→ [Systems Characterized by Linear Constant Coefficient Difference Equations

A general linear constant coefficient Difference Equn. for an LTI system with i/p $x[n]$ and o/p $y[n]$ is of the form

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k] \qquad -(**)$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega})$$

$$h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega})$$

Then, from the convolution property,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Apply Fourier Transform to both sides of (**) and using the linearity & time-shifting properties, we get

$$\sum_{k=0}^{N} a_k e^{-jk\omega}\, Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega}\, X(e^{j\omega})$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left(\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}} \right).$$

In Discrete-time, the polynomials are in the variable $(e^{j\omega})$.

→ The Difference Equn. (**) : generally referred to as an N-th order Difference Equn. as it involves delays in the o/p $y[n]$ of upto (N) time steps.

→ Also, the denominator of $H(e^{j\omega})$ is an N^{th} order polynomial in $(e^{j\omega})$.

Example: consider a causal LTI system described by the difference Equn.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

Determine the impulse response!

Taking DTFT both sides [Apply linearity + Time shift properties]

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left(\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}\right)$$

This can re-written as

$$H(e^{jw}) = \frac{2}{\left(1-\frac{1}{2}e^{-jw}\right)\left(1-\frac{1}{4}e^{-jw}\right)}$$

Expanding by method of partial fractions :-

$$H(e^{jw}) = \left(\frac{4}{1-\frac{1}{2}e^{-jw}}\right) - \left(\frac{2}{1-\frac{1}{4}e^{-jw}}\right)$$

Through inspection :-

$$\therefore h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n].$$

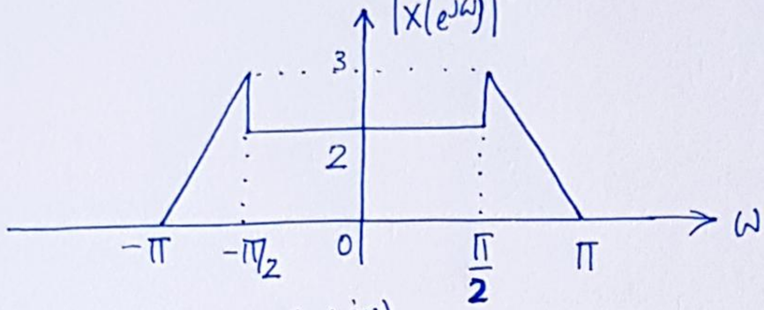
Summary of Fourier Series and Transform Expressions

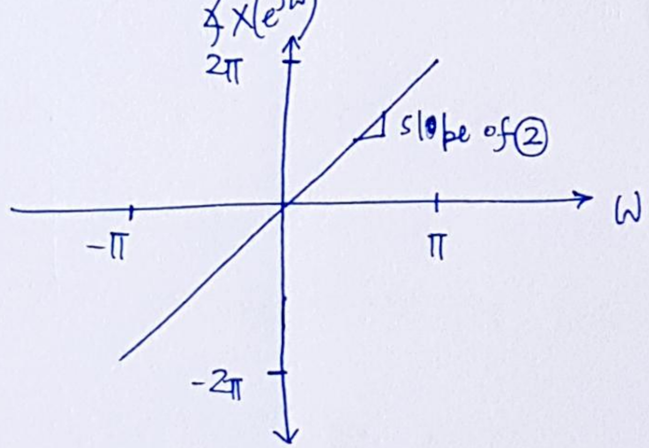
	Continuous-Time: Time Domain	Continuous-Time: Frequency Domain	Discrete-time: Time Domain	Discrete-time: Frequency Domain
Fourier series	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ continuous in time periodic in time	$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}$ Discrete Frequency aperiodic in Frequency	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$ discrete-time periodic in time	$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$ discrete frequency periodic in frequency.
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(j\omega) e^{j\omega t} d\omega$ continuous-time aperiodic in time	$x(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi} \int_{2\pi} x(e^{j\omega}) e^{j\omega n} d\omega$ discrete time aperiodic in time	$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency

Example

Consider the sequence $x[n]$ whose Fourier Transform $X(e^{j\omega})$ is depicted for $(-\pi \le \omega \le \pi)$ in the Figure

Determine whether $x[n]$ is periodic, real, even, and/or of finite energy.





Soln:

periodicity in time domain ⇒ Fourier Transform is zero except possibly for impulses located at various integer multiples of the fundamental.

Not true here!

⇒ $x[n]$ is not periodic.

From symmetry property,

real-valued sequence → Magnitude of FT even in (ω)
phase of FT odd in (ω)

True in this case $x[n] \rightarrow$ (real)

If $x[n]$ is even $\Rightarrow$ F.T. $X(e^{j\omega})$ must also be real & even

$\because X(e^{j\omega}) = |X(e^{j\omega})| \, e^{-j2\omega}$ → not real valued

$\Rightarrow x[n]$ not even.

To test for finite Energy, use Parseval's relation :-

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 \, d\omega \rightarrow \text{finite}$$

$x[n]$ has finite energy!

Example

The following four facts are given about a real signal $x[n]$ with Fourier Transform $X(e^{j\omega})$

(i) $x[n] = 0$ for $n > 0$

(ii) $x[0] > 0$

(iii) $\mathcal{Im}(X(e^{j\omega})) = (\sin\omega - \sin 2\omega)$

(iv) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 3$

Determine $x[n]$.

For a real $x[n]$, $\mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} j\,\mathcal{Im}(X(e^{j\omega})) = j\sin(\omega) - j\sin(2\omega)$

$$= \frac{1}{2}\left(e^{j\omega} - e^{-j\omega} - e^{j2\omega} + e^{-j2\omega}\right)$$

$\therefore$ $\mathcal{O}d\{x[n]\} = \mathcal{F}^{-1}\{j\,\mathcal{Im}(X(e^{j\omega}))\}$

$$= \frac{1}{2}\left(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2]\right)$$

Also, $\mathcal{O}d\{x[n]\} = \dfrac{x[n] - x^{*}[-n]}{2}$ and $x[n] = 0$ for $n > 0$

$\therefore$ For $\underline{\underline{n < 0}}$, $x[n] = 2\,\mathcal{O}d\{x[n]\} = \delta[n+1] - \delta[n+2]$

For $n=0$, $x[n] > 0$ need to find this ?

$$\frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\Rightarrow 3 = |x[0]|^2 + \underbrace{\sum_{n=-\infty}^{-1} |x[n]|^2}$$

$$|x[0]|^2 + 2$$

$$|x[0]|^2 = 1 \Rightarrow x[0] = \pm 1$$

$$x[0] = 1$$

$$\therefore x[n] = \delta[n] + \delta[n+1] - \delta[n+2].$$