Lecture #12

14.02.2022

ME 325 Control Systems
(3-0-0-6)

Instructor

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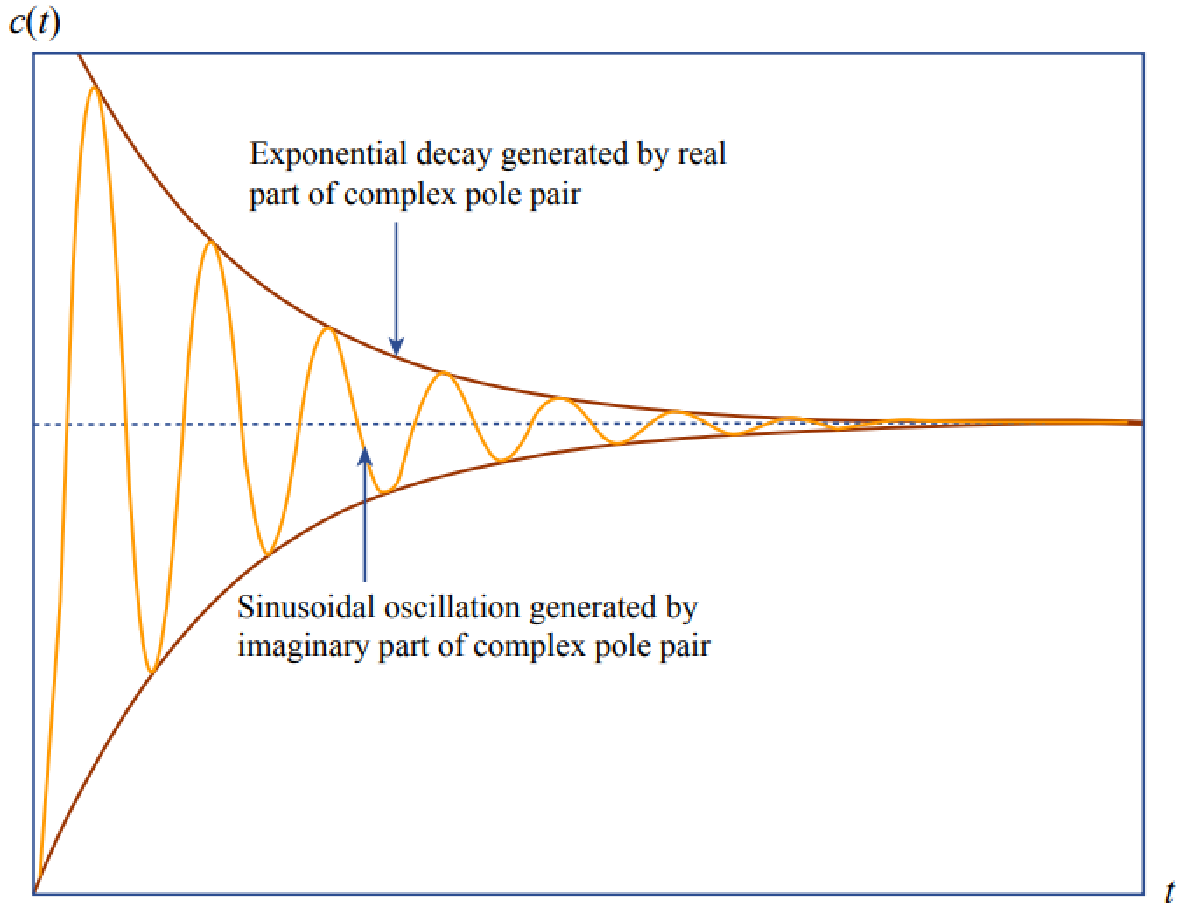
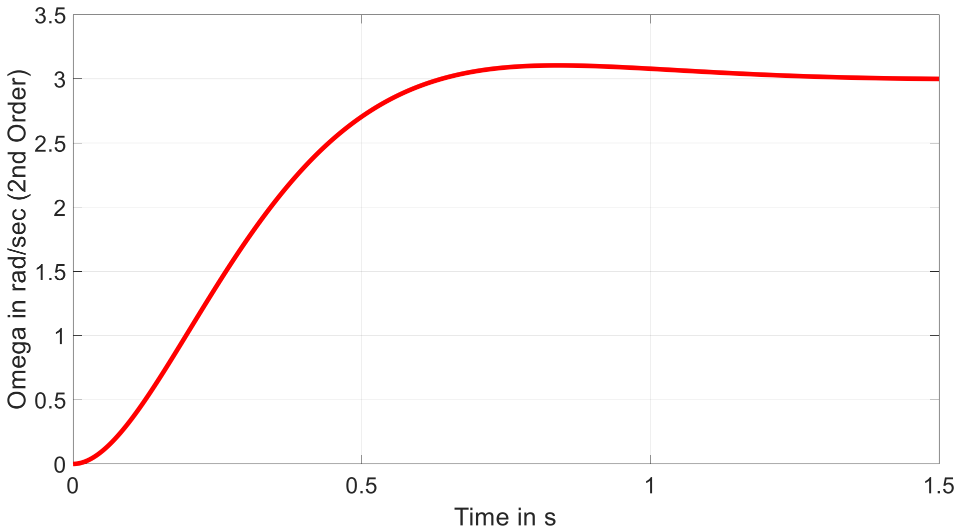
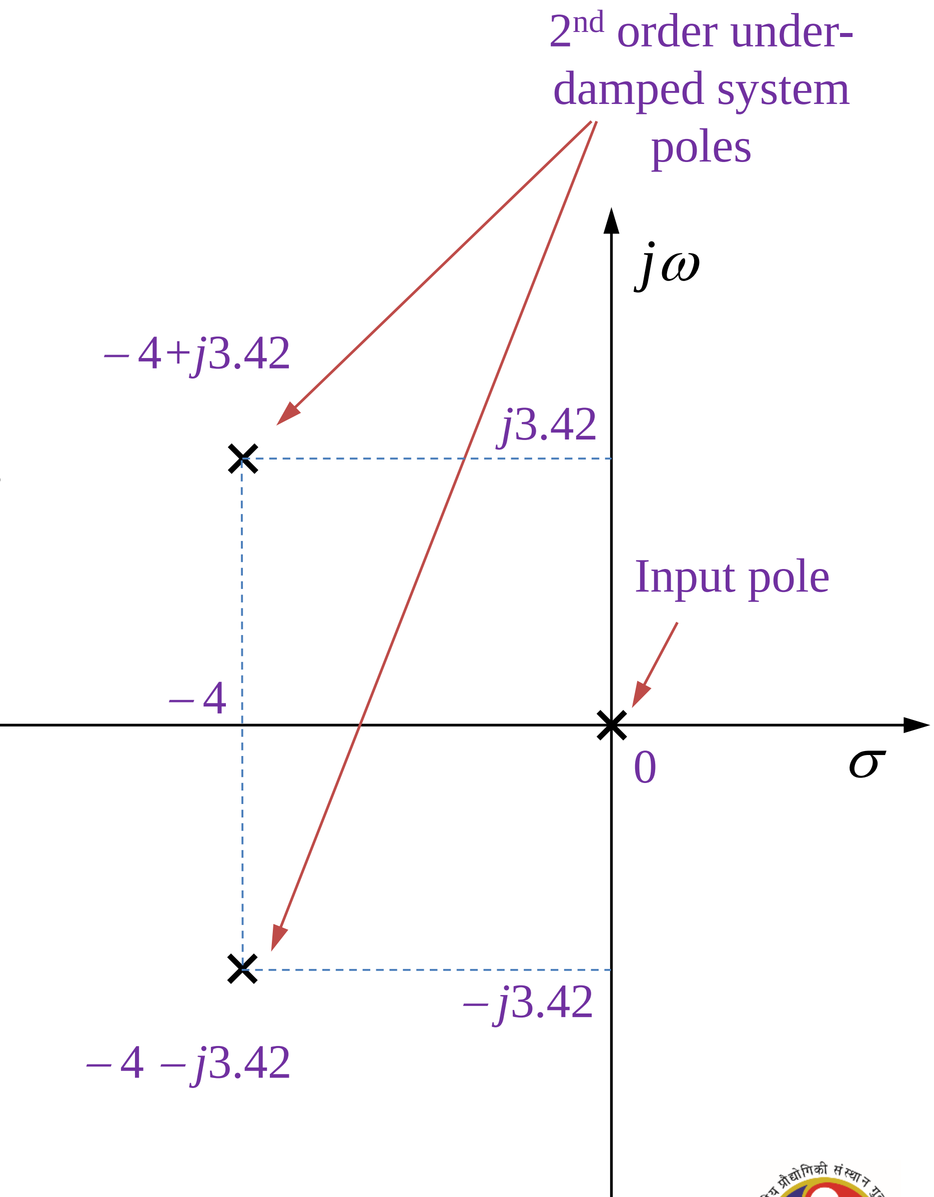
What the real and imaginary parts of the poles do?



$$\text{Damping ratio, } \xi = \frac{1}{2\pi}\frac{\text{Undamped ("natural") period}}{\text{Time constant of exponential decay}} = \frac{1}{2\pi}\frac{\left(\dfrac{2\pi}{\omega_n}\right)}{\left(\dfrac{1}{\xi\omega_n}\right)}$$

Underdamped DC motor in the s-domain





The general 2nd order system

We can write the transfer function of the general 2nd — order system with unit steady state response as follows:

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$ where

ω_n is the system's natural frequency and

ξ is the system's damping ratio

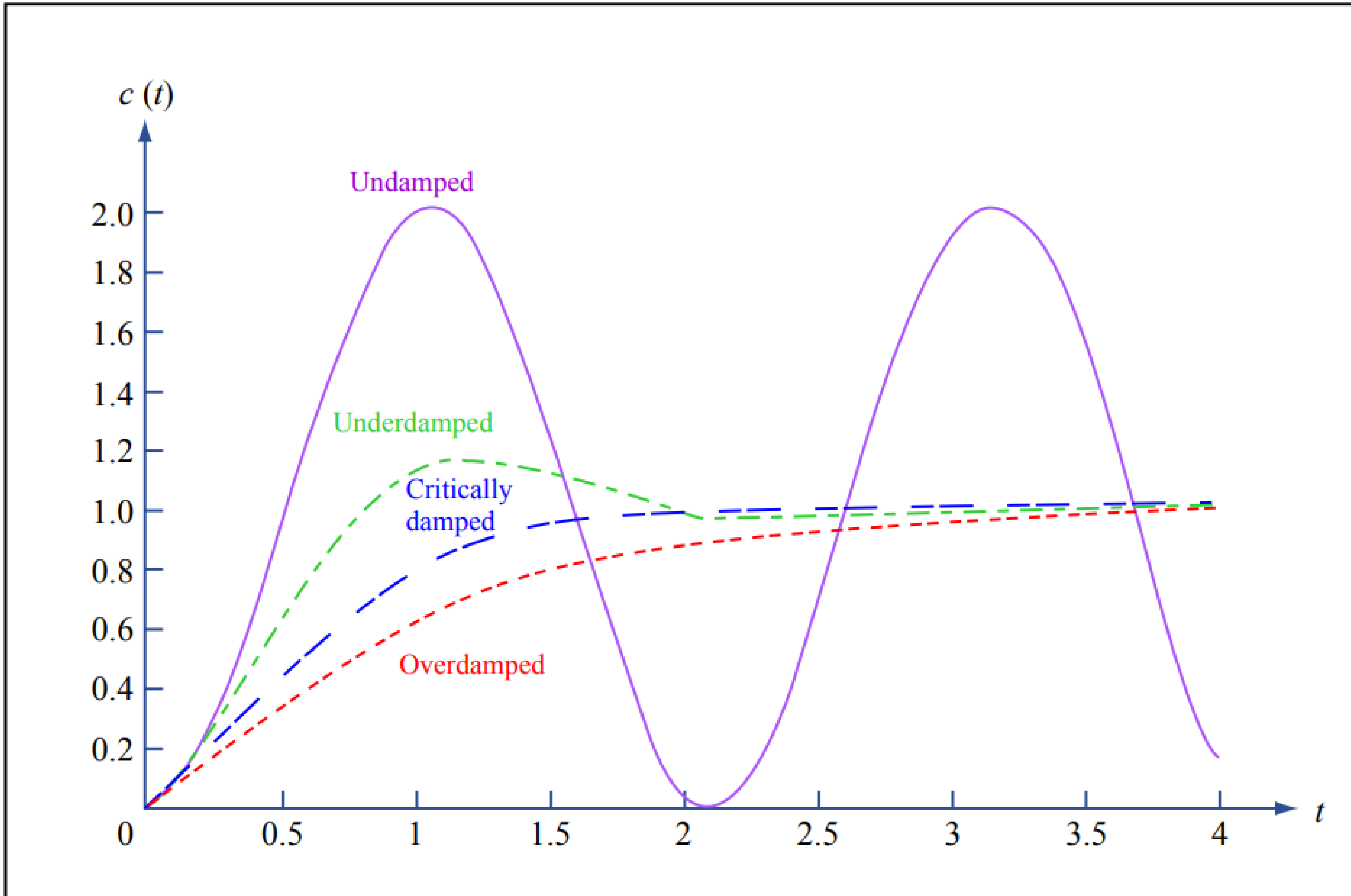
The natural frequency indicates the oscillation frequency of the undamped ("natural") system, i.e. the system with energy storage elements only and without any dissipative elements. The damping ratio denotes the relative contribution to the system dynamics by energy storage elements and dissipative elements. Recall,

$$\text{Damping ratio, } \xi = \frac{1}{2\pi}\frac{\text{Undamped ("natural") period}}{\text{Time constant of exponential decay}} = \frac{1}{2\pi}\frac{\left(\dfrac{2\pi}{\omega_n}\right)}{\left(\dfrac{1}{\xi\omega_n}\right)}$$

Depending on the damping ratio ξ, the system response is

- undamped if $\xi < 0$
- unerdamped if $0 < \xi < 1$
- critically damped if $\xi = 1$
- overdamped if $\xi > 1$

The general 2nd order system



The underdamped 2nd order system

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}, \qquad 0<\xi<1$$

The step response's Laplace transform is

$$\frac{1}{s}\times\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2\xi\omega_n s+\omega_n^2}$$

We find $$K_1=\frac{1}{\omega_n^2}, \qquad K_2=-\frac{1}{\omega_n^2}, \qquad K_3=-\frac{2\xi}{\omega_n}$$

$$\frac{1}{s}-\frac{\left(s+\xi\omega_n\right)+\frac{\xi}{\left(1-\xi^2\right)}\omega_n\sqrt{1-\xi^2}}{\left(s+\xi\omega_n\right)^2+\omega_n^2\left(1-\xi^2\right)}$$

Using the frequency shifting property of Laplace transforms we finally obtain the step response in the time domain as

$$1-e^{-\xi\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\xi^2}t\right)+\frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\omega_n\sqrt{1-\xi^2}t\right)\right]$$

The underdamped 2nd order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \qquad 0 < \xi < 1$$

$$\sigma_d = \xi\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\xi^2}, \qquad \tan\phi = \frac{\xi}{\sqrt{1-\xi^2}}$$

We can rewrite the step response as

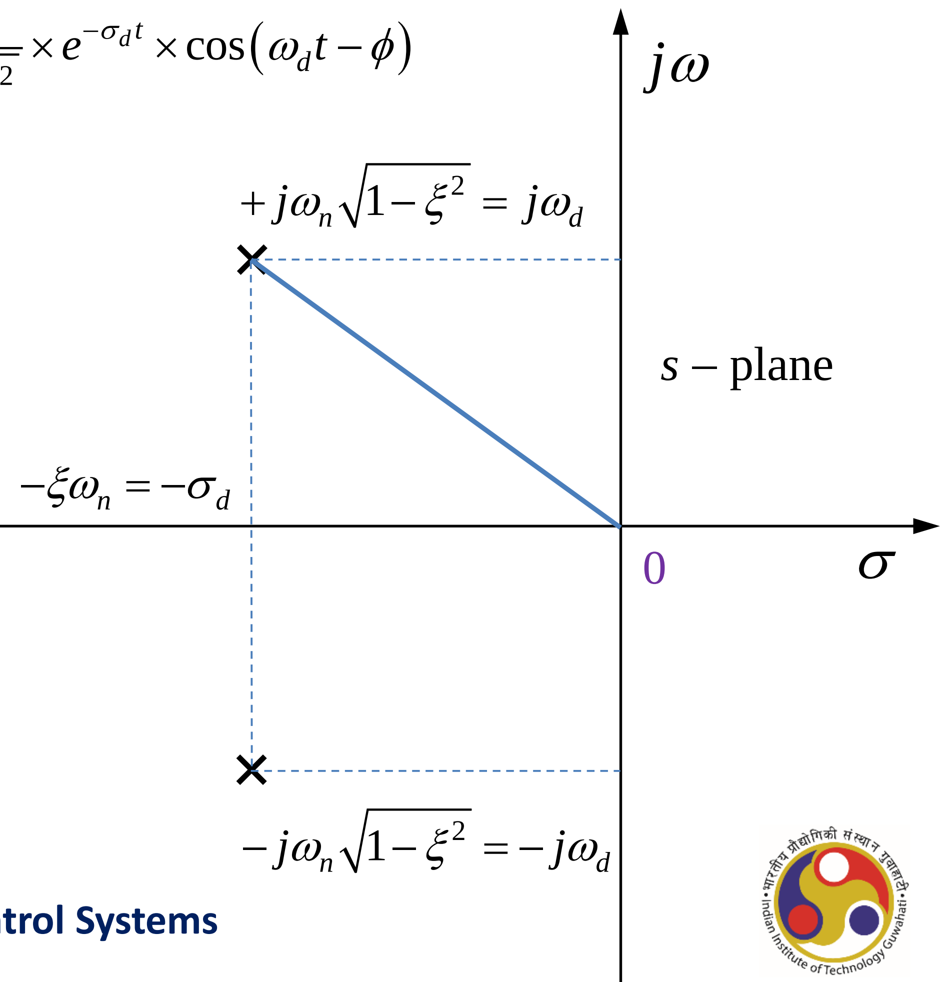
$$1 - \frac{1}{\sqrt{1-\xi^2}} \times e^{-\sigma_d t} \times \cos\left(\omega_d t - \phi\right)$$

The definition above can be re-written

$$\xi = \frac{\sigma_d}{\omega_n}$$

$$\sqrt{1-\xi^2} = \frac{\omega_d}{\omega_n}$$

$$\tan\theta = \frac{\omega_d}{\sigma_d} = \frac{\sqrt{1-\xi^2}}{\xi}$$



The underdamped 2nd order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \qquad 0 < \xi < 1$$

$$\sigma_d = \xi\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\xi^2}, \qquad \tan\phi = \frac{\xi}{\sqrt{1-\xi^2}}$$

