Queuing System:

Objective: The objective of queuing analysis is to offer a reasonable satisfactory service to waiting customer.

* It determines measures of performance of waiting times.
    * avg waiting time in queue
    * productivity of the service level. facility
    * can be used to design service installation.

Elements of Queuing system:

* customer and services
* arrival of customer is represented by interarrival time between successive customers.
* service is described by the service time per customer
* Queue size : finite or infinite
* Queue discipline : FCFS, LCFS, SIRO service in random
* service facility → Single server
    → Parallel server

* Server can be arranged in (1) Series (sequencely)
(2) Network (Router - Network)

* Source : (1) Finite
(2) Infinite

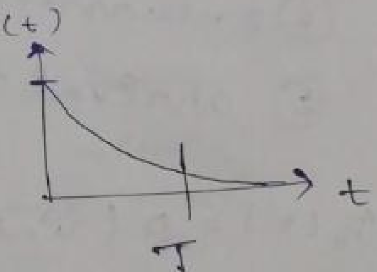
Role of Exponential Distribution:

* Arrival of customers is totally random event.

* Means occurence of an event is not influenced by length of time that has elapsed since the occurence of last event.

* Random interarrival and service time are described by exponential distribution

$$f(t) = \lambda e^{-\lambda t}, \; t > 0$$



Mean $E\{t\} = \frac{1}{\lambda}$

$$P(t \leq T) = \int_0^T \lambda e^{-\lambda t} dt$$

$$= 1 - e^{-\lambda t}$$

λ: rate per unit time at which events are generated / occured

t: time between successive events

S: interval since the occurence of the last event.

proof

$P(t > T+S \mid t > S) = P(t > T)$

$= P(t > T+S \cap t > S) / P(t > S)$

$$= \frac{P(t > T+S)}{P(t > S)} = \frac{e^{-\lambda(T+S)}}{e^{-\lambda S}} = e^{-\lambda T}$$

$= P(t > T)$

Pure birth model:

- only arrivals are allowed

$P_0(t)$ - probability of no arrival during time period 't'

Given:

(1) Interarrival time is exponential

(2) arrival rate λ customers per unit time

$P_0(t) = P(\text{interarrival time} > t)$

$= 1 - P(\text{inter time} \le t)$

$= 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$

$P_n(t)$ = probability of n arrivals during t

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \quad , n = 1, 2 \ldots$$

mean $E\{n/t\} = \lambda t$

Pure Death Model:

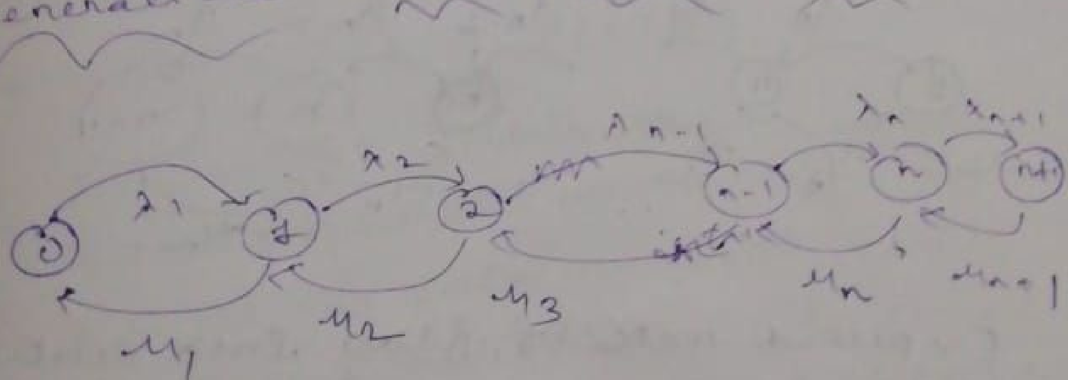
- system has N customers at time 0 and no new arrival is allowed.
- only departure can take place

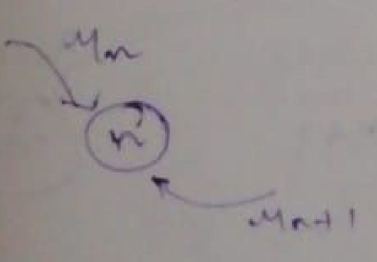
μ : departure rate of customers per unit time

$$p_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!} \quad , n = 1, 2 \ldots N$$

n : remaining no. of customers after time t'

Generalised Poisson Queuing model:





Expected rate of flow into state 'n'

Generalised Poisson Queuing Model:

→ combine both arrival and departure based on the poisson distribution

- Model is based on long-run or steady-state behaviour.

