INVERSE CURVE PROBLEMS ON DEL PEZZO SURFACES

ENIS KAYA, STEPHEN MCKEAN, SAM STREETER, AND H. UPPAL

ABSTRACT. We classify the number of k-rational lines and conic fibrations on del Pezzo surfaces over a field k in terms of relatively minimal surfaces and establish rational curve analogues of the inverse Galois problem for del Pezzo surfaces. We completely solve these problems in all degrees over all global, local and finite fields and provide new solutions of the inverse Galois problem in characteristic 2. Our results generalise well-known theorems on cubic surfaces.

CONTENTS

1. INTRODUCTION

One of the jewels of classical algebraic geometry is the *Cayley–Salmon theorem*: any smooth cubic surface over an algebraically closed field contains exactly 27 lines. Determining the number of lines becomes more subtle when keeping track of fields of definition. Segre [Seg49] proved[1] that the number of k-rational lines on a cubic surface over a field k is in $\{0, 1, 2, 3, 5, 7, 9, 15, 27\}$ and that all are realised over $\mathbb{Q}$. Schläfli [Sch58] showed that, over $\mathbb{R}$, only four counts are realised: 3, 7, 15 and 27.

Cubic surfaces are *del Pezzo surfaces* (smooth projective surfaces X with ample anti-canonical class $-K_X$, classified by degree $d := K_X^2 \in \{1, \ldots, 9\}$). We arrive at the focus of this paper: determining the possible numbers of k-rational lines on degree-d del Pezzo surfaces over an arbitrary field k. We also consider the analogue for conic fibrations on these surfaces. For definitions of lines and conics, see Definition 2.5; they coincide with the classical ones for degree at least 3 (Remark 2.6).

These problems relate to the *inverse Galois problem for del Pezzo surfaces*. The action of $G_k = \mathrm{Gal}(k_s/k)$ on the geometric Picard group $\mathrm{Pic}(\overline{X})$ of a degree-d del Pezzo surface X/k induces a homomorphism $\rho_X : G_k \to \mathrm{Aut}(\mathrm{Pic}(\overline{X}))$ to a subgroup non-canonically isomorphic to a group G_d (see Lemma 4.2). We thus obtain $[H_X] = [\rho_X(G_k)]$ in the set $C(G_d)$ of subgroup conjugacy classes of G_d.

Denote by $\mathcal{S}_d(k)$ the set of isomorphism classes of degree-d del Pezzo surfaces over k and by $C(G_d, k)$ the subset of $C(G_d)$ represented by groups isomorphic to $\mathrm{Gal}(L/k)$ for L/k some Galois extension, i.e. "splitting groups permitted by the Galois theory of k".

---

[1]Segre's claim that his proof fails when char $k = 2$ is false; see [McK22, §3.1] for more.

Problem IGP$_d(k)$ (Inverse Galois problem for del Pezzo surfaces)**.** Determine the image of the map

$$\gamma_d(k) : \mathcal{S}_d(k) \to C(G_d, k), \quad X \mapsto [H_X].$$

Note that ρ_X determines line/conic counts: there is a map $\delta_{d,1}(k) : C(G_d, k) \to \mathbb{Z}_{\geq 0}$ mapping $[H] \in C(G_d, k)$ to the number of k-lines on $X \in \mathcal{S}_d(k)$ with $[H_X] = [H]$, were such X to exist, with image in a finite set of "counts permitted by G_k" $\mathcal{C}_{d,1}(k)$ in a total set of possible counts $\mathcal{C}_{d,1}$. We have an analogous map and set $\delta_{d,2}(k) : C(G_d, k) \to \mathcal{C}_{d,2}(k) \subset \mathcal{C}_{d,2}$ for conic fibrations.

Problem ICP$_{d,e}(k)$ (Inverse curve problems for del Pezzo surfaces)**.** For $e \in \{1, 2\}$, determine the image of the map

$$\delta_{d,e}(k) \circ \gamma_d(k) : \mathcal{S}_d(k) \to \mathcal{C}_{d,e}(k).$$

We say that a solution is *surjective* if the image is surjective onto $\mathcal{C}_{d,e}(k)$. We say that we have a *simultaneous* surjective solution to both problems if $(\delta_{d,1}(k), \delta_{d,2}(k)) \circ \gamma_d(k)$ is surjective. Note that a solution to IGP$_d(k)$ gives a solution to ICP$_{d,e}(k)$.

The results above amount to solutions to ICP$_{3,1}(k)$ for k algebraically closed (where $\mathcal{C}_{3,1}(k) = \{27\}$), $k = \mathbb{Q}$ (where $\mathcal{C}_{3,1}(\mathbb{Q}) = \mathcal{C}_{3,1} = \{0, 1, 2, 3, 5, 7, 9, 15, 27\}$) and $k = \mathbb{R}$ (where $\mathcal{C}_{3,1}(\mathbb{R}) = \{3, 7, 15, 27\}$). Over separably closed fields, the analogue of Cayley–Salmon (describing $\mathcal{C}_{d,e}(k)$ for k algebraically closed) is well-known (see Theorem 2.7), and the analogue of Segre's result (describing $\mathcal{C}_{d,e}$) is fairly easy to establish (see Theorems 2.8 and 2.9). Little is known otherwise about the inverse curve problems. We develop techniques for computing rational curve counts and Galois actions to obtain new results. We completely solve the inverse curve problems over all global, local and finite fields and solve the inverse curve problems when the status of the Galois analogue is unknown.

The spirit of this work is close to that of recent advances in enumerative geometry by Kass and Wickelgren [KW21] and Larson and Vogt [LV21] on counting lines on cubic surfaces and bitangents to plane quartics (equivalent to lines on the corresponding degree-2 del Pezzo surface double-covering the plane with this branch curve) with arithmetic weights.

1.1. **Results.**

1.1.1. *Classification by blowup type.* Any del Pezzo surface is relatively minimal (Definition 2.12) or the blowup of a relatively minimal surface. Further, line and conic counts are determined by Galois action on the surface and blowup locus. Then it is natural to approach the inverse curve problems by classifying del Pezzo surfaces in terms of their *blowup types* (Definition 2.24), meaning the ways in which they can be constructed via blowups of relatively minimal surfaces. This can be thought of as a minimal model program for del Pezzo surfaces over arbitrary fields. We prove that, together with line and conic counts, the *intersection invariant* (Definition 3.1) determines blowup type.

Theorem 1.1. *For a del Pezzo surface X, denote by L_X, C_X and I_X the line count, conic count and intersection invariant of X, respectively. Then the triple (L_X, C_X, I_X) determines the blowup type of X uniquely.*

The invariant I_X appears to be novel in the del Pezzo setting and is a powerful tool in distinguishing blowup types, otherwise computationally untenable in low degree.

Using Theorem 1.1, we build tables of all blowup types with line and conic counts (Algorithm 1), found in Appendix A. One may view this as an extension of Iskovskikh's classification of relatively minimal del Pezzo surfaces over an arbitrary field (Theorem 2.14). With these tables, we attack the inverse curve problems via Algorithm 2.

1.1.2. *Finite fields.* Over finite fields, much is known about the inverse Galois problem: it was solved for $d \geq 2$ by Loughran and Trepalin [LT20]. However, the methods break down in degree 1 and one encounters additional difficulties (see Remark 5.4).

Despite this, Banwait, Fité and Loughran [BFL19] determined the possibilities for the trace of Frobenius on del Pezzo surfaces of degree 1 over finite fields. Trace of Frobenius and curve counts are both "shadows" of the Galois action, but the latter admits more possible values. By generalising their computational and theoretical techniques, we completely solve the inverse curve problems over all finite fields.

In Tables 1 and 2 below, the presence of a count in a column means that it is possible exactly when the condition labelling the column is true.

Theorem 1.2. *Over the finite field $\mathbb{F}_q$, the inverse line problem $\mathrm{ICP}_{d,1}(\mathbb{F}_q)$ admits a surjective solution for all q when $d \geq 5$. The solutions for $d \leq 4$ are given in Table 1.*

TABLE 1. Line counts over $\mathbb{F}_q$

Degree	All q	$q \geq 3$	$q \geq 4$	$q \geq 5$	$q \geq 7$	$q \geq 9$	$q \geq 11$	$q = 16$ or $q \geq 19$
4	0, 1, 2, 4, 8	–	16	–	–	–	–	–
3	0, 1, 2, 3, 5, 9, 15	–	7	–	27	–	–	–
2	0, 2, 4, 6, 8	12, 20	–	16, 32	–	56	–	–
1	0, 2, 4, 6, 12	8, 14, 20	24, 30	40	26, 72	–	126	240

Theorem 1.3. *Over the finite field $\mathbb{F}_q$, the inverse conic problem $\mathrm{ICP}_{d,2}(\mathbb{F}_q)$ admits a surjective solution for all q when $d \geq 5$. The solutions for $d \leq 4$ are given in Table 2.*

TABLE 2. Conic counts over $\mathbb{F}_q$

Degree	All q	$q \geq 3$	$q \geq 4$	$q \geq 5$	$q \geq 7$	$q \geq 9$	$q \geq 11$	$q = 16$ or $q \geq 19$
4	0, 2, 4, 6	–	10	–	–	–	–	–
3	0, 1, 2, 3, 5, 9, 15	–	7	–	27	–	–	–
2	0, 2, 4, 6	8, 12, 14	–	26, 60	24	126	–	–
1	0, 2, 4, 6	12, 18, 30, 36	24, 90	–	72, 252, 270	–	756	2160

1.1.3. *Infinite fields.* Having completely settled the problems for finite fields, we turn to infinite fields, for which we solve the inverse curve problems for $d \geq 4$.

Theorem 1.4. *For k an infinite field, the problems $\mathrm{ICP}_{d,e}(k)$ admit surjective solutions for all $d \geq 4$.*

The main content of Theorem 1.4 is for $d = 4$: $\mathrm{IGP}_4(k)$, thus $\mathrm{ICP}_{4,e}(k)$, is solved for infinite k with $\mathrm{char}(k) \neq 2$ by Kunyavskiĭ, Skorobogatov and Tsfasman [KST89, §6]. However, the representation-theoretic methods and conic bundle constructions of *loc. cit.* break down in characteristic 2. By careful construction of Artin–Schreier analogues, we solve new cases of the inverse Galois problem and arrive at our solution.

Under modest assumptions on the ground field, we obtain *simultaneous* surjective solutions to the inverse curve problems in all degrees.

Theorem 1.5. *Let k be an infinite field with separable extensions of degrees 2 to 8. Then $\mathrm{ICP}_{d,e}(k)$ has a surjective solution for all d and e. If $\mathrm{char}(k) \neq 2$ and k admits a Galois extension of degree 3, then the problems admit simultaneous surjective solutions.*

This generalises a result of Kulkarni and Vemulapalli [KV24, Thm. 3.1.1] on bitangents of plane quartics (equivalent to $\mathrm{ICP}_{2,1}(k)$) by removing characteristic and Galois assumptions. Further, it completely solves the inverse curve problems for local fields. Since the archimedean local fields $\mathbb{R}$ and $\mathbb{C}$ are already settled, we restrict to non-archimedean ones (e.g., the field of p-adic numbers $\mathbb{Q}_p$).

Corollary 1.6. *For k a non-archimedean local field, the problems $\mathrm{ICP}_{d,e}(k)$ admit surjective solutions for all d and e. If $\mathrm{char}(k) \neq 2$, these solutions are simultaneous.*

1.1.4. *Hilbertian fields.* We can say even more over Hilbertian fields (Definition 7.1), including global fields and function fields in finitely many variables. We show that all blowup types exist, thus solving the inverse curve problems simultaneously.

Theorem 1.7. *For all $1 \leq d \leq 9$ and k a Hilbertian field, all blowup types of degree-d del Pezzo surfaces exist. In particular, the inverse curve problems $\mathrm{ICP}_{d,e}(k)$ admit simultaneous surjective solutions.*

The properties of Hilbertian fields imply the existence of relatively minimal del Pezzo surfaces in each degree, but there remains the problem of determining whether there exist certain closed points in general position (Definition 2.21) on these surfaces. Through consideration of the geometric and arithmetic properties of symmetric powers, we are able to show that such points do in fact exist.

Remark 1.8. The only fields for which the inverse curve problems are not completely solved are infinite non-Hilbertian non-local non-closed fields which fail to have low-degree separable extensions. We also extend Russo's work [Rus02] over $\mathbb{R}$ to real closed fields (Remark 4.27), thus dispensing with virtually all of the fields usually encountered in arithmetic geometry.

1.2. **Related literature.** We now give a brief summary of the literature.

Arbitrary fields: For $d \geq 5$, Zaitsev [Zai23, Thms. 1.2, 1.7] showed that $\mathrm{IGP}_d(k)$ has a surjective solution for any field k.

Finite fields: Work of Trepalin [Tre20] and Loughran–Trepalin [LT20] solves $\mathrm{IGP}_d(\mathbb{F}_q)$ for all q when $d \in \{2, 3, 4\}$, following earlier work of Rybakov [Ryb05] and Rybakov–Trepalin [RT17]. Banwait, Fité and Loughran [BFL19, Thm. 1.7] determined the possibilities for the trace of Frobenius acting on the geometric Picard group and proved a quantitative form of $\mathrm{IGP}_d(\mathbb{F}_q)$ for $d \leq 6$ as $q \to \infty$; later cohomological results of Das [Das20], Bergvall and Gounelas [BG23] and Bergvall [Ber24] imply quantitative results. For more, we direct the reader to [Ber24, §1].

Infinite fields: A surjective solution to $\mathrm{IGP}_4(k)$ for k infinite with char $k \neq 2$ is given by Kunyavskiĭ, Skorobogatov and Tsfasman [KST89, Cor. 6.10] (see the final line before [KST89, Prop. 6.4]). This work also features a surjective solution of the analogous problem for conic bundles. Elsenhans and Jahnel gave a surjective solution to $\mathrm{IGP}_3(\mathbb{Q})$ [EJ15] and partial results for $\mathrm{IGP}_2(k)$ for k infinite with $\mathrm{char}(k) \neq 2$ [EJ19a, EJ19b]. Kulkarni and Vemulapalli [KV24] gave partial results on $\mathrm{ICP}_2(k)$, while Kulkarni [Kul21] established partial results for $\mathrm{IGP}_1(\mathbb{Q})$. Work of Russo [Rus02] implies $\mathrm{IGP}_d(\mathbb{R})$ for all $d \in \{1, \ldots, 9\}$.

Enriched counts: The last decade has seen a surge of activity in enumerative geometry over non-closed fields, in part due to the *enriched enumerative geometry* program. In enriched enumerative geometry, one uses tools from motivic homotopy theory to give weighted counts of geometric objects, analogous to the signed counts that are characteristic of real enumerative geometry. Kass and Wickelgren gave an enriched count of lines on cubic surfaces [KW21], while Larson and Vogt proved some results on enriching the count of bitangents to plane quartics [LV21] (which comes with additional subtleties and limitations due to issues of orientability).

While our work and enriched enumerative geometry share the goal of counting curves "arithmetically," our methods and results are distinct. For example, the enriched count of lines on a cubic surface X over $\mathbb{F}_q$ is a sum of quadratic forms over $\mathbb{F}_q$, with each line L contributing a quadratic form depending on the geometry of the inclusion $L \subset X$ and the trace form of the field of definition of L over $\mathbb{F}_q$. Over most fields, these trace forms prevent one from deducing anything about the number of k-rational lines from the enriched count of lines. A notable exception is $k = \mathbb{R}$, over which the trace form of any non-trivial extension has signature 0. In this case, the signature of the enriched count recovers the signed count of 3 lines on a real cubic surface (proved by Segre [Seg42] but only first noticed much later by Finashin–Kharlamov and Okonek–Teleman [FK12, OT14]). In the context of our results, this signed count of 3 just implies that a real cubic surface must have at least 3 real lines.

1.3. **Outline.** In Section 2 we give background on del Pezzo surfaces. In Section 3 we present results for distinguishing blowup types and Algorithm 1 for constructing tables of blowup types with line and conic counts, also proving Theorem 1.1. In Section 4 we review previous results and present Algorithm 2. In Section 5 we prove Theorems 1.2 and 1.3. In Section 6 we prove Theorems 1.4 and 1.5, and in Section 7 we prove Theorem 1.7. Appendix A contains our tables for arbitrary fields and special tables for finite fields; Appendix B contains computations on points in general position, and Appendix C contains open questions.

Our `Magma` [BCP97] code is available at

$$\text{https://github.com/HappyUppal/Lines-on-del-Pezzo-surfaces.git}.$$

We also benefited from GroupNames, Tim Dokchitser's database of groups.

1.4. Notation and conventions.

Algebra. Unless otherwise specified, we work over an arbitrary field k with separable closure k_s and algebraic closure $\overline{k}$, setting $G_k := \mathrm{Gal}(k_s/k)$. We denote by $\mathfrak{S}_n$, $\mathfrak{A}_n$, $\mathfrak{D}_n$ and C_n the symmetric and alternating groups on n letters, dihedral group of order $2n$ and cyclic group of order $n \in \mathbb{Z}_{\geq 2}$, respectively. For a prime power q, we denote by $\mathbb{F}_q$ the finite field of q elements.

Geometry. We denote by $\mathrm{Spec}\, R$ the spectrum of a ring R and by $\mathbb{A}^n_R$ and $\mathbb{P}^n_R$ the affine and projective spaces of dimension n over R respectively, omitting the subscript when it is clear from context. A *variety* X over a field k is an integral separated scheme of finite type over $\mathrm{Spec}\, k$; we denote its k-rational points by $X(k)$ and its canonical divisor class by K_X. We write X/k when emphasising the base field. Given an extension L/k, we write X_L for the base-change $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$, reserving $\overline{X} := X_{\overline{k}}$. *Curves* and *surfaces* are varieties of dimensions 1 and 2 respectively. We denote by $C \cdot D$ the intersection product of two divisors on a surface X, setting $C^2 := C \cdot C$. We say a variety X is *rational* (resp. *unirational*) if there is a birational (resp. dominant) map $\mathbb{P}^{\dim(X)} \dashrightarrow X$ over k. Let $\pi : Y \to X$ be the blowup of a surface X/k in a point $P \in X(k)$. The *exceptional divisor* of π is the preimage $E := \pi^{-1}(P)$. We have $E \cong \mathbb{P}^1_k$ and $E^2 = -1$. The *strict transform* $\tilde{C} \subset Y$ of a curve $C \subset X$ is the closure of $\pi^{-1}(C \setminus \{P\})$, and its *pullback* or *total transform* is $\pi^*C := \tilde{C} + rE$, where r is the multiplicity of C at P [Har77, Prop. V.3.6].

2. Del Pezzo surfaces

This section is devoted to gathering relevant facts about del Pezzo surfaces.

Definition 2.1. A *del Pezzo surface* over a field k is a smooth projective geometrically integral variety X/k of dimension 2 such that the anticanonical class $-K_X$ is ample. The *degree* of X is the intersection product $d := K_X^2 \in \{1, \ldots, 9\}$.

Remark 2.2. For $d \geq 3$, the class $-K_X$ is very ample, inducing an *anticanonical embedding* $X \hookrightarrow \mathbb{P}^d$. The image is a smooth cubic surface in $\mathbb{P}^3$ for $d = 3$ and a smooth intersection of quadrics in $\mathbb{P}^4$ for $d = 4$. For degree 2 and 1, the anticanonical morphism is a double cover of $\mathbb{P}^2$ branched over a quartic curve and a rational map to $\mathbb{P}^1$ defined away from one point respectively.

Remark 2.3. The anticanonical morphism when $d = 2$ and the bianticanonical morphism when $d = 1$ are both double covers: they give rise to the *Geiser and Bertini involutions* respectively, which we will use later.

The following result classifies del Pezzo surfaces over a separably closed field.

Theorem 2.4. [Poo17, Thm. 9.4.4] *Let X be a del Pezzo surface of degree d over a field k. Then X_{k_s} is either isomorphic to $\mathbb{P}^1_{k_s} \times \mathbb{P}^1_{k_s}$ (in which case $d = 8$) or X_{k_s} is isomorphic to the blowup of $\mathbb{P}^2_{k_s}$ in $9 - d$ points in general position.*

For the definition of general position, see Definition 2.20.

2.1. **Rational curves on del Pezzo surfaces.** As described in the introduction, our primary objects of study are lines and conic families on del Pezzo surfaces.

Definition 2.5. Let X/k be a del Pezzo surface. A curve $L \subset X_{k_s}$ is called a *line* (or *exceptional curve*) if $-K_X \cdot L = 1$ and $L \cdot L = -1$. A reduced curve $C \subset X_{k_s}$ is called a *conic* if $-K_X \cdot C = 2$ and $C^2 = 0$. If X contains a k-rational conic C/k, we call it a *conic del Pezzo surface.*

Remark 2.6. We observe some consequences of this definition:

  (1) Lines and conics merit their names: they have anticanonical degrees 1 and 2 respectively, thus they correspond, for degree $d \geq 3$, to such curves in $\mathbb{P}^d$ lying on an anticanonical embedding.

  (2) A conic C induces a fibration $\pi : X \to B$ of X into conics over a genus-0 curve B (see e.g. [Str21, Proof of Lem. 3.5]), hence a *conic fibration/family* on X.

  (3) A line $L \subset X_{k_s}$ is G_k-invariant if and only if it is defined over k (since it is isolated). Further, a conic $C \subset X_{k_s}$ is G_k-invariant if and only if it represents a class in $\mathrm{Pic}(X) \subset \mathrm{Pic}(\overline{X})$. To see this latter point, note that the associated morphism $X \to \mathbb{P}^1$ is G_k-stable as soon as the class of C is.

As mentioned in the introduction, it is relatively easy to determine the number of "geometric" lines and conic families on a del Pezzo surface.

Theorem 2.7. [Der06, Table 3.1] *The number of lines and conic families[2] on a degree-d del Pezzo surface over k_s is given in the following table, in which 8q denotes degree-8 del Pezzo surfaces isomorphic to quadrics and 8b denotes blowups of the plane in a point.*

TABLE 3. Geometric counts on del Pezzo surfaces

Degree	9	8q	8b	7	6	5	4	3	2	1
Number of lines	0	0	1	3	6	10	16	27	56	240
Number of conic families	0	2	1	2	3	5	10	27	126	2160

It is also possible to compute the total set of possibilities for line and conic counts.

---

[2]Note that the class of a conic is a *ruling*, as defined in [Der06, Def. 3.1].

Theorem 2.8. *The complete list of possibilities for the number of k-rational lines on a degree-d del Pezzo surface is given in the following table.*

TABLE 4. Possible line counts on del Pezzo surfaces

Degree	Possible line counts
9	0
8	0,1
7	1,3
6	0,2,6
5	0,1,2,4,10
4	0,1,2,4,8,16
3	0,1,2,3,5,7,9,15,27
2	0,2,4,6,8,12,16,20,32,56
1	0,2,4,6,8,12,14,20,24,26,30,40,60,72,126,240

Theorem 2.9. *The complete list of possibilities for the number of k-rational conic families on a degree-d del Pezzo surface is given in the following table.*

TABLE 5. Possible conic counts on del Pezzo surfaces

Degree	Possible conic family counts
9	0
8	0,2
7	0,2
6	0,1,3
5	0,1,2,3,5
4	0,2,4,6,10
3	0,1,2,3,5,7,9,15,27
2	0,2,4,6,8,12,14,24,26,60,126
1	0,2,4,6,12,18,24,30,36,72,90,252,270,756,2160

Remark 2.10. In the case $d = 3$, note that classes of lines and conic fibrations are in bijection under the map $D \mapsto -K_X - D$.

Remark 2.11. It is easy to derive these theorems from our tables in Appendix A. It is also feasible to compute the list of all possible line-conic count pairs, but it is quite large.

2.2. **Relatively minimal del Pezzo surfaces.** Determining the existence of blowup types relies on two ingredients: existence of the underlying relatively minimal surface, and existence of suitable points to blow up. We discuss the first of these here.

Definition 2.12. A surface X over a field k is *relatively minimal* if X does not contain a G_k-orbit of pairwise skew exceptional curves.

Remark 2.13. Relatively minimal surfaces include *minimal* surfaces, i.e. relatively minimal surfaces X/k such that any birational map from a surface Y/k to X is a morphism.

Iskovskikh classified relatively minimal del Pezzo surfaces over arbitrary fields.

Theorem 2.14. [Isk79, Cor., §3] *For X a degree-d del Pezzo surface over a field k:*

$d = 9$: *X is relatively minimal and $\mathrm{Pic}(X) = \langle L \rangle \cong \mathbb{Z}$ with $L^2 = 1$.*

$d = 8$: *X is relatively minimal if and only if $X \cong Q \subset \mathbb{P}^3$ for some smooth quadric Q. Either $\mathrm{Pic}(X) = \langle H \rangle \cong \mathbb{Z}$ with $H^2 = 2$ (non-split) or $\mathrm{Pic}(X) = \langle H, V \rangle \cong \mathbb{Z} \times \mathbb{Z}$ with $H^2 = V^2 = 0$ and $H \cdot V = 1$ (split).*

$d = 7$: *X is never relatively minimal.*

$d \in \{3, 5, 6\}$: *X is relatively minimal if and only if $\mathrm{Pic}(X) = \langle -K_X \rangle \cong \mathbb{Z}$.*

$d \in \{1, 2, 4\}$: *X is relatively minimal if and only if either $\mathrm{Pic}(X) = \langle -K_X \rangle \cong \mathbb{Z}$ or $\mathrm{Pic}(X) = \langle -K_X, C \rangle \cong \mathbb{Z} \times \mathbb{Z}$ for C a conic class.*

Notation 2.15. For $d \in \{3, 5, 6, 9\}$, we denote by S_d a relatively minimal degree-d del Pezzo surface. For $d \in \{1, 2, 4, 8\}$, we denote by $S_{d,I}$ (resp. $S_{d,II}$) a relatively minimal degree-d del Pezzo surface of Picard rank 1 (resp. 2).

2.2.1. *Degree at least 5.* From Theorem 2.14, it is easy to see that there exist relatively minimal del Pezzo surfaces of degrees 9 and 8 over any field: $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. Further, relatively minimal del Pezzo surfaces of degree 7 do not exist. The situation is not too much more delicate for degrees 5 and 6, as the following result of Zaitsev shows.

Theorem 2.16. [Zai23, Thm. 1.5, Thm. 1.7] *Let k be an arbitrary field. Then there exists a relatively minimal del Pezzo surface of degree 5 over k if and only if k admits a separable extension of degree 5. There exists a relatively minimal del Pezzo surface of degree 6 over k if and only if there exist separable extensions of degrees 2 and 3.*

Proof. Necessity is immediate from consideration of the corresponding Galois actions on lines. We give a sketch of the proof of sufficiency for the sake of detailing explicit constructions. For degree 5, take a separably closed point of degree 5 on a smooth conic $C \subset \mathbb{P}^2$ and blow it up to obtain a degree-4 del Pezzo surface: the strict transform of C is a line, which we may blow down to obtain the desired relatively minimal degree-5 del Pezzo surface. For degree 6, take a separably closed point of degree 2 along a line $L \subset \mathbb{P}^2$ and a separable point of degree 3 on a smooth conic $C \subset \mathbb{P}^2$; blowing up, we again obtain a degree-4 del Pezzo surface, now with two skew rational lines (the strict transform of L and the strict transform of the conic C' through all five underlying points). Blowing these lines down, we obtain a relatively minimal degree-6 del Pezzo surface. $\square$

2.2.2. *Degree at most 4.* In degree at most 4, the situation is more complex.

Over a finite field $\mathbb{F}_q$, the existence of relatively minimal del Pezzo surfaces of degree at least 2 over $\mathbb{F}_q$ for all q is confirmed by the results of Loughran–Trepalin [LT20]. In degree 1, we are able to establish the existence of all blowup types of del Pezzo surfaces over $\mathbb{F}_q$ for all q, including the two relatively minimal kinds, which appears to be novel; see Lemma 5.5.

Over a general infinite field, one has the following result.

Theorem 2.17. *For k any infinite field with $\mathrm{char}(k) \neq 2$, there exist del Pezzo surfaces of types $S_{4,I}$, $S_{4,II}$ and $S_{2,II}$ over k as soon as they are permitted by the Galois theory of k.*

Proof. This is a direct consequence of Corollary 4.14. $\hfill\square$

Remark 2.18. Note that this leaves open the existence of del Pezzo surfaces of types S_3 (relatively minimal cubic surfaces), $S_{2,I}$, $S_{1,I}$ and $S_{1,II}$.

In degree 3, we have the following result, which furnishes a relatively minimal del Pezzo surface of degree 3 for fields which are not closed under the operation of taking cube roots.

Lemma 2.19. *Let k be a field with $\mathrm{char}(k) \neq 3$. If there exists $\alpha \in k^{\times} \setminus k^{\times 3}$, then the following cubic surface in $\mathbb{P}^3$ is relatively minimal:*

$$X_0 + X_1^3 + X_2^3 + \alpha X_3^3 = 0.$$

Proof. For k perfect containing a cubic root of unity, see [Man86, Example 21.9]. In view of Coombes' result that del Pezzo surfaces are separably split [Coo88], one may drop perfectness. Further, one may drop the assumption of a cubic root of unity, as this only enlarges the Galois orbits of lines on the resulting cubic. $\hfill\square$

Over Hilbertian fields, all relatively minimal types exist, as we shall see in Section 7.

2.3. **Points in general position.** We now turn to the second ingredient in blowup types: the existence of points in general position.

2.3.1. *Defining general position.* We begin by defining general position in the plane.

Definition 2.20. We say a G_k-stable set $\{P_1, \ldots, P_n\} \subseteq \mathbb{P}^2(k_s)$ of $n \leq 8$ points are in *general position* if no 3 lie on a line, no 6 lie on a conic, and if $n = 8$, any cubic through $P_1, \ldots, P_8$ does not have a double point at P_i for some i.

It is classical that if $P_1, \ldots, P_n \in \mathbb{P}^2$ are in general position, then $\mathrm{Bl}_{P_1,\ldots,P_n} \mathbb{P}^2$ is a del Pezzo surface. We mimic this definition for other del Pezzo surfaces (as suggested in [Man86, Rem. 26.3]).

Definition 2.21. Let X be a degree-d del Pezzo surface over k and $\{P_1, \ldots, P_n\} \subseteq X(k_s)$ be a G_k-stable set of points with $n \leq d-1$. We say that $P_1, \ldots, P_n$ are in *general position* if $\mathrm{Bl}_{P_1,\ldots,P_n} X$ is a del Pezzo surface.

Using Theorem 2.4, we can derive equivalent conditions for Definition 2.21.

