LECTURE 1: INTRODUCTION AND KEPLER'S LAWS

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By the very nature of scientific inquiry, any subject that is carefully studied will inevitably branch and splinter into disparate subfields. This phenomenon has become quite pronounced in the mathematics of the 20^{th} and 21^{st} centuries. It is therefore miraculous, or perhaps a signature of truth, that common threads can emerge to interweave and bind together topics that had long since diverged. The goal of these notes is to follow one such thread: *topological modular forms*. The story and trajectory of topological modular forms will lead us back and forth between physics, operator theory, number theory, algebraic geometry, and homotopy theory.

To set the stage, we will give a brief outline of the topics to be covered. The general narrative of this outline is inspired by Hopkins's 1994 ICM address [Hop95]. This outline will gloss over a lot of the technical details, so don't worry if this doesn't all make sense. We will spend a lot more time on the details in the coming weeks. Today, we just want to get excited about the direction we're headed.

1. A concise history of tmf

1.1. **Genera.** Manifolds are fundamental mathematical objects. As is often the case with fundamental objects, you usually want to use an equivalence relation to simplify the study of manifolds. Whenever you're trying to understand equivalence classes, you want to look at invariants of the equivalence relation. Cobordism invariants are called *genera*:

Definition 1.1. Let R be a commutative ring. A *genus* is a function $\Phi : \mathrm{Mfd} \to R$ such that

(i) $\Phi(M_1) = \Phi(M_2)$ if there exists $W \in \mathrm{Mfd}$ with $\partial W = M_1 \sqcup M_2$,

(ii) $\Phi(M_1 \sqcup M_2) = \Phi(M_1) + \Phi(M_2)$,

(iii) $\Phi(M_1 \times M_2) = \Phi(M_1)\Phi(M_2)$, and

(iv) $\Phi(\partial M) = 0$.

The two types of cobordism that we understand best are:

- MU_*, the ring of cobordism classes of stably almost complex manifolds, and
- MSO_*, the ring of cobordism classes of oriented manifolds.

We know the ring structure of MU_* and MSO_* [Tho54; Mil60; Nov60; Wal60]. The torsion-free parts of these rings admit very clean descriptions:

$$\begin{aligned} \mathrm{MU}_* \otimes \mathbb{Q} &\cong \mathbb{Q}[\mathbb{CP}^1, \mathbb{CP}^2, \ldots], \\ \mathrm{MSO}_* \otimes \mathbb{Q} &\cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]. \end{aligned} \tag{1.1}$$

This implies that if R is torsion-free, then $\Phi : \mathrm{MU}_* \to R$ is determined by the values it takes on $\mathbb{CP}^n$ for all n. Moreover, if $\Phi(\mathbb{CP}^{2n}) = 0$ for all n, then Φ factors through MSO_*.

With what we've seen so far, we could try to cook up a genus by prescribing its values on $\mathbb{CP}^n$. A better way to collect this sort of data is with a generating function. There are two important types of generating functions of genera:

$$\log_\Phi(z) = \sum_{n\geq 0} \Phi(\mathbb{CP}^n)\frac{z^{n+1}}{n+1}, \tag{logarithm}$$

$$K_\Phi(z) = \frac{z}{\exp_\Phi(z)}, \tag{characteristic series}$$

where $\exp_\Phi(z) = \log_\Phi^{-1}(z)$.

Remark 1.2. If R is torsion-free, note that $\exp_\Phi(z)$ is an odd function if and only if $\Phi(\mathbb{CP}^{2n+1}) = 0$ for all n. In particular, Φ factors through MSO_* if and only if $K_\Phi(z)$ is an even function.