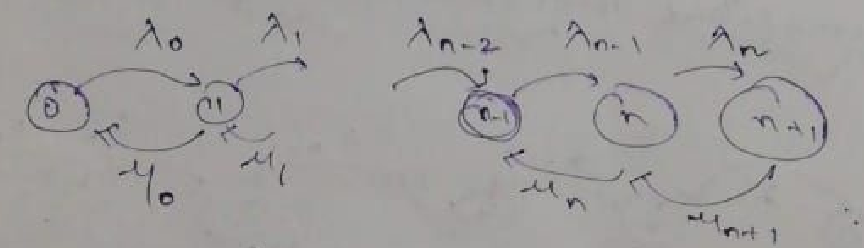
- Model assumes that both arrival and departure rates are state dependent - meaning they depend on no. of customers on facility.

n = no. of customers in system (in-queue + in service)

λ_n - arrival rate given 'n' customers in the system

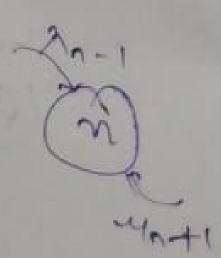
μ_n - Departure " " "

P_n = steady state probability of n customers in the system



Expected rate of flow into state 'n'

$$= \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$$



Expected rate of flow out of state 'n'

$$= \lambda_n P_n + \mu_n P_n$$

Equate:

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n$$

$n = 0$, $\lambda_0 P_0 = \mu_1 P_1 \Rightarrow P_1 = \left(\frac{\lambda_0}{\mu_1}\right) P_0$

$n = 1$ $\lambda_0 P_0 + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1$

$$\Rightarrow P_2 = \left(\frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}\right) P_0$$

$$P_n = \left(\frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1}\right) P_0$$

Value of P_0 can be estimated as,

$$\boxed{\sum_{n=0}^{\infty} P_n = 1}$$

Specialised Poisson Queue:

Notation $(a/b/c : d/e/f)$

a: arrival distribution

b: Departure (service-time) distribu[tion]

c: no of parallel servers

d: Queue discipline

e: Max. no (finite or infinite) cu[stomers] allowed in system

f: size of calling source (finite /inf[inite])

λ: arrival rate of customers per unit time

μ: departure " " " "

Arrival & departure distribution.

M = Markovian (or Poisson) distribution

D = const. time

Queue discipline - FCFS, LCFS, SIRO

GD - Generalised distribution

Eg: M/D/10 : GD/20/20.

steady state measure of performance

L_s: Expected no of customers in system

L_q: " " " queue

W_s: Expected waiting time in system

W_q: " " " queue

$\bar{c}$: Expected no. of busy servers.

$$L_s = \sum_{n=1}^{\infty} n P_n \qquad L_q = \sum_{n=c+1}^{\infty} (n-c) p_n$$

Little's formula:

$$L_s = \lambda_{eff} W_s$$

$$L_q = \lambda_{eff} W_q$$

λ_{eff} - effective arrival rate of the system $\lambda_{eff} = \lambda$, when all arriving customers can join the system

$\lambda_{eff} < \lambda$, otherwise

$\left\{ \begin{array}{l} \text{Expected} \\ \text{waiting} \\ \text{time} \\ \text{in system} \end{array} \right\} = \left\{ \begin{array}{l} \text{Expected} \\ \text{waiting} \\ \text{time in} \\ \text{queue} \end{array} \right\} + \left\{ \begin{array}{l} \text{Expected service} \\ \text{time} \end{array} \right\}$

$$W_s = W_q + \frac{1}{\mu} \qquad \lambda_{eff}$$

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

$$\bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu}$$

$$\text{Facility utilisation} = \frac{\bar{c}}{c}$$

parking space = 5

temp = 3 cars.

$\lambda = 6$ cars/hr

$\frac{1}{\mu} = 30$

$n = 5 + 3$

$$\lambda_n = 6 \text{ cars/hr}, \quad n = 0, 1, 2 \ldots 8$$

$$\mu_n = \begin{cases} n\left(\frac{60}{30}\right) = 2n, & n = 1, 2 \ldots 5 \\ 5\left(\frac{60}{30}\right) = 10, & n = 6, 7, 8 \end{cases}$$

$$p_n = \left(\frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_1}\right) p_0$$

$$\sum_{n=0}^{8} P_n = 1$$

case I: $n=1$, $P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{6}{2n} = \frac{3}{m} \cdot P_0$

$n=2$, $P_2 \left(\frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}\right) P_0 = \frac{6^2}{2\times1\times2\times2} P_0 = \frac{3^2 P_0}{2!}$

$n=3$, $P_3 = \frac{3^3}{3!} P_0$

$P_n = \frac{3^n}{n!} P_0 \; \forall \; n = 1, 2, 3 \cdots 4, 5.$

$n=6$, $n_6 = \left(\frac{\lambda_5 - \lambda_0}{\mu_6 - \mu_1}\right) P_0 = \frac{6^6}{10\times2^5\times5!} P_0$

