RESEARCH STATEMENT

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Introduction. An *algebraic variety* is the set at which a given collection of polynomials simultaneously vanish. Algebraic geometry is the study of algebraic varieties. Going back to antiquity, many central problems in mathematics are concerned with **counting** and **constructing** algebraic varieties. During the last century, algebraic geometry has benefited from various tools and perspectives offered by *algebraic topology*, the qualitative study of shape. *Motivic homotopy theory* (also called $\mathbb{A}^1$-homotopy theory) is a discipline within algebraic topology that has especially close ties to algebraic geometry. Many of my research projects revolve around the role of motivic homotopy theory in counting and constructing varieties.

Counting varieties is the subject of enumerative geometry. The problem of Apollonius is a famous result in the area: given three circles, how many circles are tangent to all three? Over the complex numbers, there are eight such circles (Figure 1). However, this answer may change over more general fields. The classical tools used in enumerative geometry break down over non-algebraically closed fields, making questions in this context much harder to address. Tools from motivic homotopy theory, especially analogs of the Brouwer degree and Euler classes, can be used to rectify this problem. The resulting *enriched enumerative geometry* program replaces classical integer-valued enumerative geometric equations with enumerative equations valued in quadratic forms.

One feature of enriched enumerative geometry is that objects are counted with a weight that carries more geometric information than the intersection multiplicity. For example, lines on smooth cubic surfaces are always of multiplicity one in the classical setting but are counted according to their involutive type in the enriched setting. So far, there is not any clear way to satisfactorily organize and classify the geometric weights that arise in general enriched enumerative questions. This poses a *geometricity problem* for weights.

Constructing varieties with prescribed properties is a problem that takes many forms in algebraic geometry. In anabelian geometry, one is interested in varieties that can be reconstructed (whether explicitly or implicitly) from certain group-theoretic data, namely the étale homotopy type. Within anabelian geometry, Grothendieck's *section conjecture* asserts that rational points on any hyperbolic curve X over a finitely generated field $k/\mathbb{Q}$ are in bijection with splittings of the étale fundamental group $\pi_1^{\text{ét}}(X) \to \mathrm{Gal}(k)$. The section conjecture has implications in algorithmic arithmetic geometry and remains a significant open problem in anabelian geometry. Because the étale homotopy type factors through the motivic homotopy type, motivic homotopy theory should be a useful tool for studying anabelian phenomena. However, motivic homotopical approaches to anabelian geometry are still uncommon in the literature.

Enriched enumerative geometry. I have proved enrichments of Bézout's theorem, the problem of Apollonius, and, joint with Darwin, Galimova, and Gu, the count of conic sections meeting eight lines in projective 3-space. My enrichment of Bézout's theorem serves as a convenient example for describing these results. Let k be a field. For each $a \in k^\times$, let $\langle a \rangle$ be the bilinear form $(x, y) \mapsto axy$. Let $\mathrm{Tr}_{L/k} : L \to k$ be the field trace of a finite, separable extension.

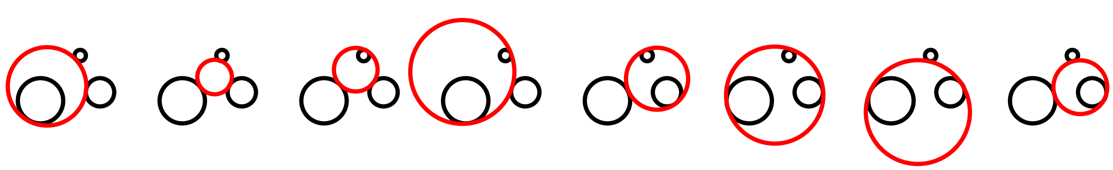

FIGURE 1. Circles of Apollonius

Theorem 1 (McKean)**.** *Let $X_1,\ldots,X_n \subset \mathbb{P}^n_k$ be hypersurfaces of degree $d_1,\ldots,d_n$, and let* $\mathrm{Vol}(p)$ *be the volume of the gradient parallelepiped of $X_1,\ldots,X_n$ at a point $p \in \bigcap_i X_i$. Then*

$$\sum_{p\in\bigcap_i X_i} \mathrm{Tr}_{k(p)/k}\langle \mathrm{Vol}(p)\rangle = \frac{d_1\cdots d_n}{2}(\langle 1\rangle + \langle -1\rangle). \tag{1}$$

Taking the rank of Equation 1 recovers the classical version of Bézout's theorem: over an algebraically closed field, there are $\prod_i d_i$ intersection points (when counted with multiplicity). Over $\mathbb{R}$, the volumes $\mathrm{Vol}(p)$ are signed, and the signature of Equation 1 implies that there are an equal number of positively and negatively oriented intersection points. This *real Bézout's theorem* is a novel corollary of Theorem 1. One obtains analogous variants of Bézout's theorem over other fields by applying other invariants to Equation 1.

