Properties of the Discrete-Time Fourier Transform

↓

Provide Insights into the transform

Reduce the complexity of evaluation of Fourier Transforms

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} \qquad \boxed{x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

① **Periodicity of DTFT**: The DTFT is always periodic in (ω) with period (2π) i.e. $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

This is in contrast to the CTFT, which in general is not periodic.

Proof:-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi n} = X(e^{j\omega})$$

② **Linearity of DTFT**:- If $x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

then $a x_1[n] + b x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$

(a)

3 Time shifting

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})\, e^{-j\omega n_0}$$

Proof:- $\mathcal{F}\{x[n-n_0]\}$

$$= \sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\omega n}$$

Put $(n-n_0) = u$, $\boxed{n = n_0 + u}$

$$\therefore \mathcal{F}\{x[n-n_0]\} = \sum_{u=-\infty}^{\infty} x[u]\, e^{-j\omega u}\, e^{-j\omega n_0}$$

$$= X(e^{j\omega})\, e^{-j\omega n_0}$$

(b)

Frequency shifting

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$e^{j\omega_0 n}\, x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Proof: $\mathcal{F}^{-1}\{X(e^{j(\omega-\omega_0)})\} =$

$$= \frac{1}{2\pi} \int_{2\pi} X(e^{j(\omega-\omega_0)})\, e^{j\omega n}\, d\omega$$

$\Rightarrow$ Put $(\omega-\omega_0) = u$

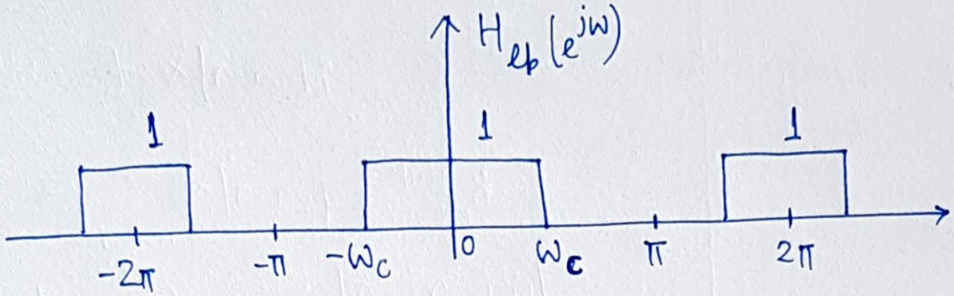
$$= \frac{1}{2\pi} \int_{2\pi} X(e^{ju})\, e^{jun}\, e^{j\omega_0 n}\, d\omega$$

$2\pi \rightarrow$ interval of integration (width $= 2\pi$)

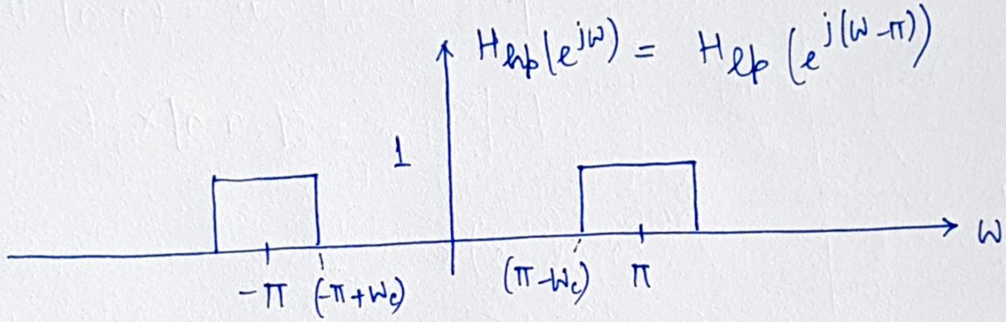
$$= \left(\frac{1}{2\pi} \int_{2\pi} X(e^{ju})\, e^{jun}\, du\right) e^{j\omega_0 n}$$

$$= e^{j\omega_0 n} \cdot x[n]$$

As a consequence of the periodicity and frequency shifting properties of the DTFT, there exists as a special relationship between ideal LPF and ideal high pass Discrete-time filters.



