LECTURE 4: COBORDISM AND GENERA

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Last time, we computed $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$ and $\Omega_1^{\mathrm{SO}} = \Omega_2^{\mathrm{SO}} = 0$ by hand. I also told you that $\Omega_3^{\mathrm{SO}} = 0$, but we didn't prove this. Each of these facts uses explicit geometric input: in dimensions 0, 1, and 2, we completely understand the set of manifolds up to homeomorphism, from which we can deduce the set (and even group) of manifolds up to oriented cobordism. In dimension 3, we apply something called surgery theory to construct oriented cobordisms from any 3-manifold to S^3.

Clearly, we can't keep playing this game forever. In each new dimension, it seems like you need a new geometric idea. You can't just "induct on dimension". So how can we possibly completely characterize Ω_*^{SO}? Later in the lecture, we'll see how homotopy theory lets us characterize $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ in a dramatically beautiful way. Wall combined this understanding of $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ with more geometric input to fully characterize Ω_*^{SO} [Wal60], although we won't have time for that story in this class.

But before we get to the homotopy theory, we'll do just a little more geometry to prove that Ω_4^{SO} is not trivial.

1. Signature of $4d$-manifolds

On the first day of class, I told you that genera were important and useful. I'll prove it by using a genus to show that Ω_4^{SO} is not trivial. In other words, I will use some function (which is analytic in nature) to imply a non-obvious fact about 4-dimensional topology.

Remark 1.1. We are about to use the *cup product* define a bilinear form on cohomology. This is best defined algebraically (in terms of singular simplices), but for our purposes it suffices to recall the analytic formulation. Given a (de Rham cohomology class of a) p-form $[\alpha] \in H^p_{\mathrm{dR}}(X;\mathbb{R})$ and a q-form $[\beta] \in H^q_{\mathrm{dR}}(X;\mathbb{R})$, the *cup product* is the class of the form $[\alpha] \smile [\beta] := [\alpha \wedge \beta] \in H^{p+q}_{\mathrm{dR}}(X;\mathbb{R})$.[1]

Whenever you hear "cohomology of compact manifolds", you should think "Poincaré duality!" Here's a corollary of Poincaré duality in even dimensions:

Theorem 1.2. *Let M be a compact, connected $2n$-manifold without boundary. Then the composition of the cup product $\smile\colon H^n(M;\mathbb{R}) \otimes H^n(M;\mathbb{R}) \to H^{2n}(M;\mathbb{R})$ with the duality isomorphism $H^{2n}(M;\mathbb{R}) \cong \mathbb{R}$ induces a non-degenerate bilinear form*

$$\beta : H^n(M;\mathbb{R}) \otimes H^n(M;\mathbb{R}) \to \mathbb{R}.$$

Moreover, β is symmetric if n is even and skew-symmetric if n is odd.

[1] The LaTeX for $\smile$ is `\smile`.

Exercise 1.3. Prove the claim about (skew-)symmetry in Theorem 1.2.

When $n = 2d$ is even (so M is a $4d$-manifold), β is a symmetric, non-degenerate bilinear form over $\mathbb{R}$. These are classified by two invariants: rank and signature.

Definition 1.4. Let V be a real vector space. Let $\beta : V \times V \to \mathbb{R}$ be a symmetric, non-degenerate bilinear form. The *rank* of β is defined as $\operatorname{rank}(\beta) := \dim(V)$. The *signature* of β is defined as $\operatorname{sign}(\beta) := n_+ - n_-$, where n_+ is the number of positive eigenvalues and n_- is the number of negative eigenvalues.

Definition 1.5. Let M be a compact, connected $4d$-manifold without boundary. The *signature* of M is defined to be

$$\sigma(M) := \operatorname{sign}(\beta) \in \mathbb{Z},$$

where β is the bilinear form determined by the cup product on middle cohomology.

We also define $\sigma(M_1 \sqcup M_2) := \sigma(M_1) + \sigma(M_2)$, which extends the definition to non-connected manifolds.

The previous definition implies that if $\sigma(M_1) = \sigma(M_2)$ whenever M_1 and M_2 are cobordant, then $\sigma : \Omega_{4d}^{\mathrm{SO}} \to \mathbb{Z}$ will be a group homomorphism. We'll prove it in dimension 4:

Lemma 1.6. *If M_1 and M_2 are oriented 4-manifolds that are cobordant, then $\sigma(M_1) = \sigma(M_2)$.*

Proof. Let W be a cobordism from M_1 to M_2. Thanks to the group structure on Ω_n^{SO}, this will follow from proving that $\sigma(M_1 \sqcup -M_2) = 0$. By the additivity of σ over connected components, we may also assume that M_1, M_2, and W are all connected.

