Algebraic Sensitivity Analysis-Change in RHS

TOYCO assembles three types of toys—trains, trucks, and cars—using three operations. The daily limits on the available times for the three operations are 430, 460, and 420 minutes, respectively, and the revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train at the three operations are 1, 3, and 1 minutes, respectively. The corresponding times per train and per car are (2, 0, 4) and (1, 2, 0) minutes (a zero time indicates that the operation is not used).

Algebraic Sensitivity Analysis-Change in RHS

- The optimal tableau is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Algebraic Sensitivity Analysis-Change in RHS

Determination of Dual Prices. The constraints of the model after adding the slack variables x_4, x_5, and x_6 can be written as follows:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 &= 430 \quad \text{(Operation 1)} \\
3x_1 + 2x_3 + x_5 &= 460 \quad \text{(Operation 2)} \\
x_1 + 4x_2 + x_6 &= 420 \quad \text{(Operation 3)}
\end{aligned}
$$

Algebraic Sensitivity Analysis-Change in RHS

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350

$$z + 4x_1 + x_4 + 2x_5 + 0x_6 = 1350$$

This equation can be written as

$$
\begin{aligned}
z &= 1350 - 4x_1 - x_4 - 2x_5 - 0x_6 \\
&= 1350 - 4x_1 + 1(-x_4) + 2(-x_5) + 0(-x_6)
\end{aligned}
$$

Given that a *decrease* in the value of a slack variable is equivalent to an *increase* in its operation time, we get

$$
\begin{aligned}
z = 1350 - 4x_1 &+ 1 \times \text{(increase in operation 1 time)} \\
&+ 2 \times \text{(increase in operation 2 time)} \\
&+ 0 \times \text{(increase in operation 3 time)}
\end{aligned}
$$

Algebraic Sensitivity Analysis-Change in RHS

This equation reveals that (1) a one-minute increase in operation 1 time increases z by \$1, (2) a one-minute increase in operation 2 time increases z by \$2, and (3) a one-minute increase in operation 3 time does not change z.

To summarize, the z-row in the optimal tableau:

Dual price

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350

- Dual prices

Resource	Slack variable	Optimal z-equation coefficient of slack variable	Dual price
Operation 1	x_4	1	\$1/min
Operation 2	x_5	2	\$2/min
Operation 3	x_6	0	\$0/min

Algebraic Sensitivity Analysis-Change in RHS

- Zero dual price of operation 3 means there is no economic advantage in allocating more production time to this operation.
  - Resource is already abundant. Slack variable is positive (+20) at the optimal solution.
- Operation 1 can improve revenue by \$1
- Operation 2 can improve revenue by \$2
- More resources can be allocated to operation 2 in priority than operation 1.

Determination of Feasible Ranges

- Let D_1, D_2 and D_3 be the changes (positive or negative) in the daily manufacturing time allotted to operations 1, 2 and 3, respectively.

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 430 + D_1 \quad &\text{(Operation 1)} \\ 3x_1 + 2x_3 &\le 460 + D_2 \quad &\text{(Operation 2)} \\ x_1 + 4x_2 &\le 420 + D_3 \quad &\text{(Operation 3)} \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Determination of Feasible Ranges

- Starting Simplex tableau

								Solution		
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	−3	−2	−5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_6	1	4	0	0	0	1	420	0	0	1

- Follow Simplex method steps

Determination of Feasible Ranges

- The optimal simplex tableau

				D1	D2	D3	
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	1/2	-1/4	0	100
x_3	$\frac{3}{2}$	0	1	0	1/2	0	230
x_6	2	0	0	-2	1	1	20

- Solve the original problem using simplex method
- Introduce D_1, D_2, D_3.

The new optimum tableau provides the following optimal solution:

$$z = 1350 + D_1 + 2D_2$$
$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2$$
$$x_3 = 230 + \tfrac{1}{2}D_2$$
$$x_6 = 20 - 2D_1 + D_2 + D_3$$

- Optimality: Entering variable
- Feasibility: Leaving variable

Determination of Feasible Ranges

- Check feasibility of basic variables

$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2 \geq 0$$
$$x_3 = 230 + \tfrac{1}{2}D_2 \geq 0$$
$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

Example:
Modified RHS
Operation 1: 480
Operation 2: 440
Operation 3: 410

D_1 = 480 – 430 = 50
D_2 = 440 – 460 = -20
D_3 = 410 – 420 = -10

$$x_2 = 100 + \tfrac{1}{2}(50) - \tfrac{1}{4}(-20) = 130 > 0 \quad \text{(feasible)}$$
$$x_3 = 230 + \tfrac{1}{2}(-20) = 220 > 0 \quad \text{(feasible)}$$
$$x_6 = 20 - 2(50) + (-20) + (-10) = -110 < 0 \quad \text{(infeasible)}$$

