4. Artificial Starting Solution

- Constraints are (≤) with nonnegative right hand sides offers a convenient all-slack starting basic feasible solution.
- Models with ≥ or = constraints do not.
- Artificial Variable: Starting "ill-behaved" LPs with ≥ or = constraints is to use artificial variable that play the role of slacks at the first iteration, and then dispose them legitimately at a later iteration.
- Two methods
  - M-method
  - Two phase method

M-Method

- Use x_3 surplus with constraint 2 and slack variable x_4 with constraint 3

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\Rightarrow$$

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

- Constraint 1 and constraint 2 do not have slack variable
- Add artificial variable R_1 and R_2 and penalize them in the objective function

M-Method

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

Z – 4x1 – x2 –MR1 –MR2 = 0

$$\begin{aligned} 3x_1 + x_2 \quad\quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + R_2 &= 6 \\ x_1 + 2x_2 \quad + x_4 \quad &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

Minimization problem: Add MR_i

- Basic variables: (R_1, R_2, x_4)
- What should be the value of M?
  - It should be large enough relative to the original objective coefficient
  - For the given problem, M = 100

M-Method

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
Z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Inconsistency: Non zero coefficient of R_1 and R_2

Z – 4x1 – x2 –MR1 –MR2 = 0

$$\begin{aligned} 3x_1 + x_2 \quad\quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + R_2 &= 6 \\ x_1 + 2x_2 \quad + x_4 \quad &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

- Substitution such that coefficient of R_1 and R_2 becomes zero
  - For the given problem:

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

M-Method

Minimization problem

Pivot column

Pivot row

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	Ratio
Z	696	399	-100	0	0	0	900	
R_1	3	1	0	1	0	0	3	1
R_2	4	3	-1	0	1	0	6	3/2
x_4	1	2	0	0	0	1	4	4

- Apply simplex method steps
  - **Entering variable:**
    - x_1 (most positive coefficient in z for minimization objective function)
  - **Leaving variable:**
    - R_1 (Minimum nonnegative ratio)

M-Method

- Apply Gauss-Jordon row operations

Pivot column

Pivot row

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	Ratio
Z	0	167	-100	-232	0	0	204	
x_1	1	1/3	0	1/3	0	0	1	3
R_2	0	5/3	-1	-4/3	1	0	2	6/5
x_4	0	5/3	0	-1/3	0	1	3	9/5

- **Entering variable:** x_2
- **Leaving variable:** R_2

M-Method

- Apply Gauss-Jordon row operations

Pivot column

Pivot row

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	Ratio
Z	0	0	1/5	-492/5	-501/5	0	18/5	
x_1	1	0	1/5	3/5	-1/5	0	3/5	3
x_2	0	1	-3/5	-4/5	3/5	0	6/5	-2
x_4	0	0	1	1	-1	1	1	1

- **Entering variable:** x_3
- **Leaving variable:** x_4

M-Method

- Apply Gauss-Jordon row operations

Any entering Variable?

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
Z	0	0	0	-493/5	-100	-1/5	17/5
x_1	1	0	0	2/5	0	-1/5	2/5
x_2	0	1	0	-1/5	0	3/5	9/5
x_3	0	0	1	1	-1	1	1

- $x_1 = 2/5$, $x_2 = 9/5$ and $z = 17/5$

Two Phase Method

- M-method uses penalty M
  - Possibility of round-off error that may impair the accuracy of simplex calculations
- Two phase method
  - Phase I attempts to find starting basic feasible solution
  - Phase II is invoked to solve the original problem
- Problem solved in the last section

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Phase-I of Two Phase Method

$$\text{Minimize } r = R_1 + R_2$$

$$\begin{aligned} 3x_1 + x_2 \qquad + R_1 \qquad\qquad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad + R_2 \qquad &= 6 \\ x_1 + 2x_2 \qquad\qquad\qquad + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

- Simplex tableau

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Inconsistence

$$\text{New } r\text{-row} = \text{Old } r\text{-row} + (1 \times R_1\text{-row} + 1 \times R_2\text{-row})$$

Phase-I of Two Phase Method

Pivot column

Pivot row

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	7	4	-1	0	0	0	9
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	5/3	-1	-7/3	0	0	2
x_1	1	1/3	0	1/3	0	0	1
R_2	0	5/3	-1	-4/3	1	0	2
x_4	0	5/3	0	-1/3	0	1	3

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
x_1	1	0	1/5	3/5	-1/5	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
x_4	0	0	1	1	-1	1	1

No entering Variable, stop.

Optimal solution

Phase-I of Two Phase Method

- Substitution $\quad$ New r-row = Old r-row + $(1 \times R_1\text{-row} + 1 \times R_2\text{-row})$

- Apply simplex steps and Gauss-Jordon row operation
- After 2 iterations, the optimum solution of Phase I is

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	-1	1	1

- r=0, basic feasible solution $x_1 = 3/5$, $x_2 = 6/5$, $x_4 = 1$
- Eliminate columns of artificial variables for Phase II

Phase-II of Two Phase Method

- Eliminate columns of artificial variables for Phase II

Minimize $z = 4x_1 + x_2$

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
x_4	0	0	1	1	1

Inconsistence

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	1/5	0	18/5
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
x_4	0	0	1	1	1

Try yourself: The optimal Solution is, x1 = 2/5, x2 = 9/5, and z = 17/5