EE322M: Tutorial

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DTFT

1. Consider the signal

$$x(n) = \left\{-1, 2, \underset{\uparrow}{-3}, 2, -1\right\}$$

with Fourier transform $X(\omega)$. Compute the following quantities, without explicitly computing $X(\omega)$:

(a) $X(0)$

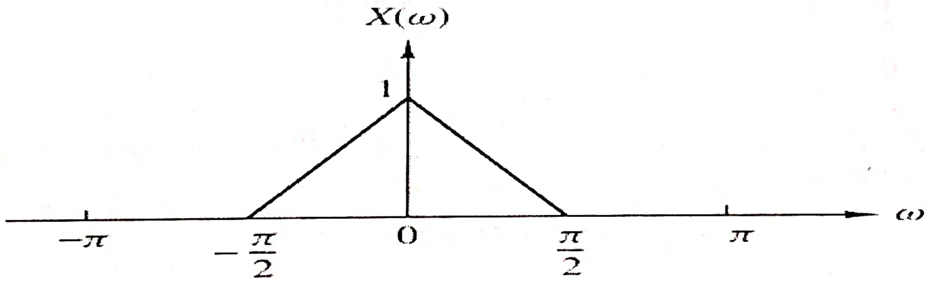
(b) $\int_{-\pi}^{\pi} X(\omega)\, d\omega$

(c) $X(\pi)$

(d) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

DTFT

2. Let $x(n)$ be a signal with Fourier transform as shown in the following Figure. Determine and sketch Fourier transform of the following signals.



(a) $x_1(n) = x(n)\cos(\pi n/4)$

(b) $x_2(n) = x(n)\cos(\pi n/2)$

(c) $x_3(n) = x(n)\sin(\pi n/2)$

(d) $x_4(n) = x(n)\cos \pi n$ lol!

DFT

3. The first 6 points of the 8-point DFT of a real valued sequence are $5, 1-j3, 0, 3-j4, 2+j4, 3+j4$. Find the last two points of the DFT.

DFT

4. The DFT of a vector $[a\ b\ c\ d]$ is the vector $[\alpha\ \beta\ \gamma\ \delta]$. Consider the product

$$[p\ \ q\ \ r\ \ s] = [a\ \ b\ \ c\ \ d]\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

Find the DFT of $[p\ \ q\ \ r\ \ s]$.

DFT 5. Let $X(k) = k+1, k = 0, 1, \ldots, 7$ be the 8-point DFT of a sequence $x(n)$ such that

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$$

Find the value of the following term without computing IDFT:

$$\sum_{n=0}^{3} x(2n)$$

DTFT 6. Determine and sketch the Fourier transforms $X_1(\omega), X_2(\omega)$, and $X_3(\omega)$ of the following signals.

(a) $x_1(n) = \left\{1, 1, \underset{\uparrow}{1}, 1, 1\right\}$

(b) $x_2(n) = \left\{1, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1, 0\right\}$

(c) $x_3(n) = \left\{1, 0, 0, 1, 0, 0, \underset{\uparrow}{1}, 0, 0, 1, 0, 0, 1, 0, 0\right\}$

expanding in time (Interpolation)!

(d) Is there any relation between $X_1(\omega), X_2(\omega)$, and $X_3(\omega)$? What is its physical meaning?

(e) Show that if

$$x_k(n) = \begin{cases} x\left(\frac{n}{k}\right), & \frac{n}{k} \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

then

$$X_k(\omega) = X(k\omega)$$

7. The DFT of a 4-point sequence $x(n) = (3, 2, 3, 4)$ is $X(k) = (12, 2j, 0, -2j)$. If $X_1(k)$ is the DFT of the 12-point sequence

$$x_1(n) = (3, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0)$$

then find the value of $\left|\frac{X_1(8)}{X_1(11)}\right|$.

Tutorial

periodic in (w) with period 2π

$x(n) \rightarrow X(w)$

$X(w+2\pi) = X(w)$

DTFT :→ $X(w) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$

IDTFT :→ $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(w)\, e^{jwn} \cdot dw$ $\left.\begin{matrix} -\pi \rightarrow \pi \\ 0 \rightarrow 2\pi \end{matrix}\right\}$

(duration has to be 2π)

(Q1) $x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$, with indices $-2, -1, 0, 1, 2$

(zero-location)

(a) $X(0) = \sum_{n=-\infty}^{\infty} x(n)(1) = -1+2-3+2-1 = -1$

(b) $\int_{-\pi}^{\pi} X(w)\, dw = 2\pi\, x(0)$

$= 2\pi(-3) = -6\pi$

(c) $X(\pi) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\pi n} = \sum_{n=-\infty}^{\infty} x(n)(-1)^n$

$= +(-1) - (+2) + (-3) - (+2) + (-1)$

$= -9$

(d) "Parseval's relation"

Energy of the signal in the time domain :→ ; in the freq. domain

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(w)|^2\, dw$$

$$\therefore \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = (1^2 + 2^2 + 3^2 + 2^2 + 1^2) \cdot 2\pi$$
$$= 38\pi$$

(Q2) $x(n) \xrightarrow{\text{DTFT}} X(\omega)$

(a) $x(n)\cos(\pi n/4) \xrightarrow{\text{DTFT}} ?$

$x_1(n) \Rightarrow$

Eg:- $x(n)\, e^{j\alpha n} \xrightarrow{\text{DTFT}} X(\omega - \alpha)$ $\Big\}$ (frequency shifting property!)

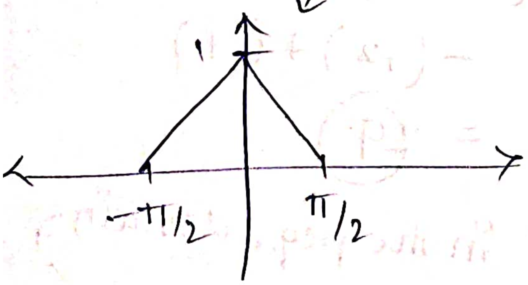
$e^{-j\omega n} \cdot e^{j\alpha n} = e^{-jn(\omega - \alpha)}$

$\therefore (\omega' = \omega - \alpha)$

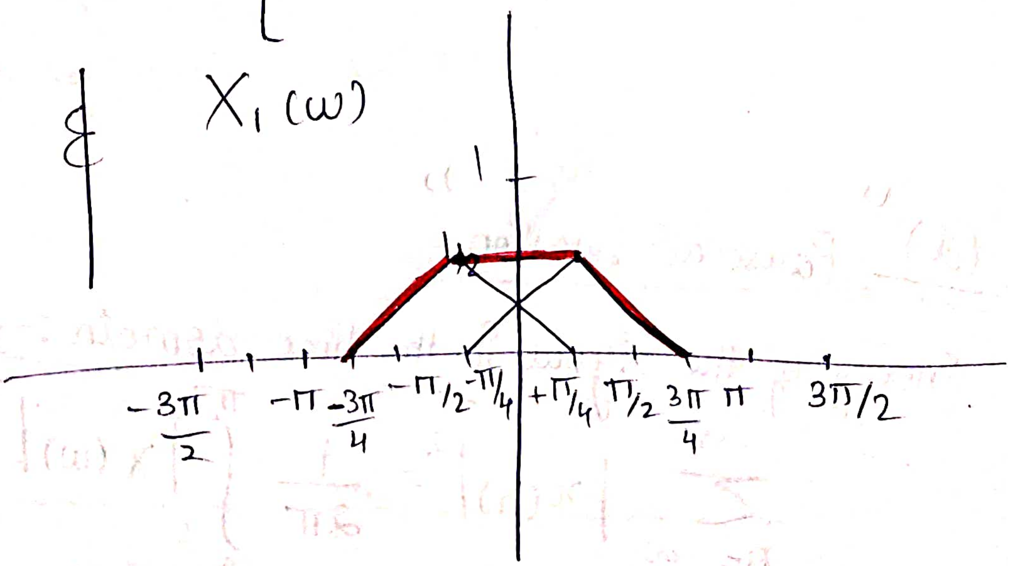
$$\therefore x_1(n) = x(n)\left[\frac{e^{j\pi n/4} + e^{-j\pi n/4}}{2}\right]$$

$$X_1(\omega) = \text{DTFT}(x_1(n)) = \frac{1}{2}\left[X(\omega - \pi/4) + X(\omega + \pi/4)\right]$$

As $X(\omega)$ ↴



& $X_1(\omega)$



(Q3) ⇒ (N = 8)

(missing)

$x(n)$ is real.

$$X_1(k) = (\underset{0}{5}, \underset{1}{1-j3}, \underset{2}{0}, \underset{3}{3-j4}, \underset{4}{2+j4}, \underset{5}{3+j4}), \underset{6}{\_\_\_}, \underset{7}{\_\_\_}$$

if $x(n)$ is real, then $\left[X(k) = X^*(N-k)\right]$

$$X(6) = X^*(8-6) = X^*(2) = (0)$$

$$X(7) = X^*(8-7) = X^*(1) = (1+j3)$$

if not mentioned, then assume the 1st as the zero point,

but then also, we will get the same answers

do check! ha ha! ☺

(Q4) $[a\ b\ c\ d] \xrightarrow{DTFT} [\alpha\ \beta\ \gamma\ \delta]$

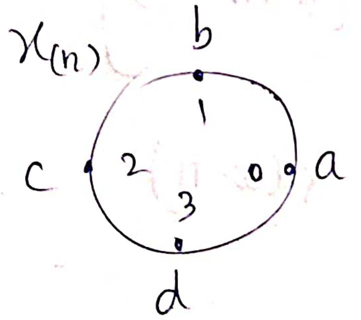
time domain signal vector

$x(n) \xrightarrow{DFT} X(k)$

$$y(n) = [p\ q\ r\ s] = [a\ b\ c\ d]\underbrace{\begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}}_{\text{(circular shifting!)}}$$

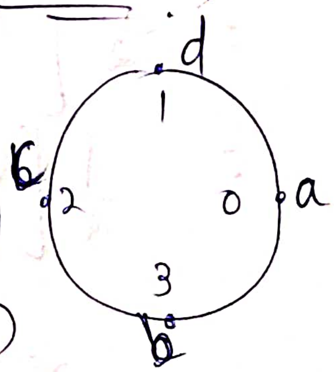
$y(n) \xrightarrow{DTFT} Y(k) = ?$

$x(n)$



$x((-n)_4)$

$(x(0), x(3), x(2), x(1)) = (a, d, c, b)$



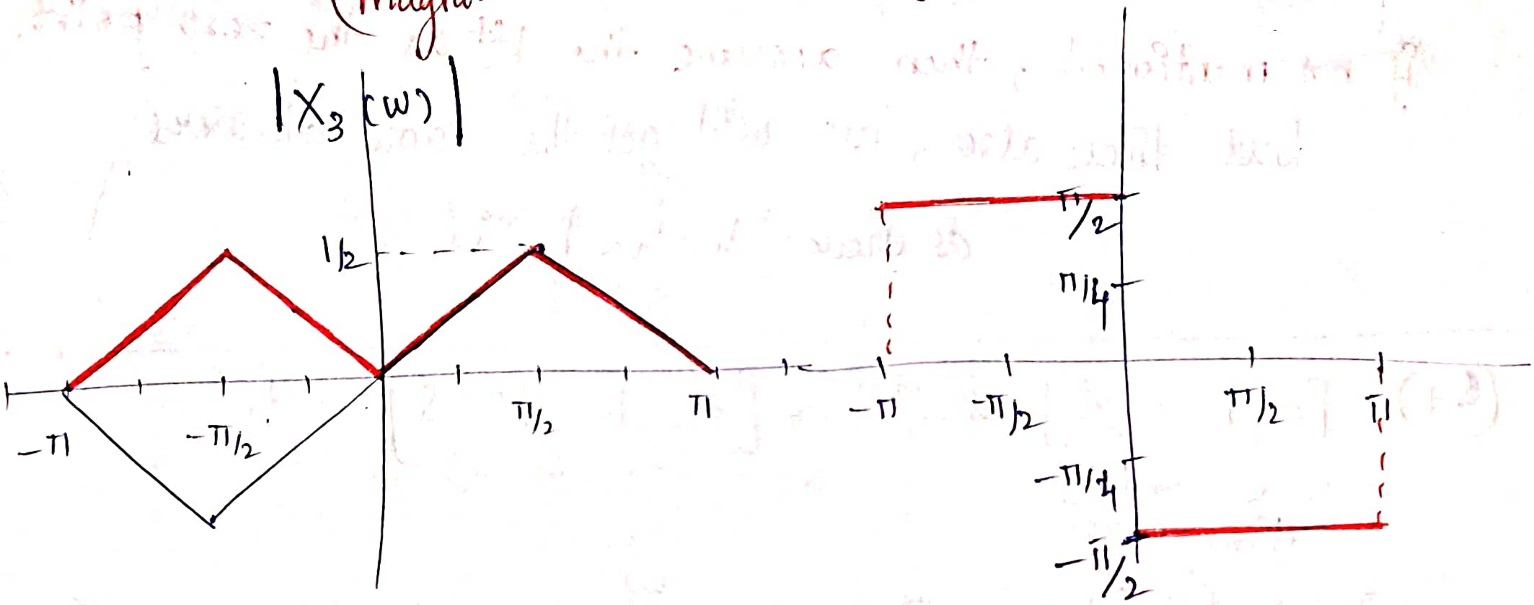
(c) $x_3(n) = x(n)\sin(\pi n/2)$

$$= \frac{x(n)}{2j}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right]$$

$$\therefore X_3(\omega) = \frac{1}{2j}\left[X\left(\omega - \frac{\pi}{2}\right) - X\left(\omega + \frac{\pi}{2}\right)\right]$$

(will have to show both magnitude as well as the angle)

$|X_3(\omega)|$



(d) $x_4(n) = x(n)\cos(\pi n)$

$$= x(n)\,\frac{1}{2}\left[e^{j\pi n} + e^{-j\pi n}\right]$$

$$X_4(\omega) = \frac{1}{2}\left[X(\omega-\pi) + X(\omega+\pi)\right]$$

$$= \frac{1}{2}\left[X(\omega-\pi) + X(\omega+\pi-2\pi)\right]$$

$$= \frac{1}{2}\left[2X(\omega-\pi)\right] = X(\omega-\pi)$$

$y(n) = x(n) \circledast x(n)$

$\rightarrow Y(k) = X(k)\, X(k) = X^2(k)$

$= \underline{\underline{(\alpha^2, \beta^2, \gamma^2, \delta^2)}}$ ✓.