→ Frequency response of an ideal LPF (with cutoff frequency (ω_c))



(Frequency response of an ideal HPF)

(high pass filter) with cut off $(\pi-\omega_c)$.

Recall! high frequencies in discrete-time are concentrated near (π) and odd multiples of (π)

⇒ The frequency response of an LTI system is the Fourier Transform of its impulse response.

$$\because \; H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

$$\therefore \; \underbrace{h_{hp}[n]}_{\text{[impulse response (HPF)]}} = e^{j\pi n}\, h_{lp}[n] = (-1)^n \underbrace{h_{lp}[n]}_{\text{[impulse response (LPF)]}}$$

4 Conjugation and Conjugate Symmetry :-

If $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

then $x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$

Remarks (i) if $x[n]$ is real-valued, its F.T. $X(e^{j\omega})$ is conjugate-symmetric

i.e. $X(e^{j\omega}) = X^*(e^{-j\omega})$

(ii) $Re\{X(e^{j\omega})\}$ is an even funcn. of (ω)

$Im\{X(e^{j\omega})\}$ is an odd funcn. of (ω)

$|X(e^{j\omega})|$ is an even funcn. of (ω)

$\measuredangle X(e^{j\omega})$ is an odd funcn. of (ω)

(iii) $Ev\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} Re\{X(e^{j\omega})\}$ (real even)

$Od\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\,Im\{X(e^{j\omega})\}$ (purely imaginary & odd)

⑤ <u>Time Reversal</u> :→ $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

$$y[n] = x[-n] \xleftrightarrow{\mathcal{F}} \underline{\underline{\overset{Y(e^{j\omega}) =}{X(e^{-j\omega})}}}$$

Proof: $$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[-n]\, e^{-j\omega n}$$

substituting $(m = -n)$,

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]\, e^{j\omega m} = \sum_{m=-\infty}^{\infty} x[m]\, e^{-j(-\omega)m}$$

$$= X(e^{-j\omega})$$

$$\therefore \boxed{x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})}$$

6 Time Expansion: → (For discrete-time signals, the time index is discrete in nature)

∴ The relation between time and frequency scaling in discrete-time takes on a somewhat different form from its continuous-time counterpart.

$$\boxed{x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)} \quad \text{in C-T.}$$

→ However, if we try to ~~...~~ define the signal $x[an]$ we run into difficulties if (a) is not an integer.

∴ we cannot slow down the signal by choosing $(a<1)$ [reduce the speed]

On the other hand, if we let (a) to be an integer, other than ± 1, for example, if we consider $x[2n]$ we donot merely speed up the original signal.

∵ (n) can take on only integer values.

