LECTURE 5: PONTRYAGIN–THOM ISOMORPHISM

STEPHEN MCKEAN

Most of last lecture was spent proving that Ω_4^{SO} is not trivial. We did this by defining the *signature* of a $4n$-manifold, which is the signature of the symmetric non-degenerate bilinear form (on the $\mathbb{R}$-vector space $H^{2n}(M;\mathbb{R})$) induced by the cup product. We then showed that the signature gives us a well-defined group homomorphism $\sigma : \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$, and I left you the task of computing $\sigma(\mathbb{CP}^2) = 1$.

Today, our goal is to see the magic of homotopy theory. Instead of having to solve some geometric question to prove something about Ω_*^{SO}, we will translate to a more algebraic computations.

1. Basics of homotopy theory

Let's recall some definitions from last time.

Definition 1.1. Given a pointed space (X, x), the *loop space* is the topological space $\Omega_x X := [(S^1, 0), (X, x)]$. Here, we give this set of continuous maps the compact open topology.

Remark 1.2. Looping gives a functor $\Omega : \mathrm{ho}(\mathrm{Top}_*) \to \mathrm{ho}(\mathrm{Top}_*)$. This is actually the right adjoint in an adjoint pair (Σ, Ω), where Σ is *suspension*. That is,

$$[\Sigma X, Y]_{\mathrm{ho}(\mathrm{Top}_*)} \cong [X, \Omega Y]_{\mathrm{ho}(\mathrm{Top}_*)}.$$

Recall that the *smash product* of two pointed spaces X and Y is defined as the cofiber (think "quotient") $X \wedge Y := (X \times Y)/(X \vee Y)$. Given a pointed space X, the *suspension* is defined as $\Sigma X := S^1 \wedge X$.

Often, one sets up spectra in terms of Σ instead of Ω, but loops are a little more intuitively accessible if you've never thought about suspensions before. Anyway, I cannot overstate how fundamentally important (Σ, Ω) are in homotopy theory.

Exercise 1.3. Prove that $[\Sigma X, Y]_{\mathrm{ho}(\mathrm{Top}_*)} \cong [X, \Omega Y]_{\mathrm{ho}(\mathrm{Top}_*)}$ as sets.

Now we can introduce *spectra*. We'll do these in more detail later, so take this as a first approximation of a richer story.

Definition 1.4. A *spectrum* E is a sequence of pointed topological spaces $\{E_n\}_{n\geq 0}$, together with *structure maps* $e_n : E_n \to \Omega E_{n+1}$.

A morphism of spectra $\varphi : E \to F$ is a sequence of maps of pointed topological spaces $\varphi_n : E_n \to F_n$ such that the following diagrams all commute:

$$\begin{tikzcd} E_n \arrow[r, "\varphi_n"] \arrow[d, "e_n"] & F_n \arrow[d, "f_n"] \\ \Omega E_{n+1} \arrow[r, "\Omega\varphi_{n+1}"] & \Omega F_{n+1}. \end{tikzcd}$$

Remark 1.5. Recall that a map of topological spaces is called a *weak equivalence* if it induces an isomorphism on homotopy groups. We have the same notion for spectra, where we replace homotopy groups with *stable* homotopy groups

$$\pi_n^s E := \operatorname*{colim}_{k\to\infty} \pi_k E_{k+n}.$$

Remark 1.6. What is going on here? There's a theorem known as *Brown representability*, which very roughly says that cohomology should be representable by a sequence of spaces. That is, there should be spaces $\{E_n\}$ such that $H^n(X) = [X, E_n]$. But cohomology isn't just a bucket of groups — there should be some connection between H^n and H^{n+1}. Spectra naturally come out of this story, and the structure maps tie all the different dimensions together. We'll talk about this more rigorously later.

Now that I've given you a definition, we need to see an example. Here's the preview, which we'll go through more carefully in a moment. Associated to the classifying space $BG(n)$ of some Lie group $G(n)$, we will build a topological space known as the *Thom space*. This will be the one-point compactification $\mathrm{Th}(\xi_n) := \xi_n \cup \{\infty\}$ of the universal bundle $\xi_n \to BG(n)$. We will see that this construction comes with very natural structure maps $\mathrm{Th}(\xi_n) \to \Omega\mathrm{Th}(\xi_{n+1})$, so that the Thom spaces fit together to form the *Thom spectrum* MG. Finally, we will talk about the *Pontryagin–Thom isomorphism*, which relates the homotopy groups of the spectrum MG to the cobordism ring Ω_*^G.