Example 1.3. Here are some examples of genera:

- The *Todd genus*, given by $\Phi(\mathbb{CP}^n) = 1$ for all n.
- The $\hat{A}$ *genus*, which has characteristic series $K(z) = \frac{z}{e^{z/2}-e^{-z/2}}$. This genus is an invariant of oriented manifolds. (Can you see why?)
- The *elliptic genus* [Och87], which has logarithm
$$\log_\Phi(z) = \int_0^z \frac{1}{\sqrt{1-2\delta t^2+\varepsilon t^4}}\, dt.$$
Here, δ and ε are parameters. When $\delta^2 \neq \varepsilon$ and $\varepsilon \neq 0$, $\log_\Phi(z)$ is an *elliptic function*.[1] We'll talk about these next time.
- The *Witten genus* [Wit87; Wit88], which has characteristic series
$$K(z) = \frac{z}{e^{z/2}-e^{-z/2}} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n e^z)(1-q^n e^{-z})}.$$
Note that the Witten genus takes values in $\mathbb{Q}[\![q]\!]$. Setting $q = 0$ recovers the $\hat{A}$ genus.

  On certain classes of manifolds, the Witten genus takes values in $\mathbb{Z}[\![q]\!]$, along with some other nice properties that make the Witten genus a *modular form*.

[1] When $\delta^2 = \varepsilon$ or $\varepsilon = 0$, can you see why you might the term *logarithm* is appropriate?

Exercise 1.4. Compute the logarithm and characteristic series of the Todd genus. Compute the $\hat{A}$ genus of $\mathbb{CP}^{2n}$. If you are feeling very ambitious, try computing the characteristic series of the elliptic genus, the logarithm of the Witten genus, or the elliptic and Witten genera of $\mathbb{CP}^n$.

Exercise 1.5. When $\delta = \varepsilon = 1$, the elliptic genus recovers the *signature* of a manifold (we'll talk about this next week). It turns out that for a particular choice of δ and ε, the elliptic genus also recovers that $\hat{A}$ genus. Find these values of δ and ε.

Remark 1.6. Why does anyone care about genera? Well, assuming that you have a way to compute $\Phi(M)$ for a given manifold, you can use a genus to measure whether M and N are cobordant. Note that the Todd genus cannot distinguish between projective spaces, while the $\hat{A}$ genus can distinguish between some projective spaces.

A general mathematical philosophy is that a good invariant is the shadow of some richer structure. Ochanine's definition of the elliptic genus was guided by some ideas in physics.[2] Witten took this physical inspiration even further in defining the Witten genus. This was a very exciting development in the mathematical community — ideas from physics led to new invariants of manifolds. Are these new genera the shadows of some richer geometric theory? Topological modular forms are part of the answer to this question, and part of a new question that arises.

1.2. **From genera to cohomology.** I've claimed that genera are interesting invariants, and that interesting invariants should be artifacts of rich theories. So let's see what sort of theory comes out of genera. Our angle of attack is to *work in families*. This is very similar to the *relative* philosophy that revolutionized algebraic geometry in the mid-1900s: algebraic varieties over a field are interesting, but schemes over some base are even more powerful. Looking fiber-by-fiber over the base, a scheme is a family of varieties parameterized by the base.

Recall that MU_* and MSO_* are rings (whose elements are complex cobordism classes and oriented cobordism classes, respectively). We saw nice formulas for these rings after removing their torsion (Equation 1.1). In fact, more is true. Any form of cobordism determines a generalized homology theory and a generalized cohomology theory. Such theories satisfy the familiar axioms of (co)homology (like homotopy invariance, exactness along inclusions, excision, and additivity over disjoint unions). However, the dimension axiom need not hold for generalized (co)homology theories: you can get non-trivial values in negative degrees, and (co)homology of a point need not be just the ring of coefficients. Indeed,

$$\begin{aligned}\mathrm{MU}_* &= \mathrm{MU}_*(\mathrm{pt}),\\ \mathrm{MSO}_* &= \mathrm{MSO}_*(\mathrm{pt}),\end{aligned}$$

and we have seen that just the torsion-free parts of these rings are already loaded with stuff. Another familiar type of generalized cohomology theory is K-theory. The study of

[2]The elliptic genus is a *partition function* of a type of *superstring*. I don't know what any of that means, but maybe we'll learn about it in Dan Freed's quantum theory class.

these things will lead us to the world of spectra[3], which we'll talk about more officially later in the course.

