SYMMETRIC POWERS OF NULL MOTIVIC EULER CHARACTERISTIC

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ABSTRACT. Let k be a field of characteristic not 2. We conjecture that if X is a quasi-projective k-variety with trivial motivic Euler characteristic, then $\mathrm{Sym}^n X$ has trivial motivic Euler characteristic for all n. Conditional on this conjecture, we show that the Grothendieck–Witt ring admits a power structure that is compatible with the motivic Euler characteristic and the power structure on the Grothendieck ring of varieties. We then discuss how these conditional results would imply an enrichment of Göttsche's formula for the Euler characteristics of Hilbert schemes.

1. INTRODUCTION

With an eye towards applications in enumerative geometry over non-closed fields, we investigate a few properties of the motivic Euler characteristic. We begin with a brisk summary of our conjecture and results. We will explain the context of this paper in Section 1.1. Let k be a field of characteristic not 2. Let $K_0(\mathrm{Var}_k)$ and $\mathrm{GW}(k)$ denote the Grothendieck ring of varieties over k and the Grothendieck–Witt ring of bilinear forms over k, respectively. Let

$$\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$$

denote the *motivic* or *compactly supported Euler characteristic* (see [Rön16, AMBO$^+$22, Azo22] and Section 2). Finally, let $\mathrm{Sym}^n : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$ denote the nth symmetric power map, defined on classes of quasi-projective varieties by $[X] \mapsto [\mathrm{Sym}^n X]$. We conjecture that Sym^n preserves $\ker \chi^c$ for all n.

Conjecture 1.1. *Let k be a field of characteristic not 2. If X is a quasi-projective k-variety such that $[X] \in \ker \chi^c$, then $[\mathrm{Sym}^n X] \in \ker \chi^c$ for all $n \geq 1$.*

The complex and real analogs of Conjecture 1.1 are true. Indeed, $X^n \to X^n/S_n$ is étale-locally trivial after a suitable stratification, and the rank and signature of χ^c (which correspond to the compactly supported Euler characteristic of $X(\mathbb{C})$ and $X(\mathbb{R})$ by Remark 4.2 and Proposition 4.4) factor over étale-locally trivial fiber bundles. In particular, if $\chi^c(X) = 0$, then the rank and signature[1] of $\chi^c(\mathrm{Sym}^n X)$ both vanish for all $n \geq 1$. (This allows us to prove Conjecture 1.1 over pythagorean fields; see Theorem 9.4). In general, χ^c only factors over locally trivial fiber bundles in topologies coarser than the étale topology.

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[1]Some extra care is needed due to the discrepancy between $(\mathrm{Sym}^n X)(\mathbb{R})$ and $\mathrm{Sym}^n(X(\mathbb{R}))$; see Corollary 9.2 for details.

Conjecture 1.1 is closely tied to the power structure on $K_0(\mathrm{Var}_k)$ [GZLMH04], the motivic Euler characteristic, and a desirable power structure on $\mathrm{GW}(k)$. We make this precise in the following theorem.

Theorem 1.2. *Let k be a field of characteristic not 2. Let*

$$\mu_0 : (1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]) \times K_0(\mathrm{Var}_k) \to 1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]$$

be the power structure defined in Section 6. Then Conjecture 1.1 is true if and only if there exists a power structure

$$\mu_{\mathrm{GW}} : (1 + t \cdot \mathrm{GW}(k)[\![t]\!]) \times \mathrm{GW}(k) \to 1 + t \cdot \mathrm{GW}(k)[\![t]\!]$$

such that

$$\chi^c(\mu_0(A(t), M)) = \mu_{\mathrm{GW}}(\chi^c(A(t)), \chi^c(M))$$

for all $A(t) \in 1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]$ and $M \in K_0(\mathrm{Var}_k)$.

In fact, one can explicitly work out a power structure on $\mathrm{GW}(k)$ that matches the conjectural power structure μ_{GW} (see [PP23] for details). The difficulty in Conjecture 1.1 is verifying the compatibility of μ_0 and μ_{GW} under the motivic Euler characteristic.

We also prove a few results that do not depend on Conjecture 1.1. For example, we give the following enrichment of Göttsche's formula for the Euler characteristic of the Hilbert scheme of points on a surface; this will be stated and proved in the body of the article as Theorem 8.7. An alternative form of this theorem, which depends on Conjecture 1.1, is given in Theorem 8.6.

Theorem 1.3. *Let k be a field of characteristic not 2. Then for any smooth quasi-projective k-surface X, we have*

$$\sum_{g \geq 0} \chi^c(\mathrm{Hilb}^g X) \cdot t^g = \prod_{n \geq 1} \left(1 + \sum_{m \geq 1} \langle -1 \rangle^{m(n-1)} \chi^c(\mathrm{Sym}^m X) \cdot t^{mn} \right)$$

in $\mathrm{GW}(k)[\![t]\!]$.

Along the way, we collect various properties of χ^c. Many of these already exist in the literature. An example of a new result is that χ^c is insensitive to universal homeomorphisms (Corollary 5.4). Recently, Pajwani–Rohrbach–Viergever showed that cellular varieties, linear varieties, and elliptic curves satisfy Conjecture 1.1 in characteristic 0 [PRV24]. Corollary 5.4 implies that their results extend to char $k \neq 2$.

1.1. **Motivation.** An influential result in enumerative geometry, formulated by Yau–Zaslow [YZ96] and proved by Beauville [Bea99], gives the generating function of certain counts of rational curves on K3 surfaces. Given a projective K3 surface X, there exists $g \geq 1$ such that X admits a g-dimensional linear system of curves of genus g. Denote the (appropriately weighted)[2] number of rational fibers of this linear system by $n(g)$. Set

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[2]The weight of a nodal rational curve is 1 so if all the rational curves in the linear series are nodal, the number is unweighted.

$n(0) := 1$. The *Yau–Zaslow formula* states that

$$(1.1) \qquad \sum_{g \geq 0} n(g) \cdot t^g = \prod_{n \geq 1} (1 - t^n)^{-24}.$$

As with many counts in algebraic geometry, Equation 1.1 is only valid over $\mathbb{C}$. One can often recover enumerative theorems over the reals by counting with an appropriate sign. Kharlamov–Răsdeaconu gave a signed analog of the Yau–Zaslow formula over $\mathbb{R}$ [KR15]. Each real rational curve has a Welschinger invariant ± 1. Given a g-dimensional linear system of curves of genus g on a real K3 surface X, let $n_+(g)$ and $n_-(g)$ denote the number of rational fibers with Welschinger invariant $+1$ and -1, respectively. Set $w(g) := n_+(g) - n_-(g)$. Then the *real Yau–Zaslow formula* states that

$$(1.2) \qquad \sum_{g \geq 0} w(g) \cdot t^g = \prod_{m \geq 1} (1 + t^m)^{-e_\mathbb{R}} \cdot \prod_{n \geq 1} (1 - t^{2n})^{-\frac{e_\mathbb{C} - e_\mathbb{R}}{2}},$$

where $e_\mathbb{C} = 24$ and $e_\mathbb{R}$ are the Euler characteristics of $X(\mathbb{C})$ and $X(\mathbb{R})$, respectively.

In *enriched, quadratic,* or $\mathbb{A}^1$-*enumerative geometry*, one applies tools from motivic homotopy theory to enumerative geometry over a wider class of base fields than just $\mathbb{C}$ or $\mathbb{R}$. Rather than integer-valued counts, enriched enumerative theorems are equations in the Grothendieck–Witt ring of the base field. Rank and signature of bilinear forms induce ring homomorphisms rank : $\mathrm{GW}(\mathbb{C}) \to \mathbb{Z}$ and sign : $\mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$. Applying these two maps to an enriched enumerative equation then yields a classical count over $\mathbb{C}$ and a signed count over $\mathbb{R}$.

We now expand on the previous paragraph in the context of the Yau–Zaslow formula. Let k be a field, possibly subject to some assumptions. Let X be a K3 surface over k that admits a g-dimensional linear system of curves of genus g. To each rational curve C, we would like to find an isomorphism class of bilinear forms $q(C) \in \mathrm{GW}(k)$ such that $\mathrm{sign}\, q(C)$ is the Welschinger invariant of C. Moreover, $\mathrm{rank}\, q(C)$ should recover the counting weight of C over $\mathbb{C}$ (i.e. 1 if C is nodal). Let $q(g)$ denote the sum of $q(C)$ over the rational fibers in the linear series on X. An *enriched Yau–Zaslow formula* should be an equality

$$(1.3) \qquad \sum_{g \geq 0} q(g) \cdot t^g = G(t),$$

where $G(t) \in \mathrm{GW}(k)[\![t]\!]$ is a power series such that the rank and signature of Equation 1.3 recover Equations 1.1 and 1.2, respectively. Applying other field invariants, such as discriminant or Hasse–Witt invariants, to Equation 1.3 would then give Yau–Zaslow theorems in other interesting cases, such as over finite fields or number fields.

Pajwani–Pál gave the first steps towards Equation 1.3 [PP22]. Their formula is given in characteristic 0 and holds up to rank, signature, and discriminant. Our results streamline various steps in op. cit., as well as give a conjectural approach to the global side of Yau–Zaslow formulae in positive characteristic. We will not address the local contributions of Yau–Zaslow formulae in this article.

The general roadblock to proving Conjecture 1.1 and treating the local terms of the Yau–Zaslow formula over fields other than $\mathbb{C}$ and $\mathbb{R}$ is the unwieldy nature of χ^c over étale-locally trivial fibrations. In contrast, compactly supported Euler characteristics factor over étale- or analytic-locally trivial fibrations, a fact which is crucial to proving Yau–Zaslow formulae over algebraically closed fields [Bea99] or $\mathbb{R}$ [KR15].

1.2. **Outline.** The first portion of this article is devoted to discussing the motivic Euler characteristic χ^c. We recall the definition of χ^c over fields of characteristic not 2 in Section 2. The motivic Euler characteristic was first defined in characteristic 0 in [Rön16, AMBO+22], and later over fields of characteristic not 2 in [Azo22]. However, we will take a slightly different approach than that of [Azo22], which will make it easier to study χ^c as a motivic measure.

In Sections 3, 4, and 5, we collect and prove some properties of the motivic Euler characteristic. We then recall the power structure on $K_0(\mathrm{Var}_k)$ in Section 6, discuss Conjecture 1.1 and power structures on $\mathrm{GW}(k)$ in Section 7, and apply our results to Göttsche's formula for Hilbert schemes in Section 8. Finally, we prove Conjecture 1.1 over pythagorean fields in Section 9.

1.3. **Related work.** Some details from this work are present in [PP23], which was split off from an earlier collaboration. In that article, the authors construct a power structure on $\mathrm{GW}(k)$ that conjecturally matches the one predicted in Theorem 1.2. The key difficulty is verifying that this power structure is compatible with the power structure on $K_0(\mathrm{Var}_k)$ and χ^c. They show that these two power structures agree under χ^c on varieties of dimension 0. In [PRV24], a larger class of varieties is given on which χ^c takes the power structure of $K_0(\mathrm{Var}_k)$ to the power structure of $\mathrm{GW}(k)$.

2. Motivic Euler characteristic

In this section, we recall the definition of the *motivic* or *compactly supported Euler characteristic*

$$\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k).$$

This definition will hold over fields of characteristic not 2. The starting point is the *categorical Euler characteristic* χ^{cat} in motivic homotopy theory, first defined by Levine [Lev19] (building on the work of Hoyois [Hoy14]).

The role of the Euler characteristic in Beauville's approach to the Yau–Zaslow formula cruicially depends on χ being a motivic measure. The motivic Euler characteristic χ^c is the result of forcing χ^{cat} to become a motivic measure. In characteristic 0, this can be done explicitly using Bittner's presentation of $K_0(\mathrm{Var}_k)$ [Rön16, AMBO+22]. In characteristic not 2, one can define χ^c categorically [Azo22]. We will give a slightly different approach than that of [Azo22], focusing on the nature of χ^c as a motivic measure.

2.1. **Background.** Let S be a scheme. Let $\mathrm{SH}(S)$ be the stable motivic homotopy category over S. The category $\mathrm{SH}(S)$ is symmetric monoidal, where the monoidal structure is given by a smash product $\wedge := \wedge_S$ and the unit object is the sphere spectrum $\mathbb{1}_S$. In this context, one can speak of strongly dualizable objects (see e.g. [Lev19, p. 257]).

Definition 2.1. The *dual* of a motivic spectrum $X \in \mathrm{SH}(S)$ is the object

$$X^\vee := \mathrm{Hom}_S(X, \mathbb{1}_S).$$

This gives rise to the evaluation map $\mathrm{ev} : X^\vee \wedge X \to \mathbb{1}_S$, which induces a map $\mathrm{can}_A : X^\vee \wedge A \to \mathrm{Hom}_S(X, A)$ for each $A \in \mathrm{SH}(S)$. We say that X is *strongly dualizable* if $\mathrm{can}_X : X^\vee \wedge X \to \mathrm{Hom}_S(X, X)$ is an isomorphism.

With this definition in hand (and building on [Hoy14]), Levine defines the Euler characteristic of a strongly dualizable motivic spectrum via the categorical definition [Lev19, Definition 1.3].

Definition 2.2. Let $X \in \mathrm{SH}(S)$ be strongly dualizable. The *categorical Euler characteristic* of X is the endomorphism

$$\mathbb{1}_S \to X \wedge X^\vee \to X^\vee \wedge X \to \mathbb{1}_S,$$

denoted $\chi_S^{\mathrm{cat}}(X) \in \mathrm{End}(\mathbb{1}_S)$. When X is an S-variety, we denote $\chi_S^{\mathrm{cat}}(X) := \chi_S^{\mathrm{cat}}(\Sigma_{\mathbb{P}^1}^\infty X_+)$. We will generally omit the base S from the notation unless confusion about the base may arise.

When $S = \mathrm{Spec}\, k$ is the spectrum of a field, Morel's $\mathbb{A}^1$-Brouwer degree theorem [Mor12, Corollary 1.24] gives an isomorphism $\mathrm{End}(\mathbb{1}_k) \cong \mathrm{GW}(k)$, so that $\chi^{\mathrm{cat}}(X) \in \mathrm{GW}(k)$. Under the further assumption that k is perfect of characteristic not 2, Levine–Raksit prove that $\chi^{\mathrm{cat}}(X)$ can be computed in terms of the de Rham complex for any smooth projective k-scheme X [LR20]. This was extended to smooth proper schemes over arbitrary fields by [BW23, Theorem 1.1]

Definition 2.3. Let k be a field. Given a smooth proper k-scheme X of dimension n, let $\mathrm{H}^i(X; \Omega_{X/k}^j)$ denote the Hodge cohomology groups of X for $0 \le i, j \le n$. By taking 0 maps as differentials, we get a perfect complex

$$\mathrm{Hdg}(X) := \bigoplus_{i,j=0}^n \mathrm{H}^i(X; \Omega_{X/k}^j)[j - i]$$

of k-vector spaces. Coherent duality defines a trace map $\eta : \mathrm{H}^n(X; \Omega_{X/k}^n) \to k$, so we can use the cup product to define perfect pairings

$$\beta_{i,j} : \mathrm{H}^i(X; \Omega_{X/k}^j) \otimes \mathrm{H}^{n-i}(X; \Omega_{X/k}^{n-j}) \xrightarrow{\smile} \mathrm{H}^n(X; \Omega_{X/k}^n) \xrightarrow{\eta} k.$$

The sum $\sum_{i,j=0}^n (-1)^{i+j} \beta_{i,j}$ determines a non-degenerate symmetric bilinear form on $\mathrm{Hdg}(X)$ (see [LR20, §8D] or [BW23, §1]). Denote the isomorphism class of this bilinear form by

$$\chi^{\mathrm{dR}}(X) \in \mathrm{GW}(k).$$

Theorem 2.4 (Levine–Raksit, Bachmann–Wickelgren). *If X is a smooth proper scheme over a field k, then $\chi^{\mathrm{cat}}(X) = \chi^{\mathrm{dR}}(X)$.*

Theorem 2.4 enables one to compute the categorical Euler characteristic of smooth proper schemes using more classical invariants. Unfortunately, outside of the set of smooth proper schemes, the categorical Euler characteristic fails to satisfy a desirable property: χ^{cat} is not a motivic measure. That is, if X is a variety with closed subvariety Z, then $\chi^{\mathrm{cat}}(X) \neq \chi^{\mathrm{cat}}(Z) + \chi^{\mathrm{cat}}(X \backslash Z)$ in general [Lev20, Proposition 2.4 (3)]. However, over fields of characteristic 0, the restriction of χ^{cat} to smooth projective varieties extends to a motivic measure [AMBO$^+$22, Theorem 1.13]. Moreover, this extension is unique [PP22, Theorem 2.10].

Theorem 2.5 (Arcila-Maya–Bethea–Opie–Wickelgren–Zakharevich, Pajwani–Pál). *Let k be a field of characteristic 0. There is a unique ring homomorphism*

$$\chi^{\mathrm{mot}} : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$$

such that $\chi^{\mathrm{mot}}([X]) = \chi^{\mathrm{cat}}(X)$ (or equivalently, $\chi^{\mathrm{mot}}([X]) = \chi^{\mathrm{dR}}(X)$) for all smooth projective irreducible k-varieties.

The *motivic Euler characteristic* χ^{mot} appeared previously in the work of Röndigs [Rön16, Theorem 5.2]. The assumption that char $k = 0$ allows one to use Bittner's presentation of $K_0(\mathrm{Var}_k)$, which is a crucial aspect of [AMBO$^+$22, PP22].

2.2. **Extending the motivic Euler characteristic.** We now extend the definition of χ^{mot} to more general bases by mimicking the categorical definition. We will start by working over a noetherian scheme S, which we will eventually assume to be the spectrum of a field. We will also eventually invert the characteristic of the base field whenever resolution of singularities is not known. Our extension of χ^{mot} to fields of characteristic at least 3 recovers Azouri's generalization of the compactly supported Euler characteristic in this setting [Azo22, Section 2], although at times we take a different route towards the desired definition.

To begin, we need to introduce two subcategories of $\mathrm{SH}(S)$.

Definition 2.6. A motivic spectrum $X \in \mathrm{SH}(S)$ is *compact* if $\mathrm{Hom}_S(X, -)$ commutes with arbitrary direct sums. Let

  (i) $\mathrm{SH}^\omega(S)$ be the full subcategory of $\mathrm{SH}(S)$ whose objects are compact, and

  (ii) $\mathrm{SH}^{\mathrm{rig}}(S)$ be the full subcategory of $\mathrm{SH}(S)$ whose objects are strongly dualizable.

By [Rio05], there is an inclusion of subcategories

$$(2.1) \qquad\qquad\qquad \mathrm{SH}^{\mathrm{rig}}(S) \subseteq \mathrm{SH}^\omega(S),$$

which is an equality when $S = \mathrm{Spec}\, k$ for k a field admitting resolution of singularities. In general, this inclusion may be strict. For example, $\mathrm{SH}^{\mathrm{rig}}(S) \neq \mathrm{SH}^\omega(S)$ when S is noetherian of positive dimension [Cis18].

The key insight that allows us to define the motivic Euler characteristic categorically is that there is a unique realization map from $K_0(\mathrm{Var}_S)$ to the Grothendieck ring of $\mathrm{SH}^\omega(S)$ [CLNS18, Chapter 2, Lemma 3.7.3]. We will state this more precisely in Lemma 2.9.

Definition 2.7. Let $\mathcal{C}$ be a triangulated category with a biadditive tensor product $\otimes$. The *Grothendieck group of* $\mathcal{C}$, denoted $K_0^\triangle(\mathcal{C})$, is the quotient of free abelian group generated by the set of isomorphism classes of objects in $\mathcal{C}$ by the set of all objects of the form
$$[X] - [Y] + [Z],$$
where $X \to Y \to Z \to X[1]$ is a distinguished triangle. The Grothendieck group can be promoted to the *Grothendieck ring of* $\mathcal{C}$, also denoted $K_0^\triangle(\mathcal{C})$, by imposing the relation $[X \otimes Y] = [X] \cdot [Y]$ for all objects X, Y in $\mathcal{C}$.