Determination of Feasible Ranges

- Example:

D_1 = -30
D_2 = -12
D_3 = 10

$$z = 1350 + D_1 + 2D_2$$
$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2$$
$$x_3 = 230 + \tfrac{1}{2}D_2$$
$$x_6 = 20 - 2D_1 + D_2 + D_3$$

$$x_2 = 100 + \tfrac{1}{2}(-30) - \tfrac{1}{4}(-12) = 88 > 0 \quad \text{(feasible)}$$
$$x_3 = 230 + \tfrac{1}{2}(-12) = 224 > 0 \quad \text{(feasible)}$$
$$x_6 = 20 - 2(-30) + (-12) + (10) = 78 > 0 \quad \text{(feasible)}$$

- Objective function value

$$z = 1350 + D_1 + 2D_2$$
$$z = 3x_1 + 2x_2 + 5x_3$$

Use any equation

  - z = $1296

Determination of Feasible Ranges

Case 1. Change in operation 1 time from 460 *to* 460 + D_1 *minutes.* This change is equivalent to setting $D_2 = D_3 = 0$ in the simultaneous conditions, which yields

D_1
D_2 = 0
D_3 = 0

$$\left.\begin{array}{l} x_2 = 100 + \tfrac{1}{2}D_1 \geq 0 \Rightarrow D_1 \geq -200 \\ x_3 = 230 > 0 \\ x_6 = 20 - 2D_1 \geq 0 \Rightarrow D_1 \leq 10 \end{array}\right\} \Rightarrow -200 \leq D_1 \leq 10$$

Determination of Feasible Ranges

Resource	Dual price	Feasibility range	Resource amount (minutes)		
			Minimum	*Current*	*Maximum*
Operation 1	1	$-200 \le D_1 \le 10$	230	430	440
Operation 2	2	$-20 \le D_2 \le 400$	420	440	860
Operation 3	0	$-20 \le D_3 < \infty$	400	420	∞

- Simultaneous changes

$D_1 = 30$
$D_2 = -12$
$D_3 = 100$

$$x_2 = 100 + \tfrac{1}{2}(30) - \tfrac{1}{4}(-12) = 118 > 0 \quad \text{(feasible)}$$
$$x_3 = 230 + \tfrac{1}{2}(-12) = 224 > 0 \quad \text{(feasible)}$$
$$x_6 = 20 - 2(30) + (-12) + (100) = 48 > 0 \quad \text{(feasible)}$$

  - z = 1350 + 1(30) + 2(-12) + 0(100) = $1350

Determination of Feasible Ranges

1. The dual prices remain valid so long as the changes D_i, $i = 1, 2, \ldots, m$, in the right-hand sides of the constraints satisfy all the feasibility conditions when the changes are simultaneous or fall within the feasibility ranges when the changes are made individually.

2. For other situations where the dual prices are not valid because the simultaneous feasibility conditions are not satisfied or because the individual feasibility ranges are violated, the recourse is to either re-solve the problem with the new values of D_i or apply the post-optimal analysis .

Algebraic Sensitivity Analysis-Objective Function

- Definition of <u>reduced cost</u>
  - In TYOCO model, the optimal z-equation on the optimal tableau

$$z + 4x_1 + x_4 + 2x_5 = 1350 \quad \Rightarrow \quad z = 1350 - 4x_1 - x_4 - 2x_5$$

  - The optimal solution says $x_1 = 0$, otherwise z will reduce by \$4.
  - x_1 can be think of unit cost as it reduces revenue.
  - It comes from the revenue per unit and the cost of consumed resources by one unit.

Optimal simplex table

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350

Algebraic Sensitivity Analysis-Objective Function

- Definition of <u>reduced cost</u>
  - In TYOCO model, the optimal z-equation on the optimal tableau

$$z + 4x_1 + x_4 + 2x_5 = 1350 \quad \Rightarrow \quad z = 1350 - 4x_1 - x_4 - 2x_5$$

  - The optimal solution says $x_1 = 0$, otherwise z will reduce by \$4.
  - x_1 can be think of unit cost as it reduces revenue.
  - It comes from the revenue per unit and the cost of consumed resources by one unit.
- **Reduced cost per unit = (The cost of consumed resources by one unit) – (Revenue per unit)**
- Make unprofitable variable (x_1) into profitable
  - By increasing the unit revenue (market driven, so difficult to change)
  - By decreasing the unit cost of consumed resources