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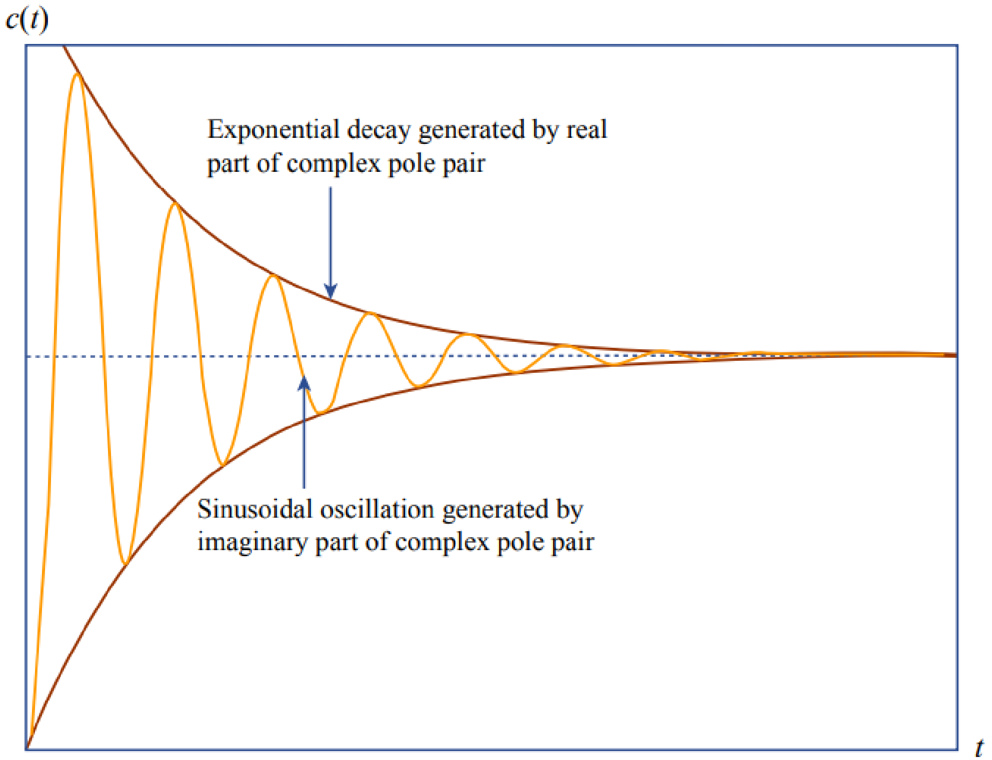
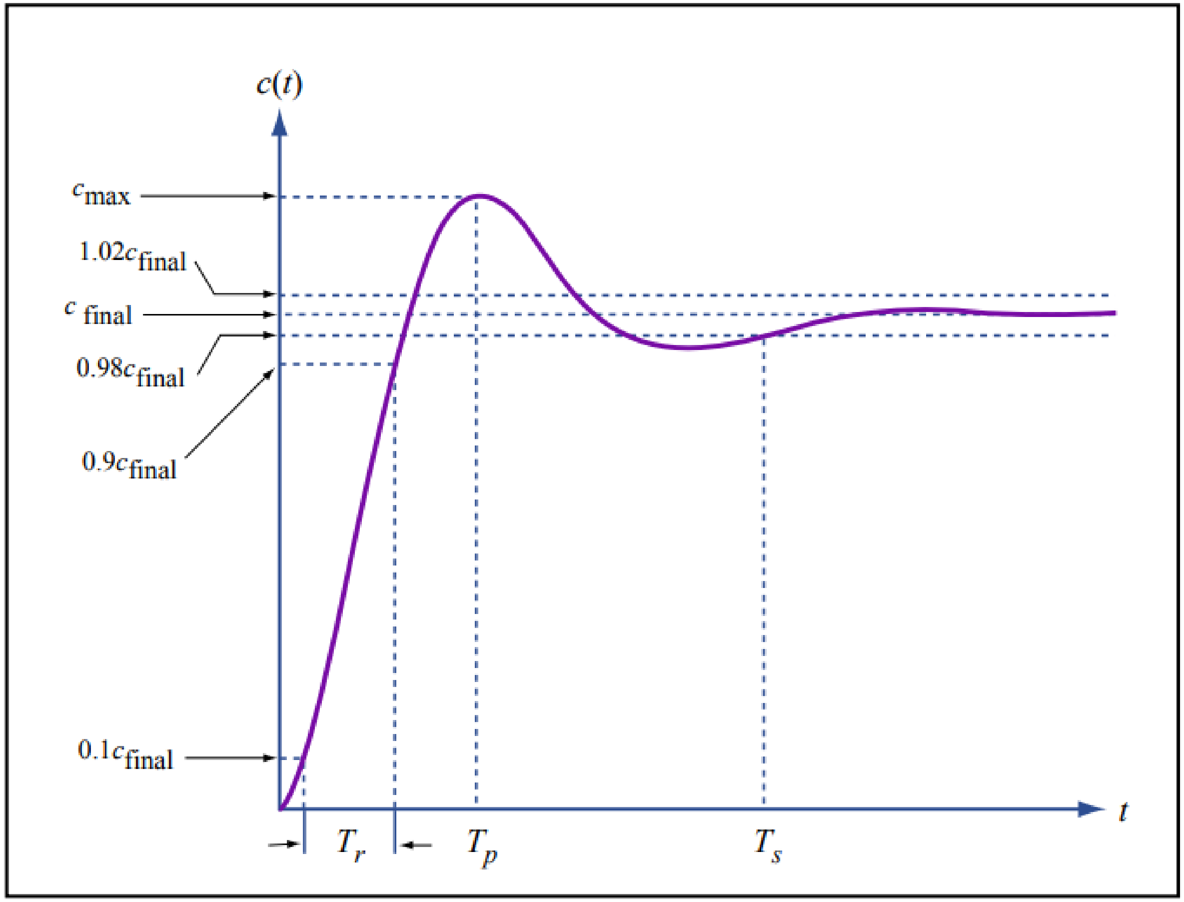


Transients in the underdamped 2nd order system



Peak time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Percent overshoot (%OS)

$$\%OS = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)\times 100$$

$$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$$

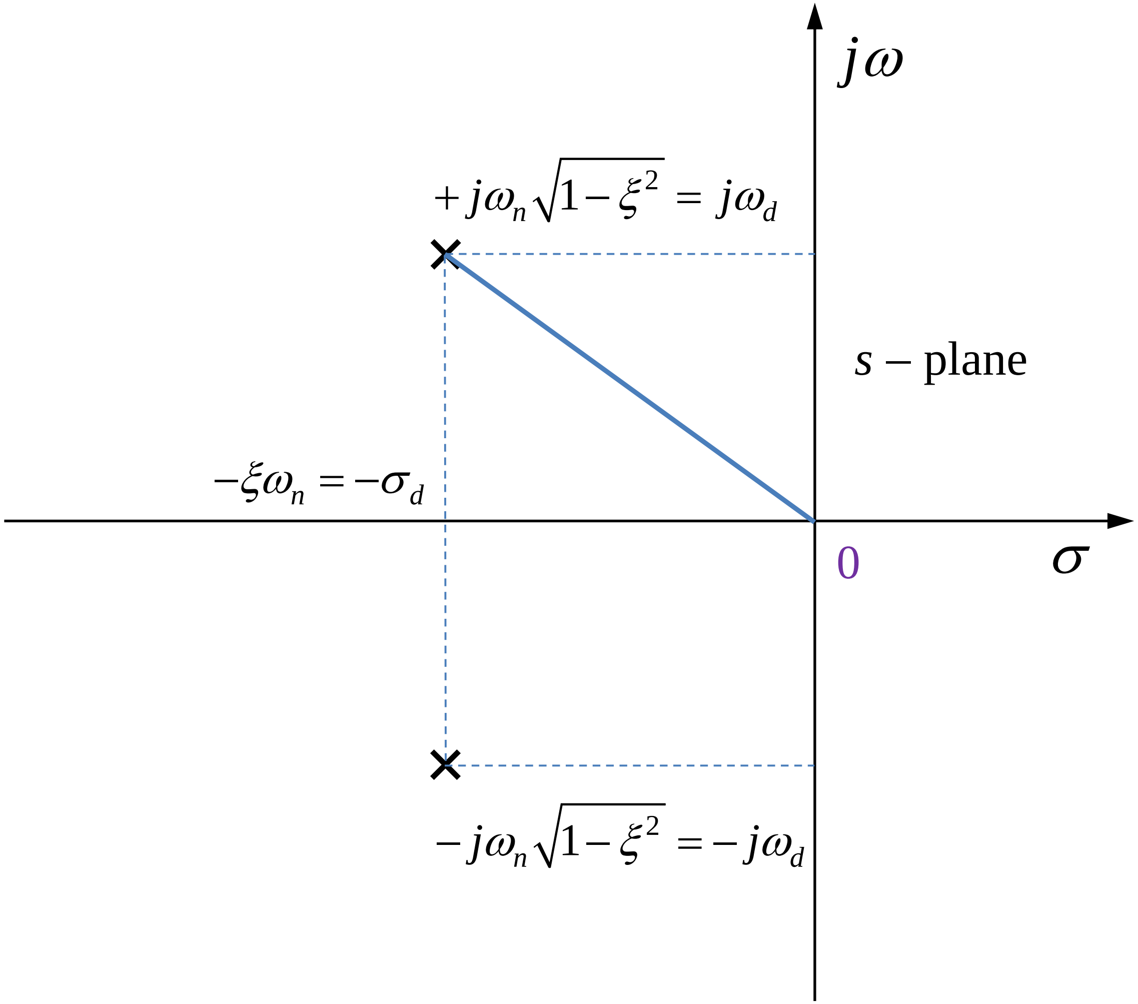
Settling time

(to within ± 2% of steady state)

$$T_s = -\frac{\ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi\omega_n} \approx \frac{4}{\xi\omega_n}$$

(approximation valid for $0<\xi<0.9$)

1. Rise time, T_r. The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. Peak time, T_P. The time required to reach the first, or maximum, peak.
3. Percent overshoot, $\%OS$. The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. Settling time, T_s. The time required for the transient's damped oscillations to reach and stay within 2% of the steady-state value.