1.1. **Classifying spaces.** If you've never seen classifying spaces before, here's a quick overview of what they are. When we were talking about spectra, I mentioned Brown representability, which says that the cohomology of a space X should actually come from maps *into* some other space Y. This is a powerful idea, because it turns an algebraic construction into something more geometric.

Classifying spaces arise from trying to apply this idea to the theory of vector bundles.

Definition 1.7. Let G be a topological group. A *principal G-bundle* over a topological space X is a bundle $P \to X$ with a G-action $\rho : G \times P \to P$ such that $(\mathrm{proj}_1, \rho) : P \times G \to P \times_X P$ is an isomorphism.

Definition 1.8. Let G be a topological group. A *classifying space* of G is a topological space BG such that there is a natural isomorphism of sets

$$\{\text{principal } G\text{-bundles over } X\}/\mathrm{iso} \cong [X, BG]$$

for sufficiently nice spaces X.

Remark 1.9. This is sort of an aspirational definition. We want this sort of space to exist, but why should it? It turns out that classifying spaces of topological groups exist and are unique up to homotopy. In fact, you can construct them as the *delooping* of G. That is, BG can be defined as the space such that $\Omega BG \simeq G$.[1]

A key player in this story is the *universal line bundle* $\xi \to BG$. This is how we get our representability result — given a map $f : X \to BG$, we get a principal G-bundle $f^*\xi$ on X. Moreover, every principal G-bundle takes this form.

This is about all we'll say on the subject for now, but we'll come back to it later. It's also important to know that we get a sequence of inclusions $B\mathrm{SO}(n) \to B\mathrm{SO}(n+1)$. Moreover, the pullback of the universal bundle $\xi_{n+1} \to B\mathrm{SO}(n+1)$ is $\xi_n \oplus \mathbb{R}$.

1.2. **Thom spaces and Thom spectra.** We are about to build a *Thom spectrum*, which is an extremely nice sort of spectrum. Today, we'll just do this story for $G(n) = \mathrm{SO}(n)$. If you want *complex cobordism*, you repeat this story with $G(n) = \mathrm{U}(n)$. From the great zoo of Lie groups, we get a great zoo of Thom spectra, which in turn tie right back to the various cobordism theories. We'll come back to this later.

Definition 1.10. If ξ is a finite-dimensional vector bundle over a compact space, the *Thom space* is the one point compactification of the total space: $\mathrm{Th}(\xi) := \xi \cup \{\infty\}$.

If ξ is finite-dimensional over a non-compact space, then $\mathrm{Th}(\xi)$ is given by one point compactifying each fiber of ξ, and then identifying the point at ∞ across all fibers.

In a moment, we'll put a sequence of Thom spaces together to form a *Thom spectrum*. However, Thom spaces are interesting in their own right, as will be evidenced by the following theorem. Unfortunately, we won't have time to prove this one, so I'll leave it as a hard exercise for the ambitious.

Theorem 1.11 (Thom isomorphism)**.** *Let X be a simply connected CW complex. Let $\pi : V \to X$ be a vector bundle of rank r. Let R be a commutative ring. Then there exists a cohomology class $u \in H^r(\mathrm{Th}(V); R)$ that induces an isomorphism*

$$H^*(X;R) \xrightarrow{\cong} \tilde{H}^{*+r}(\mathrm{Th}(V);R)$$
$$x \mapsto u \smile \pi^* x.$$

Exercise 1.12. Prove Theorem 1.11 when $R = \mathbb{Z}/2$. Hint: try proving the theorem for a trivial bundle. Then show that if the theorem is true on open subsets $U, V, U \cap V \subset X$, then it is also true on $U \cup V$.[2] Use this to prove the theorem when X is compact. When X is not compact, you'll need to apply a limit argument (which is where assuming field coefficients instead of arbitrary coefficients comes in handy).

[1] In general, you can't just deloop some random space. It turns out that having a group structure on G is precisely what allows us to deloop once. If you wanted to deloop again, you'd better hope for nice structure on G!

[2] This sort of approach is often called a *Mayer–Vietoris* argument.

Now back to Thom spaces. Let's compute a couple examples.

Example 1.13. Take the trivial line bundle $\mathbb{R} \times S^1$ on S^1. What is $\mathrm{Th}(\mathbb{R} \times S^1)$?

