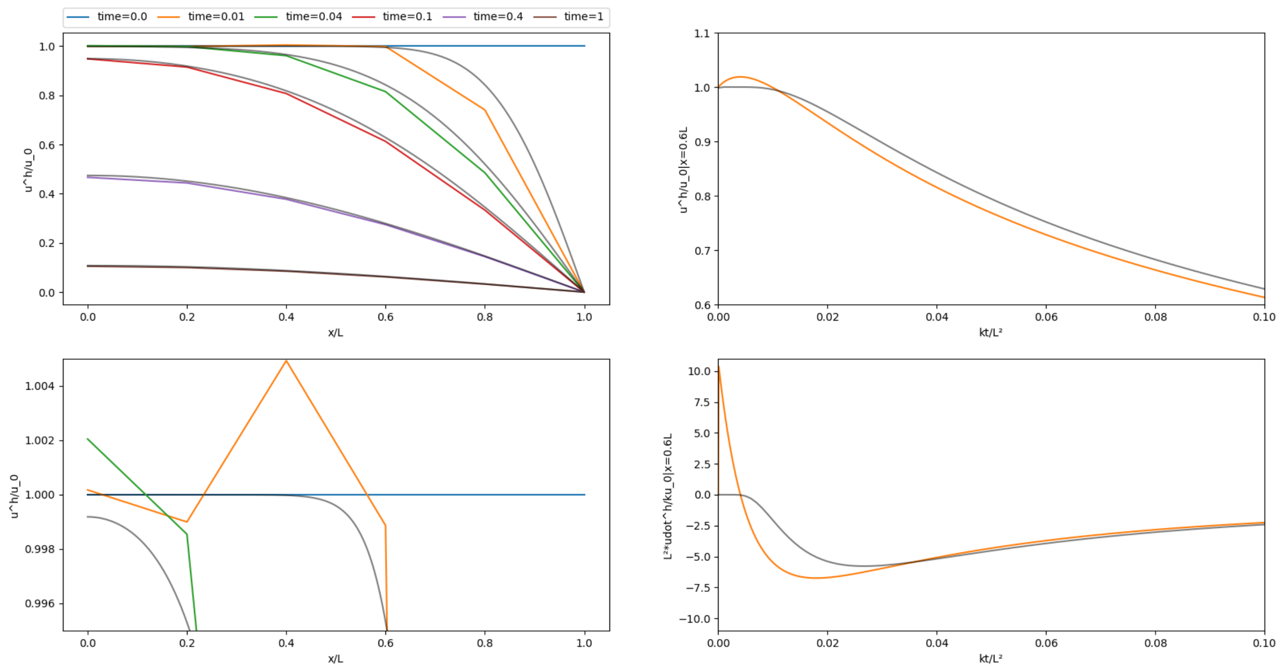


$\alpha = 1 \;\; \Delta t = 0.25h^2/k$

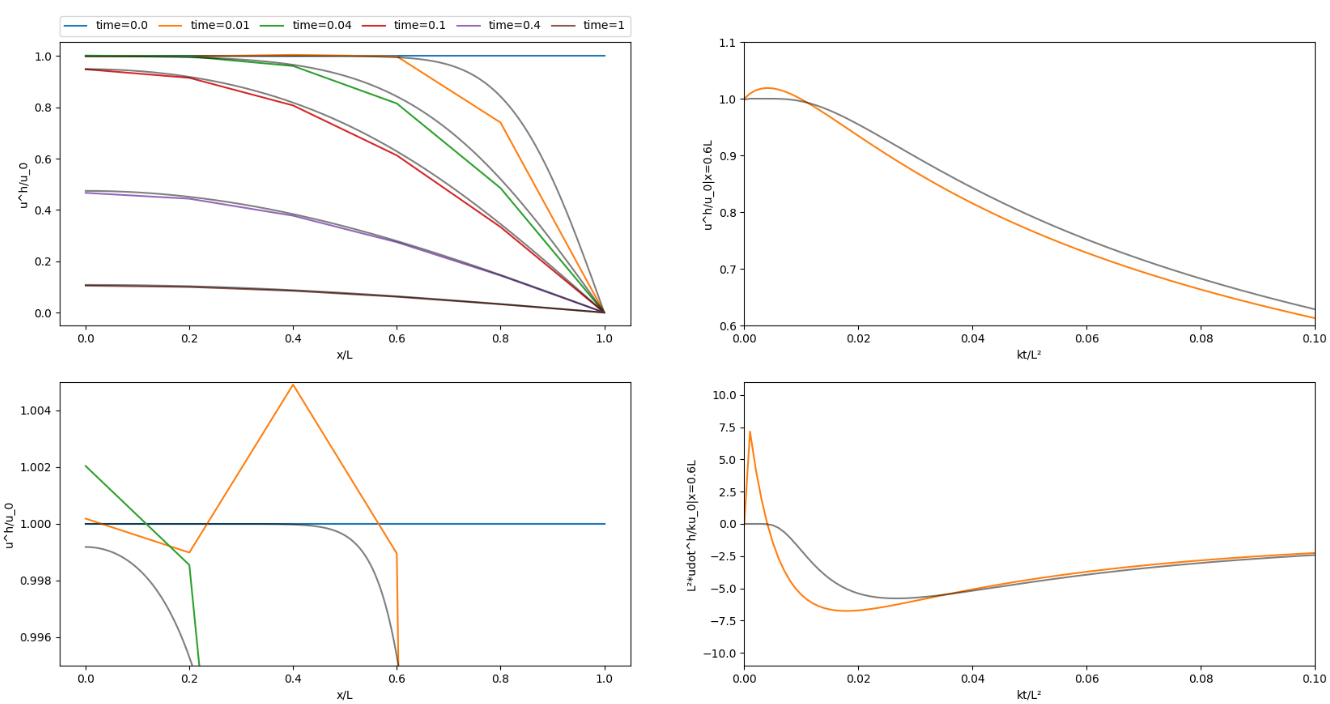


2) Solutions with Crank-Nicolson

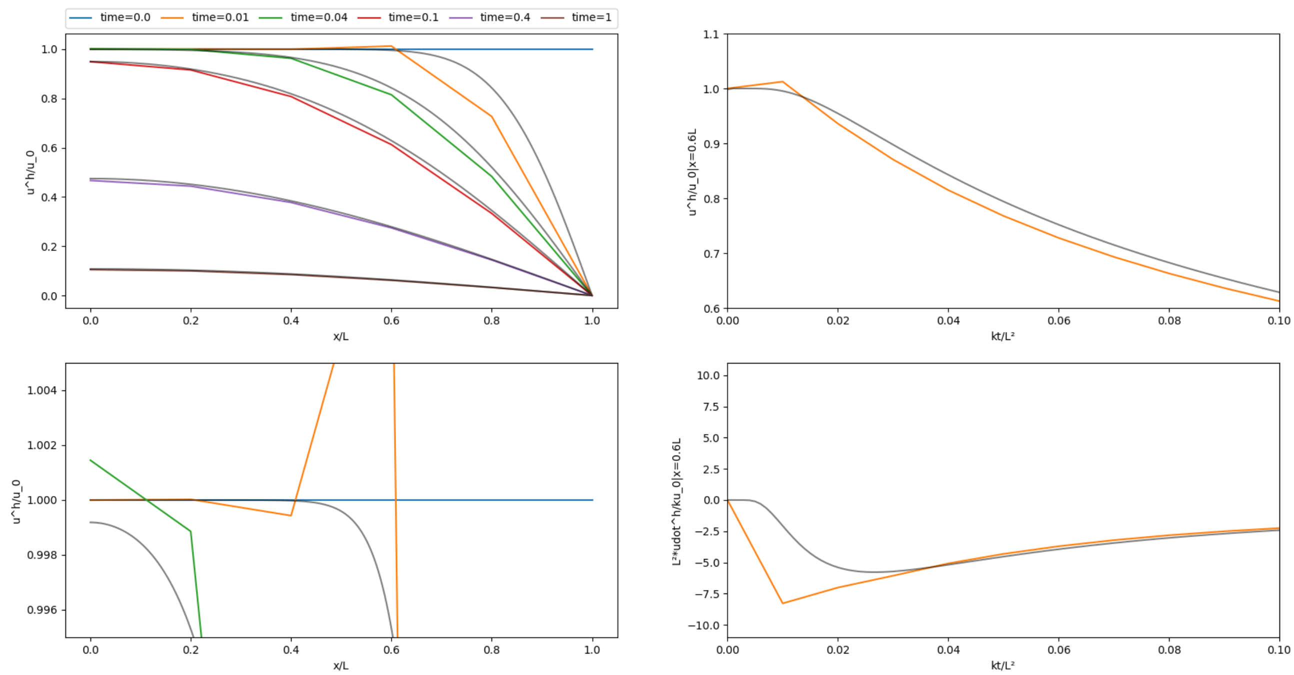
