Sampling Theorem

→ Let us say we have equally spaced samples of $x(t)$

$$x(nT), \quad n = 0, \pm 1, \pm 2 \ldots$$

→ $x(t)$ is band limited

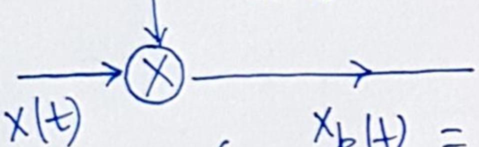
i.e. $X(j\omega) = 0, \quad |\omega| > \omega_M$

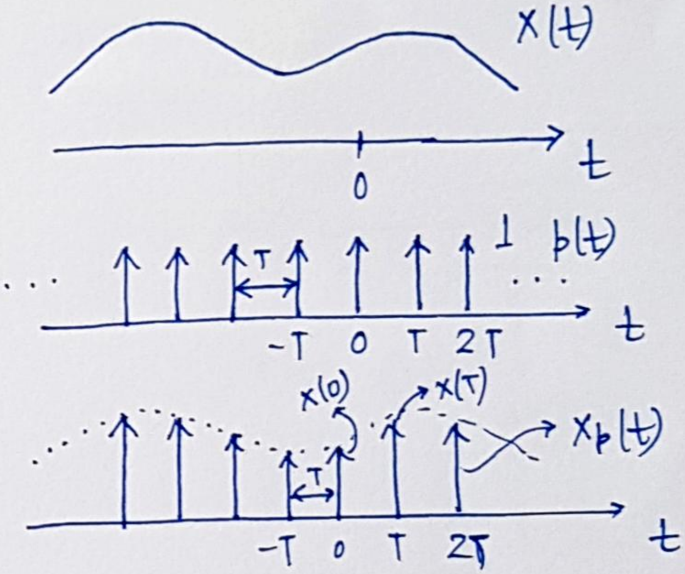
If $\omega_s > 2\omega_M$, where $\omega_s = \frac{2\pi}{T}$

Then $x(t)$ can be uniquely determined from its samples

$$x(nT), \quad n = 0, \pm 1, \pm 2 \ldots$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$





Time Domain

$$x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

↓ sampled signal

$$= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)$$

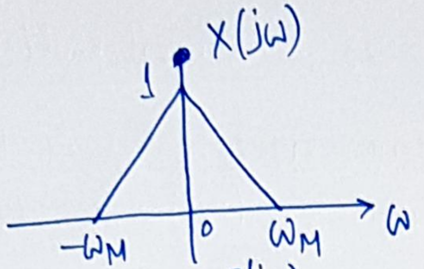
In the frequency-domain:-

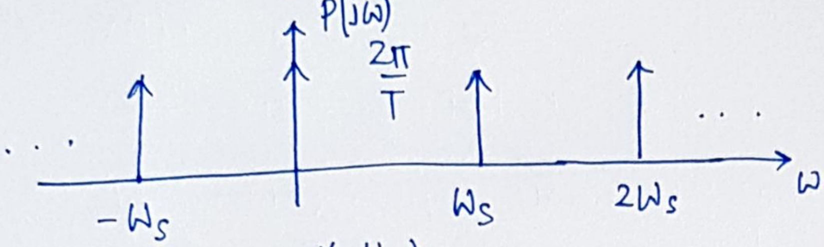
$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right]$$

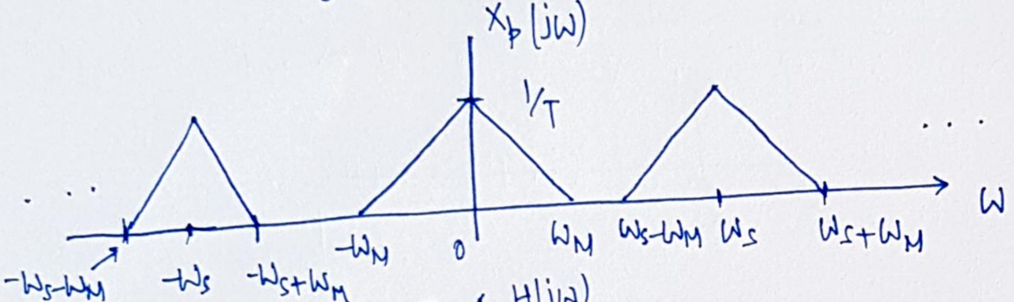
↓ Fourier Transform of the sampled signal $x_p(t)$

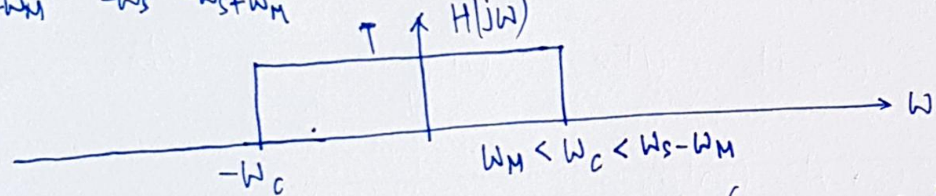
$$= \frac{1}{2\pi} X(j\omega) * \left(\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right) \quad \text{where } \left(\omega_s = \frac{2\pi}{T}\right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

