TOLERANTS

SWECHCHHA ADHIKARI, BRENT HALL, AND STEPHEN MCKEAN

ABSTRACT. We study a generalization of the discriminant of a polynomial, which we call the tolerant. The tolerant differs by multiplication by a square from the duplicant, which was discovered in recent work on $\mathbb{P}^1$-loop spaces in motivic homotopy theory. We show that the tolerant is rational by deriving a formula in terms of discriminants. This allows us to formulate a conjectural unstable Poincaré–Hopf formula over an arbitrary locus of points. We also show that the tolerant satisfies many of the same properties as the discriminant. A notable difference between the two is that the discriminant is inversion invariant for all polynomials, whereas the tolerant is only inversion invariant on a proper multiplicative subset of polynomials.

1. INTRODUCTION

In topology, May's recognition principle characterizes finite loop spaces as algebras over the little cubes operad [May72]. An analogous recognition principle for finite $\mathbb{P}^1$-loop spaces in the motivic homotopy category has remained an open question for a quarter century [Voe00]. A motivic finite loop space recognition principle would have spectacular applications, such as potentially simplifying Asok–Bachmann–Hopkins's seminal work on the motivic Freudenthal suspension theorem [ABH24], which in turn resolved an important conjecture of Murthy [Mur99, p. 173].

Recently, Igieobo–McKean–Sanchez–Taylor–Wickelgren discovered a family of configuration space operations on the $\mathbb{P}^1$-loop space $\Omega_{\mathbb{P}^1}\mathbb{P}^1$ [IMS+25]. These operations present a tantalizing shadow of a potential operadic structure on $\mathbb{P}^1$-loop spaces. In this article, we will prove some algebraic properties of these configuration space operations. To begin, we need a definition.

Definition 1.1. Let k be a field with algebraic closure $\overline{k}$. Let $f = a_n x^n + \ldots + a_0$ be a polynomial over k. Write

$$f = a_n \prod_{i=1}^{s} (x - r_i)^{m_i}$$

for the factorization of f over $\overline{k}$, where $r_1, \ldots, r_s$ are distinct. The *duplicant* of f is defined as

$$\mathrm{dupl}(f) := a_n^{2n} \prod_{1 \leq i < j \leq s} (r_i - r_j)^{2m_i m_j}.$$

Note that when $m_i = 1$ for all i (i.e. when f is separable), then the duplicant is equal to a_n^2 times the discriminant of f. However, in contrast to the discriminant, the duplicant does not vanish when f has a repeated root.

The duplicant lies at the heart of the operations discussed in [IMS$^+$25, Theorem 1.1]. Given a rational configuration $D := \{r_1, \ldots, r_n\} \subseteq \mathbb{A}^1(k)$ and a family of pointed maps $f_1, \ldots, f_n : \mathbb{P}^1 \to \mathbb{P}^1$ (that is, n points in the loop space $\Omega_{\mathbb{P}^1}\mathbb{P}^1$), we obtain a natural sum

$$\sum_D (f_1, \ldots, f_n) : \mathbb{P}^1 \to \mathbb{P}^1$$

that is determined by the duplicant of $\prod_{i=1}^n (x - r_i)^{\deg(f_i)}$.

It is desireable, both to better understand a possible operadic structure on $\Omega_{\mathbb{P}^1}\mathbb{P}^1$ and for applications to enumerative geometry, to remove the assumption that the points in the configuration D be rational. This article contributes to that goal by proving, among other things, that the duplicant of any polynomial over k is again an element of k. This proof will pass through the following definition.

Definition 1.2. Assume the notation of Definition 1.1. The *tolerant*[1] of f is defined as

$$\mathrm{tol}(f) := \frac{\mathrm{dupl}(f)}{a_n^2} = a_n^{2n-2} \prod_{1 \leq i < j \leq s} (r_i - r_j)^{2m_i m_j}.$$

We will work with the tolerant rather than the duplicant, because it is slightly more well-behaved. Of course, the tolerant and duplicant coincide for any monic polynomial, such as those needed to define the configuration space operation Σ_D.

Our main results are the following two theorems about the tolerant.

Theorem 1.3 (Corollary 2.10). *If $f(x) \in k[x]$, then $\mathrm{tol}(f) \in k^\times$.*

Theorem 1.4 (Theorem 2.15). *Consider $k[x] - (x) = \{f(x) \in k[x] : f(0) \neq 0\}$. Given $f(x) \in k[x] - (x)$, denote the reciprocal polynomial by $f^*(x)$. Let*
$$T = \{f \in k[x] - (x) : \mathrm{tol}(f) = \mathrm{tol}(f^*)\}.$$
Then T is a proper multiplicative subset of $k[x] - (x)$.