Lemma 2.22. *Let X be a degree-d del Pezzo surface over k. The elements of a G_k-stable set of points $\{P_1, \ldots, P_n\} \subseteq X(k_s)$ (with $n \leq d-1$) are in general position if and only if the following conditions are both satisfied:*

(i) *For all $1 \leq r \leq \min\{4, n\}$, no r of the P_i lie on a curve of self-intersection $r-2$ and arithmetic genus 0.*

(ii) *For $n = d-1$, the P_i do not all lie on an anticanonical curve with a double point at one of them.*

Proof. All curves on a del Pezzo surface have self-intersection at least -1, and self-intersection of the strict transform decreases by at least 1 when we blow up along a curve (and by at least 4 when the blowup is at a double point, with the arithmetic genus decreasing by 1 in this case), so necessity is immediate.

For sufficiency when X is not relatively minimal of degree 8, choose a collection of $9 - d$ skew k_s-lines in X_{k_s} and blow down to $\mathbb{P}^2_{k_s}$. Denote the blowdowns of the skew lines by $Q_1, \ldots, Q_{9-d}$, and identify the points $P_1, \ldots, P_n$ with their images under blowing down. Set $A := \{P_i\}_{i=1}^n \cup \{Q_j\}_{j=1}^{9-d}$. The points in $A \subset \mathbb{P}^2(k_s)$ are in general position (according to Definition 2.20) if and only if the P_i are in general position on X (according to Definition 2.21). Note that:

(1) The strict transform of a line through three of the points in A is an r-curve of arithmetic genus 0 through $r - 2$ of the P_i for some $r \leq 3$.

(2) The strict transform of a conic through six of the points in A is an r-curve of arithmetic genus 0 through $r - 2$ of the P_i for some $r \leq 4$.

(3) In the case $n = d - 1$, the strict transform of a cubic through all of the points in A with a double point at one of them is either:

  (a) A $(d-3)$-curve of arithmetic genus 0 through all $d-1$ of the P_i (if the double point is at some Q_i), or

  (b) An anticanonical curve of arithmetic genus 1 through all $d-1$ of the P_i with a double point at one of them (if the double point is at some P_i).

For sufficiency when X is relatively minimal of degree 8, we may simply blow up at one of the P_i and then argue as in the case $d = 7$. Alternatively, we may argue by using the isomorphism between the blowup of the plane in two points and the blowup of a quadric in one point (see Section 3.3). $\square$

Notation 2.23. For S a relatively minimal del Pezzo surface, denote by $\mathrm{Bl}_{d_1,\ldots,d_r} S$ the blowup of S in closed points $P_1, \ldots, P_r$ in general position with $\deg(P_i) = d_i$.

Definition 2.24. We say that a del Pezzo surface X has *blowup type* $\mathrm{Bl}_{d_1,\ldots,d_r} S$ if $X \cong \mathrm{Bl}_{d_1,\ldots,d_r} S$ for some relatively minimal del Pezzo surface S.

With the above definitions in hand, we begin to consider the question of existence of closed points in general position.

2.3.2. *General position over infinite fields.* Here we discuss closed points in general position over infinite fields; among these are the Hilbertian fields, for which we can say more (see Proposition 7.9). We postpone finite fields until Section 5.

The existence of *k-rational* points in general position is straightforward in most cases.

Lemma 2.25. *Let k be an infinite field, and let X be a del Pezzo surface over k of degree $d \geq 2$. Assume that X contains a k-rational point; if $d = 2$, further assume that the rational point does not lie on four exceptional curves or the ramification locus of X. Then there exists a $(d-1)$-tuple of rational points in general position on X.*

Proof. Such X is k-unirational (by [Man86] and [Kol02, Thm. 1.1] for $d \geq 3$, and [STVA14, Cor. 3.3] for $d = 2$). Since k is infinite, unirationality implies that $X(k)$ is

dense. It remains to show that density of rational points is sufficient for the result, which we now do by inductively building a tuple.

Let Z be the union of all lines on X and set $U := X \setminus Z$. Since Z is proper closed, there exists $P \in U(k)$. Note that P is a rational point in general position on X.

Now assume that there exist $P_1, \ldots, P_n \in X(k)$ in general position for some $n < d - 2$. By definition, $P_1, \ldots, P_n \in U_n(k)$, where $U_n = X \setminus Z_n$ for Z_n the proper closed subset of all lines on X, all conics on X passing through one of the P_i, and, for each $r \leq n$, all curves on X of self-intersection $r - 2$ and arithmetic genus 0. Let Z_{n+1} denote the union of all curves of arithmetic genus 0, self-intersection $n - 1$, and passing through the points $P_1, \ldots, P_n$. Set $U_{n+1} := U_n \cap (X \setminus Z_{n+1})$. Since Z_{n+1} is again a proper closed subset of X, we have $U_{n+1} \neq \varnothing$ and thus there exists $P_{n+1} \in U_{n+1}(k)$. Note that $P_1, \ldots, P_n \notin U_{n+1}(k)$, as these points were removed with the locus Z_{n+1}. It follows that $P_1, \ldots, P_{n+1}$ is an $(n+1)$-tuple of points in general position.

We can apply the previous inductive step until $n = d - 2$, in which case we must additionally ensure that P_{n+1} is chosen so that there is no anticanonical curve through all of the $P_1, \ldots, P_{n+1}$ with a double point at one of them. To see that this may be done, blow up at $P_1, \ldots, P_{d-2}$ to obtain a del Pezzo surface Y/k of degree 2 with dense rational points. Note that there exists a point $P_{d-1} \in U_{d-2}(k)$ and not on the ramification divisor. Moreover, note that for any P_i on the ramification divisor, it suffices to choose P_{d-1} to further avoid the unique anticanonical curve with a double point at P_i. We thus obtain a general $(d-1)$-tuple of rational points on X. $\qquad\square$

For our purposes, we will need to blow up del Pezzo surfaces in sets of not-necessarily-rational points in general position. This means that we need a more general result than Lemma 2.25. We first use combinatorial arguments to construct closed points in general position on $\mathbb{P}^2$ over arbitrary infinite fields.

Definition 2.26. Let $1 \leq n \leq 8$. Let $\lambda = (n_1, \ldots, n_8)$ denote the partition $n = \sum_{i=1}^8 i \cdot n_i$, and let $|\lambda| = \sum_{i=1}^8 n_i$ denote the length of the partition. We say that a set of closed points $M = \{P_1, \ldots, P_{|\lambda|}\} \subset \mathbb{P}^2_k$ *corresponds to the partition* λ if there exist disjoint subsets $M_1, \ldots, M_8 \subseteq M$ such that

  (i) $\bigsqcup_{i=1}^8 M_i = M$,

  (ii) $|M_i| = n_i$, and

  (iii) every element of M_i is a closed point of degree i.

Theorem 2.27. *Let k be an infinite field. Let $1 \leq n \leq 8$. Let $\lambda = (n_1, \ldots, n_8)$ be a partition of n. If k admits a separable extension of degree i for each $n_i > 0$, then there exists a set of closed points in general position in $\mathbb{P}^2_k$ corresponding to λ.*

Proof. See Appendix B. $\qquad\square$

Remark 2.28. Our proof of Theorem 2.27 holds for many finite fields as well. See Appendix B for sufficient (but not necessarily necessary) lower bounds on $|k|$.

If X is a rational del Pezzo surface over an infinite field k, then we obtain points in general position on X from Theorem 2.27. This allows us to get points in general position on all blowup types of del Pezzo surfaces of degree $d \geq 5$.

Proposition 2.29. *Let k be an infinite field. Let $5 \leq d \leq 9$ and $1 \leq n \leq d-1$. Let $\lambda = (n_1, \ldots, n_8)$ be a partition of n, and assume that k admits a separable extension of degree i for each $n_i > 0$. Every blowup type of degree-d del Pezzo surface contains a surface X with a set of closed points in general position corresponding to λ.*

Proof. Every del Pezzo surface is birational to some relatively minimal del Pezzo surface, so we may restrict our attention to relatively minimal blowup types. Recall that for $d \geq 5$, any degree-d del Pezzo surface containing a k-rational point is rational [Poo17, Thm. 9.4.8]. Such a surface then contains a set of closed points in general position corresponding to λ by Theorem 2.27. Thus, given a relatively minimal degree-d del Pezzo surface X over k, it suffices to find a relatively minimal degree-d del Pezzo surface Y over k with $Y(k) \neq \varnothing$. We will treat each degree in turn.

The case $d = 9$ follows from Theorem 2.27. The case $d = 8$ follows from the case $d = 9$ as the blowup of a quadric surface in a rational point is isomorphic to the blowup of the plane in two points (see Proposition 3.12). The case $d = 7$ is vacuous, as there are no relatively minimal degree-7 del Pezzo surfaces.

Finally, as discussed in Theorem 2.16 and its proof, whenever a relatively minimal del Pezzo surface of degree $d \in \{5, 6\}$ exists, one may be constructed via blowup of the plane. Thus the result follows easily from the case $d = 9$. $\square$

For $d \leq 4$, we can no longer rely on rationality for points in general position. We can, however, progress with assumptions on the ground field. We will address Hilbertian fields, finite fields, and local fields in their respective sections.

3. Classification by blowup type

3.1. **Intersection invariant.** As we construct new del Pezzo surfaces by blowing up, we must determine whether new blowups are isomorphic to old ones. Where coincidences occur, we will give explicit geometric proofs. It will then remain to prove that no further coincidences occur. By calculating line and conic counts and Picard ranks for certain blowups, we can sometimes distinguish them. We then narrow down to small sets of "potentially coincident" surfaces. Given such a pair of surfaces, we may use the additional structure of the intersection form on their Picard groups to distinguish them. For this, we introduce the following invariant.

Definition 3.1. Let X be a smooth projective surface such that $\mathrm{Pic}(\overline{X})$ is torsion-free and finitely generated (e.g. a del Pezzo surface) and $\mathcal{B} = \{B_1, \ldots, B_r\}$ be a basis of the Picard group of X. Denote by $G_X(\mathcal{B}) \in M_r(\mathbb{Z})$ the Gram matrix associated to $\mathcal{B}$, i.e. the matrix with ij^{th} entry the intersection product $B_i \cdot B_j$, and set $I_X = \det(G_X(\mathcal{B}))$.

Remark 3.2. In the previous definition, as the notation suggests, I_X is independent of the choice of basis $\mathcal{B}$. Indeed, given a new basis $\mathcal{C} = \{C_1, \ldots, C_r\}$ of $\mathrm{Pic}(X)$, there exist integers $a_{ij} \in \mathbb{Z}$, $1 \leq i, j \leq r$, such that $C_i = \sum_{j=1}^{r} a_{ij} B_j$ for all $i \in \{1, \ldots, r\}$ and that $A = (a_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$. Note that $C_i \cdot C_j = \mathbf{v} G_X(\mathcal{B}) \mathbf{w}^{\mathrm{T}}$, where $\mathbf{v} = (a_{i1}, \ldots, a_{ir})$ and $\mathbf{w} = (a_{j1}, \ldots, a_{jr})$. It readily follows that $G_X(\mathcal{C}) = A G_X(\mathcal{B}) A^{\mathrm{T}}$. Taking determinants, we deduce independence, since $\det(A) = \det(A^{\mathrm{T}}) = \pm 1$.

Essentially the same argument shows that I_X is an isomorphism invariant.

Lemma 3.3. *If X is a smooth projective surface with torsion-free finitely generated geometric Picard group and Y is a surface isomorphic to X, then $I_X = I_Y$.*

Proof. The isomorphism $X \cong Y$ induces an isomorphism $\varphi : \mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$ preserving intersection products. Choosing bases for each Picard group, φ is represented by a matrix $A \in \mathrm{GL}_r(\mathbb{Z})$, and one has $v \cdot w = vG_X w^{\mathrm{T}}$ (surpressing the choice of basis in $\mathrm{Pic}(X)$). By preservation of intersection multiplicities, we deduce that $G_X = AG_Y A^{\mathrm{T}}$. Since $\det(A) = \pm 1$, we deduce that $I_X = I_Y$. $\qquad\square$

Note that I_X takes the following values for X a relatively minimal del Pezzo surface:

$$I_X = \begin{cases} 1 \text{ if } d = 9, \\ 2 \text{ if } d = 8 \text{ and } \mathrm{rank}\,\mathrm{Pic}(X) = 1, \\ d \text{ if } d \leq 6 \text{ and } \mathrm{rank}\,\mathrm{Pic}(X) = 1, \\ -1 \text{ if } d = 8 \text{ and } \mathrm{rank}\,\mathrm{Pic}(X) = 2, \\ -4 \text{ if } d \in \{1, 2, 4\} \text{ and } \mathrm{rank}\,\mathrm{Pic}(X) = 2. \end{cases}$$

Note also that I_X transforms nicely under blowing up:

Lemma 3.4. *Let X be a smooth projective surface with torsion-free finitely generated geometric Picard group, and let $Y = \mathrm{Bl}_{d_1,\ldots,d_n} X$. Then $I_Y = (-1)^n I_X \prod_{i=1}^n d_i$.*

Proof. Let $\{B_1, \ldots, B_r\}$ be a basis of $\mathrm{Pic}(X)$. Note that
$$\mathrm{Pic}(Y) = \langle B_1', \ldots, B_r', E_1, \ldots, E_n \rangle \cong \mathbb{Z}^{r+n},$$
where B_i' is the pullback of B_i and the E_j are the (Galois orbits of) exceptional divisors over closed points in the blowup locus. Then the intersection matrix M_Y of $\mathrm{Pic}(Y)$ with respect to this presentation is a block diagonal matrix with blocks the intersection matrix M_X of X and $-\mathrm{diag}(d_1, \ldots, d_n)$. Then $I_Y = (-1)^n I_X \prod_{i=1}^n d_i$. $\qquad\square$

Corollary 3.5. *Let X and Y be two relatively minimal del Pezzo surfaces over a field k, and suppose that two blowups $X' = \mathrm{Bl}_{d_1,\ldots,d_r} X$, $Y' = \mathrm{Bl}_{e_1,\ldots,e_s} Y$ satisfy*
$$(\mathrm{rank}\,\mathrm{Pic}(X'), L_{X'}, C_{X'}) = (\mathrm{rank}\,\mathrm{Pic}(Y'), L_{Y'}, C_{Y'})$$
for L_X (respectively, C_X) the line (respectively, conic) count for X. The surfaces X' and Y' are not isomorphic in the case that
$$(-1)^r I_X \prod_{i=1}^r d_i \neq (-1)^s I_Y \prod_{j=1}^s e_j.$$

3.2. Counting lines and conics on blowups.
We now give formulae for counting lines and conic fibrations on blowups of del Pezzo surfaces of degrees 8 and 9.

Notation 3.6. For X/k a del Pezzo surface, we denote by L_X and C_X the number of k-rational lines and k-rational conic families on X respectively.

In principle we require counts on blowups of lower-degree del Pezzo surfaces, but we will see that these counts offer us enough information to distinguish all blowup types up to isomorphism, at which point we are able to run through a list of possible Galois actions to compute the remaining counts.

Notation 3.7. Let $(P_1, \ldots, P_r)$ be a tuple of separable closed points on a del Pezzo surface. Denote by $\mathbf{d} = (d_1, \ldots, d_r)$ the tuple of their degrees, i.e. $d_i = \deg(P_i)$.

- Let $\mathbf{d}(n)$ be the number of sub-multisets of $\{d_1, \ldots, d_r\}$ with sum n.

- Let $\mathbf{d}(m; n)$ be the number of pairs of disjoint sub-multisets of $\{d_1, \ldots, d_r\}$ with sums m and n.

In Tables 6, 7, 8, we assign numerical labels for each source of curves. When positive, these numbers coincide with the degree of the associated complete linear system. We use one or more asterisks to differentiate sources associated to the same linear system.

Lemma 3.8. *Let $\{P_1, \ldots, P_r\} \in \mathbb{P}^2(k_s)$ be $r \leq 8$ points in general position and set $X = \mathrm{Bl}_{P_1, \ldots, P_r} \mathbb{P}^2_{k_s}$. Denote by $[n \mid m_1, \ldots, m_r]$ the system $|\mathcal{O}(n) - \sum_{i=1}^{r} m_i P_i|$ of curves in $\mathcal{O}(n)$ with a singularity of order at least m_i at each P_i, indicating repeated entries via exponents, e.g. $[2 \mid 1, 1] = [2 \mid 1^2]$. The origins of the lines and conics on X_{k_s} in $\mathbb{P}^2$, together with k-rational counts, are described by Tables 6 and 7 respectively. In both cases, Geiser duals and Bertini duals are identified.*

TABLE 6. Origin of lines on the blowup of S_9

Number	System	k-rational count	Geiser	Bertini
0	(Exceptional divisors)	$\mathbf{d}(1)$	3	6
1	$[1\mid 1^2]$	$\mathbf{d}(2)$	2	5
2	$[2\mid 1^5]$	$\mathbf{d}(5)$	1	4
3	$[3\mid 2^1, 1^6]$	$\mathbf{d}(1; 6)$	0	3
4	$[4\mid 2^3, 1^5]$	$\mathbf{d}(8)\mathbf{d}(3)$	–	2
5	$[5\mid 2^6, 1^2]$	$\mathbf{d}(8)\mathbf{d}(2)$	–	1
6	$[6\mid 3^1, 2^7]$	$\mathbf{d}(8)\mathbf{d}(1)$	–	0

TABLE 7. Origin of conics on the blowup of S_9

Number	System	k-rational count	Geiser	Bertini
1	$[1\mid 1^1]$	$\mathbf{d}(1)$	5	11
2	$[2\mid 1^4]$	$\mathbf{d}(4)$	4	10
3	$[3\mid 2^1, 1^5]$	$\mathbf{d}(1; 5)$	3	9
4	$[4\mid 2^3, 1^4]$	$\mathbf{d}(3; 4)$	2	8
5	$[5\mid 2^6, 1^1]$	$\mathbf{d}(1; 6)$	1	7
7	$[7\mid 4^1, 3^1, 2^6]$	$\mathbf{d}(8)\mathbf{d}(1; 1)$	–	5
8	$[8\mid 4^1, 3^4, 2^3]$	$\mathbf{d}(8)\mathbf{d}(1; 3)$	–	4
9	$[9\mid 4^2, 3^5, 2^1]$	$\mathbf{d}(8)\mathbf{d}(1; 2)$	–	3
10	$[10\mid 4^4, 3^4]$	$\mathbf{d}(8)\mathbf{d}(4)$	–	2
11	$[11\mid 4^7, 3^1]$	$\mathbf{d}(8)\mathbf{d}(1)$	–	1
4*	$[4\mid 3^1, 1^7]$	$\mathbf{d}(8)\mathbf{d}(1)$	–	8*
5*	$[5\mid 3^1, 2^3, 1^4]$	$\mathbf{d}(8)\mathbf{d}(1; 3)$	–	7*
6*	$[6\mid 3^2, 2^4, 1^2]$	$\mathbf{d}(8)\mathbf{d}(2; 2)$	–	6*
7*	$[7\mid 3^4, 2^3, 1^1]$	$\mathbf{d}(8)\mathbf{d}(1; 3)$	–	5*
8*	$[8\mid 3^7, 1^1]$	$\mathbf{d}(8)\mathbf{d}(1)$	–	4*

Proof. We have the following equalities:

(1) For the dimension of the linear system, we have

$$\dim\left|\mathcal{O}(n) - \sum_{i=1}^{r} m_i P_i\right| = \frac{n(n+3)}{2} - \sum_{i=1}^{r} \binom{m_i + 1}{2}.$$

(2) For $C \sim \mathcal{O}(n)$ and $\widetilde{C}$ the strict transform of C on $\mathrm{Bl}_{P_1,\ldots,P_r} \mathbb{P}^2$,

$$p_a(\widetilde{C}) = \binom{n-1}{2} - \sum_{i=1}^{r} \binom{m_i}{2}, \quad \widetilde{C}^2 = n^2 - \sum_{i=1}^{r} m_i^2.$$

Then lines are given by solutions of

$$\frac{n(n+3)}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2} = \binom{n-1}{2} - \sum_{i=1}^{r} \binom{m_i}{2} = n^2 - \sum_{i=1}^{r} m_i^2 + 1 = 0,$$

while conic families are given by solutions to

$$\frac{n(n+3)}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2} - 1 = \binom{n-1}{2} - \sum_{i=1}^{r} \binom{m_i}{2} = n^2 - \sum_{i=1}^{r} m_i^2 = 0.$$

Relevant computations can be found in Derenthal's thesis [Der06, Ch. 3]. □

Corollary 3.9. *We have the following formulas for* $X = \mathrm{Bl}_{d_1,\ldots,d_r} S_9$:

$$L_X = \mathbf{d}(1) + \mathbf{d}(2) + \mathbf{d}(5) + \mathbf{d}(1;6) + \mathbf{d}(8)(\mathbf{d}(1) + \mathbf{d}(2) + \mathbf{d}(3)),$$
$$C_X = \mathbf{d}(1) + \mathbf{d}(4) + \mathbf{d}(1;5) + \mathbf{d}(1;6) + \mathbf{d}(3;4)$$
$$+ \mathbf{d}(8)(3\mathbf{d}(1) + \mathbf{d}(4) + \mathbf{d}(1;1) + \mathbf{d}(1;2) + 3\mathbf{d}(1;3) + \mathbf{d}(2;2)).$$

Lemma 3.10. *Let* S_8/k *be a degree-8 del Pezzo surface of quadric type, and set* $\delta = \mathrm{rank}\,\mathrm{Pic}(S_8) - 1$. *Let* X/k *be the blowup of* S_8 *in points* $P_1,\ldots,P_r \in S_8(k_s)$ *in general position. Let us denote by* $[a, b \mid m_1,\ldots,m_r]$ *the system* $|\mathcal{O}(a,b) - \sum_{i=1}^{r} m_i P_i|$. *The origins of the lines on* X_{k_s} *in* S_8, *together with their* k-*rational counts, are described by Table 8, and the origins of the conic families are described in Table 9. In both cases, Geiser duals and Bertini duals are identified.*

TABLE 8. Origin of lines on the blowup of S_8

Number	System	k-rational count	Geiser	Bertini
0	(Exceptional divisors)	$\mathbf{d}(1)$	4	8
1	$[1,0|1^1]$, $[0,1|1^1]$	$2\delta\mathbf{d}(1)$	3	7
2	$[1,1|1^3]$	$\mathbf{d}(3)$	2	6
3	$[2,1|1^5]$, $[1,2|1^5]$	$2\delta\mathbf{d}(5)$	1	5
4	$[2,2|2^1,1^5]$	$\mathbf{d}(1;5)$	0	4
4*	$[3,1|1^7]$, $[1,3|1^7]$	$2\delta\mathbf{d}(7)$	–	4*
5	$[3,2|2^2,1^5]$, $[2,3|2^2,1^5]$	$2\delta\mathbf{d}(7)\mathbf{d}(2)$	–	3
6	$[3,3|1^3,2^4]$	$\mathbf{d}(7)\mathbf{d}(3)$	–	2
7	$[4,3|2^6,1^1]$, $[3,4|2^6,1^1]$	$2\delta\mathbf{d}(7)\mathbf{d}(1)$	–	1
8	$[4,4|3^1,2^6]$	$\mathbf{d}(7)\mathbf{d}(1)$	–	0

TABLE 9. Origin of conics on the blowup of S_8

Number	System	k-rational count	Geiser	Bertini
1	$[1,0]$, $[0,1]$	2δ	7	15
2	$[1,1|1^2]$	$\mathbf{d}(2)$	6	14
3	$[2,1|1^4]$, $[1,2|1^4]$	$2\delta\mathbf{d}(4)$	5	13
4	$[2,2|2^1,1^4]$	$\mathbf{d}(1;4)$	4	12
5	$[3,2|2^2,1^4]$, $[2,3|2^2,1^4]$	$2\delta\mathbf{d}(2;4)$	3	11
6	$[3,3|1^2,2^4]$	$\mathbf{d}(2;4)$	2	10
7	$[4,3|2^6]$, $[3,4|2^6]$	$2\delta\mathbf{d}(6)$	1	9
8	$[4,4|3^2,2^3,1^2]$	$\mathbf{d}(7)\mathbf{d}(2;2)$	–	8
9	$[5,4|4^1,2^6]$, $[4,5|4^1,2^6]$	$2\delta\mathbf{d}(7)\mathbf{d}(1)$	–	7
10	$[5,5|4^1,3^2,2^4]$	$\mathbf{d}(7)\mathbf{d}(1;2)$	–	6
11	$[6,5|4^1,3^4,2^2]$, $[5,6|4^1,3^4.2^2]$	$2\delta\mathbf{d}(7)\mathbf{d}(2;4)$	–	5
12	$[6,6|4^1,3^4,2^1]$	$\mathbf{d}(7)\mathbf{d}(1;2)$	–	4
13	$[2,1|1^4]$, $[1,2|1^4]$	$2\delta\mathbf{d}(4)$	–	3
14	$[7,7|4^5,3^2]$	$\mathbf{d}(7)\mathbf{d}(2)$	–	2
15	$[8,7|4^7]$, $[7,8|4^7]$	$2\delta\mathbf{d}(7)$	–	1
6*	$[3,3|3^1,2^1,1^5]$	$\mathbf{d}(7)\mathbf{d}(1;1)$	–	10*
6**	$[4,2|2^3,1^4]$, $[2,4|2^3,1^4]$	$2\delta\mathbf{d}(7)\mathbf{d}(3)$	–	10**
7*	$[4,3|3^1,2^3,1^3]$, $[3,4|3^1,2^3,1^3]$	$2\delta\mathbf{d}(7)\mathbf{d}(1;3)$	–	9*
8*	$[5,3|3^1,2^5,1^1]$, $[3,5|3^1,2^5,1^1]$	$\mathbf{d}(7)\mathbf{d}(1;1)$	–	8*
9*	$[5,4|3^3,2^3,1^1]$, $[4,5|3^3,2^3,1^1]$	$2\delta\mathbf{d}(7)\mathbf{d}(1;3)$	–	9*
10*	$[5,5|3^5,2^1,1^1]$	$\mathbf{d}(7)\mathbf{d}(1;1)$	–	6*
10**	$[6,4|3^4,2^3]$, $[4,6|3^4,2^3]$	$2\delta\mathbf{d}(7)\mathbf{d}(3)$	–	6**

Proof. We replicate our proof when blowing up from the plane. For a quadric Q over a separably closed ground field, we have $\mathrm{Pic}(Q) = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathcal{O}(1,0), \mathcal{O}(0,1) \rangle \cong \mathbb{Z} \times \mathbb{Z}$. We have the following formulas:

$$D := \dim \left| \mathcal{O}(a,b) - \sum_{i=1}^{r} m_i P_i \right| = ab + a + b - \sum_{i=1}^{r} \binom{m_i + 1}{2},$$

$$I := \widetilde{C}^2 = 2ab - \sum_{i=1}^{r} m_i^2,$$

where $\widetilde{C}$ is the strict transform of a curve in $|\mathcal{O}(a,b) - \sum_{i=1}^{r} m_i P_i|$. Solving for $D = 0$ and $I = -1$ gives the lines, while solving for $D = 1$ and $I = 0$ gives the conics. $\square$

Corollary 3.11. *We have the following formulas for $X = \mathrm{Bl}_{d_1,\dots,d_r} S_8$:*

$$\begin{aligned}
L_X &= \mathbf{d}(1) + \mathbf{d}(3) + \mathbf{d}(1;5) + 2\delta(\mathbf{d}(1) + \mathbf{d}(5)) \\
&\quad + \mathbf{d}(7)(\mathbf{d}(1) + \mathbf{d}(3) + 2\delta(1 + \mathbf{d}(1) + \mathbf{d}(2))), \\
C_X &= \mathbf{d}(2) + \mathbf{d}(1;4) + \mathbf{d}(2;4) + 2\delta(1 + \mathbf{d}(4) + 2\mathbf{d}(6) + \mathbf{d}(2;4)) \\
&\quad + \mathbf{d}(7)(\mathbf{d}(2) + 2\mathbf{d}(1;1) + 2\mathbf{d}(1;2) + \mathbf{d}(2;2)) \\
&\quad + 2\delta\mathbf{d}(7)(1 + 2\mathbf{d}(1) + 3\mathbf{d}(3) + \mathbf{d}(1;1) + \mathbf{d}(1;2) + 2\mathbf{d}(1;3)).
\end{aligned}$$

3.3. **Coincidence of blowups.** In this section, we note the following isomorphisms of blowups between blowups of relatively minimal del Pezzo surfaces.

Proposition 3.12. *We have the following coincidences of blowups:*

(1) $\mathrm{Bl}_{2n-1} S_{8,II} \cong \mathrm{Bl}_{2n-1,1} S_9$ *for* $n \in \{1, 2, 3, 4\}$.

(2) $\mathrm{Bl}_1 S_{8,I} \cong \mathrm{Bl}_2 S_9$.

(3) $\mathrm{Bl}_1 S_6 \cong \mathrm{Bl}_3 S_{8,I}$.

(4) $\mathrm{Bl}_1 S_5 \cong \mathrm{Bl}_5 S_9$.

(5) $\mathrm{Bl}_2 S_5 \cong \mathrm{Bl}_5 S_{8,I}$.

Proof. We deal with each case in turn.

(1) Blowing up the projective plane in a rational point P and a closed point Q of degree $2n - 1$ with underlying geometric points $Q_1, \ldots, Q_{2n-1}$ such that the set of points $\{P, Q_1, \ldots, Q_{2n-1}\}$ is in general position, note that the strict transforms of the lines joining P and each Q_i form an orbit of $2n - 1$ pairwise skew lines; contracting these, one obtains $S_{8,II}$. The converse direction follows from the description of the origins of lines given in Lemma 3.10 readily imply the converse.

(2) The proof is essentially the same as the previous case: blowing up a conjugate pair of points in the plane, the strict transform of the line between them is an exceptional curve which may be blown down; the two exceptional divisors of this blowup map down to the (conjugate) fibres of the geometric rulings through the point contracted onto.

(3) Blowing up $S_{8,I}$ in a general-position cubic point, the strict transform of the unique plane section containing this point is an exceptional curve which one may contract to obtain S_6; the six lines on the blowdown correspond to the three pairs of conic fibres through the blowup points, and so the converse and relative minimality are easily seen.

(4) As discussed in the proof of Theorem 2.16, blowing up the plane in a general-position quintic point, the strict transform of the unique conic containing this point is a line; contracting it, we obtain S_5.

(5) Suppose that we have a general-position quintic point on $S_{8,I}$. It follows from Lemma 3.10 that the strict transforms of the unique curves with classes $\mathcal{O}(2, 1)$ and $\mathcal{O}(1, 2)$ through the blowup locus on the base-change to k_s form a conjugate pair of lines; blowing them down, we obtain S_5 with a general-position quadratic point, and the converse is easily seen.   □

We also have a result of Manin and Iskovskikh, which implies that there are no blowup coincidences involving relatively minimal del Pezzo surfaces of degree at most 4.

Proposition 3.13. *Let k be a field. Consider relatively minimal del Pezzo surfaces X/k of degree $1, 2, 3$, and also 4 under the condition that $X(k)$ is empty. If two such surfaces of the same degree are birational, then they are isomorphic.*

Proof. If the degree is $1, 2$ or 3, this follows from [Man67, Thm. 5.8]. The remaining case follows from [Isk70, Thm. 3.3]). See also the introduction of [Sko86]. Perfectness in all cases is unnecessary due to [Coo88].   □

3.4. **Algorithm 1.** We now detail an algorithm for classifying del Pezzo surfaces up to blowup type.

