CONICS MEETING EIGHT LINES OVER PERFECT FIELDS

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ABSTRACT. Over the complex numbers, there are 92 plane conics meeting 8 general lines in projective 3-space. Using the Euler class and local degree from motivic homotopy theory, we give an enriched version of this result over any perfect field. This provides a weighted count of the number of plane conics meeting 8 general lines, where the weight of each conic is determined the geometry of its intersections with the 8 given lines. As a corollary, real conics meeting 8 general lines come in two families of equal size.

1. INTRODUCTION

The space of plane conics in $\mathbb{P}^3$ is 8 dimensional. If we require that a conic intersects a given line, we impose one condition and lose one degree of freedom on the space of plane conics. As a result, the space of plane conics meeting 8 general lines is a 0 dimensional Noetherian scheme and is therefore a finite set. A classical theorem of enumerative geometry gives the cardinality of this set.

Theorem 1.1. *Let k be an algebraically closed field with char $k \neq 2$. Given 8 general lines in $\mathbb{P}^3_k$, there are 92 plane conics meeting all 8 lines. Moreover, each of these plane conics is smooth. (See e.g. [Ful98, Ex. 3.2.22] or [EH16, Theorem 9.26].)*

As with many results in classical enumerative geometry, this theorem is only true over an algebraically closed field. The $\mathbb{A}^1$-*enumerative geometry* program seeks to generalize such theorems using various tools from motivic homotopy theory.[1] We give an $\mathbb{A}^1$-enumerative generalization of Theorem 1.1.

We start with some notation. Let k be a perfect field with char $k \neq 2$. Given a conic $q \subset \mathbb{P}^3_k$, let $k(q)$ be its field of definition. Let $\mathrm{GW}(k)$ be the Grothendieck–Witt group of isomorphism classes of symmetric, non-degenerate bilinear forms over k. Given $a \in k^\times$, let $\langle a \rangle \in \mathrm{GW}(k)$ be the bilinear form given by $(x, y) \mapsto axy$. Finally, let $\mathrm{Tr}_{k(q)/k} : \mathrm{GW}(k(q)) \to \mathrm{GW}(k)$ be induced by the field trace.

Theorem 1.2. *Let $L_1, \ldots, L_8$ be general lines in $\mathbb{P}^3_k$. Let Q be the set of all plane conics in $\mathbb{P}^3_k$ meeting $L_1, \ldots, L_8$. Then*

$$(1.1) \qquad 46\langle 1 \rangle + 46\langle -1 \rangle = \sum_{q \in Q} \mathrm{Tr}_{k(q)/k}\langle a_q \rangle,$$

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[1]See [KW21, Lev20, LR20, BKW20, SW21, LV19, McK21, Pau20, CDH20] for some examples or [Bra20, PW20] for a survey.

where $a_q \in k(q)^\times$ is a constant determined by the conic q, the intersections $L_i \cap q$, and the tangent lines $T_{L_i \cap q} q$ for $1 \leq i \leq 8$.

Theorem 1.2 gives some insight into real conics meeting 8 general lines. Hauenstein and Sottile showed that over $\mathbb{R}$, there can be $2n$ real conics meeting 8 lines for $0 \leq n \leq 45$ [HS12, Table 6]. Griffin and Hauenstein completed this result by constructing 8 lines over $\mathbb{R}$ such that all 92 conics meeting these lines are real [GH15, Theorem 1]. Our work illuminates a small amount of extra structure on this set of $2n$ conics. Taking the signature of Equation 1.1 yields Theorem 5.3, which states that the $2n$ real conics meeting 8 general lines fall into two families of n conics.

1.1. **General approach and outline.** Our goal is to prove an equality in $\mathrm{GW}(k)$, the Grothendieck–Witt group of isomorphism classes of non-degenerate symmetric bilinear forms over k. One side of this equation will be given by an Euler class [KW21, BW20], which is valued in $\mathrm{GW}(k)$ in the context of motivic homotopy theory. The other side of this equation will consist of a sum of local contributions, which are analogs of the local Brouwer degree [Mor12, KW19, KW21]. The final step is to find a formula for these local contributions in terms of the geometry at hand – in our case, the geometry of lines meeting a plane conic.

In order to make use of Euler classes and local degrees, we need to phrase our enumerative problem in terms of a vector bundle over a scheme parameterizing conics in $\mathbb{P}^3$. This has been done classically [EH16, Chapter 9.7]. We will recall the relevant details here.

Let k be a field. Let $\mathbb{G}(2,3)$ be the Grassmannian of 2-planes in $\mathbb{P}^3_k$. Let $\mathcal{S}$ be the universal subbundle of $\mathbb{G}(2,3)$. Consider the symmetric bundle $\mathrm{Sym}^2(\mathcal{S}^\vee)$ of planar quadratic forms, which is a rank 6 vector bundle over $\mathbb{G}(2,3) \cong \mathbb{P}^3_k$. The projective bundle $X = \mathbb{P}\,\mathrm{Sym}^2(\mathcal{S}^\vee)$ is the moduli scheme of all plane conics in $\mathbb{P}^3_k$. That is, points in X are of the form (H, q), where $H \subset \mathbb{P}^3_k$ is a 2-plane and q is the projective equivalence class of a quadratic form on H corresponding to the conic section $\mathbb{V}(q) \subset H$.

Now consider the line bundle $\mathcal{O}_X(1) \to X$. Sections of $\mathcal{O}_X(1)$ are given by linear forms on X, which are linear forms on the space of quadratic forms on 2-planes in $\mathbb{P}^3$. Given a line $L \subset \mathbb{P}^3_k$, we get a section $\sigma_L : X \to \mathcal{O}_X(1)$ given by evaluating q at the point $H \cap L$. That is, $\sigma_L(H, q) = ((H, q), q(H \cap L))$. Note that $\sigma_L(H, q) = ((H, q), 0)$ if and only if $H \cap L$ lies on the conic section cut out by q. Given 8 general lines $L_1, \ldots, L_8 \subset \mathbb{P}^3_k$, the section $\sigma := \bigoplus_{i=1}^8 \sigma_{L_i} : X \to \mathcal{O}_X(1)^{\oplus 8}$ vanishes if and only if $L_1, \ldots, L_8$ all meet the conic section cut out by q.

The classical count of conics meeting 8 general lines in $\mathbb{P}^3_\mathbb{C}$ is given by $\int_X c_1(\mathcal{O}_X(1))^8 = 92$ [Ful98, Ex 3.2.22]. The enriched count of conics meeting 8 general lines over a field k is given by the Euler class $e(\mathcal{O}_X(1)^{\oplus 8}) \in \mathrm{GW}(k)$, which we can compute using a result of Srinivasan and Wickelgren [SW21, Proposition 19]. This Euler class is equal to a sum of local information over the set of conics (H, q) meeting $L_1, \ldots, L_8$ [KW21, Theorem 3]:

$$(1.2) \qquad\qquad e(\mathcal{O}_X(1)^{\oplus 8}) = \sum_{(H,q) \in \sigma^{-1}(0)} \mathrm{ind}_{(H,q)}(\sigma).$$