Remark 2.8. Recall that $\mathrm{SH}(S)$ admits the structure of a triangulated category, where the distinguished triangles are given by (co)fiber sequences. The fiber product $\times_S$ is biadditive with respect to the smash product $\wedge_S$, so we can take the Grothendieck ring of $\mathrm{SH}(S)$. Compact objects in a triangulated category are stable under extensions and retracts, and [May01, Theorem 0.1] implies the same for strongly dualizable objects. Thus both $\mathrm{SH}^\omega(S)$ and $\mathrm{SH}^{\mathrm{rig}}(S)$ are thick subcategories of $\mathrm{SH}(S)$, so the triangulated structure on $\mathrm{SH}(S)$ descends to triangulated structures on $\mathrm{SH}^\omega(S)$ and $\mathrm{SH}^{\mathrm{rig}}(S)$. In particular, we can form $K_0^\triangle(\mathrm{SH}^\omega(S))$ and $K_0^\triangle(\mathrm{SH}^{\mathrm{rig}}(S))$.

Lemma 2.9 (Chambert-Loir–Nicaise–Sebag)**.** *There is a unique ring homomorphism*
$$r_{\mathrm{mot}} : K_0(\mathrm{Var}_S) \to K_0^\triangle(\mathrm{SH}^\omega(S))$$
such that $r_{\mathrm{mot}}([X]) = [q_! \mathbb{1}_X]$ *for all quasi-projective varieties* $q : X \to S$ *(as the classes of such varieties generate* $K_0(\mathrm{Var}_S)$ *[CLNS18, Chapter 2, Corollary 2.6.6(a)]).*

Remark 2.10. In [CLNS18, Chapter 2, Lemma 3.7.3], the target of r_{mot} is actually $K_0^\triangle(\mathrm{SH}^c(S))$, where $\mathrm{SH}^c(S) \subseteq \mathrm{SH}^\omega(S)$ is the subcategory of *constructible* motivic spectra. However, $\mathrm{SH}(S)$ is constructibly generated (and therefore compactly generated) by [Hoy14, Proposition C.12], so the constructible objects in $\mathrm{SH}(S)$ are precisely the compact objects [CD19, Proposition 1.4.11]. It follows that $\mathrm{SH}^c(S) = \mathrm{SH}^\omega(S)$.

Note that for Lemma 2.9 to make sense, we need $q_! \mathbb{1}_X$ to be compact whenever $q : X \to S$ is quasi-projective. Quasi-projective implies separated of finite type [Sta18, Lemma 01VX], so $q_!$ preserves compact objects [CD19, Corollary 4.2.12] (the assumption that S is noetherian can be dropped, see e.g. [Kha21]). Since $\mathbb{1}_X \in \mathrm{SH}(X)$ is compact, it follows that $q_! \mathbb{1}_X$ is compact as well.

Notation 2.11. Given a set of primes $\mathcal{P}$, let $\mathrm{SH}(S)[\mathcal{P}^{-1}]$ denote the localization of $\mathrm{SH}(S)$ at the maps $p : E \to E$ for all $E \in \mathrm{SH}(S)$ and $p \in \mathcal{P}$. If $\mathcal{P} = \{p\}$, then we will write $\mathrm{SH}(S)[p^{-1}] := \mathrm{SH}(S)[\mathcal{P}^{-1}]$. We denote the localization functor by
$$L_\mathcal{P} : \mathrm{SH}(S) \to \mathrm{SH}(S)[\mathcal{P}^{-1}].$$

Since localization simply introduces more morphisms to our category, we can restrict to compact or strongly dualizable objects to obtain localization maps $L_{\mathcal{P}} : \mathrm{SH}^{\bullet}(S) \to \mathrm{SH}^{\bullet}(S)[\mathcal{P}^{-1}]$, where $\bullet = \omega$ or rig.

Corollary 2.12. *For any set of primes $\mathcal{P}$, there is a unique ring homomorphism*

$$r_{\mathrm{mot}} : K_0(\mathrm{Var}_S) \to K_0^{\triangle}(\mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}])$$

such that $r_{\mathrm{mot}}([X]) = [q_! \mathbb{1}_X]$ for all quasi-projective varieties $q : X \to S$.

Proof. It suffices to construct a ring homomorphism

$$L_{\mathcal{P}} : K_0^{\triangle}(\mathrm{SH}^{\omega}(S)) \to K_0^{\triangle}(\mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}])$$

taking the desired values on quasi-projective varieties. On objects, this map is given by $L_{\mathcal{P}}([X]) = [L_{\mathcal{P}} X]$. This is well-defined, as two isomorphic motivic spectra remain isomorphic after localization. Localization is an exact functor, which implies that $L_{\mathcal{P}}$ preserves:

- exact triangles, so $L_{\mathcal{P}} X \to L_{\mathcal{P}} Y \to L_{\mathcal{P}} Z$ is a (co)fiber sequence in $\mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}]$ for any (co)fiber sequence $X \to Y \to Z$ in $\mathrm{SH}^{\omega}(S)$, and

- finite limits, so that $L_{\mathcal{P}}(X \times_S Y) \cong L_{\mathcal{P}} X \times_{L_{\mathcal{P}} S} L_{\mathcal{P}} Y$.

It follows that $L_{\mathcal{P}}$ is a ring homomorphism. The localization $\mathrm{SH}^{\omega}(S) \to \mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}]$ is unique up to unique isomorphism by the universal property of localizations, so the induced map $L_{\mathcal{P}}$ on Grothendieck rings is uniquely determined. It now follows from Lemma 2.9 that

$$K_0(\mathrm{Var}_S) \xrightarrow{r_{\mathrm{mot}}} K_0^{\triangle}(\mathrm{SH}^{\omega}(S)) \xrightarrow{L_{\mathcal{P}}} K_0^{\triangle}(\mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}])$$

is uniquely determined. To simplify notation, we will generally conflate r_{mot} and $L_{\mathcal{P}} \circ r_{\mathrm{mot}}$. $\square$

Our next goal is to show that there is a unique Euler characteristic on $K_0^{\triangle}(\mathrm{SH}^{\omega}(S))$ that satisfies the expected categorical definition on $K_0^{\triangle}(\mathrm{SH}^{\mathrm{rig}}(S))$. We will begin by constructing a ring homomorphism $K_0^{\triangle}(\mathrm{SH}^{\mathrm{rig}}(S)) \to \mathrm{End}(\mathbb{1}_S)$ (as well as localizations thereof). We will then conclude with results of Riou and Elmanto–Khan [Rio05, EK20], which equate certain localizations of $\mathrm{SH}^{\omega}(k)$ and $\mathrm{SH}^{\mathrm{rig}}(k)$.

Lemma 2.13. *Let $\mathcal{P}$ be a (possibly empty) set of primes. Then the map*

$$\chi^{\mathrm{rig}} : K_0^{\triangle}(\mathrm{SH}^{\mathrm{rig}}(S)[\mathcal{P}^{-1}]) \to \mathrm{End}_{\mathrm{SH}(S)}(\mathbb{1}_S)[\mathcal{P}^{-1}]$$

defined by

$$\chi^{\mathrm{rig}}([X]) = [\mathbb{1}_S \to X \wedge X^{\vee} \to X^{\vee} \wedge X \to \mathbb{1}_S]$$

is a ring homomorphism.

Proof. Note that $\mathrm{End}_{\mathrm{SH}(S)[\mathcal{P}^{-1}]}(\mathbb{1}_S) \cong \mathrm{End}_{\mathrm{SH}(S)}(\mathbb{1}_S)[\mathcal{P}^{-1}]$, as the only additional endomorphisms in the localized category are precisely the maps $p : \mathbb{1}_S \to \mathbb{1}_S$. To begin, we show that χ^{rig} is well-defined. If $\phi : X \to Y$ is an isomorphism in $\mathrm{SH}^{\mathrm{rig}}(S)[\mathcal{P}^{-1}]$ (or even in $\mathrm{SH}(S)[\mathcal{P}^{-1}]$), we get an induced isomorphism

$$\phi^* : X^\vee := \mathrm{Hom}_S(X, \mathbb{1}_S) \xrightarrow{-\circ\phi^{-1}} \mathrm{Hom}_S(Y, \mathbb{1}_S) =: Y^\vee.$$

It follows that the diagram

commutes, where τ is the swap map (i.e. tausch map) and coev denotes coevaluation. In particular, $\chi^{\mathrm{rig}}([X])$ is independent of our choice of representative of $[X]$, so the map χ^{rig} is well-defined.

It remains to show that χ^{rig} is a ring homomorphism. Since $\mathbb{1}_S^\vee \cong \mathbb{1}_S$, we find that $\chi^{\mathrm{rig}}([\mathbb{1}_S]) = \mathrm{id}_{\mathbb{1}_S}$, which is the identity element of the ring $\mathrm{End}_{\mathrm{SH}(S)}(\mathbb{1}_S)[\mathcal{P}^{-1}]$. The additivity of χ^{rig} over (co)fiber sequences follows from [May01]. Note that $\chi^{\mathrm{rig}}([X]\cdot[Y]) = \chi^{\mathrm{rig}}([X \wedge Y])$ by definition, so multiplicativity of χ^{rig} follows from the multiplicativity of the Euler characteristic in any symmetric monoidal category. $\square$

Our next goal is to extend this *rigid Euler characteristic* χ^{rig} to a ring homomorphism $K_0^\triangle(\mathrm{SH}^\omega(S)[\mathcal{P}^{-1}]) \to \mathrm{End}(\mathbb{1}_S)[\mathcal{P}^{-1}]$ when $S = \mathrm{Spec}\, k$ and $\mathcal{P} = \{\exp(k)\}$.

Lemma 2.14. *Let k be a field of exponential characteristic e. Then there is a unique ring homomorphism*

$$\chi^\omega : K_0^\triangle(\mathrm{SH}^\omega(k)[e^{-1}]) \to \mathrm{GW}(k)[e^{-1}]$$

such that $\chi^\omega([X]) = \chi^{\mathrm{rig}}([X])$ for all $X \in \mathrm{SH}^{\mathrm{rig}}(S)$.

Proof. By [BH21, Theorem 10.12], $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{1}_k) \cong \mathrm{GW}(k)$ for any field k (generalizing Morel's result for perfect fields [Mor12, Corollary 1.24]). The lemma now follows from the fact that $\mathrm{SH}^{\mathrm{rig}}(k)[e^{-1}] = \mathrm{SH}^\omega(k)[e^{-1}]$ when k is a perfect field [Rio05] or in fact any field [EK20, Theorem 3.2.1]. $\square$

Remark 2.15. Riou further proved that $\mathrm{SH}^\omega(k) = \mathrm{SH}^{\mathrm{rig}}(k)$ for any field k admitting a weak form of resolution of singularities [Rio05]. For a more detailed account of this proof, see [RØ08, Theorem 52]. It follows that if resolution of singularities holds over k, then Lemma 2.14 remains true even without inverting the exponential characteristic.

We can now define a $\mathrm{GW}(k)[e^{-1}]$-valued Euler characteristic that is a motivic measure over k.

Definition 2.16. Let k be a field of exponential characteristic e. Define the *compactly supported Euler characteristic* over k as the composite

$$\chi^c := \chi^\omega \circ r_{\mathrm{mot}} : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)[e^{-1}].$$

Remark 2.17. We will later see that rank χ^c is always an integer (Proposition 4.1). If $\mathrm{char}\, k \neq 2$, it will follow that the image of χ^c is always contained in $\mathrm{GW}(k) \subseteq \mathrm{GW}(k)[e^{-1}]$ (Corollary 4.3). In light of Corollary 4.3, we may conflate notation and write

$$\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$$

over any field of odd exponential characteristic. For now, we will continue to specify that χ^c lands inside $\mathrm{GW}(k)[e^{-1}]$. However, starting in Section 7, it will become necessary for us to restrict our study to fields of odd exponential characteristic and think of χ^c as taking values in $\mathrm{GW}(k)$.

Since we know the value of r_{mot} and χ^ω on quasi-projective varieties, we get a nice formula for χ^c on quasi-projective varieties.

Proposition 2.18. *The compactly supported Euler characteristic satisfies*

$$\chi^c([X]) = [\mathbb{1}_k \to q_! \mathbb{1}_X \wedge (q_! \mathbb{1}_X)^\vee \to (q_! \mathbb{1}_X)^\vee \wedge q_! \mathbb{1}_X \to \mathbb{1}_k]$$

for all quasi-projective $q : X \to \mathrm{Spec}\, k$.

Proof. If $q : X \to \mathrm{Spec}\, k$ is quasi-projective, then $r_{\mathrm{mot}}([X]) = [q_! \mathbb{1}_X]$ by Lemma 2.9. After localizing at $\exp(k)$, we have $q_! \mathbb{1}_X \in \mathrm{SH}^\omega(k)[e^{-1}] = \mathrm{SH}^{\mathrm{rig}}(k)[e^{-1}]$, so Lemmas 2.13 and 2.14 give the desired result. $\square$

Remark 2.19. By construction, χ^c is uniquely determined by its image on quasi-projective varieties, which implies that our construction coincides with that of [Azo22, §2].

When $q : X \to \mathrm{Spec}\, k$ is proper, we have a natural isomorphism $q_! \cong q_*$. It follows that $\chi^c = \chi^{\mathrm{cat}}[e^{-1}]$ on smooth proper schemes and hence $\chi^c = \chi^{\mathrm{dR}}[e^{-1}]$ on smooth proper varieties over fields.

Proposition 2.20. *If* $q : X \to \mathrm{Spec}\, k$ *is smooth and proper, then* $\chi^c([X]) = \chi^{\mathrm{cat}}(X)[e^{-1}]$.

Proof. If $q : X \to \mathrm{Spec}\, k$ is smooth and proper, then $q_! \mathbb{1}_X := q_! q^* \mathbb{1}_k$ is strongly dualizable with dual $q_* q^! \mathbb{1}_k \simeq \Sigma_{\mathbb{P}^1}^\infty X_+ \in \mathrm{SH}(k)$ (see e.g. [Hoy14, §3]). Thus $\chi^c([X]) = \chi^{\mathrm{cat}}(X^\vee)[e^{-1}]$ on (infinite suspensions of) smooth proper k-schemes, and we conclude by noting that $\chi^{\mathrm{cat}}(X^\vee) = \chi^{\mathrm{cat}}(X)$. $\square$

This gives us the desired connection to the motivic Euler characteristic over fields of characteristic 0.

Corollary 2.21. *Let* k *be a field of characteristic 0. Then* $\chi^c = \chi^{\mathrm{mot}}$.

Proof. By [PP22, Theorem 2.10],

$$\chi^{\mathrm{mot}} : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$$

is the unique ring homomorphism such that $\chi^{\mathrm{mot}}([X]) = \chi^{\mathrm{dR}}(X)$ for all smooth projective irreducible varieties X. But any such variety is smooth and proper, so $\chi^c([X]) = \chi^{\mathrm{cat}}(X) = \chi^{\mathrm{dR}}(X)$ by Proposition 2.20 and Theorem 2.4. Thus

$$\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{End}(\mathbb{1}_k) \cong \mathrm{GW}(k)$$

is a ring homomorphism such that $\chi^c([X]) = \chi^{\mathrm{dR}}(X)$ for all smooth projective irreducible varieties X, so $\chi^c = \chi^{\mathrm{mot}}$. $\square$

Remark 2.22. Recall that the motivic Euler characteristic χ^{mot} was called the compactly supported Euler characteristic in [AMBO$^+$22]. Corollary 2.21 justifies calling χ^c the compactly supported Euler characteristic. In Section 4, we will see further justification for calling χ^c the compactly supported Euler characteristic.

Alternatively, one could call χ^c the *motivic Euler characteristic*, now defined over a more general base than just fields of characteristic 0. The name *motivic* Euler characteristic is justified both by χ^c being a motivic measure (i.e. a homomorphism out of $K_0(\mathrm{Var}_k)$) and by its construction via motivic homotopy theory.

In favor of brevity, we will generally use the term *motivic* rather than *compactly supported*.

3. Properties and computations of the motivic Euler characteristic

We will now prove various properties of and formulas for χ^c, some of which were shown in characteristic 0 in [PP22]. Most of the formulas in Sections 3.1, 3.2, 3.3, and 4 are either inspired by computations of χ^{cat} by Hoyois [Hoy14] and Levine [Lev20] or are direct consequences of χ^c being a motivic measure. In many cases (e.g. whenever all schemes under consideration are smooth and proper), these formulas follow directly from Hoyois and Levine. Regardless, we include the proofs in each case for the reader's convenience.

3.1. **Basic properties.** We begin by recording a few instances where χ^c is additive or multiplicative. Because $\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)[e^{-1}]$ is a ring homomorphism, the motivic Euler characteristic inherits nice properties with respect to the additive and multiplicative structure in $K_0(\mathrm{Var}_k)$.

Proposition 3.1. *Let X be a k-scheme, with $U \subset X$ an open subscheme. Then $\chi^c(X) = \chi^c(U) + \chi^c(X - U)$.*

Proof. We have the identity $[X] = [U] + [X - U]$ in $K_0(\mathrm{Var}_k)$, and χ^c respects the additive structure of $K_0(\mathrm{Var}_k)$. $\square$

Proposition 3.2. *Let X and Y be k-schemes. Then $\chi^c(X \times_k Y) = \chi^c(X) \cdot \chi^c(Y)$.*

Proof. We have the identity $[X \times_k Y] = [X] \cdot [Y]$ in $K_0(\mathrm{Var}_k)$, and χ^c respects the mutliplicative structure of $K_0(\mathrm{Var}_k)$. $\square$

As a consequence, we find that χ^c is multiplicative over any Zariski-locally trivial fiber bundle.

Corollary 3.3. *Let X be a connected k-scheme. Let $p : Y \to X$ be a Zariski-locally trivial fiber bundle with fiber F. Then $\chi^c(Y) = \chi^c(X) \cdot \chi^c(F)$.*

Proof. This follows from the well-known identity $[Y] = [X] \cdot [F]$ in $K_0(\mathrm{Var}_k)$. To verify this identity, we take a locally closed stratification $X = \coprod_i U_i$ such that each $p^{-1}(U_i) \to U_i$ is a trivial fiber bundle. By Proposition 3.2, we have $[p^{-1}(U_i)] = [U_i] \cdot [F]$. Now

$$[Y] = \sum_i [p^{-1}(U_i)]$$
$$= [F] \sum_i [U_i]$$
$$= [F] \cdot [X]. \qquad \square$$

Finally, we will prove a few propositions showing that χ^c is compatible with field extensions. If L/k is a field extension, let χ_k^c and χ_L^c denote the motivic Euler characteristic over k and L, respectively.

Proposition 3.4. *Let $i : k \to L$ be a field extension, and let $i_* : \mathrm{GW}(k)[e^{-1}] \to \mathrm{GW}(L)[e^{-1}]$ be the induced homomorphism on Grothendieck–Witt rings. Given $X \in \mathrm{Var}_k$, denote the base change $X_L := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L \in \mathrm{Var}_L$. Then $i_* \chi^c(X) = \chi^c(X_L)$.*

Proof. Let $\pi : \mathrm{Spec}\, L \to \mathrm{Spec}\, k$ be the map of schemes induced by i. Then we have the pullback functor $\pi^* : \mathrm{SH}(k)[e^{-1}] \to \mathrm{SH}(L)[e^{-1}]$, which is exact and symmetric monoidal. Note that $\mathrm{End}_{\mathrm{SH}(L)[e^{-1}]}(\mathbb{1}_L) \cong \mathrm{End}_{\mathrm{SH}(L)}(\mathbb{1}_L)[e^{-1}]$, as $e : \mathbb{1}_L \to \mathbb{1}_L$ is the only additional endomorphism introduced by localization. We thus have $\pi^* : \mathrm{End}_{\mathrm{SH}(k)[e^{-1}]}(\mathbb{1}_k) \to \mathrm{End}_{\mathrm{SH}(L)[e^{-1}]}(\mathbb{1}_L)$, which is equivalent to $i_* : \mathrm{GW}(k)[e^{-1}] \to \mathrm{GW}(L)[e^{-1}]$ by Morel's isomorphism $\mathrm{End}_{\mathrm{SH}(F)}(\mathbb{1}_F) \cong \mathrm{GW}(F)$.