Algorithm 1: Classifying del Pezzo surfaces up to blowup type

---

Input: A degree $d \in \{1, \ldots, 9\}$

Output: Tables T in Appendix A.1

(1) Create an empty list C, an empty multiset S and an empty table T with columns labelled "Type", "(L_X, C_X)","I_X","G_X" and "$\#C$".

(2) Add to C each conjugacy class of G_d with a representative.

(3) For each blowup type of a del Pezzo surface of degree d, add to S the triple consisting of the blowup type and its intersection invariant and Picard rank.

(4) Sort the multiset S into sub-multisets consisting of types S' with equal intersection invariant and Picard rank.

(5) For each sub-multiset S' in S:
  (a) If S' has size 1, then remove S' from S and add to T a corresponding row for the blowup type.
  (b) Otherwise, pass to the next sub-multiset.

(6) For each remaining sub-multiset S' in S:
  (a) If all elements in S' are blow-ups of a del Pezzo surface of degree 8 or 9, compute line and conic counts; use these to distinguish and use the coincidence result (Proposition 3.12) to check for isomorphisms. Add corresponding rows to T with line and conic counts and remove the sub-multiset.
  (b) Otherwise, distinguish by comparing possible conic counts. Remove S' and add corresponding rows to T.

(7) For each class c in C, compute the intersection invariant and Picard rank.
  (a) If this pair corresponds to only one row, add 1 to that row's $\#C$ entry and add teh representative to the G_X column. If that row does not have line and conic counts, compute them via Galois action.
  (b) If this pair corresponds to multiple rows, compute the line and conic count to determine the correct row, and add the line and conic counts, and 1 to the $\#C$ entry, in that row and the representative to the G_X entry.

Remark 3.14. We make some observations on Algorithm 1.

(1) Note that Algorithm 1 can easily be adjusted to store all subgroup conjugacy classes for each blowup type, not just the number of them; however, this necessitates notation for these classes and is infeasible to represent for low degree, when the number of conjugacy classes is incredibly large.

(2) That the above algorithm runs successfully, in particular at Step (6)(b), is not obvious, but it amounts to distinguishing the two types $X = \mathrm{Bl}_{6,2} S_9$ and $Y = \mathrm{Bl}_3 S_{4,II}$: by Corollary 3.9, we have $C_X = 0$, while $C_Y > 0$ as Y is the blowup of a surface with a rational conic family.

(3) We deduce that line and conic count is determined by blowup type. This can be seen more directly via an adaptation of the arguments of Section 3.2.

4. INVERSE PROBLEMS

We now discuss the inverse problems. Recall the following Galois representation.

Definition 4.1. For X a smooth projective surface over a field k, we denote by $\rho_X : G_k \to \mathrm{Aut}(\mathrm{Pic}(\overline{X}))$ the associated Galois representation on the geometric Picard group. The image $H_X := \rho_X(G_k)$ of ρ_X is called the *splitting group* of X.

For X/k a del Pezzo surface, H_X is contained in the following subgroup G_d.

Lemma 4.2. *For $1 \le d \le 9$ and X/k a degree-d del Pezzo surface, let $G_d \le \mathrm{Aut}(\mathrm{Pic}(\overline{X}))$ denote the subgroup of automorphisms fixing the canonical class $K_{\overline{X}}$ and preserving intersection products. We then have*

$$G_d \cong \begin{cases} 0, & d = 9, \\ C_2, & d = 8, \\ W(R_{9-d}), & d \le 7, \end{cases}$$

where $W(R_{9-d})$ is the Weyl group of a root system R_{9-d} as in Table 10.

Proof. For $d \ge 8$, the result is trivial; for $d \le 7$, see [Man86, Thm. 23.9(ii)]. $\qquad\square$

TABLE 10. Root systems and their Weyl groups

d	7	6	5	4	3	2	1
R_{9-d}	A_1	$A_2 + A_1$	A_4	D_5	E_6	E_7	E_8
$W(R_{9-d})$	C_2	$\mathfrak{D}_6$	$\mathfrak{S}_5$	$C_2^4 \rtimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$

4.0.1. *Galois action on blowups.* Let X be a degree-d del Pezzo surface over k. By the fundamental theorem of Galois theory, $H_X \cong \mathrm{Gal}(\ell_X/k)$ for some field extension ℓ_X/k.

Now suppose that $Z = \{P_1, \ldots, P_n\} \subseteq X(k_s)$ is a G_k-stable set of points in general position, so that $Y := \mathrm{Bl}_Z X$ is a del Pezzo surface over k; form ρ_Y and ℓ_Y/k as before. In addition to the homomorphisms ρ_X and ρ_Y, we have another homomorphism

$$\rho_Z : G_k \to \mathfrak{S}_n$$

via the action of G_k on $P_1, \ldots, P_n$. Again, $\rho_Z(G_k) \cong \mathrm{Gal}(\ell_Z/k)$ for some extension ℓ_Z/k.

Proposition 4.3. *In the setting above, we have isomorphisms*

$$\mathrm{Gal}(\ell_Y/k) \cong \mathrm{Gal}(\ell_X\ell_Z/k) \cong \mathrm{Gal}(\ell_X/k) \times_{\mathrm{Gal}(\ell_X \cap \ell_Z/k)} \mathrm{Gal}(\ell_Z/k),$$

where $\ell_X\ell_Z$ denotes the compositum of the fields ℓ_X and ℓ_Z as subfields of k_s.

Proof. The second isomorphism is a standard result in Galois theory, so it suffices to address the first. Let E_i be the exceptional divisor associated to P_i. Then

$$\mathrm{Pic}\,\overline{Y} \cong \mathrm{Pic}\,\overline{X} \oplus \langle E_1, \ldots, E_n \rangle.$$

Further, for the canonical inclusions $\iota_X : \mathrm{Aut}(\mathrm{Pic}\,\overline{X}) \hookrightarrow \mathrm{Aut}(\mathrm{Pic}\,\overline{Y})$ and $\iota_Z : \mathfrak{S}_n \hookrightarrow \mathrm{Aut}(\mathrm{Pic}\,\overline{Y})$, the homomorphism ρ_Y factors as

$$\rho_Y : G_k \xrightarrow{(\rho_X, \rho_Z)} \mathrm{Aut}(\mathrm{Pic}\,\overline{X}) \oplus \mathfrak{S}_n \xrightarrow{\iota_X \iota_Z} \mathrm{Aut}(\mathrm{Pic}\,\overline{Y}).$$

Note that the product $\iota_X \iota_Z$ is injective, since the images of ι_X and ι_Z act on disjoint subsets of a basis of $\mathrm{Pic}\,\overline{Y}$. Together, this implies that

$$\ker(\rho_Y) = \ker(\rho_X) \cap \ker(\rho_Z) \cong \mathrm{Gal}(k_s/\ell_X) \cap \mathrm{Gal}(k_s/\ell_Z) \cong \mathrm{Gal}(k_s/\ell_X\ell_Z),$$

and it follows that $\mathrm{im}(\rho_Y) \cong \mathrm{Gal}(\ell_X\ell_Z/k)$. $\qquad\square$

4.1. **Inverse Galois problem for del Pezzo surfaces.** We henceforth identify G_d with $W(R_{9-d})$ for $d \leq 7$. The isomorphism of Lemma 4.2 is well-defined up to conjugation. Consequently, we obtain a map

$$\gamma_d(k) : \mathcal{S}_d(k) \to C(G_d, k), \quad X \mapsto [H_X],$$

where $\mathcal{S}_d(k)$ denotes the set of isomorphism classes of degree-d del Pezzo surfaces over k and $C(G_d, k)$ denotes the subset of the set of conjugacy classes of subgroups $C(G_d)$ of G_d represented by Galois groups over k. Given a class $[H] \in C(G_d, k)$, we say that $X/k \in \mathcal{S}_d(k)$ *realises* $[H]$ if $[H_X] = [H]$.

Recall that IGP$_d(k)$ asks whether the map $\gamma_d(k)$ is surjective.

Remark 4.4. While conjugate subgroups of G_d are isomorphic, the converse does not hold in general, and so multiple conjugacy classes may be represented by the same group.

4.2. **Inverse curve problems.** Since the Galois action sends lines to lines, knowledge of the Galois action determines the line count on a del Pezzo surface. There is a partial converse: given full knowledge of the Galois action on lines (i.e. not just the number of size-1 orbits, but all orbits), we can recover the Galois action, in particular the Galois action on higher-degree rational curves, from the following result.

Lemma 4.5. [Man86, Prop. 25.1] *Let X/k be a del Pezzo surface of degree d. If $d = 8$, assume that $X_{k_s} \not\cong \mathbb{P}^1_{k_s} \times \mathbb{P}^1_{k_s}$. Then there exists a basis $\{L_0, \ldots, L_{9-d}\}$ of $\operatorname{Pic} \overline{X}$, with $L_1, \ldots, L_{9-d}$ exceptional curves and $-K_X = 3L_0 - \sum_{i=1}^{9-d} L_i$.*

An explicit instance of this determination is given in the following proposition.

Proposition 4.6. [FLS18, Prop. 5.2] *Let X/k be a del Pezzo surface of degree $d \leq 7$. Then X admits a k-rational conic fibration if and only if there exist two (possibly identical) non-empty Galois orbits of lines $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfying $(\mathfrak{L}_1 + \mathfrak{L}_2)^2 = 0$.*

Remark 4.7. Our definition of conic fibration differs from the one in [FLS18] in that we do not require the base curve of the associated projective morphism to contain a rational point. This is why [FLS18, Prop. 5.2] gives an equivalence in our setting but only a one-way implication in their setting.

In particular, for each $e \in \mathbb{Z}_{\geq 1}$, we have a well-defined map

$$\delta_{d,e}(k) : C(G_d, k) \to \mathcal{C}_{d,e}(k) \subset \mathbb{Z}_{\geq 0}$$

which associates to a class $[H] \in C_d(G, k)$ the number of families of degree-e rational curves on $X \in \mathcal{S}_d(k)$ with $[H_X] = [H]$ when such X exists. That $\mathcal{C}_{d,e}(k)$ is a finite set follows from the fact that $\operatorname{Pic}(\overline{X})$ is finitely generated and $-K_X$ is ample.

Recall that ICP$_{d,e}(k)$ asks whether $\delta_{d,e}(k)$ is surjective. In view of Lemma 4.5, a surjective solution of IGP$_d(k)$ over k implies a surjective solution of ICP$_{d,e}$ for all $e \geq 1$.

4.3. **Previous results.** We now review what is known about these inverse problems.

4.3.1. *Degree at least 5.* In high degree, the inverse Galois problem is settled.

Theorem 4.8. *For $d \geq 5$ and any field k,* $\mathrm{IGP}_d(k)$ *admits a surjective solution.*

Proof. Let X be a degree-d del Pezzo surface over k. We separate into degrees.

$d = 9$: Here, $\mathrm{Pic}(\overline{X}) \cong \mathbb{Z}$ has trivial G_k-action. This is always realised by $\mathbb{P}^2$.

$d = 8$: First, assume that $\overline{X} \cong \mathrm{Bl}_1 S_9$. Then $\mathrm{Pic}(\overline{X})$ has trivial action. This is realised by $\mathrm{Bl}_1 \mathbb{P}^2$. Now assume that $\overline{X} \cong \mathbb{P}^1_{k_s} \times \mathbb{P}^1_{k_s}$. Then $\mathrm{Pic}(\overline{X})$ has Galois action by C_2 if and only if X is non-split and trivial action otherwise. The split quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ exists over any field. The existence of a non-split quadric is equivalent to the existence of a degree-2 separable extension L/k; given such an extension, a non-split quadric is realised by the Weil restriction $R_{L/k}(\mathbb{P}^1_L)$.

$d = 7$: The possible Galois actions are trivial and C_2: the trivial action is realised by $\mathrm{Bl}_{1,1} \mathbb{P}^2$, while the C_2-action requires a degree-2 separable extension L/k and is realised by $\mathrm{Bl}_2 \mathbb{P}^2$, where the blown-up degree-2 point is defined over L.

For the cases $d = 5, 6$, see the proof of Theorem 2.16, which reduces them to $d = 9$.   □

4.3.2. *Degree 4.* The next result is due to Kunyavskiĭ, Skorobogatov and Tsfasman and was later proved independently by Elsenhans and Jahnel [EJ19b, Cor. 7.2].

Theorem 4.9. [KST89, Cor. 6.10] $\mathrm{IGP}_4(k)$ *has a surjective solution for any infinite field k with* $\mathrm{char}(k) \neq 2$.

The proof goes via a related result on conic bundles (Theorem 4.22).

Remark 4.10. A possible alternative strategy to prove Theorem 4.9 for all infinite fields is to adapt a description of degree-4 del Pezzo surfaces over fields of characteristic zero developed by Flynn [Fly09], later explained by Skorobogatov [Sko10] and employed recently by Morgan and Skorobogatov [MS24], which says that any del Pezzo surface of degree 4 is the Galois twist of one with a line, which is then rational.

4.3.3. *Degree 3.* For degree 3, we have the following results of Elsenhans and Jahnel.

Theorem 4.11. [EJ15, Thm. 0.1] $\mathrm{IGP}_3(\mathbb{Q})$ *has a surjective solution.*

The proof of Theorem 4.11 is specific to $\mathbb{Q}$ since it involves a $\mathbb{Q}$-rational point search on certain moduli spaces.

Theorem 4.12. [EJ19b, Thms. 6.1 & 7.1] *Let k be an infinite field with $\mathrm{char}\, k \neq 2$. Denote the (maximal) subgroups of $W(E_6)$ of indices 36 and 27 by U_{ds} and U_1, respectively. Then for any class $[H]$ lying in $U_{ds} \cap C(W(E_6), k)$ or $U_1 \cap C(W(E_6), k)$, there exists a degree-3 del Pezzo surface X/k with $[H_X] = [H]$.*

The subscripts ds and 1 indicate that surfaces with splitting group in these classes are exactly those containing a Galois-invariant double-six and rational line respectively.

4.3.4. *Degree 2.* In [EJ19a, EJ19b], Elsenhans and Jahnel described a construction of plane quartics with prescribed Galois action on the 28 bitangents in certain cases. Using the relationship between plane quartics and del Pezzo surfaces of degree 2 (see e.g. [Kol96, Thm. III.3.5.2]), they established partial results for the inverse Galois problem for degree-2 del Pezzo surfaces, as summarised in the following theorem.

Theorem 4.13. [EJ19a, Thm. 3.2]*, [EJ19b, Cor. 5.3]*, *Let k be an infinite field with* char $k \neq 2$. *Consider the quotient map*

$$\pi \colon W(E_7) \twoheadrightarrow W(E_7)/Z(W(E_7)) \cong \mathrm{Sp}_6(\mathbb{F}_2),$$

and denote the (maximal) subgroups of $\mathrm{Sp}_6(\mathbb{F}_2)$ *of indices 63 and 36 by* U_{63} *and* U_{36}, *respectively. Then for any class* $[H]$ *in* $\pi^{-1}(U_{63}) \cap C(W(E_7), k)$ *or* $\pi^{-1}(U_{36}) \cap C(W(E_7), k)$, *there exists a degree-2 del Pezzo surface* X/k *with* $[H_X] = [H]$.

We obtain the following important corollary.

Corollary 4.14. *For k an infinite field with* $\mathrm{char}(k) \neq 2$, *the inverse Galois problem for conic degree-2 del Pezzo surfaces admits a surjective solution.*

Proof. Theorem 4.13 solves the inverse Galois problem for degree-2 del Pezzo surfaces whose branch curve has a Galois-invariant *Steiner hexad* among its 28 bitangents; according to [EJ19b, Prop. 2.3(d)] and [EJ19b, Lem. 2.11], any Galois action resulting in a k-rational conic bundle implies a Steiner hexad for the branch curve.   □

Related work of Kulkarni and Vemulapalli [KV24] leads to the following result.

Theorem 4.15. [KV24, Thm. 3.1.1] *Let k be an infinite field with* char $k \notin \{2, 3\}$. *If k admits degree-d Galois extensions for $2 \leq d \leq 8$, then* $\mathrm{ICP}_{2,1}(k)$ *has a surjective solution.*

4.3.5. *Degree 1.* Kulkarni [Kul21] provided a partial solution to the inverse Galois problem for del Pezzo surfaces of degree one over $\mathbb{Q}$.

Theorem 4.16. [Kul21, Thm. 1.0.1] *For all* $[H] \in C(W(D_8), \mathbb{Q}) \subset C(W(E_8), \mathbb{Q})$, *there exists a degree-1 del Pezzo surface* $X/\mathbb{Q}$ *with* $[H_X] = [H]$.

In particular, we deduce the following result.

Corollary 4.17. *Over $\mathbb{Q}$, the inverse Galois problem for conic del Pezzo surfaces has a surjective solution.*

Proof. For X/k a del Pezzo surface of degree 1 with a conic bundle, we have $H_X \leq W(D_7) \leq W(D_8) \leq W(E_8)$ by Lemma 4.21.   □

Lemma 4.18. *Theorem 4.16 extends to any infinite field k with* $\mathrm{char}(k) = 0$.

Proof. Following the arguments in the proof of [Kul21, Thm. 1.0.1], it is easily seen that the properties of the ground field k are invoked in precisely the following ways:

(1) Construction 2.1.1 makes use of the fact that $\mathrm{char}(k) \neq 2$ in order to characterise quadratic extensions as radical.

(2) Lemma 2.2.1 uses that k is infinite in order to ensure Zariski-density of rational points in affine space.

(3) Lemma 3.1.3 is the point at which k must be assumed infinite with $\mathrm{char}(k) = 0$: Construction 3.1.1 gives a procedure for generating genus-4 curves over an affine base, and the locus of smooth curves is open, but one must show that it is non-empty. Example 3.1.2 does this by lifting a smooth curve from $\mathbb{F}_{101}$.

(4) The remainder of the proof goes through for k infinite with $\mathrm{char}(k) \neq 2$. $\qquad\square$

Corollary 4.19. *Over any infinite field k with $\mathrm{char}(k) = 0$, the inverse Galois problem for conic del Pezzo surfaces has a surjective solution.*

Remark 4.20. Corollary 4.19 can be extended to infinite fields k with $\mathrm{char}(k) \notin \{2, 3\}$ if one can show that [Kul21, Construction 3.1.1] yields at least one smooth curve.

4.3.6. *Conic bundles.* We briefly discuss a related inverse problem solved by Kunyavskiĭ, Skorobogatov and Tsfasman mentioned above.

Lemma 4.21. [KST89, Cor. 2.13] *Let $\phi : Y \to C$ be a (standard) conic bundle with $n \leq 7$ singular fibres. Then the subgroup of $\mathrm{Aut}(\mathrm{Pic}(\overline{Y}))$ fixing K_Y and the class of fibres of ϕ is isomorphic to $W(D_n)$.*

One may then define the *splitting group* of Y to be the image of G_k in $W(D_n)$.

Theorem 4.22. [KST89, Thm. 6.3] *The inverse Galois problem for conic bundles with $n \leq 7$ singular fibres over any infinite field k with $\mathrm{char}\, k \neq 2$ has a surjective solution: given any subgroup $G \leq W(D_n)$, there exists a conic bundle $Y \to C$ with splitting group conjugate to G if and only if G is a Galois group over k.*

Remark 4.23. It is not automatic that the conic bundles with prescribed Galois action obtained from Theorem 4.22 are del Pezzo surfaces. Indeed, the proof of *loc. cit.* notes that, in degree 4, the obtained surface may be an Iskovskikh surface, but that one can make an additional transformation to obtain a del Pezzo surface.

Despite the previous remark, the inverse Galois problem for conic del Pezzo surfaces of degree $d \geq$ over k infinite with $\mathrm{char}(k) \neq 2$ has a surjective solution (Corollary 4.14).

4.3.7. *Finite fields.* We now summarise previous work over finite fields.

Remark 4.24. Over finite fields, the inverse Galois problem for del Pezzo surfaces in the literature takes on a superficially different form: since $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is generated by the Frobenius automorphism, $\rho_X(G_{\mathbb{F}_q})$ from Definition 4.1 is determined by the image of Frobenius, which is well-defined in the associated Weyl group up to conjugacy. Thus, the inverse Galois problem for del Pezzo surfaces over finite fields is traditionally nominally concerned with which conjugacy classes of *elements* can be obtained as the image of Frobenius. However, since conjugacy classes of elements are in bijection with conjugacy classes of cyclic subgroups, this is equivalent to the version of the problem which we state.

Work of Trepalin [Tre17, Tre20] and Loughran–Trepalin [LT20] gives the next theorem.

Theorem 4.25. $\text{IGP}_d(\mathbb{F}_q)$ *admits a surjective solution for all finite fields* $\mathbb{F}_q$ *and all* $d \geq 2$.

We do not state specific results here, which necessitates a welter of notation for conjugacy classes; rather, we direct the interested reader to consult these papers or the tables in Section A.2 for finite fields. We will say more about the methods behind these results in Section 5. A generalisation of these results to degree 1, as stressed in [Tre20, §7], is a great challenge.

4.3.8. *Real closed fields.* Work of Russo [Rus02] solves $\text{IGP}_d(\mathbb{R})$.

Theorem 4.26. *For all degrees* d, $\text{IGP}_d(\mathbb{R})$ *admits a surjective solution.*

Relevant tables are found in [Rus02]: see, in *loc. cit.*, Proposition 1.2 for $d = 8$ relatively minimal, Corollary 2.4 for $d \geq 5$ except the previous case, Corollary 3.2 for $d = 4$, Corollary 3.3 for $d = 2$ and Corollary 5.3 for $d = 1$. While Russo does not directly show existence of all listed surfaces, it follows from his descriptions of their twists (see the next remark).

Remark 4.27. One may also consider *real closed fields*, i.e. fields k with $1 \neq G_k$ finite; by the Artin–Schreier theorem [Lan02, Cor. 9.3], we then have $G_k \cong C_2$. Examples aside from $\mathbb{R}$ include the field of definable numbers and the Levi–Civita field. It is easily seen that Russo's result generalises into the real closed setting. Indeed, by combining Theorem 4.8 and Theorem 4.9, it is immediate for $d \geq 4$. Further, since $G_k \cong C_2$ and a degree-3 del Pezzo surface has 27 geometric lines, at least one is G_k-stable and may be blown down, so $\text{IGP}_3(k)$ reduces to $\text{IGP}_4(k)$. For $d \leq 2$, Russo observes (see [Rus02, Cors. 4.3, 4.4, 5.3]) that any surface with no rational lines has a quadratic twist with a rational line, which ultimately gives the same reduction and relies only on the formal properties of such twists (see Section 5.3 for these twists in the finite field setting).

4.4. **Algorithm 2.** We now detail our approach to the inverse curve problems. Specifically, we explain how to handle each individual possible count.

Algorithm 2: Solving the inverse curve problems

Input: Integers $d \in \{1, \ldots, 9\}$ and $e \in \{1, 2\}$ and an integer $n \in \mathcal{C}_{d,e}$.
Output: Either a surjective solution for the count n or a list of "ugly" groups.
   (1) Create an empty list L and empty lists G, B and U of "good", "bad" and "ugly" groups.
   (2) By consulting the degree-d table, add all types T associated to the count n to L.
   (3) For each blowup type T in L, find representatives for all of its associated conjugacy classes:
      (a) If T is associated to a relatively minimal surface of degree 5 or higher, add these representatives to the list G of good groups.
      (b) Otherwise, add them to B.
   (4) If B is empty, then declare that the subproblem admits a surjective solution for n. Otherwise, continue.
   (5) For each $H \in B$, compute all quotient groups of H:
      (a) If a quotient of H lies in G, then remove H from B.
      (b) Otherwise, remove H from B and add H to U.
   (6) If U is empty, then declare that the subproblem admits a surjective solution for n. Otherwise, output U.

At the termination of Algorithm 2, we have either reached a surjective solution for the count n or else U contains groups corresponding to the count n which are associated to blowing up a relatively minimal surface of degree at most 4 and do not imply the existence of a blowup from degree at least 5 achieving the same count.

We may then employ additional strategies. For example, if char $k \neq 2$ and $d \geq 2$ and each group in U is "potentially conic" (in the sense that the associated surfaces either possess conic fibrations or do after blowing up), then we can attempt to apply Corollary 4.14; if char $k = 2$, then we can attempt an explicit Artin–Schreier version of their construction. We can similarly attempt to employ any of the results in Section 4.3 under additional hypotheses on the field k.

5. Finite fields

In this section we explain the construction of more detailed versions of our tables for the finite field case and prove Theorem 1.2 and Theorem 1.3.

5.1. **Frobenius.** Any finite extension $k/\mathbb{F}_q$ satisfies $k \cong \mathbb{F}_{q^n}$ for some $n \in \mathbb{Z}_{\geq 1}$, and we have $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \langle \mathrm{Frob}_q \rangle \cong C_n$, where Frob_q is the *Frobenius automorphism* which is defined by $\mathrm{Frob}_q(x) = x^q$. In fact, Frob_q is a topological generator of $G_{\mathbb{F}_q}$.

Given a del Pezzo surface $X/\mathbb{F}_q$, the *trace of Frobenius* $a(X)$ on X is the trace of Frob_q acting on $\mathrm{Pic}(\overline{X}) \cong \mathbb{Z}^{10-d}$. One has

$$\#X(\mathbb{F}_q) = q^2 + a(X)q + 1$$

(see, e.g. [Man67, Thm. 27.1]). It is also relatively easy to see that, for a relatively minimal conic bundle $X \to \mathbb{P}^1_{\mathbb{F}_q}$ with r singular fibres, we have

$$a(X) = 2 - r.$$

Further, by an appeal to class field theory, it can be shown that such a conic bundle has singular fibres over an even number of closed points (see [BFL19, Lem. 2.6]).

One may capture more data via the *Frame symbol* [Fra51, Eq. 3.2] (see also [Ura96, §2]).

Definition 5.1. Let $X/\mathbb{F}_q$ be a del Pezzo surface. The *Frame symbol* associated to X is a symbol $\prod_{n \geq 1} n^{a_n}$, where $\psi(t) = \prod_{n \geq 1} (t^n - 1)^{a_n} \in \mathbb{F}_q[t]$ is the characteristic polynomial of Frob_q acting on $\mathrm{Pic}(\overline{X})$. The exponent a_1 of 1 in the Frame symbol for X is thus $a(X)$.

5.2. **Cyclic conjugacy classes.** Since $G_{\mathbb{F}_q}$ acts on $\mathrm{Pic}(\overline{X})$ via a cyclic subgroup of the group G_d, there are drastically fewer conjugacy classes to consider when it comes to the inverse Galois problem. The number $\mathfrak{C}_d$ of cyclic conjugacy classes (equivalently, conjugacy classes of elements) is given in the following table.

TABLE 11. Number of cyclic conjugacy classes in G_d

d	9	8	7	6	5	4	3	2	1
$\mathfrak{C}_d$	1	2	2	6	7	18	25	60	112

These relatively manageable numbers make it feasible to produce more detailed versions of our tables with a row for each conjugacy class and more attendant data.

We are aided by the long-standing interest in the conjugacy classes of Weyl groups, going back to work of Frame on $W(E_n)$ for $n \in \{6, 7, 8\}$ [Fra51], followed by Carter [Car72]. In [Man86, Table 1], Manin, building on earlier work of Swinnerton-Dyer [SD67], produced a

version for $W(E_6)$ with data pertinent to the arithmetic of the associated cubic surfaces such as the trace of Frobenius. Urabe [Ura96] produced analogous tables for $W(E_7)$ and $W(E_8)$. A revised version of Manin's table was constructed by Banwait, Fité and Loughran [BFL19, Table 7.1].

By calculation of invariants, we identify each row in Urabe's tables with a blowup type and so produce our tables, which we use to prove Theorem 1.2 and Theorem 1.3. Our proofs will occasionally make use of the *orbit type*, meaning the number of Galois orbits of lines partitioned by configurations, as given in Urabe's tables.

Remark 5.2. During the construction of our tables, we noticed an error in Urabe's table for $W(E_7)$: the first row is not in fact of index 0, i.e. it does not correspond to a relatively minimal del Pezzo surface of degree 2. It instead corresponds to $\mathrm{Bl}_{2^3} S_{8,I}$, which has index 6.

5.3. **Geiser and Bertini twists.** Let $X/\mathbb{F}_q$ be a del Pezzo surface of degree d. When $d = 2$, the Geiser involution on X (swapping the two sheets of X as a double cover of $\mathbb{P}^2$) gives rise to a quadratic twist known as its *Geiser twist* (see [BFL19, §4.1.2]). Similarly, when $d = 1$, we may associate to X a *Bertini twist*. Geiser twists were used by Trepalin [Tre20] and Loughran and Trepalin [LT20] in order to verify the existence of certain types of del Pezzo surfaces of degree 2. These twists satisfy the following properties.

Lemma 5.3. *Let $X/\mathbb{F}_q$ be a del Pezzo surface of degree $d \leq 2$, and denote by X' its Geiser/Bertini twist.*

(1) *[Tre20, Lem. 4.4] If $d = 2$ and X has an $\mathbb{F}_q$-rational line, then X' does not.*

(2) *[Tre17, Rem. 3.3] The Frame symbol of X' is obtained from the Frame symbol of X via the following sequence of operations on the Frame symbol $\prod_{n \geq 1} n^{a_n}$:*

    (a) *Replace 1^{a_1} by $1^{a_1 - 1}$.*

    (b) *Replace each n^{a_n} with n odd by $n^{-a_n}(2n)^{a_n}$ and simplify to a new symbol $\prod_{n \geq 1} n^{b_n}$.*

    (c) *Replace 1^{b_1} by $1^{b_1 + 1}$.*

(3) *We have $a(X') = 2 - a(X)$.*

Proof. (1) is proved by Trepalin in [Tre20, Lem. 4.4]. For (2), note that the Geiser/Bertini involution acts by -1 on $K_{\overline{X}}^{\perp} \subset \mathrm{Pic}(\overline{X})$, hence, for ψ_X and $\psi_{X'}$ the characteristic polynomials of Frobenius for X and X' respectively, we have $\psi_{X'}(t) = \psi_X(-t)$, from which the result follows. Note that (3) follows immediately from (2) since $a_1 = a(X)$. $\square$

Remark 5.4. We note, as Trepalin does in [Tre20, §7], that the Bertini twist exhibits pathologies which the Geiser twist does not. In particular:

(1) The 60 classes of degree-2 del Pezzo surfaces split into 30 pairs of twists, while some degree-1 del Pezzo surfaces are in the same class as their Bertini twist.

(2) The twist of any degree-2 del Pezzo surface of Picard rank 1 is of Picard rank at least 2, while the twists of the 30 classes of degree-1 del Pezzo surfaces of Picard rank 1 include 5 pairs of twists and 7 classes with twist in the same class. It is the existence of these "persistently rank-1" del Pezzo surfaces of degree 1 that puts $\mathrm{IGP}_1(\mathbb{F}_q)$ out of current reach.