(Q5) $X(k) = k+1 \; ; \; (k=0,1 \cdots 7)$

$$\sum_{n=0}^{3} x(2n) = ?$$

$$X(0) = 1 = \sum_{n=0}^{N-1} x(n)\, W_N^0 = x(0) + x(1) + \cdots x(7)$$

$$X(4) = x(0) - x(1) + x(2) \cdots - x(7)$$

as

$$W_N^{nN/2} = e^{-\frac{j2\pi}{N} \cdot \frac{N}{2} k} = (-1)^k$$

$\left(\frac{N}{2}\right)$ ''

$(5'')$

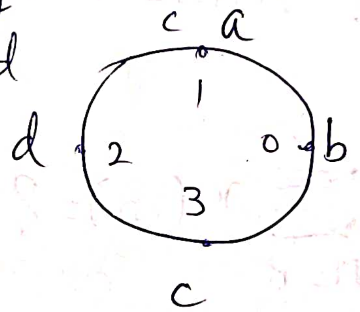
$$\text{Sum} \Rightarrow 2\left[x(0) + (x(2) + x \cdots x(6)\right] = 6$$

$$\therefore \Rightarrow \underline{3}$$

$$y(0) = p = \sum_{k=0}^{3} x(k)\, x((-k)_4) \;;$$

$$y(1) = q = \sum_{k=0}^{3} x(k)\, x((1-k)_4) \;;$$

$$x((1-n)_4) = \big(\underset{b}{x(1)},\ \underset{a}{x(0)},\ \underset{d}{x(3)},\ \underset{c}{x(2)}\big)$$



$$y(2) = r = \sum_{k=0}^{3} x(k)\, x((2-k)_4)$$

$$y(3) = s = \sum_{k=0}^{3} x(k)\, x((3-k)_4)$$

circular

$$x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2((n-k)_N)$$

linear

$$\underbrace{x_1(n)}_{L_1 \text{ length}} * \underbrace{x_2(n)}_{L_2 \text{ length}} = \underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)}_{L_1+L_2-1 \text{ length}}$$

$$\therefore \boxed{y(n) = x(n) \otimes x(n)}$$

(Q6) :→ Interpolation in time ⟺ freq. replication.

(a) $x_1(n) = \{1, 1, 1, 1, 1\}$ at $n = -2, -1, 0, 1, 2$

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n} = 1 + e^{j\omega} + e^{-j\omega} + e^{j2\omega} + e^{-j2\omega}$$

($x_1(n) \to 1$)

$$= (1 + 2\cos\omega + 2\cos 2\omega)$$

(b) $x_2(n) = \{1, 0, 1, 0, 1, 0, 1, 0, 1, 0\}$ at $n = -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$

$$X_2(\omega) = (1 + 2\cos 2\omega + 2\cos 4\omega)$$

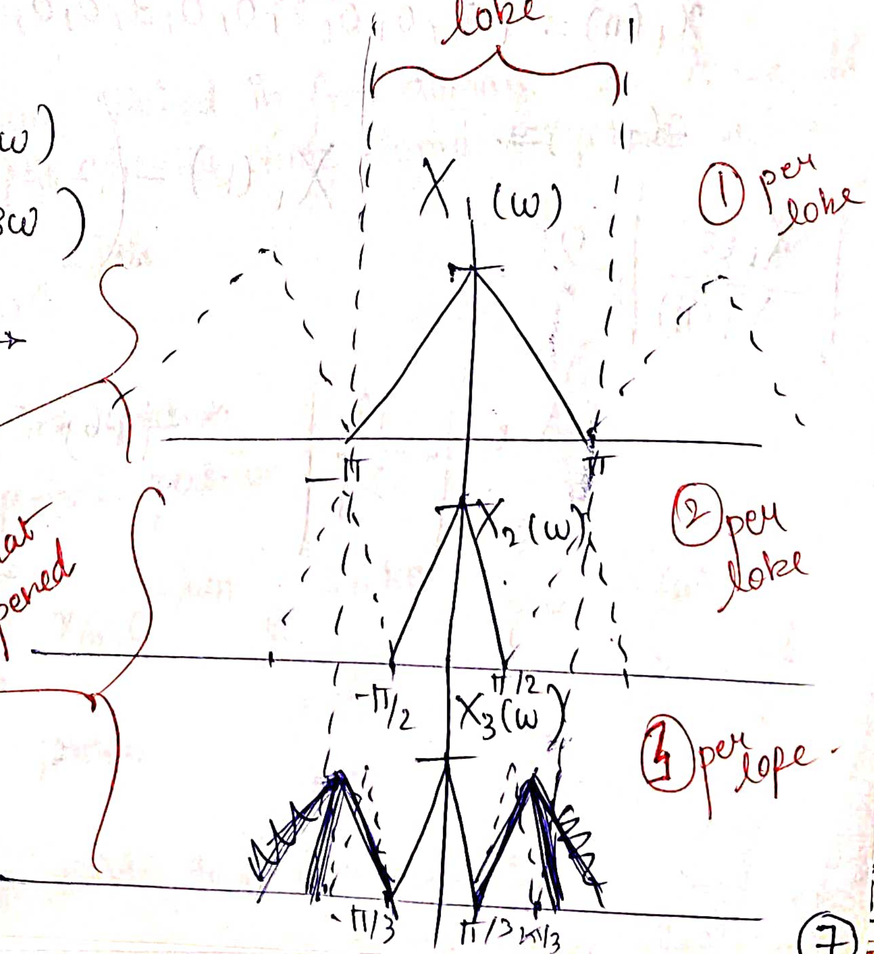
(c) $\{1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0\}$ (the middle 1 at $n = 0$)

$$X_3(\omega) = (1 + 2\cos 2\omega + 2\cos 6\omega)$$

(d) $X_2(\omega) = X_1(2\omega)$

& $X_3(\omega) = X_1(3\omega)$

physical significance :→



Just an example. If $X_1(\omega)$ looked like this, then what would have happened

(e) $x_k(n) = \begin{cases} x(n/k) \; ; & \frac{n}{k} \text{ is an integer} \\ 0 \; ; & \text{otherwise.} \end{cases}$

$$X_k(\omega) = \sum_n x(n/k) \cdot e^{-j\omega n}$$

$\frac{n}{k} \Rightarrow$ integer

$$= \sum_n x(n) \cdot e^{-j\omega k n}$$

$$= \underline{\underline{X(k\omega)}}.$$

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(07). $x(n) = (3, 2, 3, 4)$

$X(k) = (12, 2j, 0, -2j)$

$x_1(n) = (3, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0)$

~~$x_1(n) =$~~

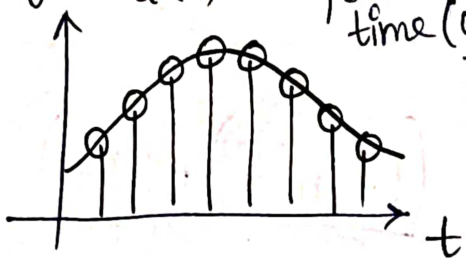
$X_1(k) = (12, 2j, 0, -2j, 12, 2j, 0, -2j, 12, 2j, 0, -2j)$

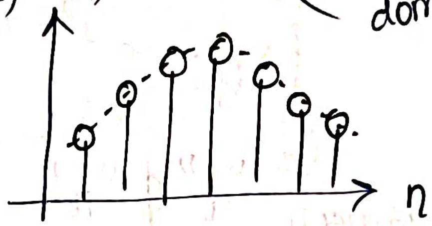
$\left| \frac{X_1(8)}{X_1(11)} \right| = ?$

$\left| \frac{12}{-2j} \right| = |+6j| = \underline{6}$

EE Minor → HARSH AJAY RANA

Sampling $x_a(t)$ periodic time (continuous) time domain $x(n)$ (discrete time domain)





$$x(n) = x_a(nT)$$ $(T < \text{some number})$

sampling frequency $\left(F_s = \frac{1}{T}\right)$ also then $(F_s > \text{some number})$

Spectrum of $x_a(t)$	Spectrum of $x(n)$
$X_a(F)$ or $X(\Omega)$	$X(\omega)$ or $X(f)$

$$= \int_{-\infty}^{\infty} x_a(t)\, e^{-j2\pi Ft}\, dt$$ } fourier transform

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F)\, e^{j2\pi Ft}\, dF$$ } inverse of fourier transform

→ if one signal is band limited in freq domain then it would be time unlimited in the time domain & vice-versa.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

{ periodic signal → discrete spectrum
all discrete time → periodic spectrum }

$$X(\omega + 2\pi K) = \sum_{-\infty}^{\infty} x(n)\, e^{-j\omega n} \underbrace{e^{-2\pi kn}}_{=1} \Big\} = X(\omega)$$

$(\omega = 2\pi f)$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi fn}$$

→ periodic in f with period (1).

$$x_{(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{(\omega)} \cdot e^{j\omega n}\, d\omega$$

$$= \int_{-1/2}^{1/2} X_{(f)} \cdot e^{j2\pi f n}\, df.$$

Relation between "t" & "n" $\rightarrow$ $(t = nT) = \left(\frac{n}{F_s}\right)$ $\rightarrow$ sampling frequency.

Relation between "F" & "f" $\rightarrow$

eg $\Rightarrow$ $x_a(t) = A\cos(2\pi F t + \theta)$

$$x(n) = x_a(nT) = A\cos(2\pi F (nT) + \theta)$$

$$= A\cos(2\pi (F/F_s) n + \theta)$$

$$\therefore x_{(n)} = A\cos(2\pi f n + \theta)$$

comparing

$$\left(f = \frac{F}{F_s}\right)$$

$$x_{(n)} = x_a(nT) = x_a(t)\Big|_{t=nT}$$

$$= \int_{-\infty}^{\infty} X_a(F) \cdot e^{j2\pi \frac{F(n)T}{F_s}}\, dF$$

$$\int_{-1/2}^{1/2} X_{(f)} \cdot e^{j2\pi f n}\, df = \int_{-\infty}^{\infty} X_a(F)\, e^{j2\pi \frac{F(n)T}{F_s}}\, dF$$

$$(f = F/F_s)$$

$$\therefore \frac{1}{F_s}\int_{-F_s/2}^{+F_s/2} X\left(\frac{F}{F_s}\right) e^{j2\pi (F/F_s)}\, d(F)$$

I am understanding 100% yet 0% knowledge of the subject.

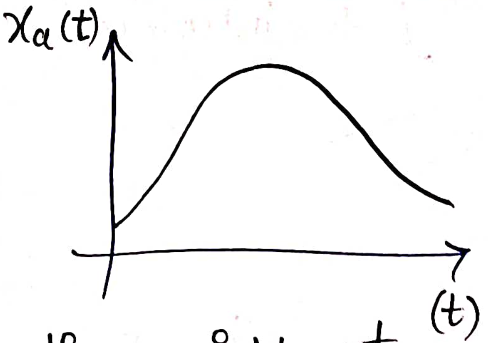
R.H.S. $\displaystyle\sum_{K=-\infty}^{\infty} \int_{(K-1/2)F_s}^{(K+1/2)F_s} X_a(F).e^{j2\pi nF/F_s} dF$

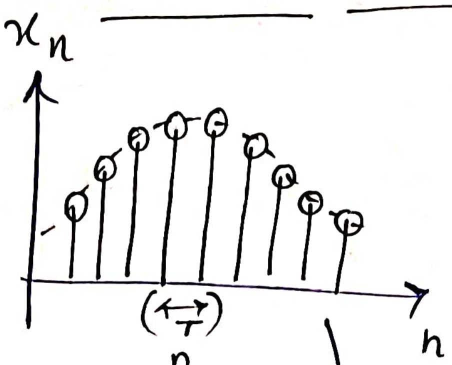
$$= \sum_{K=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} X_a(F'+KF_s) e^{j2\pi n \times \left(\frac{F'+KF_s}{F_s}\right)} dF' \quad \overline{(F'=F-KF_s)}$$

$$(K'=-K) \Rightarrow \sum_{K=\cancel{+}\infty}^{+\infty} \int_{-F_s/2}^{+F_s/2} X_a(F'-KF_s).\,e^{j2\pi n\left(\frac{F'}{F_s}\right)} dF'$$

again variable change $\displaystyle = \int_{-F_s/2}^{F_s/2} \left\{ \sum_{K=-\infty}^{\infty} X_a(F-KF_s).e^{j2\pi n\frac{F}{F_s}} .dF. \right.$

Continuous time | Discrete Time | Relation





$x(n) = x_a(nT)$

$t = nT = \left(\frac{n}{F_s}\right)$

$\left(f = \frac{F}{F_s}\right)$

time variable $t \longrightarrow n$

freq variable $F \longrightarrow f$

spectrum $X_a(F) \longrightarrow X(f)$

$$\boxed{x(n) = x_a(t)\Big|_{t=nT}}$$

★ (because computers can run only on discrete time domain) ★ (DSP processors)

now you know why

IDTFT (of $X(f)$) $=$ IFT of $X_a(F)\Big|_{t=nT}$

(inv. discrete time fourier trans) (inv. fourier trans)

Objective
↳ find out the minimum sample freq so that we can recover $x_a(t)$ from $x(n)$.

$$= \frac{1}{F_s}\int_{-F_s/2}^{F_s/2} X\left(\frac{F}{F_s}\right) e^{j2\pi n\left(\frac{F}{F_s}\right)} dF = \int_{-F_s/2}^{F_s/2} \left\{ \sum_{k=-\infty}^{\infty} X_a(F-2kF_s) \right.$$

{as it's (discrete)}

{continuous time ∴ we break down as a sum of integrals}

devesh bhai ka awesome correction

$(F' = F - kF_s)$

$\therefore (F' - kF_s) \rightarrow (F - 2kF_s)$

{$X(f)$ is periodic over a period of 1}

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$$X\left(\frac{F}{F_s}\right) = F_s \sum_{K=-\infty}^{\infty} X_a(F-2KF_s)$$ } after comparing the integrals.

$$X(f) = F_s \sum_{K=-\infty}^{\infty} X_a(fF_s - 2KF_s)$$

or $(F' - KF_s)$

Also we can see that

$$x_n = \sum_{K=-\infty}^{\infty} \delta(t-KT)\, x_a(t)$$ } time domain.

fourier transform (for getting in frequency domain)

→ similar approach

[Gausian } retains itself also after a fourier transform] information point.

direc delta train }

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} x_a(nT)\, \underbrace{\delta(nT-KT)}_{\text{we need to put } n=K}\, e^{-j2\pi f n}$$

we need to put $n=K$ so that (δ) function (impulse) gives us the value at (nT).

