TOYCO Problem: Revision of few Terms

TOYCO assembles three types of toys—trains, trucks, and cars—using three operations. The daily limits on the available times for the three operations are 430, 460, and 420 minutes, respectively, and the revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train at the three operations are 1, 3, and 1 minutes, respectively. The corresponding times per train and per car are (2, 0, 4) and (1, 2, 0) minutes (a zero time indicates that the operation is not used).

Letting x_1, x_2, and x_3 represent the daily number of units assembled of trains, trucks, and cars, respectively, the associated LP model is given as:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 430 \text{ (Operation 1)} \\ 3x_1 + 2x_3 &\le 460 \text{ (Operation 2)} \\ x_1 + 4x_2 &\le 420 \text{ (Operation 3)} \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Using x_4, x_5, and x_6 as the slack variables for the constraints of operations 1, 2, and 3, respectively, the optimum tableau is

TOYCO Problem

- The optimal tableau is

Dual price

Reduced cost per unit

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

- Unit worth of resources (\$/time)
  - Change in the optimal value per unit change in the availability of the resource
  - Commonly known as dual or shadow price
- Reduced cost per unit = (The cost of consumed resources by one unit) – (Revenue per unit)

= (imputed cost of all resources to produce a unit) – (Revenue per unit)

Definition of Dual Problem

3. RHS for j-th constraint

	Primary Variables						
	x_1	x_2	…	x_j	…	x_n	
Dual Variables	c_1	c_2	…	c_j	…	C_n	Right-hand side
y_1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}	b_1
y_2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}	b_2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
y_m	a_{m1}	a_{m2}	…	a_{mj}	…	a_{mn}	b_m

1. Dual variables for each constraint

2. j-th dual constraint for primal variable

4. Objective coefficients

4. Economic Interpretation of Duality

- If LP problem is viewed as the resources allocation problem
  - Maximize revenue by allocating limited resources

Primal	Dual
Maximize $z = \sum_{j=1}^{n} c_j x_j$	Minimize $w = \sum_{i=1}^{m} b_i y_i$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \ge c_j,\ j = 1, 2, \ldots, n$
$x_j \ge 0,\ j=1, 2, \ldots, n$	$y_i \ge 0,\ i = 1, 2, \ldots, m$

  - n: no. of variables or economic activities
  - m: no. of constraints or resources
  - c_j: revenue per unit of activity j
  - Resource i, whose maximum availability is b_i, is consumed at thre rate a_{ij} units per unit of activity j

Economics Interpretation of Dual Variables

- $z = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i = w$
- Given the problem represents a resource allocation problem
  - z = revenue (in $)
  - $z = w \rightarrow \$ = \sum_i (units\ of\ resource\ i) \times (\$\ per\ unit\ of\ resource\ i)$
- y_i represents the worth per unit of the resource *i*
  - Dual (or shadow) price of the resource *i*

$$\boxed{w = \sum_{i=1}^{m} b_i y_i}$$

- *z* < *w*
  - Revenue < worth of resources
- Total revenue from all activities is less than the worth of the resources
- At optimum, resources have been exploited completely.
  - Input (worth of resources) equals output (revenue)

Economics Interpretation of Dual Variables

- Example

Reddy Mikks primal	Reddy Mikks dual
Maximize $z = 5x_1 + 4x_2$	Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$
subject to	subject to
$6x_1 + 4x_2 \leq 24$ (resource 1, $M1$)	$6y_1 + y_2 - y_3 \geq 5$
$x_1 + 2x_2 \leq 6$ (resource 2, $M2$)	$4y_1 + 2y_2 + y_3 + y_4 \geq 4$
$-x_1 + x_2 \leq 1$ (resource 3, market)	$y_1, y_2, y_3, y_4 \geq 0$
$x_2 \leq 2$ (resource 4, demand)	
$x_1, x_2 \geq 0$	
Optimal solution:	Optimal solution:
$x_1 = 3, x_2 = 1.5, z = 21$	$y_1 = .75, y_2 = 0.5, y_3 = y_4 = 0, w = 21$

  - Dual price y_1 = $750 per ton, y_2 = $500 per ton (**for specific range**).
  - Dual prices y_3 and y_4 = 0, resources are abundant. Worth price is zero.

