CONSTANT BÉZOUTIAN IMPLIES INJECTIVITY

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Abstract. We prove that an endomorphism f of affine space is injective on rational points if its Bézoutian is constant. Similarly, f is injective at a given rational point if its reduced Bézoutian is constant. We also show that if the Jacobian determinant of f is invertible, then f is injective at a given rational point if and only if its reduced Bézoutian is constant.

1. Introduction

Let k be a field, and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial morphism. We will study the injectivity of f at the origin using the *multivariate Bézoutian.*

Definition 1.1. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. The *(multivariate) Bézoutian* of $f := (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is the determinant

$$\text{Béz}(f) := \det(\Delta_{ij}) \in k[\mathbf{x}, \mathbf{y}],$$

where

$$\Delta_{ij} = \frac{f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) - f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j}.$$

The *reduced Bézoutian* of f is $\overline{\text{Béz}}(f) := \text{Béz}(f) \bmod (f(\mathbf{x}), f(\mathbf{y}))$.

Definition 1.2. Let R be a polynomial ring over a field k. Let I be an ideal of R. If I is a proper ideal, then $k \subseteq R/I$. We say that an element $c \in R/I$ is *constant* if (i) I is a proper ideal and $c \in k$, or if (ii) I is not proper, in which case $c = 0$.

Multivariate Bézoutians generalize the classical Bézoutian of a univariate polynomial. They naturally arise in the study of global residues (see e.g. [SS75, BCRS96]). We will show that f is injective at the origin if $\overline{\text{Béz}}(f)$ is constant.

Theorem 1.3. *Let k be a field, and let $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial morphism with finite fibers. If $\overline{\text{Béz}}(f)$ is constant, then $|f^{-1}(0)| \leq 1$.*

By translating, we get a criterion for injectivity at any k-rational point. We will also see that the non-reduced Bézoutian detects global injectivity.

Corollary 1.4. *Let $q = (q_1, \ldots, q_n)$ be a k-rational point of $\mathbb{A}^n_k$. If $\overline{\text{Béz}}(f-q)$ is constant, then $|f^{-1}(q)| \leq 1$.*

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Corollary 1.5. *If* $\mathrm{Béz}(f)$ *is constant, then* f *is injective on* k*-rational points.*

In general, Corollaries 1.4 and 1.5 describe sufficient but not necessary criteria for injectivity. In Section 4, we will mention an injective morphism with non-constant Bézoutian. We will also describe circumstances under which Corollary 1.4 gives a necessary and sufficient condition for injectivity at a rational point.

Bass, Connell, and Wright have shown that if k has characteristic 0 and $\mathrm{Jac}(f) \in k^\times$, then f is invertible if and only if f is injective on k-rational points [BCW82, Theorem 2.1]. In particular, Theorem 1.3 gives a reformulation of the Jacobian conjecture in characteristic 0.

Corollary 1.6. *Let* k *be a field of characteristic 0, and let* $f = (f_1, \ldots, f_n)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ *be a polynomial morphism. If* $\mathrm{Jac}(f) \in k^\times$ *and (i)* $\mathrm{Béz}(f)$ *is constant, or (ii)* $\overline{\mathrm{Béz}}(f-q)$ *is constant for all* $q \in \mathbb{A}^n_k(k)$*, then the Jacobian conjecture is true for* f*.*

The key observation leading to Theorem 1.3 is that $\overline{\mathrm{Béz}}(f)$ records information about the dimension of $k[\mathbf{x}]/(f)$ as a k-vector space. We will recall the relevant details about Bézoutians in Section 2. We will then prove Theorem 1.3 in Section 3. Finally, we discuss Theorem 1.3 in the context of the Jacobian conjecture in Section 4.

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2. Bézoutians

As noted by Scheja–Storch [SS75, p. 182] and Becker–Cardinal–Roy–Szafraniec [BCRS96], the Bézoutian records information about the dimension of $k[\mathbf{x}]/(f)$ as a k-vector space. In particular, consider the isomorphism

$$\mu\colon \frac{k[\mathbf{x}]}{(f)} \otimes_k \frac{k[\mathbf{x}]}{(f)} \to \frac{k[\mathbf{x}, \mathbf{y}]}{(f(\mathbf{x}), f(\mathbf{y}))}$$

defined by $\mu(a(\mathbf{x}) \otimes b(\mathbf{x})) = a(\mathbf{x})b(\mathbf{y})$. There is an element $B \in k[\mathbf{x}]/(f) \otimes_k k[\mathbf{x}]/(f)$ such that $\mu(B) = \overline{\mathrm{Béz}}(f)$. Moreover, given a basis $\{c_i\}$ for $k[\mathbf{x}]/(f)$, there exists a basis $\{d_i\}$ for $k[\mathbf{x}]/(f)$ such that $B = \sum_i c_i \otimes d_i$.