$$\cancel{\sum_{K=-\infty}^{\infty} x_a(nT)\, e^{-j2\pi f n}}$$

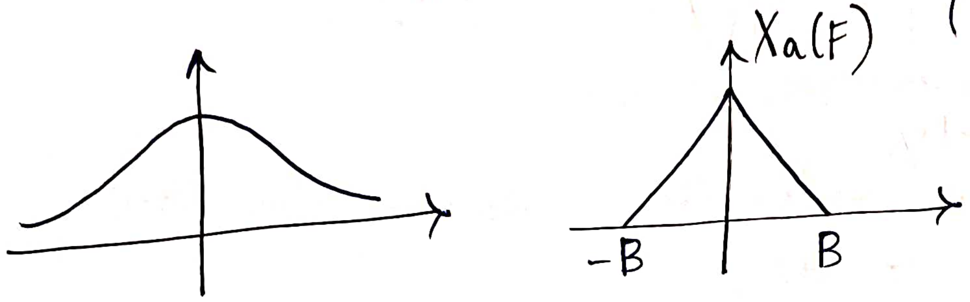
$$\sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} \int_{-\infty}^{\infty} X_a(F)\, e^{j2\pi F n T}\, dF \cdot \delta(nT-KT) \cdot e^{j2\pi f n}$$

our impulse train ⇒ $\delta_T = \sum_{K=-\infty}^{\infty} \delta(t-KT)$

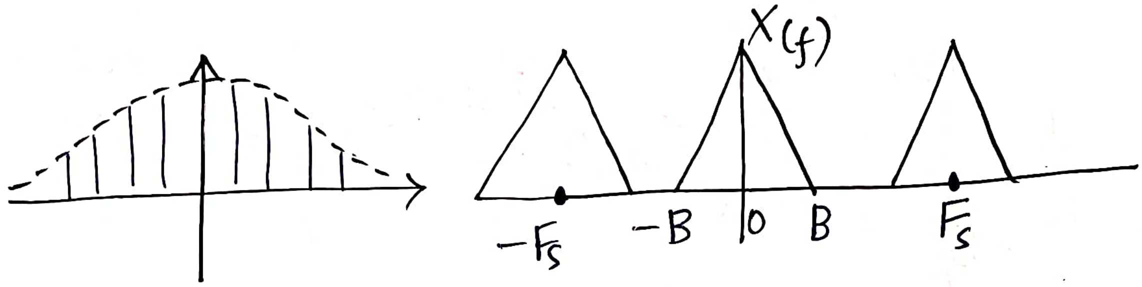
∴ it's fourier transform → $\delta_{F_s} = \sum_{K=-\infty}^{\infty} \delta(F-KF_s)$

fourier trans (convolution) = multiplication of individual fourier transforms

$$\therefore X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

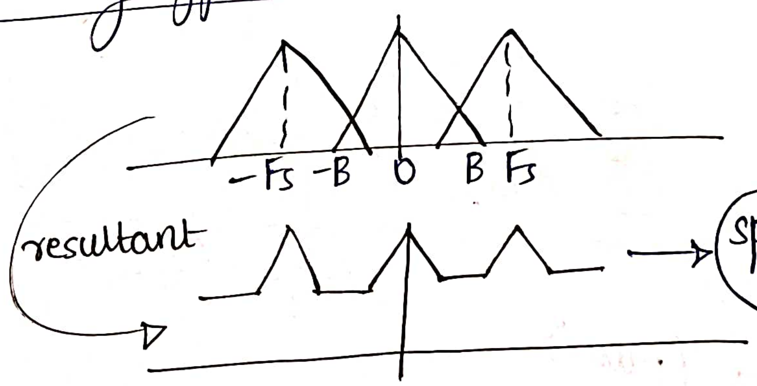


Example of an original signal



$\boxed{F_s > 2B}$ if $(F_s < 2B)$ → then overlapping will happen.

∴ (aliasing effect would happen) ✱



→ spectrum is distorted

{ if this happens then we would not be able to retrieve the original signal }

<u>In absence of aliasing :-</u>

$$X_a(F) = \begin{cases} \frac{1}{F_s} X\left(\frac{F}{F_s}\right) & ; \ |F| \leq \frac{F_s}{2} \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F)\, e^{j2\pi nFt} (dF)$$

$$= \frac{1}{F_s} \int_{-F_s/2}^{F_s/2} X(F/F_s)\, e^{j2\pi nFt} (dF)$$

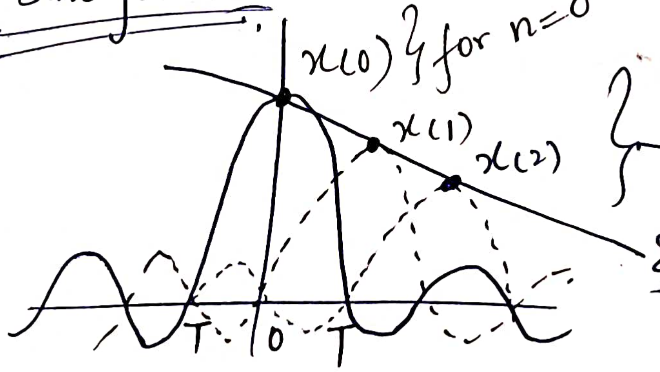
{Now we need to retrieve the continuous time signal from the discrete one}

$$= \frac{1}{F_s} \int_{-F_s/2}^{F_s/2} \left\{ \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi n\left(\frac{F}{F_s}\right)} \right\} e^{j2\pi Ft} \cdot dF$$

$$= \frac{1}{F_s} \sum_{n=-\infty}^{\infty} x(n) \int_{-F_s/2}^{F_s/2} e^{j2\pi F\left(t - \frac{n}{F_s}\right)} dF$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left\{\frac{\pi}{T}(t-nT)\right\}}{\left(\frac{\pi}{T}\right)(t-nT)}$$

sinc function (pulses)



& for $n=1 \Rightarrow \dfrac{\sin \frac{\pi}{T}(t-T)}{\frac{\pi}{T}(t-T)}$

→ interpolation

→ our main curve!!

$$g(t) = \frac{\sin \pi t / T}{(\pi / T) t}$$

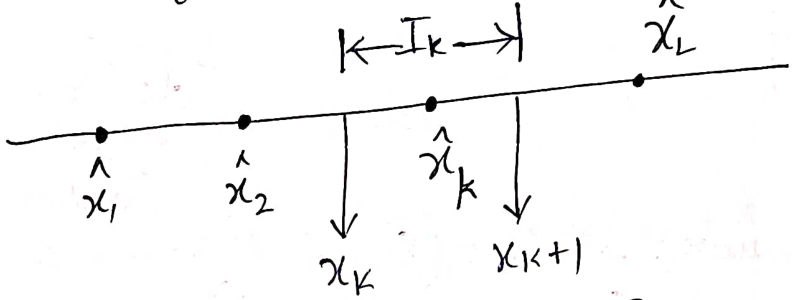
$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)\, g(t-nT) \quad \Big\}\ \text{interpolation formula!}$$

⇒ (tomorrow → how to discritise the amplitude axis)

→ Quantization :-

Only finite set of amplitudes allowed.

$$\{\hat{x}_1, \hat{x}_2, \ldots \hat{x}_L\}$$



$$I_k = \{x_k \le x(n) \le x_{k+1}\}$$

$$x \longrightarrow \boxed{Q(x)} \longrightarrow \hat{x}$$

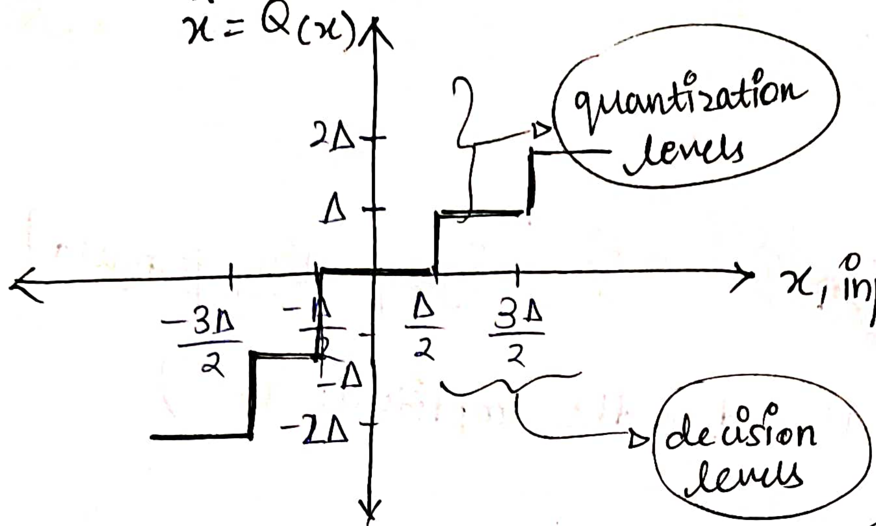
$$x_q(n) = Q[x(n)] = \hat{x}_k \ ; \text{ if } x(n) \in I_k$$

→ uniform Quantizer ;→ $(\hat{x}_{k+1} - \hat{x}_k = \Delta)$ } this difference is fixed.

& $(x_{k+1} - x_k = \Delta)$

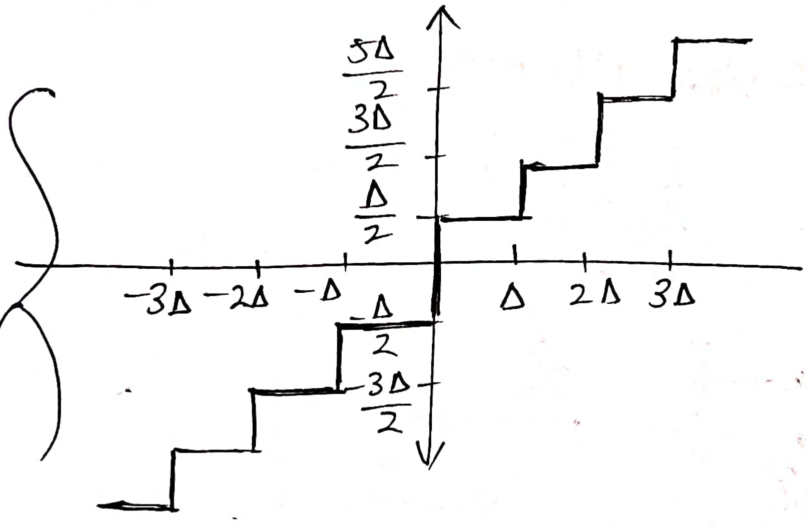
$\hat{x}_k$ } are called quantization levels

& x_k } are called decision levels.



mid-tread quantizer

quantizat^n levels include zero



Mid-rise quantizer.

decision levels include zero

$\rightarrow$ Here total quantizat^n levels $= 8$

(zero not included!)

$\therefore$ we need $b+1 = 3$ bits $\Big\}$ total no. of bits used in quantization.

Quantization Error

$\hookrightarrow$ $\underbrace{e(n)}_{\text{error}} = \underbrace{x(n)}_{\text{true value}} - \underbrace{Q[x(n)]}_{\text{quantization output}}$ $\Bigg\}$ $\therefore$ $-\frac{\Delta}{2} \leq e(n) \leq \frac{\Delta}{2}$

(you can see this from the above diagram too!)

$|x(n)| < x_m \rightarrow$ maximum value

assume :-

$-x_m < x(n) < x_m$

$b+1$ = total bits used in quantization

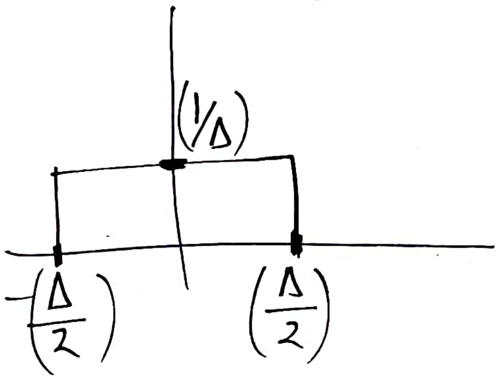
$\therefore \Delta = \frac{2x_m}{L} \rightarrow$ totals levels $=$ ~~$2x$~~ $\frac{2x_m}{2^{b+1}} = \frac{x_m}{2^b}$

6

$$\left(-\frac{\Delta}{2} \leq e \leq \frac{\Delta}{2}\right)$$

here $Q[x(n)]$ → Q is fixed ; but $x(n)$ keeps on varying

Thus we need something to characterise this property!



Quantization noise power

$$\sigma_e^2 = \int_{-\Delta/2}^{\Delta/2} e^2 f_E(e)\, d(e) = \left(\frac{\Delta^2}{12}\right)$$

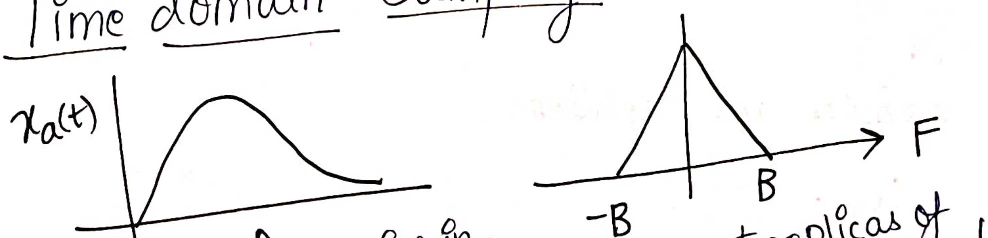
$$= \frac{(X_m)^2}{2^{2b} \times (12)}$$

→ Signal to quantization noise ratio (SQNR)

$$= 10 \log \frac{(\sigma_x)^2}{10(\sigma_e)^2} = 10 \log\left(\frac{(12)\, 2^{b} \sigma_x^2}{X_m^2}\right)$$

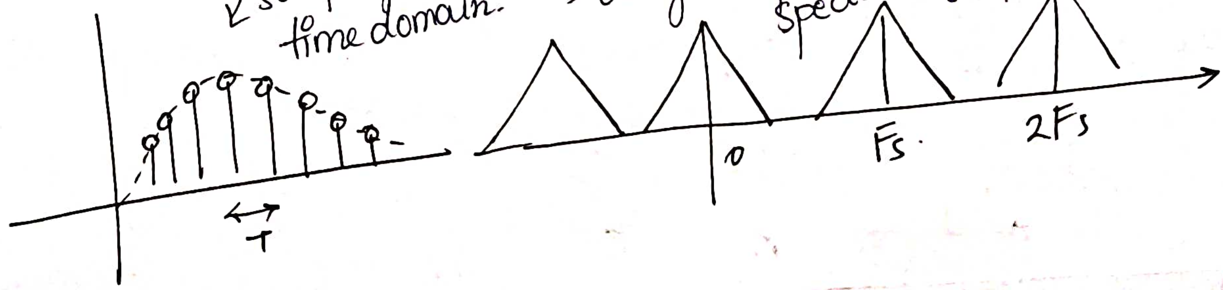
$$= \boxed{6.02b + 10.8 - 20 \log\left(\frac{X_m}{\sigma_x}\right)}$$

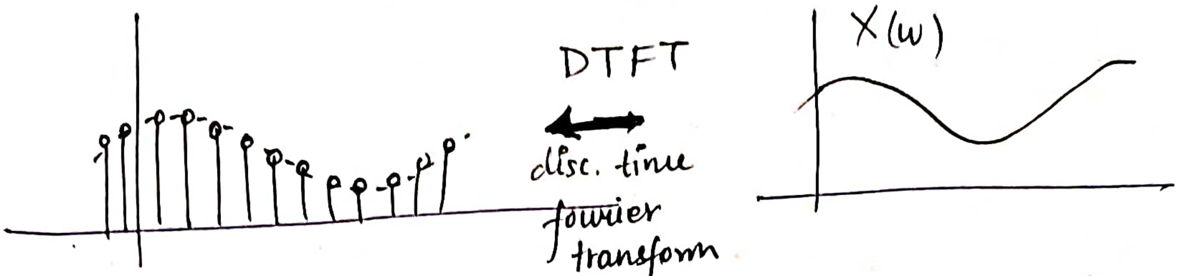
Time domain Sampling :-



$F_s \geq 2B$

sampling in time domain → we get replicas of spectrum!!