Determination of Optimal Ranges

- Find the condition such that the optimal solution remain unchanged.
- In TYOCO model, let d_1, d_2, and d_3 represent the change in unit revenue for toy trucks, trains, and cars, respectively.

$$\text{Maximize } z = (3 + d_1)x_1 + (2 + d_2)x_2 + (5 + d_3)x_3$$

- Starting tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-3 - d_1$	$-2 - d_2$	$-5 - d_3$	0	0	0	0

- The optimal tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$	0	0	$1 + \frac{1}{2}d_2$	$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3$	0	$1350 + 100d_2 + 230d_3$
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	$-\frac{1}{4}$	0	0	-2	1	1	20

Determination of Optimal Ranges

- The tableau is same last optimal tableau, except z-equation.
- The changes in z-equation coefficients can affect the optimality of the problem only.
- It means there is no need to perform simplex steps for reduced cost

Add new leftmost column corresponding to basic variables

d_j = 0 for slack variables

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

Add this new row On the top

Determination of Optimal Ranges

- Reduced cost

Left column	x_1 (d_1)	(x_1-column × left-column)
1	4	4×1
d_2	$-\frac{1}{4}$	$-\frac{1}{4}d_2$
d_3	$\frac{3}{2}$	$\frac{3}{2}d_3$
0	2	2×0
Reduced cost for $x_1 = 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$		

- Apply same rule for solution column, $z = 1350 + 100d_2 + 230d_3$
- The current optimal solution remains optimal when the new reduced cost is nonnegative for all nonbasic variables **(maximization problem)**
- Nonbasic variables x_1, x_4, x_5

$$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0$$
$$1 + \frac{1}{2}d_2 \geq 0$$
$$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0$$

How about basic variables?

Determination of Optimal Ranges

- Change TYOCO objective function

Maximize $z = 3x_1 + 2x_2 + 5x_3$ Maximize $z = 2x_1 + x_2 + 6x_3$

$d_1 = 2 - 3 = -1$, $d_2 = 1 - 2 = -1$, $d_3 = 6 - 5 = 1$

$$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0 \qquad 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 = 4 - \frac{1}{4}(-1) + \frac{3}{2}(1) - (-1) = 6.75 > 0 \text{ (satisfied)}$$
$$1 + \frac{1}{2}d_2 \geq 0 \qquad 1 + \frac{1}{2}d_2 = 1 + \frac{1}{2}(-1) = .5 > 0 \text{ (satisfied)}$$
$$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0 \qquad 2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 = 2 - \frac{1}{4}(-1) + \frac{1}{2}(1) = 2.75 > 0 \text{ (satisfied)}$$

- The optimal solution remains same $x_1 = 0$, $x_2 = 100$, $x_3 = 230$
- z = $1480 (Changed!)
- **For minimization problem, the reduced cost of z should be ≤ 0 to maintain optimality.**

Determination of Optimal Ranges

- Change d_j one at a time

Case 1. Set $d_2 = d_3 = 0$ in the simultaneous conditions, which gives

$$4 - d_1 \geq 0 \Rightarrow -\infty < d_1 \leq 4$$

- Unit revenue of the toy truck

$$2 + (-2) \leq 2 + d_2 \leq 2 + 8$$

$$\$0 \leq (\text{Unit revenue of toy truck}) \leq \$10$$

This condition assumes that the unit revenues for toy trains and toy cars remain fixed at \$3 and \$5, respectively.

Determination of Optimal Ranges

- Simultaneously change in d_1, d_2, and d_3 may not satisfy the condition

$d_1 = 6 - 3 = 3$, $d_2 = 8 - 2 = 6$, $d_3 = 3 - 5 = -2$

$$4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 = 4 - \tfrac{1}{4}(6) + \tfrac{3}{2}(-2) - 3 = -3.5 < 0 \text{ (not satisfied)}$$

$$1 + \tfrac{1}{2}d_2 = 1 + \tfrac{1}{2}(6) = 4 > 0 \text{ (satisfied)}$$

$$2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 = 2 - \tfrac{1}{4}(6) + \tfrac{1}{2}(-2) = -.5 < 0 \text{ (not satisfied)}$$

1. The optimal values of the variables remain unchanged so long as the changes d_j, $j = 1, 2, \ldots, n$, in the objective function coefficients satisfy all the optimality conditions when the changes are simultaneous or fall within the optimality ranges when a change is made individually.
2. For other situations where the simultaneous optimality conditions are not satisfied or the individual feasibility ranges are violated, the recourse is to either re-solve the problem with the new values of d_j or apply the post-optimal analysis

Thank you.