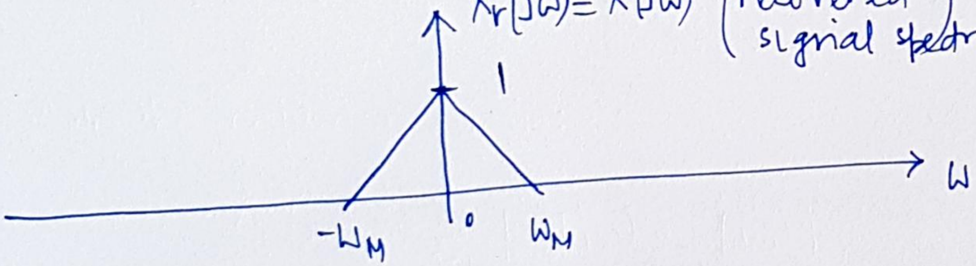








$X_r(j\omega) = X(j\omega)$ (recovered signal spectrum)



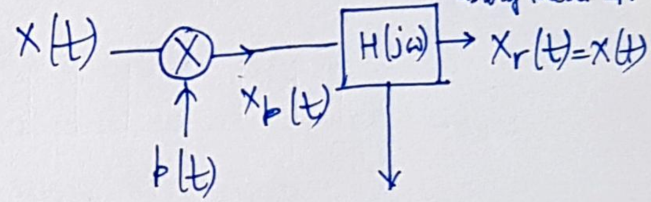
LPF extracts the desired portion of the spectrum!

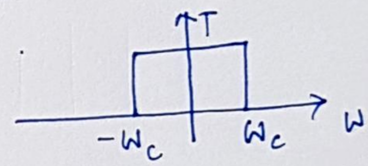
If $\omega_s - \omega_M > \omega_M$

or $\omega_s > 2\omega_M$

[shifted replicas do not overlap

reconstruction using ideal LPF





freq. response of LPF

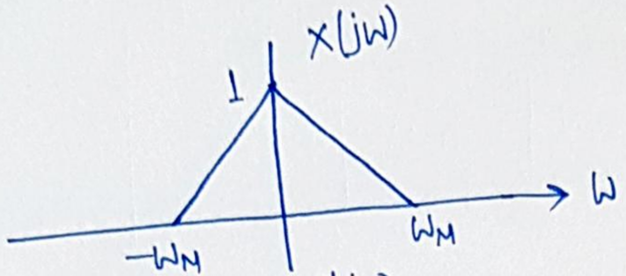
$\boxed{\omega_M < \omega_c < \omega_s - \omega_M}$

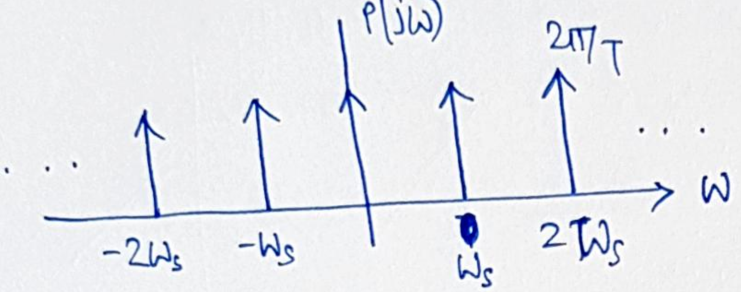
cutoff frequency

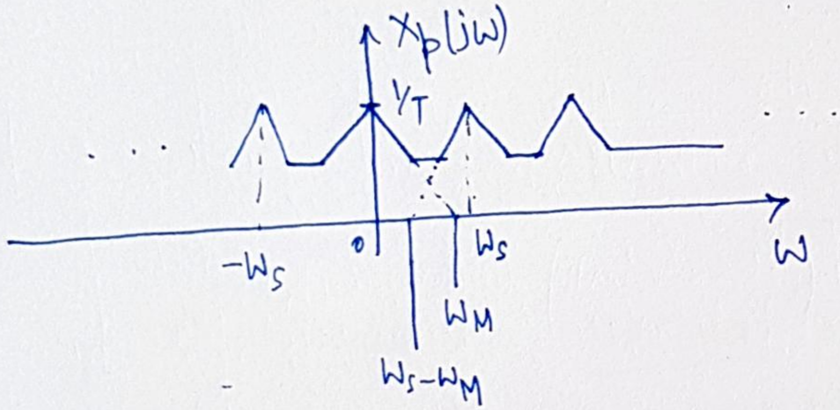
However, if $\omega_s - \omega_M < \omega_M$, $\omega_s < 2\omega_M$ [sampling frequency is too low].

then the shifted replicas of $X(j\omega)$ overlap

↳ Distortion [ALIASING].