Commutative algebra and motivic homotopy theory. Some of my research involves deriving commutative algebraic formulas for constructions in motivic homotopy theory. Together with Brazelton, Burklund, Montoro, and Opie, I proved that the local $\mathbb{A}^1$-degree respects base change and field trace for separable extensions.

Theorem 2 (Brazelton–Burklund–McKean–Montoro–Opie)**.** *Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with $p \in \mathbb{A}^n_k$ an isolated zero. Assume $k(p)/k$ is a finite, separable field extension, and let $\tilde{p} \in \mathbb{A}^n_{k(p)}$ be the canonical $k(p)$-rational point above p. Then* $\mathrm{Tr}_{k(p)/k}\deg_{\tilde{p}}(f_{k(p)}) = \deg_p(f)$.

The local and global versions of the $\mathbb{A}^1$-degree are of central importance in enriched enumerative geometry. Until recently, the state-of-the-art for computing the local $\mathbb{A}^1$-degree was limited to separable extensions, and there was no general method for computing global $\mathbb{A}^1$-degrees. Brazelton, Pauli, and I resolved these issues by giving a complete commutative algebraic description of both the local and global $\mathbb{A}^1$-degree (Theorem 3). Building on Theorem 3, I gave a sufficient criterion for the injectivity of f at a point $q \in \mathbb{A}^n_k$ in terms of $\mathrm{Béz}(f-q)$. By a theorem of Bass, Connell, and Wright, my work gives a Bézoutian criterion for the famous Jacobian conjecture (Theorem 4).

Theorem 3 (Brazelton–McKean–Pauli)**.** *Given $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ (potentially with an isolated zero p), the coefficients of the multivariate Bézoutian with respect to any basis of $k[x_1,\ldots,x_n]/(f)$ (respectively, $k[x_1,\ldots,x_n]_{\mathfrak{m}_p}/(f)$) determine* $\deg(f)$ *(respectively,* $\deg_p(f)$*).*

Theorem 4 (McKean)**.** *Let k be a field of characteristic 0. Let $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with* $\mathrm{Jac}(f) \in k^\times$*. Then f is invertible if and only if* $\mathrm{Béz}(f-q) \bmod (f(x)-q, f(y)-q)$ *lies in k for all $q \in \mathbb{A}^n_k$.*

Constructing varieties. Together with Minahan and Zhang, I explicitly constructed all 27 lines on any smooth cubic surface in terms of the cubic surface and three skew lines. Over $\mathbb{R}$, our construction completely determine the number of real lines on the cubic surface (Theorem 5). I applied these equations to recover Segre's classification of rational line counts for smooth cubic surfaces over $\mathbb{Q}$. I also proved this classification for more general fields (Theorem 6). The existence proof for Theorem 6 is constructive — for each line count, I gave a collection of six points $\{p_1,\ldots,p_6\} \subset \mathbb{P}^2_k$ such that the blow-up of $\mathbb{P}^2_k$ at $\{p_1,\ldots,p_6\}$ is a smooth cubic surface over k realizing that line count.

Theorem 5 (McKean–Minahan–Zhang)**.** *Let X be a smooth cubic surface over $\mathbb{R}$ containing three skew lines L_1, L_2, L_3. Then (X, L_1, L_2, L_3) explicitly determine a cubic polynomial and a quadratic polynomial whose factorizations over $\mathbb{R}$ characterize the number of real lines on X.*

Theorem 6 (McKean)**.** *Let k be a finitely generated extension of $\mathbb{Q}$. Every smooth cubic surface over k contains 0, 1, 2, 3, 5, 7, 9, 15, or 27 k-rational lines, and each of these line counts occurs.*