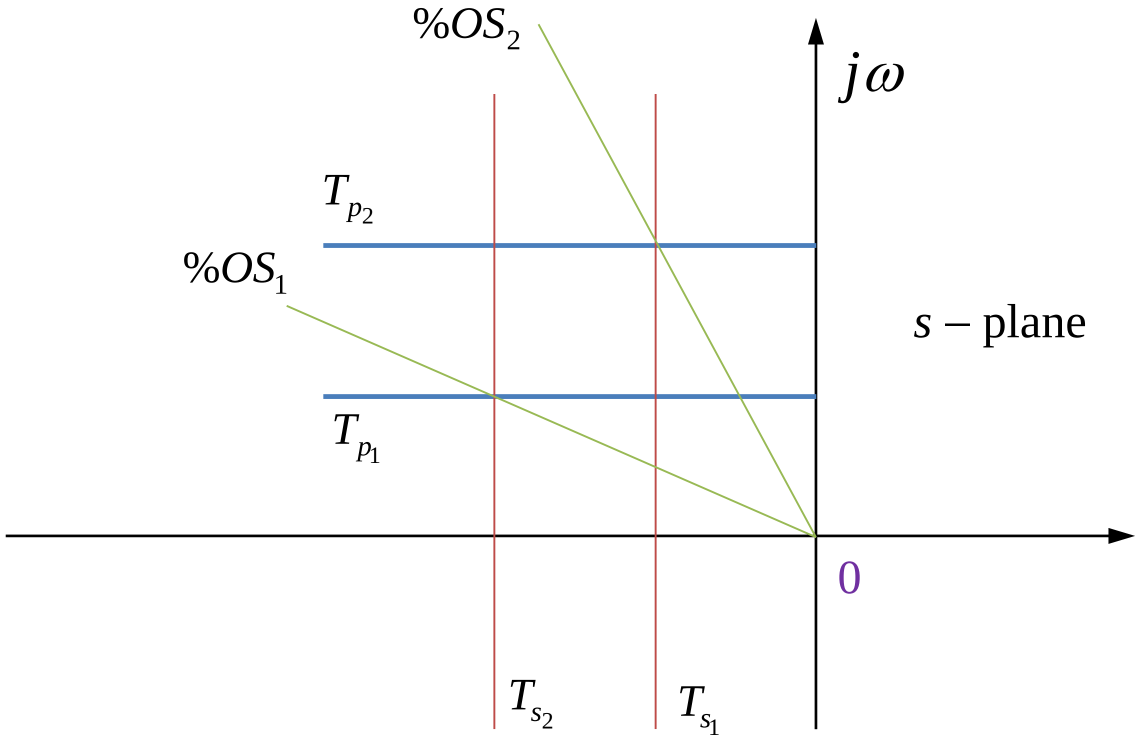


Pole plot for an underdamped second-order system

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$T_s = \frac{4}{\xi\omega_n}$$

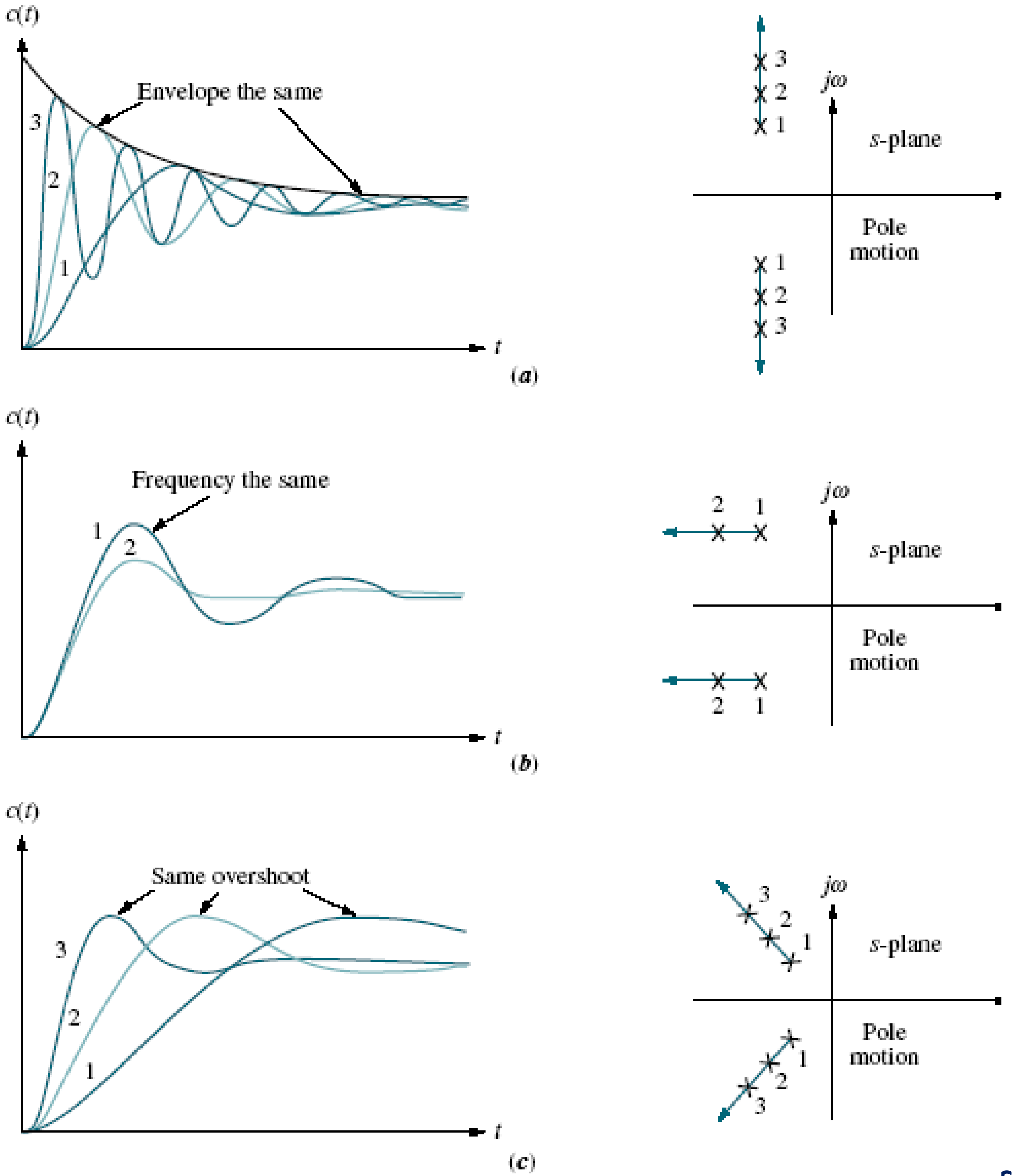
where ω_d is the imaginary part of the pole and is called the damped frequency of oscillation, and σ_d is the magnitude of the real part of the pole and is the exponential damping frequency.



T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the s–plane are lines of constant imaginary value, they are also lines of constant peak time

Settling time is inversely proportional to the real part of the pole. Since vertical lines on the s – plane are lines of constant real value, they are also lines of constant settling time.

Since $\xi = \cos\,\theta$, radial lines are lines of constant ξ. Since percent overshoot is only a function of ξ, radial lines are thus lines of constant percent overshoot, $\%OS$.