Let $i : \partial W \hookrightarrow W$. The key is relative Poincaré duality, which gives us a commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(W;\mathbb{R}) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^2(\partial W;\mathbb{R}) \arrow[r, "\delta^*"] \arrow[d, "\cong"] & H^3(W,\partial W;\mathbb{R}) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_3(W;\mathbb{R}) \arrow[r, "\delta_*"] & H_2(\partial W;\mathbb{R}) \arrow[r, "i_*"] & H_2(W;\mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

Next, we will show that β vanishes on $i^*(H^2(W;\mathbb{R}))$. Let $[\partial W]$ and $[W,\partial W]$ denote the fundamental class and relative fundamental class, respectively. Then

$$\begin{aligned}
\beta(i^*a, i^*b) &= \langle i^*a \smile i^*b, [\partial W]\rangle \\
&= \langle i^*(a \smile b), [\partial W]\rangle \\
&= \langle i^*(a \smile b), \delta_*[W,\partial W]\rangle \\
&= \langle \delta^* i^*(a \smile b), [W,\partial W]\rangle \\
&= 0.
\end{aligned}$$

Here, $\langle -, - \rangle$ denotes the intersection pairing induced by cap products. The steps here are naturality of $\smile$ under pullbacks, the interaction of relative fundamental classes and cap products, and exactness ($\delta^* i^* = 0$).

Next, we will show that $\dim H^2(\partial W; \mathbb{R}) = 2 \dim i^*(H^2(W; \mathbb{R}))$. The above diagram tells us that $\dim H^2(\partial W; \mathbb{R}) = \dim i^*(H^2(W; \mathbb{R})) + \dim \ker(\delta^*)^\perp$, and that $\ker(\delta^*)^\perp \cong \ker(i_*)^\perp$. Now the universal coefficient theorem gives a diagram

$$\begin{tikzcd} H^2(W;\mathbb{R}) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^2(\partial W;\mathbb{R}) \arrow[d, "\cong"] \\ H_2(W;\mathbb{R}) & H_2(\partial W;\mathbb{R}). \arrow[l, "i_*"] \end{tikzcd}$$

In particular, $\ker(\delta^*)^\perp \cong \ker(i_*)^\perp \cong \operatorname{im}(i^*)$.

Now to conclude, we diagonalize β to be a bilinear form on $P \oplus N$, where P and N are the positive and negative eigenspaces, respectively. If $\dim(P) > \dim(N)$, then $\dim(P) \geq \dim i^*(H^2(W; \mathbb{R})) + 1$, so $P \cap i^*(H^2(W; \mathbb{R}))$ must be non-empty. But this contradicts the vanishing of β on $i^*(H^2(W; \mathbb{R}))$. The same argument holds if $\dim(P) < \dim(N)$, so we conclude that $\dim(P) = \dim(N)$ and hence $\sigma(\partial W) = \dim(P) - \dim(N) = 0$. $\square$

Here's an exercise for you:

Exercise 1.7. Prove that $\sigma(\mathbb{CP}^2) = 1$, and deduce that $\sigma : \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$ is surjective.

2. Thom spaces, Thom spectra, and the Pontryagin–Thom construction

On the first day of class, I mentioned that $\Omega_n^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]$. Based on what we've done so far, you should be utterly amazed by this sort of theorem. How can one possibly get at all of this topology in every dimension? What would a proof even look like?

Well, this is the perfect opportunity to introduce the magic of René Thom. The motto is:

> *Turn a question about the geometry or topology of an object X into a question about the homotopy theory of an object Y.*

Remark 2.1. This motto has been shockingly effective over the last 70 years. In recent years, arithmetic questions have also been successfully attacked using homotopy theory. A reason for all of this is that homotopy theory is an excellent organizational tool, and there are lots of powerful computational tools present in the subject.

As some basic language, we need to introduce the notion of a spectrum. First, recall the definition of a *loop space*:

Definition 2.2. A *pointed space* is a pair (X, x), where X is a topological space and $x \in X$ is a chosen base point. The *loop space* $\Omega_x X$ (or just ΩX) of (X, x) is the topological space of continuous maps $[(S^1,0),(X,x)]$, where $0 \in S^1$ is a chosen base point. Here, we topologize the set $[(S^1,0),(X,x)]$ via the compact-open topology.