Previously, Theorem 1.3 was only known for polynomials whose roots are all elements of k (which holds by definition) [IMS$^+$25]. We prove this theorem by deriving a formula for the tolerant in terms of the irreducible factors of f (Lemma 2.9). The primary wrinkle comes from inseparable irreducible polynomials, which we deal with in Lemma 2.6. An amusing output of this lemma is that over any field of characteristic exponent p, and given any irreducible polynomial $f(x) = f_{\mathrm{sep}}(x^{p^e})$ with degree of inseparability e, we have

$$\mathrm{tol}(f) = \mathrm{disc}(f_{\mathrm{sep}})^{p^e}.$$

---

[1]The term tolerant was suggested by the first two authors because the tolerant *tolerates* repeated roots, whereas the discriminant *discriminates* against repeated roots.

Theorem 1.4 is interesting because the discriminant is always invariant under polynomial inversion. It is perhaps not so surprising that this property does not hold for the tolerant, but it is curious that the set of polynomials for which the tolerant is inversion invariant is multiplicative. Proposition 2.14 characterizes the set T in terms of the roots and coefficients of its constituent elements, but it would be interesting to give a more qualitative description of T.

After proving these theorems, we derive a formula for $\mathrm{tol}(f)$ in terms of the coefficients of f; this is done in Section 3. This relies on [DDR$^+$22, Proposition 2.4], where the authors give such a formula for their *generalized discriminant*. In characteristic 0, *loc. cit.* also shows that the generalized discriminant and the tolerant are equal up to a sign. We will generalize this proof to positive characteristic as well.

To conclude, we state a conjecture about a Poincaré–Hopf theorem for the unstable local degree at non-rational points (Conjecture 4.1). This conjecture is known to hold at rational points [IMS$^+$25].

2. Properties of the tolerant

Because the tolerant resembles the discriminant (and even recovers the discriminant for separable polynomials), it is natural to ask which properties of the discriminant also hold for the tolerant. We will focus on four of these properties:

  (i) Translation invariance.

 (ii) Homothety invariance.

(iii) Rationality.

(iv) Inversion invariance.

2.1. **Translation and homothety invariance.** Translation and homothety invariance follow from straightforward calculations, so we will prove these properties first.

Proposition 2.1 (Translation invariance). *Let $f \in k[x]$. For any $\alpha \in k$, we have*
$$\mathrm{tol}(f(x+\alpha)) = \mathrm{tol}(f(x)).$$

Proof. Let $f = a_n \prod_{i=1}^{s}(x - r_i)^{m_i}$ over $\overline{k}$. Then $f(x+\alpha) = a_n \prod_{i=1}^{s}(x + \alpha - r_i)^{m_i}$, so
$$\mathrm{tol}(f(x+\alpha)) = a_n^{2n-2} \prod_{i<j}((r_i - \alpha) - (r_j - \alpha))^{2m_i m_j}$$
$$= a_n^{2n-2} \prod_{i<j}(r_i - r_j)^{2m_i m_j}$$
$$= \mathrm{tol}(f(x)). \qquad \square$$

Proposition 2.2 (Homothety invariance)**.** *Let $f \in k[x]$, and let $n := \deg(f)$. Over $\overline{k}$, write $f = a_n \prod_{i=1}^{s}(x - r_i)^{m_i}$. For any $\alpha \in k - \{0\}$, we have*

$$\text{tol}(f(\alpha x)) = \alpha^{n^2 - 2n + \sum_i m_i^2} \text{tol}(f(x)).$$

Proof. We begin by computing

$$f(\alpha x) = a_n \prod_{i=1}^{s}(\alpha x - r_i)^{m_i}$$

$$= a_n \alpha^n \prod_{i=1}^{s}(x - \tfrac{r_i}{\alpha})^{m_i}.$$

Note that $2 \sum_{i<j} m_i m_j = n^2 - \sum_i m_i^2$. It follows that

$$\text{tol}(f(\alpha x)) = a_n^{2n-2} \alpha^{2n^2 - 2n} \prod_{i<j}(\tfrac{r_i}{\alpha} - \tfrac{r_j}{\alpha})^{2m_i m_j}$$

$$= a_n^{2n-2} \alpha^{n^2 - 2n + \sum_i m_i^2} \prod_{i<j}(r_i - r_j)^{2m_i m_j}. \qquad \square$$

Remark 2.3. If f is separable (i.e. $m_i = 1$ for all i), then $\text{tol}(f(\alpha x)) = \alpha^{n(n-1)} \text{tol}(f(x))$, recovering the degree of homothety of the discriminant. This is not surprising, as $\text{tol}(f) = \text{disc}(f)$ for separable polynomials. In general, we have

$$n^2 - 2n + \sum_{i=1}^{s} m_i^2 = n(n - 1) + \sum_{i=1}^{s} m_i(m_i - 1).$$

Rationality and inversion invariance are more interesting, so we will treat each of these properties in their own subsections.