Economic Interpretation of Dual Constraints

- At any iteration

$$\text{Objective coefficient of } x_j = \begin{pmatrix}\text{Left-hand side of}\\ \text{dual constraint } j\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ \text{dual constraint } j\end{pmatrix}$$
$$= \sum_{i=1}^{m} a_{ij}y_i - c_j$$

- Dimensional analysis to interpret the results
  - c_j: revenue per unit cost of activity *j* in \$ per unit.
  - Quantity $\sum_{i=1}^{m} a_{ij}y_i$ also must be in \$ per unit. Also represent cost.

$$\$\text{ cost} = \sum_{i=1}^{m} a_{ij}y_i = \sum_{i=1}^{m} \begin{pmatrix}\text{usage of resource } i\\ \text{per unit of activity } j\end{pmatrix} \times \begin{pmatrix}\text{cost per unit}\\ \text{of resource } i\end{pmatrix}$$

  - y_i: imputed cost per unit of resource *i*.
  - $\sum_{i=1}^{m} a_{ij}y_i$ is imputed cost of all resources to produce one unit of activity *j*.

Economic Interpretation of Dual Constraints

- $\sum_{i=1}^{m} a_{ij}y_i - c_j$ reduced cost of activity *j*.
- For the maximization optimality condition of the simplex method
  - An increase in the level of unused (nonbasic) activity j can improve revenue only if its reduced cost is negative.

$$\begin{pmatrix}\text{Imputed cost of}\\ \text{resources used by}\\ \text{one unit of activity } j\end{pmatrix} < \begin{pmatrix}\text{Revenue per unit}\\ \text{of activity } j\end{pmatrix}$$

- The maximization optimality condition thus says that it is economically advantageous to increase an activity to a positive level if its unit revenue exceeds its unit imputed cost.

5. Post Optimality Analysis

- Sensitivity Analysis
  - It deals with the sensitivity of the optimum solution by determining the ranges of the different LP parameters that would keep the optimum basic variables unchanged.
- Post-optimality analysis
  - It deals with making changes in the parameters of the model and finding new optimum solution
- 5.1 Changes affecting feasibility
  - The feasibility of the current optimum solution is affected when
    - RHS of constrains is changed
    - New constraint is added

5.1.1 Changes in RHS of Constraints

$$\begin{pmatrix}\text{New right-hand side of} \\ \text{tableau in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in} \\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{New right-hand} \\ \text{side of constraints}\end{pmatrix}$$

- TOYCO example

TOYCO primal	TOYCO dual
Maximize $z = 3x_1 + 2x_2 + 5x_3$	Minimize $z = 430y_1 + 460y_2 + 420y_3$
subject to	subject to
$x_1 + 2x_2 + x_3 \leq 430$ (Operation 1)	$y_1 + 3y_2 + y_3 \geq 3$
$3x_1 + 2x_3 \leq 460$ (Operation 2)	$2y_1 + 4y_3 \geq 2$
$x_1 + 4x_2 \leq 420$ (Operation 3)	$y_1 + 2y_2 \geq 5$
$x_1, x_2, x_3 \geq 0$	$y_1, y_2, y_3 \geq 0$
Optimal solution:	Optimal solution:
$x_1 = 0, x_2 = 100, x_3 = 230, z = \$1350$	$y_1 = 1, y_2 = 2, y_3 = 0, w = \$1350$

5.1.1 Changes in RHS of Constraints

- **Q1 (Change in RHS)** Suppose that TOYCO is increasing the daily capacity of operations 1, 2 and 3 to 602, 644, and 588 minutes, respectively. How would this change affect the total revenue?

Optimal simplex tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 602 \\ 644 \\ 588 \end{pmatrix} = \begin{pmatrix} 140 \\ 322 \\ 28 \end{pmatrix}$$

Maximize $z = 3x_1 + 2x_2 + 5x_3$

$Z_{new} = 1890$

5.1.1 Changes in RHS of Constraints

- **Q2 (Change in RHS)** Another proposal is to shift the slack capacity of operation 3 ($x_6 = 20$) to the capacity of operations 1. How would this change impact the optimum solution?
  - Earlier capacity of operation 1 was 430 and capacity of operation 3 was 420

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 460 \\ 400 \end{pmatrix} = \begin{pmatrix} 110 \\ 230 \\ -40 \end{pmatrix}$$

Infeasible solution

- How can we make it feasible?
  - Dual simplex method

5.1.1 Changes in RHS of Constraints

- Dual simplex method
  - Re-calculate z for new values of x_2, x_3, and x_6, that is z = 1370

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1370
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	110
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	-40
Ratio	--	---	---	½	---	-----	

Leaving variable

Entering variable

5.1.1 Changes in RHS of Constraints

- The optimal simplex tableau is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	5	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1350
x_2	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_4	-1	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	20

  - Still $x_1 = 0$,
  - Proposed shift is not advantageous
  - Operation 2 is bottleneck because the current shift from operation 3 to operation 1 is not useful.