Remark 2.3. Looping gives a functor $\Omega : \mathrm{ho}(\mathrm{Top}_*) \to \mathrm{ho}(\mathrm{Top}_*)$. This is actually the right adjoint in an adjoint pair (Σ,Ω), where Σ is *suspension.* That is,

$$[\Sigma X, Y]_{\mathrm{ho}(\mathrm{Top}_*)} \cong [X, \Omega Y]_{\mathrm{ho}(\mathrm{Top}_*)}.$$

Recall that the *smash product* of two pointed spaces X and Y is defined as the cofiber (think "quotient") $X \wedge Y := (X \times Y)/(X \vee Y)$. Given a pointed space X, the *suspension* is defined as $\Sigma X := S^1 \wedge X$.

Often, one sets up spectra in terms of Σ instead of Ω, but loops are a little more intuitively accessible if you've never thought about suspensions before. Anyway, I cannot overstate how fundamentally important (Σ,Ω) are in homotopy theory.

Exercise 2.4. Prove that $[\Sigma X, Y]_{\mathrm{ho}(\mathrm{Top}_*)} \cong [X, \Omega Y]_{\mathrm{ho}(\mathrm{Top}_*)}$ as sets.

Now we can introduce *spectra.* We'll do these in more detail later, so take this as a first approximation of a richer story.

Definition 2.5. A *spectrum* E is a sequence of pointed topological spaces $\{E_n\}_{n\geq 0}$, together with *structure maps* $e_n : E_n \to \Omega E_{n+1}$.

A morphism of spectra $\varphi : E \to F$ is a sequence of maps of pointed topological spaces $\varphi_n : E_n \to F_n$ such that the following diagrams all commute:

$$\begin{tikzcd} E_n \arrow[r, "\varphi_n"] \arrow[d, "e_n"] & F_n \arrow[d, "f_n"] \\ \Omega E_{n+1} \arrow[r, "\Omega\varphi_{n+1}"] & \Omega F_{n+1}. \end{tikzcd}$$

Remark 2.6. Recall that a map of topological spaces is called a *weak equivalence* if it induces an isomorphism on homotopy groups. We have the same notion for spectra, where we replace homotopy groups with *stable* homotopy groups

$$\pi_n^s E := \operatorname*{colim}_{k\to\infty} \pi_k E_{k+n}.$$

Remark 2.7. What is going on here? There's a theorem known as *Brown representability*, which very roughly says that cohomology should be representable by a sequence of spaces. That is, there should be spaces $\{E_n\}$ such that $H^n(X) = [X, E_n]$. But cohomology isn't just a bucket of groups — there should be some connection between H^n and H^{n+1}. Spectra naturally come out of this story, and the structure maps tie all the different dimensions together. We'll talk about this more rigorously later.

Now that I've given you a definition, we need to see an example. Here's the preview, which we'll go through more carefully in a moment. Associated to the classifying space $BG(n)$ of some Lie group $G(n)$, we will build a topological space known as the *Thom* *space*. This will be the one-point compactification $\mathrm{Th}(\xi_n) := \xi_n \cup \{\infty\}$ of the universal bundle $\xi_n \to BG(n)$. We will see that this construction comes with very natural structure maps $\mathrm{Th}(\xi_n) \to \Omega\mathrm{Th}(\xi_{n+1})$, so that the Thom spaces fit together to form the *Thom spectrum* MG. Finally, we will talk about the *Pontryagin–Thom isomorphism*, which relates the homotopy groups of the spectrum MG to the cobordism ring Ω_*^G.

2.1. **Classifying spaces.** If you've never seen classifying spaces before, here's a quick overview of what they are. When we were talking about spectra, I mentioned Brown representability, which says that the cohomology of a space X should actually come from maps *into* some other space Y. This is a powerful idea, because it turns an algebraic construction into something more geometric.

Classifying spaces arise from trying to apply this idea to the theory of vector bundles.

Definition 2.8. Let G be a topological group. A *principal G-bundle* over a topological space X is a bundle $P \to X$ with a G-action $\rho : G \times P \to P$ such that $(\mathrm{proj}_1, \rho) : P \times G \to P \times_X P$ is an isomorphism.

