Lecture 4, 5, 6, & 7: Simplex Method and Sensitivity Analysis

1. LP Model in Equation Form

- LP Model in Equation Form
  - Two requirements

1. All the constraints (with the exception of the nonnegativity of the variables) are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

- Converting Inequalities into Equations with <u>Nonnegative RHS</u>
  - Slack Variable $6x_1 + 4x_2 \leq 24 \qquad 6x_1 + 4x_2 + s_1 = 24, s_1 \geq 0$
  - Surplus Variable $x_1 + x_2 \geq 800 \qquad x_1 + x_2 - S_1 = 800, S_1 \geq 0$
  - Nonnegative RHS $-x_1 + x_2 \leq -3 \qquad -x_1 + x_2 + s_1 = -3, s_1 \geq 0$

$$x_1 - x_2 - s_1 = 3$$

LP Model in Equation Form

- Unrestricted Variable

$$y_{i+1} = y_{i+1}^- - y_{i+1}^+, \text{ where } y_{i+1}^- \geq 0 \text{ and } y_{i+1}^+ \geq 0$$

- Example

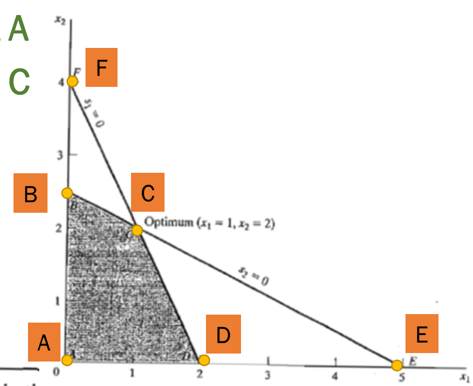
$$\text{Maximize } z = 2x_1 + 3x_2$$

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x_1 + x_2 + s_1 \quad\; &= 4 \\ x_1 + 2x_2 \quad + s_2 &= 5 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

  - m = 2 equations and n = 4 variables
  - Corner points can be found by putting n-m = 2 variables zero.

2. Transition from Graphical to Algebraic Solution

- Put $x_1 = 0$, $x_2 = 0$, and s1=4, s2 = 5 Point A
- Put s1=0, s2=0, and $x_1 = 1$, $x_2 = 2$ Point C
- Basic variables = m,
- Nonbasic variables = n-m



$$\boxed{\begin{aligned} 2x_1 + x_2 + s_1 \quad\; &= 4 \\ x_1 + 2x_2 \quad + s_2 &= 5 \end{aligned}}$$

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	*A*	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, −3)	*F*	No	—
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	*B*	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	*D*	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, −6)	*E*	No	—
(s_1, s_2)	(x_1, x_2)	**(1, 2)**	*C*	**Yes**	**8 (optimum)**

Maximum number of corner points

$$C_m^n = \frac{n!}{m!(n-m)!}$$

If m = 10, n = 20, then 184,756 corner points

Simplex Method

- Selectively investigate few corner points and locate the optimum solution
- Reddy Mikks Model

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

- Rewrite objective function $z - 5x_1 - 4x_2 = 0$

$$\begin{aligned}
&\text{Maximize } z = 5x_1 + 4x_2 \\
&6x_1 + 4x_2 \le 24 \\
&x_1 + 2x_2 \le 6 \\
&-x_1 + x_2 \le 1 \\
&x_2 \le 2 \\
&x_1, x_2 \ge 0
\end{aligned}$$

$$\begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 \quad \text{(Raw material } M1) \\
x_1 + 2x_2 + s_2 &= 6 \quad \text{(Raw material } M2) \\
-x_1 + x_2 + s_3 &= 1 \quad \text{(Market limit)} \\
x_2 + s_4 &= 2 \quad \text{(Demand limit)} \\
x_1, x_2, s_1, s_2, s_3, s_4 &\ge 0
\end{aligned}$$

Transition from Graphical to Algebraic Solution

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

$$\text{Maximize } z - 5x_1 - 4x_2 = 0$$

$$\begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 \quad \text{(Raw material } M1) \\
x_1 + 2x_2 + s_2 &= 6 \quad \text{(Raw material } M2) \\
-x_1 + x_2 + s_3 &= 1 \quad \text{(Market limit)} \\
x_2 + s_4 &= 2 \quad \text{(Demand limit)} \\
x_1, x_2, s_1, s_2, s_3, s_4 &\ge 0
\end{aligned}$$

Nonbasic (zero) variables: (x_1, x_2)

Basic variables: (s_1, s_2, s_3, s_4)

nonbasic variables $(x_1, x_2) = (0, 0)$

$$z = 0,\quad s_1 = 24,\quad s_2 = 6,\quad s_3 = 1,\quad s_4 = 2$$

3. Simplex Tableau

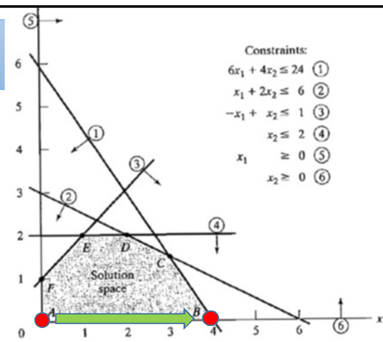
- Entering nonbasic variable
  - Which nonbasic variable (x_1 or x_2) should enter such that the objective function should improve maximally?
  - Most negative coefficient of the maximization objective function
    - Optimality condition