α = 0.5 Δt = 0.0025h²/k



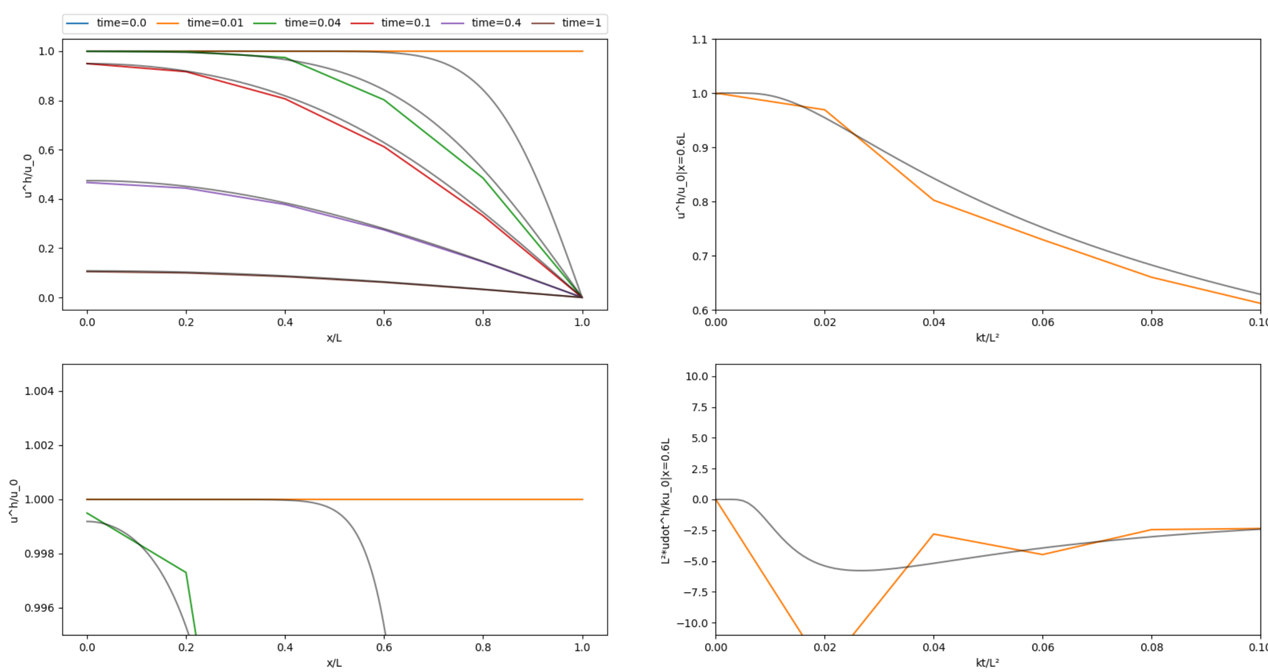
α = 0.5 Δt = 0.025h²/k



α = 0.5 Δt = 0.25h²/k



α = 0.5 Δt = 0.5h²/k



For $\alpha = 0.5$, the solution is stable and the oscillation limit defined by $\Delta t_{\mathrm{maximal}} = \frac{2}{\lambda_n^h}$, any higher time increment will cause oscillations of the last or more modal solutions.