5.4. **Proofs of Theorems 1.2 and 1.3.** In the following proves, we will make use of the Hasse–Weil bound for the number of $\mathbb{F}_q$-rational points on the ramification divisor of the anticanonical morphism of a degree-2 del Pezzo surface over $\mathbb{F}_q$: as such a curve is of genus 3, it has at most $F(q)$ such points, where

$$F(q) := \begin{cases} q + 1 + 6\sqrt{q} & \text{if } q \text{ is odd,} \\ 2(q+1) & \text{if } q \text{ is even.} \end{cases}$$

We also employ the following additional notation:

- For $\sum_{i=1}^{r} a_i n_i = 8 - d$ and $d \in \{1, 2, 4\}$, we denote by $S_{d,II}^{(n_1^{a_1}, \ldots, n_r^{a_r})}$ a degree-d relatively minimal del Pezzo surface which is a conic bundle with exactly a_i singular fibres over closed points of degree n_i for all $1 \le i \le r$.

- For $\sum_{i=1}^{r} a_i n_i = 5$, we denote by $S_{4,II}^{(n_1^{a_1}, \ldots, n_r^{a_r})}$ a degree-4 minimal del Pezzo surface of Picard rank 1 whose blowup at a rational point in general position (when one exists) is a conic bundle with exactly a_i singular fibres over closed points of degree n_i for all $1 \le i \le r$. In particular, among the 18 types I–XVIII of minimal del Pezzo surfaces of degree 4, the 6 which correspond to cyclic subgroups are given in the following table:

Type	I	II	IV	V	XVIII
del Pezzo	$S_{4,I}^{(3,2)}$	$S_{4,II}^{(3,1)}$	$S_{4,II}^{(1^4)}$	$S_{4,II}^{(2^2)}$	$S_{4,I}^{(4,1)}$

Proof of Theorem 1.2. It suffices to consider the case $d = 1$. We consider each of the possible line counts from Theorem 2.8:

$$0, 2, 4, 6, 8, 12, 14, 20, 24, 26, 30, 40, 60, 72, 126, 240.$$

0: This count is possible for all q. Indeed, $\text{Bl}_8 S_9$ always exists by [BFL19, §5.2, $a = 1$] and has $(L_X, C_X) = (0, 0)$.

2: This count is possible for all q. Indeed, $\text{Bl}_{7,1} S_9$ always exists by [BFL19, §5.2, $a = 2$] and has $(L_X, C_X) = (0, 0)$.

4: This count is possible for all q. Consider type 44, which is $\text{Bl}_1 S_{2,II}^{(4,2)}$. It is clearly enough to show that $X = S_{2,II}^{(4,2)}$ (a type 6 degree-2 del Pezzo surface) has a rational point not on any line and not on the ramification divisor. Note that $\#X(\mathbb{F}_q) = q^2 + 2q + 1$ and $(L_X, C_X) = (0, 2)$. The orbit type of X is $4^2 8^2 8^2 8^2$. Note that the 4^2 corresponds to the singular fibre above the closed point of degree 2 and its Geiser dual; in particular, it contributes no rational points, and the other orbits are too large to contribute rational points. Then there are no rational points on the lines of X. Then it suffices to note that $q^2 + 2q + 1 > F(q)$ for all q.

6: This count is possible for all q. It is enough to construct $\text{Bl}_{5,2,1} S_9$ explicitly. First, choose a Galois orbit of 5 points $P_1, \ldots, P_5$ on a smooth plane conic $C/\mathbb{F}_q \subset \mathbb{P}_{\mathbb{F}_q}^2$, which always exists (take $xz = y^2$ when $2 \nmid q$ and $xz = y^2 + y$ when $q = 2$). Now choose a line $L/\mathbb{F}_q \subset \mathbb{P}_{\mathbb{F}_q}^2$ and a Galois orbit of two points Q_1, Q_2 on L (this always exists). Finally, choose a rational point R in $\mathbb{P}_{\mathbb{F}_q}^2 \setminus L \cup C$. This is possible since $\#\mathbb{P}^2(\mathbb{F}_q) = q^2 + q + 1$, while $\#L(\mathbb{F}_q) = \#C(\mathbb{F}_q) = q + 1$ and $q^2 + q + 1 - 2(q + 1) = q^2 - q - 1 > 0$ for all prime powers q. The blowup of $\mathbb{P}_{\mathbb{F}_q}^2$ furnishes the sought surface. Indeed, no three are collinear, no six are on a

common conic, and the points are not all on a singular cubic, as can be seen by considering intersection numbers of the containing curves.

8: This count is possible exactly for $q \geq 3$. It can only arise from $\mathrm{Bl}_{2^3,1} \, S_{8,I}$ (type 98), which has $(L_X, C_X) = (8, 24)$ and $\mathrm{Bl}_{6,1^2} \, S_9$ (type 107), which has $(L_X, C_X) = (8, 12)$. The former is the blowup of a type-1 degree-2 del Pezzo surface, which only exists for $q \geq 3$ by [LT20, Thm. 1.2]. The latter only exists for $q \geq 3$ by [BFL19, §5.2, $a = 3$]. Thus we conclude.

12: This count is possible for all q. It can only arise from $\mathrm{Bl}_{1^3} \, S_{4,I}$ (types 70, 73 and 74), $\mathrm{Bl}_{3,2^2,1} \, S_9$ (type 99) or $\mathrm{Bl}_{3^2,1^2} \, S_9$ (type 100). First, we show that it is realisable via $\mathrm{Bl}_{3^2,1^2} \, S_9$ for $q \geq 3$ (combined with the bound for the blowdown to a type-54 degree-2 del Pezzo surface, this shows that type 100 exists exactly when $q \geq 3$). Note that this surface is the blowup of a type-54 degree-2 del Pezzo surface $X = \mathrm{Bl}_{3^2,1} \, S_9$. We have $\#X(\mathbb{F}_q) = q^2 + 2q + 1$ and $(L_X, C_X) = (2, 6)$. Further, the orbit type is $1^2 3^6 3^{12}$. By consideration of the origin of the 56 lines in the plane, it is clear that the orbits 3^{12} represent pairwise disjoint lines, so there are at most $2(q+1) + 6 + F(q)$ points in the bad locus. Note that $q^2 - 7 > F(q)$ for $q \geq 7$. On the other hand, it can be checked computationally that it occurs for $q \in \{3, 4, 5\}$. Computer search reveals that the following degree-1 del Pezzo surface over $\mathbb{F}_2$ has exactly 12 lines:

$$w^2 + w(x^3 + y^3 + xy^2) = x^4 z + xyz^2 + z^3 + y^2 z^2.$$

Then we conclude.

14: This count is possible exactly when $q \geq 3$. It can only arise from $\mathrm{Bl}_{4,2,1^2} \, S_9$ (type 101). We first show that this exists for $q \geq 7$. It is equivalent to show the same for its Bertini twist, $\mathrm{Bl}_{2,1} \, S_{4,I}^{(2,1^3)}$. In turn, it is equivalent to show that a type-34 degree-2 del Pezzo surface $X = \mathrm{Bl}_2 \, S_{4,I}^{(2,1^3)}$ has a rational point in general position for $q \geq 7$. Such X exists for $q \geq 3$. Note that $\#X(\mathbb{F}_q) = q^2 - 2q + 1$ and $(L_X, C_X) = (0, 0)$. Further, the orbit type is $2^2 2^6 4^{10}$. Since $C_X = 0$, the conjugate pairs of lines contain no rational points. Then there are at most $F(q)$ points in the bad locus. Note that $q^2 - 2q + 1 > F(q)$ for $q \geq 7$. Computation reveals that this also exists for $q \in \{3, 4, 5\}$, but not $q = 2$.

20: This line count is possible exactly when $q \geq 3$. The only way it can arise is as $\mathrm{Bl}_{5,1^3} \, S_9$ (type 102); the existence bound follows from [BFL19, §5.2, $a = 4$].

24: This line count is possible exactly when $q \geq 4$. It can only arise from $\mathrm{Bl}_{1^3} \, S_{4,II}$. We first show that this exists for $q \geq 8$. It suffices to show that the type-27 degree-2 del Pezzo surface $X = \mathrm{Bl}_{1^2} \, S_{4,II}^{(2^2)}$ has a rational point in general position for $q \geq 8$. Such X only exists for $q \geq 3$. We have $\#X(\mathbb{F}_q) = q^2 + 4q + 1$ and $(L_X, C_X) = (8, 6)$. The orbit type is $1^8 4^{12}$. The 8 rational lines contain at most $8q$ rational points: let us now show this. he two exceptional divisors contain exactly $2q + 2$ rational points. There are two fibrations on $S_{4,II}$: denote the fibre of the ith fibration through a point R by $C_i(R)$, and let P, Q be the two blowup points. The (strict transforms of) the curves $C_1(P), C_2(Q)$ form a Geiser-dual pair and contribute at most $2q$ more rational points; the same is true for $C_2(P)$ and $C_1(Q)$. The final two rational lines come from T_P, an anticanonical curve with a double point at P through Q, and T_Q, the same but with P and Q playing the opposite roles. Each of these contributes at most $2q$ more points (one can slightly improve this bound to $8q - 2$, but it will not change the outcome). The orbits of size 4 all

have the configuration of pairs of singular fibres from the conic fibration, hence they contain no rational points. Then there are at most $8q + F(q)$ points in the bad locus. Note that $q^2 - 4q + 1 > F(q)$ for $q \geq 8$. Now we investigate $q \leq 8$. Since $\mathrm{Bl}_{1^3} S_{4,II}$ only exists for $q \geq 3$, we know that a line count of 24 is not possible over $\mathbb{F}_2$. Moreover, the only type of $\mathrm{Bl}_{1^3} S_{4,II}$ that could possibly exist over $\mathbb{F}_3$ is type 68, which has trace 5. It is shown in [Li10, Thm. 3.1.3] that there is just one degree-1 del Pezzo surface with trace 5 over $\mathbb{F}_3$:

$$w^2 = z^3 + (2x^4 + x^2y^2 + 2y^4)z + (x^6 + 2x^4y^2 + y^6).$$

Computer search shows that this surface has no rational lines, hence there is no degree-1 del Pezzo surface with 24 lines over $\mathbb{F}_3$. It remains to search over $\mathbb{F}_4$, $\mathbb{F}_5$ and $\mathbb{F}_7$.

Over $\mathbb{F}_4$, the following surface has exactly 24 lines:

$$w^2 + wy^3 = x^5y + xy^5 + z^3.$$

Over $\mathbb{F}_5$, the following surface has 24 lines:

$$x^6 + 4x^2y^4 + z^3 + 3zy^4 + w^2 + wy^3 + 4y^6.$$

Over $\mathbb{F}_7$, the following surface has 24 lines:

$$x^5y + 5x^3y^3 + 2xy^5 + z^3 + w^2 + wy^3 = 0.$$

26: This count can only arise from $\mathrm{Bl}_{2^3,1^2} S_9$ (type 95). We claim that it exists exactly when $q \geq 7$. First, we show its existence for $q \geq 7$. It is enough to show existence of the Bertini twist, $\mathrm{Bl}_{2,1} S_{4,II}^{(1^4)}$ (type 69), for $q \geq 7$. In turn, it is enough to determine when the type-28 degree-2 del Pezzo surface $X = \mathrm{Bl}_2 S_{4,II}^{(1^4)}$ has a rational point in general position. Such X only exists for $q \geq 5$. We have $\#X(\mathbb{F}_q) = q^2 - 2q + 1$ and $(L_X, C_X) = (0, 4)$. The orbit type is $2^4 2^8 2^{16}$. The singular fibres of a chosen conic bundle structure lie over four rational points and a double point. Then we see that the 2^8 corresponds to 8 intersecting pairs of conjugate lines. The remaining size-2 orbits consist either of disjoint lines or Geiser duals; in either case, we get no more rational points off the ramification divisor, so there are at most $8 + F(q)$ points in the bad locus. Then note that $q^2 - 2q - 7 > F(q)$ for $q \geq 7$. Computational search reveals that it does not exist for $q \leq 5$.

30: This count is possible exactly when $q \geq 4$. It can only arise from $\mathrm{Bl}_{3,2,1^3} S_9$ (type 96). We first show that this exists for $q \geq 7$. It is enough to verify existence of its Bertini twist $\mathrm{Bl}_1 S_{2,I}^5$ (type 48) for $q \geq 7$. In turn, enough to show that the type-12 degree-2 del Pezzo surface $X = S_{2,I}^5$ has a rational point in general position for $q \geq 7$. This surface only exists for $q \geq 3$. We have $\#X(\mathbb{F}_q) = q^2 - 3q + 1$ and $(L_X, C_X) = (0, 0)$. The orbit type is $2^{10} 6^6$. Since there are no rational lines and no conjugate pairs of lines which meet away from the ramification curve, hence there are at most $F(q)$ rational points in the bad locus. We see that $q^2 - 3q + 1 > F(q)$ for $q \geq 7$. A computational search reveals existence for $q = 4$ and $q = 5$ and non-existence for $q = 2$ and $q = 3$.

40: This count is only possible for $q \geq 5$. The only way it can arise is from $\mathrm{Bl}_{4,1^4} S_9$ (type 97), which only exists for $q \geq 5$ by [BFL19, §5.2, $a = 5$].

60: This count is possible exactly when $q \geq 7$. It can only arise from $\mathrm{Bl}_{2^2,1^4} S_9$ (type 93). First we show that this exists when $q \geq 8$. It is enough to show the same for

the Bertini twist $\mathrm{Bl}_1 \, S_{2,II}^{(1^6)}$ (type 38). In turn, it is enough to show that the type-2 degree-2 del Pezzo surface $X = S_{2,II}^{(1^6)}$ has a rational point in general position for $q \geq 8$. This exists only for $q \geq 5$. We have $\#X(\mathbb{F}_q) = q^2 - 4q + 1$ and $(L_X, C_X) = (0, 2)$. The orbit type is $2^{12}2^{16}$. It is clear that the $2^{1}2$ corresponds to the 6 singular fibres of the conic fibration and its dual; these contribute at most 12 rational points to the bad locus. Since the other conjugate pairs are not of the same configuration, they cannot meet in a point away from the ramification curve. Then there are at most $12 + F(q)$ points in the bad locus. Then we note that $q^2 - 4q - 11 > F(q)$ for $q \geq 8$. Computer search reveals existence when $q = 7$ and non-existence when $q \leq 5$.

72: This count is possible exactly when $q \geq 7$: it arises only from $\mathrm{Bl}_{3,1^5} \, S_9$ (type 94), which exists exactly when $q \geq 7$ by [BFL19, §5.2, $a = 6$].

126: This count is possible exactly when $q \geq 11$: it arises only from $\mathrm{Bl}_{2,1^6} \, S_9$ (type 92), which exists exactly when $q \geq 11$ by [BFL19, §5.2, $a = 7$].

240: This count is possible exactly when $q = 16$ or $q \geq 19$: it arises only from $\mathrm{Bl}_{1^8} \, S_9$, which exists exactly in this range by [BFL19, §5.2, $a = 9$]. □

Proof of Theorem 1.3. It suffices to consider the case $d = 1$. We consider each of the possible conic counts from Theorem 2.9:

$$0, 2, 4, 6, 12, 18, 24, 30, 36, 72, 90, 252, 270, 756, 2160.$$

0: This count is possible for all q. Indeed, $\mathrm{Bl}_8 \, S_9$ always exists by [BFL19, §5,2, $a = 1$] and has $(L_X, C_X) = (0, 0)$.

2: This conic count is possible for all q. First we show that it is possible for $q \geq 3$. Note that $\mathrm{Bl}_{2,1} \, S_{4,I}^{(2,1^3)}$ (type 76) has conic count 2, so it suffices to show that its Bertini twist $\mathrm{Bl}_{4,2,1^2} \, S_9$ (type 101) exists for $q \geq 3$. Consider its blowdown, the type-55 degree-2 del Pezzo surface $X = \mathrm{Bl}_{4,2,1} \, S_9$, which exists for $q \geq 3$. We have $\#X(\mathbb{F}_q) = q^2 + 2q + 1$ and $(L_X, C_X) = (4, 4)$. Here, we are blowing up the plane in a rational point P, a pair of conjugate points Q_1, Q_2 and a 4-tuple of conjugate points $R_1, \ldots, R_4$. The 56 geometric lines on X correspond to the 7 exceptional divisors, the strict transforms of the 21 lines joining these points and the Geiser duals. It is easily seen that, among the 21 lines, there is only one orbit of lines which all contain a rational point: namely, letting R_{i+1} be the image of R_i under Frobenius, the lines $L_{R_1 R_3}$ and $L_{R_2 R_4}$. Altogether, we deduce that there are at most $4(q+1) + 2 + F(q)$ points in the bad locus on X. Since $q^2 - 2q - 1 > F(q)$ for $q \geq 7$, we verify existence of X for $q \geq 7$. Computational search reveals that $\mathrm{Bl}_{4,2,1^2} \, S_9$ also exists for $q \in \{3, 4, 5\}$. It remains to show that it is possible over $\mathbb{F}_2$. Computer search reveals that the following surface is of type 5 over $\mathbb{F}_2$:

$$x^5 y + x^3 yz + xyz^2 + xy^2 w + z^3 + w^2 + y^6 = 0.$$

4: This count is possible for all q. Indeed, $\mathrm{Bl}_{7,1} \, S_9$ always exists by [BFL19, §5,2, $a = 2$] and has $(L_X, C_X) = (2, 4)$.

6: This count is possible for all q. Let us first show that it is possible for $q \geq 7$. It is enough to show that $\mathrm{Bl}_{1^3} \, S_{4,I}$ (type 74) exists for $q \geq 7$. In turn, it is enough to show that the type-33 degree-2 del Pezzo surface $X = \mathrm{Bl}_{1^2} \, S_{4,I}^{(3,2)}$ has a rational point in general position for $q \geq 7$. Note that $\#X(\mathbb{F}_q) = q^2 + 3q + 1$

and $(L_X, C_X) = (4, 2)$. The orbit type is $1^4 4^2 4^2 6^2 12^2$. There are at most $4(q + 1)$ points on the 4 rational lines. Further, at most one 4^2 contributes points, hence there are at most $4q + 6$ points on lines, hence at most $4q + 6 + F(q)$ points in the bad locus. Note that $q^2 - q - 5 > F(q)$ for $q \geq 7$. It now suffices to note, via computer search, that $\mathrm{Bl}_{5,2,1} S_9$, which has conic count 6, exists for $2 \leq q \leq 7$.

12: This count is possible exactly when $q \geq 3$. Note that $\mathrm{Bl}_{6,1^2} S_9$ (type 107) has $(L_X, C_X) = (8, 12)$ and exists for $q \geq 3$ by [BFL19, §5,2, $a = 3$]. It remains to show that conic count 12 is impossible over $\mathbb{F}_2$. Considering restrictions from blowing up degree-2 del Pezzo surfaces, we see that it can only arise from type 71, which has 6 lines and trace 2. Further, the twist contains 2 lines over $\mathbb{F}_2$ and 240 lines over $\mathbb{F}_{64}$, and these properties distinguish type 71. Computational search reveals that these properties are not simultaneously satisfied, hence we conclude.

18: This count is possible exactly when $q \geq 3$. It can only arise from $\mathrm{Bl}_{3,2^2,1} S_9$ (type 99) which has Bertini twist $\mathrm{Bl}_1 S_{2,II}^{(3,1^3)}$ (type 40). First we show that a type-4 degree-2 del Pezzo surface $X = S_{2,II}^{(3,1^3)}$ has a rational point in general position for $q \geq 7$. Such X only exists for $q \geq 3$. We have $\#X(\mathbb{F}_q) = q^2 - q + 1$ and $(L_X, C_X) = (0, 2)$. Further, the orbit type is $2^4 2^6 6^2 6^4$. The 2^6 corresponds to singular points on the three singular fibres over rational points and their Geiser duals. Then there are at most $6 + F(q)$ points in the bad locus. Note that $q^2 - q - 5 > F(q)$ for $q \geq 7$. Computer search reveals that $\mathrm{Bl}_{3,2^2,1} S_9$ exists for $q \in \{3, 4, 5\}$ and does not exist when $q = 2$.

24: This count is possible exactly when $q \geq 4$. First we show that it is possible for $q \geq 4$. It is enough to show that $\mathrm{Bl}_{2^4} S_9$ (type 98) exists in this range (computer search reveals that it does not exist for $q \leq 3$). Since $\mathrm{Bl}_{2^4} S_9 \cong \mathrm{Bl}_{2^3,1} S_{8,I}$, it is enough to show that the type-1 degree-2 del Pezzo surface $X = \mathrm{Bl}_{2^3} S_{8,I}$ has a rational point in general position when $q \geq 4$. We have $\#X(\mathbb{F}_q) = q^2 + 1$ and $(L_X, C_X) = (0, 6)$. Further, the orbit type is $2^4 2^{24}$. Since there are more than 4 orbits of disjoint pairs of lines, it is clear that the lines contain at most 4 rational points. Note that $q^2 - 3 > F(q)$ when $q \geq 4$. On the other hand, computer search shows that $\mathrm{Bl}_{2^4} S_9$ does not exist for $q \leq 3$. Now we show non-existence of the other types (types 66, 67 and 68, all $\mathrm{Bl}_{1^3} S_{4,II}$) with 24 conics when $q \leq 3$. This is dealt with in the argument for line count 24.

30: This count is possible exactly when $q \geq 3$. It can only arise from $\mathrm{Bl}_{4,2,1^2} S_9$ (type 101) and $\mathrm{Bl}_{5,1^3} S_9$ (type 102). The former is the blowup of a type-55 degree-2 del Pezzo surface, which only exists for $q \geq 3$, while the latter only exists for $q \geq 3$ by [BFL19, §5,2, $a = 4$].

36: This count is possible exactly when $q \geq 3$. It can only arise from $\mathrm{Bl}_{3^2,1^2} S_9$ (type 100), which has twist type 16, which is the blowup of a type-54 degree-2 del Pezzo surface $X = \mathrm{Bl}_{3^2,1} S_9$, which only exists for $q \geq 3$. We have $\#X(\mathbb{F}_q) = q^2 + 2q + 1$ and $(L_X, C_X) = (2, 6)$. The orbit type is $1^2 3^6 3^{12}$. There are at most $2(q + 1)$ points on the rational lines. It is not too hard to see that the singular fibres of one of the conic fibrations lie above two cubic points, accounting for 3^2 in each case, ultimately accounting for the 3^{12}. It is also easy to see that the remaining 3^6 corresponds to conjugate triples of disjoint lines, hence there are at most $2q + 2 + F(q)$ points in the bad locus. Note that $q^2 - 1 > F(q)$ for $q \geq 4$. A computer search reveals existence for $q = 3$.

72: This count is possible exactly when $q \geq 7$. It can only arise from type 95. We showed in the argument for $L_X = 26$ that this exists exactly when $q \geq 7$.

90: This count is possibly exactly when $q \geq 4$. It can only arise from $\mathrm{Bl}_{3,2,1^3} S_9$ (type 96) and $\mathrm{Bl}_{4,1^4} S_9$ (type 97). We showed above in the argument for $L_X = 30$ that the former exists exactly when $q \geq 4$, and the latter exists exactly when $q \geq 5$ by [BFL19, §5,2, $a = 5$].

252: This count is possible exactly when $q \geq 7$. It can only arise from $\mathrm{Bl}_{2^2,1^4} S_9$ (type 93), whose Bertini twist has type 38, which is the blowup of a type-2 degree-2 del Pezzo surface $X = S_{2,II}^{(16)}$, which only exists for $q \geq 5$. We have $\#X(\mathbb{F}_q) = q^2 - 4q + 1$ and $(L_X, C_X) = (0, 2)$. The orbit type is $2^{12}2^{16}$. The 2^{12} clearly arises from the singular fibres of the conic fibration and its dual, and so the bad locus contains at most $12 + F(q)$ points. Note that $q^2 - 4q - 11 > F(q)$ for $q \geq 8$. A computational search reveals existence for $q = 7$ and non-existence for $q \leq 5$.

270: This count is possible exactly when $q \geq 7$. It can only arise from $\mathrm{Bl}_{3,1^5} S_9$ (type 94), which exists exactly when $q \geq 7$ by [BFL19, §5,2, $a = 6$].

756: This count is possible exactly when $q \geq 11$. It can only arise from $\mathrm{Bl}_{2,1^6} S_9$ (type 92), which exists exactly when $q \geq 11$ by [BFL19, §5,2, $a = 7$].

2160: This count is possible exactly when $q \geq 16$ or $q \geq 19$. It can only arise from $\mathrm{Bl}_{1^8} S_9$ (type 91), which exists exactly when $q = 16$ or $q \geq 19$ by [BFL19, §5,2, $a = 9$]. $\square$

5.5. **Existence of relatively minimal del Pezzo surfaces over finite fields.** From the table for degree-1 del Pezzo surfaces and the calculations above, we deduce the following result.

Lemma 5.5. *Let $\mathbb{F}_q$ be any finite field.*

   (1) For all $d \in \{1, \ldots, 9\} \setminus \{7\}$, a surface of blowup type $S_{d,I}$ exists over $\mathbb{F}_q$.

   (2) For $d \in \{1, 2, 4\}$, a surface of blowup type $S_{d,II}$ exists over $\mathbb{F}_q$.

Proof. For $d \geq 2$, both results follow from [LT20]. Further, since $S_{1,I}^{13}$ (a type-20 degree-1 del Pezzo surface) is the twist of $\mathrm{Bl}_{7,1} S_9$ (type 111) and the latter exists for all q by [BFL19, §5.2, $a = 2$], so does the former, hence $S_{1,I}$ exists for all q. Then it only remains to verify the existence of $S_{1,II}$. In the $C_X = 2$ case of the proof of Theorem 1.3, we saw that $S_{1,II}^{(6,1)}$ exists over $\mathbb{F}_2$; since the twist of $S_{1,II}^{(3,2,1^2)}$ (type 2), namely $\mathrm{Bl}_{4,3,1} S_9$ (type 105) exists for $q \geq 3$, we deduce that $S_{1,II}$ always exists. $\square$

6. Infinite fields

In this section we use our algorithms to prove Theorems 1.4 and 1.5.

We will see that this comes down to constructing conic bundles with three prescribed Galois groups in characteristic 2. Accordingly, we begin with the following well-known result from Artin–Schreier theory (see [Sta25, Tag 09I7] for more).

Lemma 6.1. *Every separable extension of degree 2 of a field k of characteristic 2 is of the form $k(\phi(\alpha))$ for some $\alpha \in k^\times$, where $\phi(\alpha)$ is a root of the polynomial $x^2 + x + \alpha$.*

Proof. Any separable quadratic extension of k arises from adjoining a root of some irreducible quadratic polynomial $ax^2 + bx + c$ where $a \in k^\times$ and, for separability, $b \in k^\times$. Without loss of generality, we may assume $a = 1$; dividing by b^2 and considering the resulting expression as a polynomial in x/b, we obtain the result. $\square$

6.1. **Proof of Theorem 1.4.** As a preliminary for the proof of Theorem 1.4, we will solve the inverse Galois problem for degree-4 del Pezzo surfaces for two specific groups over infinite fields of characteristic 2, beginning with the following proposition.

Proposition 6.2. *Let $G \leq W(D_5)$ be isomorphic to $C_2^4 \rtimes G_5$, where $G_5 \in \{C_5, A_5\}$, and suppose that k is an infinite field of characteristic 2 with a Galois extension L/k such that $\mathrm{Gal}(L/k) \cong G$. Then there exists a quintic polynomial $f(x) \in k[x]$ and a Galois extension L'/k such that $\mathrm{Gal}(L'/k) \cong G$ and $L' = k(\phi(\alpha_1), \dots, \phi(\alpha_4))$, where the α_i are four distinct roots of $f(x)$ and $\phi(\alpha)$ is a root of the polynomial $x^2 + x + \alpha$.*

Proof. Let L/k be a Galois extension with $\mathrm{Gal}(L/k) \cong C_2^4 \rtimes G_5$. Since G has a normal subgroup $H = C_2^4 \rtimes \{1\} \cong C_2^4$ with quotient isomorphic to G_5, there exists a subextension $L/M/k$ with $\mathrm{Gal}(L/M) \cong C_2^4$ and $\mathrm{Gal}(M/k) \cong G_5$.

Further, note that H is contained in a subgroup H' of the form $C_2^4 \rtimes H_5$, where $H_5 \leq G_5$ is index-5 (so $H_5 = \{1\}$ for $G_5 = C_5$ and $H_5 = A_4$ for $G_5 = A_5$), and H' corresponds to an extension $k(\alpha_1)/k$, where α_1 is one of the five distinct roots $\alpha_1, \dots, \alpha_5$ of a quintic polynomial $f(x) \in k[x]$ (the distinctness of the α_i follows from separability of $k(\alpha_1)/k$ as a subextension of a separable extension).

Note that $L' := k(\phi(\alpha_1), \dots, \phi(\alpha_5))$ is the splitting field of $f(x^2 + x)$, hence it is Galois over k. We will show that, in either case, $\mathrm{Gal}(L'/k) \cong G$.

First, note that $H' = C_2^4 \rtimes H_5$ contains an index-2 subgroup H'' but H_5 does not: in the case $G_5 = C_5$, we may take $H'' = C_2^3 \rtimes \{1\}$, and in the case $G_5 = A_5$, we may take $H'' = C_2^3 \rtimes A_4$, and the absence of index-2 subgroups of H_5 itself is clear since H_5 is simple. Then $k(\alpha_1)/k$ permits a quadratic extension $k''/k(\alpha_1)$ which is a subextension of L/k but not of M/k. By Lemma 6.1, we have $k'' = k(\alpha_1)\big(\phi(\sum_{i=0}^4 a_i \alpha_1^i)\big)$ for some $a_0, \dots, a_4 \in k$. Note that $k\big(\sum_{i=0}^4 a_i \alpha_1^i\big)$ is a subextension of a degree-5 extension, hence it is either trivial or of degree 5, but it cannot be trivial since k'' is a degree-10 extension of k, hence it is of degree 5. Then, without loss of generality, we have $k'' = k(\phi(\alpha_1))$. Then L' is a subfield of L (as the Galois closure of $k(\phi(\alpha_1)) \subset L$), and the Galois group of L' is isomorphic to a normal subgroup of G containing H'', but the only such subgroup is G itself, as may be verified in each case, hence we are done. $\square$

Theorem 6.3. *Let k be an infinite field of characteristic 2 and let $G \in \{C_2^4 \rtimes C_5, C_2^4 \rtimes A_5\}$. Then a G-extension of k exists if and only if there exists a degree-4 del Pezzo surface with splitting group G.*

Proof. It suffices to prove sufficiency. By Proposition 6.2, we see that there exists a degree-5 polynomial $f(x) \in k[x]$ such that $f(x)$ has Galois group C_5 or A_5 and $f(x^2 + x)$ has Galois group G.