Definition 2.9. Let G be a topological group. A *classifying space* of G is a topological space BG such that there is a natural isomorphism of sets

$$\{\text{principal } G\text{-bundles over } X\}/\text{iso} \cong [X, BG]$$

for sufficiently nice spaces X.

Remark 2.10. This is sort of an aspirational definition. We want this sort of space to exist, but why should it? It turns out that classifying spaces of topological groups exist and are unique up to homotopy. In fact, you can construct them as the *delooping* of G. That is, BG can be defined as the space such that $\Omega BG \simeq G$.[2]

A key player in this story is the *universal line bundle* $\xi \to BG$. This is how we get our representability result — given a map $f : X \to BG$, we get a principal G-bundle $f^*\xi$ on X. Moreover, every principal G-bundle takes this form.

This is about all we'll say on the subject for now, but we'll come back to it later. It's also important to know that we get a sequence of inclusions $B\mathrm{SO}(n) \to B\mathrm{SO}(n+1)$. Moreover, the pullback of the universal bundle $\xi_{n+1} \to B\mathrm{SO}(n+1)$ is $\xi_n \oplus \mathbb{R}$.

2.2. **Thom spaces and Thom spectra.** We are about to build a *Thom spectrum*, which is an extremely nice sort of spectrum. Today, we'll just do this story for $G(n) = \mathrm{SO}(n)$. If you want *complex cobordism*, you repeat this story with $G(n) = \mathrm{U}(n)$. From the great zoo of Lie groups, we get a great zoo of Thom spectra, which in turn tie right back to the various cobordism theories. We'll come back to this next time.

[2]In general, you can't just deloop some random space. It turns out that having a group structure on G is precisely what allows us to deloop once. If you wanted to deloop again, you'd better hope for nice structure on G!

Definition 2.11. If ξ is a finite-dimensional vector bundle over a compact space, the *Thom space* is the one point compactification of the total space: $\mathrm{Th}(\xi) := \xi \cup \{\infty\}$.

If ξ is finite-dimensional over a non-compact space, then $\mathrm{Th}(\xi)$ is given by one point compactifying each fiber of ξ, and then identifying the point at ∞ across all fibers.

Example 2.12. Take the trivial line bundle $\mathbb{R} \times S^1$ on S^1. What is $\mathrm{Th}(\mathbb{R} \times S^1)$?

Example 2.13. Take the trivial line bundle $\mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$. What is $\mathrm{Th}(\mathbb{R} \times \mathbb{R})$?

These examples lead us to an important lemma.

Lemma 2.14. *Let V be a finite-dimensional vector bundle. Then $\mathrm{Th}(V \oplus \mathbb{R})$ is homotopy equivalent to $\Sigma \mathrm{Th}(V)$.*

Exercise 2.15. Prove Lemma 2.14.

Recall that the universal bundles $\xi_n \to B\mathrm{SO}(n)$ satisfy a nice pullback relation: ξ_{n+1} pulls back to $\xi_n \oplus \mathbb{R}$ under the inclusion $B\mathrm{SO}(n) \to B\mathrm{SO}(n+1)$. We can now define our first Thom spectrum.

Definition 2.16. The Thom spectrum MSO is defined as the spectrum with spaces $\mathrm{Th}(\xi_n)$, where ξ_n is the universal bundle on $B\mathrm{SO}(n)$. The structure maps are given by $\Sigma \mathrm{Th}(\xi_n) \to \mathrm{Th}(\xi_{n+1})$, which are equivalent to maps $\mathrm{Th}(\xi_n) \to \Omega \mathrm{Th}(\xi_{n+1})$ under the loops-suspension adjunction.

To close the day, we'll state the theorem that we'll prove next time.

Theorem 2.17 (Thom)**.** *There is a ring isomorphism $\Omega_*^{\mathrm{SO}} \cong \pi_* \mathrm{MSO}$.*

The upshot will be that we can compute cobordism now as the homotopy groups of some spectrum. More on this next time.

Next time: Proving Pontryagin–Thom, more genera, index theory, and maybe spin geometry.

Daily exercises: I decided to stop collecting the exercises here. If you really want me to put them at the end of the notes like before, let me know!

References

[Wal60] C. T. C. Wall. "Determination of the cobordism ring". In: *Ann. of Math. (2)* 72 (1960), pp. 292–311. URL: `https://doi.org/10.2307/1970136`.

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