After computing the Euler class in Lemma 3.2, we will address the local indices $\mathrm{ind}_{(H,q)}(\sigma)$. Given a conic (H, q) in an affine neighborhood $U \subset X$, we will give affine coordinates $\varphi_U : \mathbb{A}_k^8 \to U$ and local trivializations (post-composed with projection) $\psi_U : \mathcal{O}_X(1)^{\oplus 8}|_U \to U \times \mathbb{A}_k^8 \to \mathbb{A}_k^8$ in Section 3.2. The local index $\mathrm{ind}_{(H,q)}(\sigma)$ is equal to the local $\mathbb{A}^1$-degree $\deg_{\varphi_U^{-1}(H,q)}^{\mathbb{A}^1}(\Phi_U)$ of the composite

$$\Phi_U := \psi_U \circ \sigma \circ \varphi_U : \mathbb{A}_k^8 \to \mathbb{A}_k^8$$

at a zero $(H, q) \in U$. In Section 4, we give an alternate formula for $\deg_{\varphi_U^{-1}(H,q)}^{\mathbb{A}^1}(\Phi_U)$ in terms of the intersection points $L_i \cap H$ and tangent lines of the conic $\mathbb{V}(q) \cap H$. By replacing $\mathrm{ind}_{(H,q)}(\sigma)$ in Equation 1.2 with our alternate formula in terms of geometric information, we recover Theorem 1.2. We conclude with a discussion of real conics in Section 5.

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2. Background in $\mathbb{A}^1$-enumerative geometry

In classical enumerative geometry, one is interested in (possibly weighted) integer-valued counts of geometric objects. For example, if Q is the set of plane conics meeting 8 general lines in $\mathbb{P}_{\mathbb{C}}^3$, then

$$(2.1) \qquad\qquad 92 = \sum_{q \in Q} 1.$$

In $\mathbb{A}^1$-enumerative geometry, we replace such integer-valued counts with bilinear form-valued counts. We will show that if Q is the set of plane conics meeting 8 general lines in $\mathbb{P}_k^3$ over a perfect field k, then

$$(2.2) \qquad\qquad 46\langle 1 \rangle + 46\langle -1 \rangle = \sum_{q \in Q} B_q.$$

Here, the bilinear form $\langle a \rangle : k \times k \to k$ is given by $(x, y) \mapsto axy$. The weight B_q is a bilinear form determined by geometric information associated to the plane conic q (see Section 4). By taking field invariants, we can recover enumerative equations over specific fields. For example, taking the rank of Equation 2.2 recovers Equation 2.1, while taking the signature of Equation 2.2 yields a new theorem (Theorem 5.3) giving a weighted count of conics meeting 8 general lines over $\mathbb{R}$.

2.1. **Grothendieck–Witt groups.** The significance of bilinear forms in $\mathbb{A}^1$-enumerative geometry stems from Morel's calculation of the Brouwer degree in $\mathbb{A}^1$-homotopy theory (also known as motivic homotopy theory):

Theorem 2.1. *[Mor12, Corollary 1.24] For $n \geq 2$, there is a group (and in fact, ring) isomorphism*

$$\deg^{\mathbb{A}^1} : [\mathbb{P}_k^n/\mathbb{P}_k^{n-1}, \mathbb{P}_k^n/\mathbb{P}_k^{n-1}]_{\mathbb{A}^1} \xrightarrow{\cong} \mathrm{GW}(k),$$

where $[-,-]_{\mathbb{A}^1}$ denotes $\mathbb{A}^1$-homotopy classes of maps, $\mathbb{P}_k^n/\mathbb{P}_k^{n-1}$ is a motivic space playing the role of the sphere, and $\mathrm{GW}(k)$ is the Grothendieck–Witt group of isomorphism classes of symmetric non-degenerate bilinear forms over k.

Morel's degree map is analagous to the Brouwer degree

$$\deg : [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}.$$

One can apply Morel's degree to endomorphisms of $\mathbb{A}_k^n$ to obtain bilinear forms. The goal of $\mathbb{A}^1$-enumerative geometry is to perform this process in such a way that the resulting bilinear forms encode enumerative information. Later in this section, we will discuss how to circumvent explicitly using motivic homotopy theory in $\mathbb{A}^1$-enumerative geometry. First, we briefly discuss $\mathrm{GW}(k)$. The Grothendieck–Witt group $\mathrm{GW}(k)$ is actually a ring that admits a nice presentation.

Proposition 2.2. *[Lam05, II Theorem 4.1] Let k be a field. Given $a \in k^\times$, let $\langle a \rangle$ be the isomorphism class of the bilinear form $k \times k \to k$ defined by $(x, y) \mapsto axy$. Then $\mathrm{GW}(k)$ is the ring generated by all such $\langle a \rangle$, subject to the following relations.*

(i) $\langle ab^2 \rangle = \langle a \rangle$ for all $a, b \in k^\times$.

(ii) $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for all $a, b \in k^\times$.

(iii) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for all $a, b \in k^\times$ such that $a + b \neq 0$.

(iv) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$ for all $a \in k^\times$.

Relation (iv) actually follows from relations (i) and (iii). Indeed, (iii) implies that $\langle -a \rangle + \langle a - 1 \rangle = \langle a^2 - a \rangle + \langle -1 \rangle$ and that $\langle a \rangle + \langle a^2 - a \rangle = \langle a^2 \rangle + \langle a^2(a-1) \rangle$. Thus by (i), we have

$$\langle a \rangle + \langle -a \rangle = \langle a \rangle + \langle a^2 - a \rangle + \langle -1 \rangle - \langle a - 1 \rangle$$
$$= \langle a^2 \rangle + \langle a^2(a-1) \rangle + \langle -1 \rangle - \langle a - 1 \rangle$$
$$= \langle 1 \rangle + \langle a - 1 \rangle + \langle -1 \rangle - \langle a - 1 \rangle$$
$$= \langle 1 \rangle + \langle -1 \rangle.$$

Definition 2.3. The isomorphism class $\mathbb{H} := \langle 1 \rangle + \langle -1 \rangle$ is called the *hyperbolic form.*

In order to obtain enumerative statements over a given field, we apply *field invariants* to our enumerative equation in $\mathrm{GW}(k)$. Field invariants can be thought of as group homomorphisms $\mathrm{GW}(k) \to G$ for some group G. For example:

(i) The rank of a bilinear form induces an isomorphism $\mathrm{rank} : \mathrm{GW}(\mathbb{C}) \to \mathbb{Z}$.

(ii) The signature of a bilinear form (the number of $+1$s minus the number of -1s on the diagonal) induces a homomorphism $\mathrm{sign} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$.

(iii) The discriminant of a bilinear form induces a homomorphism disc : $\mathrm{GW}(\mathbb{F}_q) \to \mathbb{Z}/2\mathbb{Z}$ when q is a power of an odd prime.

See [Lam05] for a discussion on field invariants for various fields. See [KW21, LV19, McK21, CDH20] for examples of applying these field invariants to obtain enumerative statements over specific fields.

2.2. **Local degrees and Euler classes.** The local $\mathbb{A}^1$-degree of a map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ at an isolated zero p is the $\mathbb{A}^1$-degree of a map $f_p : \mathbb{A}^n_k \to \mathbb{A}^n_k$ whose only zero is p. While the $\mathbb{A}^1$-degree and local $\mathbb{A}^1$-degree are homotopically defined, they admit a convenient commutative algebraic formulation [BMP21] (see also [KW19, BBM$^+$21]). For our purposes, we will be able to compute all relevant local $\mathbb{A}^1$-degrees in terms of the Jacobian by [KW19, Lemma 9] (which stems from [SS75, (4.7) Korollar]).