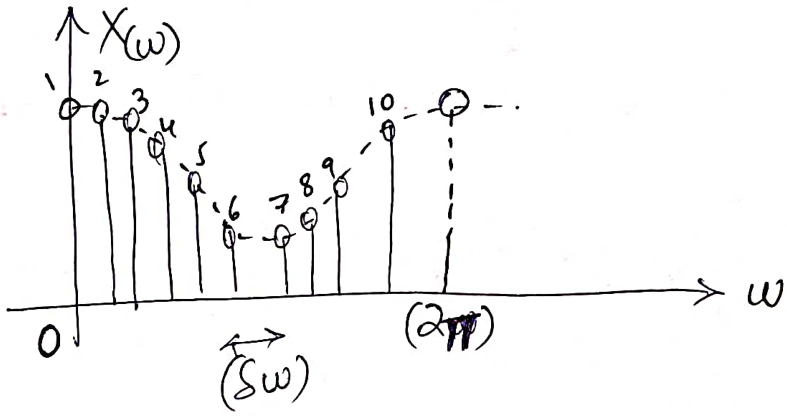


[whenever the signal is periodic → series (fourier)
whenever the signal is non-periodic → transform (fourier)]

→ Frequency Domain Sampling :-

DTFT of $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$



(N=10 samples from 0 → 2π)

$X(\omega)$ is periodic with period (2π).
we take N samples between (zero) & (2π).

$$\delta\omega = \left(\frac{2\pi}{N}\right)$$

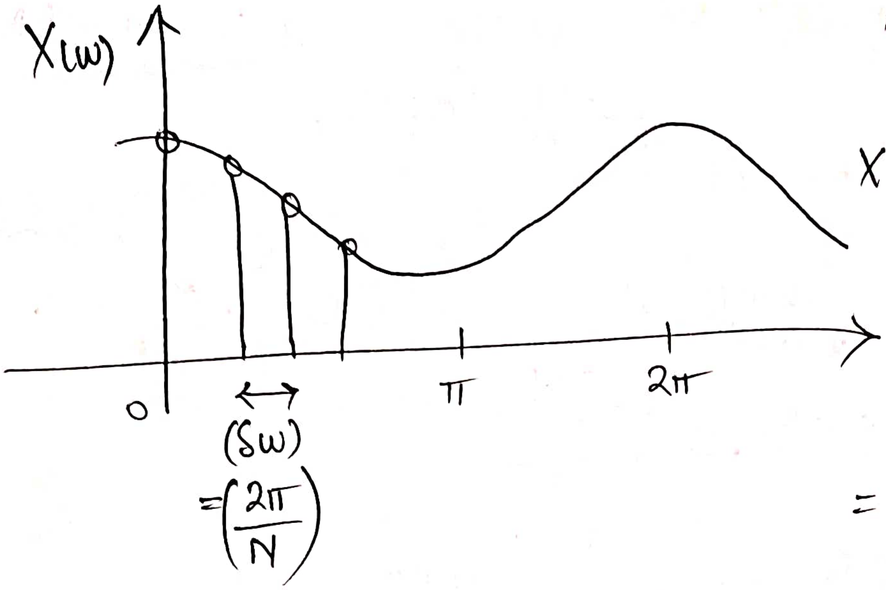
(Q)2 → How many samples are sufficient?

k^{th} sampling point ⇒ $\omega_k = \left(\frac{2\pi}{N}\right)k$; $(k = 0, 1, \dots N-1)$

$$X(\omega_k) = X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi \frac{k}{N}(n)} \;;\; (k=0,1\dots N-1)$$

$$X(\omega_k) = \left(\dots\dots + \sum_{n=-N}^{-1} + \sum_{n=0}^{N-1} + \sum_{n=N}^{2N-1} + \dots\dots \right)$$

Frequency domain Sampling :→



$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X\left(\frac{2\pi k}{N}\right) = X(\omega_k) \quad (A)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi k}{N} n}$$

$$= \left(\sum_{n=0}^{N-1} \left\{ \sum_{l=-\infty}^{\infty} x(n-lN) \right\} \times e^{-j\frac{2\pi kn}{N}} \right)$$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}} \rightarrow (1)\ \underline{Eq}$$

$x_p(n)$

$$\left[x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \right] \Rightarrow$$ these are replicas located at integer multiples of (l).

$$\left[\underbrace{x_p(n+pN) = x_p(n)}_{\text{(periodic with period } N)} \right]$$

<u>CTFS</u> :- $x(t)$ is periodic with period T

$$\phi_k(t) = e^{j\frac{2\pi k}{T} t}$$

$\phi_k(t)$ is periodic with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, \phi_k(t)$$

$$a_k = \frac{1}{T} \int_0^T x(t)\, e^{-\frac{j2\pi kt}{T}}\, dt$$

<u>DTFS</u> :- $x(n)$ is periodic with period N.

$$\phi_k(n) = e^{j2\pi kn/N} ; (k = 0, 1)$$

$$\phi_k(n + rN) = \phi_k(n)$$

$$\phi_N(n) = e^{j2\pi n} = 1$$

$$= \phi_0(n)$$

$$\phi_{N+1}(n) = e^{j2\pi \frac{(N+1)n}{N}}$$

$$= 1.e^{\frac{j2\pi n}{N}}$$

$$\boxed{\phi_{N+1}(n) = \phi_1(n)}$$

$$\Rightarrow x(n) = \sum_{k=0}^{N-1} a_k\, e^{j2\pi kn/N} ; (n = 0, 1, 2 \ldots)$$

here $$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} ; (k = 0, 1 \ldots N-1)$$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j2\pi kn/N}$$

$n \rightarrow n + lN$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+lN)\, e^{-j\frac{2\pi k}{N}(n+lN)}$$

$$e^{-j\frac{2\pi k}{N}(n+lN)} \rightarrow e^{-j\frac{2\pi kn}{N}} \cdot \underbrace{e^{-j2\pi kl}}_{1}$$

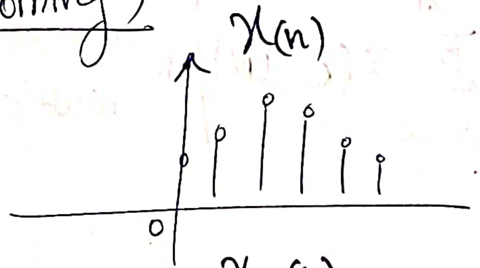
changed (no-issues)

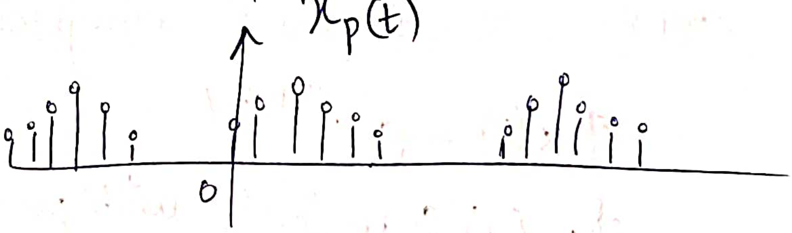
$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) \cdot e^{-\left(j\frac{2\pi kn}{N}\right)}$$

$$= \sum_{(n=0)}^{(N-1)} \left[\underbrace{\sum_{l=-\infty}^{+\infty} x(n-lN)}_{\text{same kind of replicas coming}} \right] \cdot e^{-j\frac{2\pi kn}{N}}$$

$$x_p(t) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

→ periodic discrete time signal with period N.





P → C (CTFS), D (DTFS)

A → C (CTFT), D (DTFT)

$x_p(t)$ can be represented by DTFS

→ The DTFS coefficients for $x_p(n)$:-

$$\left(a_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}\right) \rightarrow (2)\ \text{eq}$$

→ comparing (1) eq & (2) eq ; we get

$$\boxed{a_k = \frac{1}{N} X(k)}$$

$$x_p(n) = \sum_{k=0}^{N-1} a_k \cdot e^{\frac{j2\pi kn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot e^{\frac{j2\pi kn}{N}} ; \quad (n=0, \ldots\ldots)$$

→ (3) eq

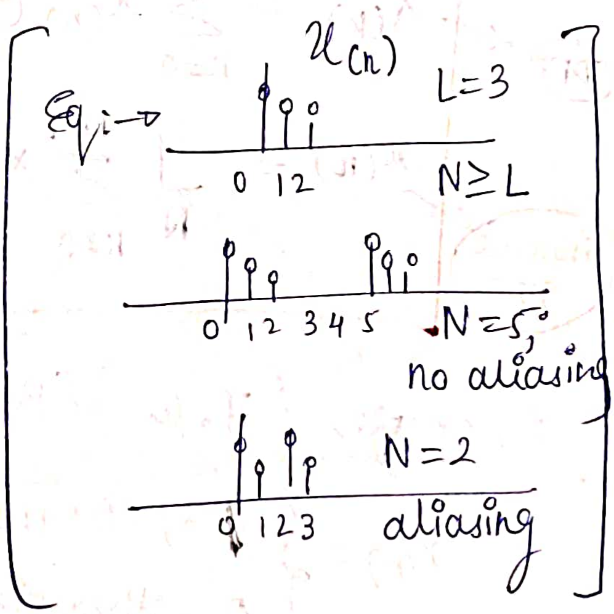
→ Flow chart

$$X(\omega) \xrightarrow[N]{\text{sampling}} X\left(\frac{2\pi k}{N}\right) \xrightarrow{(3)} x_p(n) \longrightarrow x(n)$$

$$\downarrow$$

$$X(k)$$

{ ★ the no. of samples (N) ≥ the (L) }



Suppose ; $N \geq L$

$$x(n) = \begin{cases} x_p(n) ; & n = 0, 1, 2, \ldots N-1 \\ 0 ; & \text{otherwise} \end{cases}$$

[DFT → discrete fourier transform]

A $\Rightarrow$ $X\left(\frac{2\pi k}{N}\right) = X(k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}$; $(k=0,1,\cdots N-1)$

$= \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N} \Rightarrow$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}}$$

$x(n) = x_p(n)$; $(n=0,1,\cdots N-1)$

$= \sum_{k=0}^{N-1} a_k e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$

$\rightarrow$ DFT pair ★★

DFT $\leftarrow$ $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$; $(k=0,1\cdots N-1)$

inverse DFT $\leftarrow$ $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$; $(n=0,1\cdots N-1)$

Example (7.12) (Proakis)

$x(n) = \begin{cases} 1 ; & 0 \le n \le L-1 \\ 0 ; & \text{otherwise} \end{cases}$; determine ~~(N-1)~~ N point DFT of this sequence for $N \ge L$

DTFT $\Rightarrow$

$X(\omega) = \sum_{n=0}^{L-1} 1 \cdot e^{-j\omega n}$

$= \sum_{n=0}^{L-1} (e^{-j\omega})^n = \left(\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right)$

DFT $\Rightarrow$ sampled DTFT (but sampling is done in the freq domain)

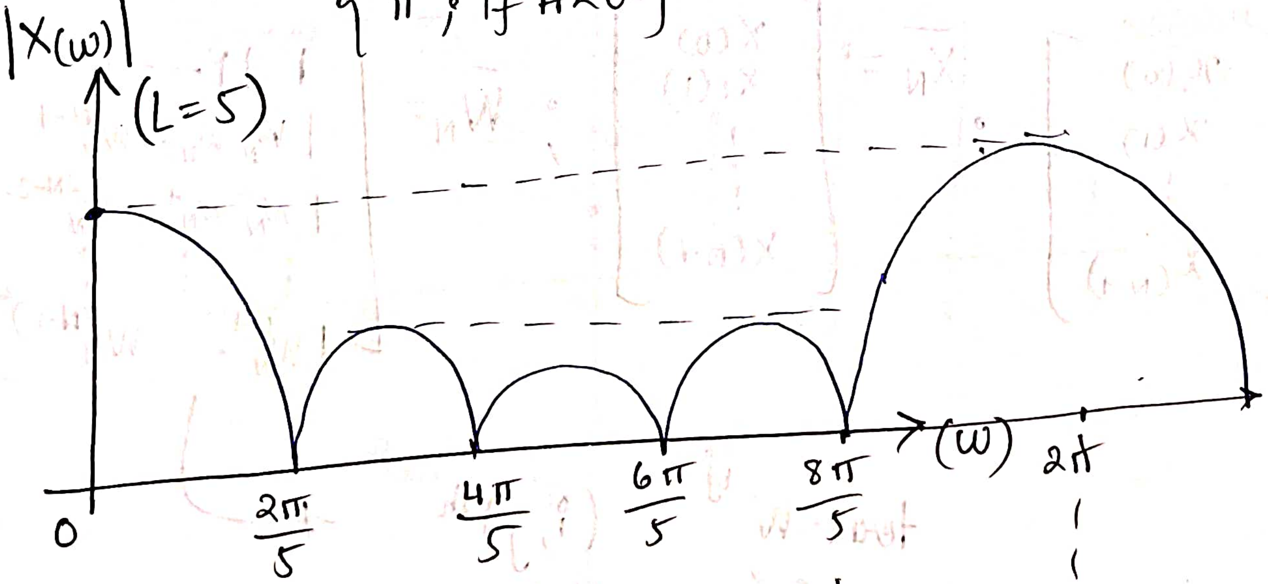
$$= \frac{e^{-\frac{j\omega L}{2}}\left\{e^{\frac{j\omega L}{2}} - e^{-\frac{j\omega L}{2}}\right\}}{e^{-\frac{j\omega}{2}}\left\{e^{j\omega/2} - e^{-j\omega/2}\right\}}$$