2.2. **Rationality.** If $f \in k[x]$, then $\text{disc}(f) \in k$. This is usually proved as a consequence of the formula $\text{disc}(f) = \text{res}(f, f')$; this resultant is the determinant of matrix with entries determined by the coefficients of f and f', all of which are elements of k. Because we do not know any analogous resultant formulas for the tolerant, we will have to take a different approach to proving $\text{tol}(f) \in k$. The strategy is to derive a formula for $\text{tol}(f)$ in terms of discriminants.

Let $f(x) = a_n \prod_{i=1}^{d} f_i(x)^{m_i}$ be the monic irreducible factorization of f over k. (So each $f_i(x) \in k[x]$ is monic and irreducible.) Distinct monic irreducible polynomials over a field must be coprime, so the roots of the factors $f_1, \ldots, f_d$ partition the roots of f. If each of the irreducible factors is separable, then we can compute $\text{tol}(f)$ in terms of discriminants of the form $\text{disc}(f_i f_j)$ and $\text{disc}(f_i)$.

Lemma 2.4. *Given $f(x) \in k[x]$, let $f(x) = a_n \prod_{i=1}^{d} f_i(x)^{m_i}$ be an irreducible factorization, so that each f_i is monic and irreducible over k, each pair f_i, f_j are coprime for*

$i \neq j$, and $n = \sum_{i=1}^{d} m_i \cdot \deg(f_i)$. Let $N = \sum_{i=1}^{d} m_i$. If each f_i is separable, then

$$\mathrm{tol}(f) = a_n^{2n-2} \prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i(2m_i - N)} \cdot \prod_{i<j} \mathrm{disc}(f_i f_j)^{m_i m_j}.$$

Proof. The proof of this lemma is given by computing the right hand side of this equation and comparing to the defining formula for $\mathrm{tol}(f)$. Since each irreducible factor of f is separable, each root of f is a root of exactly one of $f_1, \ldots, f_n$. If r_i is a root of f_i, then r_i is a root of f of multiplicity m_i. Next, we compute

$$\mathrm{disc}(f_i f_j)^{m_i m_j} = \mathrm{disc}(f_i)^{m_i m_j} \cdot \mathrm{disc}(f_j)^{m_i m_j} \cdot \prod_{\substack{f_i(r_i)=0 \\ f_j(r_j)=0}} (r_i - r_j)^{2m_i m_j}.$$

The third term is a product over all the roots of f_i and f_j and correctly accounts for factors of the form $(r_i - r_j)^{2m_i m_j}$ in $\mathrm{tol}(f)$. However, two distinct roots $r_{i,1}, r_{i,2}$ of f_i contribute a factor of $(r_{i,1} - r_{i,2})^{2m_i^2}$ to $\mathrm{tol}(f)$, whereas the same pair of roots contributes a factor of $(r_{i,1} - r_{i,2})^{2m_i m_j}$ to $\mathrm{disc}(f_i)^{m_i m_j} \cdot \mathrm{disc}(f_j)^{m_i m_j}$. We therefore need to compute the product

$$\prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i(2m_i - N)} \cdot \prod_{i<j} \mathrm{disc}(f_i)^{m_i m_j} \mathrm{disc}(f_j)^{m_i m_j}$$

$$= \prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i(2m_i - N)} \cdot \prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i \sum_{j \neq i} m_j}$$

$$= \prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i(2m_i - N) + m_i(N - m_i)}$$

$$= \prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i^2}.$$

The factors of $\mathrm{disc}(f_i)^{m_i^2}$ take the form $(r_{i,1} - r_{i,2})^{2m_i^2}$, as desired. Thus

$$\prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i(2m_i - N)} \cdot \prod_{i<j} \mathrm{disc}(f_i f_j)^{m_i m_j}$$

$$= \prod_{i=1}^{d} \mathrm{disc}(f_i)^{m_i^2} \cdot \prod_{i<j} \prod_{\substack{f_i(r_i)=0 \\ f_j(r_j)=0}} (r_i - r_j)^{2m_i m_j}$$

$$= \prod_{\substack{f(r)=0 \\ f(r')=0}} (r - r')^{2mm'},$$

where m and m' are the multiplicities of r and r' as roots of f. In particular, multiplying this product by a_n^{2n-2} gives $\mathrm{tol}(f)$. $\square$

It follows that if the irreducible factors of f are separable (for example, if k is perfect), then the tolerant of f is rational.