Throughout this section, let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a morphism with finite fibers. This ensures that $(f_1, \ldots, f_n)$ is a complete intersection ideal, which allows us to utilize the multivariate Bézoutian [BCRS96, Section 3]. If $\overline{\mathrm{Béz}}(f)$ is an element of k, then $k[\mathbf{x}]/(f)$ is at most one dimensional.

Proposition 2.1. *Suppose that* $\overline{\mathrm{Béz}}(f) \in k$*. Then* $\dim_k k[\mathbf{x}]/(f) \leq 1$*.*

Proof. Given any basis $\{c_1, \ldots, c_m\}$ of $k[\mathbf{x}]/(f)$, write $\overline{\text{Béz}}(f) = \sum_{i,j} B_{ij} c_i(\mathbf{x}) c_j(\mathbf{y})$, where $B_{ij} \in k$. The $m \times m$ matrix (B_{ij}) is non-singular by [BMP21b, Theorem 1.2], so (B_{ij}) must contain at least m non-zero entries. In particular, the number of non-zero terms of $\sum_{i,j} B_{ij} c_i(\mathbf{x}) c_j(\mathbf{y})$ is at least $m = \dim_k k[\mathbf{x}]/(f)$. If $\overline{\text{Béz}}(f) \in k^\times$, then there exists a basis $\{c_1, \ldots, c_m\}$ of $k[\mathbf{x}]/(f)$ such that $\sum_{i,j} B_{ij} c_i(\mathbf{x}) c_j(\mathbf{y})$ consists of a single non-zero term, so $1 \geq \dim_k k[\mathbf{x}]/(f)$. If $\overline{\text{Béz}}(f) = 0$, then $m = \dim_k k[\mathbf{x}]/(f) = 0$. □

We can recover the Jacobian of f from $\text{Béz}(f)$. Let $\text{Jac}(f) := \det(\frac{\partial f_i}{\partial x_j})$, and let $\delta\colon k[\mathbf{x}, \mathbf{y}] \to k[\mathbf{x}]$ be given by $\delta(a(\mathbf{x}, \mathbf{y})) = a(\mathbf{x}, \mathbf{x})$. Then δ sends $\text{Béz}(f)$ to $\text{Jac}(f)$.

Proposition 2.2. *We have* $\delta(\text{Béz}(f)) = \text{Jac}(f)$.

Proof. Note that δ is a ring homomorphism, so it suffices to show that $\delta(\Delta_{ij}) = \frac{\partial f_i}{\partial x_j}$. The result follows by taking a formal partial derivative, as we now explain. Let

$$f_{ij}(\mathbf{x}, y_j) = \frac{f_i(\mathbf{x}) - f_i(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j},$$

so that $\delta(\Delta_{ij}) = f_{ij}(\mathbf{x}, x_j)$. Since Δ_{ij} is a polynomial, $\delta(\Delta_{ij})$ and f_{ij} are polynomials as well. Now

$$\begin{aligned} \frac{\partial f_i}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(f_i(\mathbf{x}) - f(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n)\right) \\ &= \frac{\partial}{\partial x_j}\left(f_{ij}(\mathbf{x}, y_j) \cdot (x_j - y_j)\right) \\ &= \frac{\partial f_{ij}}{\partial x_j} \cdot (x_j - y_j) + f_{ij}(\mathbf{x}, y_j). \end{aligned}$$

Thus

$$\begin{aligned} \delta\left(\frac{\partial f_i}{\partial x_j}\right) &= \delta\left(\frac{\partial f_{ij}}{\partial x_j} \cdot (x_j - y_j) + f_{ij}(\mathbf{x}, y_j)\right) \\ &= 0 + f_{ij}(\mathbf{x}, x_j) \\ &= \delta(\Delta_{ij}). \end{aligned}$$

Since $\frac{\partial f_i}{\partial x_j} \in k[\mathbf{x}]$, we have $\delta(\frac{\partial f_i}{\partial x_j}) = \frac{\partial f_i}{\partial x_j}$, which proves the desired result. □

Remark 2.3. If $\text{Jac}(f) \in k^\times$ and $\overline{\text{Béz}}(f)$ is constant, then Proposition 2.2 implies that $\overline{\text{Béz}}(f) \in k^\times$.