Example 1.14. Take the trivial line bundle $\mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$. What is $\mathrm{Th}(\mathbb{R} \times \mathbb{R})$?

These examples lead us to an important lemma.

Lemma 1.15. *Let V be a finite-dimensional vector bundle. Then $\mathrm{Th}(V \oplus \mathbb{R})$ is homotopy equivalent to $\Sigma\mathrm{Th}(V)$.*

Exercise 1.16. Prove Lemma 1.15.

Recall that the universal bundles $\xi_n \to B\mathrm{SO}(n)$ satisfy a nice pullback relation: ξ_{n+1} pulls back to $\xi_n \oplus \mathbb{R}$ under the inclusion $B\mathrm{SO}(n) \to B\mathrm{SO}(n+1)$. We can now define our first Thom spectrum.

Definition 1.17. The Thom spectrum MSO is defined as the spectrum with spaces $\mathrm{Th}(\xi_n)$, where ξ_n is the universal bundle on $B\mathrm{SO}(n)$. The structure maps are given by $\Sigma\mathrm{Th}(\xi_n) \to \mathrm{Th}(\xi_{n+1})$, which are equivalent to maps $\mathrm{Th}(\xi_n) \to \Omega\mathrm{Th}(\xi_{n+1})$ under the loops-suspension adjunction.

2. Pontryagin–Thom isomorphism

We can now state the big theorem.

Theorem 2.1 ((Pontryagin–)Thom)**.** *There is a ring isomorphism $\Omega_*^{\mathrm{SO}} \cong \pi_*\mathrm{MSO}$.*

To prove this theorem, we need to do the following:

(i) Construct functions $f_n : \Omega_n^{\mathrm{SO}} \to \pi_n\mathrm{MSO}$ for all $n \geq 0$.

(ii) Construct functions $g_n : \pi_n\mathrm{MSO} \to \Omega_n^{\mathrm{SO}}$ for all $n \geq 0$.

(iii) Prove that $(f_n)^{-1} = g_n$ for all $n \geq 0$.

(iv) Prove that f_* and g_* are ring homomorphisms.

We will do steps (i) and (ii). Step (iii) is a matter of working through the constructions that show up in (i) and (ii) to check that f_n and g_n are mutually inverse. Step (iv) then boils down to checking that f_* and g_* are additive over disjoint unions, factor over Cartesian products, and preserve the multiplicative identity.

Exercise 2.2. Verify steps (iii) and (iv).

2.1. **The Pontryagin–Thom construction.** We'll first construct $f_n : \Omega_n^{\mathrm{SO}} \to \pi_n\mathrm{MSO}$ via the *Pontryagin–Thom construction*. Given an oriented n-manifold M, use the Whitney embedding theorem to embed M in $\mathbb{R}^{n+k}$ for some k. The *normal bundle* N_M of $i : M \hookrightarrow \mathbb{R}^{n+k}$ is the quotient $\mathbb{R}^{n+k}/TM$, which is a vector bundle of rank k. The *tubular neighborhood theorem* gives us an embedding $j : N_M \hookrightarrow \mathbb{R}^{n+k}$ such that the zero section of j is the embedding $i : M \to \mathbb{R}^{n+k}$.

Note that if we collapse all of $\mathbb{R}^{n+k} - j(N_M)$ to a point, we are taking a one point compactification of N_M. In other words, this gives us the Thom space $\mathrm{Th}(N_M)$. We can extend this map to $\mathbb{R}^{n+k} \cup \{\infty\}$ by sending ∞ to the compactifying point as well. Altogether, we have a composite map

$$S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - j(N_M)) = \mathrm{Th}(N_M).$$

Finally, since $N_M \to M$ is a rank k vector bundle, it arises as the pullback of $\xi_k \to B\mathrm{SO}(k)$ under map $p : M \to B\mathrm{SO}(k)$. This setup induces a map $\mathrm{Th}(N_M) \to \mathrm{Th}(\xi_k)$. Note that the homotopy class of the composite

$$S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - j(N_M)) = \mathrm{Th}(N_M) \to \mathrm{Th}(\xi_k)$$

is an element of $\pi_{n+k}(\mathrm{MSO}(k))$, which in turn gives us an element of $\pi_n\mathrm{MSO}$. Define $f_n(M) \in \pi_n\mathrm{MSO}$ to be this element.