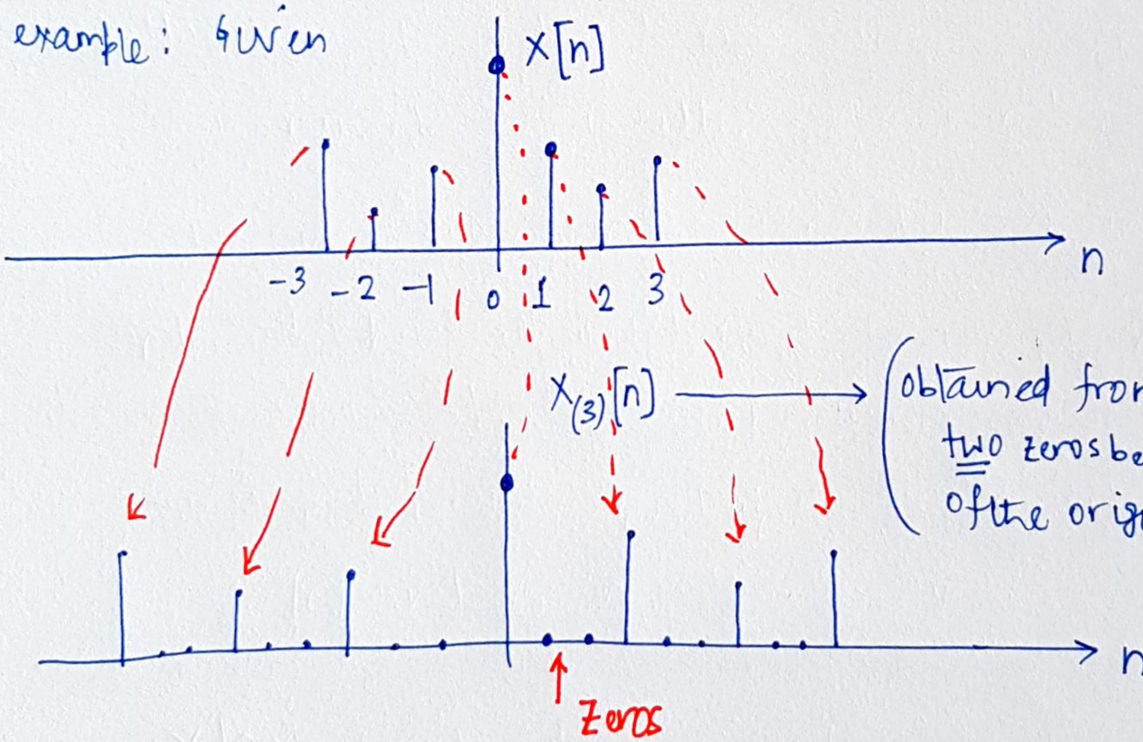
– $x[2n]$ consists of even samples of $x[n]$ alone.

Let (k) be a positive integer and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } (n) \text{ is a multiple of } (k) \\ 0, & \text{o.w. if } (n) \text{ is not a multiple of } (k). \end{cases}$$

$\Rightarrow$ $x_{(k)}[n]$ is obtained from $x[n]$ by placing $(k-1)$ zeros between successive values of the original signal.

For example: Given



(obtained from $x[n]$ by inserting two zeros between successive values of the original signal.)

Remarks (i) $x_{(k)}[n]$ can be thought of as a slowed down version of the original signal $x[n]$.

(ii) $x_{(k)}[n]$ equals 0, unless (n) is a multiple of (k) i.e. unless $(n = rk)$.

$\therefore \mathcal{F}\{x_{(k)}[n]\} = X_{(k)}(e^{j\omega})$

$$= \sum_{n=-\infty}^{\infty} x_{(k)}[n]\, e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk]\, e^{-j\omega rk}$$

$$\therefore \boxed{x_{(k)}[rk] = x[r]}$$

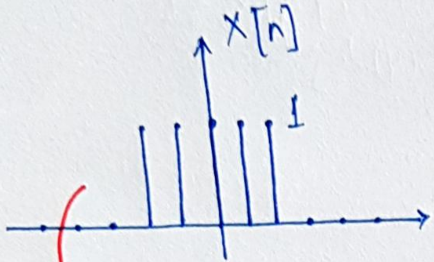
$$\therefore X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j(k\omega)r} = X(e^{jk\omega})$$

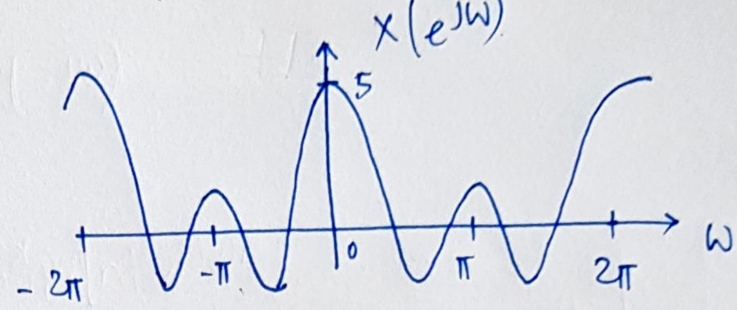
* Note that as the signal is spread out and slowed down in time by taking $(k>1)$, its Fourier Transform is compressed.