Let $p : X \to \mathrm{Spec}\, k$ and $q : X_L \to \mathrm{Spec}\, L$ be the structure maps, fitting inside the cartesian square

$$\begin{array}{ccc} X_L & \xrightarrow{\ f\ } & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Spec}\, L & \xrightarrow{\ \pi\ } & \mathrm{Spec}\, k. \end{array}$$

The exchange morphism $\pi^* p_! \to q_! f^*$ is an isomorphism by [CD19, Proposition 2.2.14], since proper base change holds in the stable motivic homotopy category and π is separated (being a morphism between affine schemes). Thus the monoidality of f^* implies that $\pi^* p_! \mathbb{1}_X \cong q_! \mathbb{1}_{X_L}$. In particular, $i_* \chi_k^c(X) = \chi_L^c(X_L)$ whenever $p : X \to \mathrm{Spec}\, k$ is quasi-projective. We thus have two ring homomorphisms

$$\chi_L^c(-_L), i_* \chi_k^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(L)[e^{-1}]$$

that coincide on quasi-projective varieties. It follows from [CLNS18, Chapter 2, Corollary 2.6.6] that $\chi_L^c(-_L) = i_* \chi_k^c$. $\qquad \square$

Given a field extension L/k, we get an induced structure map $p : \operatorname{Spec} L \to \operatorname{Spec} k$. Given an L-variety $q : X \to \operatorname{Spec} L$, we can thus view X as a k-variety $p \circ q : X \to \operatorname{Spec} k$. This induces a group (but not ring) homomorphism $\pi_{L/k} : K_0(\operatorname{Var}_L) \to K_0(\operatorname{Var}_k)$.

Proposition 3.5. *Let L/k be a finite separable field extension. Then the following diagram of groups commutes:*

$$
\begin{array}{ccc}
K_0(\operatorname{Var}_L) & \xrightarrow{\pi_{L/k}} & K_0(\operatorname{Var}_k) \\
\chi_L^c \downarrow & & \downarrow \chi_k^c \\
\operatorname{GW}(L)[e^{-1}] & \xrightarrow{\operatorname{Tr}_{L/k}} & \operatorname{GW}(k)[e^{-1}].
\end{array}
$$

Proof. Let $p : \operatorname{Spec} L \to \operatorname{Spec} k$ be the structure map, which is étale by our separability assumption. We have a left adjoint $p_\# : \operatorname{SH}(L) \to \operatorname{SH}(k)$ to p^*. Given an endomorphism $\omega \in \operatorname{End}(\mathbb{1}_L)$, [Hoy14, Proposition 5.2] states that $\operatorname{tr}(p_\#\omega) = \operatorname{Tr}_{L/k}\omega$, where tr denotes the categorical trace. Setting $\omega = \chi_L^c$, we have

(3.1) $\qquad \operatorname{Tr}_{L/k}\chi_L^c(X) = [\mathbb{1}_k \to p_\# q_! \mathbb{1}_X \wedge (p_\# q_! \mathbb{1}_X)^\vee \to (p_\# q_! \mathbb{1}_X)^\vee \wedge p_\# q_! \mathbb{1}_X \to \mathbb{1}_k]$

for every quasi-projective $q : X \to \operatorname{Spec} L$. Since p is étale, we have a natural isomorphism $p_\# \simeq p_!$, so $p_\# q_! \mathbb{1}_X \cong (p \circ q)_! \mathbb{1}_X$. Equation 3.1 thus simplifies to

$\operatorname{Tr}_{L/k}\chi_L^c(X) = [\mathbb{1}_k \to (p \circ q)_! \mathbb{1}_X \wedge ((p \circ q)_! \mathbb{1}_X)^\vee \to ((p \circ q)_! \mathbb{1}_X)^\vee \wedge (p \circ q)_! \mathbb{1}_X \to \mathbb{1}_k]$,

which is equal to $\chi_k^c \pi_{L/k}(X)$. We thus have two group homomorphisms

$$\operatorname{Tr}_{L/k}\chi_L^c, \chi_k^c \pi_{L/k} : K_0(\operatorname{Var}_L) \to \operatorname{GW}(k)[e^{-1}]$$

that coincide on quasi-projective varieties. It follows from [CLNS18, Chapter 2, Corollary 2.6.6.] that $\operatorname{Tr}_{L/k}\chi_L^c = \chi_k^c \pi_{L/k}$ as group homomorphisms. $\qquad \square$

3.2. **Computations for various schemes.** We now compute the motivic Euler characteristic of points, projective spaces, affine spaces, and a few other interesting schemes.

Notation 3.6. Given a positive integer n, let

$$n_\varepsilon := \underbrace{\langle 1 \rangle + \langle -1 \rangle + \langle 1 \rangle + \cdots + \langle -1 \rangle^{n-1}}_{n \text{ times}},$$

which can be considered as an element of $\operatorname{GW}(k)$ or $\operatorname{GW}(k)[e^{-1}]$. We also define $0_\varepsilon := 0$.

Proposition 3.7. *We have $\chi^c(\operatorname{Spec} k) = \langle 1 \rangle$ and $\chi^c(\mathbb{P}_k^n) = (n+1)_\varepsilon$.*

Proof. Since $\operatorname{Spec} k$ and $\mathbb{P}_k^n$ are smooth and proper over k, it follows that $\chi^c(X) = \chi^{\operatorname{cat}}(X)[e^{-1}]$ when X is one of these schemes. The result now follows from [Hoy14, Example 1.7]. $\qquad \square$

As a corollary, we can compute the motivic Euler characteristic of affine space.

Corollary 3.8. *We have $\chi^c(\mathbb{A}_k^n) = \langle -1 \rangle^n$.*
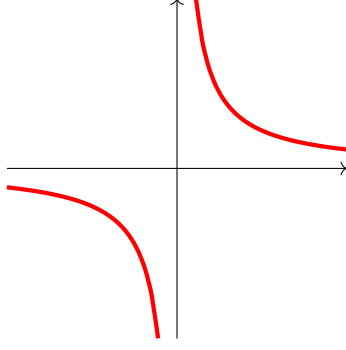
FIGURE 1. $\mathbb{G}_m(\mathbb{R})$ with compactly supported Euler characteristic -2

Proof. We use the decomposition $[\mathbb{P}_k^n] = [\mathbb{A}_k^n] + [\mathbb{P}_k^{n-1}]$ in $K_0(\text{Var}_k)$. By Propositions 3.1 and 3.7, we thus have

$$\chi^c(\mathbb{A}_k^n) = \chi^c(\mathbb{P}_k^n) - \chi^c(\mathbb{P}_k^{n-1})$$
$$= (n+1)_\varepsilon - n_\varepsilon$$
$$= \langle -1 \rangle^n.$$

Alternatively, one can compute $\chi^c(\mathbb{A}_k^1) = \langle -1 \rangle$ in this manner and then apply Proposition 3.2 to $\mathbb{A}_k^n = \prod_{i=1}^n \mathbb{A}_k^1$. $\qquad\square$

We can now obtain various important classes in $\text{GW}(k)[e^{-1}]$ as the motivic Euler characteristic of relatively simple schemes.

Corollary 3.9. *Let* $p_1, \ldots, p_r \in \mathbb{A}_k^n(k)$ *be distinct* k-*rational points. Then* $\chi^c(\mathbb{A}_k^n - \{p_1, \ldots, p_r\}) = \langle -1 \rangle^n - r\langle 1 \rangle$. *In particular,*

   (i) $\chi^c(\mathbb{G}_m) = \langle -1 \rangle - \langle 1 \rangle$,

   (ii) $\chi^c(\mathbb{P}_k^1 - \{0, 1, \infty\}) = \langle -1 \rangle - 2\langle 1 \rangle$,

   (iii) $\chi^c(\mathbb{A}_k^{2m} - \{0\}) = 0$, *and*

   (iv) $\chi^c(\mathbb{A}_k^{2m} - \{0, 1\}) = -\langle 1 \rangle$.

Proof. Since $p_1, \ldots, p_r$ are distinct, this follows immediately from Proposition 3.1 after computing $\chi^c(\mathbb{A}_k^n)$ (Corollary 3.8) and $\chi^c(\text{Spec } k)$ (Proposition 3.7). $\qquad\square$

Example 3.10. We will later prove that the signature of $\chi^c(X)$ corresponds to the compactly supported Euler characteristic of the real locus $X(\mathbb{R})$ (see Proposition 4.4). The real locus of $\mathbb{G}_m$ is the affine hyperbola illustrated in Figure 1, which has Euler characteristic $-2 = \text{sign}(\langle -1 \rangle - \langle 1 \rangle)$.

Example 3.11. By Proposition 2.20, we can compute the motivic Euler characteristic of any smooth proper scheme by computing its categorical Euler characteristic. Here, we record a few more computations from the literature.

(i) If L/k is a finite separable field extension, then $\chi^c(\operatorname{Spec} L) = \operatorname{Tr}_{L/k}\langle 1 \rangle$, where $\operatorname{Tr}_{L/k} : \operatorname{GW}(L)[e^{-1}] \to \operatorname{GW}(k)[e^{-1}]$ is induced by the field trace [Hoy14, Theorem 1.9].

(ii) If X is a Brauer–Severi variety of dimension n over k, then $\chi^c(X) = (n+1)_\varepsilon$ [Lev20, Examples 2.6]. Note that we generally do not have an equality of $[X]$ and $[\mathbb{P}^n]$ in $K_0(\operatorname{Var}_k)$ [Lit15, Theorem 7], so the computation $\chi^c(X) = \chi^c(\mathbb{P}^n)$ shows that the class $[X] - [\mathbb{P}^n]$ lies in $\ker \chi^c$. Since χ^c is a ring homomorphism, we also find that $\chi^c(P) = 1$, where $P \in K_0(\operatorname{Var}_k)$ is the class mentioned in [Lit15, Theorem 7].

(iii) Let $\operatorname{Gr}_k(r, n)$ be the Grassmannian of r-planes in k^n. Set $n_\mathbb{C} := \binom{n}{r}$ and $n_\mathbb{R} := \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor}$. Then $\chi^c(\operatorname{Gr}_k(r,n)) = \frac{n_\mathbb{C} + n_\mathbb{R}}{2}\langle 1 \rangle + \frac{n_\mathbb{C} - n_\mathbb{R}}{2}\langle -1 \rangle$. Indeed, since Grassmannians are smooth and projective over $\mathbb{Z}$, this follows from [BW23, Theorem 5.11] and the computations $\operatorname{rank} \chi^{\operatorname{cat}}(\operatorname{Gr}_k(r,n)) = \chi(\operatorname{Gr}_\mathbb{C}(r,n)) = n_\mathbb{C}$ and $\operatorname{sign} \chi^{\operatorname{cat}}(\operatorname{Gr}_k(r,n)) = \chi(\operatorname{Gr}_\mathbb{R}(r,n)) = n_\mathbb{R}$. Alternatively, one can compute $\chi^{\operatorname{cat}}(\operatorname{Gr}_k(r,n))$ using the cellular structure of $\operatorname{Gr}_k(r,n)$ (see e.g. [BMP23, Proposition 8.3 and Theorem 8.4]).

Remark 3.12. At this point, we have given enough examples to prove that χ^c surjects onto $\operatorname{GW}(k)$ when $\operatorname{char} k \neq 2$. This fact, which we prove in Proposition 7.5, will be essential to the broader goals of the article, so we postpone its proof until the surjectivity becomes pertinent.

3.3. **Computations for bundles.** Thanks to Corollary 3.3, we can compute the motivic Euler characteristic of vector bundles, projective bundles, and blow ups.

Corollary 3.13. *Let $V \to X$ be a vector bundle of rank r. Then $\chi^c(V) = \langle -1 \rangle^r \cdot \chi^c(X)$.*

Proof. A vector bundle of rank r is Zariski-locally trivial with fiber $\mathbb{A}_k^r$, so the result follows from Corollaries 3.3 and 3.8. $\qquad\square$

We also get a direct computation of the motivic Euler characteristic of projective bundles.

Corollary 3.14. *Let $V \to X$ be a vector bundle of rank r. Then $\chi^c(\mathbb{P}V) = r_\varepsilon \cdot \chi^c(X)$.*

Proof. The projective bundle $\mathbb{P}V$ is a Zariski-locally trivial fiber bundle with fibers $\mathbb{P}_k^{r-1}$, so the result follows from Corollary 3.3 and Proposition 3.7. $\qquad\square$

Since the exceptional divisor of a blow up along a smooth subscheme is a projective bundle, the motivic Euler characteristic satisfies a fairly simple formula in this setting:

Proposition 3.15. *Let X be a k-scheme. Let $Z \subset X$ be an lci closed subscheme of codimension d. Let $\tilde{X}$ denote the blow up of X along Z. Then*
$$\chi^c(\tilde{X}) = \chi^c(X) + \langle -1 \rangle \cdot (d-1)_\varepsilon \cdot \chi^c(Z).$$
In particular, if Z is of codimension 1, then $\chi^c(\tilde{X}) = \chi^c(X)$.[3]

---

[3]If Z is reduced of codimension 1, then $\tilde{X} \cong X$, which directly implies $\chi^c(\tilde{X}) = \chi^c(X)$.

Proof. Let $N_{Z/X} \to Z$ be the normal bundle of Z inside X. Let $E = \mathbb{P}N_{Z/X} \to Z$ be the exceptional divisor of $\tilde{X}$, so that $X - Z \cong \tilde{X} - E$. Since $N_{Z/X}$ has rank d, Proposition 3.1 and Corollary 3.14 thus imply

$$\chi^c(X) - \chi^c(Z) = \chi^c(\tilde{X}) - \chi^c(E)$$
$$= \chi^c(\tilde{X}) - d_\varepsilon \cdot \chi^c(Z).$$

We conclude by noting that

$$d_\varepsilon \cdot \chi^c(Z) - \chi^c(Z) = (d_\varepsilon - \langle 1 \rangle) \cdot \chi^c(Z)$$
$$= \langle -1 \rangle \cdot (d-1)_\varepsilon \cdot \chi^c(Z). \qquad \square$$

4. Recovering the étale Euler characteristic with compact support

An essential feature of the categorical Euler characteristic $\chi^{\mathrm{cat}} : \mathrm{SH}^{\mathrm{rig}}(S) \to \mathrm{End}(\mathbb{1}_S)$ is its connection to the étale Euler characteristic. When S is a subfield of $\mathbb{C}$, we have $\mathrm{rank}\,\chi^{\mathrm{cat}}(\Sigma^\infty_{\mathbb{P}^1} X_+) = \chi(X(\mathbb{C})^{\mathrm{an}})$. If S is a subfield of $\mathbb{R}$, we also have $\mathrm{sign}\,\chi^{\mathrm{cat}}(\Sigma^\infty_{\mathbb{P}^1} X_+) = \chi(X(\mathbb{R})^{\mathrm{an}})$. That is, the rank and signature of the categorical Euler characteristic of a smooth scheme are given by the singular Euler characteristic of its complex and real loci, respectively [Lev20, Remarks 2.3 (1)].

Over any field of characteristic not 2 the rank of χ^{cat} is equal to the étale Euler characteristic [Lev20, Section 1]. In this subsection, we will expand on Levine's remarks in *loc. cit.* to show that the rank of χ^c is equal to the étale Euler characteristic with compact support. This will imply that the image of χ^c is contained in $\mathrm{GW}(k) \subseteq \mathrm{GW}(k)[e^{-1}]$ (see Corollary 4.3).

Proposition 4.1. *Let k be a field of characteristic not 2. Let ℓ be a prime that not equal to* $\mathrm{char}\,k$. *If X is a k-variety, then* $\mathrm{rank}\,\chi^c([X]) = \chi^{\mathrm{ét}}(X)$, *where $\chi^{\mathrm{ét}}$ is the étale Euler characteristic with compact support:*

$$(4.1) \qquad \chi^{\mathrm{ét}}(X) := \sum_i (-1)^i \dim \mathrm{H}^i_c(X_{k^{\mathrm{sep}}}; \mathbb{Q}_\ell).$$

In particular, we obtain a ring homomorphism $\mathrm{rank}\,\chi^c : K_0(\mathrm{Var}_k) \to \mathbb{Z}$.

Proof. The proof is essentially identical to Levine's argument for $\mathrm{rank}\,\chi^{\mathrm{cat}}$ given in [Lev20, Section 1]. We will expand on the details for the reader's convenience. Let $q : X \to \mathrm{Spec}\,k$ be a quasi-projective k-variety. Let $\mathrm{D}^b_c(-, \mathbb{Q}_\ell)$ denote the derived category of constructible sheaves with $\mathbb{Q}_\ell$-coefficients. For any prime ℓ not equal to e, the ℓ-adic realization functor $r_\ell : \mathrm{SH}(k)[e^{-1}] \to \mathrm{DM}(k)[e^{-1}] \to \mathrm{D}^b_c(k, \mathbb{Q}_\ell)$ sends the motivic spectrum $q_! \mathbb{1}_X \in \mathrm{SH}(k)[e^{-1}]$ to the complex $q_! \mathbb{Q}_{\ell, X} \in \mathrm{D}^b_c(k, \mathbb{Q}_\ell)$. The key point is that the complex $q_! \mathbb{Q}_{\ell, X} = q_! q^* \mathbb{Q}_\ell$ represents ℓ-adic cohomology with compact supports (see e.g. [Del77, Théorème (5.4)], or [Gal21, §1.4] for a modern survey).

Now we apply the previous discussion to the base change $\bar{q} : X_{k^{\mathrm{sep}}} \to \mathrm{Spec}\,k^{\mathrm{sep}}$ to the separable closure to see that $\bar{q}_! \mathbb{Q}_{\ell, X_{k^{\mathrm{sep}}}} \in \mathrm{D}^b_c(k^{\mathrm{sep}}, \mathbb{Q}_\ell)$ represents $\mathrm{H}^\bullet_c(X_{k^{\mathrm{sep}}}; \mathbb{Q}_\ell)$ and so the categorical trace $\mathrm{tr}(\mathrm{id}_{\bar{q}_! \mathbb{Q}_{\ell, X}})$ is the alternating sum $\chi^{\mathrm{ét}}(X)$ in Equation 4.1. On the other hand, we can identify the map rank with the base extension map $i_* : \mathrm{GW}(k) \to$

$\mathrm{GW}(k^{\mathrm{sep}}) \cong \mathbb{Z}$, as k^{sep} is quadratically closed when char $k \neq 2$. By Proposition 3.4, we have rank $\chi^c(X) = \chi^c(X_{k^{\mathrm{sep}}})$ and by Proposition 2.18, $\chi^c(X_k^{\mathrm{sep}}) = \mathrm{tr}(\mathrm{id}_{\bar{q}_! \mathbb{1}_{X_{k^{\mathrm{sep}}}}})$.

Now note that the ℓ-adic realization functor r_ℓ induces a canonical map $\mathrm{GW}(k^{\mathrm{sep}})[e^{-1}] \rightarrow \mathrm{End}_{\mathrm{D}^b_c(k^{\mathrm{sep}}, \mathbb{Q}_\ell)}(\mathbb{Q}_\ell) = \mathbb{Q}_\ell$ which may be identified with the inclusion $\mathbb{Z}[e^{-1}] \rightarrow \mathbb{Q}_\ell$. Moreover, categorical traces are compatible with symmetric monoidal functors, so under this inclusion $\mathrm{tr}(\mathrm{id}_{\bar{q}_! \mathbb{1}_{X_{k^{\mathrm{sep}}}}})$ gets identified with $\mathrm{tr}(\mathrm{id}_{\bar{q}_! \mathbb{Q}_{\ell, X}})$. Thus rank $\chi^c(X) = \chi^{\mathrm{ét}}(X)$ as required.

Next, we note that both rank χ^c and $\chi^{\mathrm{ét}}$ induce ring homomorphisms

$$\mathrm{rank}\, \chi^c, \chi^{\mathrm{ét}} : K_0(\mathrm{Var}_k) \rightarrow \mathbb{Z}[e^{-1}].$$

We have proved that these two ring homomorphisms agree on quasi-projective varieties, so rank $\chi^c = \chi^{\mathrm{ét}}$ by [CLNS18, Chapter 2, Corollary 2.6.6].