5.1.2 Addition of New Constraint

- If constraint is redundant, it will have no effect on the current optimum solution
- Otherwise, the new optimum solution has to be found using the dual simplex method
- **Q3 (New Constraint)** Suppose that TOYCO is changing the design of its toys and that the change will require the additional fourth assembly operation. The daily capacity of the new operation is 500 minutes and the times per unit for the three products on this operation are 3, 1, and 1 minutes, respectively. What is the optimum solution?

The constraint for operation 4 is

$$3x_1 + x_2 + x_3 \leq 500$$

5.1.2 Addition of New Constraint

- At the current optimum solution, the value of constraint is

Optimal solution:
$x_1 = 0,\ x_2 = 100,\ x_3 = 230$

0 + 100 +230 < 500

- The constraint is redundant.
- The optimum solution remains the same.

- **Q4 (New Constraint)** Suppose TOYCO unit times on the fourth operation are changed to 3, 3, and 1 minutes, respectively. What is the optimum solution?

$$3x_1 + 3x_2 + x_3 \leq 500$$

The value of constraint at the current optimum solution is 0 +300 + 230 > 500 (infeasible)

5.1.2 Addition of New Constraint

- Since the current optimum solution becomes infeasible due to the new constraint, the constraint should be added to the optimum simplex tableau.
- Use x_7 as slack variable for the new constraint

Basic	X1	X2	X3	X4	X5	X6	X7	Solution
z	4	0	0	1	2	0	0	1370
X2	-1/4	1	0	½	-1/4	0	0	100
X3	3/2	0	1	0	½	0	0	230
X6	2	0	0	-2	1	1	0	20
X7	3	3	1	0	0	0	1	500

- Coefficients under basic variables in x_7-row should be zero. (Remember identity matrix)

$$\text{New } x_7\text{-row} = \text{Old } x_7\text{-row} - \{3 \times (x_2\text{-row}) + 1 \times (x_3\text{-row})\}$$

5.1.2 Addition of New Constraint

- The modified simplex tableau is

Basic	X1	X2	X3	X4	X5	X6	X7	Solution
z	4	0	0	1	2	0	0	1370
X2	-1/4	1	0	½	-1/4	0	0	100
X3	3/2	0	1	0	½	0	0	230
X6	2	0	0	-2	1	1	0	20
X7	9/4	0	0	-3/2	1/4	0	1	-30

Infeasible solution

- Follow dual simplex rules and find the optimum solution
- The optimum solution is $x_1 = 0$, $x_2 = 90$, $x_3 = 230$ and $z = 1330 < 1350$.
- Worsen the revenue

5.2 Changes Affecting Optimality

- Change in objective function coefficient
- Addition of new activity (variable)

- 5.2.1 Change in the objective function coefficient
  - Step 1 Compute dual variable values using

Method 2.

$$\begin{pmatrix}\text{Optimal values}\\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal}\\ \text{inverse}\end{pmatrix}$$

5.2.1 Change in the objective function coefficient

- Step1: Compute dual variable values using

Method 2.

$$\begin{pmatrix}\text{Optimal values}\\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal}\\ \text{inverse}\end{pmatrix}$$

- Step 2: Use the new dual values to determine the new reduced cost (z-row coefficients)

Formula 2: Objective *z*-row Computations. In any simplex iteration, the objective equation coefficient (reduced cost) of x_j is computed as follows:

$$\begin{pmatrix}\textit{Primal } z\text{-equation}\\ \text{coefficient of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix}$$

5.2.1 Change in the objective function coefficient

- **Q1 (Objective coefficients)** In the TOYCO model, suppose that the company has a new pricing policy to meet the competition. The unit revenues are \$2, \$3, and \$4 for train, truck, and car toys, respectively. What is the new optimum solution and revenue?
- New z = 2x1 + 3x2 + 4x3
- Row vector of original objective coefficients of optimal primal basic variables = $(x_2, x_3, x_6) = (3, 4, 0)$

Method 2.

$$\begin{pmatrix}\text{Optimal values}\\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal}\\ \text{inverse}\end{pmatrix}$$

- Step1:

$$(y_1, y_2, y_3) = (3, 4, 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = \left(\tfrac{3}{2}, \tfrac{5}{4}, 0\right)$$