The eigen values are calculated here by a python function:

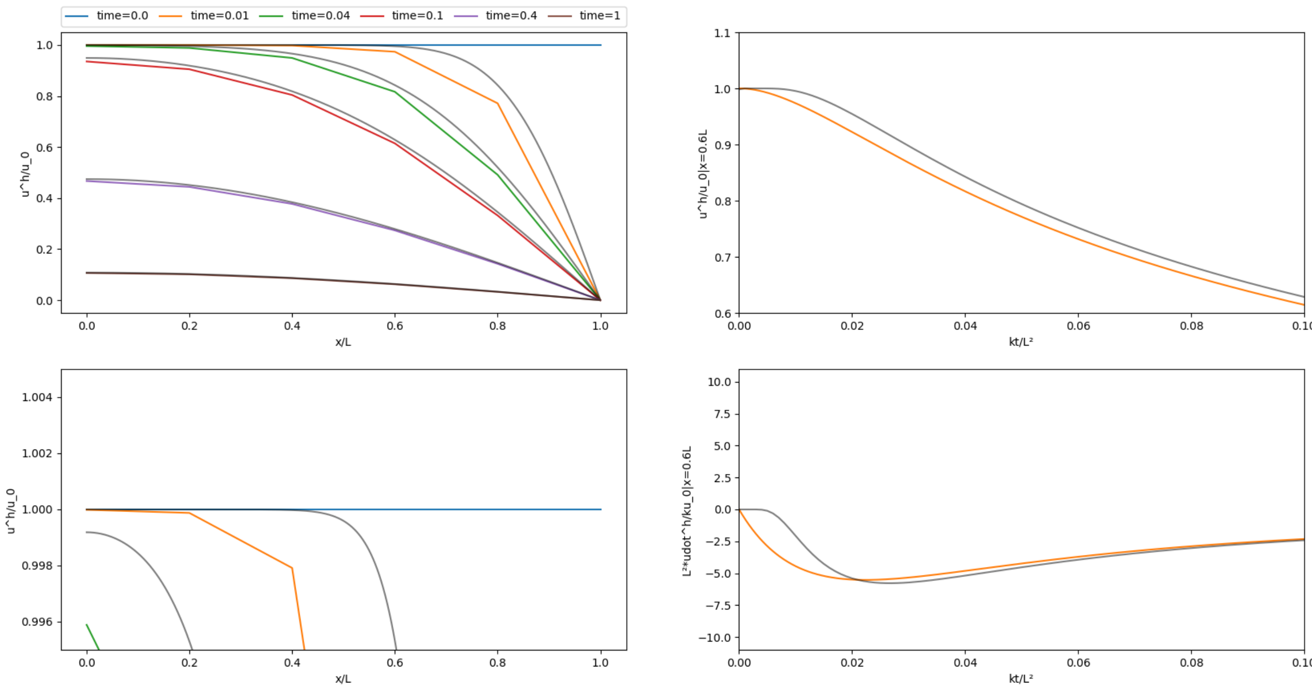
λ_1 =2.49 λ_2 =23.89 λ_3 =75. λ_4 =168.65 λ_5 =279.0