The Euler classes that we work with were introduced in [KW21] and further studied in [BW20]. See [KW21, Section 1.1] and the introduction of [BW20] for a discussion of related notions of Euler classes in arithmetic geometry.

3. COORDINATES, TRIVIALIZATIONS, AND RELATIVE ORIENTABILITY

Let $X = \mathbb{P}\operatorname{Sym}^2(\mathcal{S}^\vee)$, as described in Section 1.1. We first prove that $\mathcal{O}_X(1)^{\oplus 8} \to X$ is relatively orientable.

Lemma 3.1. *The vector bundle $\mathcal{O}_X(1)^{\oplus 8} \to X$ is relatively orientable.*

Proof. In order to show that $\mathcal{O}_X(1)^{\oplus 8} \to X$ is relatively orientable, we need to show that $\det \mathcal{O}_X(1)^{\oplus 8} \otimes \omega_X$ is the tensor square of a line bundle, where ω_X is the canonical bundle of X. Since $\mathcal{O}_X(1)^{\oplus 8}$ is a direct sum of line bundles, we have $\det \mathcal{O}_X(1)^{\oplus 8} \cong \mathcal{O}_X(8)$.

Given a vector bundle $V \to Y$ of rank r, the canonical bundle of $\pi : \mathbb{P}V \to Y$ is given by

(3.1) $$\omega_{\mathbb{P}V} \cong \mathcal{O}_{\mathbb{P}V}(-r) \otimes \pi^* \det V^\vee \otimes \pi^* \omega_Y.$$

This can be computed via the short exact sequence of tangent bundles

$$0 \to T_{\mathbb{P}V/Y} \to T_{\mathbb{P}V} \to \pi^* T_Y \to 0$$

and the tautological exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}V}(-1) \to \pi^* V \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the tautological quotient bundle of $\mathbb{P}V \to Y$.

Before applying Equation 3.1 to $X = \mathbb{P}\operatorname{Sym}^2(\mathcal{S}^\vee) \to \mathbb{G}(2,3)$, we need to calculate $\det \operatorname{Sym}^2(\mathcal{S}^\vee)^\vee$ and $\omega_{\mathbb{G}(2,3)}$. Since $\mathbb{G}(2,3) \cong \mathbb{P}^3$, we have $\omega_{\mathbb{G}(2,3)} \cong \mathcal{O}_{\mathbb{G}(2,3)}(-4)$. Recall that if $\mathcal{E}$ is a vector bundle of rank r, then $\det \operatorname{Sym}^n(\mathcal{E}) = (\det \mathcal{E})^{\otimes \binom{r+n-1}{r}}$. Since $\mathcal{S}^\vee$ has rank 3 and $\det \mathcal{S} \cong \mathcal{O}_{\mathbb{G}(2,3)}(1)$, we have $\det \operatorname{Sym}^2(\mathcal{S}^\vee)^\vee = (\det \mathcal{S})^{\otimes 4} \cong \mathcal{O}_{\mathbb{G}(2,3)}(4)$. Equation 3.1 thus gives us

$$\begin{aligned} \omega_X &\cong \mathcal{O}_X(-6) \otimes \pi^* \mathcal{O}_{\mathbb{G}(2,3)}(4) \otimes \pi^* \mathcal{O}_{\mathbb{G}(2,3)}(-4) \\ &\cong \mathcal{O}_X(-6). \end{aligned}$$

Thus $\det \mathcal{O}_X(1)^{\oplus 8} \otimes \omega_X \cong \mathcal{O}_X(2) \cong \mathcal{O}_X(1)^{\otimes 2}$, as desired. $\qquad\square$

Since $\mathcal{O}_X(1)^{\oplus 8}$ is relatively orientable, its Euler class is well-defined. We now compute the Euler class $e(\mathcal{O}_X(1)^{\oplus 8})$ using [SW21, Lemma 5 and Proposition 19], which will constitute the fixed global count of conics meeting 8 general lines.

Lemma 3.2. *The vector bundle $\mathcal{O}_X(1)^{\oplus 8} \to X$ has Euler class $e(\mathcal{O}_X(1)^{\oplus 8}) = 46 \cdot \mathbb{H}$.*

Proof. Apply [SW21, Proposition 19] with $\mathcal{E} := \mathcal{O}_X(1)^{\oplus 7}$ and $\mathcal{E}' := \mathcal{O}_X(1)$. It follows that $e(\mathcal{O}_X(1)^{\oplus 8})$ is of the form $n \cdot \mathbb{H}$, which has rank $2n$. Taking the rank of this Euler class recovers the classical count $\int_X c_1(\mathcal{O}_X(1))^8 = 92$ [SW21, Lemma 5], so we have $e(\mathcal{O}_X(1)^{\oplus 8}) = 46 \cdot \mathbb{H}$. $\qquad\square$

3.1. **Local decomposition.** The Euler class $e(\mathcal{O}_X(1)^{\oplus 8})$ gives us half of our desired enumerative formula. In order to complete this enumerative formula, we need to express $e(\mathcal{O}_X(1)^{\oplus 8})$ as a sum of local contributions (see [KW21, Theorem 3]). To start, we will describe the section $\sigma : X \to \mathcal{O}_X(1)^{\oplus 8}$ whose vanishing locus corresponds to the set of conics meeting 8 general lines $L_1, \ldots, L_8 \subset \mathbb{P}_k^3$. Our enumerative formula will then come from computing the local indices $\mathrm{ind}_{(H,q)}(\sigma)$ in the decomposition

$$e(\mathcal{O}_X(1)^{\oplus 8}) = \sum_{(H,q) \in \sigma^{-1}(0)} \mathrm{ind}_{(H,q)}(\sigma).$$

Sections of $\mathcal{O}_X(1)$ are given by linear forms on X. Since X is the space of quadratic forms on planes in $\mathbb{P}_k^3$, we obtain sections of $\mathcal{O}_X(1)$ be evaluating quadratic forms. Let $[y_0 : y_1 : y_2 : y_3]$ be coordinates on $\mathbb{P}_k^3$. Let $L_1, \ldots, L_8 \subset \mathbb{P}_k^3$ be general lines. For $1 \le i \le 8$, let $p_i = [p_{i0} : p_{i1} : p_{i2} : p_{i3}]$ and $s_i = [s_{i0} : s_{i1} : s_{i2} : s_{i3}]$ be points in $\mathbb{P}_k^3$ such that

$$L_i = \{tp_i + (1-t)s_i : t \in \mathbb{P}_k^1\}.$$

Given $\mathbf{a} = [a_0 : a_1 : a_2 : a_3] \in \mathbb{P}_k^3$, let $H_{\mathbf{a}} = \mathbb{V}(\sum_{i=0}^3 a_i y_i)$. If $L_i \not\subset H_{\mathbf{a}}$, then this plane and line intersect in $L_i \cap H_{\mathbf{a}} = [x_{i0} : x_{i1} : x_{i2} : x_{i3}]$, where

$$(3.2) \qquad\qquad x_{ij}(\mathbf{a}) = x_{ij} := \sum_{\ell=0}^3 a_\ell (p_{i\ell} s_{ij} - p_{ij} s_{i\ell}).$$

Definition 3.3. We define the section $\sigma_i : X \to \mathcal{O}_X(1)$ by $\sigma_i(H, q) = ((H, q), q(L_i \cap H))$. In particular, if q is the projective equivalence class of a quadratic form on the plane $H_{\mathbf{a}}$, then $\sigma_i(H_{\mathbf{a}}, q) = ((H_{\mathbf{a}}, q), q(x_{i0}, x_{i1}, x_{i2}, x_{i3}))$. We define $\sigma := \bigoplus_{i=1}^8 \sigma_i : X \to \mathcal{O}_X(1)^{\oplus 8}$.