For now, here's a rough approximation of what the cohomology theory of a cobordism theory means. Let Ω denote a cobordism theory (so MU, or MSO, or something fancier if you've seen this before). Take a base space S. Over each point $s \in S$, associate an n-manifold M_s, with the requirement that M_s and $M_{s'}$ are Ω-cobordant for all $s, s' \in S$. So as you follow a path in S, the fibers M_s trace out a cobordism from M_{s_0} to M_{s_1}. This family is an element of the cohomology group $\Omega^{-n}(S)$. So $\Omega^{-n}(S)$ tells you something about families (over S) of Ω-cobordant manifolds.

Question 1.7. What we just did was a little hand-wavy, but can you see how to turn the set $\Omega^{-n}(S)$ into a group? What about ring structure? And what should $\Omega^*(S)$ mean in positive degrees?

We'll make this whole story rigorous in the coming weeks. For now, the key is to think of cobordism as a generalized cohomology theory $\Omega : \mathrm{Mfd} \to \mathrm{Groups}$. If we keep track of the grading coming from dimension, we might even get $\Omega : \mathrm{Mfd} \to \mathrm{GradedRings}$. This suggests how we should define genera in families:

Definition 1.8. Let Ω be the generalized cohomology theory associated to a cobordism theory. Let E be another generalized cohomology theory. An *E-valued genus* is a multiplicative map

$$\Omega \to E$$

of generalized cohomology theories.

Remark 1.9. This will all be a lot more convenient to say in terms of spectra. I'll talk more about spectra next week, and then probably again halfway through the semester.

Example 1.10. In Example 1.3, we saw various examples of genera. What do these look like in families? In other words, what is the generalized cohomology theory E and the map $\Omega \to E$ such that $\Omega(\mathrm{pt}) \to E(\mathrm{pt})$ recovers the functions in Example 1.3?

- The Todd genus can be promoted to $\mathrm{MU} \to H\mathbb{Q}$, or better yet to $\mathrm{MU} \to \mathrm{KU}$.
- When restricted to spin manifolds, the $\hat{A}$ genus can be promoted to $\mathrm{MSpin} \to \mathrm{KO}$. This was proved by Atiyah–Singer [AS63], which we'll learn about later in the semester.
- The elliptic genus can be promoted to $\mathrm{MSO} \to \mathrm{Ell}$, where
$$\mathrm{Ell}^*(-) = \mathrm{MSO}^*(-) \otimes_{\mathrm{MSO}_*(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{6}, \delta, \varepsilon, \Delta, \Delta^{-1}]/(64\varepsilon(\delta - \varepsilon^2)^2 - \Delta).$$
Here, Δ is the *modular discriminant*. Ell is known as *elliptic cohomology*, and we'll learn more about it later in the semester. For now, it suffices to point out

[3]Many words are overloaded in math, but few suffer from this problem so much as spectra. This overloading will be especially pronounced in this course, since we will be talking about homotopy theory, algebraic geometry, and operator theory.

that we have some bizarre formula, but we still do not know how to describe Ell geometrically! This is in contrast with most generalized cohomology theories (like ordinary cohomology, K-theory, cobordism), where we have a good geometric story to tell. The *Stolz–Teichner conjecture* says that the geometric nature of elliptic cohomology should come from quantum field theory.

- Promoting the Witten genus is where our story really kicks in. The natural domain of the family-level Witten genus is MString, but it was unknown what cohomology theory X should fit into the Witten genus $\mathrm{MString} \to X$. Topological modular forms are the answer to that question:
$$\mathrm{MString} \to \mathrm{tmf}.$$

Remark 1.11. The Witten genus is often called the *universal elliptic genus* — you can recover any elliptic genus (coming from some choice of δ and ε) by choosing an appropriate value of q. Similarly, topological modular forms are the *universal elliptic cohomology theory*. Justifying this statement will be one of the main highlights of this class.

Just as we don't know how to geometrically describe elliptic cohomology, we don't have a geometric description for tmf. But the Stolz–Teichner conjecture again says that this has something to do with quantum field theory.