$$\Delta t_{maximal} = \frac{2}{\lambda_n^h} = \frac{2}{279} = 0.007$$

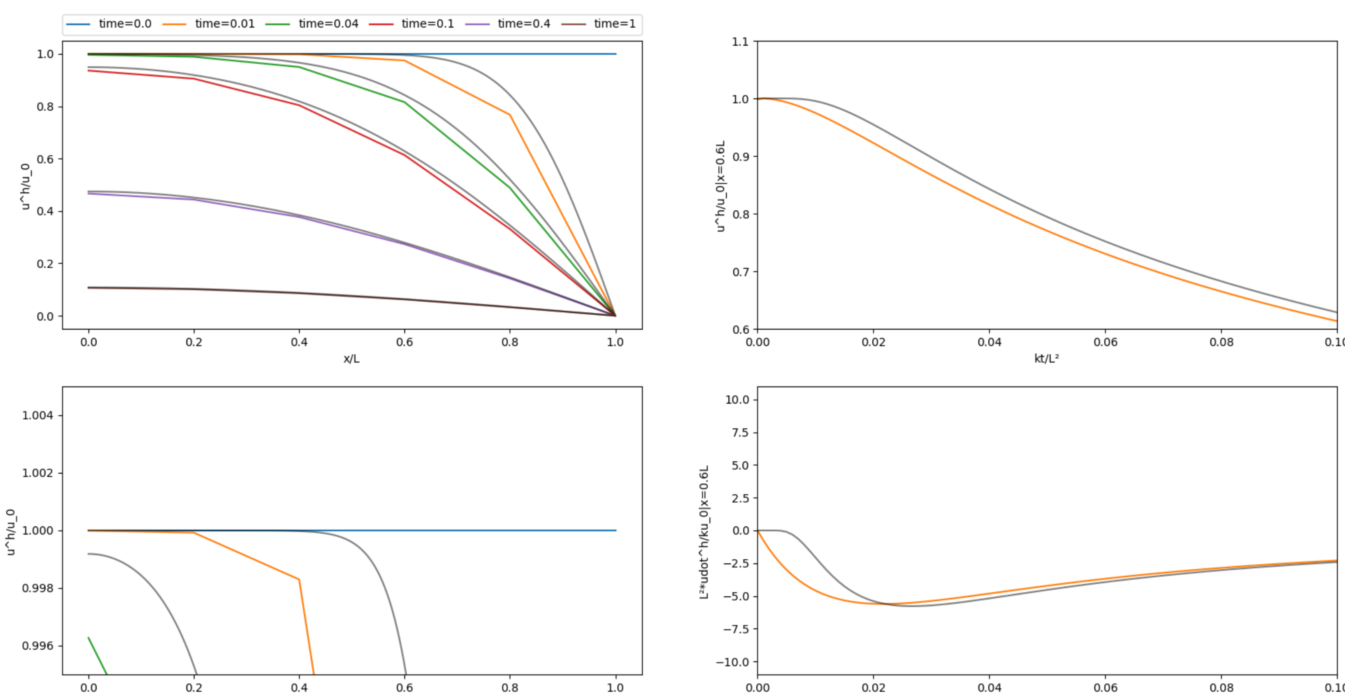
$\Delta t = \frac{0.5h^2}{k}$ requires $\Delta t = 0.02$ which is above the oscillation limit for our problem.