The compactly supported ℓ-adic Euler characteristic of any k-variety is a sum of integers, so rank $\chi^c([X]) \in \mathbb{Z}$ for any k-variety X. Since $K_0(\mathrm{Var}_k)$ is generated as a ring by classes of k-varieties, the image of rank χ^c is generated by its value on classes of k-varieties. In particular, rank $\chi^c(M) \in \mathbb{Z}$ for any $M \in K_0(\mathrm{Var}_k)$. □

Remark 4.2. If $k \subseteq \mathbb{C}$, the Betti realization functor

$$\mathrm{re}_\mathbb{C} : \mathrm{SH}(\mathbb{C}) \rightarrow \mathrm{SH}$$
$$\Sigma^\infty_{\mathbb{P}^1} X_+ \mapsto \Sigma^\infty X(\mathbb{C})_+$$

sends the map of varieties $q : X \rightarrow \mathrm{Spec}\, k$ to the map $q(\mathbb{C}) : X(\mathbb{C}) \rightarrow \{\mathrm{pt}\}$ of topological spaces. Singular cohomology with compact supports of X is represented by the functor $q(\mathbb{C})_!$, so we find that $\mathrm{re}_\mathbb{C}$ sends $\chi^c(X_\mathbb{C})$ to the compactly supported Euler characteristic of the topological space $X(\mathbb{C})$. Together with the observation that

$$\mathrm{re}_\mathbb{C} : \mathrm{GW}(\mathbb{C}) \cong \mathrm{End}_{\mathrm{SH}(\mathbb{C})}(\mathbb{1}_\mathbb{C}) \rightarrow \mathrm{End}_{\mathrm{SH}}(\mathbb{S}) \cong \mathbb{Z}$$

is the rank homomorphism (cf. [Lev20, Remarks 2.3 (1)]), we get a slightly different version of Proposition 4.1 in this context.

Because rank χ^c is $\mathbb{Z}$-valued, we can deduce that χ^c is $\mathrm{GW}(k)$-valued when char $k \neq 2$.

Corollary 4.3. *If k is a field with odd exponential characteristic e, then the image of $\chi^c : K_0(\mathrm{Var}_k) \rightarrow \mathrm{GW}(k)[e^{-1}]$ is contained in $\mathrm{GW}(k) \subseteq \mathrm{GW}(k)[e^{-1}]$.*

Proof. This follows from the discussion in [Lev20, Remarks 2.1 (2)], which we briefly explain here. When char $k \neq 2$, the map $\mathrm{GW}(k) \rightarrow \mathrm{GW}(k)[e^{-1}]$ is injective, fitting into the bi-cartesian commutative diagram

$$
\begin{array}{ccc}
\mathrm{GW}(k) & \longhookrightarrow & \mathrm{GW}(k)[e^{-1}] \\
{\scriptstyle \mathrm{rank}} \downarrow & & \downarrow {\scriptstyle \mathrm{rank}} \\
\mathbb{Z} & \longhookrightarrow & \mathbb{Z}[e^{-1}].
\end{array}
$$

Since the image of rank χ^c is contained in $\mathbb{Z} \subseteq \mathbb{Z}[e^{-1}]$, this implies that the image of χ^c is contained in $\mathrm{GW}(k) \subseteq \mathrm{GW}(k)[e^{-1}]$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As one might hope, the signature of χ^c recovers the compactly supported Euler characteristic of the real locus.

Proposition 4.4. *Let k be a field admitting an embedding $\sigma : k \to \mathbb{R}$. If X is a k-variety, then $\mathrm{sign}_\sigma \chi^c([X]) = \chi^t(X_\sigma(\mathbb{R}))$, where χ^t is the Euler characteristic with compact support for topological spaces, $X_\sigma(\mathbb{R})$ is the real locus of X under the embedding σ, and sign_σ is the signature with respect to the embedding σ.*

Proof. The real embedding $\sigma : k \to \mathbb{R}$ induces a base change functor[4] $\sigma^* : \mathrm{SH}(k) \to \mathrm{SH}(\mathbb{R})$, which is symmetric monoidal. There is a real Betti realization functor

$$\mathrm{re}_\mathbb{R} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}$$
$$\Sigma_{\mathbb{P}^1}^\infty X_+ \mapsto \Sigma^\infty X(\mathbb{R})_+.$$

Using the same argument outlined in Remark 4.2, the composite $\mathrm{re}_\mathbb{R} \circ \sigma^*$ sends $\chi^c(X)$ to $\chi^t(X_\sigma(\mathbb{R}))$. Because σ^* is monoidal, the real embedding induces $\sigma^* : \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{1}_k) \to \mathrm{End}_{\mathrm{SH}(\mathbb{R})}(\mathbb{1}_\mathbb{R})$, which is the homomorphism $\mathrm{GW}(k) \to \mathrm{GW}(\mathbb{R})$ used to compute sign_σ. We will thus have the desired result as soon as we know that

$$\mathrm{re}_\mathbb{R} : \mathrm{GW}(\mathbb{R}) \cong \mathrm{End}_{\mathrm{SH}(\mathbb{R})}(\mathbb{1}_\mathbb{R}) \to \mathrm{End}_{\mathrm{SH}}(\mathbb{S}) \cong \mathbb{Z}$$

is the signature homomorphism. This is an observation of Morel, recorded in [Lev20, Remarks 2.3 (1)]: the automorphism $[x : y] \mapsto [x : \pm y]$ of $\mathbb{P}_k^1$ corresponds to $\langle \pm 1 \rangle \in \mathrm{GW}(k)$ under Morel's degree map, and the real realization of this automorphism is the map

$$S^1 \mapsto S^1$$
$$\theta \mapsto \pm \theta,$$

which has Brouwer degree ± 1. Thus $\mathrm{re}_\mathbb{R}$ induces the map

$$\mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$$
$$\langle \pm 1 \rangle \mapsto \pm 1,$$

which is the signature homomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.5. A key observation of Pajwani–Pál is that in characteristic 0, the discriminant of χ^c recovers Saito's determinant of ℓ-adic cohomology, appropriately scaled [PP22, Theorem 2.27]. This follows from work of Saito (see e.g. [Sai94, Sai97, Sai12, Ter18]).

The necessary results of Saito are known for smooth proper varieties, whose classes generate $K_0(\mathrm{Var}_k)$ in characteristic 0. In positive characteristic, one would need analogous results for all quasi-projective varieties (which may be assumed to be smooth if one works over a perfect field and inverts e.). Alternatively, one could search for a realization functor sending the trace $\mathrm{tr}(\mathrm{id}_{q_! \mathbb{1}_X})$ to the determinant of the complex represented by $q_! \mathbb{1}_X$. Potentially relevant articles in this direction include [Bei07, Bre11, MTW15].

---

[4]Here, we conflate notation and also write $\sigma : \mathrm{Spec}\,\mathbb{R} \to \mathrm{Spec}\,k$.

Remark 4.6. There are many other invariants of quadratic forms beyond rank, signature, and discriminant. Taking any such invariant of $\chi^c(X)$ gives an invariant of X, and one can ask whether such an invariant already occurs in the literature. For example, if k is a number field, then the Hasse–Witt invariants of $\chi^c(X)$ have likely been studied previously in some guise.

5. Modifications of $K_0(\mathrm{Var}_k)$

There are various ways that $K_0(\mathrm{Var}_k)$ can be modified. For example, one can invert universal homeomorphisms, localize at $[\mathbb{A}^1_k]$, complete with respect to the dimension filtration, or localize at a topology finer than the Zariski topology. In this section, we will discuss a few modifications of $K_0(\mathrm{Var}_k)$ through which χ^c factors.

5.1. **Inverting universal homeomorphisms.**
In positive characteristic, the lack of resolution of singularities poses a problem for many arguments that we might hope to make. A common workaround is to invert those maps that might prevent an alteration, which exist over any field, from being a resolution of singularities. To this end, we will recall a certain modification of the Grothendieck ring of varieties and show that χ^c factors through this modification.

Definition 5.1. A morphism $f : X \to Y$ is called a *universal homeomorphism* if $f_{Y'} : X \times_Y Y' \to Y'$ is a homeomorphism (of underlying topological spaces) for every scheme $Y' \to Y$. Equivalently, f is integral, surjective, and *universally injective*, i.e. $f|_{X(K)} : X(K) \to Y(K)$ is injective for every field K.

Definition 5.2. Let S be a scheme. Let

$$I_S^{\mathrm{uh}} := ([X] - [Y] : \exists \text{ universal homeomorphism } f : X \to Y)$$

be the ideal in $K_0(\mathrm{Var}_S)$ generated by differences of classes of universally homeomorphic S-varieties. Define the *Grothendieck ring of S-varieties up to universal homeomorphism* as the quotient $K_0^{\mathrm{uh}}(\mathrm{Var}_S) := K_0(\mathrm{Var}_S)/I_S^{\mathrm{uh}}$.

See [CLNS18, Chapter 2, §4] for more details on $K_0^{\mathrm{uh}}(\mathrm{Var}_S)$. In characteristic 0, the ideal I_S^{uh} is trivial, so that $K_0^{\mathrm{uh}}(\mathrm{Var}_S) \cong K_0(\mathrm{Var}_S)$ [CLNS18, Chapter 2, Corollary 4.4.7]. It is unknown whether $I_k^{\mathrm{uh}} = (0)$ when char $k > 0$, but the following lemma and its corollary will be useful for us.

Lemma 5.3. *Let S be a noetherian scheme. Let $\mathcal{P}$ denote the set of primes not invertible in $\mathcal{O}_S$. Let $L_{\mathcal{P}} : K_0^{\triangle}(\mathrm{SH}^\omega(S)) \to K_0^{\triangle}(\mathrm{SH}^\omega(S)[\mathcal{P}^{-1}])$ denote localization at the set of non-invertible primes. Then $L_{\mathcal{P}} \circ r_{\mathrm{mot}} : K_0(\mathrm{Var}_S) \to K_0^{\triangle}(\mathrm{SH}^\omega(S)[\mathcal{P}^{-1}])$ factors through $K_0^{\mathrm{uh}}(\mathrm{Var}_S)$.*

Proof. By Lemma 2.9, it suffices to prove that if $p : X \to S$ and $q : Y \to S$ are quasi-projective varieties that are universally homeomorphic, then $[p_! \mathbb{1}_X] = [q_! \mathbb{1}_Y]$ in

$K_0^{\triangle}(\mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}])$. Let $f : X \to Y$ be a universal homeomorphism, so that we have a commutative triangle

$$X \xrightarrow{\quad f \quad} Y.$$
$$p \searrow \qquad \swarrow q$$
$$S$$

Any prime not invertible in $\mathcal{O}_X$ must be contained in $\mathcal{P}$, so [EK20, Proof of Theorem 2.1.1] implies that the unit $\mathrm{id} \to f_* f^*$ of the adjunction

$$f^* : \mathrm{SH}(Y)[\mathcal{P}^{-1}] \rightleftarrows \mathrm{SH}(X)[\mathcal{P}^{-1}] : f_*$$

is invertible. In particular, $q_! \mathbb{1}_Y \cong q_! f_* f^* \mathbb{1}_Y$. Since f^* is monoidal, this implies that $q_! \mathbb{1}_Y \cong q_! f_* \mathbb{1}_X$. If f were proper, then $f_* \simeq f_!$. We would then have $q_! f_* \mathbb{1}_X \cong q_! f_! \mathbb{1}_X \cong p_! \mathbb{1}_X$ and hence $q_! \mathbb{1}_Y \cong p_! \mathbb{1}_X$. This would immediately imply that $[p_! \mathbb{1}_X] = [q_! \mathbb{1}_Y]$ in $K_0^{\triangle}(\mathrm{SH}^{\omega}(S)[\mathcal{P}^{-1}])$.

It thus remains to show that a universal homeomorphism between quasi-projective schemes is proper. By [Sta18, Lemma 01VX] and [Sta18, Lemma 01T8], any morphism between quasi-projective schemes is locally of finite type. By [Sta18, Lemma 04DE] a universal homeomorphism is affine and in particular quasi-compact and separated. Finally, it is universally closed by definition so it is proper. $\qquad \square$

Corollary 5.4. *Let k be a field. Then χ^c factors through $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$.*

Proof. If $\mathrm{char}\, k = 0$, then $K_0(\mathrm{Var}_k) \cong K_0^{\mathrm{uh}}(\mathrm{Var}_k)$ and there is nothing to show. Otherwise $\mathcal{P} = \{\mathrm{char}\, k\}$. Since $\chi^c = \chi^{\omega} \circ L_e \circ r_{\mathrm{mot}}$, the result follows from Lemma 5.3. $\qquad \square$

Remark 5.5. Corollary 5.4 implies that various results in [PRV24] extend from characteristic 0 to all odd characteristics.

5.2. **Localization at the Lefschetz motive.** We now show that χ^c is insensitive to localization at $[\mathbb{A}_k^1]$ [CLNS18, Chapter 2, §4.2].

Definition 5.6. Let S be a scheme. Let $\mathbb{L}_S := [\mathbb{A}_S^1] \in K_0(\mathrm{Var}_S)$, which is called the *Lefschetz motive*. Let $\mathcal{M}_S := K_0(\mathrm{Var}_S)[\mathbb{L}_S^{-1}]$.

Proposition 5.7. *Let k be a field. Then χ^c factors through $\mathcal{M}_k$.*

Proof. By the universal property of $\mathcal{M}_k$, it suffices to verify that $\chi^c(\mathbb{L}_k)$ is invertible in $\mathrm{GW}(k)[e^{-1}]$ [CLNS18, Paragraph (4.2.1)]. We have computed $\chi^c(\mathbb{L}_k) = \langle -1 \rangle$ (Corollary 3.8), which has multiplicative inverse $\langle -1 \rangle$. $\qquad \square$

Note that we can similarly localize the Grothendieck ring up to universal homeomorphisms as well.

Definition 5.8. Let S be a scheme. Let $\mathbb{L}_S^{\mathrm{uh}} := [\mathbb{A}_S^1] \in K_0^{\mathrm{uh}}(\mathrm{Var}_S)$. We can then define $\mathcal{M}_S^{\mathrm{uh}} := K_0^{\mathrm{uh}}(\mathrm{Var}_S)[(\mathbb{L}_S^{\mathrm{uh}})^{-1}]$.

Proposition 5.9. *Let k be a field. Then χ^c factors through $\mathcal{M}_k^{\mathrm{uh}}$.*

Proof. We have already seen that χ^c factors through $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$ (Corollary 5.4) and through $\mathcal{M}_k$ (Proposition 5.7). It thus suffices to note that the following diagram

$$(5.1) \qquad \begin{array}{ccc} K_0(\mathrm{Var}_k) & \xrightarrow{-/I_k^{\mathrm{uh}}} & K_0^{\mathrm{uh}}(\mathrm{Var}_k) \\ {\scriptstyle \mathbb{L}_k^{-1}} \downarrow & & \downarrow {\scriptstyle (\mathbb{L}_k^{\mathrm{uh}})^{-1}} \\ \mathcal{M}_k & \xrightarrow{-/I_k^{\mathrm{uh}}} & \mathcal{M}_k^{\mathrm{uh}} \end{array}$$

is a pushout, since this is a diagram of commutative rings and localization commutes with quotients for commutative rings. Thus there exists a unique dashed arrow completing the following diagram:

$$(5.2)$$

Diagram 5.2 commutes, and hence χ^c factors through $\mathcal{M}_k^{\mathrm{uh}}$. $\qquad \square$

5.3. Topological localization.

As mentioned in the proof of Corollary 3.3, classes in $K_0(\mathrm{Var}_k)$ factor over Zariski-locally trivial fiber bundles. Given a topology τ finer than the Zariski topology, one can impose multiplicativity over τ-locally trivial fiber bundles to obtain the quotient ring $K_0^\tau(\mathrm{Var}_k)$. One can then ask whether a motivic measure factors through $K_0^\tau(\mathrm{Var}_k)$.

Definition 5.10. Let τ be a Grothendieck topology on Var_k. A finite type morphism $f : Y \to X$ is called a *τ-locally trivial fiber bundle with fiber F* if there exist a τ covering $\{p_i : U_i \to X\}_{i \in I}$ and isomorphisms of U_i-schemes $Y \times_X U_i \cong F \times_k U_i$ for all $i \in I$.

Now consider the ideal $I^\tau \subset K_0(\mathrm{Var}_k)$ generated by classes of the form $[E] - [F] \cdot [B]$, where $E \to B$ is a τ-locally trivial fiber bundle with fiber F. Define the τ *local Grothendieck ring of varieties* as the quotient

$$K_0^\tau(\mathrm{Var}_k) := K_0(\mathrm{Var}_k)/I^\tau.$$

A motivic measure that factors through $K_0^\tau(\mathrm{Var}_k)$ is said to be *τ local*.

The étale topology drastically culls classes in $K_0(\mathrm{Var}_k)$. Indeed, the étale Euler characteristic induces an isomorphism $K_0^{\text{ét}}(\mathrm{Var}_k) \to \mathbb{Z}$ in characteristic 0 (see e.g. [Shi17]). In particular, if a motivic measure μ is strictly more interesting than the étale Euler characteristic, then μ should not be étale local. The motivic Euler characteristic is an example of such a motivic measure.

Proposition 5.11. *Let k be a field of characteristic 0. Assume that k is not quadratically closed. Then χ^c does not factor through $K_0^{\text{ét}}(\text{Var}_k)$.*

Proof. We will later prove that χ^c surjects onto $\text{GW}(k)$ (Proposition 7.5), which surjects onto but is not isomorphic to $\mathbb{Z}$ whenever k is not quadratically closed. In particular, χ^c cannot factor through $K_0^{\text{ét}}(\text{Var}_k) \cong \mathbb{Z}$. $\qquad\square$

Sitting between the Zariski and étale topologies is the Nisnevich topology, which is built from étale covers that induce isomorphisms on residue fields of points. Recall that the unstable motivic homotopy category consists of $\mathbb{A}^1$-local Nisnevich sheaves on smooth schemes, and that χ^c factors through the Grothendieck ring of the stable motivic homotopy category. It is therefore unsurprising that χ^c is Nisnevich local. However, it turns out that all motivic measures are Nisnevich local.

Proposition 5.12. *Let Nis denote the Nisnevich topology. There is a ring isomorphism $K_0(\text{Var}_k) \cong K_0^{\text{Nis}}(\text{Var}_k)$, so any motivic measure is Nisnevich local.*

Proof. Let $f : E \to B$ be a Nisnevich-locally trivial fibration with fiber F. We will show that $[E] = [F] \cdot [B]$ in $K_0(\text{Var}_k)$, which will yield the claimed ring isomorphism. Finite type schemes over a field are noetherian,[5] so B has finitely many irreducible components $B_1, \ldots, B_n$. When a scheme has finitely many irreducible components, these components are pairwise disjoint and hence $[B] = [B_1] + \ldots + [B_n]$. Since χ^c is additive, we may therefore assume that B is irreducible. Let η_B denote the generic point of B.

Let $\mathcal{U} := \{p_i : U_i \to B\}_{i \in I}$ be a Nisnevich cover that trivializes f. Nisnevich covers are jointly surjective on all (not just closed) points, so there exists $(p, U) \in \mathcal{U}$ with $p^{-1}(\eta_B) \in U$. In particular, there exists an irreducible component V of U with generic point η_V such that p induces an isomorphism $k(\eta_V) \cong k(\eta_B)$. In other words, $p|_V : V \to B$ is a birational morphism.

By further restricting $p|_V$ to the open locus of definition, there exist non-empty open subschemes $V' \subseteq V$ and $B' \subseteq B$ such that $p|_{V'} : V' \to B'$ is an isomorphism. Since f is trivial over U by assumption and $V' \subset U$ is an open immersion, we have $E \times_B V' \cong F \times_k V'$. Since $V' \cong B'$ as k-schemes, we thus have $E \times_B B' \cong F \times_k B'$. Thus $[f^{-1}(B')] = [F] \cdot [B']$. Since B' is a non-empty open subset of B, the complement $B_1 := B \setminus B'$ is a proper closed subscheme of B. Then $[E] = [f^{-1}(B')] + [f^{-1}(B_1)]$ and the restriction of $\mathcal{U}$ to B_1 gives a Nisnevich cover which trivializes f. Thus we may repeat the argument on B_1 to obtain a non-empty open $B_1' \subset B_1$ with $[f^{-1}(B_1')] = [F] \cdot [B_1']$ and set $B_2 = B_1 \setminus B_1'$ and repeat. Continuing in this way, we get a decreasing sequence of closed subschemes $\ldots \subset B_{m+1} \subset B_m \subset \ldots \subset B_0 = B$ with complements $B_m' = B_m \setminus B_{m+1}$ (where we set $B' =: B_0'$) such that $[f^{-1}(B_m')] = [F] \cdot [B_m']$. Since B is noetherian, this chain must terminate with $B_N' = B_N$ for some $N \gg 0$ and so

$$[E] = \sum_{m=0}^{N} [f^{-1}(B_m')] = [F] \cdot \sum_{m=0}^{N} [B_m'] = [F] \cdot [B]. \qquad\square$$

---

[5]By convention, Var_S is the category of S-varieties of finite presentation, which implies finite type.