Individual replicas of the F.T. of the original signal overlap and what we end up is with some distortion

If we pass this $x_p(t)$ through a LPF to recover the original signal, we won't recover the original signal, since these individual replicas overlap.

Effect: Higher frequencies get folded down into lower frequencies, what comes out of the LPF is the reflection of some higher frequencies onto lower frequencies
This effect is termed as aliasing !

- In other words, when $W_s < 2W_M$, the spectrum of $x(t)$ is no longer replicated in $X_p(j\omega)$ and thus is no longer recoverable by lowpass filtering.

This effect in which the individual terms in

$$X_p(jW) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(W-kW_s)) \quad \text{overlap}$$

is referred to as $\boxed{\text{ALIASING}}$!

Let us examine ALIASING more closely by taking the specific example of a sinusoidal signal

Let us take $x(t) = \cos\omega_0 t$

$\boxed{\omega_M = \omega_0}$ (highest freq. content)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

TIME-DOMAIN

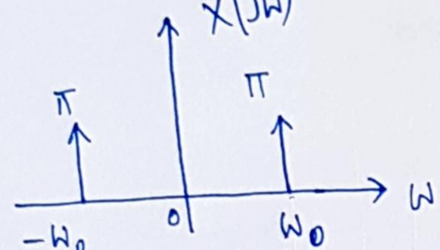
$\boxed{e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega-\omega_0)}$

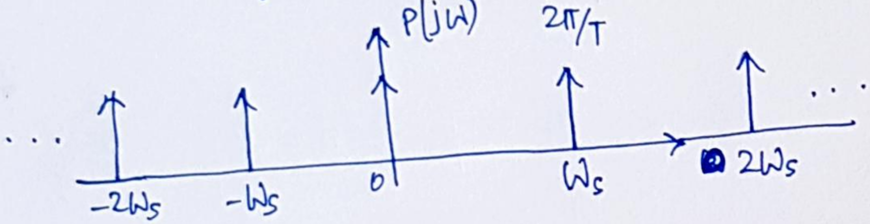
$$X(j\omega) = \cancel{2}\pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \quad \boxed{\omega_s \triangleq \frac{2\pi}{T}}$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

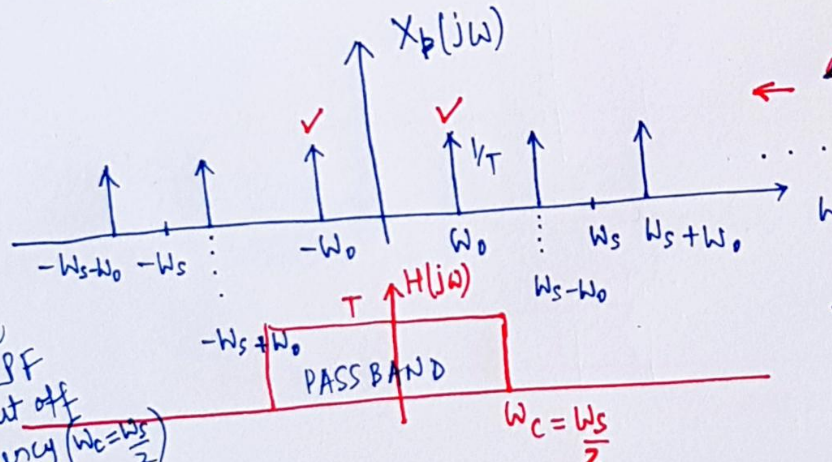
FREQUENCY-DOMAIN





Case 1

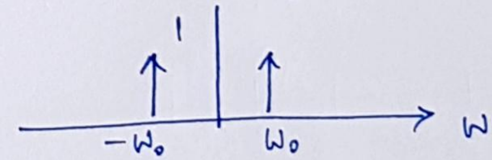
$(\omega_s = 6\omega_0)$ [oversampling]



We assume that the LPF has a cut off frequency $\left(\omega_c = \frac{\omega_s}{2}\right)$