Consider the surface

$$Z_0 = \{x^2 + xy + ty^2 + f(t)z^2 = 0\} \subset \mathbb{P}^2_{x,y,z} \times \mathbb{A}^1_t.$$

One can easily check that Z_0 is smooth. Indeed, smoothness in the patches $x \neq 0$, $y \neq 0$ is trivial, and any singular point $([x : y : z], t)$ in the patch $z \neq 0$ would imply $f(t) = f'(t) = 0$, which contradicts the separability of f.

We now seek to patch this affine conic bundle with another to produce a conic bundle over $\mathbb{P}^1$. First, consider the assignments

$$u = \frac{1}{t}, \quad X = x, \quad Y = \frac{y}{u}, \quad Z = \frac{z}{u^3}.$$

Substituting these into the equation for Z_0, we arrive at the surface

$$V = \{X^2 + uXY + uY^2 + ug(u)Z^2 = 0\} \subset \mathbb{P}^2_{X,Y,Z} \times \mathbb{A}^1_u,$$

where $g(T) := T^5 f(1/T)$. That is, if $f(t) = t^5 + c_4 t^4 + \cdots + c_1 t + c_0$, then $g(t) = c_0 t^5 + c_1 t^4 + \cdots + c_4 t + 1$.

This surface patches with Z_0, but it is singular at the point $([0 : 1 : 1], 0)$. We are thus motivated to transform the equation by moving this singular point to the origin in the affine patch $z \neq 0$, blowing up and taking the strict transform. This amounts to the following additional change of variables:

$$X' = \frac{X}{u}, \quad Y' = \frac{Y + Z}{u}, \quad Z' = Z.$$

Substituting these new variables in for the previous ones, we arrive at the surface

$$Z_\infty = \{X'^2 + uX'Y' + X'Z' + uY'^2 + (c_4 + \cdots + c_0 u^4)Z'^2 = 0\} \subset \mathbb{P}^2_{X',Y',Z'} \times \mathbb{A}^1_u.$$

It is easily seen from the above that $Z_0 \cap \{t \neq 0\} \cong Z_\infty \cap \{u \neq 0\}$; we claim that Z_∞ is, moreover, smooth. Again, we check in affine coordinate patches.

First consider the case $X' \neq 0$, and normalise so that $X' = 1$. Then we have

$$F = 1 + uY' + Z' + uY'^2 + (c_4 + \cdots + c_0 u^4)Z'^2,$$
$$J = (u, 1, Y' + Y'^2 + (c_3 + c_1 u^2)Z'^2).$$

Since the second entry of J is never zero, there are no singularities in this patch.

Next consider the case $Y' \neq 0$, and normalise so that $Y' = 1$. Then we have

$$F = X'^2 + uX' + X'Z' + u + (c_4 + \cdots + c_0 u^4)Z'^2,$$
$$J = (u + Z', X', X' + 1 + (c_3 + c_1 u^2)Z'^2).$$

Any singularity must have $X' = 0$, but then the first entry of J gives $u = Z'$, and then the equation gives $u + (c_4 + \cdots + c_0 u^4)u^2 = 0$ while the third entry of the Jacobian gives $1 + c_3 u^2 + c_1 u^4 = 0$. These imply $g(u) = g'(u) = 0$, but it is easily seen that g is separable. Then there are no singularities in this patch.

Finally consider the case $Z' \neq 0$, and normalise so that $Z' = 1$. Then we have

$$F = X'^2 + uX'Y' + X' + uY'^2 + (c_4 + \cdots + c_0 u^4),$$
$$J = (uY' + 1, uX', X'Y' + Y'^2 + c_3 + c_1 u^2).$$

The second entry of J gives $u = 0$ or $X' = 0$. If $u = 0$, then the first entry of J gives $1 = 0$, a contradiction, so we must have $X' = 0$. Then the first entry of J gives $Y' = \frac{1}{u}$, and the third entry of J and the equation then give $g(u) = g'(u) = 0$, which is impossible by separability of g, hence there are no singularities in this patch. We deduce that Z_∞ is smooth.

Lastly, note that the fibre of Z_∞ over $u = 0$ is given by

$$X'^2 + X'Z' + c_4 Z'^2 = 0.$$

We claim that this fibre is split over k, i.e. that $\phi(c_4) \in k$. To see this, note that $\phi(\alpha_5) = \phi(c_4) + \sum_{i=1}^4 \phi(\alpha_i)$, but $\phi(\alpha_5) \in k(\phi(\alpha_1), \ldots, \phi(\alpha_4))$, hence $k(\phi(c_4)) \subset L$, but L contains no index-2 subgroups, hence $k(\phi(c_4)) = k$.

Denote by Z the surface obtained by gluing Z_0 and Z_∞ along $\{t \neq 0\}$ and $\{u \neq 0\}$ and contracting the fibre over $u = 0$: we see that Z is a conic bundle over $\mathbb{P}^1$ with five singular fibres and splitting group G, and we are done. $\qquad\square$

Before we prove Theorem 1.4, we need one more auxiliary result.

Proposition 6.4. *Let k be an infinite field with $\mathrm{char}(k) = 2$ admitting a separable extension of degree 2. Then there exists a degree-4 del Pezzo surface X/k of type $S_{4,II}$ with splitting group C_2.*

Proof. Let $L = k(\phi(\alpha))$ be a separable extension of degree 2 for some $\alpha \in k^\times$. Choose $\beta \in k^\times \setminus \{\alpha\}$, and consider

$$Z_0 := \{x^2 + xy + ty^2 + (t^2 + t + \alpha)(t^2 + t + \alpha + \beta)z^2 = 0\} \subset \mathbb{P}^2_{x,y,z} \times \mathbb{A}^1_t.$$

Set $f(t) := (t^2 + t + \alpha)(t^2 + t + \alpha + \beta)$ and $g(t) := t^4 f(1/t)$. Analogously to the proof of Theorem 6.3, we consider first the change of variables

$$u := \frac{1}{t}, \quad X := x, \quad Y := \frac{y}{u}, \quad Z := z,$$

and upon finding singularities on the associated surface, further consider

$$X' := \frac{Y}{u}, \quad Y' := \frac{X + Z}{u}, \quad Z' := Z,$$

obtaining the surface

$$Z_\infty := \{Y'^2 + uX'Y' + X'Z' + h(u)Z'^2 = 0\} \subset \mathbb{P}^2_{X',Y',Z'} \times \mathbb{A}^1_u.$$

It is easily seen that this surface is smooth and glues with Z_0. Further, note that the singular fibres of Z_0 as a conic bundle under projection to $\mathbb{A}^1_t$ are over the rational points $\alpha, \alpha + 1, \beta, \beta + 1$ and are non-split over k, each with splitting field $k(\alpha)$. Further, Z_∞ contributes no new singular fibres, hence the gluing Z is a relatively minimal conic bundle with splitting group C_2. As in the proof of Theorem 6.3, we deduce the result. $\qquad\square$

Corollary 6.5 (Theorem 1.4). *The inverse line and conic problems $\mathrm{ICP}_{4,e}(k)$ admit surjective solutions for all infinite fields k.*

Proof. We first consider $\mathrm{ICP}_{4,1}(k)$. It is easily seen from the tables in Section A that a degree-4 del Pezzo surface with $n \in \{1, 2, 4, 8, 16\}$ lines exists as soon as one is permitted by Galois theory, since these counts correspond exclusively to blowups of del Pezzo surfaces of degree 5 or greater, for which the inverse Galois problem is known. Then the only count left to deal with is 0.

Assume that the Galois theory of k permits the existence of a degree-4 del Pezzo surface with zero lines. Since the blowup $\mathrm{Bl}_{2^2} S_{8,I}$ is a degree-4 del Pezzo surface with 0 lines, we may assume without loss of generality that k admits no separable extensions of degree 2.

By explicitly analysing the list of possible splitting groups for a degree-4 del Pezzo surface with zero lines, we see that all except the two groups in Proposition 6.2 contain subgroups

of index 2 and so imply a separable quadratic extension. Then we reduce to assuming that we know only that k admits a Galois extension with one of these two groups as its Galois group. The result now follows from Theorem 6.3.

It remains to consider $\mathrm{ICP}_{4,2}(k)$. Here, the only problematic conic counts are 0 and 2. For a conic count of 2, we see from Proposition 6.4 that a quadratic extension is sufficient, as is a quartic extension; all minimal subgroups of $W(D_4)$ admit normal subgroups of index 2 or 4, hence this count can be dealt with. For the count 0, we see that a separable extension of degree 4 or 5 is sufficient; all minimal subgroups of $W(D_5)$ contain a subgroup of index either 4 or 5, so we are done. $\qquad\square$

6.1.1. *Cubic surfaces.*
Applying Algorithm 2 for $d = 3$ (recalling $\mathrm{ICP}_{3,1}(k)$ and $\mathrm{ICP}_{3,2}(k)$ are equivalent), we obtain a surjective solution for each positive count. For the count 0, our algorithm terminates in $U = \{G\}$ for G the unique index-2 subgroup in $W(E_6)$, and B contains C_3, $\mathfrak{D}_6$ and $\mathfrak{S}_3$. Since G is simple, our reduction strategy cannot be employed; further, none of the results in Section 4.3 are able to handle this case. We thus obtain the following result.

Proposition 6.6. *For k an infinite field with a separable cubic extension, $\mathrm{ICP}_{3,1}(k)$ and $\mathrm{ICP}_{3,2}(k)$ admit positive solutions.*

Proof. As discussed above, it suffices to consider the count 0; a cubic extension permits the existence of $\mathrm{Bl}_{3^2} S_9$, which has no lines. $\qquad\square$

6.2. **Proof of Theorem 1.5.**

Proof of Theorem 1.5. Let k be an infinite field admitting separable extensions of degrees 1 to 8. Proposition 2.29 implies any blowup type coming from a relatively minimal del Pezzo surface of degree at least 5 exists over k. Inspection of the tables in Section A reveals that the only line or conic count unobtainable from such blowups is 24 in degree 1, both as a line count and a conic count, which can only be obtained from the type $\mathrm{Bl}_{1^3} S_{4,II}$. Then one deduces $\mathrm{ICP}_d(k)$ for all d immediately from Theorem 4.9 for $\mathrm{char}(k) \neq 2$ and Proposition 6.4 otherwise.

Now assume that $\mathrm{char}(k) \neq 2$ and k admits a Galois extension of degree 3. Inspection of the tables in Section A reveals that the only pairs of line and conic counts (L_X, C_X) unobtainable from these blowup types are as follows:

(1) $d \geq 3$: None.

(2) $d = 2$: $(0, 2)$ (from $S_{2,II}$), $(0, 4)$ (from $\mathrm{Bl}_2 S_{4,II}$) and $(8, 6)$ (from $\mathrm{Bl}_{1^2} S_{4,II}$).

(3) $d = 1$: $(2, 2)$ (from $\mathrm{Bl}_{2,1} S_{4,I}$), $(12, 6)$ (from $\mathrm{Bl}_{1^3} S_{4,I}$), $(0, 6)$ (from $\mathrm{Bl}_3 S_{4,II}$), $(6, 12)$ (from $\mathrm{Bl}_{2,1} S_{4,II}$), $(24, 24)$ (from $\mathrm{Bl}_{1^3} S_{4,II}$) and $(4, 4)$ (from $\mathrm{Bl}_1 S_{2,II}$).

We immediately deduce the result for $d \geq 3$. In the case $d = 2$, note that all of the blowup types have rational conic fibrations, among which some have splitting group C_2 in each degree, hence the result follows from Corollary 4.14. Finally we consider $d = 1$. Of the six listed blowup types, note that all except $\mathrm{Bl}_3 S_{4,II}$ arise as the blowup of a degree-2 del Pezzo surface Y with a rational conic fibration. Further, it is easily seen that these surfaces can be assumed unirational by Theorem 4.9, hence Y has a rational point in general position, hence the blowup type exists. It only remains to deal with $\mathrm{Bl}_3 S_{4,II}$. Note that here, one may construct such a surface with splitting group C_6 (see

type 75 in the degree-1 table for finite fields). Note then that one may Bertini twist to obtain a surface of the form $\mathrm{Bl}_{1^3} S_{4,II}$, hence the blowup type exists. $\square$

This result applies in particular to non-archimedean local fields.

Definition 6.7. A field k is a *local field* if it is isomorphic to one of:

(1) $\mathbb{R}$, $\mathbb{C}$ (k is *archimedean*).

(2) A finite extension of $\mathbb{Q}_p$, $\mathbb{F}_q((u))$ for some prime p, prime power q (k is *non-archimedean*).

Since a non-archimedean local field has a cyclic (unramified) Galois extension of every degree, we obtain Corollary 1.6.

7. Hilbertian fields

In this section, we give background on Hilbertian fields and prove Theorem 1.7.

Definition 7.1. A field k is *Hilbertian* if $\mathrm{char}(k) = 0$ and, for any finite collection of finite morphisms $\phi_i : V_i \to \mathbb{P}^1$, $i = 1, \ldots, n$, the set $\mathbb{P}^1(k) \setminus \bigcup_{i=1}^n \phi_i(V_i(k))$ is Zariski-dense.

Example 7.2. It essentially follows from Hilbert's Irreducibility Theorem that any number field K is Hilbertian. Further, fields finitely generated over their prime field and function fields are Hilbertian; see [FJ23, §13] for more on this. On the other hand, algebraically closed fields, finite fields and local fields are not Hilbertian.

Lemma 7.3. *Let k be a Hilbertian field and let $d \in \{1, \ldots, 9\}$.*

(1) A relatively minimal degree-d del Pezzo surface with splitting group G_d exists over k.

(2) If $d \in \{2, 4\}$, relatively minimal conic and non-conic degree-d del Pezzo surfaces with maximal Galois action exist over k.

Proof. Part (i) follows from [JL15, Prop. 4.6] (while the cited result is given for number fields, its proof holds for arbitrary Hilbertian fields). The extension to conic del Pezzo surfaces follows in the same way, noting that the existence of such surfaces with maximal action over number fields follows from Theorem 4.9 and Corollary 4.19. $\square$

Lemma 7.4. *A relatively minimal conic del Pezzo surface of degree 1 with maximal action (splitting group $W(D_7)$) exists over $\mathbb{Q}$.*

Proof. Follows directly from Corollary 4.17. $\square$

Lemma 7.5. *For each $n \in \mathbb{Z}_{\geq 1}$ and $e \in \mathbb{Z}_{\geq 1}$, there exists a general position degree-e closed point in $\mathbb{P}_k^2$ for k a Hilbertian field.*

Proof. First note that, for every abelian group A, there exists a Galois extension l/k with $\mathrm{Gal}(l/k) \cong A$ [FJ23, Prop. 16.3.5]. In particular, $\mathbb{P}_k^1$ admits closed points of degree e for all $e \in \mathbb{Z}_{\geq 1}$, thus so does $\mathbb{P}_k^2$. Then observe that the condition that the underlying points are in general position can be phrased in terms of non-linear polynomial equations in the coefficients of the underlying geometric points as in [BFL19, Lem. 2.5]. The result then follows upon noting that $\mathrm{Sym}^e(\mathbb{P}^2)$ is a rational variety, thus has the Hilbert property. $\square$

Proposition 7.6. *Let k be a Hilbertian field and let $d \in \{1, \ldots, 9\} \setminus \{7\}$.*

(1) A relatively minimal degree-d del Pezzo surface with splitting group G_d and a k-rational point exists over k.

(2) For $d \geq 3$, a surface as in part (i) exists with a rational point in general position for all $e \geq 1$.

Proof. First we prove (i). As the result is trivial for degree $d \geq 8$, we assume henceforth that $d \leq 6$. Consider the Hilbert scheme $\mathcal{H}_{d,k}$ parametrising anticanonically embedded degree-d del Pezzo surfaces over k as in [JL15, §4]; by [JL15, Proof of Lem. 4.1], this scheme is a rational quasi-projective variety, thus $\mathcal{H}_{d,k}(k)$ is non-thin. Meanwhile, the scheme $\mathcal{L}_{d,k}$ parametrising the lines on these surfaces is a finite étale cover of $\mathcal{H}_{d,k}$ of degree at least 6; the Galois action on the finite fibres corresponds to the Galois action on lines of the associated surfaces, and an application of Hilbert's Irreducibility Theorem ensures that this action is by the full G_d outside of a thin locus T.

Next we prove (ii). First we show that a non-thin collection of surfaces contain a rational point. Note that we may form a fibration $\pi_d : X_d \to \mathcal{H}_{d,k}$ with the fibre above $P \in \mathcal{H}_{d,k}$ isomorphic to the associated del Pezzo surface. Note further that π_d has geometrically integral generic fibre, thus the image $\pi_d(A) \subset \mathcal{H}_{d,K}$ of a non-thin set $A \subset X_d(k)$ is non-thin. We first observe that $\pi_d(X_d(k))$ is itself non-thin. To see this, note that X_d has a second fibration (the "coordinate fibration") with base a projective space and fibres isomorphic to linear hypersurfaces. Then by a standard fibration argument for the Hilbert property [BSFP14, Thm. 1.1], $X_d(k)$ is non-thin, thus $\pi_d(X_d(k))$ is non-thin, so $\pi_d(X_d(k)) \setminus T$ is non-thin. It remains only to consider the general position requirement, which is non-vacuous only for $d = 2$. It is enough to observe that all rational points not in general position are contained in a one-dimensional subspace of each fibre; by density of rational points in the total space, it follows that there exists at least one fibre, hence surface, with a rational point in general position. $\square$

We now set our sights on proving that closed points in general position exist on del Pezzo surfaces with a rational point over Hilbertian fields.

Our approach hinges on working along *tangent sections*, which we now define. In what follows, we denote birational equivalence by $\sim$.

Definition 7.7. Let X be a del Pezzo surface of degree 3 or 4 and $P \in X(k)$ not on a line of X. A *tangent section* $C_P \subset X$ is the intersection of a hyperplane $T_P X$ tangent to X at P with X, which is a singular curve. This is an irreducible curve of geometric genus zero with a rational point, hence $C_P \sim \mathbb{P}^1$.

Note that C_P is unique if $d = 3$ and a member of a one-dimensional family if $d = 4$.

Lemma 7.8. *For $P \in X(k)$ a rational point on a del Pezzo surface X of degree 3 or 4, we have $\mathrm{Sym}^2(C_P) \sim \mathbb{P}_k^2$ and $\mathrm{Sym}^3(C_P) \sim \mathbb{P}^3$.*

Proof. Since C_P is birational to $\mathbb{P}^1$, it suffices to show that $\mathrm{Sym}^2(\mathbb{P}_k^1) \cong \mathbb{P}_k^2$ and $\mathrm{Sym}^3(\mathbb{P}_k^1) \cong \mathbb{P}_k^3$. Note that we have the maps

$$\mathrm{Sym}^2(\mathbb{A}_k^1) \xrightarrow{\sim} \mathbb{A}_k^2, \quad \{x, y\} \quad \mapsto \quad (x + y, xy),$$

$$\mathrm{Sym}^3(\mathbb{A}_k^1) \xrightarrow{\sim} \mathbb{A}_k^3, \quad \{x, y, z\} \quad \mapsto \quad (x + y + z, xy + yz + zx, xyz).$$

By interpretation of the coefficients of a univariate polynomial in terms of symmetric functions on its roots, it is clear that these two maps are isomorphisms. These isomorphisms induce the desired ones. $\square$

We are now ready to prove the existence of closed points in general position.

Proposition 7.9. *Suppose given a positive integer $1 \leq d \leq 9$, positive integers $e_1, \ldots, e_r$ such that $\sum_{i=1}^r e_i \leq d-1$ and a Hilbertian field k. Then on any degree-d del Pezzo surface X/k with $X(k) \neq \varnothing$ (and a rational point in general position for $d = 2$), there closed points $P_1, \ldots, P_r$ on X with $\deg(P_i) = e_i$ in general position.*

Proof. First assume that $d \geq 5$. Then such X is k-rational, and the proof is immediate from the existence of points in general position in the plane.

Now assume that $d \leq 4$. We separate into cases depending on $d, E := \{e_i\}$. Note that, by Lemma 2.25, we can (and do) assume that at least one e_i is greater than 1.

(1) If $d = 4, E = \{2\}$, take $P \in X(k)$ and consider a tangent section $C_P \subset X$, which is birational to $\mathbb{P}^1$. Note that $\mathrm{Sym}^2(C_P) \sim \mathbb{P}^2$. Note that each of the 16 lines on X intersects C_P in 1 point. Then there are at most 16 points on C_P lying on a line on X, thus at most 16 curves in $\mathrm{Sym}^2(C_P) \sim \mathbb{P}^2$ corresponding to a pair of points such that one of them is on a line. Denote the 16 associated curves in $\mathbb{P}^2$ by $L_1, \ldots, L_{16}$. Next, for each of the 10 families of conics $\pi_i : X \to \mathbb{P}^1$, a general member intersects C_P in 2 points; we thus obtain morphisms $\phi_i : \mathbb{P}^1 \to \mathrm{Sym}^2(C_P)$, $i = 1, \ldots, 10$, sending $Q \in \mathbb{P}^1$ to the set of points $\pi_i^{-1}(Q) \cap C_P$. We obtain 10 associated curves $C_1, \ldots, C_{10} \subset \mathbb{P}^2$. Lastly, denote by T the locus of points in $\mathrm{Sym}^2(C_P)(k) \sim \mathbb{P}^2(k)$ corresponding to pairs of rational points, which is thin, coming from the degree-2 cover $C_P \times C_P \to \mathrm{Sym}^2(C_P)$. Choosing $R \in \mathbb{P}^2(k) \backslash \left(\bigcup_{i=1}^{16} L_i(k) \cup \bigcup_{j=1}^{10} C_j(k) \cup T \right)$ (which we may do since $\mathbb{P}^2(k)$ is non-thin), the associated quadratic point on X is in general position.

(2) If $d = 4, E = \{2, 1\}$, choose a conjugate pair of points Q_1, Q_2 as in case (1) and blow up to a degree-2 del Pezzo surface. Again, by density of rational points, we may choose a rational point R' not on any line and not on the ramification divisor. Denoting by $R \in X(k)$ the associated point, $\{Q_1, Q_2, R\}$ is in general position.

(3) If $d = 4, E = \{3\}$, take $P \in X(k)$ and a tangent section $C_P \subset X$, and consider $Z := \mathrm{Sym}^3(C_P) \sim \mathbb{P}^3$. There are 16 curves in Z corresponding to a point on a line and two more points, 10 surfaces from two points on a conic in a particular family and one other point, and one surface corresponding to intersection with a tangent section of a point on C_P (the intersection multiplicity is 4, but the underlying intersection is always at most 3 points away from P). Choose a rational point away from these subvarieties in $\mathbb{P}^3$ and away from the thin locus corresponding to reducible cubic points; then the associated triple of conjugate points $\{Q_1, Q_2, Q_3\}$ are in general position.

(4) If $d = 3, E = \{2\}$, choose $P \in X(k)$ and construct the tangent section C_P. Again, consider $Y := \mathrm{Sym}^2(C_P) \sim \mathbb{P}^2$. Each of the 27 lines intersects C_P in 1 points, so pairs with a point on a line are parametrised by 27 curves. Further, pairs of points both contained in a singular anticanonical curve with a singularity at one of them are parametrised by the curve C_P, hence by a curve in $\mathrm{Sym}^2 C_P \sim \mathbb{P}^2$.

Choose a rational point away from these curves and away from the thin locus corresponding to pairs of rational points.   □

Corollary 7.10 (Theorem 1.7). *Over a Hilbertian field, all blowup types exist in all degrees. Thus the inverse line and conic problems admit simultaneous surjective solutions.*

Appendix A. Tables

A.1. **General fields.** In this section, we give tables detailing properties of del Pezzo surfaces of each degree.

Denoting by X a del Pezzo surface:

- The "Type" column gives its description as blowups of relatively minimal surfaces.
- The "(L_X, C_X)" column gives its line and conic counts respectively.
- The "I_X" column gives the intersection invariant.
- The "G_X" column gives its possible splitting groups.
- The "$\#C$" column gives the number of associated conjugacy classes in $G_{\deg(X)}$.

We denote by $T_n \leq \mathfrak{S}_n$ a transitive subgroup of $\mathfrak{S}_n$. We denote by $M_3 \leq W(E_6)$ and $M_6 \leq W(A_2 \times A_1)$ the splitting groups of relatively minimal degree-3 and degree-6 del Pezzo surfaces respectively. For $d \in \{1, 2, 4\}$, we denote by $M_{d,I} \leq W(R_d)$ and $M_{d,II} \leq W(D_{8-d})$ the splitting groups of relatively minimal degree-d del Pezzo surfaces of Picard ranks 1 and 2 respectively. Note that:

(1) $M_6 \in \{C_6, \mathfrak{S}_3, \mathfrak{D}_6\}$.

(2) $T_3 \in \{C_3, \mathfrak{S}_3\}$.

(3) $T_4 \in \{C_2^2, C_4, \mathfrak{D}_8, \mathfrak{A}_4, \mathfrak{S}_4\}$.

(4) $T_5 \in \{C_5, \mathfrak{D}_{10}, \mathrm{GA}(1, 5), \mathfrak{A}_5, \mathfrak{S}_5\}$.

A.1.1. *Degree 9.* 1 relatively minimal class, 0 non-relatively minimal classes; 1 class in total.

TABLE 12. Relatively minimal del Pezzo surfaces of degree 9

Type	(L_X, C_X)	I_X	G_X	$\#C$
S_9	$(0,0)$	1	1	1

A.1.2. *Degree 8.* 2 relatively minimal classes, 1 non-relatively minimal class; 3 classes in total.

TABLE 13. Relatively minimal del Pezzo surfaces of degree 8

Type	(L_X, C_X)	I_X	G_X	$\#C$
$S_{8,II}$	$(0,2)$	-1	1	1
$S_{8,I}$	$(0,0)$	2	C_2	1

TABLE 14. Non-relatively minimal del Pezzo surfaces of degree 8

Type	(L_X, C_X)	I_X	G_X	$\#C$
$\mathrm{Bl}_1 S_9$	$(1, 1)$	-1	1	1

A.1.3. *Degree 7.* 0 relatively minimal classes, 2 non-relatively minimal classes; 2 classes in total.

TABLE 15. Non-relatively minimal del Pezzo surfaces of degree 7

Type	(L_X, C_X)	I_X	G_X	$\#C$
$\mathrm{Bl}_1 S_{8,II}$, $\mathrm{Bl}_{1,1} S_9$	$(3, 2)$	1	1	1
$\mathrm{Bl}_1 S_{8,I}$, $\mathrm{Bl}_2 S_9$	$(1, 0)$	-2	C_2	1

A.1.4. *Degree 6.* 3 relatively minimal classes, 7 non-relatively minimal classes; 10 classes in total.

TABLE 16. Relatively minimal del Pezzo surfaces of degree 6

Type	(L_X, C_X)	I_X	G_X	$\#C$
S_6	$(0, 0)$	6	M_6	3

TABLE 17. Non-relatively minimal del Pezzo surfaces of degree 6

Type	(L_X, C_X)	I_X	G_X	$\#C$
$\mathrm{Bl}_3 S_9$	$(0, 0)$	-3	T_3	2
$\mathrm{Bl}_{2,1} S_9$ $\mathrm{Bl}_{1^2} S_{8,I}$	$(2, 1)$	2	C_2	1
$\mathrm{Bl}_{1^3} S_9$ $\mathrm{Bl}_{1^2} S_{8,II}$	$(6, 3)$	-1	1	1
$\mathrm{Bl}_2 S_{8,I}$	$(0, 1)$	-4	C_2 or C_2^2	2
$\mathrm{Bl}_2 S_{8,II}$	$(0, 3)$	2	C_2	1

A.1.5. *Degree 5.* 5 relatively minimal classes, 14 non-relatively minimal classes; 19 classes in total.

TABLE 18. Relatively minimal del Pezzo surfaces of degree 5

Type	(L_X, C_X)	I_X	G_X	$\#C$
S_5	$(0, 0)$	5	T_5	5

TABLE 19. Non-relatively minimal del Pezzo surfaces of degree 5

Type	(L_X, C_X)	I_X	G_X	$\#C$
$\mathrm{Bl}_4 S_9$	$(0, 1)$	-4	T_4	5
$\mathrm{Bl}_{3,1} S_9$ $\mathrm{Bl}_3 S_{8,II}$	$(1, 2)$	3	T_3	2
$\mathrm{Bl}_{2^2} S_9$ $\mathrm{Bl}_{2,1} S_{8,I}$	$(2, 1)$	4	C_2 or C_2^2	2

$Bl_{2,1^2} S_9$ $Bl_{1^3} S_{8,I}$ $Bl_{2,1} S_{8,II}$	$(4,3)$	-2	C_2	1
$Bl_{1^4} S_9$ $Bl_{1^3} S_{8,II}$	$(10,5)$	1	1	1
$Bl_1 S_6$ $Bl_3 S_{8,I}$	$(1,0)$	-6	M_6	3

A.1.6. *Degree 4.* 148 relatively minimal classes, 49 non-relatively minimal classes; 197 classes in total.

TABLE 20. Relatively minimal del Pezzo surfaces of degree 4

Type	(L_X, C_X)	I_X	G_X	$\#C$
$S_{4,I}$	$(0,0)$	4	$M_{4,I} \leq W(D_5)$	98
$S_{4,II}$	$(0,2)$	-4	$M_{4,II} \leq W(D_4)$	50

TABLE 21. Non-relatively minimal del Pezzo surfaces of degree 4

Type	(L_X, C_X)	I_X	G_X	$\#C$
$Bl_5 S_9$ $Bl_1 S_5$	$(1,0)$	-5	T_5	5
$Bl_{4,1} S_9$	$(2,2)$	4	T_4	5
$Bl_{3,2} S_9$ $Bl_{3,1} S_{8,I}$ $Bl_{1,1} S_6$	$(2,0)$	6	M_6	3
$Bl_{3,1^2} S_9$ $Bl_{3,1} S_{8,II}$	$(4,4)$	-3	T_3	2
$Bl_{2^2,1} S_9$ $Bl_{2,1^2} S_{8,I}$	$(4,2)$	-4	C_2 or C_2^2	2
$Bl_{2,1^3} S_9$ $Bl_{1^4} S_{8,I}$ $Bl_{2,1^2} S_{8,II}$	$(8,6)$	2	C_2	1
$Bl_{1^5} S_9$ $Bl_{1^4} S_{8,II}$	$(16,10)$	-1	1	1
$Bl_4 S_{8,I}$	$(0,0)$	-8	T_4 or $C_2 \times T_4$	11
$Bl_{2^2} S_{8,I}$	$(0,2)$	8	C_2, C_2^2 or C_2^3	5
$Bl_4 S_{8,II}$	$(0,4)$	4	T_4	5
$Bl_{2^2} S_{8,II}$	$(0,6)$	-4	C_2 or C_2^2	2
$Bl_2 S_6$	$(0,0)$	-12	M_6 or $M_6 \times C_2$	7

A.1.7. *Degree 3.* 137 relatively minimal classes, 213 non-relatively minimal classes; 350 classes in total.