Remark 3.4. Note that the sections σ_i (and hence σ) depend on our choices of $p_{i0}, \ldots, p_{i3}$ and $s_{i0}, \ldots, s_{i3}$ (which give us a parameterization of L_i). However, we will later see that multiplying p_{ij} or s_{ij} by a non-zero scalar does not change the local index $\mathrm{ind}_{(H,q)}(\sigma)$, so we are free to work with any representatives of p_i and s_i.

3.2. **Affine coordinates and local trivializations.** In order to compute the local index $\mathrm{ind}_{(H,q)}(\sigma)$ of a conic (H,q) meeting the lines $L_1, \ldots, L_8$, we need to describe affine coordinates for $X = \mathbb{P}\,\mathrm{Sym}^2(\mathcal{S}^\vee)$ and local trivializations of $\mathcal{O}_X(1)^{\oplus 8}$. Since X is a $\mathbb{P}_k^5$-bundle over $\mathbb{G}(2,3) \cong \mathbb{P}_k^3$, the standard affine covers of $\mathbb{P}_k^3$ and $\mathbb{P}_k^5$ yield a convenient affine cover of X.

Notation 3.5. Given a polynomial f over k, let $[f]$ denote the equivalence class of all $k^\times$ multiples of f.

Notation 3.6. Let $\mathbf{a} = [a_0 : a_1 : a_2 : a_3] \in \mathbb{P}_k^3$, and let $\mathbf{b} = [b_0 : \cdots : b_5] \in \mathbb{P}_k^5$. Let $H_{\mathbf{a}} = \mathbb{V}(\sum_{i=0}^3 a_i y_i)$. Given coordinates z_0, z_1, z_2 on $H_{\mathbf{a}}$, let

$$q_{\mathbf{b}}(z_0, z_1, z_2) = [b_0 z_0^2 + b_1 z_1^2 + b_2 z_2^2 + b_3 z_1 z_2 + b_4 z_0 z_2 + b_5 z_0 z_1].$$

Definition 3.7. Let $0 \le i \le 3$ and $0 \le j \le 5$. Define *affine coordinates* to be the maps $\varphi_{ij} : \mathbb{A}_k^8 \to X$ given by $(a_1, a_2, a_3, b_1, \ldots, b_5) \mapsto (H_{\alpha_i(a_1,a_2,a_3)}, q_{\beta_j(b_1,\ldots,b_5)}(z_{i0}, z_{i1}, z_{i2}))$, where

$$\alpha_i(a_1, a_2, a_3) = [a_1 : \cdots : 1 : \cdots : a_3] \in \mathbb{P}_k^3,$$
$$\beta_j(b_1, \ldots, b_5) = [b_1 : \cdots : 1 : \cdots : b_5] \in \mathbb{P}_k^5$$

are the projective points given by inserting 1 in the ith and j^{th} coordinates, respectively, and

$$(3.3) \qquad (z_{i0}, z_{i1}, z_{i2}) = \begin{cases} (y_1, y_2, y_3) & i = 0, \\ (y_0, y_2, y_3) & i = 1, \\ (y_0, y_1, y_3) & i = 2, \\ (y_0, y_1, y_2) & i = 3. \end{cases}$$

Denote $U_{ij} := \varphi_{ij}(\mathbb{A}_k^8)$.

The next proposition states that our affine coordinates satisfy a technical condition necessary for computing the $\mathbb{A}^1$-degree.

Proposition 3.8. *The affine coordinates $\varphi_{ij} : \mathbb{A}_k^8 \to U_{ij}$ induce Nisnevich coordinates $\varphi_{ij}^{-1} : U_{ij} \to \mathbb{A}_k^8$ in the sense of [KW21, Definition 18].*

Proof. We need to show that φ_{ij}^{-1} is an étale map that induces an isomorphism of residue fields $k(H,q) \cong k(\varphi_{ij}^{-1}(H,q))$ for any closed point $(H,q) \in U_{ij}$. In fact, U_{ij} is isomorphic to $\mathbb{P}_3^k|_{U_i} \times \mathbb{P}_k^5|_{U_j}$, where U_i (respectively U_j) is the ith (respectively j^{th}) standard affine patch of $\mathbb{P}_k^3$ (respectively $\mathbb{P}_k^5$). Moreover, φ_{ij}^{-1} is constructed to be the isomorphism $U_{ij} \xrightarrow{\cong} \mathbb{P}_k^3|_{U_i} \times \mathbb{P}_k^5|_{U_j} \xrightarrow{\cong} \mathbb{A}_k^3 \times \mathbb{A}_k^5$ and is therefore étale and induces an isomorphism of residue fields on closed points. $\square$

Next, we give local trivializations $\psi_{ij} : \mathcal{O}_X(1)^{\oplus 8}|_{U_{ij}} \to U_{ij} \times \mathbb{A}_k^8$. We do this by describing local trivializations $\psi_{ij}' : \mathcal{O}_X(1)|_{U_{ij}} \to U_{ij} \times \mathbb{A}_k^1$ and setting $\psi_{ij} = \bigoplus_{\ell=1}^8 \psi_{ij}'$. (Later, we

will conflate ψ_{ij} with the composite $\mathcal{O}_X(1)^{\oplus 8}|_{U_{ij}} \to U_{ij} \times \mathbb{A}_k^8 \to \mathbb{A}_k^8$.) To define ψ'_{ij}, it suffices to construct a non-vanishing section $\tau_{ij} : U_{ij} \to \mathcal{O}_X(1)|_{U_{ij}}$.

Definition 3.9. For $0 \leq i \leq 3$ and $0 \leq j \leq 2$, let $T_{ij} := \mathbb{V}(\{z_{i\ell}\}_{\ell \neq j})$, so that

$$T_{ij} \cap H_{\mathbf{a}} = \begin{cases} [-a_1 : a_0 : 0 : 0] & (i,j) = (0,0), \\ [-a_2 : 0 : a_0 : 0] & (i,j) = (0,1), \\ [-a_3 : 0 : 0 : a_0] & (i,j) = (0,2), \\ \quad \vdots & \\ [a_3 : 0 : 0 : -a_0] & (i,j) = (3,0), \\ [0 : a_3 : 0 : -a_1] & (i,j) = (3,1), \\ [0 : 0 : a_3 : -a_2] & (i,j) = (3,2). \end{cases}$$

Let $\tau_{ij} : U_{ij} \to \mathcal{O}_X(1)|_{U_{ij}}$ be the section defined by

$$\tau_{ij}(H_{\mathbf{a}}, q_{\mathbf{b}}) = ((H_{\mathbf{a}}, q_{\mathbf{b}}), q_{\mathbf{b}}(T_{ij} \cap H_{\mathbf{a}})).$$

By construction, τ_{ij} is a non-vanishing section on U_{ij} and hence determines a local trivialization of $\mathcal{O}_X(1)|_{U_{ij}}$. In particular, $q_{\mathbf{b}}(T_{ij} \cap H_{\mathbf{a}}) = a_i^2 b_j$.