$\rightarrow$

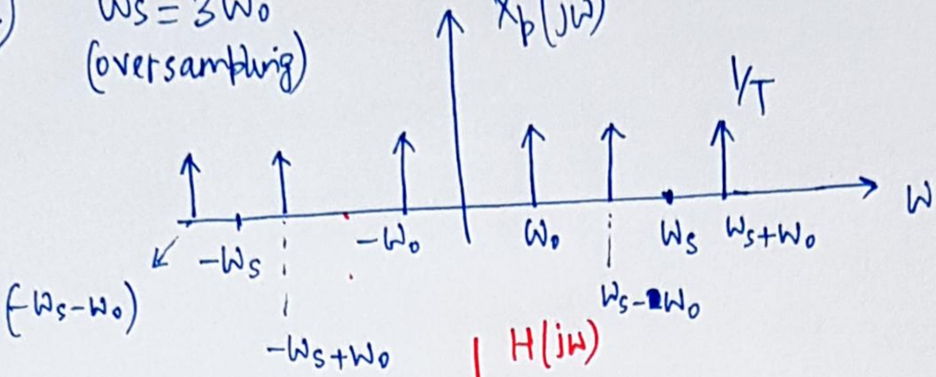
$X_r(j\omega) = X(j\omega)$

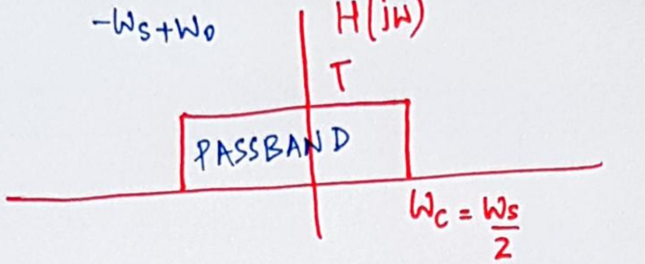


$x_r(t) = x(t) = (\cos\omega_0 t)$

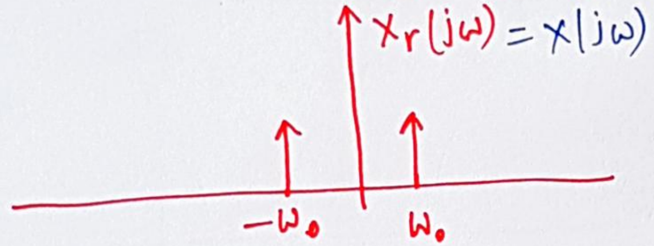
case 2

$W_s = 3W_0$ (oversampling)





~~$W_0 < W_c < W_s - W_0$~~

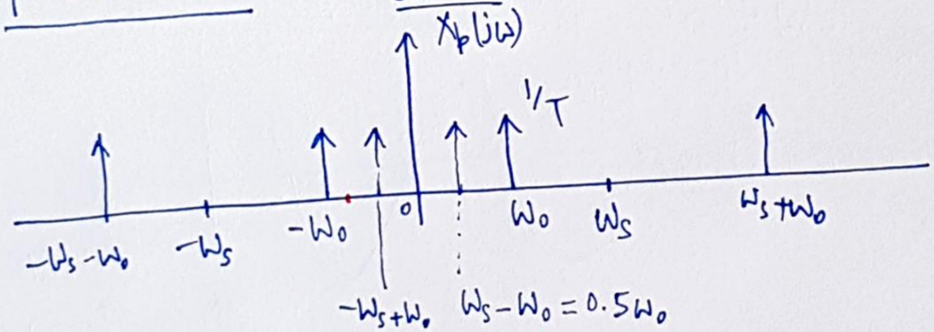


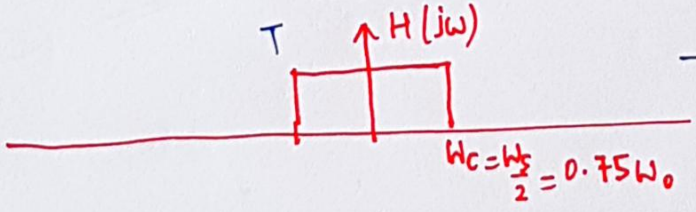
As long as $W_s > 2W_M$

$X_r(jw) = X(jw)$

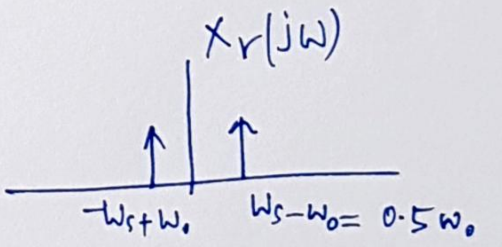
$X_r(t) = X(t)$ unique recovery possible!

case 3 $\boxed{W_s = 1.5W_0}$ undersampling





$\rightarrow$



$X_r(t) = \left(\cos \frac{W_0}{2} t\right) = \cos(W_s - W_0)t$

$\neq X(t)$

→ If a sinusoid with highest frequency (ω_0) is sampled at a rate $\omega_s = 1.5\,\omega_0$ [$< 2\omega_0$]

What you end up reconstructing [after passing the samples through a LPF with cut off $\frac{\omega_s}{2}$]

is a sinusoid with frequency $\omega_s - \omega_0 = 0.5\,\omega_0$

→ So, when aliasing occurs, the original frequency (ω_0) takes on the identity of a lower frequency ($\omega_s - \omega_0$)