Remark 1.12. Despite having the above list, extending a genus to families is not a canonical process. In order to extend a genus, one needs to really understand how that genus is defined in terms of geometry and analysis.

Some of you may wince when you hear that something is not canonical. One punchline of this course is that there is plenty of beautiful mathematics that is not canonical.

1.3. **The universal elliptic cohomology theory.** To close this section, I want to wave my hands at how tmf is defined. A *formal group law* over a ring R is just that — a formally defined rule for combining pairs of elements in R. There are a few axioms that don't matter to us right now, but the key point is that one can turn formal group laws into generalized cohomology theories. For example, the *additive* formal group law $F(x,y) = x + y$ yields ordinary cohomology, while the *multiplicative* formal group law $F(x,y) = x + y + xy$ yields K-theory.

In fact, any algebraic group determines a formal group law: you take the formal power series expansion of the product map at the identity.

Exercise 1.13. Compute the formal group laws determined by the algebraic groups $\mathbb{G}_a := (\mathbb{A}^1, +)$ and $\mathbb{G}_m := (\mathbb{A}^1 - \{0\}, \times)$.

So any algebraic group determines a formal group law. A formal group law determined by an elliptic curve in turn determines a cohomology theory, and such cohomology theories are called *elliptic cohomology* theories. So if you want a universal elliptic cohomology theory, you need some way of capturing the universal behavior of elliptic curves. This suggests that we should be working with the moduli space of elliptic curves.

It turns out that we need to take the Deligne–Mumford compactification $\overline{\mathcal{M}}_{1,1}$, but not just in ordinary schemes. We need to work in *derived* algebraic geometry, because we want to capture the higher structure that comes with a cohomology theory. In this setting, there is a structure sheaf $\mathcal{O}$ of $\overline{\mathcal{M}}_{1,1}$, and taking global sections gives us $\mathrm{Tmf} := \Gamma(\overline{\mathcal{M}}_{1,1}, \mathcal{O})$. Now we're almost there, but we have some extra stuff in negative degrees that we don't want. So we kill all the negative degrees to obtain tmf.

Setting up all this machinery and motivating where these ideas come from will be a big part of the last portion of this class. There's a sort of dance between the genus story and this construction of tmf. Lifting genera to families involves a lot of non-canonical reasoning in terms of geometry and analysis, while many definitions and constructions in homotopy theory and derived algebraic geometry use canonicity in a crucial way.

2. Kepler's laws of planetary motion

Alright, let's come back down to earth, or any planet for that matter. As we just learned, the discovery of topological modular forms was catalyzed by the introduction of the elliptic genus and its connection to elliptic cohomology. Because the term *elliptic* seems to show up everywhere, next lecture will be all about *elliptic functions*. I want to close today's lecture by explaining why elliptic functions are an inevitable piece of scientific history.[4]

The stars have long inspired the human mind. While most lights in the heavens follow a clear periodic motion, the planets wander. For millennia, the motion of the planets was a great mystery. A revolution in physics came about with Copernicus's theory of heliocentrism. Copernicus posited that the planets follow circular orbits around the sun, with the sun lying at the center.

Heliocentrism was revised by Kepler, who used Tycho Brahe's extensive astronomical measurements to show that while Mars does indeed orbit the sun, its orbit is not circular and the sun does not lie at the center. Instead, Mars's orbit is an ellipse, and the sun lies at one of the foci. These observations led Kepler to propose three laws of planetary motion:

Law 2.1 (Kepler)**.**

(1) The orbit of a planet around the sun is an ellipse, with the sun at one of the foci.

(2) A line segment joining the sun and planet sweeps out equal areas during equal intervals of time.

(3) The square of a planet's orbital period is proportional to the cube of its orbit's semi-major axis.

[4]When learning something new, I often ask myself: was this discovered by coincidence? Or would a different society on a different world eventually come up with the same concept?