Definition 3.10. For $0 \leq i \leq 3$ and $3 \leq j \leq 5$, let $T_{ij} := \mathbb{V}(z_{i,j-3}, z_{im} - z_{in})$, where $m < n$ and $\{j-3, m, n\} = \{0, 1, 2\}$. For example, $T_{03} = \mathbb{V}(z_{00}, z_{01} - z_{02}) = \mathbb{V}(y_1, y_2 - y_3)$. Then

$$T_{ij} \cap H_{\mathbf{a}} = \begin{cases} [-a_2 - a_3 : 0 : a_0 : a_0] & (i,j) = (0,3), \\ [-a_1 - a_3 : a_0 : 0 : a_0] & (i,j) = (0,4), \\ [-a_1 - a_2 : a_0 : a_0 : 0] & (i,j) = (0,5), \\ \quad \vdots & \\ [a_3 : a_3 : 0 : -a_0 - a_1] & (i,j) = (3,3), \\ [a_3 : 0 : a_3 : -a_0 - a_2] & (i,j) = (3,4), \\ [0 : a_3 : a_3 : -a_1 - a_2] & (i,j) = (3,5). \end{cases}$$

As before, we let $\tau_{ij} : U_{ij} \to \mathcal{O}_X(1)|_{U_{ij}}$ be the section defined by

$$\tau_{ij}(H_{\mathbf{a}}, q_{\mathbf{b}}) = ((H_{\mathbf{a}}, q_{\mathbf{b}}), q_{\mathbf{b}}(T_{ij} \cap H_{\mathbf{a}})),$$

which is non-vanishing on U_{ij} and hence gives a local trivialization of $\mathcal{O}_X(1)|_{U_{ij}}$. In particular, $q_{\mathbf{b}}(T_{ij} \cap H_{\mathbf{a}}) = a_i^2 b_j$.

Remark 3.11. As with our sections $\sigma_i : X \to \mathcal{O}_X(1)$, our non-vanishing sections $\tau_{ij} : U_{ij} \to \mathcal{O}_X(1)|_{U_{ij}}$ depend on our choice of representative of the projective point $T_{ij} \cap H_{\mathbf{a}}$. However, a different choice of representative coordinates for $T_{ij} \cap H_{\mathbf{a}}$ only changes τ_{ij} by a square in $k^\times$, which will not change the local index $\mathrm{ind}_{(H,q)}(\sigma)$.

3.3. **Compatibility.** Using our affine coordinates φ_{ij} and local trivializations ψ_{ij}, we will compute the local index $\mathrm{ind}_{(H,q)}(\sigma)$ in terms of the local $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}_{\varphi_{ij}^{-1}(H,q)}(\psi_{ij} \circ \sigma \circ \varphi_{ij})$ via [KW21, Definition 30 and Corollary 31]. In order to do this, we need to certify that φ_{ij} and ψ_{ij} satisfy a technical condition with respect to the relative orientation

$$\mathrm{Hom}(\det T_X, \det \mathcal{O}_X(1)^{\oplus 8})|_{U_{ij}} \cong \mathcal{O}_X(1)^{\otimes 2}|_{U_{ij}}$$

from Lemma 3.1. First, let

$$\partial_{ij} := \bigwedge_{\substack{0 \le \ell \le 3 \\ \ell \ne i}} \frac{\partial}{\partial(a_\ell/a_i)} \wedge \bigwedge_{\substack{0 \le \ell \le 5 \\ \ell \ne j}} \frac{\partial}{\partial(b_\ell/b_j)} \in \Gamma(\det T_X, U_{ij})$$

be the non-vanishing section determined by our affine coordinates on U_{ij}. Let

$$\gamma_{ij} : \mathrm{Hom}(\det T_X, \det \mathcal{O}_X(1)^{\oplus 8})|_{U_{ij}} \xrightarrow{\cong} \mathrm{Hom}(\mathcal{O}_X(6), \det \mathcal{O}_X(1)^{\oplus 8})|_{U_{ij}}$$

be the isomorphism induced by the isomorphism $\det T_X|_{U_{ij}} \cong \mathcal{O}_X(6)|_{U_{ij}}$ sending ∂_{ij} to the non-vanishing section $(a_i^2 b_j)^6 \in \Gamma(\mathcal{O}_X(6), U_{ij})$. Let

$$\delta_{ij} : \mathrm{Hom}(\mathcal{O}_X(6), \det \mathcal{O}_X(1)^{\oplus 8})|_{U_{ij}} \xrightarrow{\cong} \mathrm{Hom}(\mathcal{O}_X(6), \mathcal{O}_X(8))|_{U_{ij}} \xrightarrow{\cong} \mathcal{O}_X(1)^{\otimes 2}|_{U_{ij}}$$

be the isomorphism given by sending $[(a_i^2 b_j)^6 \mapsto \bigwedge_{\ell=1}^8 (a_i^2 b_j)]$ to $(a_i^2 b_j) \otimes (a_i^2 b_j)$.

Proposition 3.12. *The local trivializations ψ_{ij} are compatible (in the sense of [KW21, Definition 21]) with the affine coordinates φ_{ij} and the relative orientation*

$$\varpi_{ij} := \delta_{ij} \circ \gamma_{ij} : \mathrm{Hom}(\det T_X, \det \mathcal{O}_X(1)^{\oplus 8})|_{U_{ij}} \xrightarrow{\cong} \mathcal{O}_X(1)^{\otimes 2}|_{U_{ij}}.$$

Proof. Let $\eta_{ij} \in \mathrm{Hom}(\det T_X, \det \mathcal{O}_X(1)^{\oplus 8})|_{U_{ij}}$ be the map given by $\eta_{ij}(\partial_{ij}) = \bigwedge_{\ell=1}^8 (a_i^2 b_j)$. By construction, $\varpi_{ij}(\eta_{ij}) = (a_i^2 b_j) \otimes (a_i^2 b_j) \in \Gamma(\mathcal{O}_X(1)^{\otimes 2}, U_{ij})$ is a tensor square. To verify that ϖ_{ij} is well-defined, one can check that over $U_{mn} \cap U_{ij}$, the isomorphism ϖ_{ij} sends the transition function $\eta_{mn} \mapsto \eta_{ij}$ to $(\frac{a_i^2 b_j}{a_m^2 b_n})^{\otimes 2}$. $\square$

4. LOCAL CONTRIBUTIONS

We now compute the local index $\mathrm{ind}_{(H,q)}(\sigma)$ of a conic (H, q) meeting the lines $L_1, \ldots, L_8$. By [KW21, Definition 30 and Corollary 31], the local index is equal to the local $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}_{\varphi_{ij}^{-1}(H,q)}(\Phi_{ij})$ of the composite

$$\Phi_{ij} := \psi_{ij} \circ \sigma \circ \varphi_{ij}$$

for any i, j such that $(H, q) \in U_{ij}$. (Here, ψ_{ij} is the local trivialization $\mathcal{O}_X(1)^{\oplus 8}|_{U_{ij}} \to U_{ij} \times \mathbb{A}_k^8$ post-composed with the projection $U_{ij} \times \mathbb{A}_k^8 \to \mathbb{A}_k^8$.) By [EH16, Proposition 9.25], all zeros of Φ_{ij} are simple, so local degree can be computed by the Jacobian determinant of Φ_{ij} evaluated at $\varphi_{ij}^{-1}(H, q)$ [KW19, Lemma 9]. As a result, we have the following proposition.