Inverse relation between time & frequency domain

↗ $X(e^{j\omega})$ is periodic with period (2π)

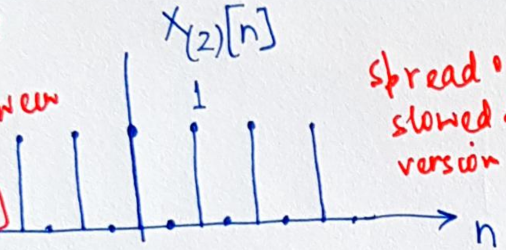
↗ $X(e^{jk\omega})$ is periodic with period $(2\pi/k)$



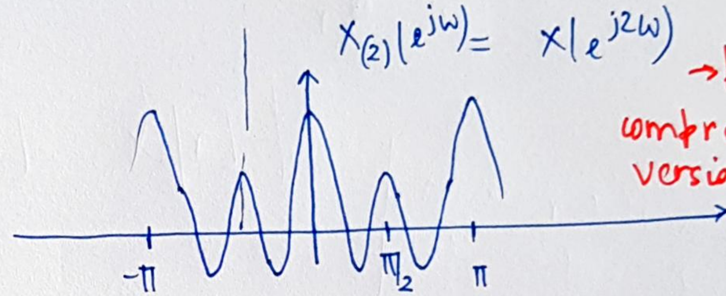


period (2π)

obtained by inserting one zero between successive values of x[n]



spread out slowed down version of x[n]



→ period $\frac{2\pi}{2} = (\pi)$
compressed version of X(e^{jw})

⑦ <u>Differentiation</u> <u>in Frequency</u> :→

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{jw})}{dw}$$

<u>Proof</u> →

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$

$$\frac{dX(e^{jw})}{dw} = \sum_{n=-\infty}^{\infty} -jn\, x[n]\, e^{-jwn}$$

$$\Rightarrow \quad -jn\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(e^{jw})}{dw}$$

$$n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{jw})}{dw}$$

Thus,

$$\boxed{n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{jw})}{dw}}$$

⑧ Parseval's relation $\sum_{n=-\infty}^{\infty} |x[n]|^2 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$

⑨ convolution property :→ $y[n] = x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega}) . H(e^{j\omega})$

(convolution)

⑩ Multiplication property :→ $y[n] = x_1[n]\, x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$

Periodic convolution of $X_1(e^{j\omega})$ & $X_2(e^{j\omega})$

Proof :-

$$y[n] = x_1[n] \,.\, x_2[n]$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$Y(e^{j\omega}) \qquad X_1(e^{j\omega}) \qquad X_2(e^{j\omega})$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_1[n] x_2[n]\, e^{-j\omega n}$$

$$\because x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n}\, d\theta$$

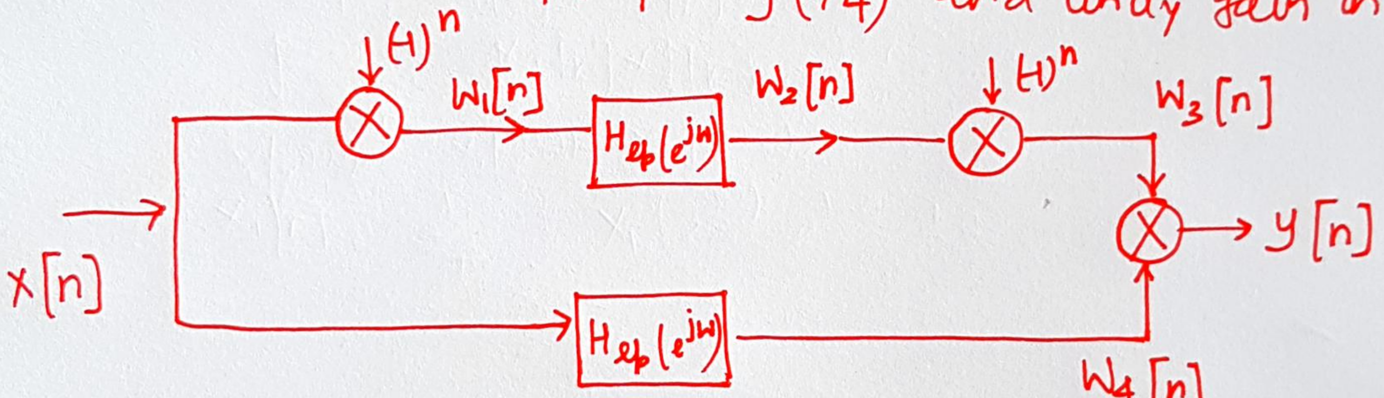
$$\therefore Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n] \left\{ \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n} d\theta \right\} e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \left[\sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j(\omega-\theta)n} \right] d\theta$$