TABLE 22. Relatively minimal del Pezzo surfaces of degree 3

Type	$L_X = C_X$	I_X	G_X	$\#C$
S_3	0	3	$M_3 \leq W(E_6)$	137

————————————————————

TABLE 23. Non-relatively minimal del Pezzo surfaces of degree 3

Type	$L_X = C_X$	I_X	G_X	#C
$Bl_6\, S_9$	0	−6	T_6	16
$Bl_{5,1}\, S_9$ $Bl_5\, S_{8,II}$ $Bl_{1^2}\, S_5$	2	5	T_5	5
$Bl_{4,2}\, S_9$ $Bl_{4,1}\, S_{8,I}$	1	8	T_4 or $C_2 \times T_4$	11
$Bl_{4,1^2}\, S_9$ $Bl_{4,1}\, S_{8,II}$	5	−4	T_4	5
$Bl_{3^2}\, S_9$	0	9	T_3, $T_3 \times C_3$ or $C_3 \ltimes \mathfrak{S}_3$	6
$Bl_{3,2,1}\, S_9$ $Bl_{3,1^2}\, S_{8,I}$ $Bl_{3,2}\, S_{8,II}$ $Bl_{1^3}\, S_6$	3	−6	M_6	3
$Bl_{3,1^3}\, S_9$ $Bl_{3,1^2}\, S_{8,II}$	9	3	T_3	2
$Bl_{2^3}\, S_9$ $Bl_{2^2,1}\, S_{8,I}$	3	−8	C_2, C_2^2 or C_2^3	4
$Bl_{2^2,1^2}\, S_9$ $Bl_{2,1^3}\, S_{8,I}$ $Bl_{2^2,1}\, S_{8,II}$	7	4	C_2 or C_2^2	2
$Bl_{2,1^4}\, S_9$ $Bl_{1^5}\, S_{8,I}$ $Bl_{2,1^3}\, S_{8,II}$	15	−2	C_2	1
$Bl_{1^6}\, S_9$ $Bl_{1^5}\, S_{8,II}$	27	1	1	1
$Bl_5\, S_{8,I}$ $Bl_2\, S_5$	0	−10	T_5 or $T_5 \times C_2$ (except C_5)	8
$Bl_{3,2}\, S_{8,I}$ $Bl_{2,1}\, S_6$	1	12	M_6 or $M_6 \times C_2$	7
$Bl_3\, S_6$	0	−18	M_6, $M_6 \times C_3$ or $M_6 \times \mathfrak{S}_3$	11
$Bl_1\, S_{4,I}$	1	−4	$M_{4,I}$	98
$Bl_1\, S_{4,II}$	3	4	$M_{4,II}$	33

A.1.8. *Degree 2.* 6877 relatively minimal classes, 1197 non-relatively minimal classes; 8074 classes in total.

TABLE 24. Relatively minimal del Pezzo surfaces of degree 2

Type	(L_X, C_X)	I_X	G_X	#C
$S_{2,I}$	(0, 0)	2	$M_{2,I} \le W(E_7)$	5565
$S_{2,II}$	(0, 2)	−4	$M_{2,II} \le W(D_7)$	1312

Number of relatively minimal classes: 6877.

TABLE 25. Non-relatively minimal del Pezzo surfaces of degree 2

Type	(L_X, C_X)	I_X	G_X	$\#C$
$\mathrm{Bl}_7 S_9$	$(0,0)$	-7	T_7	7
$\mathrm{Bl}_{6,1} S_9$	$(2,2)$	6	T_6	16
$\mathrm{Bl}_{5,2} S_9$ $\mathrm{Bl}_{5,1} S_{8,I}$ $\mathrm{Bl}_{2,1} S_5$	$(2,0)$	10	T_5 or $T_5 \times C_2$ (except C_5)	8
$\mathrm{Bl}_{5,1^2} S_9$ $\mathrm{Bl}_{5,1} S_{8,II}$ $\mathrm{Bl}_{1^3} S_5$	$(6,6)$	-5	T_5	5
$\mathrm{Bl}_{4,3} S_9$	$(0,2)$	12	$T_4 \times T_3,$ $T_4 \times_{C_3} T_3$ or $T_4 \times_{\mathfrak{S}_3} T_3$	18
$\mathrm{Bl}_{4,2,1} S_9$ $\mathrm{Bl}_{4,1^2} S_{8,I}$	$(4,4)$	-8	T_4 or $T_4 \times C_2$	11
$\mathrm{Bl}_{4,1^3} S_9$ $\mathrm{Bl}_{4,1^2} S_{8,II}$	$(12,14)$	4	T_4	5
$\mathrm{Bl}_{3^2,1} S_9$ $\mathrm{Bl}_{3^2} S_{8,II}$	$(2,6)$	-9	$T_3, T_3 \times C_2$ or $C_3 \ltimes \mathfrak{S}_3$	6
$\mathrm{Bl}_{3,2^2} S_9$ $\mathrm{Bl}_{3,2,1} S_{8,I}$ $\mathrm{Bl}_{2,1^2} S_6$	$(4,2)$	-12	M_6 or $M_6 \times C_2$	7
$\mathrm{Bl}_{3,2,1^2} S_9$ $\mathrm{Bl}_{3,1^3} S_{8,I}$ $\mathrm{Bl}_{3,2,1} S_{8,II}$ $\mathrm{Bl}_{1^4} S_6$	$(8,12)$	6	M_6	3
$\mathrm{Bl}_{3,1^4} S_9$ $\mathrm{Bl}_{3,1^3} S_{8,II}$	$(20,30)$	-3	T_3	2
$\mathrm{Bl}_{2^3,1} S_9$ $\mathrm{Bl}_{2^2,1^2} S_{8,I}$	$(8,8)$	8	C_2, C_2^2 or C_2^3	4
$\mathrm{Bl}_{2^2,1^3} S_9$ $\mathrm{Bl}_{2,1^4} S_{8,I}$ $\mathrm{Bl}_{2^2,1^2} S_{8,II}$	$(16,26)$	-4	C_2 or C_2^2	2
$\mathrm{Bl}_{2,1^5} S_9$ $\mathrm{Bl}_{1^6} S_{8,I}$ $\mathrm{Bl}_{2,1^4} S_{8,II}$	$(32,60)$	2	C_2	1
$\mathrm{Bl}_{1^7} S_9$ $\mathrm{Bl}_{1^6} S_{8,II}$	$(56,126)$	-1	1	1
$\mathrm{Bl}_6 S_{8,I}$	$(0,0)$	-12	$T_6 \times C_2$ or $T_6 \times_{C_2} C_2$	36
$\mathrm{Bl}_6 S_{8,II}$	$(0,6)$	6	T_6	16
$\mathrm{Bl}_{4,2} S_{8,I}$	$(0,2)$	16	$T_4, C_2 \times T_4$ or $(T_4 \times_{C_2} C_2) \times_{C_2} C_2$	41
$\mathrm{Bl}_{4,2} S_{8,II}$	$(0,12)$	-8	$T_4 \times C_2$ or $T_4 \times_{C_2} C_2$	11

$\mathrm{Bl}_{3^2}\, S_{8,I}$ $\mathrm{Bl}_{3,1}\, S_6$	$(2,0)$	18	M_6, $M_6 \times C_3$ or $M_6 \times \mathfrak{S}_3$	11
$\mathrm{Bl}_{2^3}\, S_{8,I}$	$(0,6)$	-16	C_2, C_2^2, C_2^3 or C_2^4	12
$\mathrm{Bl}_{2^3}\, S_{8,II}$	$(0,24)$	8	C_2, C_2^2 or C_2^3	4
$\mathrm{Bl}_4\, S_6$	$(0,0)$	-24	$T_4 \times M_6$ or $T_4 \times_{Q_6} M_6$, $Q_6 \in \{C_6, \mathfrak{S}_3, \mathfrak{D}_6\}$	55
$\mathrm{Bl}_{2^2}\, S_6$	$(0,0)$	24	C_6, $\mathfrak{S}_3$, $\mathfrak{D}_6$, $C_6 \times C_2$, $\mathfrak{D}_6 \times C_2$, $C_2^2 \times C_6$, $\mathfrak{D}_6 \times C_2^2$	16
$\mathrm{Bl}_3\, S_5$	$(0,0)$	-15	$T_5 \times T_3$, $T_5 \times_{C_3} T_3$ or $T_5 \times_{\mathfrak{S}_3} \mathfrak{S}_3$	13
$\mathrm{Bl}_2\, S_{4,I}$	$(0,0)$	-8	$M_{4,I} \times C_2$ or $M_{4,I} \times_{C_2} C_2$	435
$\mathrm{Bl}_2\, S_{4,II}$	$(0,4)$	8	$M_{4,II} \times C_2$ or $M_{4,II} \times_{C_2} C_2$	183
$\mathrm{Bl}_{1^2}\, S_{4,I}$	$(4,2)$	4	$M_{4,I}$	98
$\mathrm{Bl}_{1^2}\, S_{4,II}$	$(8,6)$	-4	$M_{4,II}$	33
$\mathrm{Bl}_1\, S_{3,M_3}$	$(2,0)$	-3	M_3	137

A.1.9. *Degree 1.* 53016 relatively minimal classes, 9076 non-relatively minimal classes; 62092 classes in total.

TABLE 26. Relatively minimal del Pezzo surfaces of degree 1

Type	(L_X, C_X)	I_X	G_X	$\#C$
$S_{1,I}$	$(0,0)$	1	$M_{1,I} \le W(E_8)$	48797
$S_{1,II}$	$(0,2)$	-4	$M_{1,II} \le W(D_8)$	4219

Number of relatively minimal classes: 53016.

TABLE 27. Non-relatively minimal del Pezzo surfaces of degree 1

Type	(L_X, C_X)	I_X	G_X	$\#C$
$\mathrm{Bl}_8\, S_9$	$(0,0)$	-8	T_8	50
$\mathrm{Bl}_{7,1}\, S_9$ $\mathrm{Bl}_7\, S_{8,II}$	$(2,4)$	7	T_7	7
$\mathrm{Bl}_{6,2}\, S_9$ $\mathrm{Bl}_{6,1}\, S_{8,I}$	$(2,0)$	12	$T_6 \times C_2$ or $T_6 \times_{C_2} C_2$	36
$\mathrm{Bl}_{6,1^2}\, S_9$ $\mathrm{Bl}_{6,1}\, S_{8,II}$	$(8,12)$	-6	T_6	16
$\mathrm{Bl}_{5,3}\, S_9$ $\mathrm{Bl}_{3,1}\, S_5$	$(2,0)$	15	$T_5 \times T_3$, $T_5 \times_{C_3} T_3$ or $T_5 \times_{\mathfrak{S}_3} \mathfrak{S}_3$	13

$\mathrm{Bl}_{5,2,1}\, S_9$ $\mathrm{Bl}_{5,1^2}\, S_{8,I}$ $\mathrm{Bl}_{2,1^2}\, S_5$	$(6,6)$	-10	T_5 or $T_5 \times C_2$ (except C_5)	8
$\mathrm{Bl}_{5,1^3}\, S_9$ $\mathrm{Bl}_{5,1^2}\, S_{8,II}$ $\mathrm{Bl}_{1^4}\, S_5$	$(20,30)$	5	T_5	5
$\mathrm{Bl}_{4^2}\, S_9$	$(0,4)$	16	$T_4 \times_{Q_4} T_4'$, $A_4, T_4' \in \{C_2^2, C_4, \mathfrak{D}_8, \mathfrak{A}_4, \mathfrak{S}_4\}$	50
$\mathrm{Bl}_{4,3,1}\, S_9$ $\mathrm{Bl}_{4,3}\, S_{8,II}$	$(4,12)$	-12	$T_4 \times T_3$, $T_4 \times_{C_3} T_3$ or $T_4 \times_{\mathfrak{S}_3} \mathfrak{S}_3$	18
$\mathrm{Bl}_{4,2^2}\, S_9$	$(4,6)$	-16	$T_4, C_2 \times T_4$ or $(T_4 \times_{C_2} C_2) \times_{C_2} C_2$	32
$\mathrm{Bl}_{4,2,1^2}\, S_9$ $\mathrm{Bl}_{4,1^3}\, S_{8,I}$	$(14,30)$	8	T_4 or $T_4 \times C_2$	11
$\mathrm{Bl}_{4,1^4}\, S_9$ $\mathrm{Bl}_{4,1^3}\, S_{8,II}$	$(40,90)$	-4	T_4	5
$\mathrm{Bl}_{3^2,2}\, S_9$ $\mathrm{Bl}_{3^2,1}\, S_{8,I}$ $\mathrm{Bl}_{3,1^2}\, S_6$	$(6,0)$	-18	$M_6, M_6 \times C_3$ or $M_6 \times \mathfrak{S}_3$	11
$\mathrm{Bl}_{3^2,1^2}\, S_9$ $\mathrm{Bl}_{3^2,1}\, S_{8,II}$	$(12,36)$	9	$T_3, T_3 \times C_2$ or $C_3 \ltimes \mathfrak{S}_3$	6
$\mathrm{Bl}_{3,2^2,1}\, S_9$ $\mathrm{Bl}_{3,2,1^2}\, S_{8,I}$ $\mathrm{Bl}_{2,1^3}\, S_6$	$(12,18)$	12	M_6 or $M_6 \times C_2$	7
$\mathrm{Bl}_{3,2,1^3}\, S_9$ $\mathrm{Bl}_{3,1^4}\, S_{8,I}$ $\mathrm{Bl}_{3,2,1^2}\, S_{8,II}$ $\mathrm{Bl}_{1^5}\, S_6$	$(30,90)$	-6	M_6	3
$\mathrm{Bl}_{3,1^5}\, S_9$ $\mathrm{Bl}_{3,1^4}\, S_{8,II}$	$(72,270)$	3	T_3	2
$\mathrm{Bl}_{2^4}\, S_9$ $\mathrm{Bl}_{2^3,1}\, S_{8,I}$	$(8,24)$	16	C_2, C_2^2, C_2^3 or C_2^4	8
$\mathrm{Bl}_{2^3,1^2}\, S_9$ $\mathrm{Bl}_{2^2,1^3}\, S_{8,I}$ $\mathrm{Bl}_{2^3,1}\, S_{8,II}$	$(26,72)$	-8	C_2, C_2^2 or C_2^3	4
$\mathrm{Bl}_{2^2,1^4}\, S_9$ $\mathrm{Bl}_{2,1^5}\, S_{8,I}$ $\mathrm{Bl}_{2^2,1^3}\, S_{8,II}$	$(60,252)$	4	C_2 or C_2^2	2
$\mathrm{Bl}_{2,1^6}\, S_9$ $\mathrm{Bl}_{1^7}\, S_{8,I}$ $\mathrm{Bl}_{2,1^5}\, S_{8,II}$	$(126,756)$	-2	C_2	1
$\mathrm{Bl}_{1^8}\, S_9$ $\mathrm{Bl}_{1^7}\, S_{8,II}$	$(240,2160)$	1	1	1
$\mathrm{Bl}_7\, S_{8,I}$	$(0,0)$	-14	$T_7 \times C_2$ or $T_7 \times_{C_2} C_2$	10
$\mathrm{Bl}_{5,2}\, S_{8,I}$ $\mathrm{Bl}_{2^2}\, S_5$	$(0,2)$	20	$G \times C_2$ or $G \times_{C_2} C_2$, $G = T_5$ (not C_5) or $G = T_5 \times C_2$	19

$\mathrm{Bl}_{4,3}\, S_{8,I}$ $\mathrm{Bl}_{4,1}\, S_6$	$(2,0)$	24	$T_4 \times M_6$ or $T_4 \times_{Q_6} M_6$, $Q_6 \in \{C_6, \mathfrak{S}_3, \mathfrak{D}_6\}$	55
$\mathrm{Bl}_{3,2^2}\, S_{8,I}$ $\mathrm{Bl}_{2^2,1}\, S_6$	$(2,6)$	-24	$C_6, \mathfrak{S}_3, \mathfrak{D}_6,$ $C_6 \times C_2, \mathfrak{D}_6 \times C_2,$ $C_2^2 \times C_6, \mathfrak{D}_6 \times C_2^2$	16
$\mathrm{Bl}_5\, S_6$	$(0,0)$	-30	$T_5 \times M_6$ or $T_5 \times_{Q_6} M_6$, $Q_6 \in \{C_6, \mathfrak{S}_3, \mathfrak{D}_6\}$	30
$\mathrm{Bl}_{3,2}\, S_6$	$(0,0)$	36	$G \times C_2$ or $G \times_{C_2} C_2$, $G \in \{M_6, M_6 \times C_3, M_6 \times \mathfrak{S}_3\}$	29
$\mathrm{Bl}_4\, S_5$	$(0,0)$	-20	$T_4 \times_{Q_4} T_5$, $q_4 \in \{C_2^2, C_4, \mathfrak{D}_8, \mathfrak{A}_4, \mathfrak{S}_4\}$	44
$\mathrm{Bl}_3\, S_{4,I}$	$(0,0)$	-12	$M_{4,I} \times T_3,$ $M_{4,I} \times_{C_3} T_3$ or $M_{4,I} \times_{\mathfrak{S}_3} \mathfrak{S}_3$	568
$\mathrm{Bl}_{2,1}\, S_{4,I}$	$(2,2)$	8	$M_{4,I} \times C_2$ or $M_{4,I} \times_{C_2} C_2$	435
$\mathrm{Bl}_{1^3}\, S_{4,I}$	$(12,6)$	-4	$M_{4,I}$	98
$\mathrm{Bl}_3\, S_{4,II}$	$(0,6)$	12	$M_{4,II} \times T_3,$ $M_{4,II} \times_{C_3} T_3$ or $M_{4,II} \times_{\mathfrak{S}_3} \mathfrak{S}_3$	150
$\mathrm{Bl}_{2,1}\, S_{4,II}$	$(6,12)$	-8	$M_{4,II} \times C_2$ or $M_{4,II} \times_{C_2} C_2$	108
$\mathrm{Bl}_{1^3}\, S_{4,II}$	$(24,24)$	4	$M_{4,II}$	33
$\mathrm{Bl}_2\, S_{3,M_3}$	$(0,0)$	-6	$M_3 \times C_2$ or $M_3 \times_{C_2} C_2$	391
$\mathrm{Bl}_{1^2}\, S_{3,M_3}$	$(6,0)$	3	M_3	137
$\mathrm{Bl}_1\, S_{2,I}$	$(2,0)$	-2	$M_{2,I}$	5565
$\mathrm{Bl}_1\, S_{2,II}$	$(4,4)$	4	$M_{2,II}$	1092

A.2. **Finite fields.** In this section, we provide tables detailing the possible Galois actions on del Pezzo surfaces in each degree, together with associated properties.

Denoting by X a del Pezzo surface over the finite field $\mathbb{F}_q$:

- The "Class" column gives the conjugacy class in the Weyl group. For degree $d \geq 4$, we use our own numbering. For degree 3, we use Manin's numbering from [Man86, Table 1]. For degrees 2 and 1, we use Urabe's numbering from [Ura96].

- The "Type" column gives all descriptions as blowups of minimal surfaces.

- The "(L_X, C_X)" column lists the line and conic counts respectively.

- The "I_X" column lists the intersection invariant.

- The "Order" column gives the order n of the cyclic splitting group C_n.

- The "$a(X)$" column gives the trace of Frobenius.

- The "Restrictions" column gives a necessary and sufficient condition on q for existence coming from either [BFL19, §5.2] or Section 5.4.

Lastly, in tables for degrees 1 and 2, we add columns giving the class and blowup type of the Bertini/Geiser twist respectively, leaving these entries blank when the class is self-dual.

A.2.1. *Degree 9.* 1 relatively minimal class, 0 non-relatively minimal classes, 1 class in total.

TABLE 28. Relatively minimal del Pezzo surfaces of degree 9

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	S_9	$(0,0)$	1	1	1	All q

A.2.2. *Degree 8.* 2 relatively minimal classes, 1 non-relatively minimal class, 3 classes in total.

TABLE 29. Relatively minimal del Pezzo surfaces of degree 8

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	$S_{8,I}$	$(0,0)$	2	2	0	All q
2	$S_{8,II}$	$(0,2)$	-1	1	2	All q

TABLE 30. Non-relatively minimal del Pezzo surfaces of degree 8

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
3	$\mathrm{Bl}_1 S_9$	$(1,1)$	-1	1	1	All q

A.2.3. *Degree 7.* 0 relatively minimal classes, 2 non-relatively minimal classes, 2 classes in total.

TABLE 31. Non-relatively minimal del Pezzo surfaces of degree 7

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	$\mathrm{Bl}_1 S_{8,I}$, $\mathrm{Bl}_2 S_9$	$(1,0)$	-2	2	1	All q
2	$\mathrm{Bl}_1 S_{8,II}$, $\mathrm{Bl}_{1,1} S_9$	$(3,2)$	1	1	3	All q

A.2.4. *Degree 6.* 1 relatively minimal class, 5 non-relatively minimal classes, 6 classes in total.

TABLE 32. Relatively minimal del Pezzo surfaces of degree 6

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	S_6	$(0,0)$	6	6	-1	All q

TABLE 33. Non-relatively minimal del Pezzo surfaces of degree 6

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
2	$\mathrm{Bl}_2 S_{8,I}$	$(0,1)$	-4	2	0	All q
3	$\mathrm{Bl}_2 S_{8,II}$	$(0,3)$	2	2	2	All q
4	$\mathrm{Bl}_3 S_9$	$(0,0)$	-3	3	1	All q

5	$\text{Bl}_{2,1}\, S_9$ $\text{Bl}_{1^2}\, S_{8,I}$	$(2,1)$	2	2	2	All q
6	$\text{Bl}_{1^3}\, S_9$ $\text{Bl}_{1^2}\, S_{8,II}$	$(6,3)$	-1	1	4	All q

A.2.5. *Degree 5.* 1 relatively minimal class, 6 non-relatively minimal classes, 7 classes in total.

TABLE 34. Relatively minimal del Pezzo surfaces of degree 5

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	S_5	$(0,0)$	5	5	0	All q

TABLE 35. Non-relatively minimal del Pezzo surfaces of degree 5

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
2	$\text{Bl}_1\, S_6$ $\text{Bl}_3\, S_{8,I}$	$(1,0)$	-6	6	0	All q
3	$\text{Bl}_4\, S_9$	$(0,1)$	-4	4	1	All q
4	$\text{Bl}_{3,1}\, S_9$ $\text{Bl}_3\, S_{8,II}$	$(1,2)$	3	3	2	All q
5	$\text{Bl}_{2^2}\, S_9$ $\text{Bl}_{2,1}\, S_{8,I}$	$(2,1)$	4	2	1	All q
6	$\text{Bl}_{2,1^2}\, S_9$ $\text{Bl}_{1^3}\, S_{8,I}$ $\text{Bl}_{2,1}\, S_{8,II}$	$(4,3)$	-2	2	3	All q
7	$\text{Bl}_{1^4}\, S_9$ $\text{Bl}_{1^3}\, S_{8,II}$	$(10,5)$	1	1	5	All q

A.2.6. *Degree 4.* 6 relatively minimal classes, 12 non-relatively minimal classes, 18 classes in total.

TABLE 36. Relatively minimal del Pezzo surfaces of degree 4

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	$S_{4,I}^{(4,1)}$	$(0,0)$	4	8	0	All q
2	$S_{4,I}^{(3,2)}$	$(0,0)$	4	12	1	All q
3	$S_{4,I}^{(2,1^3)}$	$(0,0)$	4	4	-2	All q
4	$S_{4,II}^{(3,1)}$	$(0,2)$	-4	6	1	All q
5	$S_{4,II}^{(2^2)}$	$(0,2)$	-4	4	2	$q \geq 3$
6	$S_{4,II}^{(1^4)}$	$(0,2)$	-4	2	-2	$q \geq 4$

TABLE 37. Non-relatively minimal del Pezzo surfaces of degree 4

Class	Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
7	$\text{Bl}_2\, S_6$	$(0,0)$	-12	6	-1	All q
8	$\text{Bl}_4\, S_{8,I}$	$(0,0)$	-8	4	0	$q \geq 3$

9	$\mathrm{Bl}_{2^2}\, S_{8,I}$	$(0,2)$	8	2	0	All q
10	$\mathrm{Bl}_4\, S_{8,II}$	$(0,4)$	4	4	2	All q
11	$\mathrm{Bl}_{2^2}\, S_{8,II}$	$(0,6)$	-4	2	2	$q \geq 4$
12	$\mathrm{Bl}_5\, S_9$ $\mathrm{Bl}_1\, S_5$	$(1,0)$	-5	5	1	All q
13	$\mathrm{Bl}_{4,1}\, S_9$	$(2,2)$	4	4	2	All q
14	$\mathrm{Bl}_{3,2}\, S_9$ $\mathrm{Bl}_{3,1}\, S_{8,I}$ $\mathrm{Bl}_{1,1}\, S_{6,M_6}$	$(2,0)$	6	6	1	All q
15	$\mathrm{Bl}_{3,1^2}\, S_9$ $\mathrm{Bl}_{3,1}\, S_{8,II}$	$(4,4)$	-3	3	3	All q
16	$\mathrm{Bl}_{2^2,1}\, S_9$ $\mathrm{Bl}_{2,1^2}\, S_{8,I}$	$(4,2)$	-4	2	2	$q \geq 3$
17	$\mathrm{Bl}_{2,1^3}\, S_9$ $\mathrm{Bl}_{1^4}\, S_{8,I}$ $\mathrm{Bl}_{2,1^2}\, S_{8,II}$	$(8,6)$	2	2	4	All q
18	$\mathrm{Bl}_{1^5}\, S_9$ $\mathrm{Bl}_{1^4}\, S_{8,II}$	$(16,10)$	-1	1	6	$q \geq 4$

A.2.7. *Degree 3.* 5 relatively minimal classes, 20 non-relatively minimal classes, 25 classes in total.

TABLE 38. Relatively minimal del Pezzo surfaces of degree 3

Class	Type	$L_X = C_X$	I_X	Order	$a(X)$	Restrictions
1	S_3^1	0	3	12	0	All q
2	S_3^2	0	3	6	2	All q
3	S_3^3	0	3	3	-2	All q
4	S_3^4	0	3	9	1	All q
5	S_3^5	0	3	6	-1	$q \geq 3$

TABLE 39. Non-relatively minimal del Pezzo surfaces of degree 3

Class	Type	$L_X = C_X$	I_X	Order	$a(X)$	Restrictions
6	$\mathrm{Bl}_1\, S_{4,I}^{(3,2)}$	1	-4	12	2	All q
7	$\mathrm{Bl}_1\, S_{4,I}^{(4,1)}$	1	-4	8	1	All q
8	$\mathrm{Bl}_1\, S_{4,II}^{(3,1)}$	3	4	6	2	All q
9	$\mathrm{Bl}_1\, S_{4,I}^{(2,1^3)}$	1	-4	4	-1	All q
10	$\mathrm{Bl}_1\, S_{4,II}^{(2^2)}$	3	4	4	3	$q \geq 3$
11	$\mathrm{Bl}_1\, S_{4,II}^{(1^4)}$	3	4	2	-1	$q \geq 4$
12	$\mathrm{Bl}_5\, S_{8,I}$ $\mathrm{Bl}_2\, S_5$	0	-10	10	0	All q
13	$\mathrm{Bl}_3\, S_6$	0	-18	6	-1	All q
14	$\mathrm{Bl}_{3,2}\, S_{8,I}$ $\mathrm{Bl}_{2,1}\, S_6$	1	12	6	0	All q
15	$\mathrm{Bl}_6\, S_9$	0	-6	6	1	All q

16	$\mathrm{Bl}_{5,1}\,S_9$ $\mathrm{Bl}_5\,S_{8,II}$ $\mathrm{Bl}_{1^2}\,S_5$	2	5	5	2	All q
17	$\mathrm{Bl}_{4,2}\,S_9$ $\mathrm{Bl}_{4,1}\,S_{8,I}$	1	8	4	1	$q \geq 3$
18	$\mathrm{Bl}_{3^2}\,S_9$	0	9	3	1	$q \geq 3$
19	$\mathrm{Bl}_{4,1^2}\,S_9$ $\mathrm{Bl}_{4,1}\,S_{8,II}$	5	-4	4	3	All q
20	$\mathrm{Bl}_{3,2,1}\,S_9$ $\mathrm{Bl}_{3,1^2}\,S_{8,I}$ $\mathrm{Bl}_{3,2}\,S_{8,II}$ $\mathrm{Bl}_{1^3}\,S_6$	3	-6	6	2	All q
21	$\mathrm{Bl}_{2^3}\,S_9$ $\mathrm{Bl}_{2^2,1}\,S_{8,I}$	3	-8	2	1	All q
22	$\mathrm{Bl}_{3,1^3}\,S_9$ $\mathrm{Bl}_{3,1^2}\,S_{8,II}$	9	3	3	4	All q
23	$\mathrm{Bl}_{2^2,1^2}\,S_9$ $\mathrm{Bl}_{2,1^3}\,S_{8,I}$ $\mathrm{Bl}_{2^2,1}\,S_{8,II}$	7	4	2	3	$q \geq 4$
24	$\mathrm{Bl}_{2,1^4}\,S_9$ $\mathrm{Bl}_{1^5}\,S_{8,I}$ $\mathrm{Bl}_{2,1^3}\,S_{8,II}$	15	-2	2	5	All q
25	$\mathrm{Bl}_{1^6}\,S_9$ $\mathrm{Bl}_{1^5}\,S_{8,II}$	27	1	1	7	$q \geq 7$

A.2.8. *Degree 2*. 18 relatively minimal classes, 42 non-relatively minimal classes, 60 classes in total.