Step responses of second-order underdamped systems as poles move: (a) with constant real part; (b) with constant imaginary part; (c) with constant damping ratio

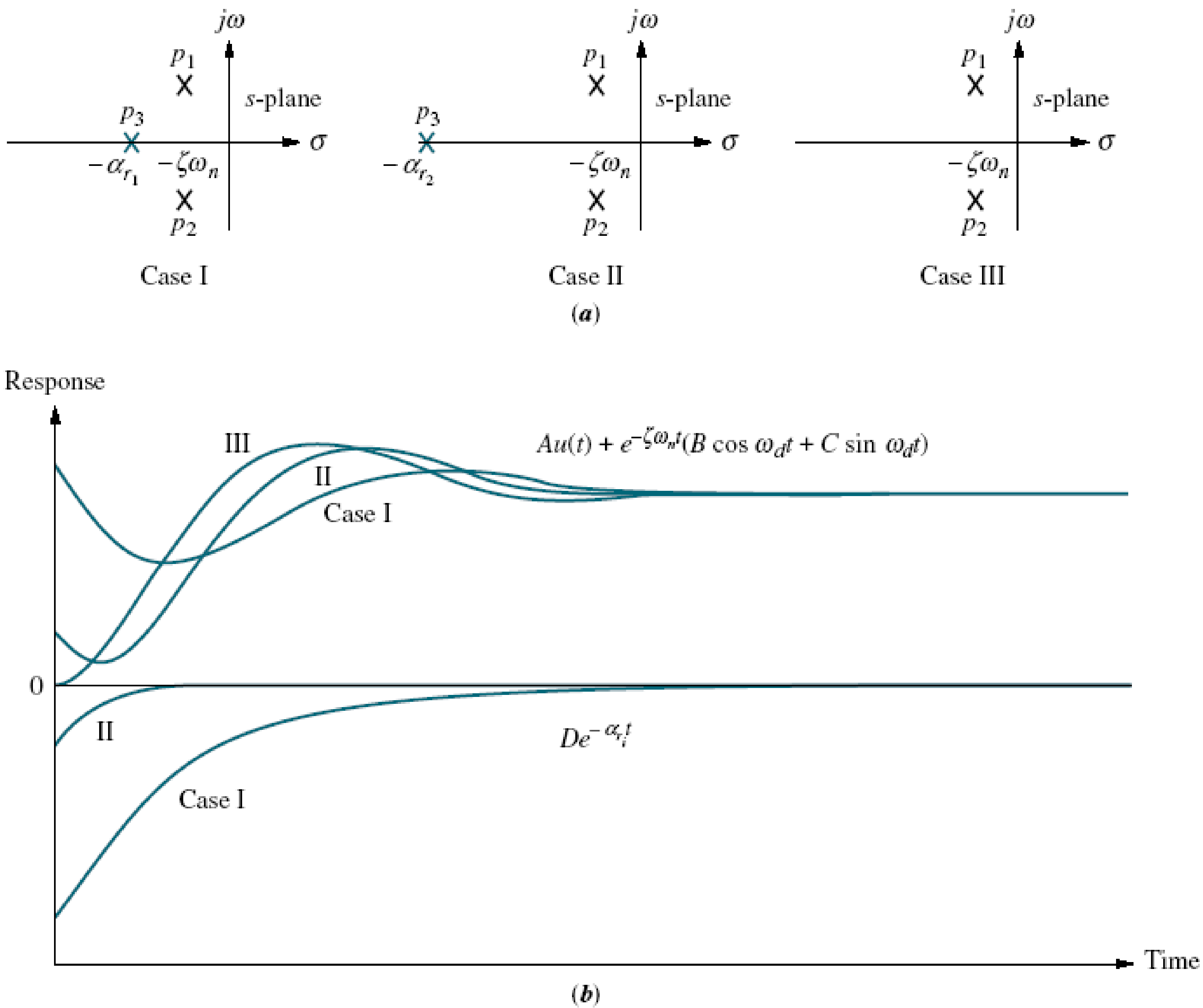
System Response with Additional Poles

Consider a three-pole system with complex poles and a third pole on the real axis. Assuming that the complex poles are at $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ and the real pole is at $-\alpha_r$, the step response of the system can be determined from a partial-fraction expansion. Thus, the output transform is

$$C(s) = \frac{A}{s} + \frac{B(s+\xi\omega_n) + C\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{D}{(s+\alpha_r)}$$

$$c(t) = Au(t) + e^{-\xi\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

The component parts of $c(t)$ are shown for three cases of α_r. For Case I, $\alpha_r = \alpha_{r1}$ and is not much larger than $\xi\omega_n$; for Case II, $\alpha_r = \alpha_{r2}$ and is much larger than $\xi\omega_n$; and for Case III, $\alpha_r = \infty$.


jω
p_1
p_3
s-plane
σ
$-\alpha_{r_1}$
$-\zeta\omega_n$
p_2
Case I
jω
p_1
p_3
s-plane
σ
$-\alpha_{r_2}$
$-\zeta\omega_n$
p_2
Case II
jω
p_1
s-plane
σ
$-\zeta\omega_n$
p_2
Case III
(a)
Response
III
II
Case I
$Au(t) + e^{-\zeta\omega_n t}(B \cos \omega_d t + C \sin \omega_d t)$
0
II
$De^{-\alpha_{r_i} t}$
Case I
Time
(b)

Lecture #13

15.02.2022

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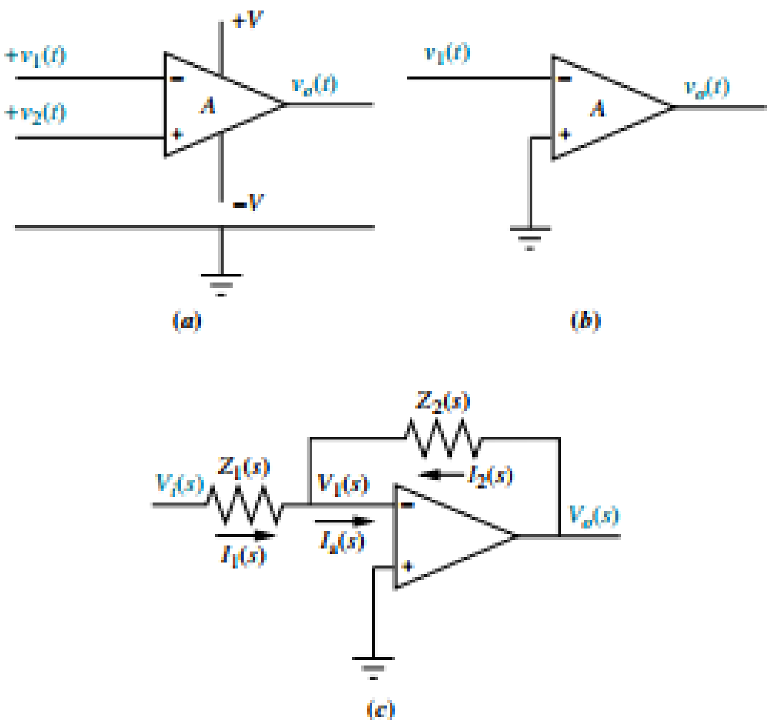
Operational Amplifiers



An operational amplifier is an electronic amplifier used as a basic building block to implement transfer functions.

It has the following characteristics:

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

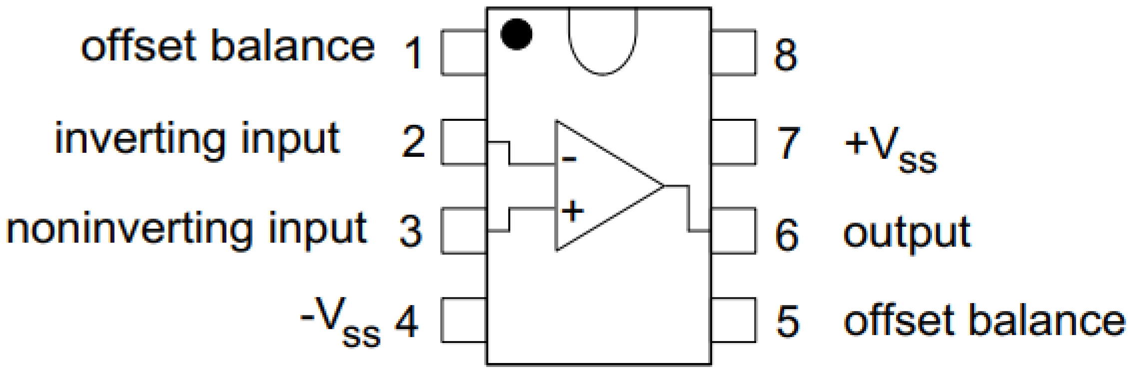
Practically, the magnitude of A is approximately $10^5 \sim 10^6$

The output, $v_o(t)$, is given by

$$v_0(t) = A\left(v_2(t) - v_1(t)\right)$$

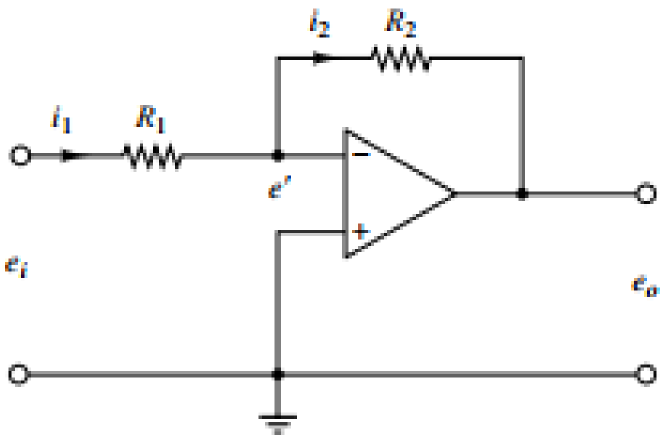
(*a*) Operational amplifier; (*b*) schematic for an inverting operational amplifier; (*c*) inverting operational amplifier configured for transfer function realization.

Op-amps come in a variety of packages. A common inexpensive op-amp, the 741, has an 8 pin DIP (dual in-line package) form. Many amps use a common basic pin-out for this package as shown below:



The pins are numbered counter-clockwise from the top left as shown above. (Note that pin 1 is identified by a notch at the top or a dot beside pin 1.) The basic amplifier is connected between pins 2, 3 and 6. The amplifier requires a pair of external supply voltages to operate, these are typically ±15 volts and are connected to pins 7 (positive) and 4 (negative). Pins 1 and 5 are usually used for optional external offset nulling circuitry - the actual connection is dependent on the type. We will not use this feature if we can get away without it.

Inverting Amplifiers



$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = \frac{e' - e_0}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2.