Finer than the Nisnevich topology (but incomparable to the étale topology) is the cdh topology. This is obtained from the Nisnevich topology by adding in covers $\{p : Y \to X, i : Z \to X\}$, where p is proper, i is a closed immersion,

(5.3)
$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ Z & \stackrel{i}{\longrightarrow} & X \end{array}$$

is a Cartesian diagram, and $Y - E \cong X - Z$. Diagram 5.3 is called an *abstract blow up square*. Since $[E] = [Y] + [Z] - [X] \in K_0(\mathrm{Var}_k)$ for any abstract blow up square, it follows that $K_0(\mathrm{Var}_k)$ cannot distinguish between the Nisnevich and cdh topologies.

Proposition 5.13. *There is a ring isomorphism $K_0^{\mathrm{Nis}}(\mathrm{Var}_k) \cong K_0^{\mathrm{cdh}}(\mathrm{Var}_k)$, so any motivic measure is* cdh *local.*

Proof. By [Kui23, Proposition 7.10], the ring $K_0(\mathrm{Var}_k)$ can be obtained using proper varieties as generators and abstract blow up squares as relations. Since every cover in the cdh topology is either an abstract blow up square or a Nisnevich cover, we find that $K_0^{\mathrm{Nis}}(\mathrm{Var}_k) \cong K_0^{\mathrm{cdh}}(\mathrm{Var}_k)$. The fact that any motivic measure is cdh local now follows from Proposition 5.12. $\square$

Remark 5.14. After a suitable stratification, $X^n \to \mathrm{Sym}^n X$ is étale-locally trivial, but this only tells us that $\mathrm{rank}\,\chi^c(\mathrm{Sym}^n X) = 0$ if $\mathrm{rank}\,\chi^c(X) = 0$. In order to attack Conjecture 1.1, one might hope to find a coarser topology in which $X^n \to \mathrm{Sym}^n X$ is locally trivial (after a suitable stratification). Since χ^c factors through $K_0^{\mathrm{Nis}}(\mathrm{Var}_k)$, the Nisnevich topology is the first place to look.

Unfortunately, $X^n \to \mathrm{Sym}^n X$ (for $n \geq 2$) is not Nisnevich-locally trivial after any stratification. Indeed, over the open stratum of $\mathrm{Sym}^n X$ parametrizing reduced subschemes of length n, the map $X^n \to \mathrm{Sym}^n X$ restricts to a degree $n!$ étale cover with irreducible source. This yields non-trivial extensions of residue fields, which prevents the étale-locally trivial fiber bundle from being Nisnevich-locally trivial. This same issue arises with any completely decomposed topology, such as the cdh topology.

6. A POWER STRUCTURE ON $K_0(\mathrm{Var}_k)$

A key input for Beauville's proof of the Yau–Zaslow formula [Bea99] is Göttsche's formula, which implies that for a smooth complex surface X, the generating series of the Euler characteristics of the Hilbert schemes $\mathrm{Hilb}^n X$ is given by $\prod_{i \geq 1}(1-t^i)^{-\chi(X)}$. The power structure on $K_0(\mathrm{Var}_\mathbb{C})$ [GZLMH04] enables one to give a very natural proof of Göttsche's formula. This power structure allows one to make sense of (and prove [GZLMH06]) the expression

$$\sum_{g \geq 0}[\mathrm{Hilb}^g X] \cdot t^g = \left(\sum_{m \geq 0}[\mathrm{Hilb}_0^m(\mathbb{A}^d)] \cdot t^m\right)^{[X]}$$

in $K_0(\mathrm{Var}_\mathbb{C})$, where d is any smooth quasi-projective variety of dimension d. Furthermore, the Euler characteristic with compact supports is compatible with this exponentiation,

so that

$$\sum_{g\geq 0}\chi(\mathrm{Hilb}^g X)\cdot t^g = \Big(\sum_{m\geq 0}\chi(\mathrm{Hilb}^m_0(\mathbb{A}^d))\cdot t^m\Big)^{\chi(X)}.$$

The key to proving this equality is that the Euler characteristic with compact support induces a surjective ring homomorphism $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ that respects the power structure on the source and target. Our ultimate goal for the next few sections is to show that over any field k with char $k \neq 2$, there is a power structure on $\mathrm{GW}(k)$ (assuming Conjecture 1.1) that is compatible with a power structure on $K_0(\mathrm{Var}_k)$ and the motivic Euler characteristic χ^c. This will enable us to prove, conditional on Conjecture 1.1, an enriched Göttsche formula en route to the arithmetic Yau–Zaslow formula.

To begin, we will provide some exposition around the details of [GZLMH04]. Rather than restricting our attention to $\mathbb{C}$, we will work over an arbitrary field, addressing any positive characteristic anomalies that might arise. Many of these details in arbitrary characteristic are present in [GZLMH06, GZLMH13] under the added assumption that k is algebraically closed, but we include and expand on them here for the reader's convenience.

Definition 6.1. Let R be a ring. Let $1+t\cdot R[\![t]\!]$ denote the (multiplicatively closed) set of formal power series of the form $1+\sum_{i=1}^{\infty} a_i t^i$ with $a_i \in R$. A *power structure* on R is a function

$$(1+t\cdot R[\![t]\!]) \times R \to 1+t\cdot R[\![t]\!],$$

denoted by $(A(t), r) \mapsto A(t)^r$, satisfying the following properties for all $A(t), B(t) \in 1+t\cdot R[\![t]\!]$ and $r, s \in R$:

  (i) $A(t)^0 = 1$.

  (ii) $A(t)^1 = A(t)$.

  (iii) $(A(t)\cdot B(t))^r = A(t)^r \cdot B(t)^r$.

  (iv) $A(t)^{r+s} = A(t)^r \cdot A(t)^s$.

  (v) $A(t)^{rs} = (A(t)^r)^s$.

  (vi) $(1+t)^r = 1 + rt + o(t^2)$.

  (vii) $A(t^i)^r = A(t)^r|_{t\mapsto t^i}$ for all $i \geq 1$.

Example 6.2. When $R = \mathbb{Z}$, the usual exponentiation of power series determines a power structure on $\mathbb{Z}$. In fact, Gusein-Zade–Luengo–Melle-Hernández use this power structure to guide their construction of the power structure on $K_0(\mathrm{Var}_{\mathbb{C}})$ [GZLMH04, p. 50]. Let $r \in \mathbb{Z}$ and $1+\sum_{i=1}^{\infty} a_i t^i \in 1+t\cdot \mathbb{Z}[\![t]\!]$. Inducting on the multinomial theorem, we find that for $r \geq 0$,

$$(6.1)\qquad \Big(1+\sum_{i=1}^{\infty} a_i t^i\Big)^r = 1+\sum_{n=1}^{\infty}\left(\sum_{\substack{b_1,\ldots,b_n\geq 0;\\ \sum_i ib_i=n}}\binom{r}{b_1,\ldots,b_n, r-\sum_i b_i}\prod_{i=1}^{n} a_i^{b_i}\right)t^n,$$

where each $\binom{r}{b_1,\ldots,b_n,r-\sum_i b_i} = \frac{r(r-1)\cdots(r-\sum_i b_i+1)}{b_1!\cdots b_n!}$ is a multinomial coefficient. Note that for the degree n term, we only need to consider the multinomial theorem for $(1 + a_1 t + \ldots + a_n t^n)^r$, since all remaining terms will have degree strictly greater than n. For $-r < 0$, we set $A(t)^{-r} := \frac{1}{A(t)^r}$. One can check by induction on polynomial division that $A(t)^{-r} \in 1 + t \cdot \mathbb{Z}[\![t]\!]$.

We now want to adapt Equation 6.1 to $R = K_0(\mathrm{Var}_k)$, where k is an arbitrary field. Since $a_i, r \in R$, these should be replaced with isomorphism classes of quasi-projective varieties $[A_i], [X] \in K_0(\mathrm{Var}_k)$. If we continue to view b_i as integers, a natural replacement of $a_i^{b_i}$ is the product variety $A_i^{b_i}$. The real work comes from correctly interpreting the multinomial coefficient $\binom{X}{b_1,\ldots,b_n,X-\sum_i b_i}$. Over an algebraically closed field, this should be a variety whose Euler characteristic is $\binom{\chi(X)}{b_1,\ldots,b_n,\chi(X)-\sum_i b_i}$. Such a variety is given by $(\prod_{i=1}^n X^{b_i} - \Delta)/\prod_{i=1}^n S_{b_i}$, where the *big diagonal* Δ is the union of the pairwise diagonals in $\prod_i X^{b_i}$. For an example of this computation in a concrete case, see e.g. [dC00].

Based on these observations, one is led to propose the following power structure on isomorphism classes of quasi-projective varieties (i.e. the Grothendieck semiring of *effective* classes).

Definition 6.3. Let k be an arbitrary field. Let X and $A_1, A_2, \ldots$ be quasi-projective varieties over k. Let $A(t) := 1 + \sum_i [A_i] t^i \in 1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]$. Define

$$\mu_{\mathrm{eff}}(A(t), [X]) := 1 + \sum_{n=1}^\infty \left(\sum_{\substack{b_1,\ldots,b_n \geq 0; \\ \sum_i ib_i = n}} \left[\frac{((\prod_{i=1}^n X^{b_i}) - \Delta) \times \prod_{i=1}^n A_i^{b_i}}{\prod_{i=1}^n S_{b_i}} \right] \right) t^n,$$

where Δ is the union of the pairwise diagonals in $\prod_i X^{b_i}$ and S_{b_i} acts by permuting the b_i factors in $\prod_i X^{b_i} - \Delta$ and $A_i^{b_i}$.

Remark 6.4. Over an algebraically closed field, one can describe Δ as the set of points in $\prod_i X^{b_i}$ with at least two coinciding coordinates. Over an arbitrary field, we no longer have a clean description of Δ as the boundary of a configuration space, so we instead construct Δ in terms of the various diagonal morphisms.

Note that μ_0 satisfies the criteria of Definition 6.1, at least when the exponents involved are classes of quasi-projective varieties. In order to turn μ_{eff} into a genuine power structure, it thus suffices to define μ_{eff} when the exponent is a general element of $K_0(\mathrm{Var}_k)$.

Rather than explicitly extending the definition of μ_{eff}, we will define $(1-t)^{-[X]}$ for every quasi-projective variety X. By [GZLMH06, Proposition 1], defining the power series $(1-t)^{-[X]}$ for all quasi-projective varieties X uniquely determines a power structure on $K_0(\mathrm{Var}_k)$. We start with a definition.

Definition 6.5. Let $n \geq 1$, and let X be a quasi-projective variety over a field k. Define

$$\mathrm{Sym}^n_\Delta X := \coprod_{\substack{a_1,\ldots,a_n \geq 0 \\ \sum_i ia_i = n}} \left(\left(\prod_{i=1}^n X^{a_i} - \Delta \right) \Big/ \prod_{i=1}^n S_{a_i} \right),$$

where Δ is the union of the pairwise diagonals in $\prod_i X^{a_i}$.

Lemma 6.6. *For each quasi-projective variety X, define*

$$\mu_0(1 - t, -[X]) := 1 + \sum_{n=1}^\infty [\mathrm{Sym}^n_\Delta X] \cdot t^n.$$

Then μ_0 determines a power structure on $K_0(\mathrm{Var}_k)$ (which we also denote by μ_0), and $\mu_0(A(t), [X]) = \mu_{\mathrm{eff}}(A(t), [X])$ for every quasi-projective variety X.

Proof. That μ_0 determines a power structure on $K_0(\mathrm{Var}_k)$ follows from [GZLMH06, Proposition 1]. Since μ_0 is a power structure, we have

$$\mu_0 \left(1 + \sum_{n \geq 1} t^n, [X] \right) = \mu_0((1 - t)^{-1}, [X])$$

$$= \mu_0(1 - t, -[X]),$$

and $\mu_0(1 - t, -[X]) = \mu_{\mathrm{eff}}(1 + \sum_{n \geq 1} t^n, [X])$ by construction. Thus μ_0 and μ_{eff} agree on pairs of the form $(1 - t, -[X])$, so these two formulas determine the same power structure by [GZLMH06, Proposition 1]. $\square$

6.1. **A power structure on $K_0(\mathrm{Var}_k)^{\mathrm{uh}}$.** Since χ^c is insensitive to universal homeomorphisms, it would suffice for our purposes to work with a power structure on $K_0(\mathrm{Var}_k)^{\mathrm{uh}}$. Analogous to Lemma 6.6, we can define a power structure μ_0^{uh} on $K_0(\mathrm{Var}_k)^{\mathrm{uh}}$ by specifying

$$\mu_0^{\mathrm{uh}}(1 - t, -[X]) := 1 + \sum_{n=1}^\infty [\mathrm{Sym}^n_\Delta X] \cdot t^n$$

for each quasi-projective variety X. If we replace $\mathrm{Sym}^n_\Delta X$ by a scheme Y_n such that $[\mathrm{Sym}^n_\Delta X] = [Y_n]$ in $K_0(\mathrm{Var}_k)^{\mathrm{uh}}$, then the formula

$$\mu(1 - t, -[X]) := 1 + \sum_{n=1}^\infty [Y_n] \cdot t^n$$

determines the same power structure on $K_0(\mathrm{Var}_k)^{\mathrm{uh}}$ as μ_0^{uh}. We list a few universally homeomorphic replacements of $[\mathrm{Sym}^n_\Delta X]$ below.

Proposition 6.7. *Let X be a quasi-projective variety over a field k. For each $n \geq 1$, the classes in $K_0(\mathrm{Var}_k)^{\mathrm{uh}}$ of the following k-schemes are equal:*

  (i) $\mathrm{Sym}^n_\Delta X$.

  (ii) $\mathrm{Sym}^n X$.

(iii) The scheme of degree n divided powers of X.

Proof. The symmetric power and divided power schemes are universally homeomorphic by [Ryd08, Paper III], so it suffices to consider $\mathrm{Sym}^n_\Delta X$ and $\mathrm{Sym}^n X$. This is explained in the proof of [CLNS18, Chapter 7, Proposition 1.1.11], but we will include the relevant details here.

The stratification of $\mathrm{Sym}^n X$ into orbit types (see e.g. [Göt01, dCM02]) gives us an equality

$$\mathrm{Sym}^n X = \coprod_{\lambda \in \mathcal{P}(n)} \mathrm{Sym}^\lambda X,$$

where $\mathcal{P}(n)$ is the set of partitions of n. Given $\lambda = (\lambda_1, \ldots, \lambda_m)$ (so that $n = \sum_{i=1}^m \lambda_i$), the closed points of the stratum $\mathrm{Sym}^\lambda X$ are 0-cycles $\sum_i \lambda_i[x_i]$, with $x_1, \ldots, x_m$ distinct closed points of X. Let $\Delta \subset X^m$ denote the diagonal consisting of m-tuples of points in which at least two coincide. Let $S_\lambda \leq S_m$ denote the subgroup consisting of permutations σ such that $\lambda_{\sigma(i)} = \lambda_i$ for all $1 \leq i \leq m$. This group acts freely on $X^m - \Delta$, and the morphism

$$X^m - \Delta \to \mathrm{Sym}^\lambda X$$

$$(x_1, \ldots, x_m) \mapsto \sum_{i=1}^m \lambda_i[x_i]$$

factors through the quotient $X^m - \Delta \to (X^m - \Delta)/S_\lambda$. The induced map $(X^m - \Delta)/S_\lambda \to \mathrm{Sym}^\lambda X$ is finite, surjective, and a bijection on geometric points, which implies that $(X^m - \Delta)/S_\lambda \to \mathrm{Sym}^\lambda X$ is a universal homeomorphism.

For our partition λ, let a_i denote the number of terms λ_j such that $\lambda_j = i$ (so that $n = \sum_{i=1}^n ia_i$ and $m = \sum_{i=1}^n a_i$). Let $\prod_{i=1}^n X^{a_i} - \Delta$, where Δ is again the diagonal consisting of m-tuples of points in which at least two coincide. The action of $\prod_i S_{a_i}$ on $\prod_i X^{a_i} - \Delta$ assigns the same weight λ_i to the points in the a_i-tuple coming from $X^{a_i} - \Delta$; this is identical to the action of S_λ on $X^m - \Delta$. In particular, we have an isomorphism

$$(X^m - \Delta)/S_\lambda \cong \left(\prod_{i=1}^n X^{a_i} - \Delta \right) / \prod_{i=1}^n S_{a_i},$$

the latter terms being the strata of $\mathrm{Sym}^n_\Delta X$. Thus

$$[\mathrm{Sym}^n X] = \sum_{\lambda \in \mathcal{P}(n)} [\mathrm{Sym}^\lambda_\Delta X]$$

$$= \sum_{\lambda \in \mathcal{P}(n)} [(X^m - \Delta)/S_\lambda]$$

$$= \sum_{\lambda \in \mathcal{P}(n)} \left[\left(\prod_{i=1}^n X^{a_i} - \Delta \right) / \prod_{i=1}^n S_{a_i} \right]$$

$$= [\mathrm{Sym}^n_\Delta X]$$

in $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$, as desired. $\qquad\square$

7. A conjectural power structure on $\mathrm{GW}(k)$

Recall over a field k of characteristic not 2, the motivic Euler characteristic is a ring homomorphism $\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$ (see Remark 2.17). In the complex setting, Göttsche's formula arises from the compatibility of the Euler characteristic (with compact support) $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ with the standard power structure on $\mathbb{Z}$ and Gusein-Zad–Luengo–Melli-Hernández's power structure on $K_0(\mathrm{Var}_{\mathbb{C}})$.

We have given an analogous Euler characteristic χ^c and recounted a power structure on $K_0(\mathrm{Var}_k)$. Over any field k with $\operatorname{char} k \neq 2$ (and conditional on Conjecture 1.1), we will use these constructions to induce a power structure on $\mathrm{GW}(k)$ such that

$$\chi^c(A(t)^M) = \chi^c(A(t))^{\chi^c(M)}$$

for all $M \in K_0(\mathrm{Var}_k)$ and $A(t) \in 1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]$. The reason that we assume $\operatorname{char} k \neq 2$ is so that the motivic Euler characteristic is a ring homomorphism $\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$ (see Corollary 4.3). In Proposition 7.5, we will prove that χ^c surjects onto $\mathrm{GW}(k)$. The next subsection will explain the role of surjective ring homomorphisms with respect to power structures.

7.1. Pre-power structures.
Recall that a power structure on a ring R is a function $(1 + t \cdot R[\![t]\!]) \times R \to 1 + t \cdot R[\![t]\!]$ that satisfies the formal properties usually associated with exponentiation (see Definition 6.1). Rather than defining the values of such a function for each $f(t) \in 1 + t \cdot R[\![t]\!]$, it will be convenient to work with a simpler subset of $1 + t \cdot R[\![t]\!]$. This leads us to the notion of a *pre-power structure*.

Definition 7.1. Let R be a ring. Let $S_R := \{1 - t^i : i \geq 0\} \subset 1 + t \cdot R[\![t]\!]$. A *pre-power structure* on R is a function

$$S_R \times R \to 1 + t \cdot R[\![t]\!],$$

denoted by $((1 - t^i), r) \mapsto (1 - t^i)^r$, satisfying the following properties for all $r, s \in R$:

(i) $(1 - t)^{-1} = \sum_{i=0}^{\infty} t^i$.

(ii) $(1 - t)^{-a} = 1 + at + o(t^2)$.

(iii) $(1 - t)^{-(a+b)} = (1 - t)^{-a} \cdot (1 - t)^{-b}$.

(iv) If $(1 - t)^a = f(t)$, then $(1 - t^i)^a = f(t^i)$ for all $i \geq 1$.

Note that any power structure restricts to a pre-power structure.