Corollary 2.5. *If every irreducible factor of f is separable, then* $\mathrm{tol}(f) \in k^{\times}$.

Proof. Each irreducible factor of f is an element of $k[x]$, so $\mathrm{disc}(f_i), \mathrm{disc}(f_i f_j) \in k$ by rationality of the discriminant. The claim now follows from Lemma 2.4.   □

Next, we aim to prove a similar formula for $\mathrm{tol}(f)$ when its irreducible factors are not all separable. Inseparable irreducible polynomials only occur over imperfect fields, which are always of positive characteristic. If $f \in k[x]$ is an inseparable and irreducible polynomial, then there exists a separable polynomial $f_{\mathrm{sep}} \in k[x]$ and an exponent $e \geq 1$ such that $f(x) = f_{\mathrm{sep}}(x^{p^e})$, where $p = \mathrm{char}\,k$. If $r_1, \ldots, r_s$ are the roots of f_{sep} (which are necessarily distinct), then the roots of f are $r_1^{p^{-e}}, \ldots, r_s^{p^{-e}}$ (each with multiplicity p^e). This allows us to compute the tolerant of irreducible, inseparable polynomials.

Lemma 2.6. *Let k be a field of characteristic p. Let $f(x)$ be a monic, irreducible, inseparable polynomial over k. Write $f(x) = f_{\mathrm{sep}}(x^{p^e})$, and let $r_1, \ldots, r_s$ be the roots of f_{sep}. Then*

$$\mathrm{tol}(f) = \prod_{i<j}(r_i - r_j)^{2p^e}$$

$$= \mathrm{disc}(f_{\mathrm{sep}})^{p^e}.$$

Proof. By the discussion above and the definition of the tolerant, we have

$$\mathrm{tol}(f) = \prod_{i<j}(r_i^{1/p^e} - r_j^{1/p^e})^{2p^e p^e}.$$

By the freshman's dream, we have $(r_i^{1/p^e} - r_j^{1/p^e})^{p^e} = r_i - r_j$, which yields the first equality. Finally, since f_{sep} is separable with roots $r_1, \ldots, r_s$, we have $\mathrm{disc}(f_{\mathrm{sep}}) = \prod_{i<j}(r_i - r_j)^2$.   □

We can unify the computation of $\mathrm{tol}(f)$ for irreducible polynomials as follows.

Definition 2.7. Given a field k, the *characteristic exponent* of k is $\mathrm{char}\,k$ if $\mathrm{char}\,k > 0$, and is 1 if $\mathrm{char}\,k = 0$.

Now let k be a field of characteristic exponent $p \geq 1$. Given a polynomial $f \in k[x]$, let e denote the *degree of inseparability* of f, which is the minimal exponent $e \geq 0$ such that there exists a separable polynomial f_{sep} with $f(x) = f_{\mathrm{sep}}(x^{p^e})$.

Corollary 2.8. *If k is a field of characteristic exponent p and $f \in k[x]$ is a monic, irreducible polynomial, then*

$$(2.1) \qquad\qquad \mathrm{tol}(f) = \mathrm{disc}(f_{\mathrm{sep}})^{p^e}.$$

Proof. If char $k > 0$, then this is Lemma 2.6. If char $k = 0$, then all irreducible polynomials are separable, and so the $\text{tol}(f) = \text{disc}(f)$ in this case. $\square$

Using Equation 2.1, we can give a formula for the tolerant of not-necessarily-separable polynomials in terms of the discriminants of their irreducible factors. We can then deduce rationality of the tolerant from this formula.

Lemma 2.9. *Let k be a field of characteristic exponent p. Given $f(x) \in k[x]$, let $f(x) = a_n \prod_{i=1}^d f_{i,\text{sep}}(x^{p^{e_i}})^{m_i}$ be an irreducible factorization, so that each $f_i = f_{i,\text{sep}}(x^{p^{e_i}})$ is monic and irreducible over k, each pair f_i, f_j are coprime for $i \neq j$, and*

$$n = \sum_{i=1}^d m_i \cdot p^{e_i} \cdot \deg(f_{i,\text{sep}}).$$

Let $N = \sum_{i=1}^d m_i p^{e_i}$. Then

$$(2.2) \qquad \text{tol}(f) = a_n^{2n-2} \prod_{i=1}^d \text{disc}(f_{i,\text{sep}})^{m_i p^{e_i}(2m_i p^{e_i} - N)} \cdot \prod_{i<j} \text{disc}(f_{i,\text{sep}} f_{j,\text{sep}})^{m_i m_j p^{e_i + e_j}}.$$