$$Y(e^{jw}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \left(\sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j(w-\theta)n} \right) d\theta$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(w-\theta)})\, d\theta$$

Example :- Consider the system shown in figure with i/p $x[n]$ and o/p $y[n]$. The LTI systems with frequency response $H_{lp}(e^{jw})$ are ideal LPF with cut off frequency ($\pi/4$) and unity gain in the passband.



What is the overall frequency response of the system?

Top path: $W_1(e^{jw}) = ?$ $\quad w_1[n] = x[n]\cdot(-1)^n = e^{j\pi n}\, x[n]$

$\therefore W_1(e^{jw}) = X(e^{j(w-\pi)})$

$W_2(e^{jw}) = H_{lp}(e^{jw}) \cdot X(e^{j(w-\pi)})$

$w_3[n] = (-1)^n w_2[n] \Rightarrow W_3(e^{jw}) = W_2(e^{j(w-\pi)}) = H_{lp}(e^{j(w-\pi)})\, X(e^{j(w-2\pi)})$

$\because$ DTFT are always periodic in (w)

$X(e^{j(w-2\pi)}) = X(e^{jw})$

$$\therefore \quad W_3(e^{j\omega}) = H_{ep}(e^{j(\omega-\pi)})\, X(e^{j\omega})$$

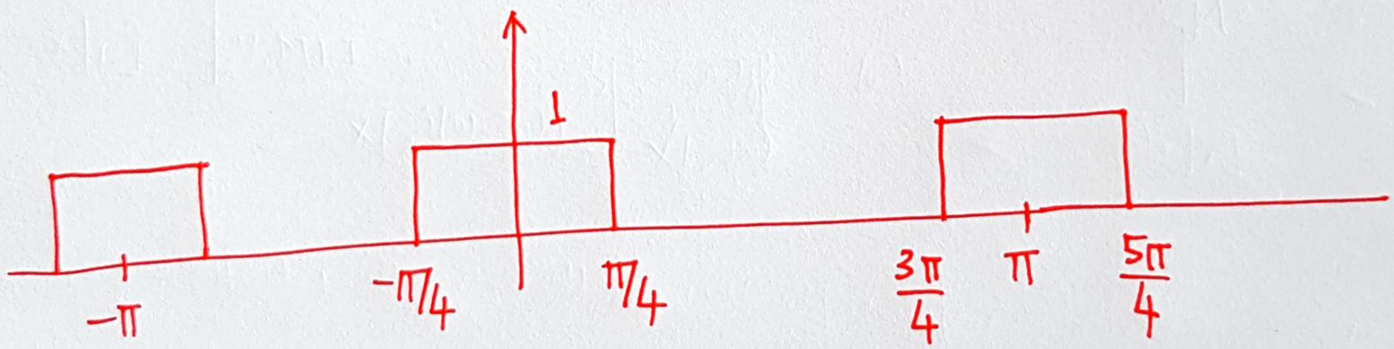
Applying convolution to the lower path,

$$W_4(e^{j\omega}) = X(e^{j\omega})\, H_{ep}(e^{j\omega})$$

From, linearity property of DTFT,

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega})$$

$$= \Big[\underbrace{H_{ep}(e^{j(\omega-\pi)})}_{HPF} + \underbrace{H_{ep}(e^{j\omega})}_{LPF}\Big] X(e^{j\omega})$$



The overall system passes both low & high frequencies & stops frequencies between these two passbands.

↓

Filter has ideal bandstop characteristics where the stopband is in the region $\left[\frac{\pi}{4} < |\omega| < \frac{3\pi}{4}\right]$