ω_0 at the i/p

↓

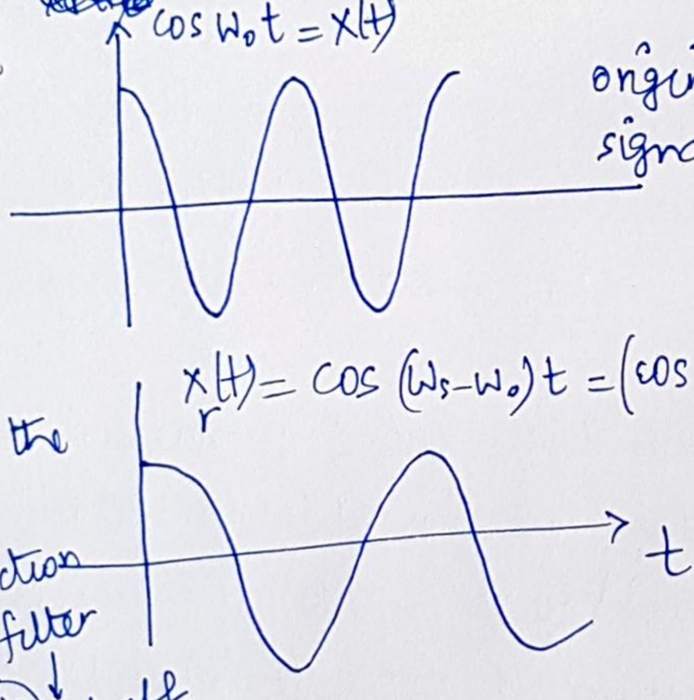
appears as

↓

$\omega_s - \omega_0$ at the o/p of reconstruction (LPF) filter

↓

$\omega_c = \frac{\omega_s}{2}$ cut off



original signal

for ($\omega_s = 1.5\,\omega_0$)

reconstructed signal

To summarize, if we sample a signal and then reconstruct back from the samples, using a lowpass filter, as long as ($\omega_s > 2\omega_M$) the sampling frequency is greater than twice the highest frequency content in the signal, we reconstruct exactly.

If, on the other hand, the sampling frequency is too low [less than twice the highest frequency then we observe aliasing!

↓ higher frequencies get folded/reflected back onto lower frequencies as they come through the LPF.

The frequency ($2\omega_M$), which under the sampling theorem must be exceeded by the sampling frequency to obtain exact/faithful recovery of $x(t)$ is called the Nyquist rate.

Interpretation of reconstruction as an interpolation process

Interpolation : Fitting of a continuous signal to a set of sample values.

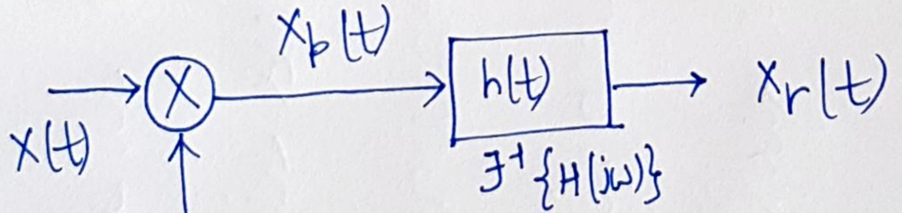
↓

In developing the sampling theorem, we based the reconstruction procedure for recovering the original signal from its samples on the use of a lowpass filter.

↓

Follows naturally from the interpretation of sampling process in the frequency domain!

→ [Let us now look at the process of impulse train sampling & reconstruction in the time-domain :



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The sampled signal

$$x_p(t) = x(t) \cdot p(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)$$

sample values at instants where the impulses are located

The sampled signal is also an impulse train.

And in reconstruction, we process this impulse train $\underline{\underline{x_p(t)}}$ with a LPF

so, in the time-domain, the reconstructed signal

$$x_r(t) = x_p(t) * h(t)$$

(impulse train of samples) convolved with the (LPF impulse response)

i.e. $$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT) * h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT)$$

* This impulse response gets reproduced at each of the locations of the impulses in $x_p(t)$ with the appropriate area!

$$\therefore\ x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT)$$

↓

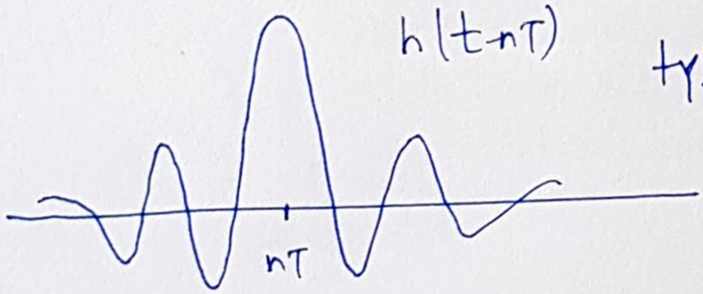
- Basic reconstruction expression in the time-domain
- This eqn. describes how to fit a continuous time curve between the sample points $x(nT)$
- represents an interpolation formula.
- The reconstructed signal is simply a linear combination of shifted versions of the impulse response with amplitudes which are the sample values.