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

Entering variable x_1

Simplex Tableau

- Leaving basic variable
  - Minimum nonnegative ratio of RHS of the equation to the corresponding constraint coefficient under the entering variable
    - Feasible condition



Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	24/6=4
s2	0	1	2	0	1	0	0	6	6/1=6
s3	0	-1	1	0	0	1	0	1	1/-1=-1
s4	0	0	1	0	0	0	1	2	2/0

Nonbasic (zero) variables at B: (s_1, x_2)

Basic variables at B: (x_1, s_2, s_3, s_4)

Leaving Variable s_1

Simplex Tableau

Pivot element

Entering Variable

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s1	0	6	4	1	0	0	0	24	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

Leaving variable

Pivot row

Pivot column

Gauss-Jordon Row Operation

1. *Pivot row*

   a. Replace the leaving variable in the *Basic* column with the entering variable.

   b. New pivot row = Current pivot row ÷ Pivot element

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

Pivot row

- Pivot element = 6

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row z: current row coefficient (1, -5, -4, 0, 0, 0, 0, 0); pivot column coefficient= -5; new pivot row coefficient (0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

- New z-row is (1, 0, -2/3, 5/6, 0, 0, 0, 20)

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row s2: current row coefficient (0, 1, 2, 0, 1, 0, 0, 6); pivot column coefficient= 1; new pivot row coefficient (0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	1	2	0	1	0	0	6	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

- For row s2: new row (0, 0, 4/3, -1/6, 1, 0, 0, 2)

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row s3: current row coefficient (0, -1, 1, 0, 0, 1, 0, 1);
  pivot column coefficient= -1;
  new pivot row coefficient (0, 0, 1, 2/3, 1/6, 0, 0, 0, 4)

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	0	4/3	-1/6	1	0	0	2	
s3	0	-1	1	0	0	1	0	1	
s4	0	0	1	0	0	0	1	2	

- For row s3: new row (0, 0, 5/3, 1/6, 0, 1, 0, 5)

Gauss-Jordon Row Operation

2. *All other rows, including z*

New Row = (Current row) − (Its pivot column coefficient) × (New pivot row)

- For row s4: current row coefficient (0, 0, 1, 0, 0, 0, 1, 2);
  pivot column coefficient= 0;
  new pivot row coefficient (0, 0, 1, 2/3, 1/6, 0, 0, 0, 4)

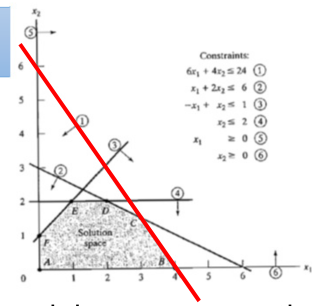
Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	
s2	0	0	4/3	-1/6	1	0	0	2	
s3	0	0	5/3	1/6	0	1	0	5	
s4	0	0	1	0	0	0	1	2	

- For row s3: new row (0, 0, 1, 0, 0, 0, 1, 2)

Simplex Tableau

- 1st iteration is over

Entering variable: Most negative coefficient in z-row for maximization problem



Leaving variable: minimum nonnegative ratio

Basic	z	x1	x2	s1	s2	s3	s4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x1	0	1	2/3	1/6	0	0	0	4	6
s2	0	0	4/3	-1/6	1	0	0	2	3/2
s3	0	0	5/3	1/6	0	1	0	5	3
s4	0	0	1	0	0	0	1	2	2

Pivot Row and Column

- Gauss-Jordon Row operations

1. New pivot x_2-row = Current s_2-row $\div \frac{4}{3}$
2. New z-row = Current z-row $- \left(-\frac{2}{3}\right) \times$ New x_2-row
3. New x_1-row = Current x_1-row $- \left(\frac{2}{3}\right) \times$ New x_2-row
4. New s_3-row = Current s_3-row $- \left(\frac{5}{3}\right) \times$ New x_2-row
5. New s_4-row = Current s_4-row $- (1) \times$ New x_2-row

Simplex Tableau

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Any entering Variable?

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is $21,000

Constraints

Resource	Slack value	Status
Raw material, $M1$	$s_1 = 0$	Scarce
Raw material, $M2$	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Steps of Simplex Method

Optimality condition	
Maximization problem	**Minimization problem**
Most negative coefficient of nonbasic variable	Most positive coefficient of nonbasic variable
Feasibility condition	
Smallest nonnegative ratio	Smallest nonnegative ratio

Gauss-Jordan row operations.

1. Pivot row

a. Replace the leaving variable in the *Basic* column with the entering variable.
b. New pivot row = Current pivot row ÷ Pivot element

2. All other rows, including z
New row = (Current row) − (pivot column coefficient) × (New pivot row)

Steps of Simplex Method

Step 1. Determine a starting basic feasible solution.

Step 2. Select an *entering variable* using the optimality condition. Stop if there is no entering variable; the last solution is optimal. Else, go to step 3.

Step 3. Select a *leaving variable* using the feasibility condition.

Step 4. Determine the new basic solution by using the appropriate Gauss-Jordan computations. Go to step 2.