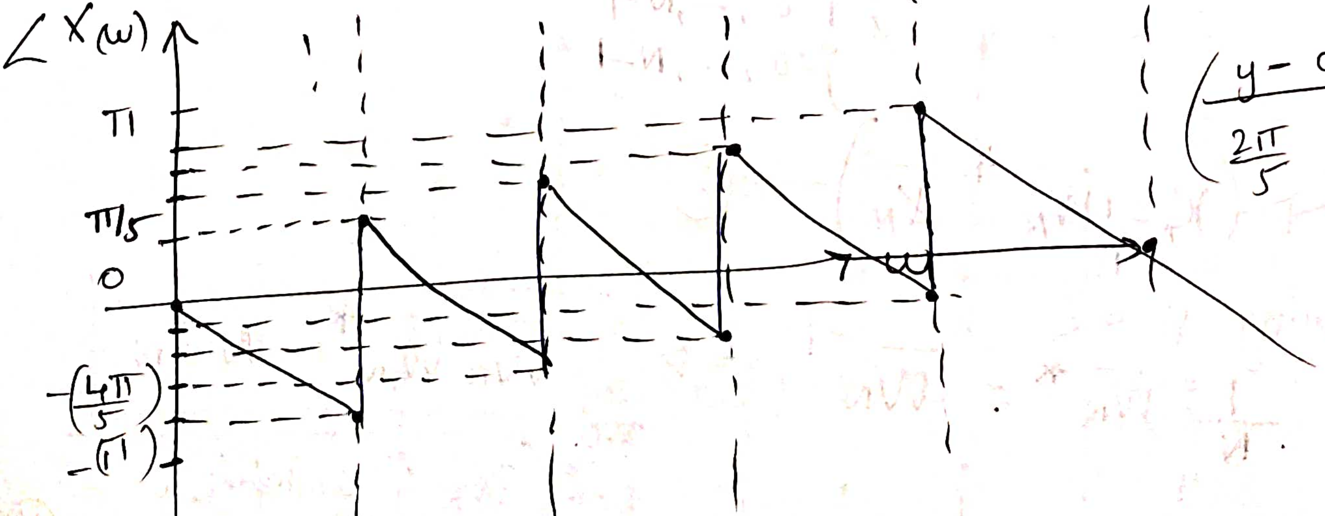
$$= \boxed{\frac{\sin(\omega L/2)}{\sin(\omega/2)} \cdot e^{-\frac{j\omega}{2}(L-1)}}$$

$$|X(\omega)| = \left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right| \quad \& \quad \left\{\angle X(\omega) = \angle\frac{\sin(\omega L/2)}{\sin(\omega/2)} - \frac{\omega}{2}(L-1)\right\}$$

Angle

$$\angle A = \begin{Bmatrix} 0 \; ; \text{if } A \geq 0 \\ \pi \; ; \text{if } A < 0 \end{Bmatrix}$$





$$\left(\frac{y-0}{\frac{2\pi}{5}} = 2\right)$$

DFT as a linear Transformation :-

DFT :→ $X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}}$; $(k=0,1,\cdots N-1)$

IDFT :→ $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$; $(n=0,1,\cdots N-1)$

$W_N = e^{-j2\pi/N}$ } Nth root of unity } $(e^{j2\pi} = 1)$

→ $= \left(\sum_{n=0}^{N-1} x(n) W_N^{kn}\right)$ ⇒ $\left(\bar{X}_N = \bar{W}_N \bar{x}_N\right)$

⇓

→ $= \left(\frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}\right)$ ⇔ $\left(\bar{x}_n = \bar{W}_N^{-1} \bar{X}_N\right)$ → ①

$\bar{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$; $\bar{X}_N = \begin{bmatrix} X(0) \\ X_1(1) \\ \vdots \\ X(N-1) \end{bmatrix}$; $\bar{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2N-2} \\ \vdots & & \vdots & & \vdots \\ 1 & W_N^{N-1} & & \cdots & W_N^{(N-1)^2} \end{bmatrix}$

term $= W^{ij}$ $(i,j)^{th}$
; $i=0,\cdots,N-1$
$j=0,\cdots,N-1$

→ $\left(\bar{x}_n = \frac{1}{N}\bar{W}_N^{*} \bar{X}_N\right)$ → ②

By comparing ① & ②

$\frac{1}{N}\bar{W}_N^{*} = \bar{W}_N^{-1}$ ⇒ $\boxed{\bar{W}_N \bar{W}_N^{*} = N I_N}$

∴ ($\bar{W}_N$ is unitary matrix)

Properties of DFT :

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

But we started with non-periodic?

(actually we are doing shifting (circular))

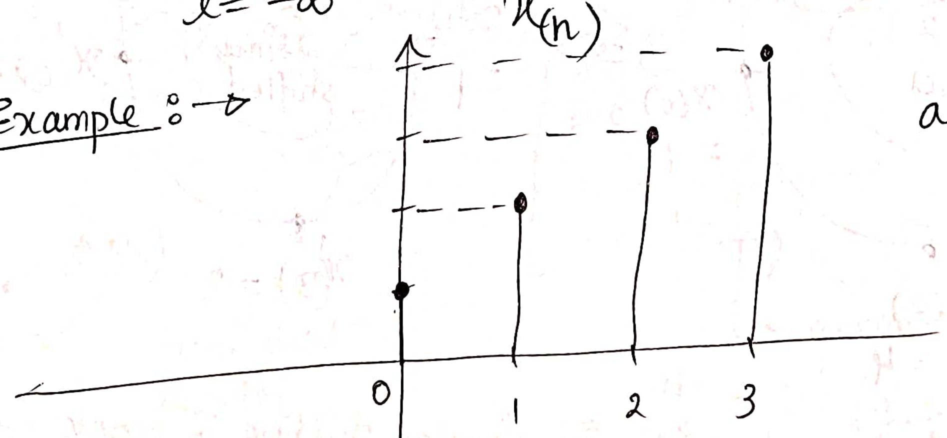
① Periodicity : $X(k+N) = X(k)$; $x(n+N) = x(n)$

② Linearity : $a_1 x_1(n) + a_2 x_2(n) \xrightarrow{DFT_N} a_1 X_1(k) + a_2 X_2(k)$

③ Circular shift :

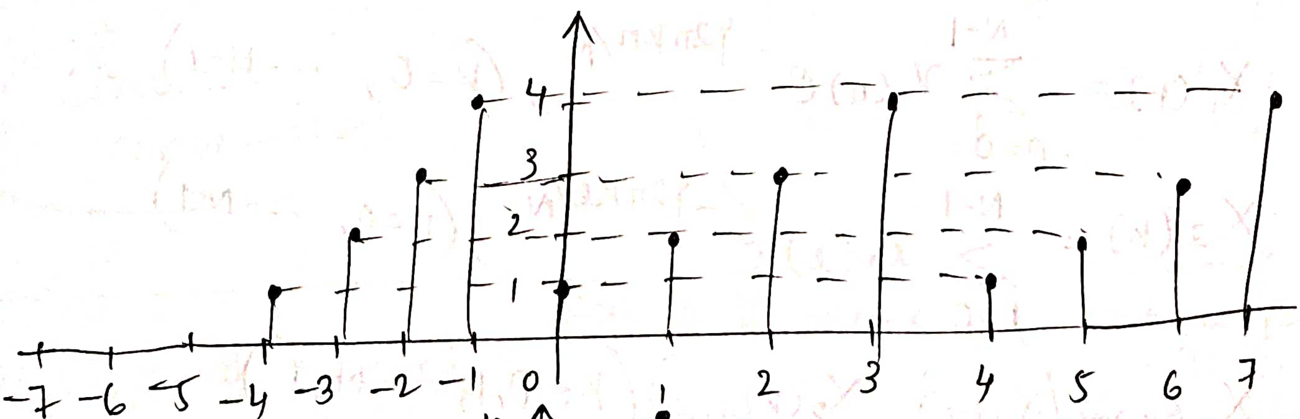
$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Example :

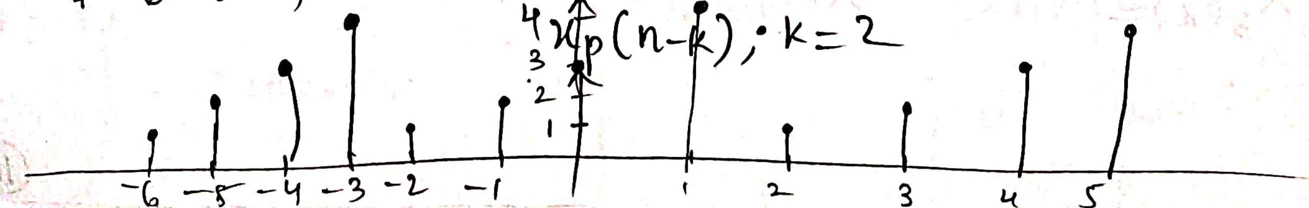


assume $(N=4)$

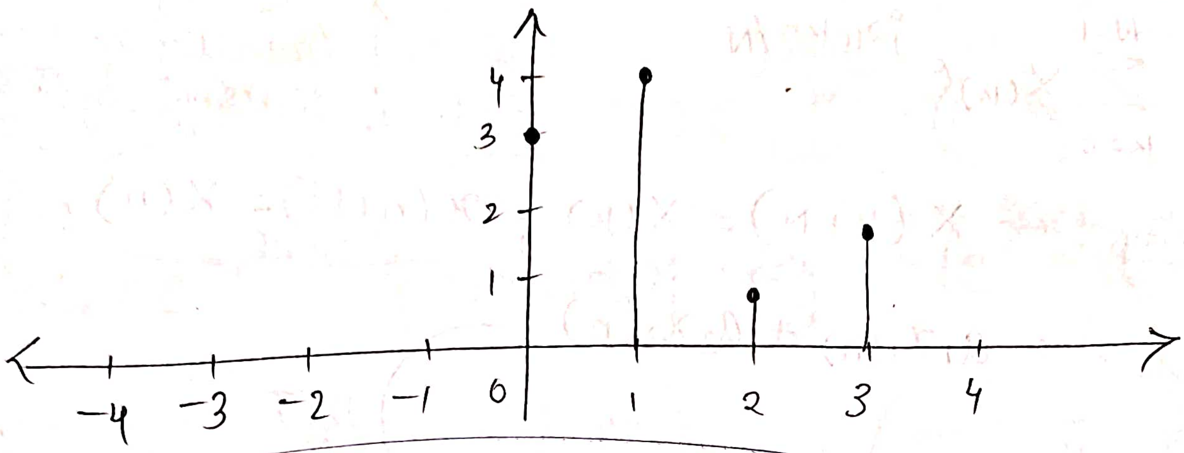
then $x_p(n) \Rightarrow$?



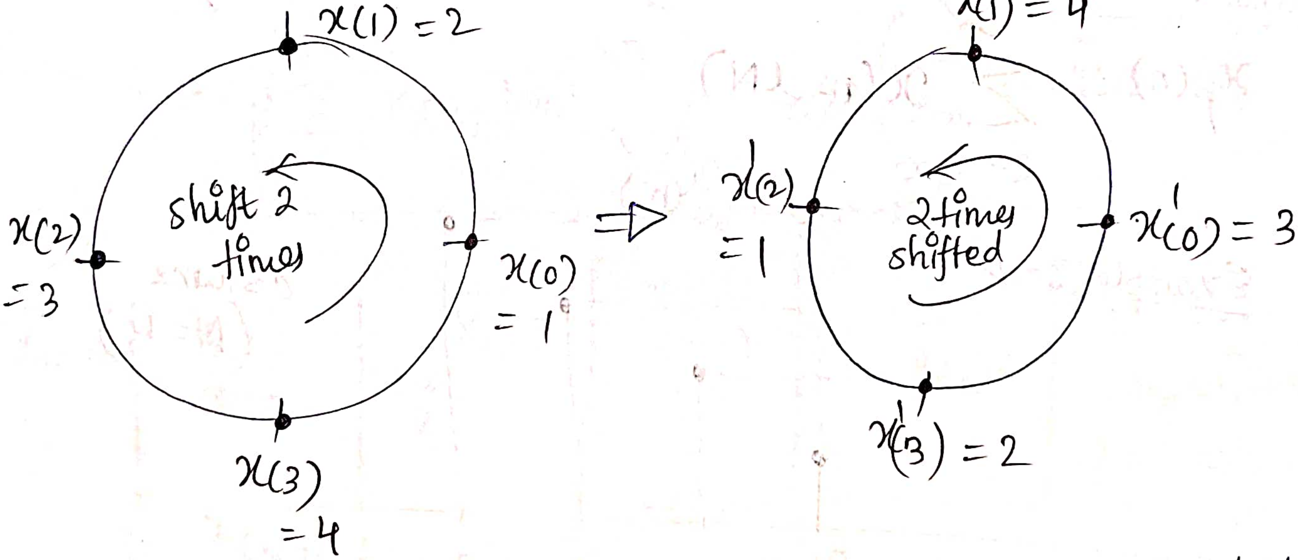
$x_p(n-k)$; $k=2$



$$x'(n) = \begin{cases} x(n) & ; 0 < n \leq N-1 \\ 0 & ; \text{otherwise} \end{cases}$$



∴ this is like circular shifting



→ Shifting towards right is circular shifting in counter-clockwise manner

(4) Multiplication of two DFTs & circular convolution

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N} \; ; (k=0, \dots N-1)$$

$$X_2(k) = \sum_{l=0}^{N-1} x_2(l)\, e^{-j2\pi kl/N} \; ; (k=0, \dots N-1)$$

$$X_3(k) = (X_1(k) \cdot X_2(k) \dots) ; (k=0,1, \dots N-1)$$

Inverse DFT of $\{X_3(k)\}$; $x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k) e^{\frac{j2\pi km}{N}}$

; $m = (0, \cdots N-1)$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{\frac{j2\pi km}{N}}$$

$$X_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} \left\{ \underbrace{\sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}}_{X_1(k)} \right\} \left\{ \underbrace{\sum_{l=0}^{N-1} x_2(l) \cdot e^{-j\frac{2\pi kl}{N}}}_{X_2(k)} \right\} \times \left(e^{j\frac{2\pi km}{N}} \right)$$

$$X_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\underbrace{\sum_{k=0}^{N-1} e^{j2\pi k\frac{(m-n-l)}{N}}}_{\text{"(T)"}} \right]$$

↳ geometric series

$$\therefore \frac{1}{\cancel{N}} \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n)_N) \cdot \cancel{N}$$

only one term that satisfies $l = (m-n)_N$

$$\Rightarrow \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n)_N)$$

Circular convolution

$$T = \sum_{k=0}^{N-1} \left\{ e^{j2\pi\left(\frac{m-n-l}{N}\right)} \right\}^k$$

$$\therefore \boxed{T = \frac{1 - e^{j2\pi\left(\frac{m-n-l}{\cancel{N}}\right) \times \cancel{N}}}{1 - e^{j\frac{2\pi(m-n-l)}{N}}}}$$

when ⇒ if $m-n-l = pN$; where $p \in \mathbb{Z}$

then $T = \begin{cases} N & ; \text{if} \; (\text{"}) \\ 0 & ; \text{otherwise} \end{cases}$

$$T = \begin{cases} N & ; \text{if } l = \underbrace{(m-n)_N}_{\text{modulus}} \\ 0 & ; \text{otherwise} \end{cases}$$

Linear convolution :-

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n)$$

→ Digital signal processing → by John Proakis.