Pictures

$x_p(t)$: sampled waveform (Train of impulses each with amplitude $x(nT)$)

$x(t)$: original signal

nT



typical response of an ideal LPF with cut off (ω_c)

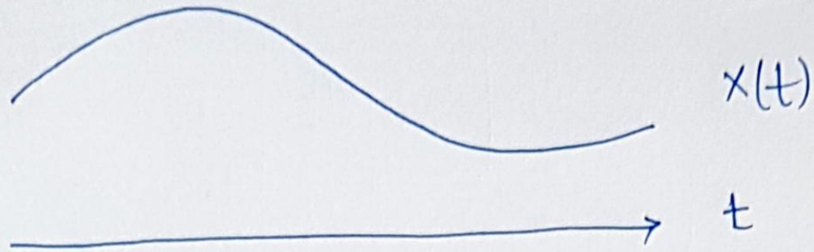
↓

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0 & \text{o.w.} \end{cases}$$

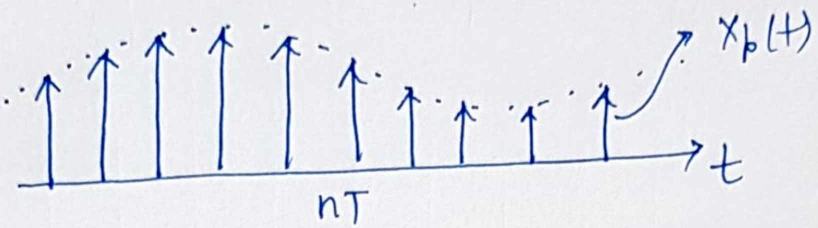
$\mathcal{F}$

$$h(t) = T \cdot \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

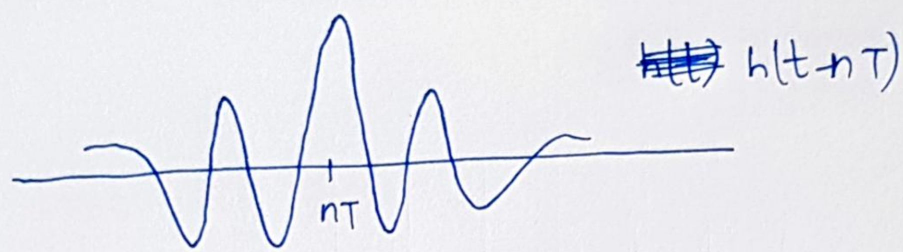
in reconstruction, we convolve $x_p(t)$ with $h(t)$, we superimpose the filter response at each of these time instants and in doing that those are then added up → to give the reconstructed signal.