Remark 2.4. Becker–Cardinal–Roy–Szafraniec [BCRS96, p. 90] show Proposition 2.2 for $\overline{\text{Béz}}(f)$. That is, they work with $\delta\colon k[\mathbf{x}, \mathbf{y}]/(f(\mathbf{x}), f(\mathbf{y})) \to k[\mathbf{x}]/(f)$ and prove that $\delta(\overline{\text{Béz}}(f)) = \text{Jac}(f) \bmod (f)$. Scheja–Storch [SS75, p. 184] note that $\delta(\Delta_{ij}) = \frac{\partial f_i}{\partial x_j}$ but then proceed to work modulo (f) as well.

Example 2.5. Let $f = (x_1^2, x_2^2, x_3^2)$. The set $\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}$ is a basis for $k[\mathbf{x}]/(f)$. Let

$$\begin{aligned} B &= x_1x_2x_3 \otimes 1 + x_2x_3 \otimes x_1 + x_1x_3 \otimes x_2 + x_1x_2 \otimes x_3 \\ &\quad + x_1 \otimes x_2x_3 + x_2 \otimes x_1x_3 + x_3 \otimes x_1x_2 + 1 \otimes x_1x_2x_3. \end{aligned}$$

By Definition 1.1, we have

$$\begin{aligned} \text{Béz}(f) &= x_1x_2x_3 + x_2x_3y_1 + x_1x_3y_2 + x_1x_2y_3 \\ &\quad + x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 + y_1y_2y_3. \end{aligned}$$

One can readily check that $\mu(B) = \text{Béz}(f)$. Moreover, $\delta(\text{Béz}(f)) = 8x_1x_2x_3$, which is equal to $\text{Jac}(f)$ (see Proposition 2.2).

3. Proof of Theorem 1.3

Let $q = (q_1, \ldots, q_n) \in \mathbb{A}_k^n$ be a k-rational point. The dimension of $k[\mathbf{x}]/(f - q)$ as a k-vector space is closely related to the fiber cardinality $|f^{-1}(q)|$.

Proposition 3.1. *Let I be an ideal in $k[\mathbf{x}]$. Suppose that $\mathbb{V}(I) = \{p_1, \ldots, p_m\}$ is a finite set of points. Let $\mathfrak{m}_i$ be the maximal ideal in $k[\mathbf{x}]$ corresponding to p_i. Then there is a natural isomorphism*

$$\frac{k[\mathbf{x}]}{I} \cong \prod_{i=1}^{m} \frac{k[\mathbf{x}]_{\mathfrak{m}_i}}{I}.$$

Proof. Since $\mathbb{V}(I)$ is a finite set, $k[\mathbf{x}]/I$ is Artinian by [Sta21, Lemma 00KH]. It follows from [Sta21, Lemma 00JA] that the desired isomorphism exists. □

Corollary 3.2. *Under the assumptions of Proposition 3.1, we have*

$$\dim_k \frac{k[\mathbf{x}]}{I} = \sum_{i=1}^{m} \dim_k \frac{k[\mathbf{x}]_{\mathfrak{m}_i}}{I}.$$

In particular, if $I = (f_1 - q_1, \ldots, f_n - q_n)$, then $\dim_k k[\mathbf{x}]/(f - q) \geq |f^{-1}(q)|$.

We are now prepared to prove Theorem 1.3 and Corollary 1.5.

Proof of Theorem 1.3. By Proposition 2.1, we have $\dim_k k[\mathbf{x}]/(f) \leq 1$, so Corollary 3.2 implies that $|f^{-1}(0)| \leq 1$. Thus f is injective at the origin. □

Proof of Corollary 1.5. Note that for any k-rational point q, we have $\text{Béz}(f - q) = \text{Béz}(f)$. Thus if $\text{Béz}(f) \in k$, then $\overline{\text{Béz}}(f - q) \in k$ for all $q \in \mathbb{A}_k^n(k)$. By Corollary 1.4, f is injective on k-rational points. □

4. Drużkowski morphisms with constant Bézoutian

If we assume that $\mathrm{Jac}(f) \in k$, then we get slightly stronger injectivity results. By the work of Bass, Connell, and Wright [BCW82, Theorem 2.1], we can study the Jacobian conjecture by studying the injectivity of morphisms with $\mathrm{Jac}(f) \in k^\times$.