Lemma 2.3. *The Pontryagin–Thom construction $f_n : \Omega_n^{\mathrm{SO}} \to \pi_n\mathrm{MSO}$ is well-defined.*

Proof. We made three choices along the way: we chose an embedding, a tubular neighborhood, and a classifying map of the normal bundle. Any two tubular neighborhoods are isotopic, and any two classifying maps are homotopic, so neither of these choices changes the homotopy class of our composite map. To show that f_n is independent of our choice of embedding, we'll just wave our hands and say that you can take the standard embedding of $\mathbb{R}^{n+k}$ into $\mathbb{R}^{n+k+1}$, apply the construction there, and show that the resulting element of $\pi_{n+k+1}(\mathrm{MSO}(k+1))$ yields the same element of $\pi_n\mathrm{MSO}$. We can thus assume that $k > n+1$, and then show that any two embeddings into $\mathbb{R}^{n+k}$ must be isotopic.

Finally, we need to show that if M_1 and M_2 are cobordant, then $f_n(M_1) = f_n(M_2)$. We'll have to wave our hands at this for time's sake and say that applying the Pontryagin–Thom construction to the cobordism W will give us a homotopy between the composite maps $f_n(M_1)$ and $f_n(M_2)$. $\square$

2.2. **Transversality.** Now we will use transversality to construct $g_n : \pi_n\mathrm{MSO} \to \Omega_n^{\mathrm{SO}}$. An element α of $\pi_n\mathrm{MSO}$ is represented by a map $S^{n+k} \to \mathrm{MSO}(k)$ for some $k \geq n+1$. This map factors through $S^{n+k} \to \mathrm{Th}(\xi_{k,\ell})$, where $\xi_{k,\ell}$ is the canonical bundle of the Grassmannian of oriented k-planes in $\mathbb{R}^\ell$. We may assume that this map is smooth and transverse to the zero section of $\xi_{k,\ell}$. This zero section is a codimension k submanifold of the total space of $\xi_{k,\ell}$, so its preimage under $S^{n+k} - \{\infty\} = \mathbb{R}^{n+k} \to \xi_{k,\ell}$ is a manifold in $\mathbb{R}^{n+k}$ of dimension n. One can show that this manifold is canonically oriented, so we call its cobordism class $g_n(\alpha)$.

Lemma 2.4. *The map* $g_n : \pi_n\mathrm{MSO} \to \Omega_n^{\mathrm{SO}}$ *is well-defined.*

Proof. For the sake of time, we'll say even less about this than the previous lemma. We need to show that g_n is independent of the choice of representative $S^{n+k} \to \mathrm{MSO}(k)$, as well as the perturbations we used to get transversality. □

3. Computing $\pi_*\mathrm{MSO} \otimes \mathbb{Q}$

Theorem 2.1 tells us that if we want to compute Ω_*^{SO}, it suffices to compute $\pi_*\mathrm{MSO}$. How hard could it be? Well it turns out that this is doable, but it's still pretty hard. If you want to know the answer, you'll have to go to the great work of Wall [Wal60].

We'll just answer the easier question of computing $\pi_*\mathrm{MSO}\otimes\mathbb{Q}$. This will be a compilation of a few lemmas, each with some content that we won't entirely prove. Before we begin, we need to define the homology of the spectrum MSO.

Definition 3.1. The *stable rational homology groups* of MSO are defined as

$$H_n(\mathrm{MSO};\mathbb{Q}) := \operatorname*{colim}_{k\to\infty} H_{n+k}(\mathrm{MSO}(k);\mathbb{Q}).$$

Lemma 3.2. *We have* $\pi_*\mathrm{MSO} \otimes \mathbb{Q} \cong H_*(\mathrm{MSO};\mathbb{Q})$.

Proof. This is an isomorphism between homotopy groups and homology groups — if you've taken algebraic topology before, your brain should be screaming, "Hurewicz!" We only need the rational version of the Hurewicz theorem, which says that if X is simply connected with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i \leq r$, then $\pi_i(X) \otimes \mathbb{Q} \to \tilde{H}_i(X;\mathbb{Q})$ is an isomorphism for $0 \leq i \leq 2r$ and a surjection for $i = 2r + 1$.