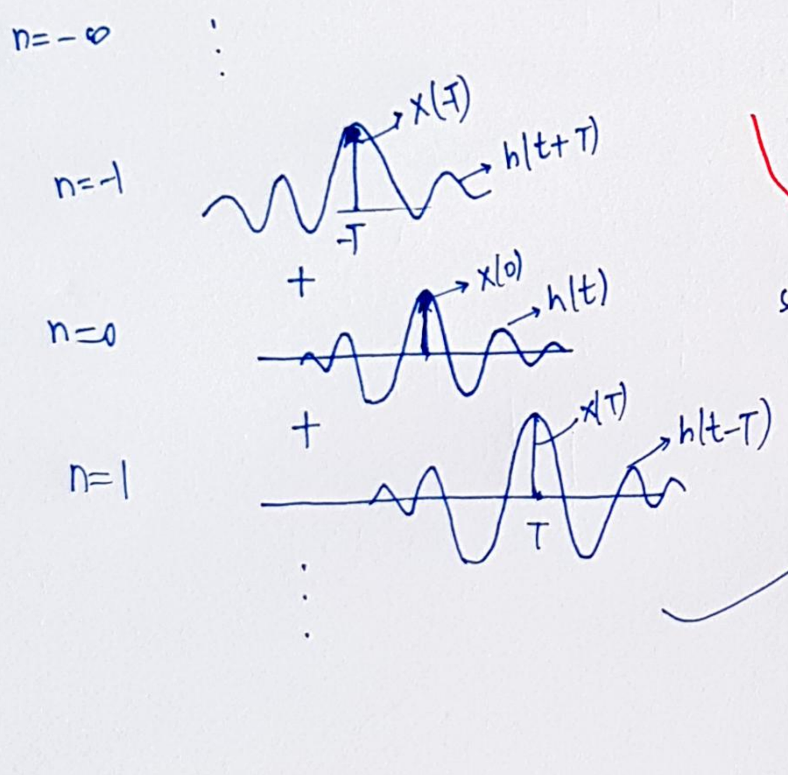
Band limited signal $x(t)$



impulse train of samples of $x(t)$



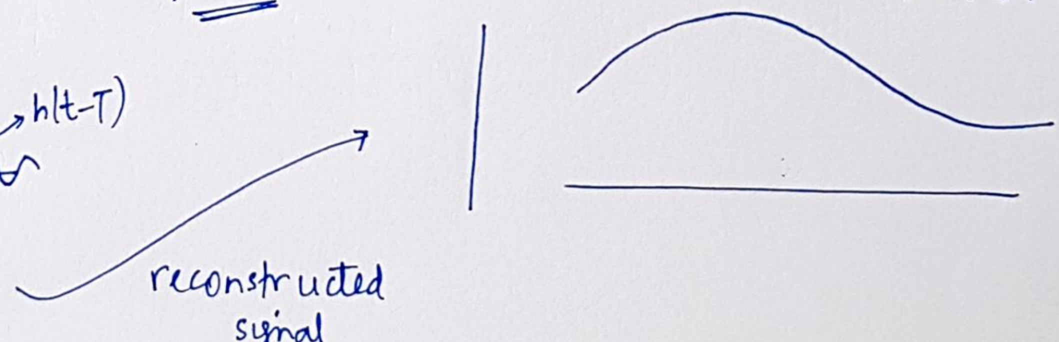
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h(t-nT)$$



ideal bandlimited interpolation in which the impulse train $x_p(t)$ is replaced by a superposition of the sinc functions to yield $x_r(t)$

superposition

$x_r(t) = x(t)$



Interpolation using the impulse response of an ideal LPF is referred to as band limited interpolation since it implements exact reconstruction if $x(t)$ is bandlimited and $\underline{\omega_s > 2\omega_M}$.

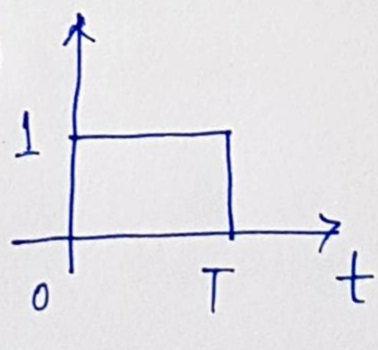
In many cases, it is preferable to use a less accurate filter but a simpler filter / a simple interpolating funcn than the function in

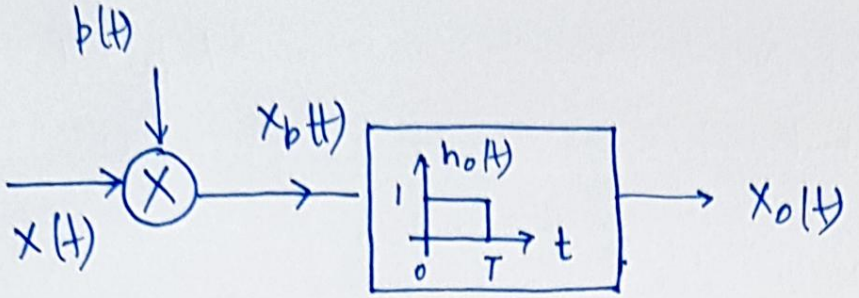
$$h(t) = \frac{T\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c T}{\pi}\right)$$ → non-causal
not practically realizable!

Zero-order Hold (ZOH):-

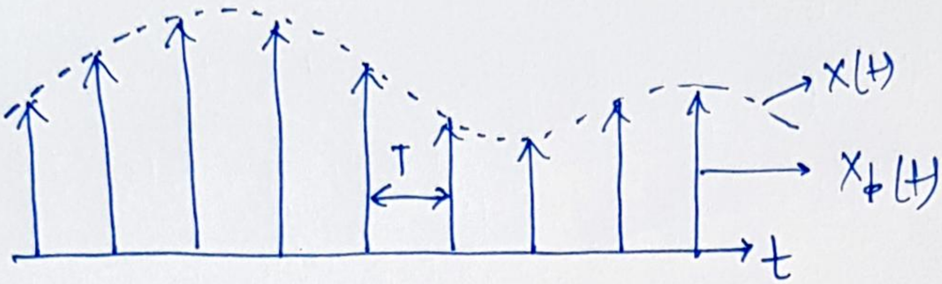
A simpler form of interpolation between sample values in which the interpolating function $h(t) = h_0(t)$ has the impulse response

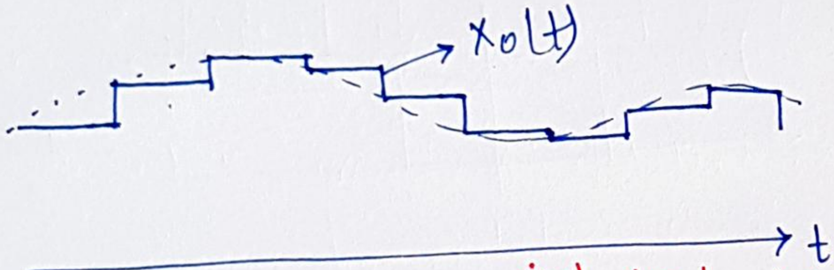
$$h(t) = h_0(t)$$





reconstruction of a sampled signal with a zero order hold!