Proposition 7.2. *Let R be a ring with a power structure $\mu : (1 + t \cdot R[\![t]\!]) \times R \to 1 + t \cdot R[\![t]\!]$. Then $\mu|_{S_R \times R}$ is a pre-power structure on R.*

Proof. Properties (ii), (iii), and (iv) of Definition 7.1 are special cases of properties (vi), (iv), and (vii), respectively, of Definition 6.1. To deduce property (i) of Definition 7.1, note that

$$1 = (1 - t)^0$$
$$= (1 - t)^1 (1 - t)^{-1}.$$

We can thus derive the formula $(1-t)^{-1} = \sum_{i=0}^{\infty} t^i$ by inductively computing $1 = (1-t)(\sum_{i=0}^{\infty} t^i)$. $\qquad\square$

The benefits of working with a pre-power structure are twofold: a pre-power structure uniquely extends to a power structure [GZLMH06, Proposition 1], and a ring homomorphism $\varphi : R_1 \to R_2$ respects given power structures μ_i on R_i if and only if φ respects the pre-power structures $\mu_i|_{S_{R_i} \times R_i}$ [GZLMH06, Proposition 2]. In the following lemma, we will utilize these results to show when a surjective ring homomorphism sends a power structure on the source to a compatible power structure on the target.

Before stating the lemma, recall that a ring homomorphism $\varphi : R_1 \to R_2$ induces a ring homomorphism $R_1[\![t]\!] \to R_2[\![t]\!]$. The ring structure on $R_i[\![t]\!]$ is given by formal addition and multiplication of power series, with the ring structure on R_i governing the addition and multiplication of coefficients.

Notation 7.3. Let $\varphi : R_1 \to R_2$ be a ring homomorphism. Let $1 + t \cdot \ker \varphi[\![t]\!]$ denote the subset of $1 + t \cdot R_1[\![t]\!]$ consisting of power series whose positive degree coefficients are all elements of $\ker \varphi \subseteq R_1$. Note that $1 + t \cdot \ker \varphi[\![t]\!]$ is a multiplicatively closed set, as φ is a ring homomorphism and all positive degree coefficients in a product of elements of $1 + t \cdot \ker \varphi[\![t]\!]$ consist of sums of products of elements of $\ker \varphi$.

Lemma 7.4. *Let $\varphi : R_1 \to R_2$ be a surjective ring homomorphism. Let μ_1 be a power structure on R_1. Assume that $\mu_1(1-t, r) \in 1 + t \cdot \ker \varphi[\![t]\!]$ for all $r \in \ker \varphi$. Then there exists a unique power structure μ_2 on R_2 such that $\varphi(\mu_1(A(t), r)) = \mu_2(\varphi(A(t)), \varphi(r))$ for all $A(t) \in 1 + t \cdot R_1[\![t]\!]$ and $r \in R_1$.*

Proof. Since φ is surjective, $\varphi^{-1}(s)$ is non-empty for each $s \in R_2$. We begin by defining $\mu_2(1 - t^i, s) := \varphi(\mu_1(1 - t^i, \varphi^{-1}(s)))$ for each $i \geq 1$ and $s \in R_2$. To see that this is well-defined, note that given $a, b \in \varphi^{-1}(s)$, there exists $r \in \ker \varphi$ such that $a = b + r$. Thus $\mu_1(1 - t^i, a) = \mu_1(1 - t^i, b) \cdot \mu_1(1 - t^i, r)$, so

$$\varphi(\mu_1(1 - t^i, a)) = \varphi(\mu_1(1 - t^i, b) \cdot \mu_1(1 - t^i, r))$$
$$= \varphi(\mu_1(1 - t^i, b)) \cdot \varphi(\mu_1(1 - t^i, r)).$$

Since μ_1 is a power structure, we have $\mu_1(1 - t^i, r) = \mu_1(1 - t, r)|_{t \mapsto t^i}$. That is, the coefficients in positive degree of $\mu_1(1 - t^i, r)$ belong to the same set as the coefficients in positive degree of $\mu_1(1 - t, r)$. In particular, $\mu_1(1 - t, r) \in 1 + t \cdot \ker \varphi[\![t]\!]$ implies that $\mu_1(1 - t^i, r) \in 1 + t \cdot \ker \varphi[\![t]\!]$ for all $i \geq 1$. Thus $\varphi(\mu_1(1 - t^i, r)) = 1$ for all $r \in \ker \varphi$, so $\mu_2(1 - t^i, \varphi^{-1}(s))$ is well-defined.

This is enough to yield a pre-power structure on R_2. Indeed:

(i) We have $\varphi(1_{R_1}) = 1_{R_2}$ and $\varphi(-1_{R_1}) = -1_{R_2}$, as is the case for any ring homomorphism. Since $\mu_1(1-t,-1) = \sum_{i=0}^{\infty} t^i$, we have

$$\begin{aligned} \mu_2(1-t,-1) &= \varphi(\mu_1(1-t,-1)) \\ &= \varphi(\textstyle\sum_{i=0}^{\infty} t^i) \\ &= \textstyle\sum_{i=0}^{\infty} t^i. \end{aligned}$$

(ii) Since $\mu_1(1-t,-a) = 1 + at + o(t^2)$, we have

$$\begin{aligned} \mu_2(1-t,-s) &= \varphi(\mu_1(1-t,-\varphi^{-1}(s))) \\ &= \varphi(1 + \varphi^{-1}(s)t + o(t^2)) \\ &= 1 + st + o(t^2). \end{aligned}$$

(iii) Since $\mu_1(1-t,-(a+b)) = \mu_1(1-t,-a) \cdot \mu_1(1-t,-b)$, we have

$$\begin{aligned} \mu_2(1-t,-(c+d)) &= \varphi(\mu_1(1-t,-(\varphi^{-1}(c) + \varphi^{-1}(d)))) \\ &= \varphi(\mu_1(1-t,-\varphi^{-1}(c))) \cdot \varphi(\mu_1(1-t,-\varphi^{-1}(d))) \\ &= \mu_2(1-t,-c) \cdot \mu_2(1-t,-d). \end{aligned}$$

(iv) Let $f(t) := \mu_2(1-t,s)$. Then $f(t) = \varphi(g(t))$, where $g(t) = \mu_1(1-t,\varphi^{-1}(s))$. Since μ_1 is a power structure on R_1, we have $g(t^i) = \mu_1(1-t^i,\varphi^{-1}(s))$. Because φ acts on power series by acting on their coefficients, we have $\varphi(g(t^i)) = f(t^i)$. Thus $f(t^i) = \mu_2(1-t^i,s)$, as desired.

By [GZLMH06, Proposition 1], μ_2 extends to a power structure on R_2. By construction, φ respects the pre-power structures underlying μ_1 and μ_2, so [GZLMH06, Proposition 2] implies that $\varphi(\mu_1(A(t),r)) = \mu_2(\varphi(A(t)),\varphi(r))$ for all $A(t) \in 1 + t \cdot R_1[\![t]\!]$ and $r \in R_1$. $\square$

7.2. **Symmetric powers of** $\ker \chi^c$**.** Let k be a field with odd exponential characteristic (i.e. $\operatorname{char} k \neq 2$). Our goal is to induce a power structure on $\mathrm{GW}(k)$ from the power structure on $K_0(\mathrm{Var}_k)$ by way of Lemma 7.4. In order to do this, we need to show that $\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$ is surjective. Remarkably, we do not need any virtual classes in $K_0(\mathrm{Var}_k)$ to prove the surjectivity of χ^c.

Proposition 7.5. *Let k be a field with $\operatorname{char} k \neq 2$. Then for each $q \in \mathrm{GW}(k)$, there exists a variety $X_q \in \mathrm{Var}_k$ such that $\chi^c(X_q) = q$. In particular,*

$$\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$$

is surjective.

Proof. Since $\mathrm{GW}(k)$ is additively generated by isomorphism classes of rank ± 1 forms [Lam05, II Theorem 4.1], there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in k^\times$ such that $q = \sum_{i=1}^{n} \langle a_i \rangle - \sum_{j=1}^{m} \langle b_j \rangle$. Since χ^c is additive on disjoint unions, it suffices to find, for each $a \in k^\times$, varieties $X_{\langle a \rangle}$ and $X_{-\langle a \rangle}$ such that $\chi^c(X_{\langle a \rangle}) = \langle a \rangle$ and $\chi^c(X_{-\langle a \rangle}) = -\langle a \rangle$.

Let $a \in k^\times/(k^\times)^2$. Since char $k \neq 2$, the degree 2 extension $k(\sqrt{a})/k$ is separable. We can thus apply Example 3.11 (i) to compute $\chi^c(\operatorname{Spec} k(\sqrt{a})) = \operatorname{Tr}_{k(\sqrt{a})/k}\langle 1\rangle = \langle 2\rangle + \langle 2a\rangle$. We now proceed in two cases.

(i) First, assume that $2 \in k$ is not a square. Let $x_a : \operatorname{Spec} k(\sqrt{a}) \to \mathbb{P}^1_k$ denote $\operatorname{Spec} k(\sqrt{a})$ as a degree 2 closed point in $\mathbb{P}^1_k$. Then

$$\chi^c(\mathbb{P}^1_k - \{x_2\}) = \mathbb{H} - \langle 2\rangle - \langle 4\rangle$$
$$= \langle 2\rangle + \langle -2\rangle - \langle 2\rangle - \langle 1\rangle$$
$$= \langle -2\rangle - \langle 1\rangle.$$

Let $X_{\langle -2\rangle} = (\mathbb{P}^1_k - \{x_2\}) \amalg \operatorname{Spec} k$, which satisfies $\chi^c(X_{\langle -2\rangle}) = \langle -2\rangle$ by Proposition 3.1 and the above calculation. Let $X_{\langle -1\rangle} = \mathbb{G}_m \amalg \operatorname{Spec} k$, which satisfies $\chi^c(X_{\langle -1\rangle}) = \langle -1\rangle$ by Proposition 3.1 and Corollary 3.9. Then $\chi^c(X_{\langle -2\rangle} \times X_{\langle -1\rangle}) = \langle 2\rangle$ by Proposition 3.2. We then have

$$\chi^c(X_{\langle -2\rangle} \times (\mathbb{P}^1_k - \{x_a\})) = \langle -2\rangle(\langle -2\rangle - \langle 2a\rangle)$$
$$= \langle 1\rangle - \langle -a\rangle.$$

Since $0 \in \mathbb{P}^1_k - \{x_a\}$ and $0 \in (\mathbb{P}^1_k - \{x_2\}) \amalg \operatorname{Spec} k$ are both k-rational, we can define $X_{-\langle -a\rangle} := X_{\langle -2\rangle} \times (\mathbb{P}^1_k - \{x_a\}) - \{(0,0)\}$, which satisfies $\chi^c(X_{-\langle -a\rangle}) = -\langle -a\rangle$. We then have $\chi^c(X_{\langle -1\rangle} \times X_{-\langle -a\rangle}) = -\langle a\rangle$ by Proposition 3.2. We also have

$$\chi^c(X_{-\langle -a\rangle} \amalg \mathbb{P}^1_k) = -\langle -a\rangle + \langle a\rangle + \langle -a\rangle$$
$$= \langle a\rangle,$$

as desired.

(ii) Second, assume that $2 \in k$ is a square. Then $\chi^c(\operatorname{Spec} k(\sqrt{a})) = \langle 1\rangle + \langle a\rangle$, so $\chi^c(\operatorname{Spec} k(\sqrt{a}) \amalg (\mathbb{A}^2_k - \{0,1\})) = \langle a\rangle$ by Proposition 3.1 and Corollary 3.9. Similarly, since $x_a : \operatorname{Spec} k(\sqrt{a}) \to \mathbb{A}^1_k - \{0\}$ is a closed point, we have $\chi^c((\mathbb{A}^1_k - \{0, x_a\}) \amalg \operatorname{Spec} k) = -\langle a\rangle$.

In either case, we have constructed (for each $a \in k^\times/(k^\times)^2$) the desired varieties $X_{\langle a\rangle}$ and $X_{-\langle a\rangle}$ with $\chi^c(X_{\langle a\rangle}) = \langle a\rangle$ and $\chi^c(X_{-\langle a\rangle}) = -\langle a\rangle$. $\square$

Now that we know that $\chi^c : K_0(\operatorname{Var}_k) \to \operatorname{GW}(k)$ is surjective, we can obtain a power structure on $\operatorname{GW}(k)$ that is compatible with χ^c and the power structure on $K_0(\operatorname{Var}_k)$ by proving $(1-t)^{\ker \chi^c} \in 1 + t \cdot \ker \chi^c[\![t]\!]$. We will start by considering virtual classes of quasi-projective varieties.

Proposition 7.6. *Let X be a quasi-projective k-variety. Assume that $[X] \in \ker \chi^c$. Then $(1-t)^{-[X]} \in 1 + t \cdot \ker \chi^c[\![t]\!]$ if and only if $[\operatorname{Sym}^n X] \in \ker \chi^c$ for all $n \geq 1$.*

Proof. By definition of the power structure on $K_0(\operatorname{Var}_k)$ (Lemma 6.6), we have $(1-t)^{-[X]} = 1 + \sum_{n\geq 1}[\operatorname{Sym}^n_\Delta X] \cdot t^n$. Thus $(1-t)^{-[X]} \in 1 + t \cdot \ker \chi^c[\![t]\!]$ if and only if $[\operatorname{Sym}^n_\Delta X] \in \ker \chi^c$ for all $n \geq 1$. Since $\operatorname{Sym}^n_\Delta X$ and $\operatorname{Sym}^n X$ are equal in $K_0^{\mathrm{uh}}(\operatorname{Var}_k)$ for all $n \geq 1$ (Proposition 6.7), the desired result follows from Corollary 5.4. $\square$

This leads us to Conjecture 1.1, which we restate here.

Conjecture 7.7. *If X is a quasi-projective k-variety with $[X] \in \ker \chi^c$, then $[\operatorname{Sym}^n X] \in \ker \chi^c$ for all $n \geq 1$ (equivalently, $[\operatorname{Sym}^n_\Delta X] \in \ker \chi^c$ for all $n \geq 1$).*

Conditional on Conjecture 7.7, we can now consider classes in $\ker \chi^c$ that are purely sums of genuine classes or purely sums of virtual classes.

Corollary 7.8. *Assume Conjecture 7.7. Let $X_1, \ldots, X_n$ be quasi-projective k-varieties. Let $M = \pm \sum_{i=1}^n [X_i]$. Assume that $M \in \ker \chi^c$. Then $(1-t)^M \in 1 + t \cdot \ker \chi^c[\![t]\!]$.*

Proof. Let $X = \coprod_{i=1}^n X_i$, so that $\sum_{i=1}^n [X_i] = [X]$. Since each X_i is quasi-projective, so is X. In the case that $-[X] = M$, the desired result is now given in Proposition 7.6.

Suppose instead that $[X] = M$. Properties (i) and (iii) of Definition 7.1 imply that $(1-t)^M \cdot (1-t)^{-M} = 1$. Let $(1-t)^M = 1 + \sum_{i \geq 1} A_i t^i$ and $(1-t)^{-M} = 1 + \sum_{i \geq 1} B_i t^i$, where $A_i, B_i \in K_0(\operatorname{Var}_k)$. Set $A_0 = B_0 = 1$. With this notation, we have

$$1 = 1 + \sum_{\ell \geq 1} \left(\sum_{\substack{i,j \geq 0 \\ i+j=\ell}} A_i B_j \right) t^\ell.$$

The first paragraph of this proof implies that $B_i \in \ker \chi^c$ for all i. We now proceed by induction. For $\ell = 1$, we have $A_1 + B_1 = 0$, so $B_1 \in \ker \chi^c$ implies that $A_1 \in \ker \chi^c$. Now assume that $A_1, \ldots, A_n \in \ker \chi^c$. Since $0 = A_{n+1} + \sum_{i=0}^n A_i B_{n+1-i}$ and $A_i B_{n+1-i} \in \ker \chi^c$ for $0 \leq i \leq n$, we find that $A_{n+1} \in \ker \chi^c$ as well. It follows that $(1-t)^M \in 1 + t \cdot \ker \chi^c[\![t]\!]$, as desired. $\square$

Finally, we can consider arbitrary classes in $\ker \chi^c$.

Proposition 7.9. *Assume Conjecture 7.7. If $M \in K_0(\operatorname{Var}_k)$ with $\chi^c(M) = 0$, then $(1-t)^M \in 1 + t \cdot \ker \chi^c[\![t]\!]$.*

Proof. Since $K_0(\operatorname{Var}_k)$ is generated as a group by classes of quasi-projective k-varieties [CLNS18, Chapter 2, Corollary 2.6.6(a)], there exist quasi projective k-varieties $\{X_i\}_{i=1}^a$ and $\{Y_j\}_{j=1}^b$ such that $M = \sum_{i=1}^a [X_i] - \sum_{j=1}^b [Y_j]$. Let $X = \coprod_{i=1}^a X_i$ and $Y = \coprod_{j=1}^b Y_j$, so that $M = [X] - [Y]$.

Since X is quasi-projective over a noetherian base, we can partition X into a finite set of locally closed affine subvarieties $\{A_i\}_{i=1}^c$, so that $[X] = \sum_{i=1}^c [A_i]$. Each of these affine varieties A_i is a subvariety of $\mathbb{A}^{n_i}$, where the dimension $n_i \geq 1$ depends on i. Using any closed embedding $\mathbb{A}^{n_i} \subset \mathbb{A}^{2n_i} - \{(1, \ldots, 1)\}$ (for example, given by the vanishing of the first n variables), we find that A_i also occurs as a subvariety of $B_i := \mathbb{A}^{2n_i} - \{(1, \ldots, 1)\}$. In particular, $B_i - A_i$ is a quasi-projective k-variety with $[B_i - A_i] = [B_i] - [A_i]$.

Let $B := \coprod_{i=1}^{c} B_i$ and $C := \coprod_{i=1}^{c}(B_i - A_i)$. By Corollary 3.9 (iii) (after translating the origin to $(1, \ldots, 1)$), we have $[B_i] \in \ker \chi^c$ and thus $[B] \in \ker \chi^c$. Moreover,

$$
\begin{aligned}
M - \sum_{i=1}^{c}[B_i] &= -[Y] + \sum_{i=1}^{c}[A_i] - \sum_{i=1}^{c}[B_i] \\
&= -[Y] - \sum_{i=1}^{c}[B_i - A_i] \\
&= -[C \amalg Y].
\end{aligned}
$$

Since $M, [B] \in \ker \chi^c$, we have $[C \amalg Y] \in \ker \chi^c$. In particular, we have $(1 - t)^M = (1 - t)^{-[C \amalg Y]} \cdot (1 - t)^{[B]}$ with $[C \amalg Y], [B] \in \ker \chi^c$. By Corollary 7.8, it follows that $(1 - t)^{[B]}, (1 - t)^{-[C \amalg Y]} \in 1 + t \cdot \ker \chi^c[\![t]\!]$. We conclude by noting that $1 + t \cdot \ker \chi^c[\![t]\!]$ is multiplicatively closed (see Notation 7.3). $\qquad\square$

We are now ready to prove Theorem 1.2, which we restate here.

Theorem 7.10. *Let k be a field of characteristic not 2. Let*

$$
\mu_0 : (1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]) \times K_0(\mathrm{Var}_k) \to 1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]
$$

be the power structure defined in Section 6. Then Conjecture 1.1 is true if and only if there exists a power structure

$$
\mu_{\mathrm{GW}} : (1 + t \cdot \mathrm{GW}(k)[\![t]\!]) \times \mathrm{GW}(k) \to 1 + t \cdot \mathrm{GW}(k)[\![t]\!]
$$

such that

$$
\chi^c(\mu_0(A(t), M)) = \mu_{\mathrm{GW}}(\chi^c(A(t)), \chi^c(M))
$$

for all $A(t) \in 1 + t \cdot K_0(\mathrm{Var}_k)[\![t]\!]$ and $M \in K_0(\mathrm{Var}_k)$.

Proof. First assume that Conjecture 1.1 is true. Since $\chi^c : K_0(\mathrm{Var}_k) \to \mathrm{GW}(k)$ is surjective (Proposition 7.5) and $\mu_0(1 - t, M) \in 1 + t \cdot \ker \chi^c[\![t]\!]$ for all $M \in \ker \chi^c$ (Proposition 7.9), the result follows from Lemma 7.4.