TABLE 40. Relatively minimal del Pezzo surfaces of degree 2

Class	Type	Geiser Class	Geiser Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
2	$S_{2,II}^{(1^6)}$	47	$\mathrm{Bl}_{2,1^5}\,S_9$ $\mathrm{Bl}_{1^6}\,S_{8,I}$ $\mathrm{Bl}_{2,1^4}\,S_{8,II}$	$(0,2)$	-4	2	-4	$q \geq 5$
3	$S_{2,II}^{(2^2,1^2)}$	31	$\mathrm{Bl}_2\,S_{4,II}^{(2^2)}$	$(0,2)$	-4	4	0	$q \geq 3$
4	$S_{2,II}^{(3,1^3)}$	51	$\mathrm{Bl}_{3,2,1^2}\,S_9$ $\mathrm{Bl}_{3,1^3}\,S_{8,I}$ $\mathrm{Bl}_{3,2,1}\,S_{8,II}$ $\mathrm{Bl}_{1^4}\,S_6$	$(0,2)$	-4	6	-1	$q \geq 3$
5	$S_{2,II}^{(5,1)}$	58	$\mathrm{Bl}_{5,2}\,S_9$ $\mathrm{Bl}_{5,1}\,S_{8,I}$ $\mathrm{Bl}_{2,1}\,S_5$	$(0,2)$	-4	10	1	All q
6	$S_{2,II}^{(4,2)}$	11	$S_{2,I}^4$	$(0,2)$	-4	8	2	All q
7	$S_{2,II}^{(3,3)}$	42	$\mathrm{Bl}_{3^2}\,S_{8,I}$ $\mathrm{Bl}_{3,1}\,S_6$	$(0,2)$	-4	6	2	All q
8	$S_{2,I}^1$	46	$\mathrm{Bl}_{1^7}\,S_9$ $\mathrm{Bl}_{1^6}\,S_{8,II}$	$(0,0)$	2	2	-6	$q \geq 9$

9	$S_{2,I}^2$	27	$\mathrm{Bl}_{1^2} S_{4,II}^{(2^2)}$	$(0,0)$	2	4	-2	$q \geq 3$
10	$S_{2,I}^3$	24	$\mathrm{Bl}_1 S_3^2$	$(0,0)$	2	6	-1	All q
11	$S_{2,I}^4$	6	$S_{2,II}^{(4,2)}$	$(0,0)$	2	8	0	All q
12	$S_{2,I}^5$	49	$\mathrm{Bl}_{3,1^4} S_9$ $\mathrm{Bl}_{3,1^3} S_{8,II}$	$(0,0)$	2	6	-3	$q \geq 3$
13	$S_{2,I}^6$	56	$\mathrm{Bl}_{5,1^2} S_9$ $\mathrm{Bl}_{5,1} S_{8,II}$ $\mathrm{Bl}_{1^3} S_5$	$(0,0)$	2	10	-1	All q
14	$S_{2,I}^7$	54	$\mathrm{Bl}_{3^2,1} S_9$ $\mathrm{Bl}_{3^2} S_{8,II}$	$(0,0)$	2	6	0	$q \geq 3$
15	$S_{2,I}^8$	23	$\mathrm{Bl}_1 S_3^4$	$(0,0)$	2	18	0	All q
16	$S_{2,I}^9$	60	$\mathrm{Bl}_7 S_9$	$(0,0)$	2	14	1	All q
17	$S_{2,I}^{10}$	22	$\mathrm{Bl}_1 S_3^1$	$(0,0)$	2	12	1	All q
18	$S_{2,I}^{11}$	37	$\mathrm{Bl}_3 S_5$	$(0,0)$	2	30	2	All q
19	$S_{2,I}^{12}$	20	$\mathrm{Bl}_1 S_3^3$	$(0,0)$	2	6	3	$q \geq 3$

TABLE 41. Non-relatively minimal del Pezzo surfaces of degree 2

Class	Type	Geiser Class	Geiser Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1 [3]	$\mathrm{Bl}_{2^3} S_{8,I}$	50	$\mathrm{Bl}_{2^3,1} S_9$ $\mathrm{Bl}_{2^2,1^2} S_{8,I}$	$(0,6)$	-16	2	0	$q \geq 3$
20	$\mathrm{Bl}_1 S_3^3$	19	$S_{2,I}^{12}$	$(2,0)$	-3	3	-1	$q \geq 3$
21	$\mathrm{Bl}_1 S_3^5$	45	$\mathrm{Bl}_6 S_{8,II}$	$(2,0)$	-3	6	0	$q \geq 3$
22	$\mathrm{Bl}_1 S_3^1$	17	$S_{2,I}^{10}$	$(2,0)$	-3	12	1	All q
23	$\mathrm{Bl}_1 S_3^4$	15	$S_{2,I}^8$	$(2,0)$	-3	9	2	All q
24	$\mathrm{Bl}_1 S_3^2$	10	$S_{2,I}^3$	$(2,0)$	-3	6	3	All q
25	$\mathrm{Bl}_{1^2} S_{4,II}^{(1^4)}$	40	$\mathrm{Bl}_{2^3} S_{8,II}$	$(8,6)$	-4	2	0	$q \geq 7$
26	$\mathrm{Bl}_{1^2} S_{4,II}^{(3,1)}$	38	$\mathrm{Bl}_{2^2} S_6$	$(8,6)$	-4	6	3	All q
27	$\mathrm{Bl}_{1^2} S_{4,II}^{(2^2)}$	9	$S_{2,I}^2$	$(8,6)$	-4	4	4	$q \geq 3$
28	$\mathrm{Bl}_2 S_{4,II}^{(1^4)}$	48	$\mathrm{Bl}_{2^2,1^3} S_9$ $\mathrm{Bl}_{2,1^4} S_{8,I}$ $\mathrm{Bl}_{2^2,1^2} S_{8,II}$	$(0,4)$	8	2	-2	$q \geq 5$
29	$\mathrm{Bl}_{1^2} S_{4,I}^{(2,1^3)}$	41	$\mathrm{Bl}_{4,2} S_{8,II}$	$(4,2)$	4	4	0	$q \geq 3$
30	$\mathrm{Bl}_2 S_{4,II}^{(3,1)}$	53	$\mathrm{Bl}_{3,2^2} S_9$ $\mathrm{Bl}_{3,2,1} S_{8,I}$ $\mathrm{Bl}_{2,1^2} S_6$	$(0,4)$	8	6	1	All q
31	$\mathrm{Bl}_2 S_{4,II}^{(2^2)}$	3	$S_{2,II}^{(2^2,1^2)}$	$(0,4)$	8	4	2	$q \geq 3$
32	$\mathrm{Bl}_{1^2} S_{4,I}^{(4,1)}$	35	$\mathrm{Bl}_2 S_{4,I}^{(4,1)}$	$(4,2)$	4	8	2	All q
33	$\mathrm{Bl}_{1^2} S_{4,I}^{(3,2)}$	39	$\mathrm{Bl}_4 S_6$	$(4,2)$	4	12	3	All q
34	$\mathrm{Bl}_2 S_{4,I}^{(2,1^3)}$	52	$\mathrm{Bl}_{4,1^3} S_9$ $\mathrm{Bl}_{4,1^2} S_{8,II}$	$(0,0)$	-8	4	-2	$q \geq 3$
35	$\mathrm{Bl}_2 S_{4,I}^{(4,1)}$	32	$\mathrm{Bl}_{1^2} S_{4,I}^{(4,1)}$	$(0,0)$	-8	8	0	All q

---

[3]This class was incorrectly identified by Urabe [Ura96] as corresponding to a minimal surface, hence the numbering.

36	$\mathrm{Bl}_2\, S_{4,I}^{(3,2)}$	57	$\mathrm{Bl}_{4,3}\, S_9$	$(0,0)$	-8	12	1	All q
37	$\mathrm{Bl}_3\, S_5$	18	$S_{2,I}^{11}$	$(0,0)$	-15	15	0	All q
38	$\mathrm{Bl}_{2^2}\, S_6$	26	$\mathrm{Bl}_{1^2}\, S_{4,II}^{(3,1)}$	$(0,0)$	-24	6	-1	All q
39	$\mathrm{Bl}_4\, S_6$	33	$\mathrm{Bl}_{1^2}\, S_{4,I}^{(3,2)}$	$(0,0)$	24	12	-1	All q
40	$\mathrm{Bl}_{2^3}\, S_{8,II}$	25	$\mathrm{Bl}_{1^2}\, S_{4,II}^{(1^4)}$	$(0,24)$	8	2	-1	$q \geq 7$
41	$\mathrm{Bl}_{4,2}\, S_{8,II}$	29	$\mathrm{Bl}_{1^2}\, S_{4,I}^{(2,1^3)}$	$(0,12)$	-8	4	-1	$q \geq 3$
42	$\mathrm{Bl}_{3^2}\, S_{8,I}$ $\mathrm{Bl}_{3,1}\, S_6$	7	$S_{2,II}^{(3,3)}$	$(2,0)$	18	6	2	All q
43	$\mathrm{Bl}_{4,2}\, S_{8,I}$	55	$\mathrm{Bl}_{4,2,1}\, S_9$ $\mathrm{Bl}_{4,1^2}\, S_{8,I}$	$(0,2)$	16	4	2	$q \geq 3$
44	$\mathrm{Bl}_6\, S_{8,II}$	59	$\mathrm{Bl}_{6,1}\, S_9$	$(0,6)$	6	6	0	All q
45	$\mathrm{Bl}_6\, S_{8,I}$	21	$\mathrm{Bl}_1\, S_3^5$	$(0,0)$	-12	6	0	$q \geq 3$
46	$\mathrm{Bl}_{1^7}\, S_9$ $\mathrm{Bl}_{1^6}\, S_{8,II}$	8	$S_{2,I}^1$	$(56,126)$	-1	1	8	$q \geq 9$
47	$\mathrm{Bl}_{2,1^5}\, S_9$ $\mathrm{Bl}_{1^6}\, S_{8,I}$ $\mathrm{Bl}_{2,1^4}\, S_{8,II}$	2	$S_{2,II}^{(1^6)}$	$(32,60)$	2	2	6	$q \geq 5$
48	$\mathrm{Bl}_{2^2,1^3}\, S_9$ $\mathrm{Bl}_{2,1^4}\, S_{8,I}$ $\mathrm{Bl}_{2^2,1^2}\, S_{8,II}$	28	$\mathrm{Bl}_2\, S_{4,II}^{(1^4)}$	$(16,26)$	-4	2	4	$q \geq 5$
49	$\mathrm{Bl}_{3,1^4}\, S_9$ $\mathrm{Bl}_{3,1^3}\, S_{8,II}$	12	$S_{2,I}^5$	$(20,30)$	-3	3	5	$q \geq 3$
50	$\mathrm{Bl}_{2^3,1}\, S_9$ $\mathrm{Bl}_{2^2,1^2}\, S_{8,I}$	1	$\mathrm{Bl}_{2^3}\, S_{8,I}$	$(8,8)$	8	2	2	$q \geq 3$
51	$\mathrm{Bl}_{3,2,1^2}\, S_9$ $\mathrm{Bl}_{3,1^3}\, S_{8,I}$ $\mathrm{Bl}_{3,2,1}\, S_{8,II}$ $\mathrm{Bl}_{1^4}\, S_6$	4	$S_{2,II}^{(3,1^3)}$	$(8,12)$	6	6	3	$q \geq 3$
52	$\mathrm{Bl}_{4,1^3}\, S_9$ $\mathrm{Bl}_{4,1^2}\, S_{8,II}$	34	$\mathrm{Bl}_2\, S_{4,I}^{(2,1^3)}$	$(12,14)$	4	4	4	$q \geq 3$
53	$\mathrm{Bl}_{3,2^2}\, S_9$ $\mathrm{Bl}_{3,2,1}\, S_{8,I}$ $\mathrm{Bl}_{2,1^2}\, S_6$	30	$\mathrm{Bl}_2\, S_{4,II}^{(3,1)}$	$(4,2)$	-12	6	1	All q
54	$\mathrm{Bl}_{3^2,1}\, S_9$ $\mathrm{Bl}_{3^2}\, S_{8,II}$	14	$S_{2,I}^7$	$(2,6)$	-9	3	2	$q \geq 3$
55	$\mathrm{Bl}_{4,2,1}\, S_9$ $\mathrm{Bl}_{4,1^2}\, S_{8,I}$	43	$\mathrm{Bl}_{4,2}\, S_{8,I}$	$(4,4)$	-8	4	2	$q \geq 3$
56	$\mathrm{Bl}_{5,1^2}\, S_9$ $\mathrm{Bl}_{5,1}\, S_{8,II}$ $\mathrm{Bl}_{1^3}\, S_5$	13	$S_{2,I}^6$	$(6,6)$	-5	5	3	All q
57	$\mathrm{Bl}_{4,3}\, S_9$	36	$\mathrm{Bl}_2\, S_{4,I}^{(3,2)}$	$(0,2)$	12	12	1	All q
58	$\mathrm{Bl}_{5,2}\, S_9$ $\mathrm{Bl}_{5,1}\, S_{8,I}$ $\mathrm{Bl}_{2,1}\, S_5$	5	$S_{2,II}^{(5,1)}$	$(2,0)$	10	10	1	All q
59	$\mathrm{Bl}_{6,1}\, S_9$	44	$\mathrm{Bl}_6\, S_{8,I}$	$(2,2)$	6	6	2	All q
60	$\mathrm{Bl}_7\, S_9$	16	$S_{2,I}^9$	$(0,0)$	-7	7	1	All q

A.2.9. *Degree 1.* 37 relatively minimal classes, 83 non-relatively minimal classes, 120 classes in total.

TABLE 42. Relatively minimal del Pezzo surfaces of degree 1

Class	Type	Bertini Class	Bertini Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
1	$S_{1,II}^{(2,1^5)}$	97	$\mathrm{Bl}_{4,1^4} S_9$ $\mathrm{Bl}_{4,1^3} S_{8,II}$	$(0,2)$	-4	4	-3	$q \geq 5$
2	$S_{1,II}^{(3,2,1^2)}$	105	$\mathrm{Bl}_{4,3,1} S_9$ $\mathrm{Bl}_{4,3} S_{8,II}$	$(0,2)$	-4	12	0	
3	$S_{1,II}^{(2^3,1)}$	–	–	$(0,2)$	-4	4	1	
4	$S_{1,II}^{(4,1^3)}$	73	$\mathrm{Bl}_{1^3} S_{4,I}^{(4,1)}$	$(0,2)$	-4	8	-1	
5	$S_{1,II}^{(6,1)}$	–	–	$(0,2)$	-4	12	1	
6	$S_{1,II}^{(5,2)}$	84	$\mathrm{Bl}_4 S_5$	$(0,2)$	-4	20	2	
7	$S_{1,II}^{(4,3)}$	82	$\mathrm{Bl}_3 S_{4,I}^{(4,1)}$	$(0,2)$	-4	24	2	
8	$S_{1,I}^1$	91	$\mathrm{Bl}_{1^8} S_9$ $\mathrm{Bl}_{1^7} S_{8,II}$	$(0,0)$	1	2	-7	$q = 16$ or $q \geq 19$
9	$S_{1,I}^2$	37	$S_{1,I}^{30}$	$(0,0)$	1	3	-3	
10	$S_{1,I}^3$	68	$\mathrm{Bl}_{1^3} S_{4,II}^{(2^2)}$	$(0,0)$	1	4	-3	
11	$S_{1,I}^4$	35	$S_{1,I}^{28}$	$(0,0)$	1	5	-1	
12	$S_{1,I}^5$	60	$\mathrm{Bl}_{1^2} S_3^2$	$(0,0)$	1	6	-2	
13	$S_{1,I}^6$	42	$\mathrm{Bl}_1 S_{2,II}^{(4,2)}$	$(0,0)$	1	8	-1	
14	$S_{1,I}^7$	33	$S_{1,I}^{26}$	$(0,0)$	1	9	0	
15	$S_{1,I}^8$	94	$\mathrm{Bl}_{3,1^5} S_9$ $\mathrm{Bl}_{3,1^4} S_{8,II}$	$(0,0)$	1	6	-4	$q \geq 7$
16	$S_{1,I}^9$	100	$\mathrm{Bl}_{3^2,1^2} S_9$ $\mathrm{Bl}_{3^2,1} S_{8,II}$	$(0,0)$	1	6	-1	
17	$S_{1,I}^{10}$	–	–	$(0,0)$	1	4	1	
18	$S_{1,I}^{11}$	78	$\mathrm{Bl}_3 S_{4,II}^{(2^2)}$	$(0,0)$	1	12	0	
19	$S_{1,I}^{12}$	102	$\mathrm{Bl}_{5,1^3} S_9$ $\mathrm{Bl}_{5,1^2} S_{8,II}$ $\mathrm{Bl}_{1^4} S_5$	$(0,0)$	1	10	-2	
20	$S_{1,I}^{13}$	111	$\mathrm{Bl}_{7,1} S_9$ $\mathrm{Bl}_7 S_{8,II}$	$(0,0)$	1	14	0	All q
21	$S_{1,I}^{14}$	–	–	$(0,0)$	1	12	1	
22	$S_{1,I}^{15}$	109	$\mathrm{Bl}_{5,3} S_9$ $\mathrm{Bl}_{3,1} S_5$	$(0,0)$	1	30	1	
23	$S_{1,I}^{16}$	–	–	$(0,0)$	1	8	1	
24	$S_{1,I}^{17}$	36	$S_{1,I}^{29}$	$(0,0)$	1	12	-1	
25	$S_{1,I}^{18}$	–	–	$(0,0)$	1	6	1	
26	$S_{1,I}^{19}$	59	$\mathrm{Bl}_{1^2} S_3^4$	$(0,0)$	1	18	-1	
27	$S_{1,I}^{20}$	58	$\mathrm{Bl}_{1^2} S_3^1$	$(0,0)$	1	12	0	
28	$S_{1,I}^{21}$	56	$\mathrm{Bl}_{1^2} S_3^3$	$(0,0)$	1	6	2	
29	$S_{1,I}^{22}$	34	$S_{1,I}^{27}$	$(0,0)$	1	30	0	
30	$S_{1,I}^{23}$	–	–	$(0,0)$	1	24	1	

31	$S_{1,I}^{24}$	–	–	$(0,0)$	1	20	1	
32	$S_{1,I}^{25}$	–	–	$(0,0)$	1	12	1	
33	$S_{1,I}^{26}$	14	$S_{1,I}^{7}$	$(0,0)$	1	18	2	
34	$S_{1,I}^{27}$	29	$S_{1,I}^{22}$	$(0,0)$	1	15	2	
35	$S_{1,I}^{28}$	11	$S_{1,I}^{4}$	$(0,0)$	1	10	3	
36	$S_{1,I}^{29}$	24	$S_{1,I}^{17}$	$(0,0)$	1	12	3	
37	$S_{1,I}^{30}$	9	$S_{1,I}^{2}$	$(0,0)$	1	6	5	

TABLE 43. Non-relatively minimal del Pezzo surfaces of degree 1

Class	Type	Bertini Class	Bertini Type	(L_X, C_X)	I_X	Order	$a(X)$	Restrictions
38	$\mathrm{Bl}_1\, S_{2,II}^{(1^6)}$	93	$\mathrm{Bl}_{2^2,1^4}\, S_9$ $\mathrm{Bl}_{2,1^5}\, S_{8,I}$ $\mathrm{Bl}_{2^2,1^3}\, S_{8,II}$	$(4,4)$	4	2	-3	
39	$\mathrm{Bl}_1\, S_{2,II}^{(2^2,1^2)}$	–	–	$(4,4)$	4	4	1	
40	$\mathrm{Bl}_1\, S_{2,II}^{(3,1^3)}$	99	$\mathrm{Bl}_{3,2^2,1}\, S_9$ $\mathrm{Bl}_{3,2,1^2}\, S_{8,I}$ $\mathrm{Bl}_{2,1^3}\, S_6$	$(4,4)$	4	6	0	
41	$\mathrm{Bl}_1\, S_{2,II}^{(5,1)}$	89	$\mathrm{Bl}_{5,2}\, S_{8,I}$ $\mathrm{Bl}_{2^2}\, S_5$	$(4,4)$	4	10	2	
42	$\mathrm{Bl}_1\, S_{2,II}^{(4,2)}$	13	$S_{1,I}^{6}$	$(4,4)$	4	8	3	
43	$\mathrm{Bl}_1\, S_{2,II}^{(3,3)}$	85	$\mathrm{Bl}_{3,2}\, S_6$	$(4,4)$	4	6	3	
44	$\mathrm{Bl}_1\, S_{2,I}^{1}$	92	$\mathrm{Bl}_{2,1^6}\, S_9$ $\mathrm{Bl}_{1^7}\, S_{8,I}$ $\mathrm{Bl}_{2,1^5}\, S_{8,II}$	$(2,0)$	-2	2	-5	$q \geq 11$
45	$\mathrm{Bl}_1\, S_{2,I}^{2}$	72	$\mathrm{Bl}_{2,1}\, S_{4,II}^{(2^2)}$	$(2,0)$	-2	4	-1	
46	$\mathrm{Bl}_1\, S_{2,I}^{3}$	65	$\mathrm{Bl}_2\, S_3^3$	$(2,0)$	-2	6	0	
47	$\mathrm{Bl}_1\, S_{2,I}^{4}$	112	$\mathrm{Bl}_8\, S_9$	$(2,0)$	-2	8	1	All q
48	$\mathrm{Bl}_1\, S_{2,I}^{5}$	96	$\mathrm{Bl}_{3,2,1^3}\, S_9$ $\mathrm{Bl}_{3,1^4}\, S_{8,I}$ $\mathrm{Bl}_{3,2,1^2}\, S_{8,II}$ $\mathrm{Bl}_{1^5}\, S_6$	$(2,0)$	-2	6	-2	
49	$\mathrm{Bl}_1\, S_{2,I}^{6}$	106	$\mathrm{Bl}_{5,2,1}\, S_9$ $\mathrm{Bl}_{5,1^2}\, S_{8,I}$ $\mathrm{Bl}_{2,1^2}\, S_5$	$(2,0)$	-2	10	0	
50	$\mathrm{Bl}_1\, S_{2,I}^{7}$	103	$\mathrm{Bl}_{3^2,2}\, S_9$ $\mathrm{Bl}_{3^2,1}\, S_{8,I}$ $\mathrm{Bl}_{3,1^2}\, S_6$	$(2,0)$	-2	6	1	
51	$\mathrm{Bl}_1\, S_{2,I}^{8}$	64	$\mathrm{Bl}_2\, S_3^4$	$(2,0)$	-2	18	1	
52	$\mathrm{Bl}_1\, S_{2,I}^{9}$	90	$\mathrm{Bl}_7\, S_{8,I}$	$(2,0)$	-2	14	2	
53	$\mathrm{Bl}_1\, S_{2,I}^{10}$	63	$\mathrm{Bl}_2\, S_3^1$	$(2,0)$	-2	12	2	
54	$\mathrm{Bl}_1\, S_{2,I}^{11}$	86	$\mathrm{Bl}_5\, S_6$	$(2,0)$	-2	30	3	
55	$\mathrm{Bl}_1\, S_{2,I}^{12}$	61	$\mathrm{Bl}_2\, S_3^3$	$(2,0)$	-2	6	4	
56	$\mathrm{Bl}_{1^2}\, S_3^3$	28	$S_{1,I}^{21}$	$(6,0)$	3	3	0	
57	$\mathrm{Bl}_{1^2}\, S_3^5$	–	–	$(6,0)$	3	6	1	

58	$\mathrm{Bl}_{1^2} S_3^1$	27	$S_{1,I}^{20}$	$(6,0)$	3	12	2	
59	$\mathrm{Bl}_{1^2} S_3^4$	26	$S_{1,I}^{19}$	$(6,0)$	3	9	3	
60	$\mathrm{Bl}_{1^2} S_3^2$	12	$S_{1,I}^5$	$(6,0)$	3	6	4	
61	$\mathrm{Bl}_2 S_3^3$	55	$\mathrm{Bl}_1 S_{2,I}^{12}$	$(0,0)$	3	6	-2	
62	$\mathrm{Bl}_2 S_3^5$	107	$\mathrm{Bl}_{6,1^2} S_9$ $\mathrm{Bl}_{6,1} S_{8,II}$	$(0,0)$	3	6	-1	$q \geq 3$
63	$\mathrm{Bl}_2 S_3^1$	53	$\mathrm{Bl}_1 S_{2,I}^{10}$	$(0,0)$	3	12	0	
64	$\mathrm{Bl}_2 S_3^4$	51	$\mathrm{Bl}_1 S_{2,I}^8$	$(0,0)$	3	18	1	
65	$\mathrm{Bl}_2 S_3^2$	46	$\mathrm{Bl}_1 S_{2,I}^3$	$(0,0)$	3	6	2	
66	$\mathrm{Bl}_{1^3} S_{4,II}^{(1^4)}$	–	–	$(24,24)$	4	2	1	
67	$\mathrm{Bl}_{1^3} S_{4,II}^{(3,1)}$	75	$\mathrm{Bl}_3 S_{4,II}^{(1^4)}$	$(24,24)$	4	6	4	
68	$\mathrm{Bl}_{1^3} S_{4,II}^{(2^2)}$	10	$S_{1,I}^3$	$(24,24)$	4	4	5	
69	$\mathrm{Bl}_{2,1} S_{4,II}^{(1^4)}$	95	$\mathrm{Bl}_{2^3,1^2} S_9$ $\mathrm{Bl}_{2^2,1^3} S_{8,I}$ $\mathrm{Bl}_{2^3,1} S_{8,II}$	$(6,12)$	-8	2	-1	
70	$\mathrm{Bl}_{1^3} S_{4,I}^{(2,1^3)}$	–	–	$(12,6)$	-4	4	1	
71	$\mathrm{Bl}_{2,1} S_{4,II}^{(3,1)}$	87	$\mathrm{Bl}_{3,2^2} S_{8,I}$ $\mathrm{Bl}_{2^2,1} S_6$	$(6,12)$	-8	6	2	
72	$\mathrm{Bl}_{2,1} S_{4,II}^{(2^2)}$	45	$\mathrm{Bl}_1 S_{2,I}^2$	$(6,12)$	-8	4	3	
73	$\mathrm{Bl}_{1^3} S_{4,I}^{(4,1)}$	4	$S_{1,II}^{(4,1^3)}$	$(12,6)$	-4	8	3	
74	$\mathrm{Bl}_{1^3} S_{4,I}^{(3,2)}$	81	$\mathrm{Bl}_3 S_{4,I}^{(2,1^3)}$	$(12,6)$	-4	12	4	
75	$\mathrm{Bl}_3 S_{4,II}^{(1^4)}$	67	$\mathrm{Bl}_{1^3} S_{4,II}^{(3,1)}$	$(0,6)$	12	6	-2	
76	$\mathrm{Bl}_{2,1} S_{4,I}^{(2,1^3)}$	101	$\mathrm{Bl}_{4,2,1^2} S_9$ $\mathrm{Bl}_{4,1^3} S_{8,I}$	$(2,2)$	8	4	-1	
77	$\mathrm{Bl}_3 S_{4,II}^{(3,1)}$	–	–	$(0,6)$	12	6	1	
78	$\mathrm{Bl}_3 S_{4,II}^{(2^2)}$	18	$S_{1,I}^{11}$	$(0,6)$	12	12	2	
79	$\mathrm{Bl}_{2,1} S_{4,I}^{(4,1)}$	–	–	$(2,2)$	8	8	1	
80	$\mathrm{Bl}_{2,1} S_{4,I}^{(3,2)}$	88	$\mathrm{Bl}_{4,3} S_{8,I}$ $\mathrm{Bl}_{4,1} S_6$	$(2,2)$	8	12	2	
81	$\mathrm{Bl}_3 S_{4,I}^{(2,1^3)}$	74	$\mathrm{Bl}_{1^3} S_{4,I}^{(3,2)}$	$(0,0)$	-12	12	-2	
82	$\mathrm{Bl}_3 S_{4,I}^{(4,1)}$	7	$S_{1,II}^{(4,3)}$	$(0,0)$	-12	24	0	
83	$\mathrm{Bl}_3 S_{4,I}^{(3,2)}$	–	–	$(0,0)$	-12	12	1	
84	$\mathrm{Bl}_4 S_5$	6	$S_{1,II}^{(5,2)}$	$(0,0)$	-20	20	0	
85	$\mathrm{Bl}_{3,2} S_6$	43	$\mathrm{Bl}_1 S_{2,II}^{(3,3)}$	$(0,0)$	36	6	-1	
86	$\mathrm{Bl}_5 S_6$	54	$\mathrm{Bl}_1 S_{2,I}^{11}$	$(0,0)$	-30	30	-1	
87	$\mathrm{Bl}_{3,2^2} S_{8,I}$ $\mathrm{Bl}_{2^2,1} S_6$	71	$\mathrm{Bl}_{2,1} S_{4,II}^{(3,1)}$	$(2,6)$	-24	6	0	
88	$\mathrm{Bl}_{4,3} S_{8,I}$ $\mathrm{Bl}_{4,1} S_6$	80	$\mathrm{Bl}_{2,1} S_{4,I}^{(3,2)}$	$(2,0)$	24	12	0	
89	$\mathrm{Bl}_{5,2} S_{8,I}$ $\mathrm{Bl}_{2^2} S_5$	41	$\mathrm{Bl}_1 S_{2,II}^{(5,1)}$	$(0,2)$	20	10	0	
90	$\mathrm{Bl}_7 S_{8,I}$	52	$\mathrm{Bl}_1 S_{2,I}^9$	$(0,0)$	-14	14	0	

91	$\mathrm{Bl}_{1^8}\,S_9$ $\mathrm{Bl}_{1^7}\,S_{8,II}$	8	$S_{1,I}^1$	$(240, 2160)$	1	1	9	$q = 16$ or $q \geq 19$
92	$\mathrm{Bl}_{2,1^6}\,S_9$ $\mathrm{Bl}_{1^7}\,S_{8,I}$ $\mathrm{Bl}_{2,1^5}\,S_{8,II}$	44	$\mathrm{Bl}_1\,S_{2,I}^1$	$(126, 756)$	-2	2	7	$q \geq 11$
93	$\mathrm{Bl}_{2^2,1^4}\,S_9$ $\mathrm{Bl}_{2,1^5}\,S_{8,I}$ $\mathrm{Bl}_{2^2,1^3}\,S_{8,II}$	38	$\mathrm{Bl}_1\,S_{2,II}^{(1^6)}$	$(60, 252)$	4	2	5	
94	$\mathrm{Bl}_{3,1^5}\,S_9$ $\mathrm{Bl}_{3,1^4}\,S_{8,II}$	15	$S_{1,I}^8$	$(72, 270)$	3	3	6	$q \geq 7$
95	$\mathrm{Bl}_{2^3,1^2}\,S_9$ $\mathrm{Bl}_{2^2,1^3}\,S_{8,I}$ $\mathrm{Bl}_{2^3,1}\,S_{8,II}$	69	$\mathrm{Bl}_{2,1}\,S_{4,II}^{(1^4)}$	$(26, 72)$	-8	2	3	
96	$\mathrm{Bl}_{3,2,1^3}\,S_9$ $\mathrm{Bl}_{3,1^4}\,S_{8,I}$ $\mathrm{Bl}_{3,2,1^2}\,S_{8,II}$ $\mathrm{Bl}_{1^5}\,S_6$	48	$\mathrm{Bl}_1\,S_{2,I}^5$	$(30, 90)$	-6	6	4	
97	$\mathrm{Bl}_{4,1^4}\,S_9$ $\mathrm{Bl}_{4,1^3}\,S_{8,II}$	1	$S_{1,II}^{(2,1^5)}$	$(40, 90)$	-4	4	5	$q \geq 5$
98	$\mathrm{Bl}_{2^4}\,S_9$ $\mathrm{Bl}_{2^3,1}\,S_{8,I}$	$-$	$-$	$(8, 24)$	16	2	1	
99	$\mathrm{Bl}_{3,2^2,1}\,S_9$ $\mathrm{Bl}_{3,2,1^2}\,S_{8,I}$ $\mathrm{Bl}_{2,1^3}\,S_6$	40	$\mathrm{Bl}_1\,S_{2,II}^{(3,1^3)}$	$(12, 18)$	12	6	2	
100	$\mathrm{Bl}_{3^2,1^2}\,S_9$ $\mathrm{Bl}_{3^2,1}\,S_{8,II}$	16	$S_{1,I}^9$	$(12, 36)$	9	3	3	
101	$\mathrm{Bl}_{4,2,1^2}\,S_9$ $\mathrm{Bl}_{4,1^3}\,S_{8,I}$	76	$\mathrm{Bl}_{2,1}\,S_{4,I}^{(2,1^3)}$	$(14, 30)$	8	4	3	
102	$\mathrm{Bl}_{5,1^3}\,S_9$ $\mathrm{Bl}_{5,1^2}\,S_{8,II}$ $\mathrm{Bl}_{1^4}\,S_5$	19	$S_{1,I}^{12}$	$(20, 30)$	5	5	4	$q \geq 3$
103	$\mathrm{Bl}_{3^2,2}\,S_9$ $\mathrm{Bl}_{3^2,1}\,S_{8,I}$ $\mathrm{Bl}_{3,1^2}\,S_6$	50	$\mathrm{Bl}_1\,S_{2,I}^7$	$(6, 0)$	-18	6	1	
104	$\mathrm{Bl}_{4,2^2}\,S_9$	?	?	$(4, 6)$	-16	4	1	
105	$\mathrm{Bl}_{4,3,1}\,S_9$ $\mathrm{Bl}_{4,3}\,S_{8,II}$	2	$S_{1,II}^{(3,2,1^2)}$	$(4, 12)$	-12	12	2	
106	$\mathrm{Bl}_{5,2,1}\,S_9$ $\mathrm{Bl}_{5,1^2}\,S_{8,I}$ $\mathrm{Bl}_{2,1^2}\,S_5$	49	$\mathrm{Bl}_1\,S_{2,I}^6$	$(6, 6)$	-10	10	2	
107	$\mathrm{Bl}_{6,1^2}\,S_9$ $\mathrm{Bl}_{6,1}\,S_{8,II}$	62	$\mathrm{Bl}_2\,S_3^5$	$(8, 12)$	-6	6	3	$q \geq 3$
108	$\mathrm{Bl}_{4^2}\,S_9$	$-$	$-$	$(0, 4)$	16	4	1	
109	$\mathrm{Bl}_{5,3}\,S_9$ $\mathrm{Bl}_{3,1}\,S_5$	22	$S_{1,I}^{15}$	$(2, 0)$	15	15	1	
110	$\mathrm{Bl}_{6,2}\,S_9$ $\mathrm{Bl}_{6,1}\,S_{8,I}$	$-$	$-$	$(2, 0)$	12	6	1	

111	$\mathrm{Bl}_{7,1}\, S_9$ $\mathrm{Bl}_7\, S_{8,II}$	20	$S_{1,I}^{13}$	$(2,4)$	7	7	2	All q
112	$\mathrm{Bl}_8\, S_9$	47	$\mathrm{Bl}_1\, S_{2,I}^4$	$(0,0)$	-8	8	1	All q

APPENDIX B. POINTS IN GENERAL POSITION

In this appendix, we prove Theorem 2.27. Before we do so, we need a lemma that states that we can find irreducible polynomials with prescribed penultimate coefficients. Over finite fields, this is a special case of the Hansen–Mullen conjecture, which was proved by Wan [Wan97] and Ham–Mullen [HM98]. Over infinite fields, the proof is straightforward, albeit a little tedious [McK22, Lem. 4.3].