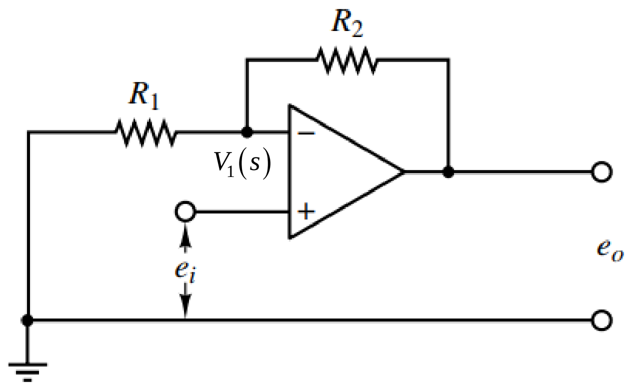
$$\frac{e_i - e'}{R_1} = \frac{e' - e_0}{R_2}$$

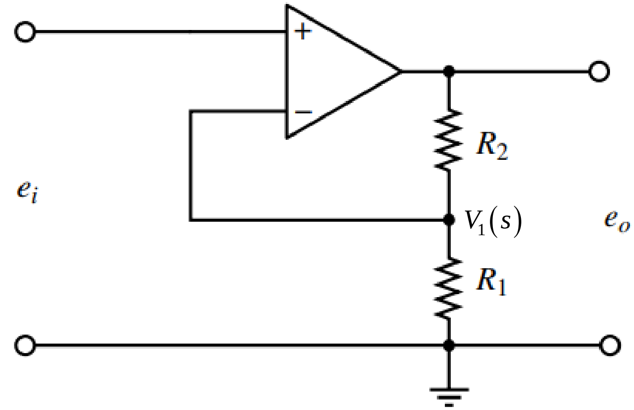
Since $K(0 - e') = e_0$ and $K \gg 1$, e' must be almost zero.

$$\frac{e_i}{R_1} = \frac{-e_0}{R_2} \Rightarrow \qquad e_0 = -\frac{R_2}{R_1} e_i$$

If $R_1 = R_2$, then the op-amp circuit acts as a sign inverter.

Noninverting Amplifiers





$$e_0(s) = K\left(e_i(s) - V_1(s)\right)$$

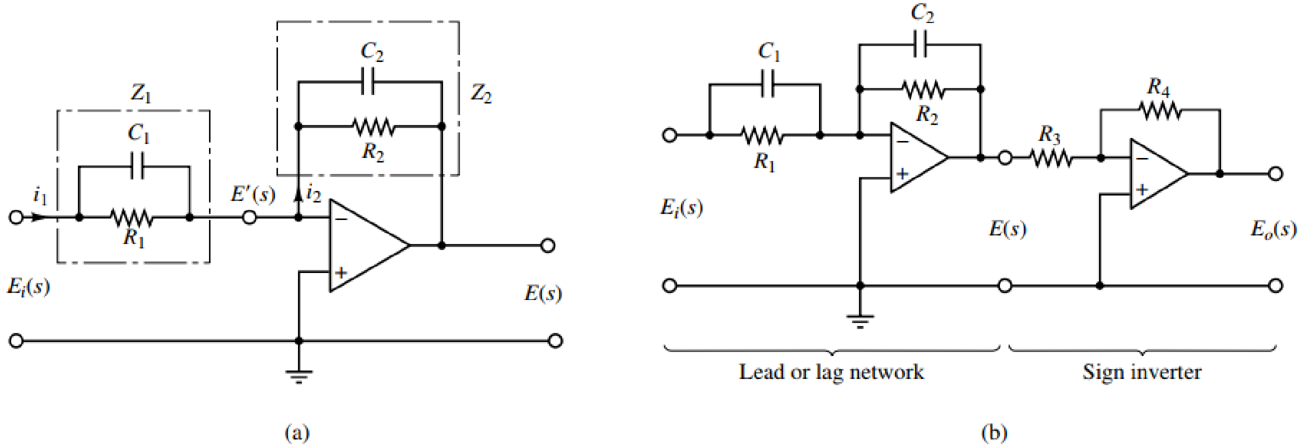
where K is the differential gain of the amplifier. Using voltage division

$$V_1(s) = \frac{R_1}{R_1 + R_2} e_0(s) \Rightarrow \quad e_0(s) = K\left(e_i(s) - \frac{R_1}{R_1 + R_2} e_0(s)\right)$$

$$e_i(s) = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K}\right) e_0(s)$$ Since $K \gg 1$, if $\frac{R_1}{R_1 + R_2} \gg \frac{1}{K}$, then

$$e_0(s) = \left(1 + \frac{R_2}{R_1}\right) e_i(s)$$

Lead or Lag Networks using Operational Amplifiers



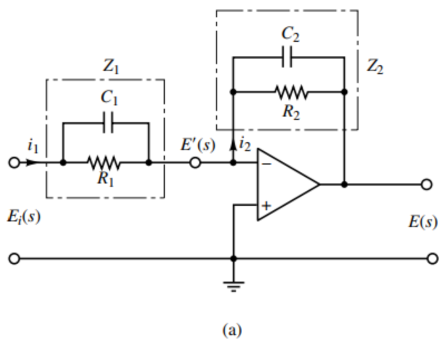
(a) Operational-amplifier circuit; (b) Operational-amplifier circuit used as a lead or lag compensator

$$Z_1(s) = \frac{R_1}{R_1C_1s+1} \qquad Z_2(s) = \frac{R_2}{R_2C_2s+1}$$

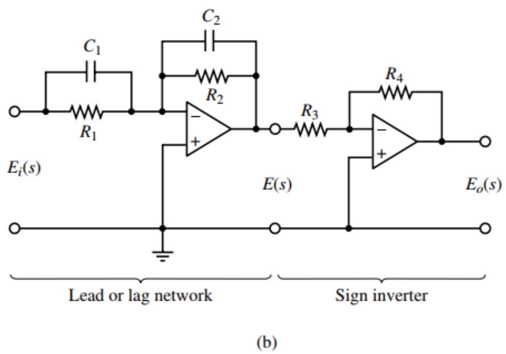
$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1} = \frac{C_1}{C_2}\frac{s+\dfrac{1}{R_1C_1}}{s+\dfrac{1}{R_2C_2}}$$

Again $$\frac{E_0(s)}{E(s)} = -\frac{R_4}{R_3}$$

Lead or Lag Networks using Operational Amplifiers



(a) Operational-amplifier circuit;



(b) Operational-amplifier circuit used as a lead or lag compensator

$$Z_1(s) = \frac{R_1}{R_1C_1s+1} \qquad \frac{1}{Z_1(s)} = \frac{1}{R_1} + C_1s$$

$$Z_2(s) = \frac{R_2}{R_2C_2s+1} \qquad \frac{1}{Z_2(s)} = \frac{1}{R_2} + C_2s$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2}{R_1}\frac{R_4}{R_3}\frac{R_1C_1s+1}{R_2C_2s+1} = \frac{R_4}{R_3}\frac{C_1}{C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

where $T = R_1C_1,\qquad \alpha T = R_2C_2,\qquad K_c = \dfrac{R_4C_1}{R_3C_2}$

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\frac{R_2C_2}{R_1C_1} = \frac{R_4}{R_3}\frac{R_2}{R_1} \qquad \alpha = \frac{R_2C_2}{R_1C_1}$$

This network has a DC gain of $K_c\alpha = \dfrac{R_4}{R_3}\dfrac{R_2}{R_1}$

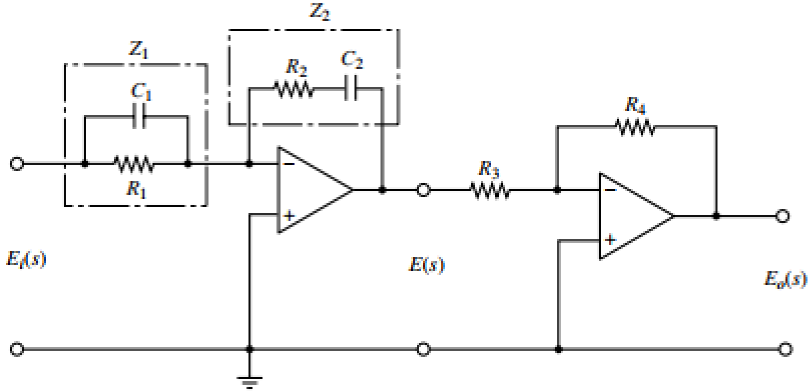
Lead network

$$R_1C_1 > R_2C_2, \quad \text{or} \quad \alpha < 1$$

Lag network

$$R_1C_1 < R_2C_2$$

PID controller using Operational Amplifiers



Electronic PID controller

$$Z_1(s) = \frac{R_1}{R_1 C_1 s + 1} \qquad \frac{1}{Z_1(s)} = \frac{1}{R_1} + C_1 s$$