Now assume that the power structure μ_{GW} exists. The idea is to use Proposition 7.6: given any quasi-projective variety X with $\chi^c(X) = 0$, we have

$$
\begin{aligned}
1 &= \mu_{\mathrm{GW}}(1 - t, 0) \\
&= \mu_{\mathrm{GW}}(\chi^c(1 - t), \chi^c(-[X])) \\
&= \chi^c(\mu_0((1 - t), -[X])) \\
&= 1 + \sum_{n \geq 1} \chi^c(\mathrm{Sym}_\Delta^n X) \cdot t^n \\
&= 1 + \sum_{n \geq 1} \chi^c(\mathrm{Sym}^n X) \cdot t^n.
\end{aligned}
$$

It follows that $\chi^c(\mathrm{Sym}^n X) = 0$ for all $n \geq 1$, which implies Conjecture 1.1. $\qquad\square$

8. ENRICHING GÖTTSCHE'S FORMULA

Now that we have developed a conjectural power structure on $\mathrm{GW}(k)$ that is compatible with the power structure on $K_0(\mathrm{Var}_k)$ via χ^c, we are prepared to return to the Yau–Zaslow formula. The relevant generating function comes from Göttsche's formula [Göt90], which states that for a smooth projective surface X over an algebraically closed field, the Hilbert schemes $\mathrm{Hilb}^g X$ satisfy

$$(8.1) \qquad \sum_{g \geq 0} \chi(\mathrm{Hilb}^g X) \cdot t^g = \prod_{n \geq 1} (1 - t^n)^{-\chi(X)}.$$

Equation 8.1 admits a generalization in terms of the power structure on $K_0(\mathrm{Var}_k)$. Given a scheme Y and a point $y \in Y$, let $\mathrm{Hilb}_y^m Y$ denote the Hilbert scheme of length m, dimension 0 subschemes of Y that are concentrated at the point y. Then for any smooth quasi-projective variety X of dimension d, there is an equality

$$(8.2) \qquad \sum_{g \geq 0} [\mathrm{Hilb}^g X] \cdot t^g = \Big(\sum_{m \geq 0} [\mathrm{Hilb}_0^m(\mathbb{A}^d)] \cdot t^m \Big)^{[X]}$$

in $K_0^{\mathrm{uh}}(\mathrm{Var}_k)[\![t]\!]$ [GZLMH06, Theorem 1]. (This result is stated in $K_0(\mathrm{Var}_\mathbb{C})$ in *loc. cit.*, but one can repeat the same argument verbatim over any field with the power structure on $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$ given in Section 6.)

To deduce Equation 8.1 from Equation 8.2, one uses the compatibility of the power structures on $K_0^{\mathrm{uh}}(\mathrm{Var}_{\overline{k}})$ and $\mathbb{Z}$ via the Euler characteristic, as well as Ellingsrud–Strømme's computation of $\chi(\mathrm{Hilb}_0^m(\mathbb{A}^2))$ as the partition number of m [ES87]:

$$\sum_{m \geq 0} \chi(\mathrm{Hilb}_0^m(\mathbb{A}^2)) \cdot t^m = \prod_{n \geq 1} (1 - t^n)^{-1}.$$

To enrich Göttsche's formula, we therefore need Conjecture 1.1 to hold (which gives the necessary power structure on $\mathrm{GW}(k)$) and a computation of $\chi^c(\mathrm{Hilb}_0^m(\mathbb{A}^2))$.

8.1. **Computing $\chi^c(\mathrm{Hilb}_0^m(\mathbb{A}^2))$.** We now set out to give a generating series for the motivic Euler characteristic of the local punctual Hilbert scheme $\mathrm{Hilb}_0^m(\mathbb{A}^2)$. We will show that the cell decomposition of $\mathrm{Hilb}_0^m(\mathbb{A}_\mathbb{C}^2)$ given in [ES88] in fact holds over arbitrary fields. This will allow us to compute $\chi^c(\mathrm{Hilb}_0^m(\mathbb{A}_k^2))$ in terms of $\chi^c(\mathbb{A}^i) = \langle -1 \rangle^i$. As a result, we will find that $\chi^c(\mathrm{Hilb}_0^m(\mathbb{A}_k^2))$ is determined entirely by its values for $k = \mathbb{C}$ and $\mathbb{R}$, which were computed by Ellingsrud–Strømme [ES87] and Kharlamov–Răsdeaconu [KR15], respectively.

Definition 8.1. A *cellular decomposition* of a scheme X is a chain $X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 = \varnothing$ of closed subschemes such that each $X_i - X_{i-1}$ is a disjoint union of locally closed subschemes isomorphic to affine spaces.

Given such a cellular decomposition of X, note that $X = [X_n - X_{n-1}] + [X_{n-1}]$ in $K_0(\mathrm{Var}_k)$. Inducting, we find that $[X] = \sum_{i=1}^n [X_i - X_{i-1}]$, as $[X_0] = 0$. We can then further decompose each class $[X_i - X_{i-1}]$ by expressing $X_i - X_{i-1}$ as a disjoint union of affine spaces.

Lemma 8.2. *Let X be a scheme over a field of characteristic not 2. If X admits a cellular decomposition over k, then $\chi^c(X) = a\langle 1 \rangle + b\langle -1 \rangle$ for some $a, b \in \mathbb{N} \cup \{0\}$.*

Proof. If X admits a cellular decomposition, then

$$[X] = \sum_{i=1}^{n} [X_i - X_{i-1}]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m_i} [\mathbb{A}_k^{d(i,j)}],$$

where $d(i,j) \in \mathbb{N} \cup \{0\}$. Applying χ^c and recalling that $\chi^c(\mathbb{A}^{d(i,j)}) = \langle \pm 1 \rangle$ proves the claim. $\square$

Since the element $a\langle 1 \rangle + b\langle -1 \rangle \in \mathrm{GW}(k)$ is completely determined by its rank and signature, Lemma 8.2 allows us to compute the motivic Euler characteristic of schemes admitting a cellular decomposition purely in terms of topological data, namely the Euler characteristic of the complex and real loci. We now apply this strategy to $\mathrm{Hilb}_0^m(\mathbb{A}^2)$.

Proposition 8.3. *If k is a field, then the scheme $\mathrm{Hilb}_0^m(\mathbb{A}_k^2)$ admits a cellular decomposition.*

Proof. Although the cellular decompositions of $\mathrm{Hilb}^m(\mathbb{A}^2)$ and $\mathrm{Hilb}_0^m(\mathbb{A}^2)$ in [ES88, §3] are only stated over $\mathbb{C}$ in loc. cit., the requisite Białynicki-Birula theory [BB73, BB76] in fact holds over arbitrary fields (see e.g. [JS19, §1, §2, and Example 7.9]). In particular, $\mathrm{Hilb}_0^m(\mathbb{A}_k^2)$ admits a cellular decomposition. $\square$

As a result, we can deduce the generating function of $\chi^c(\mathrm{Hilb}_0^m(\mathbb{A}_k^2))$ from the complex and real settings.

Corollary 8.4. *Let k be a field of characteristic not 2. Then*

$$(8.3) \qquad \sum_{m \geq 0} \chi^c(\mathrm{Hilb}_0^m(\mathbb{A}_k^2)) \cdot t^m = \prod_{n \geq 1} (1 - \langle -1 \rangle^{n-1} \cdot t^n)^{-1}.$$

Proof. Lemma 8.2 and Proposition 8.3 give us

$$\chi^c(\mathrm{Hilb}_0^m(\mathbb{A}_k^2)) = \frac{e_{\mathbb{C}} + e_{\mathbb{R}}}{2} \langle 1 \rangle + \frac{e_{\mathbb{C}} - e_{\mathbb{R}}}{2} \langle -1 \rangle,$$

where $e_{\mathbb{C}} = \chi(\mathrm{Hilb}_0^m(\mathbb{A}_{\mathbb{C}}^2))$ and $e_{\mathbb{R}} = \chi(\mathrm{Hilb}_0^m(\mathbb{A}_{\mathbb{R}}^2))$. We conclude by applying [ES87, (1.1) Theorem (iv)] (together with the partition generating series) and [KR15, Proposition 3.1], which compute

$$(8.4) \qquad \sum_{m \geq 0} \chi(\mathrm{Hilb}_0^m(\mathbb{A}_{\mathbb{C}}^2)) \cdot t^m = \prod_{n \geq 1} (1 - t^n)^{-1},$$

$$(8.5) \qquad \sum_{m \geq 0} \chi(\mathrm{Hilb}_0^m(\mathbb{A}_{\mathbb{R}}^2)) \cdot t^m = \prod_{n \geq 1} (1 + (-t)^n)^{-1} = \prod_{n \geq 1} (1 - (-1)^{n-1} t^n)^{-1}.$$

We therefore need a series in $\mathbb{Z}\{\langle 1 \rangle, \langle -1 \rangle\}[\![t]\!]$ whose rank and signature recover Equations 8.4 and 8.5, respectively (cf. Propositions 4.1 and 4.4). This is given by the product $\prod_{n \geq 1}(1 - q_n t^n)^{-1}$, where $q_n = a\langle 1 \rangle + b\langle -1 \rangle$ has rank 1 and signature $(-1)^{n-1}$. These criteria determine the form $q_n = \langle -1 \rangle^{n-1}$. $\square$

Remark 8.5. Equivalently, we could directly compute the following generating series identity over $K_0(\mathrm{Var_k})$:

$$(8.6) \qquad \sum_{m \geq 0} [\mathrm{Hilb}_0^m(\mathbb{A}_k^2)] \cdot t^m = \prod_{n \geq 1}(1 - \mathbb{L}^{n-1} \cdot t^n)^{-1}.$$

Indeed, the cells in the cell decomposition of [ES87] are indexed by partitions of n. For each partition $\lambda \vdash n$, the cell C_λ has dimension $n - l(\lambda)$ where l is the number of parts of the partition. Standard generating function arguments for partitions gives (8.6) (see [Bej23, Proposition 2.1] for more details). Applying χ^c then yields the corollary.

Combining Corollary 8.4 with Equation 8.2 and Theorem 1.2, we get a conditional enrichment of Göttsche's formula.

Theorem 8.6. *Let k be a field of characteristic not 2. Assume Conjecture 1.1. Then for any smooth quasi-projective k-surface X, we have*

$$\sum_{g \geq 0} \chi^c(\mathrm{Hilb}^g X) \cdot t^g = \prod_{n \geq 1}(1 - \langle -1 \rangle^{n-1} \cdot t^n)^{-\chi^c(X)}$$

in $\mathrm{GW}(k)[\![t]\!]$.

Proof. Theorem 1.2 and Equation 8.2 imply that

$$\sum_{g \geq 0} \chi^c(\mathrm{Hilb}^g X) \cdot t^g = \Big(\sum_{m \geq 0} \chi^c(\mathrm{Hilb}_0^m(\mathbb{A}^2)) \cdot t^m \Big)^{\chi^c(X)}.$$

By Corollary 8.4, we have

$$\Big(\sum_{m \geq 0} \chi^c(\mathrm{Hilb}_0^m(\mathbb{A}^2)) \cdot t^m \Big)^{\chi^c(X)} = \prod_{n \geq 1}(1 - \langle -1 \rangle^{n-1} \cdot t^n)^{-\chi^c(X)},$$

as desired. $\square$

Using a theorem of Totaro, we can give an alternative formulation of the generating series of $\chi^c(\mathrm{Hilb}^g X)$ that does not depend on Conjecture 1.1. This result was stated in the introduction as Theorem 1.3:

Theorem 8.7. *Let k be a field of characteristic not 2. Then for any smooth quasi-projective k-surface X, we have*

$$(8.7) \qquad \sum_{g \geq 0} \chi^c(\mathrm{Hilb}^g X) \cdot t^g = \prod_{n \geq 1} \left(1 + \sum_{m \geq 1} \langle -1 \rangle^{m(n-1)} \chi^c(\mathrm{Sym}^m X) \cdot t^{mn} \right)$$

in $\mathrm{GW}(k)[\![t]\!]$.

Proof. Since the power structure on $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$ is finitely determined, one can compute

$$(1 - \mathbb{L}^{n-1} \cdot t^n)^{-[X]} = (1 - t^n)^{-[\mathbb{A}^{n-1} \times X]}$$
$$= 1 + \sum_{m \geq 1} [\mathrm{Sym}^m(\mathbb{A}^{n-1} \times X)] \cdot t^{mn}.$$

By Equations 8.2 and 8.6, we thus have an equality

$$\sum_{g \geq 0} [\mathrm{Hilb}^g X] \cdot t^g = \prod_{n \geq 1} (1 - \mathbb{L}^{n-1} \cdot t^n)^{-[X]}$$
$$= \prod_{n \geq 1} \left(1 + \sum_{m \geq 1} [\mathrm{Sym}^m(\mathbb{A}^{n-1} \times X)] \cdot t^{mn} \right)$$

in $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$. Totaro proved that

$$[\mathrm{Sym}^m(\mathbb{A}^{n-1} \times X)] = [\mathrm{Sym}^m X] \cdot \mathbb{L}^{m(n-1)}$$

in $K_0^{\mathrm{uh}}(\mathrm{Var}_k)$ (see e.g. [CLNS18, Chapter 7, Proposition 1.1.11]), so

$$\prod_{n \geq 1} \left(1 + \sum_{m \geq 1} [\mathrm{Sym}^m(\mathbb{A}^{n-1} \times X)] \cdot t^{mn} \right) = \prod_{n \geq 1} \left(1 + \sum_{m \geq 1} [\mathrm{Sym}^m X] \cdot \mathbb{L}^{m(n-1)} \cdot t^{mn} \right).$$

The result now follows from the fact that $\chi^c : K_0^{\mathrm{uh}}(\mathrm{Var}_k)[\![t]\!] \to \mathrm{GW}(k)[\![t]\!]$ is a ring homomorphism. $\qquad\square$

8.2. **Computing $\chi^c(\mathrm{Hilb}^g X)$ directly.** The motivation behind Conjecture 1.1 is to enable a conceptual proof of Theorem 8.6 in terms of power structures. However, we can actually compute $\chi^c(\mathrm{Hilb}^g X)$ directly for any smooth projective surface X even without assuming Conjecture 1.1, as long as X is the base change of a scheme over $\mathbb{Z}$.

Theorem 8.8. *Let k be a field of characteristic not 2. Let $X \to \mathrm{Spec}\,\mathbb{Z}$ be a smooth, separated, and projective scheme of relative dimension 2. Let $e_{\mathbb{C}} := \chi(X(\mathbb{C}))$ and $e_{\mathbb{R}} := \chi(X(\mathbb{R}))$. Then*

$$(8.8) \qquad \sum_{g \geq 0} \chi^c(\mathrm{Hilb}^g X_k) \cdot t^g = \prod_{r \geq 1} (1 - \langle -1 \rangle^r \cdot t^r)^{-e_{\mathbb{R}}} \cdot \prod_{s \geq 1} (1 - t^{2s})^{-\frac{e_{\mathbb{C}} - e_{\mathbb{R}}}{2}}$$

in $\mathrm{GW}(k)[\![t]\!]$.

Proof. The Hilbert scheme of a projective scheme is again projective, and $\mathrm{Hilb}^g(X)$ is also smooth by [ES87, Proposition 7.27]. We have $\mathrm{Hilb}^g X \times_{\mathbb{Z}} \mathrm{Spec}\,k \cong \mathrm{Hilb}^g X_k$ by [FGI$^+$05, p. 112], so [BW23, Theorem 5.11] implies that

$$(8.9) \qquad \chi^c(\mathrm{Hilb}^g X_k) = \frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2} \langle 1 \rangle + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2} \langle -1 \rangle,$$

where $n_{\mathbb{C}}^g$ and $n_{\mathbb{R}}^g$ are the Euler characteristics of the complex and real loci of $\mathrm{Hilb}^g X$, respectively. These Euler characteristics fit into the generating series

$$\sum_{g \geq 0} n_{\mathbb{C}}^g \cdot t^g = \prod_{n \geq 1} (1 - t^n)^{-e_{\mathbb{C}}},$$

$$\sum_{g \geq 0} n_{\mathbb{R}}^g \cdot t^g = \prod_{r \geq 1} (1 - (-t)^r)^{-e_{\mathbb{R}}} \cdot \prod_{s \geq 1} (1 - t^{2s})^{-\frac{e_{\mathbb{C}} - e_{\mathbb{R}}}{2}}$$

by [Göt90] and [KR15, Theorem B], respectively. Taking the rank and signature of Equation 8.8 recovers these two generating series, so Equation 8.8 is the generating series for Equation 8.9.   □

Theorem 8.8 lends some credence to Conjecture 1.1, but it will not serve our goal of enriching the Yau–Zaslow formula. This is because no K3 surface is smooth and proper over $\mathbb{Z}$ [Fon93].

8.3. **Motivic Euler characteristic of K3 surfaces.** Given a specific K3 surface X, one might wish to compute the motivic Euler characteristic $\chi^c(X)$ and simplify the product $\prod_{n \geq 1} (1 - \langle -1 \rangle^{n-1} t^n)^{-\chi^c(X)}$ in terms of the conjectural power structure on $\mathrm{GW}(k)$.

Let X be a K3 surface over a field k of characteristic not 2. Since every K3 surface over a field is smooth and proper, Proposition 2.20 and Theorem 2.4 imply that $\chi^c(X) = \mathbb{H} + P_{\mathrm{dR}}(X)$, where

$$P_{\mathrm{dR}}(X) : H^1(X, \Omega_X^1) \otimes H^1(X, \Omega_X^1) \overset{\smile}{\to} H^2(X, \omega_X) \overset{\eta}{\to} k$$

is (the isomorphism class of) the bilinear form given by composing the intersection pairing with the trace map from coherent duality. Since $\omega_X \cong \mathcal{O}_X$, Serre duality implies that $H^2(X, \omega_X) \cong H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X)$. Now $\dim H^0(X, \mathcal{O}_X) = 1$ for all K3 surfaces by definition, so the trace map $\eta : H^2(X, \omega_X) \to k$, being non-degenerate, must be an isomorphism. In particular, $P_{\mathrm{dR}}(X)$ is entirely determined by the cup product on $H^1(X, \Omega_X^1)$.

Remark 8.9. Since $\mathrm{sign}\, \mathbb{H} = 0$, Proposition 4.4 implies that $\chi(X(\mathbb{R})) = \mathrm{sign}\, P_{\mathrm{dR}}(X)$ for any real K3 surface X. The Euler characteristic of the real locus of a K3 surface attains any even value between -18 and 20 (as mentioned in [KR17, §1]; see [DK00, §3.7.2] for details).

The remainder of this section is dedicated to sharing a loose end that we could not tie off. A classical invariant of a K3 surface X is the Picard lattice [Huy16, Chapter 1, Section 2], which is the finitely generated free abelian group $\mathrm{Pic}(X)$, together with the intersection form

$$P_{\mathrm{Pic}}(X) : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}.$$

Over $\mathbb{C}$, the cohomology group $H^2(X(\mathbb{C}), \mathbb{Z})$ is known to be the rank 22 lattice $\Lambda_{K3} :=$ $E_8(-1)^{\oplus 2} \oplus \mathbb{H}^{\oplus 3}$ [Huy16, Chapter 1, Proposition 3.5], where

$$
E_8(-1) = \begin{pmatrix}
-2 & 1 & & & & & & \\
1 & -2 & 1 & & & & & \\
& 1 & -2 & 1 & 1 & & & \\
& & 1 & -2 & 0 & & & \\
& & 1 & 0 & -2 & 1 & & \\
& & & & 1 & -2 & 1 & \\
& & & & & 1 & -2 & 1 \\
& & & & & & 1 & -2
\end{pmatrix}.
$$

The Picard lattice is the sublattice $H^{1,1}(X) \cap H^2(X(\mathbb{C}), \mathbb{Z})$ by the Lefschetz $(1,1)$-theorem. By the Hodge decomposition, this is uniquely determined by the line $H^{0,2} = H^2(X, \mathcal{O}_X) \subset H^2(X(\mathbb{C}), \mathbb{C}) \cong \Lambda_{K3} \otimes_{\mathbb{Z}} \mathbb{C}$. Over other fields, determining the rank $\rho(X)$ of the Picard lattice is a difficult question. Over $\mathbb{R}$, the Hodge index theorem implies that $\operatorname{sign} P_{\operatorname{Pic}}(X) = (1, \rho(X) - 1)$.