Fast Fourier Transform (FFT)

→ FFT in an iterative algorithm to compute DFT.

Complexity analysis of DFT :→

DFT of a sequence $\{x(n)\}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \quad ; k = 0, 1, \ldots N-1$$

$W_N = e^{-j2\pi/N}$ ∴ $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$

W_N : nth root of unity

$$X(k) = \sum_{n=0}^{N-1} \left\{ Re[x(n)] + j\, Im[x(n)] \right\} \times \left\{ Re[W_N^{nk}] + j\, Im[W_N^{nk}] \right\} \quad ; k = 0, 1, \ldots N-1$$

No. of complex multiplications

for one $X(k)$ ⇒ (N) complex multiplications.

⇒ $(N)^2$ for all $X(k)$ → complex multiplications.

No. of complex additions

for one $X(k)$ ⇒ $(N-1)$ complex additions

⇒ $(N)(N-1)$ for all $X(k)$ → complex additions.

∴ we can see that ⇒ complexity of DFT is very high for larger values of N.

* Direct computation of DFT does not consider the following properties of W_N

(a) Symmetry property : $W_N^{K+N/2} = -W_N^{K}$

(Hence proved!)

$$W_N^{K+N/2} = \left(e^{-j2\pi/N}\right)^{K+N/2}$$

$$= e^{-j\frac{2\pi K}{N}} \cdot e^{-j\pi} \rightarrow (-1)$$

$$= -e^{-j\frac{2\pi K}{N}} = -(W_N)^K$$

(b) Periodicity property :- $\left(W_N^{K+N} = W_N^{K}\right)$

(Hence proved!)

$$\left(e^{-j2\pi/N}\right)^{K+N} = e^{-j\frac{2\pi K}{N}} \cdot \cancel{e^{-j2\pi}} \rightarrow (1)$$

$$= e^{-j\frac{2\pi K}{N}} = (W_N)^K$$

for (FFT) :- break the DFT of longer sequence into DFTs of smaller ones.

$$8\,DFT \to \begin{cases} 4\,DFT \to \begin{cases} 2DFT \\ 2DFT \end{cases} \\ 4\,DFT \to \begin{cases} 2\,DFT \\ 2DFT \end{cases} \end{cases}$$

2DFT → 2 point DFT

$$X(K) = \sum_{n=0}^{1} x(n) e^{-j2\pi\frac{kn}{2}} \quad ; K = 0, 1$$

$$X(0) = x(0) + x(1)$$

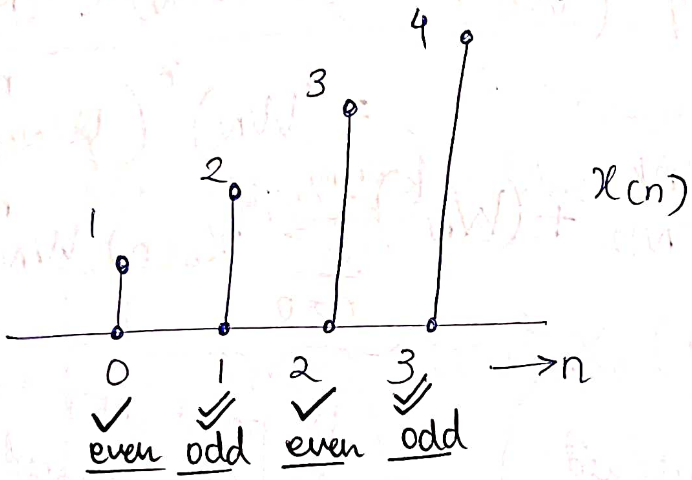
$$X(1) = x(0) - x(1)$$

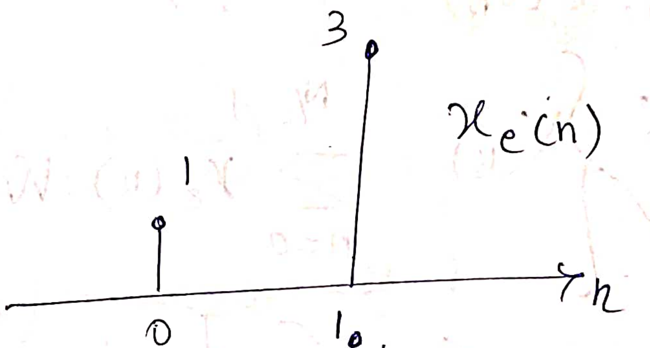
Breaking the DFT of larger seq into DFT of smaller ones.

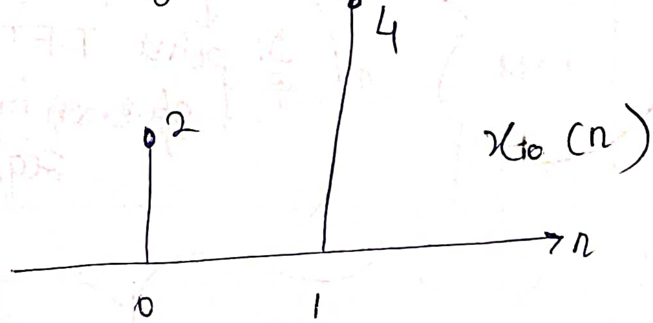
→ Dessimation in time Algorithm

suppose N is a power of 2 i.e, $N = 2^M$; M is an integer.

$$x(n) \rightarrow \begin{cases} x_e(n) \text{ even indexed} \\ x_o(n) \text{ odd indexed} \end{cases}$$







$x_e(n) = x(2n)$; $n = 0, 1 \ldots \frac{N}{2} - 1$

$x_o(n) = x(2n+1)$; $n = 0, 1 \ldots \frac{N}{2} - 1$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \; ; \quad K = 0, 1 \ldots N-1$$

(n = even) (n = odd)

$$= \sum_{\substack{n=0 \\ (n=\text{even})}}^{N-1} x(n) W_N^{nk} + \sum_{\substack{n=0 \\ (n=\text{odd})}}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk} + \sum_{n=0}^{N/2-1} x_o(n) W_N^{(2n+1)k}$$

$$(W_N)^2 = \left(e^{-j2\pi/N}\right)^2 = \left(e^{-j4\pi/N}\right) = e^{-j2\pi/(N/2)} = (W_{N/2})$$

$$\left(W_N = e^{-j2\pi/N}\right)$$

$$\left(W_{N/2} = e^{-j2\pi/N/2}\right)$$

$$\therefore (W_N)^{2nk} = \left(W_{\frac{N}{2}}\right)^{nk} \; \& \; (W_N)^{(2n+1)k} = (W_N)^k \left(W_N^{2nk}\right)$$

$$= (W_N)^k \left(W_{N/2}^{nk}\right)$$

$$\therefore \left\{ X(k) = \underbrace{\sum_{n=0}^{N/2-1} x_e(n)\, W_{N/2}^{nk}}_{\left(\frac{N}{2} \text{ point DFT of even indexed sequence}\right)} + (W_N^k) \underbrace{\sum_{n=0}^{N/2-1} x_o(n)\, W_{N/2}^{nk}}_{\left(\frac{N}{2} \text{ point DFT of odd indexed sequence}\right)} \right\}$$

$$\underbrace{X(k)}_{(k=0,1 \ldots N-1)} = \boxed{\underbrace{X_e(k)}_{(k=0,1,\ldots,\frac{N}{2}-1)} + (W_N)^k \underbrace{X_o(k)}_{(k=0,1,\ldots,\frac{N}{2}-1)}}$$

from symmetry property.

$$\boxed{W^{k+N/2} = -W_N^k}$$

suppose ; $\left(k \geq \frac{N}{2}\right)$

$$k = k' + \frac{N}{2} \; ; \left(k' = 0, 1 \ldots \frac{N}{2} - 1\right)$$

$$\therefore \cancel{X(k) = X_e\left(k - \frac{N}{2}\right) + (W_N)^k (}$$

we know:-

$$X_e\left(k'+\frac{N}{2}\right)=X_e(k') \quad \text{periodic in period} \left(\frac{N}{2}\right).$$

$$X_o\left(k'+\frac{N}{2}\right)=X_o(k')$$

$$\therefore \boxed{X(k)=X_e(k')+\left(-W_N^{k'}\right)X_o(k')} \text{ ; where } \left(k=k'+\frac{N}{2}\right) \Rightarrow k'=k-N/2$$

$$X(k)=X_e\left(k-\frac{N}{2}\right)-W_N^{(k-N/2)}X_o(k-N/2)$$

$$\Rightarrow \boxed{\begin{cases} X_e(k)+W_N^{k}X_o(k) & ; \; k=0,1\ldots\frac{N}{2}-1 \\ X_e\left(k-\frac{N}{2}\right)-W_N^{k-N/2}X_o(k-N/2) & ; \; k=\frac{N}{2}\ldots N-1 \end{cases}}$$

$\Rightarrow$ Total no. of complex multiplications

$$\Rightarrow \underbrace{(N)}_{(W\times X)}+\underbrace{\left(\frac{N}{2}\right)^2}_{\text{for } X_e}+\underbrace{\left(\frac{N}{2}\right)^2}_{\text{for } X_o}$$

$$= N+\frac{N^2}{2} \Rightarrow \text{when N is very large} \approx \frac{N^2}{2}$$

$\Rightarrow$ Total no. of complex additions

$$= N+\frac{N}{2}\left(\frac{N}{2}-1\right)+\frac{N}{2}\left(\frac{N}{2}-1\right)$$

$$= \frac{N^2}{2}$$

Example :- (N=8)

$x_e(0) = x(0)$ & $x_o(0) = x(1)$

$x_e(1) = x(2)$ $x_o(1) = x(3)$

$x_e(2) = x(4)$ $x_o(2) = x(5)$

$x_e(3) = x(6)$ $x_o(3) = x(7)$

$$X(k) = \begin{cases} X_e(k) + W_8^{k} X_o(k); & k = 0, 1 \ldots 3 \\ X_e(k-4) + W_8^{k-4} X_o(k-4); & k = 4, 5 \ldots 7 \end{cases}$$

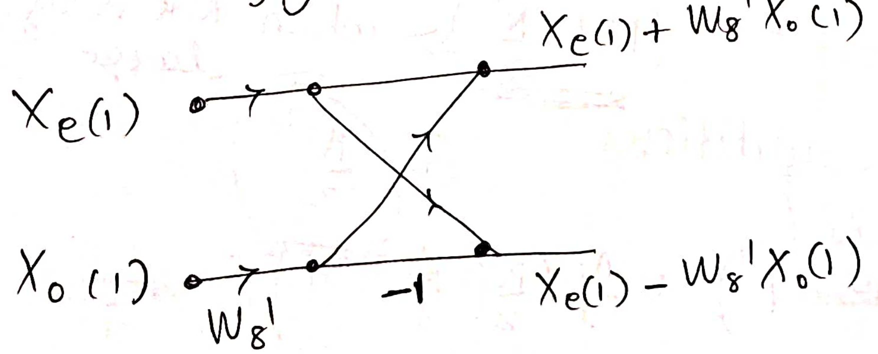
$X(0) = X_e(0) + W_8^0 X_o(0)$ $X(4) = X_e(0) - W_8^0 X_o(0)$

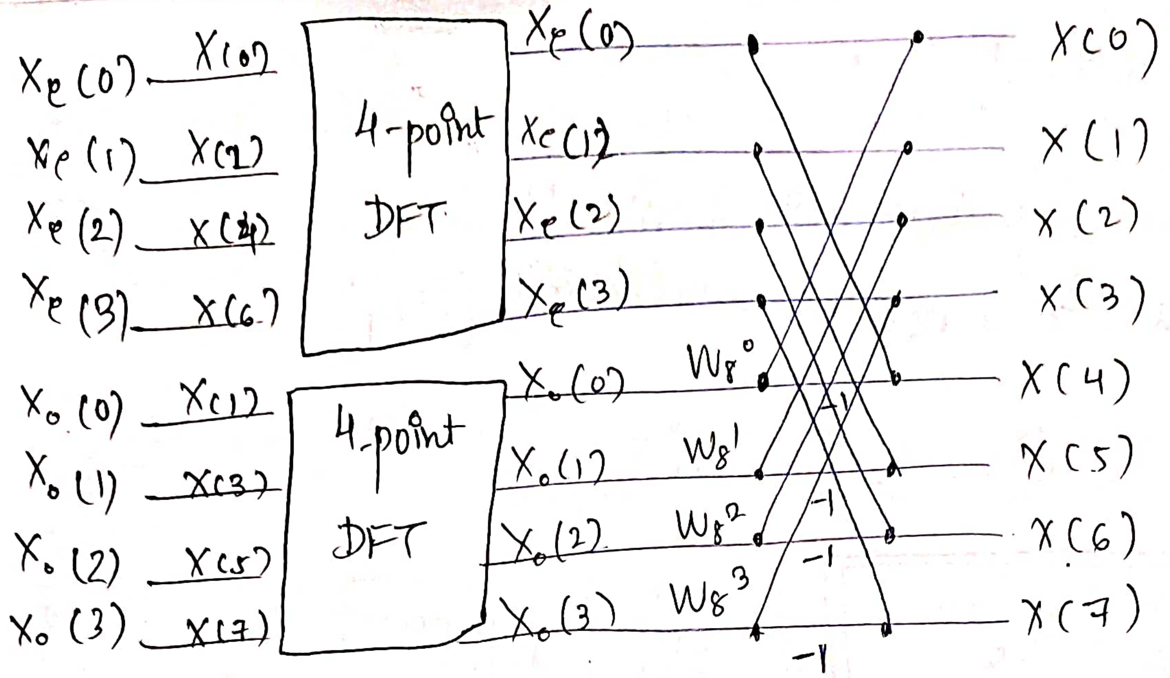
$X(1) = X_e(1) + W_8^1 X_o(1)$ $X(5) = X_e(1) - W_8^1 X_o(1)$