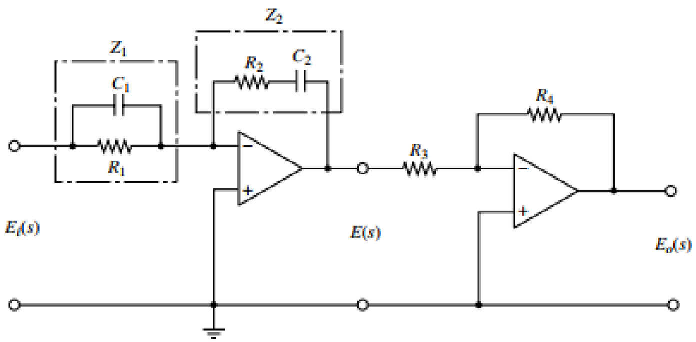
$$Z_2(s) = R_2 + \frac{1}{C_2 s} = \frac{R_2 C_2 s + 1}{C_2 s}$$

$$\frac{E(s)}{E_i(s)} = -\left(\frac{R_2 C_2 s + 1}{C_2 s}\right)\left(\frac{R_1 C_1 s + 1}{R_1}\right) \qquad \frac{E_0(s)}{E(s)} = -\left(\frac{R_4}{R_3}\right)$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_4(R_1 C_1 + R_2 C_2)}{R_3 R_1 C_2}\left[1 + \frac{1}{(R_1 C_1 + R_2 C_2)s} + \frac{R_1 C_1 R_2 C_2}{(R_1 C_1 + R_2 C_2)} s\right]$$

The second operational amplifier acts as a sign inverter as well as a gain adjuster

PID controller using Operational Amplifiers



Electronic PID controller

$$\frac{E_0(s)}{E_i(s)} = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}\left[1+\frac{1}{(R_1C_1+R_2C_2)s}+\frac{R_1C_1R_2C_2}{(R_1C_1+R_2C_2)}s\right]$$

when a PID controller is expressed as

$$\frac{E_0(s)}{E_i(s)} = K_p\left[1+\frac{T_i}{s}+T_d s\right]$$

$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}$$

$$T_i = \frac{1}{(R_1C_1+R_2C_2)}$$

$$T_d = \frac{R_1C_1R_2C_2}{(R_1C_1+R_2C_2)}$$

K_p is called the proportional gain, T_i is called the integral time, and T_d is called the derivative time.

When a PID controller is expressed as

$$\frac{E_0(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s$$

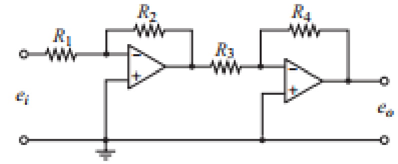
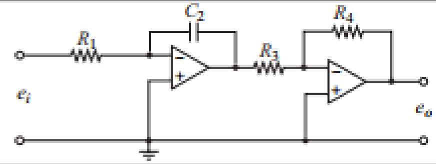
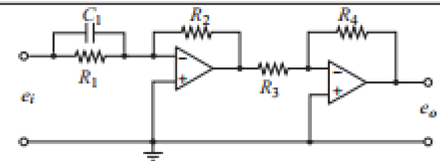
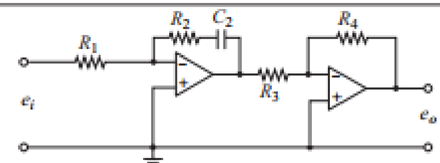
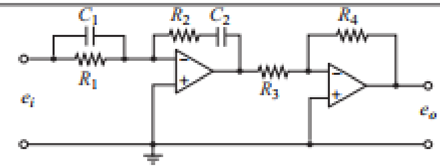
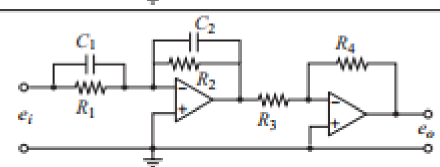
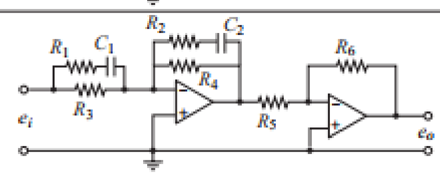
$$K_p = \frac{R_4(R_1C_1+R_2C_2)}{R_3R_1C_2}$$

$$K_i = \frac{R_4}{R_3R_1C_2}$$

$$K_d = \frac{R_4R_2C_1}{R_4}$$

K_p is called the proportional gain, K_i is called the integral gain, and K_d is called the derivative gain.

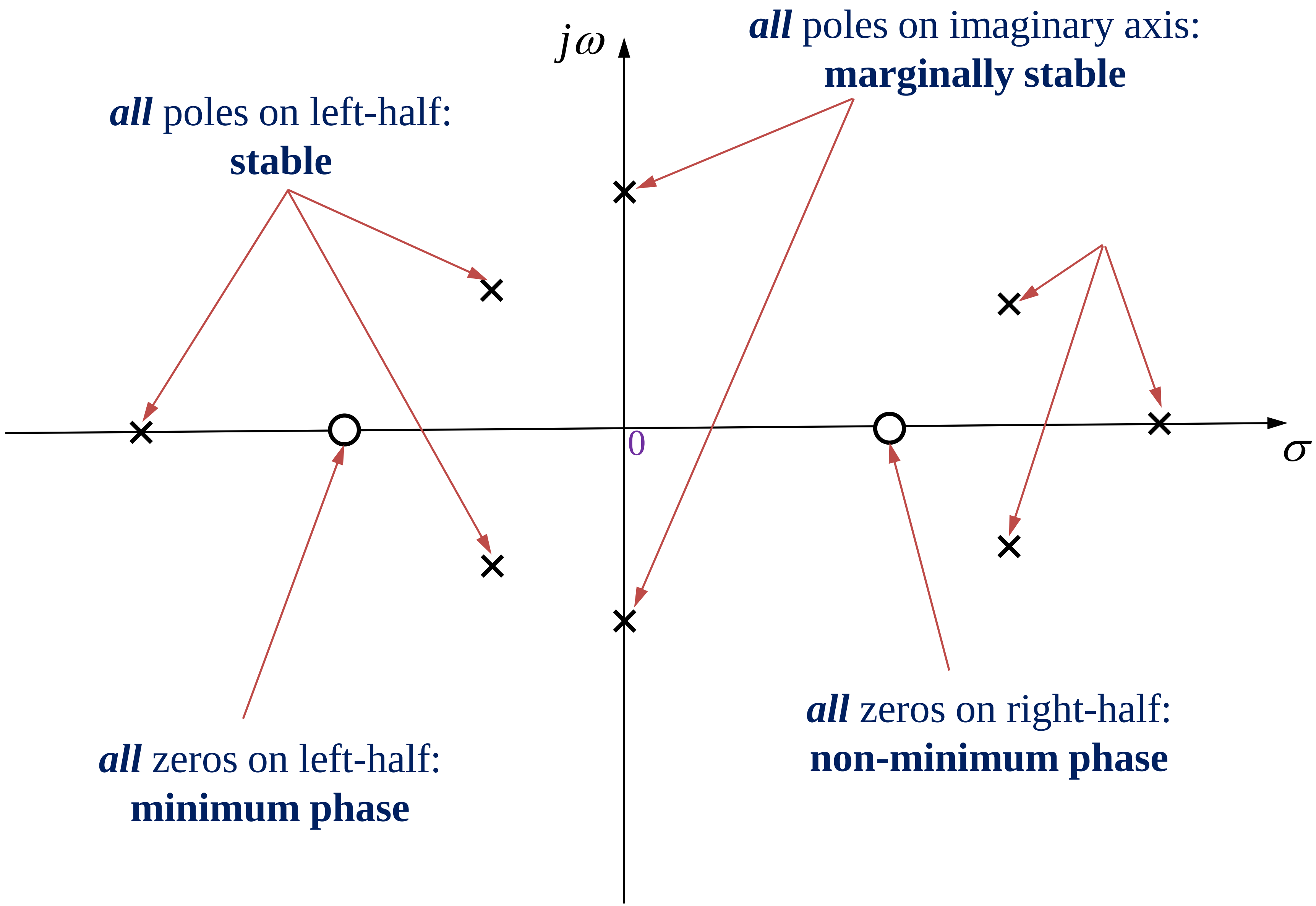
Table 3–1 Operational-Amplifier Circuits That May Be Used as Compensators

	Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational-Amplifier Circuits
1	P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
2	I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
3	PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s+1)$	
4	PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s+1}{R_2C_2s}$	
5	PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$	
6	Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}$	
7	Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1+R_3)C_1s+1](R_2C_2s+1)}{(R_1C_1s+1)[(R_2+R_4)C_2s+1]}$	

Definition of Stability

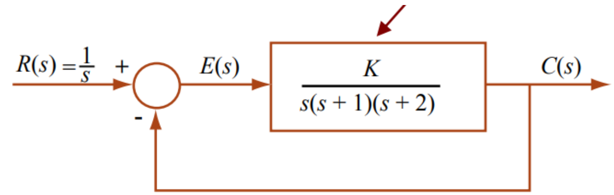
Reminder: $c(t) = c_{natural}(t) + c_{forced}(t)$

- A system is stable if
  - the natural response decays exponentially to zero as $t \rightarrow \infty$
  - for every bounded input the output is also bounded as $t \rightarrow \infty$
- A system is unstable if
  - the natural response increases exponentially as $t \rightarrow \infty$
  - there is at least one bounded input for which the output is unbounded (increases without bound) as $t \rightarrow \infty$
- A system is marginally stable if
  - the natural response oscillates as $t \rightarrow \infty$ (i.e. neither decays exponentially to zero nor increases exponentially)
  - there is at least one bounded input for which the output oscillates as $t \rightarrow \infty$ (i.e. neither decays exponentially to zero nor increases exponentially)


$j\omega$
all poles on imaginary axis:
marginally stable
all poles on left-half:
stable
0
σ
all zeros on left-half:
minimum phase
all zeros on right-half:
non-minimum phase