Proof. The proof is analogous to that of Lemma 2.4. The roots of f_i take the form $r_i^{1/p^{e_i}}$, where r_i is a root of $f_{i,\text{sep}}$, each having multiplicity $m_i p^{e_i}$. The factors of the form $\text{disc}(f_{i,\text{sep}} f_{j,\text{sep}})^{m_i m_j p^{e_i + e_j}}$ overcount factors of the form $(r_{i,1}^{1/p^{e_i}} - r_{i,2}^{1/p^{e_i}})^{2m_i^2 p^{2e_i}}$, so it remains to calculate the extent of this overcounting. The total multiplicity of such factors is

$$2m_i p^{e_i} \sum_{j \neq i} m_j p^{e_j},$$

so we need to correct by a multiplicity of

$$2m_i^2 p^{2e_i} - 2m_i p^{e_i} \sum_{j \neq i} m_j p^{e_j} = 2m_i p^{e_i} \left(m_i p^{e_i} - \sum_{j \neq i} m_j p^{e_j} \right)$$
$$= 2m_i p^{e_i}(2m_i p^{e_i} - N). \qquad \square$$

Corollary 2.10. *If $f \in k[x]$, then $\text{tol}(f) \in k^\times$.*

Proof. Each of the discriminants in Equation 2.2 is an element of $k^\times$. $\square$

2.3. **Inversion invariance.** We now come to the inversion of a polynomial.

Definition 2.11. Given a polynomial $f(x) \in k[x]$ with $f(0) \neq 0$, the *reciprocal* of f is defined as

$$f^*(x) = x^{\deg(f)} \cdot f(\tfrac{1}{x}).$$

Given a polynomial f satisfying $f(0) \neq 0$, one can readily prove that $\text{disc}(f) = \text{disc}(f^*)$ by noting that if r is a multiplicity m root of f, then r^{-1} is a multiplicity m root of f^*. However, this same observation quickly leads to examples of polynomials whose tolerant is not inversion invariant.

Example 2.12. Let $f(x) = (x-2)^2(x-3)$. Then $f^*(x) = (1-2x)^2(1-3x)$, and

$$\mathrm{tol}(f) = (2-3)^4 = 1,$$
$$\mathrm{tol}(f^*) = 12^4(\tfrac{1}{2} - \tfrac{1}{3})^4 = 2^4.$$

Example 2.13. Let $f(x) = (x-2)^2(x+\tfrac{1}{4})$. Then $f^*(x) = (1-2x)^2(1+\tfrac{1}{4}x)$, and

$$\mathrm{tol}(f) = (2+\tfrac{1}{4})^4 = (\tfrac{9}{4})^4,$$
$$\mathrm{tol}(f^*) = (\tfrac{1}{2} + 4)^4 = (\tfrac{9}{2})^4.$$

Of course, it is trivially true that $\mathrm{tol}(f) = \mathrm{tol}(f^*)$ whenever $f = f^*$; such polynomials are called *palindromic*. It follows that the set of polynomials whose tolerant is inversion invariant is a non-empty proper subset $T \subset k[x] - (x) = \{f \in k[x] : f(0) \neq 0\}$. Our next goal is to characterize the set T.

Proposition 2.14. *Let $f(x) = a_n x^n + \ldots + a_0$ with $a_n, a_0 \neq 0$. Let $r_1, \ldots, r_s$ be the roots of f with multiplicities $m_1, \ldots, m_s$. Then $\mathrm{tol}(f) = \mathrm{tol}(f^*)$ if and only if*

$$(2.3) \qquad\qquad \prod_{i<j}(r_i r_j)^{2m_i m_j} = \left(\frac{a_0}{a_n}\right)^{2n-2}.$$

Proof. We have $\mathrm{tol}(f) = \mathrm{tol}(f^*)$ if and only if

$$a_n^{2n-2}\prod_{i<j}(r_i - r_j)^{2m_i m_j} = a_0^{2n-2}\prod_{i<j}(\tfrac{1}{r_i} - \tfrac{1}{r_j})^{2m_i m_j}$$
$$= a_0^{2n-2}\prod_{i<j}\left(\frac{r_j - r_i}{r_i r_j}\right)^{2m_i m_j}$$
$$= a_0^{2n-2}\prod_{i<j}(r_i r_j)^{-2m_i m_j} \cdot \prod_{i<j}(r_i - r_j)^{2m_i m_j}. \qquad \square$$

Here are a few classes of polynomials that satisfy Equation 2.3.