Proposition 4.1. *Let $k(q)$ be the field of definition of $(H, q) \in U_{ij}$, let $\mathrm{Tr}_{k(q)/k}$ be the field trace, and let $\mathrm{Jac}(\Phi_{ij})$ be the determinant of the Jacobian matrix of Φ_{ij}. Then*

$$\mathrm{ind}_{(H,q)}(\sigma) = \mathrm{Tr}_{k(q)/k}\langle \mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}\rangle.$$

Our goal in this section is to interpret $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$ in terms of geometric information intrinsic to the conic (H, q) and the lines $L_1, \dots, L_8$. To simplify notation, let $\Phi_{ij}^n = \psi'_{ij} \circ \sigma_n \circ \varphi_{ij}$, so that $\Phi_{ij} = (\Phi_{ij}^0, \dots, \Phi_{ij}^8)$. As a function of $(a_1, a_2, a_3, b_1, \dots, b_5) \in \mathbb{A}_k^8$, we can explicitly write out Φ_{ij}^n:

$$
\begin{aligned}
\Phi_{ij}^n(a_1, \dots, b_5) &= \frac{\sigma_n(H_{\alpha_i(a_1,a_2,a_3)}, q_{\beta_j(b_1,\dots,b_5)})}{\tau_{ij}(H_{\alpha_i(a_1,a_2,a_3)}, q_{\beta_j(b_1,\dots,b_5)})} \\
&= \frac{q_{\beta_j(b_1,\dots,b_5)}(L_n \cap H_{\alpha_i(a_1,a_2,a_3)})}{q_{\beta_j(b_1,\dots,b_5)}(T_{ij} \cap H_{\alpha_i(a_1,a_2,a_3)})}
\end{aligned}
$$

(4.1)
$$
= \begin{cases}
z_0^2 + b_1 z_1^2 + b_2 z_2^2 + b_3 z_1 z_2 + b_4 z_0 z_2 + b_5 z_0 z_1 & j = 0, \\
b_1 z_0^2 + z_1^2 + b_2 z_2^2 + b_3 z_1 z_2 + b_4 z_0 z_2 + b_5 z_0 z_1 & j = 1, \\
\qquad\qquad\qquad\qquad\qquad\vdots \\
b_1 z_0^2 + b_2 z_1^2 + b_3 z_2^2 + b_4 z_1 z_2 + b_5 z_0 z_2 + z_0 z_1 & j = 5,
\end{cases}
$$

where $z_\ell = z_{i\ell}(x_{n0}(\alpha_i(a_1, a_2, a_3)), \dots, x_{n3}(\alpha_i(a_1, a_2, a_3)))$ (see Equations 3.2 and 3.3). The point (z_0, z_1, z_2) is an affine representative of the intersection $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$ in terms of the coordinates (z_{i0}, z_{i1}, z_{i2}) on $H_{\alpha_i(a_1,a_2,a_3)}$. For example, if $i = 1$, then

$$z_0 = x_{n0}(\alpha_1(a_1, a_2, a_3)) = (p_{n1}s_{n0} - p_{n0}s_{n1}) + a_2(p_{n2}s_{n0} - p_{n0}s_{i2}) + a_3(p_{n3}s_{n0} - p_{n0}s_{i3}),$$
$$z_1 = x_{n2}(\alpha_1(a_1, a_2, a_3)) = a_1(p_{n0}s_{n2} - p_{n2}s_{n0}) + (p_{n1}s_{n2} - p_{n2}s_{n1}) + a_3(p_{n3}s_{n2} - p_{n2}s_{n3}),$$
$$z_2 = x_{n3}(\alpha_1(a_1, a_2, a_3)) = a_1(p_{n0}s_{n3} - p_{n3}s_{n0}) + (p_{n1}s_{n3} - p_{n3}s_{n1}) + a_2(p_{n2}s_{n3} - p_{n3}s_{n2}).$$

Remark 4.2. Since Φ_{ij}^n is homogeneous of degree 2 in (z_0, z_2, z_2), scaling our choice of affine representative (z_0, z_1, z_2) of $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$ by $\lambda_n \in k(q)^\times$ changes Φ_{ij}^n by λ_n^2. It follows that $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$ is scaled by $\Lambda^2 := \prod_{n=1}^8 \lambda_n^2$. But this is a square in $k(q)$, so $\langle \mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}\rangle = \langle \Lambda^2 \mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}\rangle$ as elements of $\mathrm{GW}(k(q))$. That is, our choice of representatives of $L_n \cap H_{\alpha_i(a_1,\dots,a_3)}$ for $1 \le n \le 8$ do not change the value of $\mathrm{ind}_{(H,q)}(\sigma)$, as claimed in Remark 3.4.

4.1. **Geometric interpretation.** In order to provide a geometric interpretation of $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$, we will give a geometric interpretation of the partial derivatives $\frac{\partial \Phi_{ij}^n}{\partial a_\ell}$ and $\frac{\partial \Phi_{ij}^n}{\partial b_\ell}$. We will phrase this interpretation in terms of the affine geometry underlying our projective conics and projective lines; in particular, $H_{\alpha_i(a_1,a_2,a_3)}$ will be an affine 3-plane instead of a projective 2-plane, L_n will be an affine 2-plane instead of a projective line, and $\mathbb{V}(q_{\beta_j(b_1,\dots,b_5)})$ will be an affine cone instead of a projective conic. We start with $\frac{\partial \Phi_{ij}^n}{\partial b_\ell}$. Since Φ_{ij}^n are linear in b_ℓ, these partial derivatives are straightforward to compute (see Figure 1). Geometrically, the entries $\frac{\partial \Phi_{ij}^n}{\partial b_\ell}$ of the Jacobian matrix of Φ_{ij}

at $\varphi_{ij}^{-1}(H, q)$ are recording the coordinates of the intersections $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$ in terms of the coordinates (z_{i0}, z_{i1}, z_{i2}) on the plane $H_{\alpha_i(a_1,a_2,a_3)}$.

ℓ \ j	0	1	2	3	4	5
1	z_1^2	z_0^2	z_0^2	z_0^2	z_0^2	z_0^2
2	z_2^2	z_2^2	z_1^2	z_1^2	z_1^2	z_1^2
3	$z_1 z_2$	$z_1 z_2$	$z_1 z_2$	z_2^2	z_2^2	z_2^2
4	$z_0 z_2$	$z_0 z_2$	$z_0 z_2$	$z_0 z_2$	$z_1 z_2$	$z_1 z_2$
5	$z_0 z_1$	$z_0 z_1$	$z_0 z_1$	$z_0 z_1$	$z_0 z_1$	$z_0 z_2$

FIGURE 1. $\frac{\partial \Phi_{ij}^n}{\partial b_\ell}$

Now we consider $\frac{\partial \Phi_{ij}^n}{\partial a_\ell}$. Noting that z_0, z_1, z_2 are functions of (a_1, a_2, a_3), we compute $\frac{\partial \Phi_{ij}^n}{\partial a_\ell}$ via the chain rule:

$$(4.2) \qquad \frac{\partial \Phi_{ij}^n}{\partial a_\ell} = \frac{\partial \Phi_{ij}^n}{\partial z_0} \cdot \frac{\partial z_0}{\partial a_\ell} + \frac{\partial \Phi_{ij}^n}{\partial z_1} \cdot \frac{\partial z_1}{\partial a_\ell} + \frac{\partial \Phi_{ij}^n}{\partial z_2} \cdot \frac{\partial z_2}{\partial a_\ell}.$$