$P_n = \frac{3^n}{5^{n-5}\, 5!} P_0$, $n = 6, 7, 8$

$$\sum_{n=0}^{8} = P_0 + P_0 \left(\frac{3}{1!} \cdots \frac{3^6}{5\,5!} + \frac{3^8}{5^3 5!}\right)$$

$\lambda = \lambda_{eff} + \lambda_{lost}$

$\lambda_{lost} = P_8 \lambda = 0.1203$

$\lambda_{eff} = 6 - 0.1203 = 5.87$

Example:

$\frac{1}{\mu} = 30$ min or $\mu = \frac{60}{30} = 2$ car/h

$\lambda = 6$ car/h

$$\lambda_n = 6 \qquad \mu_n = \begin{cases} 2n & n = 1, \dots 5 \\ 10 & n = 6, 7, 8 \end{cases}$$

(a)
$$p_n = \begin{cases} \frac{3^n}{n!} p_0 & , n = 1, 2 \dots 5 \\ \frac{3^n}{5^{n-5}\, 5!} p_0 & , n = 6, 7, 8 \end{cases}$$

$p_0 = 0.04812$

(b)
$$\text{cars} \xrightarrow{\lambda} \quad \xrightarrow{\lambda_{eff}} \boxed{\text{system}}^{\,n=8}$$
$$\lambda_{lost}$$

$\lambda_{lost} = \lambda p_8$

$\lambda_{eff} = \lambda - \lambda_{lost}$

$= 5.8737$

(c) L_s or L_q

$$L_s = \sum_{n=0}^{8} n p_n = 3.1286 \text{ cars}$$

(d)
$$W_s = \frac{L_s}{\lambda_{eff}}$$

$$W_q = W_s - \frac{1}{\mu} = 0.03265\, h$$

(e) $\bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu} = 2.9368$

(f) utilisation $\frac{\bar{c}}{c} = 0.58736$

Single - server models:

1. (M/M/1 : GD/∞/∞)

$$\left.\begin{array}{l}\lambda_n = \lambda \\ \mu_n = \mu\end{array}\right\} \quad n = 1, 2 \ldots$$

$\lambda_{eff} = \lambda$, $\lambda_{lost} = 0$

we know, $p_n = \left(\frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1}\right) p_0$

$$= \frac{\lambda^n}{\mu^n} p_0 = \rho^n p_0$$

where $\rho = \frac{\lambda}{\mu}$

$\sum_{n=0}^{\infty} p_n = 1$ or $p_0 [1 + \rho + \rho^2 \cdots] = 1$

(1) $\rho < 1$, $p_0 \left[\frac{1}{1-\rho}\right] = 1$

$$p_0 = 1 - \rho$$

$p_n = \rho^n (1 - \rho)$, when $\rho < 1$

(2) If $\rho \geq 1$, $\lambda > \mu$

- not a steady state system

$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \rho^n (1-\rho),$

where $\rho < 1$

$\frac{d}{d\rho}(\rho^n) = n\rho^{n-1}$

$= (1-\rho)\rho \sum_{n=0}^{\infty} n \rho^{n-1}$

$= (1-\rho)\rho \sum_{n=0}^{\infty} \frac{d}{d\rho}\rho^n = (1-\rho)\rho \frac{d}{d\rho}\left(\sum_n \rho^n\right)$

$\Rightarrow \boxed{L_s = \frac{\rho}{1-\rho}}$

$W_s = \frac{L_s}{\lambda_{eff}} = \frac{1}{\lambda} \frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$

$W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$

$L_q = \lambda_{eff} W_q = \frac{\rho^2}{1-\rho}$

$\bar{c} = L_s - L_q = \rho$

(B) $(M/M/1 : GD/N/\infty)$

system capacity $= N$

$$\lambda_n = \begin{cases} \lambda & n = 0, \ldots N-1, \\ 0 & n = N, N+1, \ldots \end{cases}$$

$\mu_n = \mu$, $n = 1, 2, \ldots$

using $\rho = \lambda/\mu$

$$P_n = \begin{cases} \rho^n P_0 & n \leq N \\ 0 & n > N \end{cases}$$

$\sum_{n=0}^{N} P_n = 1$ or $P_0 [1 + \rho + \rho^2 + \ldots \rho^N] = 1$

$$P_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}} & \rho \neq 1 \\ \dfrac{1}{N+1} & \rho = 1 \end{cases}$$

$$P_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}} & \rho \neq 1 \\ \dfrac{1}{N+1} & \rho = 1 \end{cases}$$

$\lambda_{eff} = \lambda - \lambda_{lost} = \lambda - \lambda P_N = \lambda(1 - P_N)$

$$L_s = \sum_{n=0}^{N} n P_n = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \rho \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \rho \frac{d}{d\rho} \left[\frac{1-\rho^{N+1}}{1-\rho} \right]$$

$$= \rho \left[\frac{1-(N+1)\rho^{n} + N\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})} \right] , \rho \neq 1$$