Theorem 2.15. *Let $f(x) \in k[x]$, and assume that $f(0) \neq 0$. Assume that f satisfies at least one of the following criteria:*

(i) f is separable.

(ii) $f = gh$ for some coprime $g, h \in k[x]$ satisfying Equation 2.3.

Then $\mathrm{tol}(f) = \mathrm{tol}(f^*)$.

Proof. We will treat each case in turn.

(i) If f is separable, then f^* is also separable. It then follows that $\mathrm{tol}(f) = \mathrm{disc}(f) = \mathrm{disc}(f^*) = \mathrm{tol}(f^*)$.

(ii) Let $g(x) = g_a x^a + \ldots + g_0$ with roots $r_1, \ldots, r_c$ of multiplicities $m_1, \ldots, m_c$ and $h(x) = h_b x^b + \ldots + h_0$ with roots $s_1, \ldots, s_d$ of multiplicities $n_1, \ldots, n_d$. Then $gh(x) = g_a h_b x^{a+b} + \ldots + g_0 h_0$, and the set of roots (with their multiplicities) of gh is the union of the sets of roots of g and h (with their multiplicities). Denote the roots of $f = gh$ by z_i, with multiplicity ℓ_i. Since g and h both satisfy Equation 2.3, it follows that

$$\prod_{i<j}(z_i z_j)^{2\ell_i \ell_j} = \prod_{1 \le i < j \le c}(r_i r_j)^{2m_i m_j} \cdot \prod_{1 \le i < j \le d}(s_i s_j)^{2n_i n_j} \cdot \prod_{\substack{1 \le i \le c \\ 1 \le j \le d}}(r_i s_j)^{2m_i n_j}$$

$$= \left(\frac{g_0}{g_a}\right)^{2a-2} \cdot \left(\frac{h_0}{h_b}\right)^{2b-2} \cdot \prod_{i=1}^{c} r_i^{2m_i b} \cdot \prod_{j=1}^{d} s_j^{2n_j a}.$$

By Vieta's formulas, we have $\prod_i r_i^{m_i} = \pm\frac{g_0}{g_a}$ and $\prod_j s_j^{n_j} = \pm\frac{h_0}{h_b}$, so $\prod_i r_i^{2m_i b} = (\frac{g_0}{g_a})^{2b}$ and $\prod_j s_j^{2n_j a} = (\frac{h_0}{h_b})^{2a}$. Thus

$$\prod_{i<j}(z_i z_j)^{2\ell_i \ell_j} = \left(\frac{g_0}{g_a}\right)^{2a+2b-2} \cdot \left(\frac{h_0}{h_b}\right)^{2a+2b-2}$$

$$= \left(\frac{g_0 h_0}{g_a h_b}\right)^{2(a+b)-2},$$

so f satisfies Equation 2.3. $\qquad\square$

Theorem 2.15 (ii) states that the set T of polynomials for which the tolerant is inversion invariant is a multiplicative set. We note that not all elements of T are characterized by Theorem 2.15. For a simple example, $f(x) = (x - 1)^n$ and its reciprocal both have tolerant 1, as their leading coefficients both square to 1 and the product of pairs of roots is an empty product.

The previous example might lead one to guess that any power of a separable polynomial belongs to T, but this guess is incorrect. Indeed, together with Theorem 2.15 and the fundamental theorem of algebra, such a statement would imply that $T = k[x]$ whenever k is a perfect field. Such a conclusion would contradict Example 2.13. We can also directly construct an example of a polynomial that is a power of a separable polynomial and does not belong to T.

Example 2.16. Let $f(x) = (x - 1)^2(x - 2)^2$. Then $f^*(x) = (1 - x)^2(1 - 2x)^2$, and

$$\mathrm{tol}(f) = (1 - 2)^8 = 1,$$
$$\mathrm{tol}(f^*) = 4^6(1 - \tfrac{1}{2})^8 = 2^4.$$

3. Intrinsic formula for the tolerant

So far, all known formulas for the tolerant require some information about its irreducible factorization. One of the key advantages of the discriminant is that it provides information about the irreducible factors (namely, whether there is a multiple root) strictly in

terms of the coefficients of $f(x) = a_n x^n + \ldots + a_0$. This can be seen by the resultant formula

$$\operatorname{disc}(f) = \frac{(-1)^{\binom{n}{2}}}{a_n} \operatorname{res}(f, f').$$

We are interested in deriving an analogous formula for the tolerant. We had left such a formula as an open question in our first preprint of this article, but Rémi Prébet directed our attention to [DDR+22]. Using the ideas in this article, we can give the desired resultant formula for $\operatorname{tol}(f)$.