Proposition 4.1. *If $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then f is quasi-finite. In particular, f has finite fibers.*

Proof. Let $X = k[\mathbf{x}]/(f)$, and recall that the module of Kähler differentials $\Omega_{X/k}$ is the cokernel of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$. Since $\mathrm{Jac}(f) \in k^\times$, we have that $\Omega_{X/k} = 0$ and hence f is unramified. By [Sta21, Lemma 02V5], f is locally quasi-finite. Since $\mathbb{A}^n_k$ is Noetherian, $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasi-compact. Thus [Sta21, Lemma 01TJ] implies that f is quasi-finite. □

We saw in Corollary 3.2 that $\dim_k k[\mathbf{x}]/(f-q) \geq |f^{-1}(q)|$. Assuming that k is algebraically closed of characteristic 0 and $\mathrm{Jac}(f) \in k^\times$, this inequality is an equality.

Proposition 4.2. *Let k be an algebraically closed field of characteristic 0. If $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ has $\mathrm{Jac}(f) \in k^\times$, then $\dim_k k[\mathbf{x}]/(f-q) = |f^{-1}(q)|$.*

Proof. We need to show that if $p \in f^{-1}(q)$ with corresponding maximal ideal $\mathfrak{m}$, then $\dim_k k[\mathbf{x}]_\mathfrak{m}/(f-q) = 1$. Since f has finite fibers, $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a local Artin ring. Since k has characteristic 0, [SS75, (4.7) Korollar] implies that $\mathrm{Jac}(f)$ generates the socle of $k[\mathbf{x}]_\mathfrak{m}/(f-q)$, which is the annihilator of the maximal ideal $\mathfrak{m}$. That is, the maximal ideal of $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is annihilated by a scalar, so this maximal ideal must be the zero ideal. In particular, $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a field. On the other hand, $k[\mathbf{x}]_\mathfrak{m}/(f-q)$ is a finitely generated k-algebra. Since k is algebraically closed, the only finitely generated k-algebra that is a field is k, so we conclude that $k[\mathbf{x}]_\mathfrak{m}/(f-q) \cong k$. □

Let k be an algebraically closed field of characteristic 0. If $\mathrm{Jac}(f) \in k^\times$, we get a converse to Theorem 1.3.

Lemma 4.3. *If $\mathrm{Jac}(f) \in k^\times$ and f is injective at $q \in \mathbb{A}^n_k(k)$, then $\overline{\mathrm{Béz}}(f-q)$ is constant.*

Proof. Since f is injective at q, Proposition 4.2 implies that $\dim_k k[\mathbf{x}]/(f-q) \leq 1$. If $\dim_k k[\mathbf{x}]/(f-q) = 0$, then $\overline{\mathrm{Béz}}(f-q) = 0 \in k$. If $\dim_k k[\mathbf{x}]/(f-q) = 1$, then $k[\mathbf{x},\mathbf{y}]/(f(\mathbf{x})-q, f(\mathbf{y})-q) \cong k \otimes_k k \cong k$. Thus $\overline{\mathrm{Béz}}(f-q) \in k$, as desired. □

Combining Corollary 1.6, Lemma 4.3, and [BCW82, Theorem 2.1] gives us the following formulation of the Jacobian conjecture in terms of the Bézoutian.

Corollary 4.4. *If $\mathrm{Jac}(f) \in k^\times$, then the Jacobian conjecture is true for f if and only if $\overline{\mathrm{Béz}}(f-q)$ is a constant for all $q \in \mathbb{A}^n_k(k)$.*

Using Corollary 1.6, we can prove the Jacobian conjecture for any morphism whose Bézoutian is a constant. An important class of morphisms to consider are Drużkowski morphisms.

Definition 4.5. A morphism $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ is called a *Drużkowski morphism* if f is of the form $(x_1 + (\sum_{i=1}^n a_{1i}x_i)^3, \ldots, x_n + (\sum_{i=1}^n a_{ni}x_i)^3)$ with $\mathrm{Jac}(f) \in k^\times$.

It was proved by Drużkowski [Dru83, Theorem 3] that if the Jacobian conjecture is true for all Drużkowski morphisms over a field k of characteristic 0, then the Jacobian conjecture is true over k. By [BCW82, (1.1) Remark 4], the Jacobian conjecture is true over all fields of characteristic 0 if the Jacobian conjecture is true over $\mathbb{C}$.