3) Solutions with Forward Euler

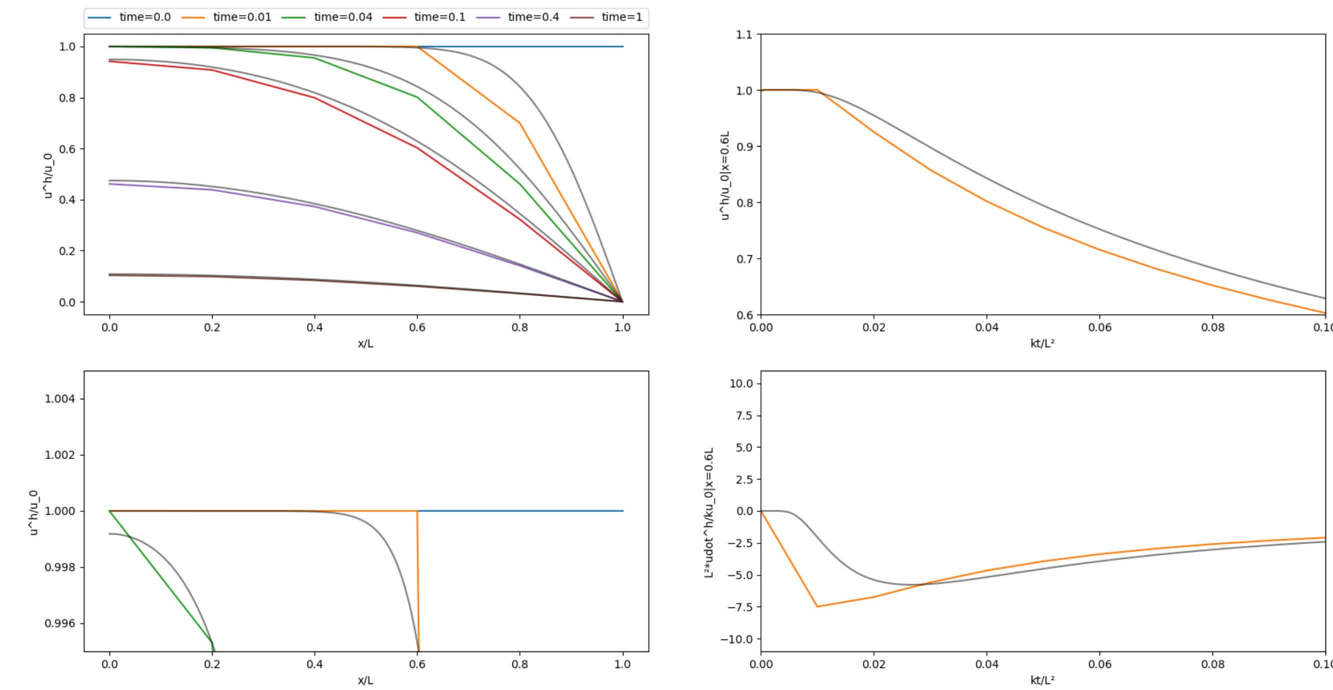
α = 0 Δt = 0.0025h²/k



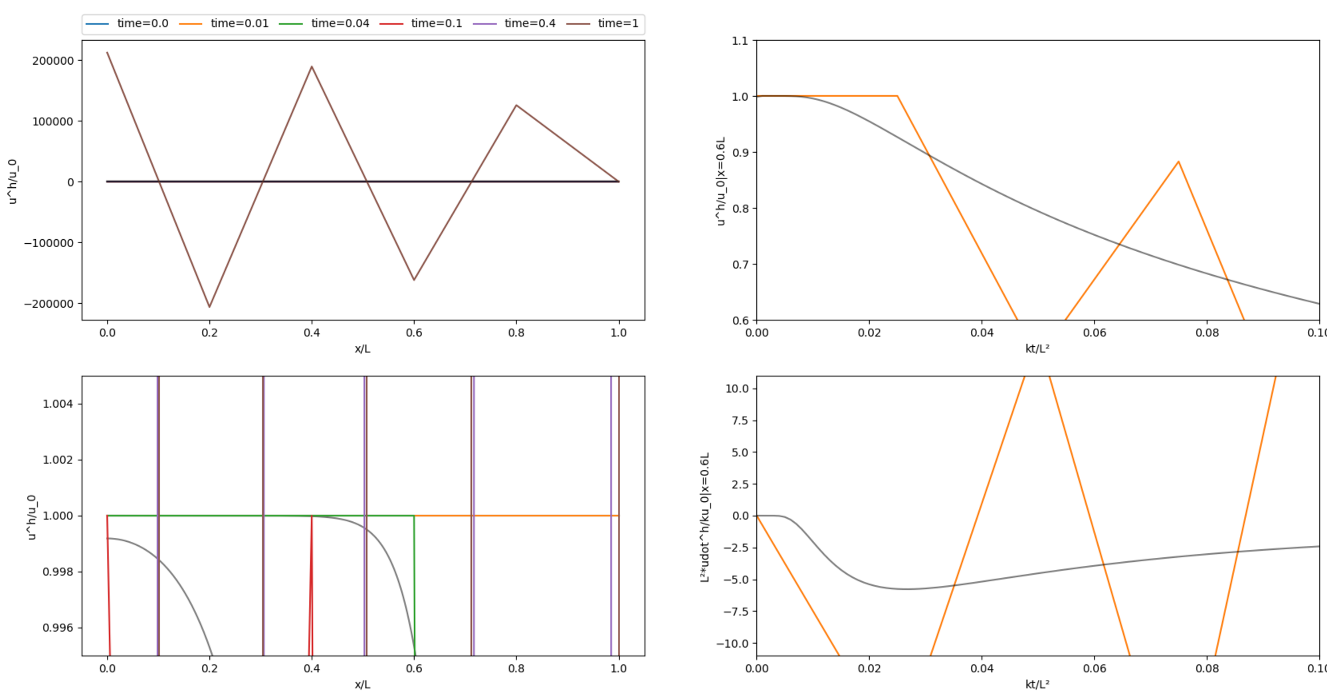
α = 0 Δt = 0.025h²/k



α = 0 Δt = 0.25h²/k



α = 0 Δt = 0.625h²/k



For $\alpha = 0$, the solution is stable if $\Delta t_{\text{stable}} < \frac{2}{\lambda_n^h}$ and the oscillation limit is defined by $\Delta t_{\text{oscillation}} = \frac{1}{\lambda_n^h}$. The eigen values are calculated here both for the whole problem and for a single element:

For a single element, the calculation is done here:

$$\det(K - \lambda M) = 0$$

$$\left|\frac{\kappa}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \lambda \cdot \frac{\rho c h}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| = \begin{vmatrix}5 - 0.1\lambda & -5\\ -5 & 5 - 0.1\lambda\end{vmatrix} = (5 - 0.1\lambda)^2 - 25 = -\lambda + 0.01\lambda^2 = 0$$

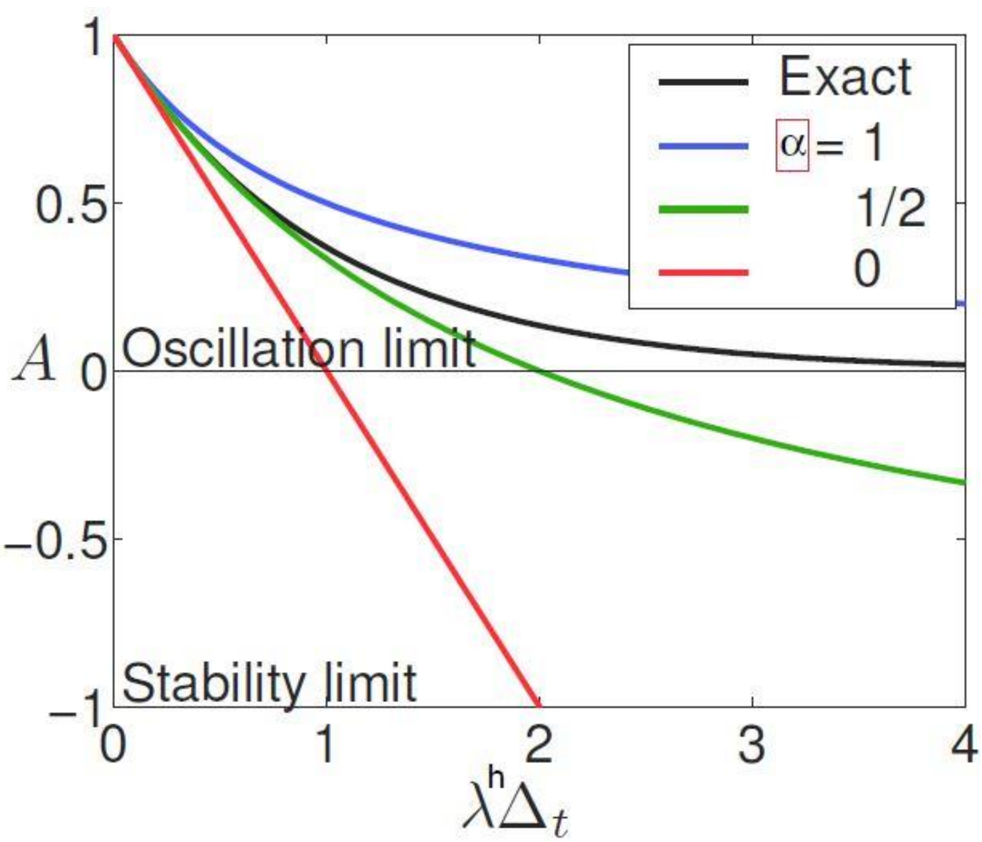
$\lambda_1 = 0, \lambda_2 = 100$

$$\Delta t_{stable} = \frac{2}{\lambda_n^h} = 0.02 \quad ; \quad \Delta t_{\text{oscillation}} = \frac{1}{\lambda_n^h} = 0.01$$

$\Delta t = \frac{0.625h^2}{k}$ is met with a step of $\Delta t = 0.025$ which is over the stability limit for our problem and the solution does not converge.

$\Delta t = \frac{0.25h^2}{k}$ is met with a step of $\Delta t = 0.01$ which is within the stability limit for our problem.

$\Delta t = \frac{0.025h^2}{k}$ is met with a step of $\Delta t = 0.001$ which is within the stability and oscillation limits for our problem.



For the whole problem, the calculation is done by a python function:

λ_1 =2.46 λ_2 =21.49 λ_3 =53.49 λ_4 =83.91 λ_5 =98.63

$$\Delta t_{stable} = \frac{2}{\lambda_n^h} = \frac{2}{98.63} = 0.00202$$

As it can be seen, the element-wise estimation is more demanding.