Definition 3.1. For each $r \geq 0$, let

$$D^r : k[x] \to k[x]$$

be the k-linear map determined by

$$D^r x^n = \begin{cases} \binom{n}{r} x^{n-r} & n \geq r, \\ 0 & n < r. \end{cases}$$

The map D^r is called the rth *Hasse derivative*. Given $f \in k[x]$, we will write

$$f^{[r]} := D^r f.$$

Note that if $\operatorname{char} k = 0$, then $D^r = \frac{1}{r!} \frac{\mathrm{d}^r}{\mathrm{d}\, x^r}$. Hasse derivatives are defined in a manner that provides an analog of Taylor's theorem over fields of arbitrary characteristic [Gol03]. In particular, for any $f \in k[x]$ and $\alpha \in \overline{k}$, one has

$$(3.1) \qquad f(x + \alpha) = \sum_{i=0}^{\deg(f)} f^{[i]}(\alpha) x^i.$$

Notation 3.2. Given a ring R and a polynomial $f \in R[z]$, let $\operatorname{lc}_z(f)$ and $\operatorname{tc}_z(f)$ denote the coefficients of the non-zero terms in f of largest and smallest degree, respectively. We will refer to $\operatorname{lc}_z(f)$ and $\operatorname{tc}_z(f)$ as the *leading* and *trailing coefficients* of f.

Definition 3.3. [DDR+22, Definition 2.3] Given $f \in k[x]$, the *generalized discriminant* of f is defined as

$$\operatorname{gdisc}(f) := \operatorname{lc}_x(f)^{-1} \cdot \operatorname{tc}_u \left(\operatorname{res}_x \left(f, \sum_{i=1}^{\deg(f)} u^{i-1} \cdot f^{[i]} \right) \right).$$

We now prove that $\operatorname{gdisc}(f) = \pm \operatorname{tol}(f)$ over any field, generalizing [DDR+22, Proposition 2.4].

Theorem 3.4. *We have* $\operatorname{gdisc}(f) = (-1)^{\binom{\deg(f)}{2}} \operatorname{tol}(f)$.

Proof. Let $g, h \in k[x]$ of degrees m and n. Let $r_1, \ldots, r_s \in \overline{k}$ denote the roots of g, and let m_i denote the multiplicity of r_i as a root of g. It is a standard fact about resultants that

$$\mathrm{res}_x(g, h) = \mathrm{lc}_x(g)^n \cdot \prod_{i=1}^{s} h(r_i)^{m_i}.$$

Applying this fact to $g = f$ and $h = \sum_{i=1}^{\deg(f)} u^{i-1} \cdot f^{[i]}$, we have

$$\mathrm{res}_x\left(f, \sum_{i=1}^{\deg(f)} u^{i-1} \cdot f^{[i]} \right) = \mathrm{lc}_x(f)^{\deg(f)-1} \cdot \prod_{j=1}^{s} \sum_{i=1}^{\deg(f)} u^{i-1} \cdot f^{[i]}(r_j)^{m_i}.$$

Note that $f^{[i]}(r_j) = 0$ for $i < m_j$ and

$$(3.2) \qquad f^{[m_j]}(r_j) = \mathrm{lc}_x(f) \cdot \prod_{i \neq j}(r_j - r_i)^{m_i} \neq 0$$

by Equation 3.1. This allows us to compute the trailing coefficient

$$\mathrm{tc}_u\left(\mathrm{lc}_x(f)^{\deg(f)-1)} \cdot \prod_{j=1}^{s} \sum_{i=1}^{\deg(f)} u^{i-1} \cdot f^{[i]}(r_j)^{m_i} \right) = \mathrm{lc}_x(f)^{\deg(f)-1} \cdot \prod_{j=1}^{s} f^{[m_j]}(r_j)^{m_j}.$$

Combined with Equation 3.2, we find that

$$\begin{aligned}
\mathrm{gdisc}(f) &= \mathrm{lc}_x(f)^{2\deg(f)-2} \cdot \prod_{i=1}^{s}\prod_{j \neq i}(r_j - r_i)^{m_i m_j} \\
&= (-1)^{\binom{\deg(f)}{2}} \mathrm{lc}_x(f)^{2\deg(f)-2} \cdot \prod_{i<j}(r_i - r_j)^{2m_i m_j} \\
&= (-1)^{\binom{\deg(f)}{2}} \mathrm{tol}(f). \qquad \qquad \square
\end{aligned}$$

The upshot of Theorem 3.4 is that we can compute the tolerant of a polynomial purely in terms of its coefficients via the resultant formula given in Definition 3.3.

4. Unstable Poincaré–Hopf at non-rational points

To conclude this article, we will apply Corollary 2.10 to state a conjecture on removing the rationality assumption from [IMS$^+$25, Proposition 6.5].