$X(2) = X_e(2) + W_8^2 X_o(2)$ $X(6) = X_e(2) - W_8^2 X_o(2)$

$X(3) = X_e(3) + W_8^3 X_o(3)$ $X(7) = X_e(3) - W_8^3 X_o(3)$

⇒ Butterfly diagram :- Example





$$X_e(k) = \sum_{n=0}^{(N/2-1)} x_e(n)\, W_{N/2}^{nk} \; ; \quad \left(k=0,1\cdots\frac{N}{2}-1\right)$$

$$= \sum_{\substack{n=0 \\ n\Rightarrow \text{even}}}^{\frac{N}{2}-1} x_e(n)\, W_{N/2}^{nk} + \sum_{\substack{n=0 \\ n\Rightarrow \text{odd}}}^{\frac{N}{2}-1} x_e(n)\, W_{N/2}^{nk}$$

$$= \sum_{n=0}^{N/4-1} x_e(2n)\, W_{N/2}^{2nk} + \sum_{n=0}^{\frac{N}{4}-1} x_e(2n+1)\, W_{N/2}^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n)\, W_{N/4}^{nk} + W_{N/2}^{k} \sum_{n=0}^{\frac{N}{4}-1} x_{eo}(n)\, W_{N/4}^{nk}$$

$\left(\frac{N}{4}\right)$ point DFT

$X_{ee}(k)$

$$\left\{ \begin{aligned} &\text{as } W_N = e^{-j2\pi/N} \\ &W_{N/2} = \left(e^{-j\frac{2\pi}{N}}\right)^2 \\ &\quad = (W_N)^2 \end{aligned} \right\}$$

$\therefore$

$\left(\frac{N}{4}\right)$ point DFT

$X_{eo}(k)$

$$= \boxed{X_{ee}(k) + W_{N/2}^{k} X_{eo}(k)}$$

Suppose ; $\left(K > \frac{N}{4}\right)$

(N) point DFT is periodic with period (N)

$\therefore$ $\left(\frac{N}{4}\right)$ point DFT is periodic with period $(N/4)$

$$\therefore \quad X_e(K) = X_{ee}(K - N/4)$$

$$\therefore \quad \boxed{W_{N/2}^{K} = W_N^{2K}} \quad \text{\& symmetry property}$$

$$W_{N/2}^{K \pm \frac{N}{4}} = -W_{N/2}^{K}$$

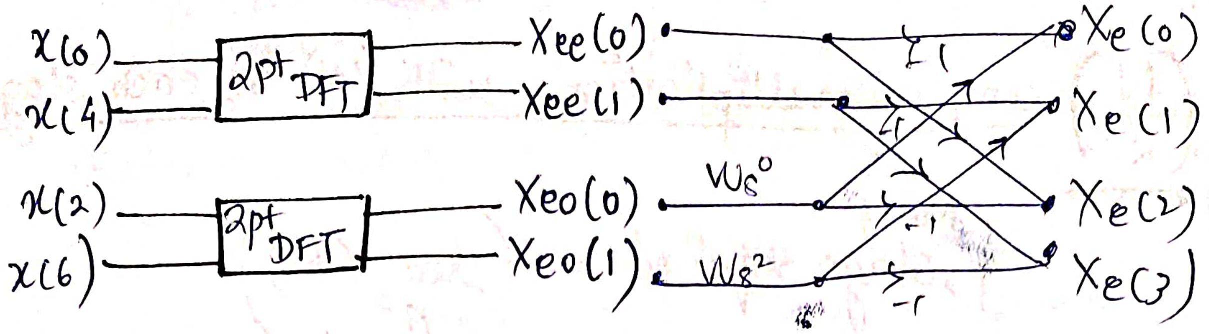
$$W_N^{2K \pm N/2} = W_N^{2K} \cdot W_N^{\pm N/2} = -W_N^{2K} = -W_{N/2}^{K}$$

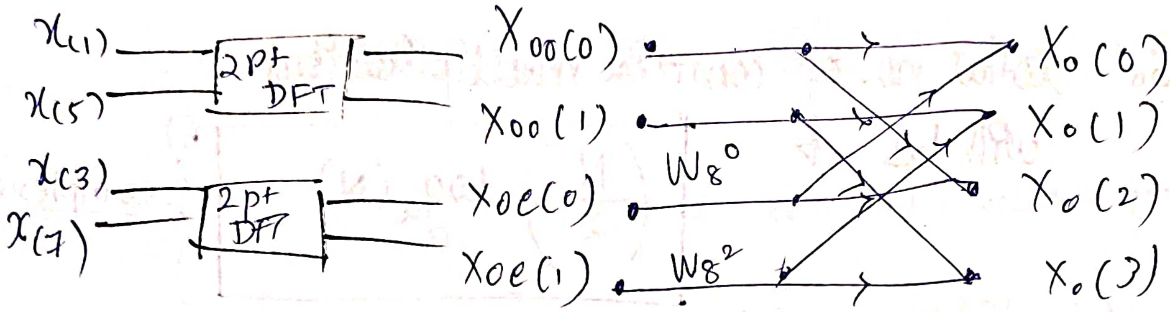
$\therefore$ for $\left(K > \frac{N}{4}\right)$

$$X_e(K) = X_{ee}\left(K - \frac{N}{4}\right) - W_{N/2}^{K - N/4} X_{eo}(K - N/4)$$

$$\therefore \quad X_e(K) = \begin{cases} X_{ee}(K) + W_{N/2}^{K} \cdot X_{eo}(K) \; ; & \left(K = 0, 1 \ldots \frac{N}{4} - 1\right) \\ X_{ee}\left(K - \frac{N}{4}\right) - W_{N/2}^{K - N/4} \cdot X_{eo}\left(K - \frac{N}{4}\right) \; ; & \left(K = \frac{N}{4} \ldots \frac{N}{2} - 1\right) \end{cases}$$

$\therefore$ ~~Xee(0)~~
~~Xee(1)~~
Xee(2)
~~Xee(~~

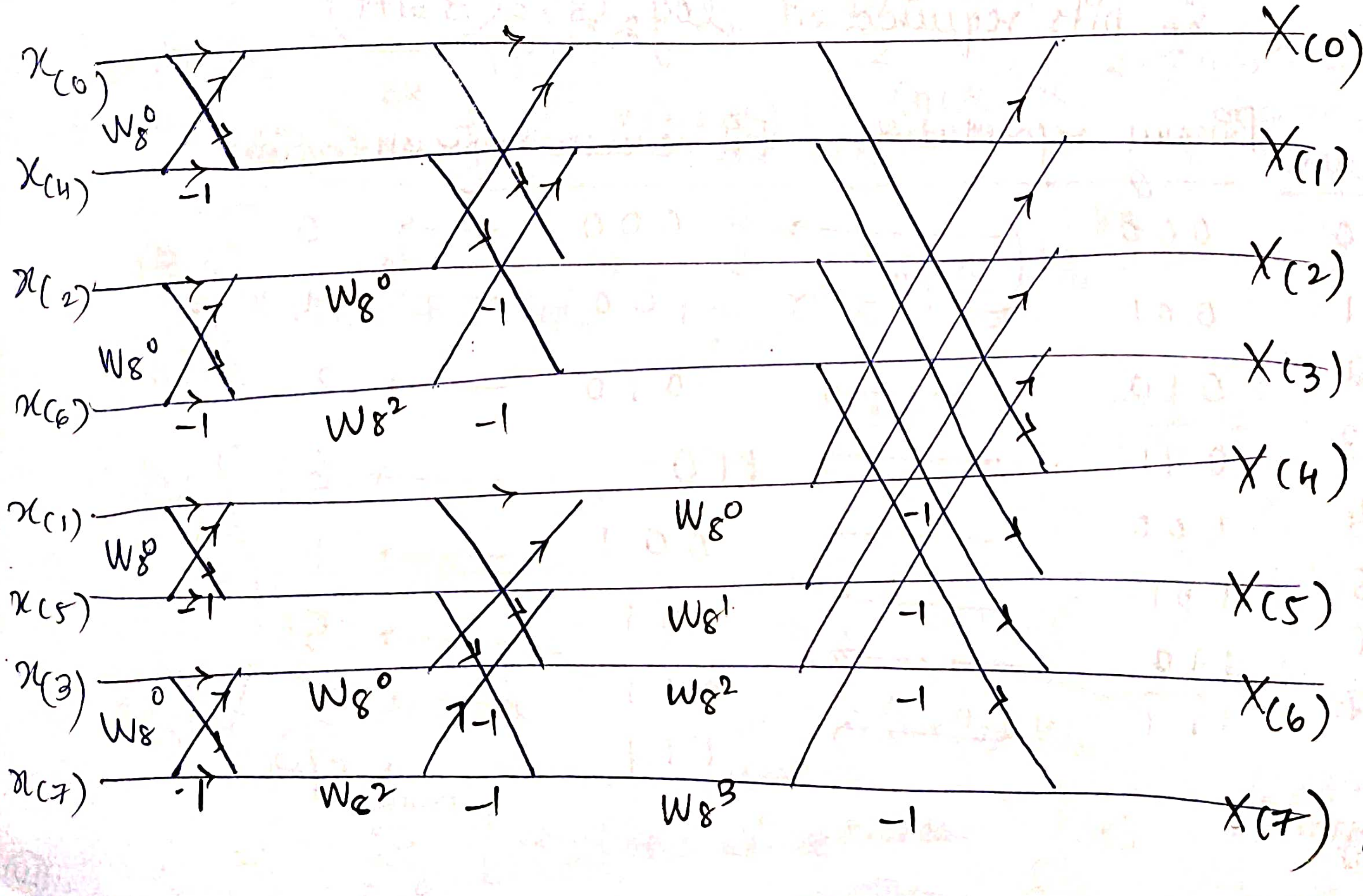




<u>What is 2pt DFT? Again</u>

$$X(0) = x(0) + x(1)$$
$$X(1) = x(0) - x(1)$$

} as $W_2^k = -1$



⇒ $\left(\frac{N}{2}\right)$ complex multiplications with (W) in each stage.

& No. of stages required = $\left(\log_2 (N)\right)$

∴ total no. of complex multiplications will be ⇒ $\boxed{\left(\frac{N}{2}\right) \log_2 (N)}$ } complexity.

→ The order of inputs are in bit-reversal sequence

$X(k)$ will be in natural order i.e $X(0), X(1) \cdots X(7)$

but $x(n)$ are in order → $x(0), x(4), x(2), x(6)$
$x(1), x(5), x(3)$ & $x(7)$

Bit reversal wrt "N" = 8

∴ bits required = $\log_2 (8) =$ 3 bits

i/p index	Binary repsentation		Bit reversed representation		
0	000	→	000	→	0
1	001	→	100	→	4
2	010	→	010	→	2
3	011	→	110	→	6
4	100	→	001	→	1
5	101	→	101	→	5
6	110	→	011	→	3
7	111	→	111	→	7

→ Dessimation in Frequency :-

Desimate :- means dividing a sequence into two subsequences.
→ one is even-indexed subsequence
→ other is odd-indexed subsequence.

→ DIT → divides $x(n)$ into even / odd subsequences
(in time)

→ DIF → divides $X(k)$ into even/odd subsequences.

x_n → 1^{st} half } $x(0), x(1), \ldots x(\frac{N}{2}-1)$

→ 2^{nd} half } $x(\frac{N}{2}), x(\frac{N}{2}+1) \ldots x(N-1)$

$$X(k) = \sum_{n=0}^{(\frac{N}{2}-1)} x(n) W_N^{nk} \rightarrow \sum_{n=0}^{(\frac{N}{2}-1)} x(n) W_N^{nk} + \sum_{n=\frac{N}{2}}^{(N-1)} x(n) W_N^{nk}$$

$(n \to n+\frac{N}{2})$

$$= \sum_{n=0}^{(\frac{N}{2}-1)} x(n) W_N^{nk} + \sum_{n=0}^{(\frac{N}{2}-1)} x(n+\tfrac{N}{2}) W_N^{(n+\frac{N}{2})k}$$

$$= \sum_{n=0}^{(\frac{N}{2}-1)} x(n) W_N^{nk} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\tfrac{N}{2}\right) W_N^{kn}$$

Here $W_N^{kN/2} = \left(e^{-j2\pi/N}\right)^{kN/2} = \left(e^{-j\frac{2\pi}{N}\frac{kN}{2}}\right)$

$= \left(e^{-j\pi}\right)^k = (-1)^k$

$$\therefore X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + (-1)^k \sum_{n=0}^{N/2-1} x(n+\tfrac{N}{2}) W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k x(n+N/2) \right] W_N^{nk} \; ; (k=0,1,\ldots N-1)$$

Let us decimate $X(K)$

Even part :- $$X(2K) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] W_N^{2kn} ; \quad \left(K = 0, 1, \ldots \frac{N}{2}-1\right)$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[\underbrace{x(n) + x\left(n+\frac{N}{2}\right)}_{g_1(n)}\right]\left(W_{N/2}^{kn}\right) \Bigg\} \text{ like } \left(\frac{N}{2}\right) \text{ point DFT}$$