Equation 4.2 is closely related to the tangent plane in $H_{\alpha_i(a_1,a_2,a_3)} \cong \mathbb{A}^3_{k(q)}$ of $\mathbb{V}(q_{\beta_j(b_1,\dots,b_5)})$ at (z_0, z_1, z_2). Indeed, replacing z_0, z_1, z_2 with z_{i0}, z_{i1}, z_{i2} in Equation 4.1, we let

$$q_{ij} := q_{\beta_j(b_1,\dots,b_5)} = \begin{cases} z_{i0}^2 + b_1 z_{i1}^2 + b_2 z_{i2}^2 + b_3 z_{i1} z_{i2} + b_4 z_{i0} z_{i2} + b_5 z_{i0} z_{i1} & j = 0, \\ b_1 z_{i0}^2 + z_{i1}^2 + b_2 z_{i2}^2 + b_3 z_{i1} z_{i2} + b_4 z_{i0} z_{i2} + b_5 z_{i0} z_{i1} & j = 1, \\ \quad\vdots \\ b_1 z_{i0}^2 + b_2 z_{i1}^2 + b_3 z_{i2}^2 + b_4 z_{i1} z_{i2} + b_5 z_{i0} z_{i2} + z_{i0} z_{i1} & j = 5 \end{cases}$$

be the defining equation for $\mathbb{V}(q_{\beta_j(b_1,\dots,b_5)}) \subset H_{\alpha_i(a_1,a_2,a_3)} \cong \mathbb{A}^3_{k(q)}$ (see Figure 2). Any conic meeting 8 general lines in $\mathbb{P}^3$ is smooth by [EH16, Lemma 9.21], so the affine cone $\mathbb{V}(q_{ij})$ (corresponding to the projective conic $\mathbb{V}(q_{ij})$) is smooth away from the cone point at the origin $(0, 0, 0) \in \mathbb{A}^3_{k(q)}$. It follows that the tangent plane $T_p \mathbb{V}(q_{ij})$ at $p := (z_0, z_1, z_2)$ is defined by the equation

$$\left. \frac{\partial q_{ij}}{\partial z_{i0}} \right|_p \cdot z_{i0} + \left. \frac{\partial q_{ij}}{\partial z_{i1}} \right|_p \cdot z_{i1} + \left. \frac{\partial q_{ij}}{\partial z_{i2}} \right|_p \cdot z_{i2} = 0.$$

Since $\frac{\partial \Phi_{ij}^n}{\partial z_\ell} = \frac{\partial q_{ij}}{\partial z_{i\ell}}\big|_p$ and p lies on the tangent plane $T_p \mathbb{V}(q_{ij})$, we have

$$\frac{\partial \Phi_{ij}^n}{\partial a_\ell} = \left. \frac{\partial q_{ij}}{\partial z_{i0}} \right|_p \left(\frac{\partial z_0}{\partial a_\ell} \right) + \left. \frac{\partial q_{ij}}{\partial z_{i1}} \right|_p \left(\frac{\partial z_1}{\partial a_\ell} \right) + \left. \frac{\partial q_{ij}}{\partial z_{i2}} \right|_p \left(\frac{\partial z_2}{\partial a_\ell} \right)$$

$$(4.3) \qquad = \left. \frac{\partial q_{ij}}{\partial z_{i0}} \right|_p \left(z_0 + \frac{\partial z_0}{\partial a_\ell} \right) + \left. \frac{\partial q_{ij}}{\partial z_{i1}} \right|_p \left(z_1 + \frac{\partial z_1}{\partial a_\ell} \right) + \left. \frac{\partial q_{ij}}{\partial z_{i2}} \right|_p \left(z_2 + \frac{\partial z_2}{\partial a_\ell} \right).$$

Geometrically, this records information about the slope of the affine plane L_n (corresponding to the projective line L_n) relative to the plane $T_p \mathbb{V}(q_{ij})$. Indeed, $z_m = (p_{ni}s_{nu} - p_{nu}s_{ni}) + \sum_{v \neq i} a_v(p_{nv}s_{nu} - p_{nu}s_{nv})$ for some u depending on i and m (see
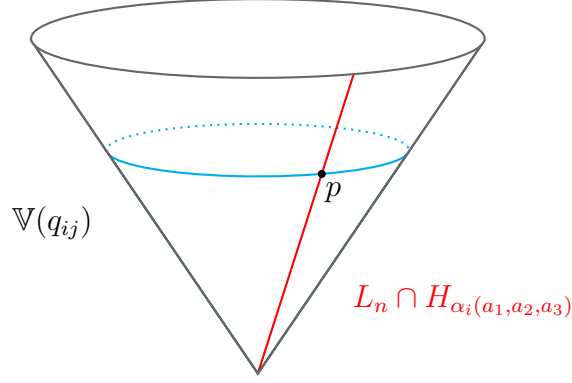
FIGURE 2. $\mathbb{V}(q_{ij})$ as a cone in $\mathbb{A}^3_{k(q)}$
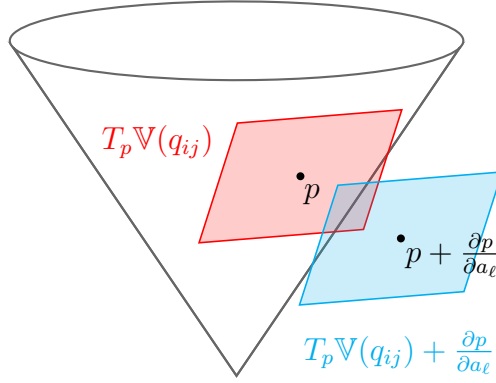
FIGURE 3. $T_p\mathbb{V}(q_{ij})$ and its $\frac{\partial \Phi^n_{ij}}{\partial a_\ell}$-level set

Equations 3.2 and 3.3); this is the z_{im}-coordinate of our chosen basis vector of the line $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$. Thus

$$z_m + \frac{\partial z_m}{\partial a_\ell} = (p_{ni}s_{nu} - p_{nu}s_{ni}) + (p_{n\ell}s_{nu} - p_{nu}s_{n\ell}) + \sum_{v \neq i} a_v(p_{nv}s_{nu} - p_{nu}s_{nv})$$

is z_m shifted by the rate of change of z_m (representing a coordinate of a basis vector in $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$) as a_ℓ changes. Equation 4.3 states that $(z_0 + \frac{\partial z_0}{\partial a_\ell}, z_1 + \frac{\partial z_1}{\partial a_\ell}, z_2 + \frac{\partial z_2}{\partial a_\ell})$ lies on the level set defined by

$$\frac{\partial q_{ij}}{\partial z_{i0}}\bigg|_p \cdot z_{i0} + \frac{\partial q_{ij}}{\partial z_{i1}}\bigg|_p \cdot z_{i1} + \frac{\partial q_{ij}}{\partial z_{i2}}\bigg|_p \cdot z_{i2} = \frac{\partial \Phi^n_{ij}}{\partial a_\ell},$$

which is parallel to the tangent plane $T_p\mathbb{V}(q_{ij})$. It follows that $\frac{\partial \Phi^n_{ij}}{\partial a_\ell}$ measures the deviation of $p + \frac{\partial p}{\partial a_\ell}$ from the tangent plane $T_p\mathbb{V}(q_{ij})$, as illustrated in Figure 3.