5.2.1 Change in the objective function coefficient

- Step 2

$$\begin{pmatrix}\textit{Primal } z\text{-equation}\\ \text{coefficient of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix}$$

$$\text{Reduced cost of } x_1 = y_1 + 3y_2 + y_3 - \mathbf{2} = \tfrac{3}{2} + 3\left(\tfrac{5}{4}\right) + 0 - \mathbf{2} = \tfrac{13}{4}$$

$$\text{Reduced cost of } x_4 = y_1 - 0 = \tfrac{3}{2}$$

$$\text{Reduced cost of } x_5 = y_2 - 0 = \tfrac{5}{4}$$

  - Reduced cost of basic variables are always zero
- Optimality condition of primal simplex method for maximization problem
  - Most negative z-row coefficient of non basic variable
  - No entering variable. Therefore, the optimum solution remains the same.
  - New z = 2*0 + 3*100 + 4*230 = 1220 < 1350
  - Not recommended because of its lower revenue.

5.2.1 Change in the objective function coefficient

- **Q2 (Objective coefficients)** Suppose that the TOYCO objective function is changed to, Maximize z = $6x_1 + 3x_2 + 4x_3$, will the optimum solution change?
- Step 1

$$(y_1, y_2, y_3) = (3, 4, 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = \left(\frac{3}{2}, \frac{5}{4}, 0\right)$$

- Step 2

$$\text{Reduced cost of } x_1 = y_1 + 3y_2 + y_3 - 6 = \frac{3}{2} + 3\left(\frac{5}{4}\right) + 0 - 6 = -\frac{3}{4}$$

$$\text{Reduced cost of } x_4 = y_1 - 0 = \frac{3}{2}$$

$$\text{Reduced cost of } x_5 = y_2 - 0 = \frac{5}{4}$$

Entering non basic variable
Using primal simplex rules

- New z value is 6*0 + 3*100 + 4*230 = 1220

5.2.1 Change in the objective function coefficient

- The modified simplex tableau is

Leaving basic variable

Basic	X1	X2	X3	X4	X5	X6	Solution
z	-3/4	0	0	3/2	5/4	0	1220
X2	-1/4	1	0	½	-1/4	0	100
X3	3/2	0	1	0	½	0	230
X6	2	0	0	-2	1	1	20

Entering non basic variable

- Follow one more iteration of primal simplex method
- The optimum solution is $x_1 = 10$, $x_2 = 102.5$, $x_3 = 215$ and z = 1227.5 < 1350
- Not profitable

5.2.2 Addition of New Activity (Variable)

- **Q3 (New activity)** TOYCO recognizes that toy trains are currently not in production because they are not profitable. The company wants to replace toy trains with a new product, a toy fire engine, to be assembled on the existing facilities. TOYCO estimates the revenue per toy fire engine to be $4 and the assemble times per unit to be 1 minute on each operations 1 and 2, and 2 minutes on operation 3. What is the new optimum solution and revenue?
  - x_7: number of toy fire engine
- Step 1: New z = $2x_2 + 5x_3 + 4x_7$

$$(y_1, y_2, y_3) = (2,5,0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = (1,2,0)$$

- Step 2

$$\text{Reduced cost of } x_7 = 1y_1 + 1y_2 + 2y_3 - 4 = 1 \times 1 + 1 \times 2 + 2 \times 0 - 4 = -1$$

Entering non basic variable

5.2.2 Addition of New Activity (Variable)

- Since x7 is introducing to the simplex tableau, its column coefficient can be found using Formula 1

Formula 1: Constraint Column Computations. In any simplex iteration, a left-hand or a right-hand side column is computed as follows:

$$\begin{pmatrix}\text{Constraint column} \\ \text{in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in} \\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{Original} \\ \text{constraint column}\end{pmatrix}$$

$$x_7\text{-constraint colum} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{pmatrix}$$

5.2.2 Addition of New Activity (Variable)

- The modified simplex tableau is

Basic	X1	X2	X3	X7	X4	X5	X6	Solution
z	4	0	0	-1	1	2	0	1350
X2	-1/4	1	0	¼	½	-1/4	0	100
X3	3/2	0	1	½	0	½	0	230
X6	2	0	0	1	-2	1	1	20

- After two more iterations of primal simplex method, we get
- The optimum solution as $x_1 = x_2 = 0$, $x_3 = 125$, $x_7 = 210$ and $z = 1465 > 1350$
- Profitable

Thank you.