Question 2.2. It turns out that all of the inner planets' orbits are almost circular, but Mars's orbit is the most eccentric these. Would it have been possible to recognize a less eccentric orbit as non-circular with the tools of Kepler's time? Would more precise tools for astronomical measurements have been developed without a knowledge of Kepler's laws? How might this have changed the history of science had we appeared on Mars instead of Earth?

Before we let this turn into an existential crisis, let's see how Newton's laws of motion imply Kepler's laws of planetary motion.

(1) Suppose we have two bodies of masses M and m. Fix our frame of reference to the larger body, so that the smaller body is moving with velocity v at distance r. Newton then gives us the total energy of the system as

$$E = \frac{mv^2}{2} - \frac{GMm}{r}.$$

Now we rewrite v in polar coordinates. The radial and tangential components of v are orthogonal, so $v^2 = r'^2 + r^2\theta'^2$.

The angular momentum of the smaller body is $L = r \times mv$. Since the vectors r and r' are parallel, we have $r \times r' = 0$. On the other hand, the vectors r and $r\theta'$ are orthogonal, so we find that $L = mr^2\theta'$. Now write $\rho = 1/r$ and note that $r' = -\frac{1}{\rho^2}\rho'$, so that $\theta = \int \frac{L\rho^2}{m} d\rho = -\int \frac{L}{mr'} d\rho$. Substituting, we get

$$\begin{aligned} E &= \frac{m}{2}\left(r'^2 + r^2\left(\frac{L}{mr^2}\right)^2\right) - \frac{GMm}{r} \\ &= \frac{mr'^2}{2} + \frac{L^2}{2mr^2} - \frac{GMm}{r}. \end{aligned}$$

Solve for r' and use the non-obvious substitutions $a = \frac{L^2}{GMm^2}$ and $e^2 = 1 + \frac{2Ea}{GMm}$ to find

$$r' = \frac{L}{m}\left(\frac{e^2}{a^2} - \left(\frac{1}{r} - \frac{1}{a}\right)^2\right)^{1/2}.$$

Since $\theta = -\int \frac{L}{mr'} d\rho$, we find that

$$\begin{aligned} \theta &= -\int \frac{1}{\sqrt{e^2/a^2 - (\rho - 1/a)^2}} d\rho \\ &= \arccos\left(\frac{\rho - 1/a}{e/a}\right). \end{aligned}$$

In other words, $r = \frac{a}{1+e\cos\theta}$, which is an ellipse with one focus at the origin.

(2) The infinitesimal area dA swept out by the planet is a right triangle with legs r and dr, so we have $dA = \frac{r}{2} \times dr$. It follows that rate of area swept out is $A' = \frac{r}{2} \times r'$. To show that this rate of area sweeping is constant, we need to show that $A'' = 0$. By the product rule, we have $A'' = \frac{1}{2}(r' \times r' + r \times r'')$. The first term vanishes, but the second term need not vanish in general. It happens to

vanish for planetary motion, because Newton's laws imply that the acceleration of the planet due to gravity is parallel to the line through the sun and planet. Thus $r \times r'' = 0$, so $A'' = 0$.

Exercise 2.3. Derive Kepler's third law from Newton's laws of motion. (Hint: the area of an ellipse with semi-major axis a and eccentricity e is $A = \pi a\sqrt{1-e^2}$. At the end of the day, you will find that $T^2 = \frac{4\pi^2 a^3}{GM}$, where T is the orbital period.)

Next time: Elliptic functions. Today's discussion on Kepler's laws will show how scientists were naturally led from observing the stars to thinking about elliptic functions.

Daily exercises: In each lecture, I will try to give at least a couple exercises for you to think about. These may range from trivial to impossible. The point is to encourage you to think about the material outside of lecture time. I'll always put a hyperlinked list of exercises at the end of the notes to make them easy to find.

- Exercise 1.4: a few genera computations.
- Exercise 1.5: recover the $\hat{A}$ genus from the elliptic genus.
- Exercise 1.13: compute the formal group laws associated to $\mathbb{G}_a$ and $\mathbb{G}_m$.
- Exercise 2.3: derive Kepler's third law.

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