Now let's think about what $\pi_*\mathrm{MSO}$ and $H_*(\mathrm{MSO})$ mean. We want to relate

$$\operatorname{colim} \pi_{n+k}\mathrm{MSO}(k) \otimes \mathbb{Q} \quad \text{and} \quad \operatorname{colim} H_{n+k}(\mathrm{MSO}(k);\mathbb{Q}).$$

I will leave it as an exercise that $\mathrm{MSO}(k)$ is simply connected with $\pi_i(\mathrm{MSO}(k)) = 0$ for $1 \leq i \leq k-1$. The rational Hurewicz theorem now tells us that $\pi_{i+k}\mathrm{MSO}(k) \otimes \mathbb{Q} \cong H_{i+k}(\mathrm{MSO}(k);\mathbb{Q})$ for $0 \leq i \leq 2k-2$. Taking the colimit as $k \to \infty$ gives us the desired result. □

Exercise 3.3. Prove that $\mathrm{MSO}(k)$ is simply connected with $\pi_i(\mathrm{MSO}(k)) = 0$ for $1 \leq i \leq k-1$. Remember that $\mathrm{MSO}(k)$ is a Thom space of a rank k vector bundle over $B\mathrm{SO}(k)$.

Lemma 3.4. *We have* $H_*(\mathrm{MSO};\mathbb{Q}) \cong H_*(B\mathrm{SO};\mathbb{Q})$.

Proof. This is an application of the Thom isomorphism from earlier. We had phrased it in terms of cohomology, but on homology we get an isomorphism $H_{i+k}(\mathrm{MSO}(k);\mathbb{Q}) \cong H_i(B\mathrm{SO}(k);\mathbb{Q})$. Here, the degree k shift comes from the fact that $\mathrm{MSO}(k)$ is the Thom space of a rank k vector bundle over $B\mathrm{SO}(k)$.

It remains to show that $H_i(B\mathrm{SO};\mathbb{Q}) \cong \operatorname{colim} H_i(B\mathrm{SO}(k);\mathbb{Q})$. To see this, it suffices to show that the inclusion $B\mathrm{SO}(k) \to B\mathrm{SO}$ is *k-connected*. It follows that $H_i(B\mathrm{SO}(k);\mathbb{Q}) \cong H_i(B\mathrm{SO};\mathbb{Q})$ for $k > i$, so the desired result holds by taking the colimit as $k \to \infty$. $\square$

Exercise 3.5. Show that the inclusion $B\mathrm{SO}(k) \to B\mathrm{SO}$ is k-connected. If you don't know what it means for a map of topological spaces to be k-connected, feel free to ask me (or consult the world wide web).

Remark 3.6. So far, we've turned a geometric question (computing $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$) into a homotopical question (computing $\pi_*\mathrm{MSO}\otimes\mathbb{Q}$), which we've now turned into an algebraic question (computing $H_*(B\mathrm{SO};\mathbb{Q})$).

We had to do some math for each of these translations, but it feels like everything we've done so far is easier than computing Ω_*^{SO} itself. By *conservation of math*, there should be as much mathematical content (whatever that means) in proving Ω_*^{SO} geometrically or via our current route. The next step is to compute $H_*(B\mathrm{SO};\mathbb{Q})$. This is a "standard" computation, but it generally involves several other theorems. We will certainly be out of time for this today, so I encourage you to look up a proof of the following lemma.

Lemma 3.7. *We have $H_*(B\mathrm{SO};\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots]$, where $|p_i| = 4i$.*

The generators p_i are typically chosen to be the *Pontryagin classes*, which belong in any basic toolkit of cohomology classes.

We're almost there. The last thing we need to do is justify the "equation" $p_i = [\mathbb{CP}^{2i}]$.

Exercise 3.8. Building on last lecture, show that the signature of a $4n$-manifold induces a group homomorphism $\sigma : \Omega_{4n}^{\mathrm{SO}} \to \mathbb{Z}$. Prove that $\sigma(\mathbb{CP}^{2n}) = 1$, and deduce that $\{\mathbb{CP}^2, \mathbb{CP}^4, \ldots\}$ generate $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$.

Next time: L-genus, the Hirzebruch signature theorem, and spin geometry.

Daily exercises: I decided to stop collecting the exercises here. If you really want me to put them at the end of the notes like before, let me know!

References

[Wal60] C. T. C. Wall. "Determination of the cobordism ring". In: *Ann. of Math. (2)* 72 (1960), pp. 292–311. URL: `https://doi.org/10.2307/1970136`.

Department of Mathematics, Harvard University

Email address: `smckean@math.harvard.edu`

URL: `shmckean.github.io`