If (a_{ij}) is strictly upper triangular or strictly lower triangular, then the Drużkowski morphism $(x_1 + (\sum_{j=1}^n a_{1j}x_j)^3, \ldots, x_n + (\sum_{j=1}^n a_{nj}x_j)^3)$ has constant Bézoutian. This allows us to recover the well-known solution of the Jacobian conjecture for such morphisms.

Proposition 4.6. *Let k be an algebraically closed field of characteristic 0. Suppose $a_{ij} = 0$ either for all $i \geq j$ or for all $i \leq j$. Then the morphism*

$$f := (x_1 + (\textstyle\sum_{j=1}^n a_{1j}x_j)^3, \ldots, x_n + (\sum_{j=1}^n a_{nj}x_j)^3)\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$$

satisfies the Jacobian conjecture.

Proof. First suppose $a_{ij} = 0$ for all $i \geq j$. Since $a_{ij} = 0$ for $i > j$, we have $\Delta_{ij} = 0$ for $i > j$. Since $a_{ii} = 0$ for all i, we have $\Delta_{ii} = 1$ for all i. Thus $\mathrm{Béz}(f) = \mathrm{Jac}(f) = 1$, so f satisfies the Jacobian conjecture by Corollary 1.6. Symmetrically, if $a_{ij} = 0$ for all $i \leq j$, then we again have $\mathrm{Béz}(f) = \mathrm{Jac}(f) = 1$. $\square$

Proposition 4.6 follows from [Dru83, Theorem 5] when the rank of (a_{ij}) is 0, 1, 2, or $n-1$. As mentioned in [Dru83, Remark 6], f is a Drużkowski morphism (in particular, $\mathrm{Jac}(f) \in k^\times$) only if $\mathrm{rank}(a_{ij}) < n$.

More strongly, Drużkowski proved that if the Jacobian conjecture is true for all Drużkowski morphisms with (a_{ij}), then the Jacobian conjecture is true in general [Dru01, Theorem 2]. Since every nilpotent matrix is similar to a strictly upper triangular matrix and an invertible matrix P determines an automorphism $P\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ given by $P(\mathbf{x}) = P\mathbf{x}^T$, Proposition 4.6 gives a potential to approach the Jacobian conjecture.

Definition 4.7. If $f := (x_1 + (\sum_{j=1}^n a_{1j}x_j)^3, \ldots, x_n + (\sum_{j=1}^n a_{nj}x_j)^3)$ is a Drużkowski morphism, then we say that f *has matrix* (a_{ij}).

Given an invertible matrix P and a Drużkowski morphism f with matrix (a_{ij}), we would like to find automorphisms $S, T\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ such that $S \circ f \circ T$ is a Drużkowski morphism with matrix $P(a_{ij})P^{-1}$. Meisters proved that not all nilpotent matrices are cubic similar to a strictly upper triangular matrix [Mei95], which suggests that finding such automorphisms S, T is not trivial. It suffices to reduce to the case where P is an elementary matrix. If P is a permutation matrix, then finding S, T is straightforward.

Proposition 4.8. *[GTGZ99, Proposition 3.1] Let P be a permutation matrix. If f is a Drużkowski morphism with matrix (a_{ij}), then $P \circ f \circ P^{-1}$ is a Drużkowski morphism with matrix $P(a_{ij})P^{-1}$.*

Proof. Let σ be the permutation of $\{1, \dots, n\}$ such that left multiplication by P permutes the rows of a matrix by σ. Right multiplication by P^{-1} permutes the columns of a matrix by σ^{-1}. We thus have

$$\begin{aligned} P \circ f \circ P^{-1} &= P(x_{\sigma^{-1}(1)} + (\textstyle\sum_{j=1}^n a_{1j} x_{\sigma^{-1}(j)})^3, \dots, x_{\sigma^{-1}(n)} + (\textstyle\sum_{j=1}^n a_{nj} x_{\sigma^{-1}(j)})^3) \\ &= P(x_{\sigma^{-1}(1)} + (\textstyle\sum_{j=1}^n a_{1\sigma(j)} x_j)^3, \dots, x_{\sigma^{-1}(n)} + (\textstyle\sum_{j=1}^n a_{n\sigma(j)} x_j)^3) \\ &= (x_1 + (\textstyle\sum_{j=1}^n a_{\sigma(1)\sigma(j)} x_j)^3, \dots, x_n + (\textstyle\sum_{j=1}^n a_{\sigma(n)\sigma(j)} x_j)^3). \end{aligned}$$

Moreover, since $\mathrm{Jac}(P) = \det(P)$, we have $\mathrm{Jac}(P \circ f \circ P^{-1}) = \mathrm{Jac}(f) \in k^\times$. Thus $P \circ f \circ P^{-1}$ is a Drużkowski morphism with matrix $P(a_{ij})P^{-1}$. ☐

Paired with Proposition 4.6, we recover [GTGZ99, Theorem 3.2]. Unfortunately, the obvious trick does not quite work when P is a row multiplication matrix, as we show in Remark 4.9. Row addition matrices seem to be even more problematic than row multiplication matrices.