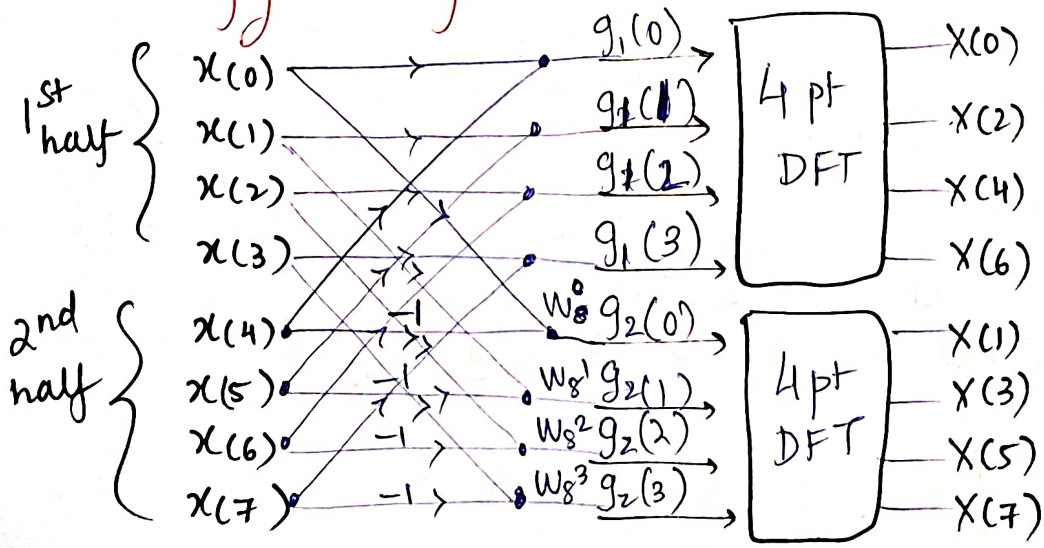
Odd part :- $$X(2K+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{(2K+1)n} ; \quad \left(K = 0, 1, \ldots \frac{N}{2}-1\right)$$

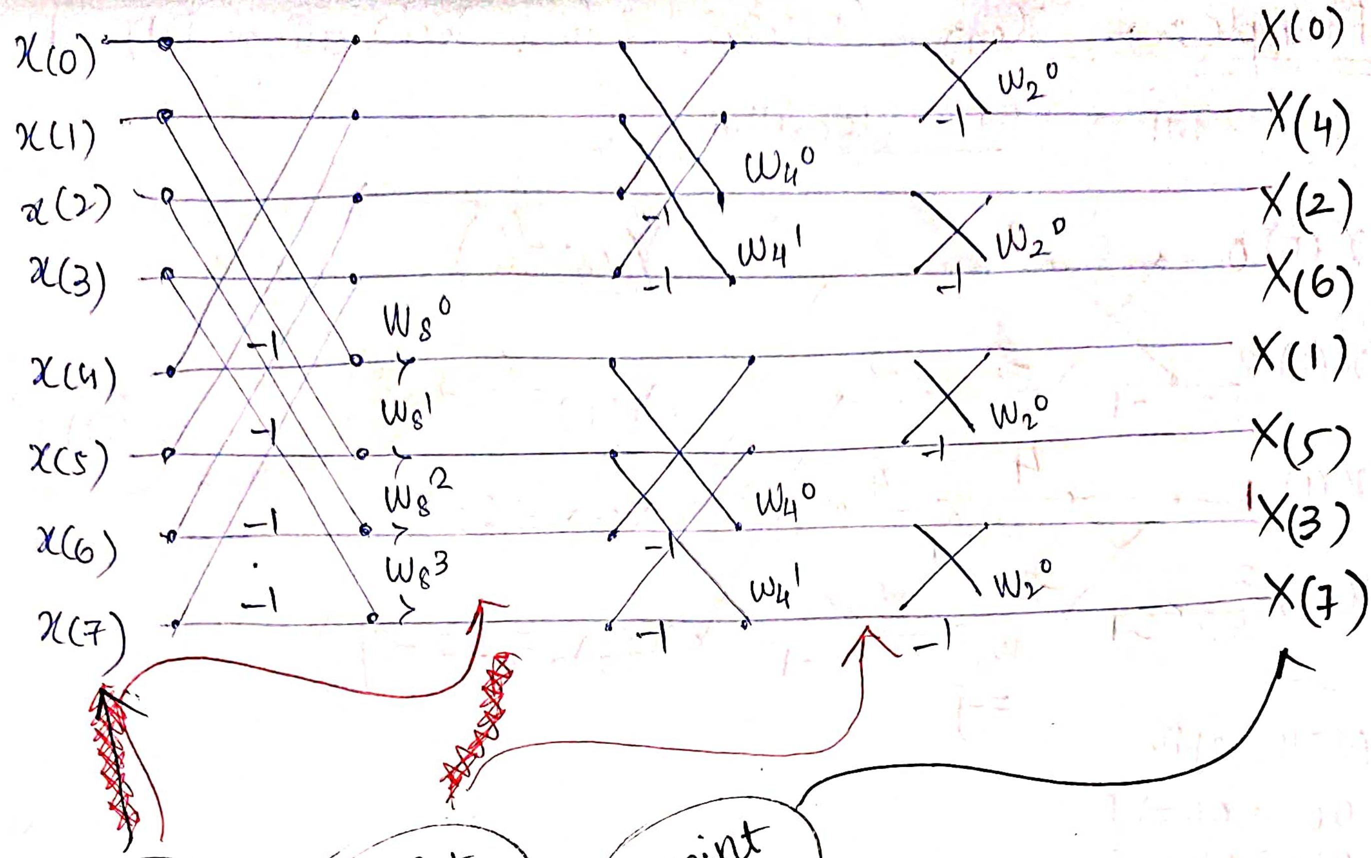
$$= \sum_{n=0}^{\frac{N}{2}-1} \left[\underbrace{\left\{x(n) - x\left(n+\frac{N}{2}\right)\right\} W_N^{n}}_{g_2(n)}\right] W_{N/2}^{nk} \Bigg\} \text{ like } \left(\frac{N}{2}\right) \text{ point DFT}$$

$\therefore$ $$\left.\begin{aligned} g_1(n) &= x(n) + x(n+N/2) \quad \& \\ g_2(n) &= \left[x(n) - x(n+N/2)\right] W_N^{n} \end{aligned}\right\} \left(n = 0, 1, \ldots \frac{N}{2}-1\right)$$

$$\boxed{X(2K) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n) \cdot W_{N/2}^{kn}} \quad \boxed{\& \quad X(2K+1) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n) \cdot W_{N/2}^{nk}}$$

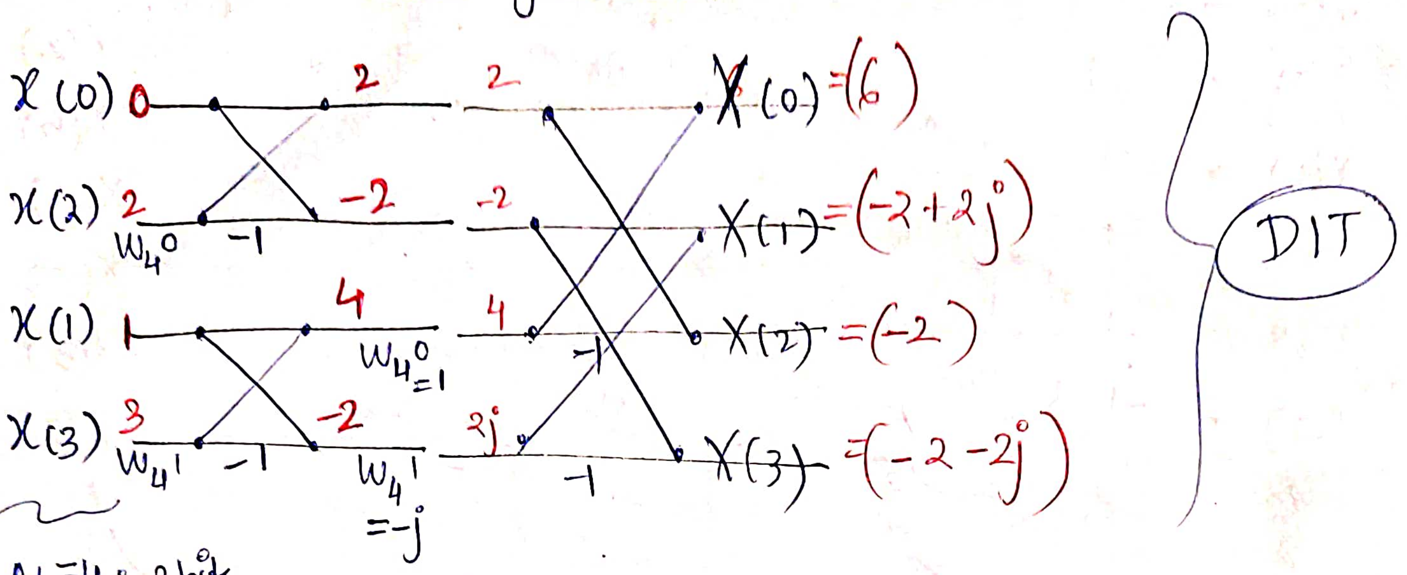
$\Rightarrow$ Butterfly :- for N = 8





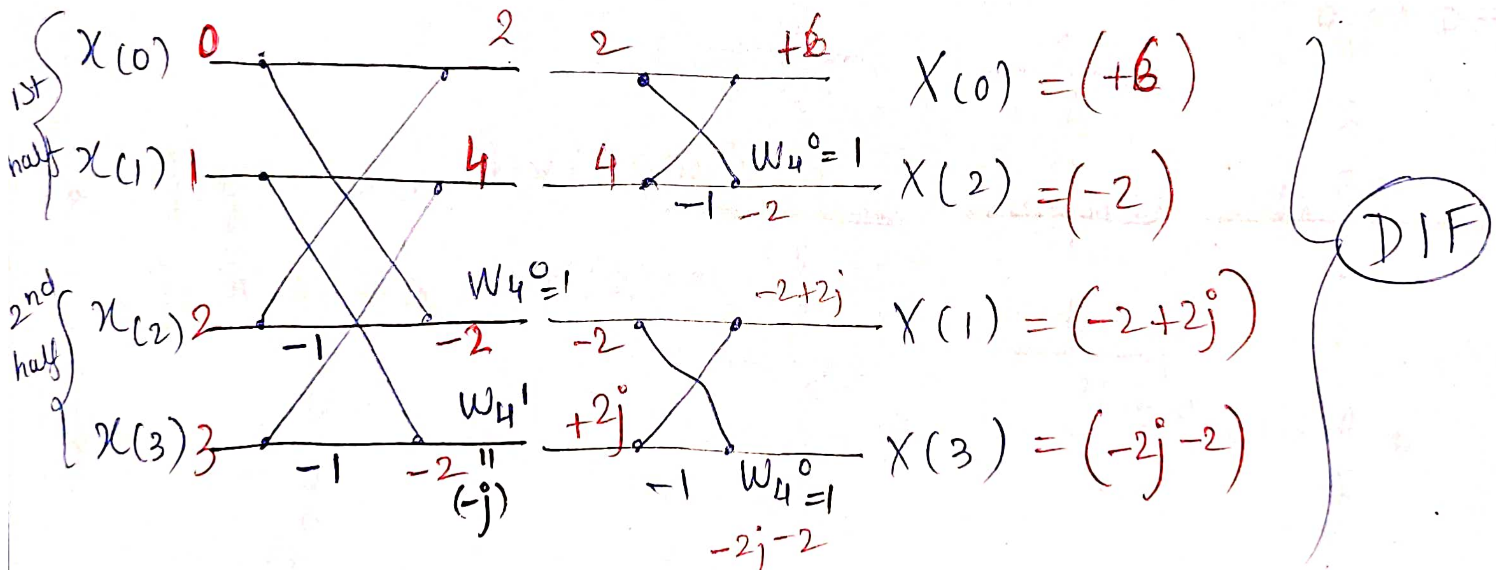
8 point DFT	4 point DFT	2point DFT
0	0	0
1	2	4
2	4	2
3	6	6
4	1	1
5	3	5
6	5	3
7	7	7

Example :→ $x(n) = (0,1,2,3)$; find $X(k)$ using DIT & DIF . FFT algorithm :→



$N = 4$ ∴ 2 bits

0 ⇒ 00 → 00 ⇒ 0
1 ⇒ 01 → 10 ⇒ 2
2 ⇒ 10 → 01 ⇒ 1
3 ⇒ 11 → 11 ⇒ 3



Inverse DFT :→ $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$

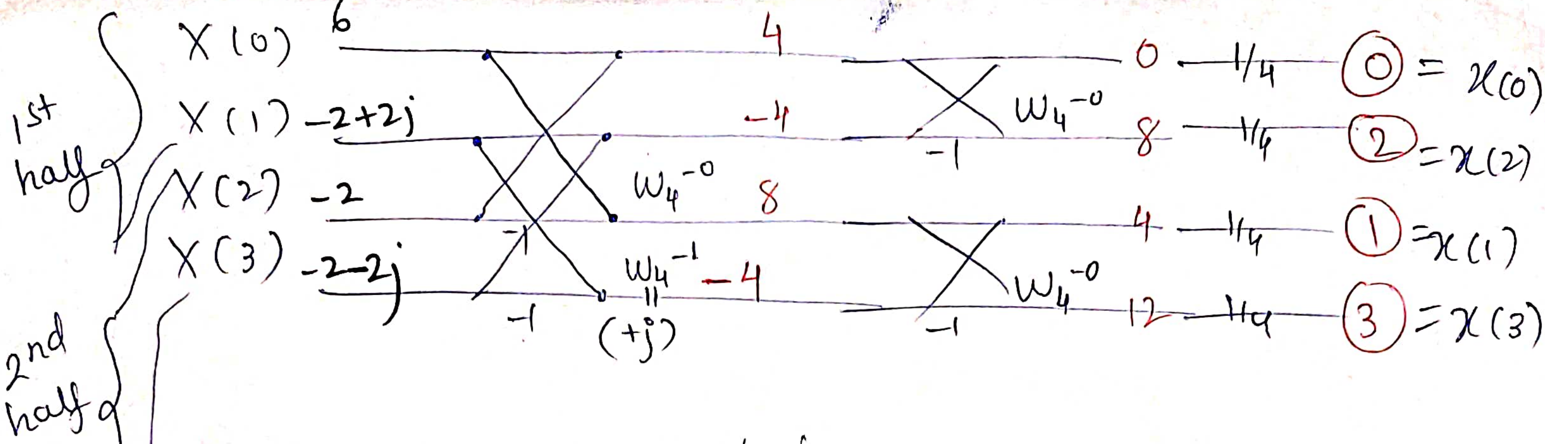
⇒ $W_N^{kn} \longrightarrow W_N^{-kn}$

⇒ Divide by $\left(\frac{1}{N}\right)$

Example

find IDFT of

$X(k) = \{6, -2+2j, -2, -2-2j\}$



~~2+2j-2j-2~~

~~2+2j-2j-2~~

$-\cancel{2}+2j-\cancel{2}+2j$

$= 4j \times j = -4$