Lemma B.1. *Let k be a field. Assume that k admits a separable extension of degree $n \geq 2$. Pick $a \in k$, which we assume is non-zero if $\mathrm{char}(k) = n = 2$. Then there is a monic, separable, irreducible polynomial $f(t) \in k[t]$ of degree n whose degree $n - 1$ coefficient is a.*

We are now ready to prove Theorem 2.27.

Proof of Theorem 2.27. For $n \leq 6$ the argument follows from [McK22, §4.2]. In *loc. cit.*, the author uses the parameterisation $\{[1 : t : t^3]\}$ of the cuspidal cubic $C = \mathbb{V}(y^3 - x^2 z) \subset \mathbb{P}^2_k$ to construct a set of closed points in general position for each partition of 6. The idea is that three distinct geometric points $\{[1 : t_i : t_i^3]\}_{i=1}^3$ lie on a line $\mathbb{V}(ax + by + cz)$ if and only if each t_i is a root of
$$a + bt + ct^3,$$
which occurs only if $t_1 + t_2 + t_3 = 0$. Similarly, six distinct geometric points $\{[1 : t_i : t_i^3]\}_{i=1}^6$ lie on a conic $\mathbb{V}(ax^2 + bxy + cy^2 + dxz + eyz + fz^2)$ if and only if each t_i is a root of
$$a + bt + ct^2 + dt^3 + et^4 + ft^6,$$
which occurs only if $\sum_{i=1}^6 t_i = 0$. Sets of closed points in general position can therefore be obtained by constructing sextic polynomials with no repeated roots, a non-zero quintic term, and such that no three roots sum to 0. The degrees of the closed points arising from this construction correspond to the degrees of the irreducible factors of the sextic polynomial. A suitable sextic for each partition of 6 can be constructed when $|k| \geq 22$, but of course there is no field with precisely 22 elements.

Every partition of $1 \leq n \leq 5$ can be obtained by some partition of 6 by omitting some terms. A set of closed points in general position stays in general position after omitting some closed points, so these cases follow from treating partitions of 6. A more direct treatment of these cases would yield a better bound on $|k|$; for example, any field will do for $n = 1$ or 2.

We now address the remaining cases.

(ii) $n = 7$. Of the 15 partitions of 7, there are 11 that take the form $\lambda' = (1 + n_1, \ldots, n_6, 0)$, where $\lambda = (n_1, \ldots, n_6)$ is a partition of 6. By case (i), there is a Galois-invariant set M of 6 geometric points (contained in the cuspidal cubic C) in general position corresponding to the partition λ. In order to construct a Galois-invariant set of 7 geometric points in general position corresponding to the partition λ', it suffices to find a k-rational point on C that does not lie on a line through any two points of M or on a conic through any five points of M.

Since each pair of points determines a unique line, and each quintuple of points determines a unique conic, there are $\binom{6}{2} = 15$ lines and $\binom{6}{5} = 6$ conics to consider. Moreover, each line meets C with multiplicity 3, and each conic meets C with multiplicity 6, so we must avoid at most $3 \cdot 6 + 6 \cdot 6 = 54$ geometric points when selecting our rational point. This is possible if C has at least 54 points in the open affine $\{[1 : x : y]\}$, i.e. if $|k| > 54$.

The remaining partitions of 7, which are

$$2 + 2 + 3, \quad 2 + 5, \quad 3 + 4, \quad 7, \tag{1}$$

can all be treated by the same argument. We will construct degree 7 polynomials with irreducible factors of the degrees corresponding to each of these partitions. To see that no six roots sum to zero in each of these cases, note that the sum of all seven roots is an element of k (in particular, the negative of the degree 6 coefficient). If any six roots summed to zero, the remaining root would have to be rational, but none of these polynomials have a rational root.

Fix a partition of 7 among those listed in Equation 1. Let $f(t)$ be a monic, separable polynomial of degree 7 whose irreducible factors over k have degrees corresponding to our fixed partition. By assumption, we are guaranteed at least one monic, separable, irreducible polynomial of the required degrees. For the partition 2+2+3, we need two distinct quadratic polynomials. If $g(t)$ is a monic, separable, irreducible quadratic over k, then so is $g(t + x)$ for all $x \in k$. To ensure that $g(t) \neq g(t + x)$, it suffice to ensure that $\{r_1, r_2\} \neq \{r_1 - x, r_2 - x\}$, where r_1, r_2 are the roots of g. This can be done by picking $x \in k - \{0, \pm(r_1 - r_2)\}$.

If no three roots of f sum to 0, then we are done. Otherwise, let $\{s_1, \ldots, s_{35}\}$ be the set of sums of all $\binom{7}{3}$ subsets of three roots of f. Pick $x \in k - \{s_1, \ldots, s_{35}\}$. If $\mathrm{char}(k) \neq 3$, then any three roots of $f(t + \frac{x}{3})$ sum to $s_i - x \neq 0$, so $f(t + \frac{x}{3})$ satisfies the desired criteria.

If $\mathrm{char}(k) = 3$, let $f^*(t) := f(0)^{-1} \cdot t^7 \cdot f(\frac{1}{t})$ be the scaled reciprocal of f, which is again monic, separable, and has irreducible factors of the same degrees as those of f. (Note that $f(0) \neq 0$, since f has no linear factors over k.) Denote the roots of f by $r_1, \ldots, r_7$. For any $\alpha \in k$, let $f_\alpha(t) = f(t - \alpha)$, so that the roots of f_α^* are given by $\frac{1}{r_1 + \alpha}, \ldots, \frac{1}{r_7 + \alpha}$. The sum of any three roots of f_α^* is given by

$$\frac{1}{r_i + \alpha} + \frac{1}{r_j + \alpha} + \frac{1}{r_\ell + \alpha} = \frac{r_i r_j + r_i r_\ell + r_j r_\ell + 2\alpha(r_i + r_j + r_\ell) + 3\alpha^2}{(r_i + \alpha)(r_j + \alpha)(r_\ell + \alpha)}$$

$$= \frac{r_i r_j + r_i r_\ell + r_j r_\ell - \alpha(r_i + r_j + r_\ell)}{(r_i + \alpha)(r_j + \alpha)(r_\ell + \alpha)}.$$

If $r_i + r_j + r_\ell = 0$, then $r_i = -r_j - r_\ell$, so that

$$r_i r_j + r_i r_\ell + r_j r_\ell = -(r_j^2 + 2r_j r_\ell + r_\ell^2) + r_j r_\ell$$
$$= -(r_j^2 - 2r_j r_\ell + r_\ell^2)$$
$$= -(r_j - r_\ell)^2.$$

Since f is separable, we have $r_j \neq r_\ell$ and hence $r_i r_j + r_i r_\ell + r_j r_\ell \neq 0$. It follows that if $r_i + r_j + r_\ell = 0$, then $\frac{1}{r_i + \alpha} + \frac{1}{r_j + \alpha} + \frac{1}{r_\ell + \alpha} \neq 0$.

If $r_i + r_j + r_\ell \neq 0$, then $r_i r_j + r_i r_\ell + r_j r_\ell - \alpha(r_i + r_j + r_\ell) \neq 0$ for all $\alpha \neq \frac{r_i r_j + r_i r_\ell + r_j r_\ell}{r_i + r_j + r_\ell}$. In particular, if we pick

$$\alpha \in k - \left\{ \frac{r_i r_j + r_i r_\ell + r_j r_\ell}{r_i + r_j + r_\ell} : r_i + r_j + r_\ell \neq 0 \right\},$$

then f_α^* will satisfy the desired criteria. There are at most 34 such triples of roots to consider (or else no three roots of f sum to 0). We may thus find a suitable polynomial whenever $|k| \geq 37$.

(iii) $n = 8$. We will follow the same strategy as in case (ii), but there is an extra criterion to worry about: we need to ensure that our eight geometric points do not lie on a singular cubic with one of the points at the singularity. We will prove in Lemma B.2 that if none of our eight points lie on the cusp of C, then no cubic through these eight points has a singular point at one of the eight points.

As in the $n = 7$ case, we would like to reduce most partitions of 8 to partitions of 7. In order to do this while still using Lemma B.2, we need to make sure that each set of six points corresponding to a partition of 6 avoids the cusp point of C; for this purpose, the partitions of 6 given in [McK22] were constructed so as to avoid the cusp point of C (see [McK22, Rem. 4.2]). The partitions of 7 of the form $(1+n_1, \ldots, n_6, 0)$ can be made to avoid the cusp point by assuming $|k| > 53+1$. The partitions of 7 of the form $(0, n_2, \ldots, n_7)$ avoid the cusp point. Given a partition $(n_1, \ldots, n_7)$ of 7, we can thus obtain a partition $(1+n_1, n_2, \ldots, n_7, 0)$ of 8 that avoids the cusp point of C by avoiding at most 8 rational points of C in the open affine $\{[1 : x : y]\}$ (the cusp point and the points from our partition of 7). This is always possible if $|k| \geq 9$.

It follows that the only remaining cases are partitions of 8 with $n_1 = 0$. As in case (ii), we begin with a monic, separable polynomial f of degree 8 whose irreducible factors over k correspond to one of the following partitions:

$$2+2+2+2, \quad 2+2+4, \quad 2+3+3, \quad 2+6, \quad 3+5, \quad 4+4, \quad 8. \tag{2}$$

In order to address the partitions with repeated summands, we again need to justify why having a single monic, separable, irreducible polynomial of a given degree over k gives us more monic, separable, irreducible polynomials of the same degree.

- Constructing two distinct quadratics proceeds exactly as in case (ii).

- For four distinct quadratics, let $g_1(t), g_2(t)$ be two distinct monic, separable, irreducible quadratics over k. Let r_1, r_2 be the roots of g_1 and s_1, s_2 be the roots of g_2. Pick $x \in k - \{0, \pm(r_1 - r_2), r_1 - s_1, r_1 - s_2, r_2 - s_1, r_2 - s_2\}$. Then the roots of $g_3(t) := g_1(t + x)$ are disjoint from $\{r_1, r_2, s_1, s_2\}$, so g_3 is a monic, separable, irreducible quadratic distinct from g_1 and g_2. To obtain a fourth distinct quadratic, we take $g_4(t + y)$ with $y \in k - \{0, \pm(r_1 - r_2)\} \cup \{r_i - s_j : i, j = 1, 2\} \cup \{r_i - w_j : i, j = 1, 2\}$, where w_1, w_2 are the roots of g_3. Now the roots of g_4 are distinct from those of g_1, g_2, g_3, so we have four distinct monic, separable, irreducible quadratics over k. This construction requires $|k| \geq 13$.

- Given a monic, separable, irreducible cubic $g(t)$ over k, we can construct another irreducible cubic by picking $x \in k - \{r_i - r_j : 1 \leq i, j \leq 3\}$, where r_1, r_2, r_3 are the roots of g, and considering $g(t + x)$. By construction, $g(r_i + x) \neq 0$ for $1 \leq i \leq 3$, so g and $g(t + x)$ are distinct monic, separable, irreducible cubics. This construction requires $|k| \geq 5$.

- We construct two distinct quartics in the same was as cubics: given an irreducible quartic $g(t)$, we consider $g(t+x)$ with $x \in k - \{r_i - r_j : 1 \leq i, j \leq 4\}$, where $r_1, \ldots, r_4$ are the roots of g. This requires $|k| \geq 8$.

We now know that given a partition among those listed in Equation 2, we can construct a monic, separable polynomial $f(t)$ of degree 8 corresponding to this partition. Moreover, by repeating the arguments in step (ii), we can guarantee that no three roots of $f(t)$ sum to 0 as long as $|k| > \binom{8}{3}$. However, in contrast to case (ii), we need to verify that no six roots of $f(t)$ sum to 0.

- If $\text{char}(k) \neq 2$ or 3, let $\{s_1, \ldots, s_{56}\}$ and $\{u_1, \ldots, u_{28}\}$ denote the sets of sums of all $\binom{8}{3}$ subsets of three roots and $\binom{8}{6}$ subsets of six roots, respectively, of f. Pick $x \in k - \{\frac{s_1}{3}, \ldots, \frac{s_{56}}{3}, \frac{u_1}{6}, \ldots, \frac{u_{28}}{6}\}$. Now any three roots of $f(t+x)$ sum to $s_i - 3x \neq 0$, and any six roots of $f(t+x)$ sum to $u_i - 6x \neq 0$. This requires $|k| \geq 89$.

- If $\text{char}(k) = 2$, any sum of three roots of f_α^* is given by $\frac{r_i r_j + r_i r_\ell + r_j r_\ell + \alpha^2}{(r_i + \alpha)(r_j + \alpha)(r_\ell + \alpha)}$, where r_i, r_j, r_ℓ are distinct roots of f. Similarly, the sum of six roots $r_1, \ldots, r_6$ of f_α^* is given by

$$\frac{\sum_{i=1}^6 \prod_{j \neq i} r_j + \alpha^2 \sum_{i,j,\ell} r_i r_j r_\ell + \alpha^4 (r_1 + \ldots + r_6)}{\prod_{i=1}^6 (r_i + \alpha)}. \tag{3}$$

In order to pick α such that no three or six roots of f_α^* sum to 0, it suffices to pick α not a root of the quadratic polynomials $r_i r_j + r_i r_\ell + r_j r_\ell + x^2$ or the quartic polynomials given by the numerators of Equation 3. We therefore have at most $2\binom{8}{3} + 4\binom{8}{6} = 224$ roots to avoid, so we can pick a suitable α if $|k| \geq 227$.

- The $\text{char}(k) = 3$ case is analogous to the $\text{char}(k) = 2$ case. We again have a linear polynomial

$$r_i r_j + r_i r_\ell + r_j r_\ell + 2\alpha(r_i + r_j + r_\ell)$$

and a quartic polynomial

$$\sum_{i=1}^6 \prod_{j \neq i} r_j + 2\alpha \sum_{i,j,\ell,n} r_i r_j r_\ell r_n + \alpha^3 \sum_{i,j} r_i r_j + 2\alpha^4 (r_1 + \ldots + r_6)$$

whose roots correspond to a sum of three or six roots, respectively, of f_α^* summing to 0. We thus have $\binom{8}{3} + 4\binom{8}{6} = 168$ potential roots to avoid. It follows that if $|k| \geq 169$, then there exists $\alpha \in k$ such that f_α^* satisfies the desired conditions. $\square$

Lemma B.2. *Let k be a field. Let $C \subset \mathbb{P}_k^2$ be a singular cubic. Let $P_1, \ldots, P_8$ be a set of distinct geometric points contained in the smooth locus of C. Then there is no singular cubic C' through $P_1, \ldots, P_8$ such that some P_i lies on the singularity of C'.*

Proof. By the Cayley–Bacharach theorem (see, e.g. [EGH96, Thm CB3]), the points $P_1, \ldots, P_8$ determine a distinct ninth point P_9 such that any cubic through $P_1, \ldots, P_8$ must pass through P_9. In particular, if a singular cubic C' passes through $P_1, \ldots, P_8$ with one of P_i at its singular point, then C and C' meet with multiplicity 10 — both curves must contain $P_1, \ldots, P_9$, and they meet with multiplicity 2 at P_i. This contradicts Bézout's theorem, so such a C' cannot exist. $\square$

APPENDIX C. OPEN QUESTIONS

While preparing this article, we encountered a few questions that we were not able to answer. The first of these pertains to points in general position on del Pezzo surfaces. We showed in Proposition 2.29 that over an infinite field k and for $d \geq 5$, every blowup type of degree-d del Pezzo surfaces admits a surface with a set of points in general position for every possible partition (i.e. those not obstructed by the Galois theory of k). The key was finding a rational surface of each blowup type.

For $d \leq 4$, this method breaks down, which is why we turned our attention to various characteristics of the base field. However, if one could show that every *unirational* del Pezzo surface contains a set of points in general position for each partition, one could likely dispense with the assumptions on the base field and treat all infinite fields uniformly. This leads to the following question.

Question C.1. Let k be an infinite field, and let X be a unirational del Pezzo surface of degree $d \geq 2$. Let $1 \leq n \leq d - 1$, and let $\lambda = (n_1, \ldots, n_8)$ be a partition of n. If k admits a separable extension of degree i for each $n_i > 0$, does X contain a set of closed points in general position corresponding to λ?

It is known that for $d \geq 3$, degree-d del Pezzo surfaces with a rational point are unirational. For unirationality of degree-2 del Pezzo surfaces, see [Man86, Thm. 29.4] and [STVA14].

Although we found a suitable way to construct points in general position on low degree del Pezzo surfaces over Hilbertian fields, we spent some time trying to prove a stronger statement that seems quite interesting.

Question C.2. Let X be a degree-d del Pezzo surface over a Hilbertian field k. Let $n < d$. Let $X^{[n]}$ denote the Hilbert scheme of n points on X. Does $X^{[n]}$ satisfy the Hilbert property? That is, is $X^{[n]}(k)$ not a thin set?

Of course, $\mathbb{P}^1$ and $\mathbb{P}^2$, along with their blowups, satisfy the Hilbert property, so the remaining open cases are relatively minimal del Pezzo surfaces of degree at most 6, as well as their blowups.

As mentioned in Remark 5.4, there are 17 classes of degree-1 del Pezzo surfaces over finite fields for which the surface and its Bertini twist are both of Picard rank 1. These are the surfaces currently obstructing the resolution of $\mathrm{IGP}_1(\mathbb{F}_q)$.

Question C.3. For any of the 17 classes of degree-1 del Pezzo surfaces over finite fields with Picard rank 1 and Bertini twist of Picard rank 1, what are the exact bounds on q for their existence over $\mathbb{F}_q$?

REFERENCES

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993). 6

[Ber24] Olof Bergvall. Frobenius actions on del Pezzo surfaces of degree 2. *Innov. Incidence Geom.*, 21(1):1–10, 2024. 5

[BFL19] Barinder Banwait, Francesc Fité, and Daniel Loughran. Del Pezzo surfaces over finite fields and their Frobenius traces. *Math. Proc. Cambridge Philos. Soc.*, 167(1):35–60, 2019. 3, 5, 26, 27, 28, 29, 30, 31, 32, 33, 38, 48

[BG23]   Olof Bergvall and Frank Gounelas. Cohomology of moduli spaces of Del Pezzo surfaces. *Math. Nachr.*, 296(1):80–101, 2023. 5

[BSFP14] Lior Bary-Soroker, Arno Fehm, and Sebastian Petersen. On varieties of Hilbert type. *Ann. Inst. Fourier (Grenoble)*, 64(5):1893–1901, 2014. 39

[Car72]   R. W. Carter. Conjugacy classes in the Weyl group. *Compositio Math.*, 25:1–59, 1972. 26

[Coo88]   Kevin R. Coombes. Every rational surface is separably split. *Comment. Math. Helv.*, 63(2):305–311, 1988. 10, 18

[Das20]   Ronno Das. Arithmetic statistics on cubic surfaces. *Res. Math. Sci.*, 7(3):Paper No. 23, 12, 2020. 5

[Der06]   Ulrich Derenthal. *Geometry of Universal Torsors*. PhD thesis, Georg-August-Universität Göttingen, 2006. 7, 16

[EGH96]   David Eisenbud, Mark Green, and Joe Harris. Cayley-Bacharach theorems and conjectures. *Bull. Am. Math. Soc., New Ser.*, 33(3):295–324, 1996. 62

[EJ15]   Andreas-Stephan Elsenhans and Jörg Jahnel. Moduli spaces and the inverse Galois problem for cubic surfaces. *Trans. Amer. Math. Soc.*, 367(11):7837–7861, 2015. 5, 22

[EJ19a]   Andreas-Stephan Elsenhans and Jörg Jahnel. On plane quartics with a Galois invariant Cayley octad. *Eur. J. Math.*, 5(4):1156–1172, 2019. 5, 23

[EJ19b]   Andreas-Stephan Elsenhans and Jörg Jahnel. Plane quartics with a Galois-invariant Steiner hexad. *Int. J. Number Theory*, 15(5):1075–1109, 2019. 5, 22, 23

[FJ23]   Michael D. Fried and Moshe Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, [2023] ©2023. Fourth edition [of 868860], Revised by Moshe Jarden. 38

[FK12]   Sergey Finashin and Viatcheslav Kharlamov. Abundance of Real Lines on Real Projective Hypersurfaces. *International Mathematics Research Notices*, 2013(16):3639–3646, 06 2012. 5

[FLS18]   Christopher Frei, Daniel Loughran, and Efthymios Sofos. Rational points of bounded height on general conic bundle surfaces. *Proc. Lond. Math. Soc. (3)*, 117(2):407–440, 2018. 21

[Fly09]   E. V. Flynn. Homogeneous spaces and degree 4 del Pezzo surfaces. *Manuscripta Math.*, 129(3):369–380, 2009. 22

[Fra51]   J. S. Frame. The classes and representations of the groups of 27 lines and 28 bitangents. *Ann. Mat. Pura Appl. (4)*, 32:83–119, 1951. 26

[Har77]   Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. 6

[HM98]   Kie H. Ham and Gary L. Mullen. Distribution of irreducible polynomials of small degrees over finite fields. *Math. Comp.*, 67(221):337–341, 1998. 59

[Isk70]   V. A. Iskovskih. Rational surfaces with a sheaf of rational curves and with a positive square of canonical class. *Mat. Sb. (N.S.)*, 83(125):90–119, 1970. 18

[Isk79]   V. A. Iskovskih. Minimal models of rational surfaces over arbitrary fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):19–43, 237, 1979. 9

[JL15]   Jörg Jahnel and Daniel Loughran. The Hasse principle for lines on del Pezzo surfaces. *Int. Math. Res. Not. IMRN*, (23):12877–12919, 2015. 38, 39

[Kol96]   János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996. 23

[Kol02]   János Kollár. Unirationality of cubic hypersurfaces. *J. Inst. Math. Jussieu*, 1(3):467–476, 2002. 11

[KST89]   B. È. Kunyavskiĭ, A. N. Skorobogatov, and M. A. Tsfasman. del Pezzo surfaces of degree four, 1989. 4, 5, 22, 24

[Kul21]   Avinash Kulkarni. Sixteen points in $\mathbb{P}^4$ and the inverse Galois problem for del Pezzo surfaces of degree one, 2021. preprint available at https://arxiv.org/abs/2109.14106. 5, 23, 24

[KV24]   Avinash Kulkarni and Sameera Vemulapalli. On intersections of symmetric determinantal varieties and theta characteristics of canonical curves. *J. Pure Appl. Algebra*, 228(5):Paper No. 107538, 42, 2024. 4, 5, 23

[KW21]   Jesse Leo Kass and Kirsten Wickelgren. An arithmetic count of the lines on a smooth cubic surface. *Compos. Math.*, 157(4):677–709, 2021. 2, 5

[Lan02]   Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002. 25

[Li10]   Shujing Li. *Rational points on del Pezzo surfaces of degree 1 and 2*. PhD thesis, Rice University, 2010. 30

[LT20]   Daniel Loughran and Andrey Trepalin. Inverse Galois problem for del Pezzo surfaces over finite fields. *Math. Res. Lett.*, 27(3):845–853, 2020. 3, 5, 9, 24, 27, 29, 33

[LV21]   Hannah Larson and Isabel Vogt. An enriched count of the bitangents to a smooth plane quartic curve. *Res. Math. Sci.*, 8(2):Paper No. 26, 21, 2021. 2, 5

[Man67]   Ju.Ĩ. Manin. Rational surfaces over perfect fields. II. *Mat. Sb. (N.S.)*, 72(114):161–192, 1967. 18, 26

[Man86]   Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel. 10, 11, 20, 21, 26, 48, 63

[McK22]   Stephen McKean. Rational lines on smooth cubic surfaces, 2022. preprint available at `https://arxiv.org/abs/2101.08217`. 1, 59, 61

[MS24]   Adam Morgan and Alexei Skorobogatov. Hasse principle for intersections of two quadrics via Kummer surfaces, 2024. preprint available at `https://arxiv.org/abs/2407.16459`. 22

[OT14]   Christian Okonek and Andrei Teleman. Intrinsic signs and lower bounds in real algebraic geometry. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(688):219–241, 2014. 5

[Poo17]   Bjorn Poonen. *Rational Points on Varieties*, volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2017. 7, 13

[RT17]   S. Yu. Rybakov and A. S. Trepalin. Minimal cubic surfaces over finite fields. *Mat. Sb.*, 208(9):148–170, 2017. 5

[Rus02]   Francesco Russo. The antibirational involutions of the plane and the classification of real del Pezzo surfaces. In *Algebraic geometry*, pages 289–312. de Gruyter, Berlin, 2002. 4, 5, 25

[Ryb05]   Sergey Rybakov. Zeta functions of conic bundles and Del Pezzo surfaces of degree 4 over finite fields. *Mosc. Math. J.*, 5(4):919–926, 974, 2005. 5

[Sch58]   L. Schläfli. An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface. *Quart. J. Pure Appl. Math. (2)*, pages 110–120, 1858. 1

[SD67]   H. P. F. Swinnerton-Dyer. The zeta function of a cubic surface over a finite field. *Proc. Cambridge Philos. Soc.*, 63:55–71, 1967. 26

[Seg42]   B. Segre. *The Non-singular Cubic Surfaces: A New Method of Investigation with Special Reference to Questions of Reality*. Clarendon Press, 1942. 5

[Seg49]   Beniamino Segre. Le rette delle superficie cubiche nei corpi commutativi. *Boll. Un. Mat. Ital. (3)*, 4:223–228, 1949. 1

[Sko86]   A. N. Skorobogatov. Birational invariants of rational surfaces. *Mat. Zametki*, 39(5):736–746, 1986. 18

[Sko10]   Alexei Skorobogatov. Del Pezzo surfaces of degree 4 and their relation to Kummer surfaces. *Enseign. Math. (2)*, 56(1-2):73–85, 2010. 22

[Sta25]   The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2025. 33

[Str21]   Sam Streeter. Hilbert property for double conic bundles and del Pezzo varieties. *Math. Res. Lett.*, 28(1):271–283, 2021. 7

[STVA14]   Cecília Salgado, Damiano Testa, and Anthony Várilly-Alvarado. On the unirationality of del Pezzo surfaces of degree 2. *J. Lond. Math. Soc. (2)*, 90(1):121–139, 2014. 11, 63

[Tre17]   Andrey Trepalin. Minimal del Pezzo surfaces of degree 2 over finite fields. *Bull. Korean Math. Soc.*, 54(5):1779–1801, 2017. 24, 27

[Tre20]   Andrey Trepalin. Del Pezzo surfaces over finite fields. *Finite Fields Appl.*, 68:101741, 32, 2020. 5, 24, 25, 27

[Ura96]   Tohsuke Urabe. Calculation of Manin's invariant for Del Pezzo surfaces. *Math. Comp.*, 65(213):247–258, S15–S23, 1996. 26, 27, 48, 53

[Wan97]   Daqing Wan. Generators and irreducible polynomials over finite fields. *Math. Comp.*, 66(219):1195–1212, 1997. 59

[Zai23]   A. V. Zaitsev. Forms of del Pezzo surfaces of degree 5 and 6. *Mat. Sb.*, 214(6):69–86, 2023. 5, 9

Department of Mathematics, Bilkent University, 06800 Ankara, Turkey

Email address: enis.kaya@bilkent.edu.tr

Department of Mathematics, Brigham Young University, Provo, UT, USA

Email address: mckean@math.byu.edu

School of Mathematics, University of Bristol, Fry Building, Woodland Road, BS8 1UG, UK

Email address: sam.streeter@bristol.ac.uk

School of Mathematics, University of Bristol, Fry Building, Woodland Road, BS8 1UG, UK

Email address: huppal.maths@gmail.com