As the cup product is compatible with the intersection product, it is natural to ask how $P_{\operatorname{dR}}(X) \in \operatorname{GW}(k)$ and $P_{\operatorname{Pic}}(X) \in \operatorname{GW}(\mathbb{Z})$ relate. Note that $P_{\operatorname{dR}}(X)$ is not simply the base change of $P_{\operatorname{Pic}}(X)$, as $\operatorname{rank} P_{\operatorname{dR}}(X) = 22$, while $\operatorname{rank} P_{\operatorname{Pic}}(X) = \rho(X)$ can vary. Similarly, $\operatorname{sign} P_{\operatorname{Pic}}(X) = 2 - \rho(X) \leq 0$ (as $\rho(X) \geq 1$), whereas $\operatorname{sign} P_{\operatorname{dR}}(X)$ can be positive.

Instead, we could try looking at the cup product on τ-cohomology for $\tau \in \{\operatorname{Zar}, \operatorname{Nis}, \operatorname{\acute{e}t}\}$ [Jar15, §8.4]. Hilbert's Theorem 90 implies that $H^1_\tau(X, \mathbb{G}_m) \cong \operatorname{Pic}(X)$ [Sta18, Theorem 03P8]. The cup product is a map

$$
\smile : H^1_\tau(X, \mathbb{G}_m) \otimes H^1_\tau(X, \mathbb{G}_m) \to H^2_\tau(X, \mathbb{G}_m \otimes \mathbb{G}_m) \cong H^2_\tau(X, \mathbb{G}_m).
$$

Now our choice of τ becomes relevant. Since K3 surfaces are noetherian, integral, and locally factorial, $H^i_{\operatorname{Zar}}(X, \mathbb{G}_m) = 0$ for all $i > 1$. On the other hand, $H^2_{\operatorname{\acute{e}t}}(X, \mathbb{G}_m)$ is interesting. Gabber proved that $H^2_{\operatorname{\acute{e}t}}(X, \mathbb{G}_m)$ is torsion if X admits an ample invertible sheaf, identifying $H^2_{\operatorname{\acute{e}t}}(X, \mathbb{G}_m)$ with the cohomological Brauer group $\operatorname{Br}(X) := H^2_{\operatorname{\acute{e}t}}(X, \mathbb{G}_m)_{\operatorname{tors}}$. If we had a "$\tau$-Brauer trace form" $\eta_\tau : H^2_\tau(X, \mathbb{G}_m) \to k$, we could define a bilinear form

$$
P_\tau(X) : H^1_\tau(X, \mathbb{G}_m) \otimes H^1_\tau(X, \mathbb{G}_m) \xrightarrow{\smile} H^2_\tau(X, \mathbb{G}_m) \xrightarrow{\eta_\tau} k
$$

and try comparing the classes $P_\tau(X), P_{\operatorname{dR}}(X) \in \operatorname{GW}(k)$. For example, if we have a rational point $x \in X(k)$, then we could try defining η_τ as the composite

$$
\operatorname{Br}(X) \xrightarrow{\ x_* \ } \operatorname{Br}(k) \xrightarrow{\quad \operatorname{norm} \quad} k
$$

$$
\alpha \longmapsto \alpha(x) \longmapsto \operatorname{norm}(\alpha(x)),
$$

but the potential dependence on x would be undesirable.

9. Proving Conjecture 1.1 over pythagorean fields

To conclude this article, we will show that if X is a quasi-projective variety over a field k of characteristic not 2, and if $\chi^c(X) = 0$, then $\chi^c(\operatorname{Sym}^n X)$ is torsion for all $n \geq 1$.

Proposition 9.1. *Let k be a field of characteristic not 2. Let $f : X \to Y$ be a morphism of k-varieties. If $\operatorname{rank} \chi^c(X) = 0$ and $\operatorname{sign}_\sigma \chi^c(X) = 0$ with respect to all embeddings $\sigma : k \to \mathbb{R}$, then $\chi^c(X) \in \operatorname{GW}(k)_{\mathrm{tors}}$.*

Proof. This is essentially Pfister's local-global principle [Sch85, Theorem 2.7.3], which characterizes the torsion subgroup of the Witt group $\operatorname{W}(k)$ as the kernel of the signature map. Since the group $\mathbb{Z}$ is torsion-free, $\operatorname{GW}(k)_{\mathrm{tors}}$ must lie in the kernel of rank : $\operatorname{GW}(k) \to \mathbb{Z}$. We then conclude by recalling that $\operatorname{W}(k) \cong \operatorname{GW}(k)/(\mathbb{H})$, so that

$$\operatorname{GW}(k)_{\mathrm{tors}} = \ker \operatorname{rank} \cap \bigcap_{\sigma : k \to \mathbb{R}} \ker \operatorname{sign}_\sigma. \qquad \square$$

Corollary 9.2. *Let k be a field of characteristic not 2. Let X be a quasi-projective variety over k. If $\chi^c(X) \in \operatorname{GW}(k)_{\mathrm{tors}}$, then $\chi^c(\operatorname{Sym}^n X) \in \operatorname{GW}(k)_{\mathrm{tors}}$ for all $n \geq 1$.*

Proof. The assumption $\chi^c(X) \in \operatorname{GW}(k)_{\mathrm{tors}}$ implies that $\chi^{\text{ét}}(X) = 0$ (by Proposition 4.1) and $\chi^t(X_\sigma(\mathbb{R})) = 0$ for any embedding $\sigma : k \to \mathbb{R}$ (by Proposition 4.4).

To show that $\operatorname{rank} \chi^c(\operatorname{Sym}^n X) = 0$, it suffices to prove that $\chi^{\text{ét}}(\operatorname{Sym}^n_\Delta X) = 0$. Let $C_m(X)$ denote $X^m \setminus \Delta$ where Δ is the big diagonal. Then $\operatorname{Sym}^n_\Delta X$ is a disjoint union of quotients of $C_m(X)$ by free group actions for various m. Since $\chi^{\text{ét}}$ is multiplicative for étale maps, it suffices to show $\chi^{\text{ét}}(C_m(X)) = 0$. Now we induct on m. For $m = 1$, we have $\chi^{\text{ét}}(X) = 0$ by assumption. Now suppose $\chi(C_n(X)) = 0$ for all $n \leq m$. The projection $C_{m+1}(X) \to C_m(X)$ forgetting the last coordinate is an étale-locally trivial fibration over a base with $\chi^{\text{ét}}(C_m(X)) = 0$, and so $\chi^{\text{ét}}(C_{m+1}(X)) = 0$.

It remains to show that if $\sigma : k \to \mathbb{R}$ is a real embedding, then $\operatorname{sign}_\sigma \chi^c(\operatorname{Sym}^n X) = 0$ for all $n \geq 1$. By Proposition 4.4, for any k-variety Y, we have

$$\operatorname{sign}_\sigma \chi^c(\operatorname{Sym}^n X) = \chi^t((\operatorname{Sym}^n X)_\sigma(\mathbb{R})),$$

where χ^t is the compactly supported Euler characteristic and $(\operatorname{Sym}^n X)_\sigma(\mathbb{R})$ is the space of real points of $\operatorname{Sym}^n X$ under the embedding σ. For the rest of the proof, we will suppress the embedding from our notation. By [KR15, Equation (3.1)], we have

$$(\operatorname{Sym}^n X)(\mathbb{R}) = \coprod_{a+2b=n} \operatorname{Sym}^a(X(\mathbb{R})) \times \operatorname{Sym}^b(X'_{\mathbb{C}}),$$

where $X'_{\mathbb{C}} := \frac{X(\mathbb{C}) - X(\mathbb{R})}{\text{conjugation}}$. Thus

$$\chi^t((\operatorname{Sym}^n X)(\mathbb{R})) = \sum_{a+2b=n} \chi^t(\operatorname{Sym}^a(X(\mathbb{R}))) \cdot \chi^t(\operatorname{Sym}^b(X'_{\mathbb{C}})).$$

It is a classical computation that

$$(9.1) \qquad \chi^t(\operatorname{Sym}^n M) = \binom{\chi^t(M) + n - 1}{n}$$

for any topological space M. This is done by stratifying $\operatorname{Sym}^n M$ so that the quotient M^n/S_n is given by a finite free action over each stratum (see e.g. [Mac62]). Since

$\chi^c(X) = 0$, we have $\chi^t(X(\mathbb{R})) = 0$ and hence $\chi^t(\mathrm{Sym}^a(X(\mathbb{R}))) = 0$ for all $a \geq 1$. It follows that

$$\chi^t((\mathrm{Sym}^n X)(\mathbb{R})) = \begin{cases} 0 & n \text{ odd}, \\ \chi^t(\mathrm{Sym}^{n/2}(X'_{\mathbb{C}})) & n \text{ even}. \end{cases}$$

It remains to prove that $\chi^t(\mathrm{Sym}^{n/2}(X'_{\mathbb{C}})) = 0$. Since $\chi^c(X) = 0$ and hence $\mathrm{rank}\,\chi^c(X) = \chi^{\text{ét}}(X) = \chi^t(X(\mathbb{C})) = 0$ (see Remark 4.2), we find that $\chi^t(X(\mathbb{C}) - X(\mathbb{R})) = 0$. Complex conjugation is a free $\mathbb{Z}/2\mathbb{Z}$ action on $X(\mathbb{C}) - X(\mathbb{R})$, so $\chi^t(X'_{\mathbb{C}}) = \chi^t(X(\mathbb{C}) - X(\mathbb{R}))/2 = 0$. The desired result now follows from Equation 9.1. $\qquad\square$

As a corollary, we also obtain the following compatibility of $\chi^{\text{ét}}$ and χ^t with power structures.

Theorem 9.3. *Let k be a field of characteristic not 2.*

(i) The ring homomorphism

$$\chi^{\text{ét}} = \mathrm{rank}\,\chi^c : K_0(\mathrm{Var_k}) \to \mathbb{Z}$$

is compatible with the natural power structures.

(ii) Let $\sigma : k \to \mathbb{R}$ be a real embedding. Then the ring homomorphism

$$\mathrm{sign}_\sigma \chi^c : K_0(\mathrm{Var_k}) \to \mathbb{Z}$$

is compatible with the natural power structures.

Proof. Recall the natural power structure on $\mathbb{Z}$ is given by exponentiation of power series. Let φ denote one of the homomorphisms in the theorem. By Lemma 7.4 and the proof of Proposition 7.9, it suffices to check that $\varphi(\mathrm{Sym}^n X) = 0$ for all X with $\varphi(X) = 0$. This is precisely Corollary 9.2. $\qquad\square$

Finally, Corollary 9.2 implies that Conjecture 1.1 is true over any field for which $\mathrm{GW}(k)$ is torsion-free.

Theorem 9.4. *Let k be a field of characteristic not 2 such that $\mathrm{GW}(k)_{\text{tors}} = \{0\}$ (such as a pythagorean field of characteristic not 2). Then Conjecture 1.1 is true over k.*

Proof. Let X be a quasi-projective variety over k. If $\chi^c(X) = 0$, then Corollary 9.2 implies that $\chi^c(\mathrm{Sym}^n X) \in \mathrm{GW}(k)_{\text{tors}}$ for all $n \geq 1$. By assumption, $\mathrm{GW}(k)_{\text{tors}} = \{0\}$, giving the desired result. $\qquad\square$

It follows that all of the conditional results presented in this article hold over such fields.

References

[AMBO$^+$22] Niny Arcila-Maya, Candace Bethea, Morgan Opie, Kirsten Wickelgren, and Inna Zakharevich. Compactly supported $\mathbb{A}^1$-Euler characteristic and the Hochschild complex. *Topology Appl.*, 316:Paper No. 108108, 24, 2022.

[Azo22]   Ran Azouri. Motivic Euler characteristic of nearby cycles and a generalized quadratic conductor formula. arXiv:2101.02686, 2022.

[BB73]   A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[BB76]   A. Białynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 24(9):667–674, 1976.

[Bea99]   Arnaud Beauville. Counting rational curves on $K3$ surfaces. *Duke Math. J.*, 97(1):99–108, 1999.

[Bei07]   Alexander A. Beilinson. Topological ε-factors. *Pure Appl. Math. Q.*, 3(1):357–391, 2007.

[Bej23]   Dori Bejleri. The Hilbert zeta function of a planar n-fold thickening of a smooth curve. `https://math.umd.edu/~dbejleri/non_reduced_hilbert_zeta.pdf`, 2023.

[BH21]   Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, (425):ix+207, 2021.

[BMP23]   Thomas Brazelton, Stephen McKean, and Sabrina Pauli. Bézoutians and the $\mathbb{A}^1$-degree. *Algebra Number Theory*, 17(11):1985–2012, 2023.

[Bre11]   Manuel Breuning. Determinant functors on triangulated categories. *J. K-Theory*, 8(2):251–291, 2011.

[BW23]   Tom Bachmann and Kirsten Wickelgren. Euler classes: six-functors formalism, dualities, integrality, and linear subspaces of complete intersections. *J. Inst. Math. Jussieu*, 22(2):681—746, 2023.

[CD19]   Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives*. Springer Monographs in Mathematics. Springer, Cham, [2019] ©2019.

[Cis18]   Denis-Charles Cisinski. Which motivic spectra are dualizable? MathOverflow, 2018. URL:https://mathoverflow.net/q/304313 (version: 2018-07-06).

[CLNS18]   Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag. *Motivic integration*, volume 325 of *Progress in Mathematics*. Birkhäuser/Springer, New York, 2018.

[dC00]   Mark Andrea A. de Cataldo. Hilbert schemes of a surface and Euler characteristics. *Arch. Math. (Basel)*, 75(1):59–64, 2000.

[dCM02]   Mark Andrea A. de Cataldo and Luca Migliorini. The Chow groups and the motive of the Hilbert scheme of points on a surface. *J. Algebra*, 251(2):824–848, 2002.

[Del77]   P. Deligne. *Cohomologie étale*, volume 569 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1977. Séminaire de géométrie algébrique du Bois-Marie SGA $4\frac{1}{2}$.

[DK00]   A. I. Degtyarev and V. M. Kharlamov. Topological properties of real algebraic varieties: Rokhlin's way. *Uspekhi Mat. Nauk*, 55(4(334)):129–212, 2000.

[EK20]   Elden Elmanto and Adeel A. Khan. Perfection in motivic homotopy theory. *Proc. Lond. Math. Soc. (3)*, 120(1):28–38, 2020.

[ES87]   Geir Ellingsrud and Stein Arild Strømme. On the homology of the Hilbert scheme of points in the plane. *Invent. Math.*, 87(2):343–352, 1987.

[ES88]   Geir Ellingsrud and Stein Arild Strømme. On a cell decomposition of the Hilbert scheme of points in the plane. *Invent. Math.*, 91(2):365–370, 1988.

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[Fon93]   Jean-Marc Fontaine. Schémas propres et lisses sur **Z**. In *Proceedings of the Indo-French Conference on Geometry (Bombay, 1989)*, pages 43–56. Hindustan Book Agency, Delhi, 1993.

[Gal21]   Martin Gallauer. An introduction to six-functor formalisms. arXiv:2112.10456, 2021.

[Göt90]   Lothar Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. *Math. Ann.*, 286(1-3):193–207, 1990.

[Göt01]   Lothar Göttsche. On the motive of the Hilbert scheme of points on a surface. *Math. Res. Lett.*, 8(5-6):613–627, 2001.

[GZLMH04] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. A power structure over the Grothendieck ring of varieties. *Math. Res. Lett.*, 11(1):49–57, 2004.

[GZLMH06] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points. *Michigan Math. J.*, 54(2):353–359, 2006.

[GZLMH13] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. On the pre-λ-ring structure on the Grothendieck ring of stacks and the power structures over it. *Bull. Lond. Math. Soc.*, 45(3):520–528, 2013.

[Hoy14]   Marc Hoyois. A quadratic refinement of the Grothendieck-Lefschetz-Verdier trace formula. *Algebr. Geom. Topol.*, 14(6):3603–3658, 2014.

[Huy16]   Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[Jar15]   John F. Jardine. *Local homotopy theory*. Springer Monographs in Mathematics. Springer, New York, 2015.

[JS19]   Joachim Jelisiejew and Ł ukasz Sienkiewicz. Białynicki-Birula decomposition for reductive groups. *J. Math. Pures Appl. (9)*, 131:290–325, 2019.

[Kha21]   Adeel Khan. Voevodsky's criterion for constructible categories of coefficients. `https://www.preschema.com/papers/six.pdf`, 2021.

[KR15]   Viatcheslav Kharlamov and Rareş Răsdeaconu. Counting real rational curves on $K3$ surfaces. *Int. Math. Res. Not. IMRN*, (14):5436–5455, 2015.

[KR17]   Viatcheslav Kharlamov and Rareş Răsdeaconu. Qualitative aspects of counting real rational curves on real K3 surfaces. *Geom. Topol.*, 21(1):585–601, 2017.

[Kui23]   Josefien Kuijper. A descent principle for compactly supported extensions of functors. *Ann. K-Theory*, 8(3):489–529, 2023.

[Lam05]   T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Lev19]   Marc Levine. Motivic Euler characteristics and Witt-valued characteristic classes. *Nagoya Math. J.*, 236:251–310, 2019.

[Lev20]   Marc Levine. Aspects of enumerative geometry with quadratic forms. *Doc. Math.*, 25:2179–2239, 2020.

[Lit15]   Daniel Litt. Zeta functions of curves with no rational points. *Mich. Math. J.*, 64(2):383–395, 2015.

[LR20]   Marc Levine and Arpon Raksit. Motivic Gauss-Bonnet formulas. *Algebra Number Theory*, 14(7):1801–1851, 2020.

[Mac62]   I. G. Macdonald. The Poincaré polynomial of a symmetric product. *Proc. Cambridge Philos. Soc.*, 58:563–568, 1962.

[May01]   J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.

[Mor12]   Fabien Morel. $\mathbb{A}^1$-*algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[MTW15]   Fernando Muro, Andrew Tonks, and Malte Witte. On determinant functors and K-theory. *Publ. Mat., Barc.*, 59(1):137–233, 2015.

[PP22]   Jesse Pajwani and Ambrus Pál. An arithmetic Yau–Zaslow formula. arXiv:2210.15788, 2022.

[PP23]   Jesse Pajwani and Ambrus Pál. Power structures on the Grothendieck–Witt ring and the motivic Euler characteristic. arXiv:2309.03366, 2023.

[PRV24] Jesse Pajwani, Herman Rohrbach, and Anna M. Viergever. Compactly supported $\mathbb{A}^1$-euler characteristics of symmetric powers of cellular varieties. arXiv:2404.08486, 2024.

[Rio05] Joël Riou. Dualité de Spanier-Whitehead en géométrie algébrique. *C. R. Math. Acad. Sci. Paris*, 340(6):431–436, 2005.

[RØ08] Oliver Röndigs and Paul Arne Østvær. Modules over motivic cohomology. *Adv. Math.*, 219(2):689–727, 2008.

[Rön16] Oliver Röndigs. The Grothendieck ring of varieties and algebraic K-theory of spaces. arXiv:1611.09327, 2016.

[Ryd08] David Rydh. *Family of cycles and the Chow scheme*. PhD thesis, KTH Royal Institute of Technology, 2008.

[Sai94] Takeshi Saito. Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies. *J. Algebr. Geom.*, 3(3):411–434, 1994.

[Sai97] Takeshi Saito. Determinant of l-adic cohomology. *RIMS Kokyuroku*, 999:39–43, 1997.

[Sai12] Takeshi Saito. The discriminant and the determinant of a hypersurface of even dimension. *Math. Res. Lett.*, 19(4):855–871, 2012.

[Sch85] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[Shi17] Evgeny Shinder. Etale local fibrations in the Grothendieck ring of varieties. MathOverflow, 2017. URL:https://mathoverflow.net/q/282309 (version: 2017-09-29).

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[Ter18] Yasuhiro Terakado. The determinant and the discriminant of a complete intersection of even dimension. *Math. Res. Lett.*, 25(1):255–280, 2018.

[YZ96] Shing-Tung Yau and Eric Zaslow. BPS states, string duality, and nodal curves on $K3$. *Nuclear Phys. B*, 471(3):503–512, 1996.

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