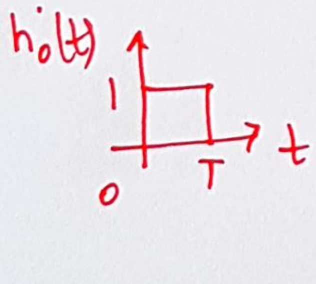




$$x_o(t) = x_p(t) * h_o(t)$$

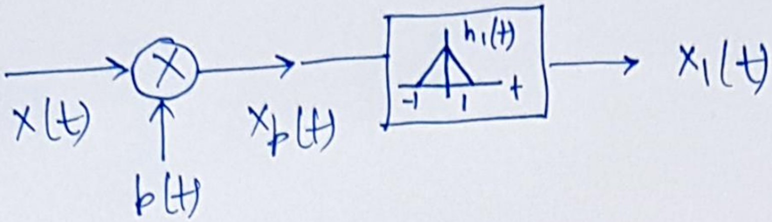
$$= \sum_{n=-\infty}^{\infty} x(nT)\, h_o(t-nT)$$

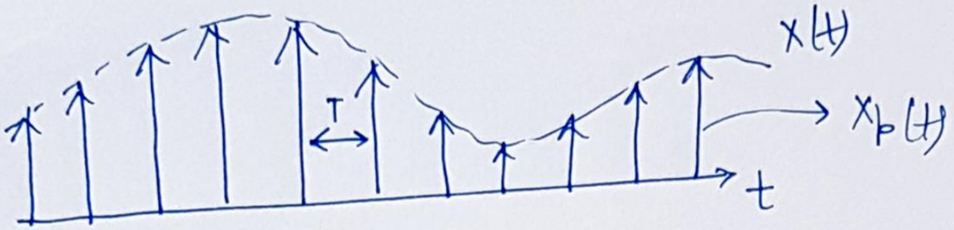
impulse train is replaced by superposition of rectangular functions

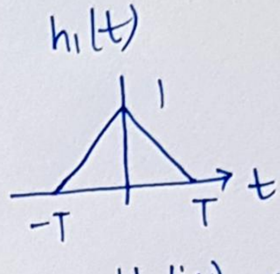


$$\xleftrightarrow{\mathcal{F}} \quad H_o(j\omega) = 2\left(e^{\frac{-j\omega T}{2}}\right) \frac{\sin\left(\frac{\omega T}{2}\right)}{\omega}$$

Reconstruction of sampled signal with first order hold / Linear Interpolation



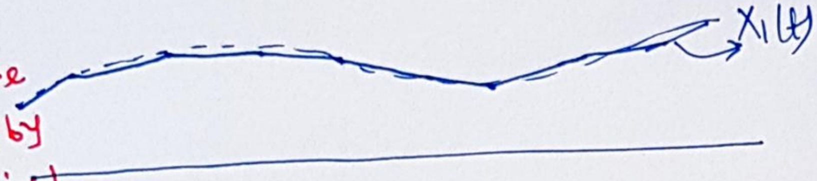




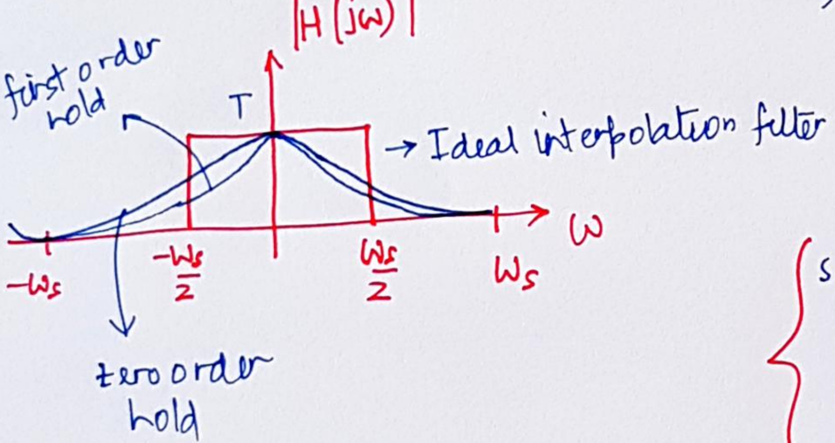
$$H_1(j\omega) = \frac{1}{T}\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\omega/2}\right)^2$$

(work this out)! HW

Adjacent samples are connected by straight lines!



first order hold applied to the sampled signal



second & higher order holds produce reconstructions with a higher degree of smoothness!