Conjecture 4.1. *Let $f/g : \mathbb{P}^1 \to \mathbb{P}^1$ be a pointed rational function with vanishing locus $D = \{r_1, \ldots, r_n\} \subset \mathbb{A}^1_k$. Let $m_i(x) \in k[x]$ denote the (monic) minimal polynomial of r_i for each i. At each zero, let the unstable local degree be given by $\deg_{r_i}(f/g) = (\beta_i, d_i) \in \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times$, where $\mathrm{GW}(k)$ denotes the Grothendieck–Witt group over k. Then we have the following decomposition of the unstable degree of f/g:*

$$\deg(f/g) = \left(\bigoplus_{i=1}^{n} \beta_i, \prod_{i=1}^{n} d_i \cdot \mathrm{tol}\left(\prod_{i=1}^{n} m_i(x) \right) \right).$$

More generally, we expect that

$$\bigoplus_D (\beta_i, d_i) = \left(\bigoplus_{i=1}^n \beta_i, \prod_{i=1}^n d_i \cdot \mathrm{tol} \left(\prod_{i=1}^n m_i(x) \right) \right)$$

for any $(\beta_1, d_1), \ldots, (\beta_n, d_n) \in \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times$. This more general statement should follow from Conjecture 4.1 by applying the same inductive method used in [IMS$^+$25]. In *op. cit.*, the known case of Conjecture 4.1 forms the base case of the induction.

In order to prove Conjecture 4.1, one will probably first need to derive a formula for the unstable local degree at non-rational points. At rational points, the unstable local degree is given by something called the *local Newton matrix*. The local Newton matrix has only been defined at rational points. To extend the definition to a non-rational point r, and to employ the change-of-basis techniques used to prove [IMS$^+$25, Proposition 6.5], one needs a suitable choice of k-basis for the residue field $k(r)$. A promising candidate for such a basis is the Horner basis [BM23].

5. CONCLUSION

The discriminant is a classical invariant of polynomials. The utility of the discriminant is that it vanishes on polynomials with repeated roots but can be computed without knowing the roots of the given polynomial. In this article, we investigated the *tolerant*, which coincides with the discriminant for polynomials with no repeated roots. In contrast to the discriminant, the tolerant never vanishes.

Like the discriminant, the tolerant is invariant under translations and homothety, and it is always valued in the field of definition of the given polynomial. Unlike the discriminant, the tolerant is not always inversion invariant. Interestingly, the set of polynomials for which the tolerant *is* inversion invariant is multiplicative.

The tolerant is almost identical to two other notions in the literature. The first, which was the inspiration for this article, is the *duplicant*, which naturally arises in unstable motivic homotopy theory. The second is the *generalized discriminant*, which was discovered in [DDR$^+$22]. The tolerant, duplicant, and generalized discriminant all differ from each other by multiplication by a scalar.

REFERENCES

[ABH24]   Aravind Asok, Tom Bachmann, and Michael J. Hopkins. On $\mathbb{P}^1$-stabilization in unstable motivic homotopy theory. arXiv:2306.04631, 2024.

[BM23]   Thomas Brazelton and Stephen McKean. Lifts, transfers, and degrees of univariate maps. *Math. Scand.*, 129(1):5–38, 2023.

[DDR$^+$22] Daouda Niang Diatta, Sény Diatta, Fabrice Rouillier, Marie-Françoise Roy, and Michael Sagraloff. Bounds for polynomials on algebraic numbers and application to curve topology. *Discrete Comput. Geom.*, 67(3):631–697, 2022.

[Gol03]   David M. Goldschmidt. *Algebraic functions and projective curves*, volume 215 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[IMS$^+$25] John Igieobo, Stephen McKean, Steven Sanchez, Dae'Shawn Taylor, and Kirsten Wickelgren. Motivic configurations on the line. *Adv. Math.*, 482:Paper No. 110637, 2025.

[May72] J. P. May. *The geometry of iterated loop spaces*, volume Vol. 271 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972.

[Mur99] M. Pavaman Murthy. A survey of obstruction theory for projective modules of top rank. In *Algebra, K-theory, groups, and education (New York, 1997)*, volume 243 of *Contemp. Math.*, pages 153–174. Amer. Math. Soc., Providence, RI, 1999.

[Voe00] V. Voevodsky. My view of the current state of the motivic homotopy theory. Online note, 2000.

DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY

Email address: adhikar6@student.byu.edu

DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY

Email address: brentah@student.byu.edu

DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY

Email address: mckean@math.byu.edu

URL: shmckean.github.io