Remark 4.3. Using the Laplace expansion of the Jacobian determinant

$$\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)} = \begin{vmatrix} \frac{\partial \Phi_{ij}^1}{\partial a_1} & \cdots & \frac{\partial \Phi_{ij}^8}{\partial a_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_{ij}^1}{\partial b_5} & \cdots & \frac{\partial \Phi_{ij}^8}{\partial b_5} \end{vmatrix}$$

with respect to the first three and last five rows, we get a convenient expression for $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$. We will look at the product of minors

$$\begin{vmatrix} \frac{\partial \Phi_{ij}^1}{\partial a_1} & \frac{\partial \Phi_{ij}^2}{\partial a_1} & \frac{\partial \Phi_{ij}^3}{\partial a_1} \\ \frac{\partial \Phi_{ij}^1}{\partial a_2} & \frac{\partial \Phi_{ij}^2}{\partial a_2} & \frac{\partial \Phi_{ij}^3}{\partial a_2} \\ \frac{\partial \Phi_{ij}^1}{\partial a_3} & \frac{\partial \Phi_{ij}^2}{\partial a_3} & \frac{\partial \Phi_{ij}^3}{\partial a_3} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial \Phi_{ij}^4}{\partial b_1} & \cdots & \frac{\partial \Phi_{ij}^8}{\partial b_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_{ij}^4}{\partial b_5} & \cdots & \frac{\partial \Phi_{ij}^8}{\partial b_5} \end{vmatrix},$$

with all other minors in the Laplace expansion following our analysis analogously. Let

$$X_n = (X_{n0}, X_{n1}, X_{n2}) := (z_{i0}, z_{i1}, z_{i2})\Big|_{(x_{n0}, x_{n1}, x_{n2}, x_{n3})}$$

be our chosen representative of $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$ (see Equation 3.2). The unique conic in $H_{\alpha_i(a_1,a_2,a_3)}$ passing through $L_n \cap H_{\alpha_i(a_1,a_2,a_3)}$ for $4 \leq n \leq 8$ is defined by the quadratic polynomial $f(z_{i0}, z_{i1}, z_{i2}) = \det F$, where

$$(4.4) \qquad F = \begin{pmatrix} z_{i0}^2 & z_{i1}^2 & z_{i2}^2 & z_{i1}z_{i2} & z_{i0}z_{i2} & z_{i0}z_{i1} \\ X_{40}^2 & X_{41}^2 & X_{42}^2 & X_{41}X_{42} & X_{40}X_{42} & X_{40}X_{41} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{80}^2 & X_{81}^2 & X_{82}^2 & X_{81}X_{82} & X_{80}X_{82} & X_{80}X_{81} \end{pmatrix}.$$

Since $\mathbb{V}(q_{\beta_j(b_1,\dots,b_5)})$ passes through these same 5 points, it follows that $f = c \cdot q_{\beta_j(b_1,\dots,b_5)}$ for some $c \neq 0$. In fact,

$$(4.5) \qquad c = (-1)^j \begin{vmatrix} \frac{\partial \Phi_{ij}^4}{\partial b_1} & \cdots & \frac{\partial \Phi_{ij}^8}{\partial b_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_{ij}^4}{\partial b_5} & \cdots & \frac{\partial \Phi_{ij}^8}{\partial b_5} \end{vmatrix}.$$

Indeed, the 5×5 determinant in Equation 4.5 corresponds to the 5×5 minor $F_{1,j+1}$ (see Figure 1 and Equation 4.4). It follows that the coefficient in f of the $(1, j+1)^{\mathrm{st}}$ entry of F is c, while the coefficient in $q_{\beta_j(b_1,\dots,b_5)}$ of the $(1, j+1)^{\mathrm{st}}$ entry of F is 1 by Equation 4.1. Thus

$$\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)} = \sum_{u<v<w} (-1)^{u+v+w+j} \begin{vmatrix} \frac{\partial \Phi_{ij}^u}{\partial a_1} & \frac{\partial \Phi_{ij}^v}{\partial a_1} & \frac{\partial \Phi_{ij}^w}{\partial a_1} \\ \frac{\partial \Phi_{ij}^u}{\partial a_2} & \frac{\partial \Phi_{ij}^v}{\partial a_2} & \frac{\partial \Phi_{ij}^w}{\partial a_2} \\ \frac{\partial \Phi_{ij}^u}{\partial a_3} & \frac{\partial \Phi_{ij}^v}{\partial a_3} & \frac{\partial \Phi_{ij}^w}{\partial a_3} \end{vmatrix} F^j_{(u,v,w)},$$

where $F^j_{(u,v,w)}$ is the coefficient in f (determined by the 5 points $\{X_\ell\}_{\ell \neq u,v,w}$) of the $(1, j+1)^{\mathrm{st}}$ entry of F.

5. Real conics meeting 8 lines

Let $k = \mathbb{R}$. If a non-real conic meets 8 general lines in $\mathbb{P}^3_{\mathbb{R}}$, then its complex conjugate meets these 8 lines as well. In particular, the number of real conics meeting the 8 lines must be an even integer between 0 and 92. By Hauenstein–Sottile [HS12, Table 6] and Griffin–Hauenstein [GH15, Theorem 1], there exist 8 general lines for each $2n \in \{0, 2, \ldots, 92\}$ realizing the count of $2n$ real conics. It follows from Theorem 1.2 that these $2n$ conics come in two families of order n.

Definition 5.1. Let $L_1, \ldots, L_8$ be 8 general lines in $\mathbb{P}^3_{\mathbb{R}}$. A real conic (H, q) meeting $L_1, \ldots, L_8$ is called *positive* (respectively, *negative*) if $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$ is positive (respectively, negative). Note that this definition implicitly depends on the order of $L_1, \ldots, L_8$; permuting these lines by an odd permutation turns a positive conic into a negative conic (and vice versa).

Remark 5.2. By [EH16, Proposition 9.25], all zeros of Φ_{ij} are simple and hence

$$\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)} \neq 0.$$

Moreover, since we have assumed that (H, q) is real, it follows that $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$ is real. Finally, Proposition 3.12 implies that $\mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)}$ depends on the choice of $U_{ij} \ni (H, q)$ only up to squares in $\mathbb{R}$. In particular, the sign of this value does not depend on the choice of U_{ij}, so Definition 5.1 is well-defined.

Theorem 5.3. *Given 8 general lines $L_1, \ldots, L_8$ in $\mathbb{P}^3_{\mathbb{R}}$, there are an equal number of positive and negative real conics meeting $L_1, \ldots, L_8$.*

Proof. The result follows from Theorem 1.2 by taking the signature of

$$46 \cdot \mathbb{H} = \sum_{(H,q) \in \sigma^{-1}(0)} \mathrm{Tr}_{\mathbb{R}(q)/\mathbb{R}} \langle \mathrm{Jac}(\Phi_{ij})|_{\varphi_{ij}^{-1}(H,q)} \rangle.$$

The signature of $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \langle c \rangle$ is 0 for any $c \in \mathbb{C}^\times$, and the signature of $46 \cdot \mathbb{H}$ is likewise 0. We thus have

$$0 = \sum_{\substack{\text{real positive} \\ (H,q)}} 1 + \sum_{\substack{\text{real negative} \\ (H,q)}} (-1)$$

$$= \#\{\text{positive real conics}\} - \#\{\text{negative real conics}\},$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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