Remark 4.9. Let $\mathbf{e}_i$ be the i^{th} standard column vector. Let $D_i = (\mathbf{e}_1 \ \cdots \ m\mathbf{e}_i \ \cdots \ \mathbf{e}_n)$ for some $m \in k^\times$. If f is a Drużkowski morphism with matrix (a_{ij}), then $D_i^3 \circ f \circ D_i^{-1}$ has matrix $D_i(a_{ij})D_i^{-1}$ but is not quite a Drużkowski morphism. Indeed, we compute

$$\begin{aligned} D_i^3 \circ f \circ D_i^{-1} = (x_1 + (m^{-1} a_{1i} x_i + &\textstyle\sum_{j \neq i} a_{1j} x_j)^3, \dots, \\ m^2 x_i + (a_{ii} x_i + &\textstyle\sum_{j \neq i} m a_{ij} x_j)^3, \dots, \\ x_n + (m^{-1} a_{ni} x_i + &\textstyle\sum_{j \neq i} a_{nj} x_j)^3). \end{aligned}$$

This is a Drużkowski morphism if and only if $m = \pm 1$, since we need $m^2 x_i = x_i$.

By Propositions 4.6 and 4.8 and Remark 4.9, we get the following corollary.

Corollary 4.10. *If (a_{ij}) is conjugate to a strictly upper triangular matrix via permutation matrices and ± 1 row multiplications, then the Jacobian conjecture is true for the Drużkowski morphism with matrix (a_{ij}).*

If all entries of (a_{ij}) are non-negative real numbers, then (a_{ij}) is nilpotent if and only if it is permutation-similar to a strictly upper triangular matrix. Unfortunately, Corollary 4.10 does not cover all Drużkowski morphisms. For example, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is not conjugate to a strictly upper triangular matrix via permutation matrices and ± 1 row multiplications [GTGZ99, (1.6)]. Moreover, the Drużkowski morphism f_A with matrix A has $\mathrm{Béz}(f_A) \notin k$, so one cannot hope for Corollary 1.5 to be of use for general

Drużkowski morphisms. Nevertheless, the Jacobian conjecture is true for f_A, so $\overline{\text{Béz}}(f_A - q) = 1$ for all q.

It is known that Drużkowski morphisms are injective at the origin [Dru83, Proposition 1], so $\overline{\text{Béz}}(f) = 1$ for any Drużkowski morphism f by Lemma 4.3. Indeed, $\text{Jac}(f) = 1$ for any Drużkowski morphism, so $1 \equiv \text{Jac}(f) \mod (f)$. Since $\overline{\text{Béz}}(f) \in k$ and δ is injective on k (see Section 2), we have $\overline{\text{Béz}}(f) = 1$. Similarly, $\text{Jac}(f - q) = 1$ for any k-rational point $q \in \mathbb{A}^n_k$, so if $\overline{\text{Béz}}(f - q)$ is constant, then $\overline{\text{Béz}}(f - q) = 1$.

Question 4.11. *Given a Drużkowski morphism f, is $\overline{\text{Béz}}(f - q) = 1$ for all $q \in \mathbb{A}^n_k(k)$?*

Since $\overline{\text{Béz}}(f) = 1$, we have $\text{Béz}(f) \equiv 1 \mod (f(\mathbf{x}), f(\mathbf{y}))$. Since $\text{Béz}(f) = \text{Béz}(f - q)$ for any q, Question 4.11 is asking if $\text{Béz}(f) \equiv 1 \mod (f(\mathbf{x}) - q, f(\mathbf{y}) - q)$ for all q.

We include a basic Sage script [McK21] that takes as input a matrix A and a k-rational point $q \in \mathbb{A}^n_k$ and returns as output $\text{Jac}(f_A - q)$ and $\overline{\text{Béz}}(f_A - q)$ of the corresponding Drużkowski morphism f_A. This script is based off of code written jointly with Thomas Brazelton and Sabrina Pauli for computing the $\mathbb{A}^1$-degree via the Bézoutian [BMP21a].

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