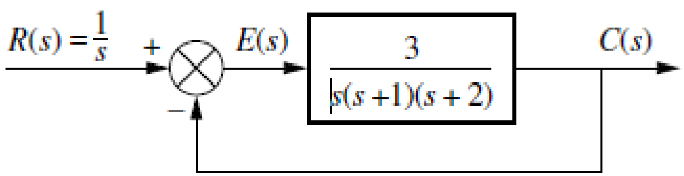
Stability and feedback

feedback gain

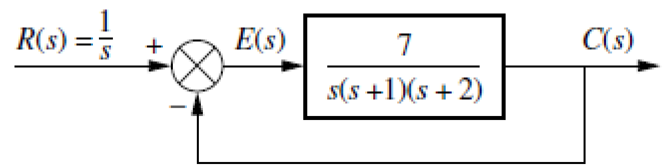


Small gain: *stable*

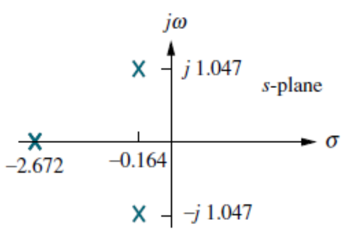
Large gain: *unstable*



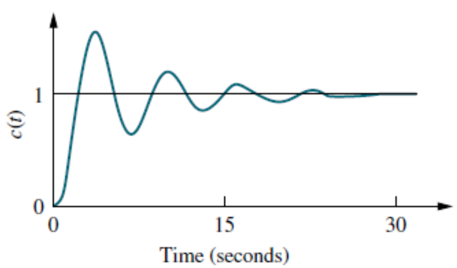
Stable system

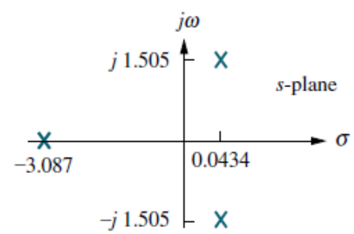


Unstable system

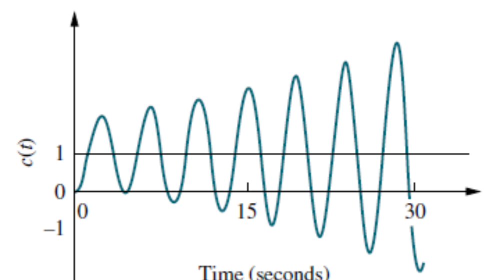


Stable system's
closed-loop poles
(not to scale)





Unstable system's
closed-loop poles
(not to scale)



(b)

Routh-Hurwitz Criterion

The method requires two steps:

(1) Generate a data table called a Routh table and

(2) Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$ - axis.

Routh-Hurwitz Criterion: the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

Generating a basic Routh Table

$$R(s) \longrightarrow \boxed{\dfrac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow C(s)$$

1.

s^4:	a_4	a_2	a_0
s^3:	a_3	a_1	0
s^2:			
s^1:			
s :			

2.

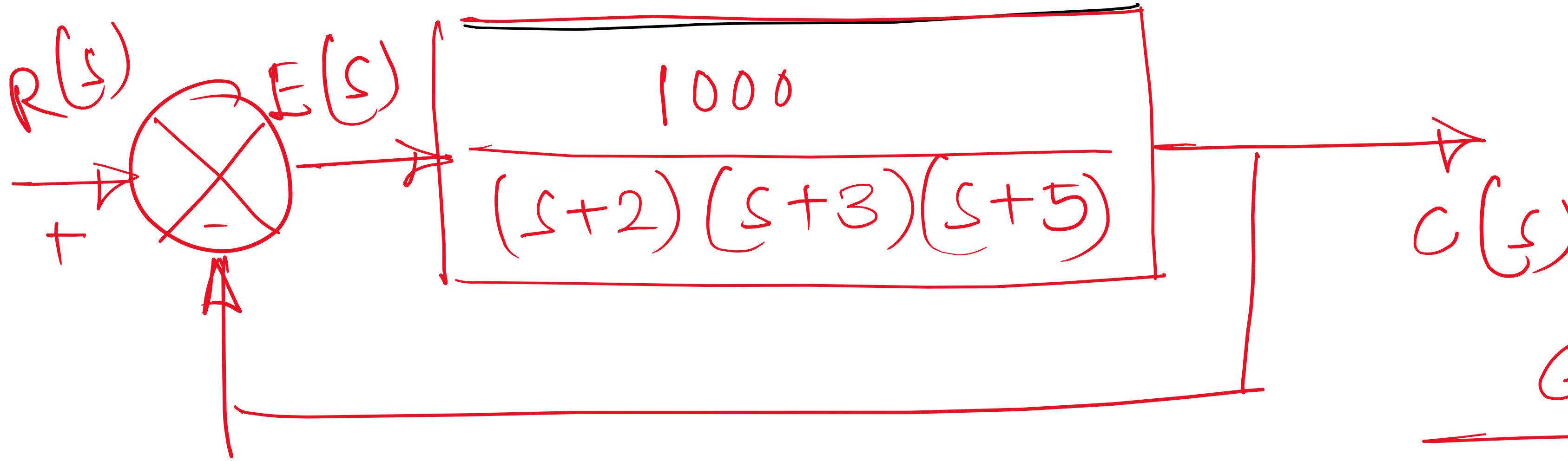
$s^4: \quad a_4 \quad a_2 \quad a_0$

$s^3: \quad a_3 \quad a_1 \quad 0$

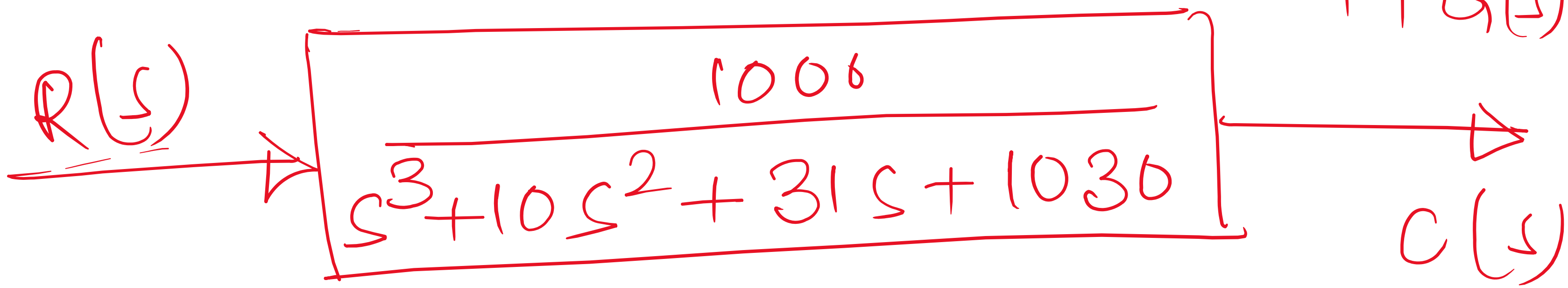
$s^2: \quad \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 \qquad \dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 \qquad \dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$

$s^1: \quad \dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 \qquad \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = c_2 \qquad \dfrac{-\begin{vmatrix} a_3 & 0 \\ a_1 & 0 \end{vmatrix}}{b_1} = c_3$

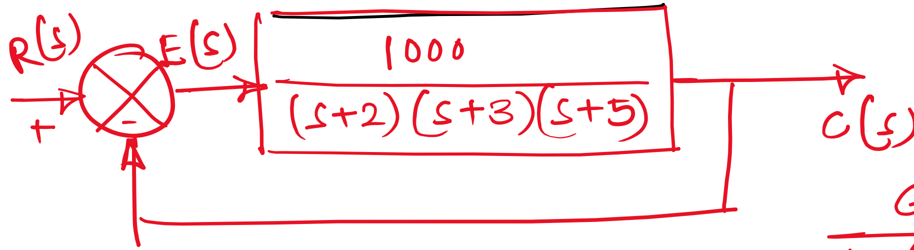
$s^0: \quad \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 \qquad \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_2 \qquad \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_3$



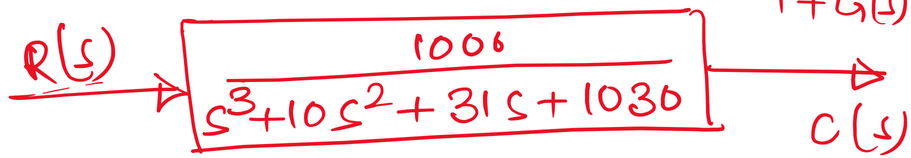
$$\frac{G(s)}{1+G(s)H(s)}$$

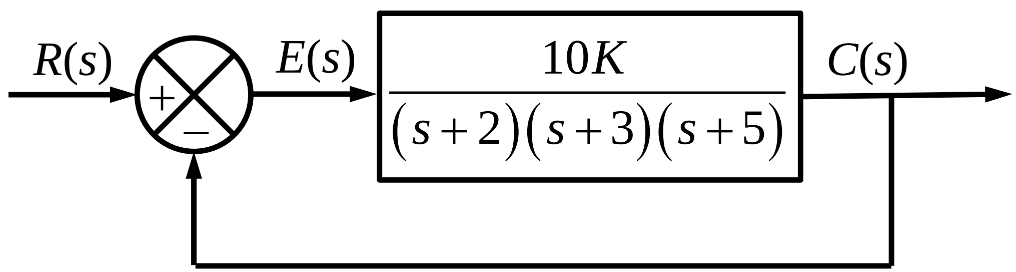


roots([1 10 31 1030])



$$\frac{G(s)}{1+G(s)H(s)}$$



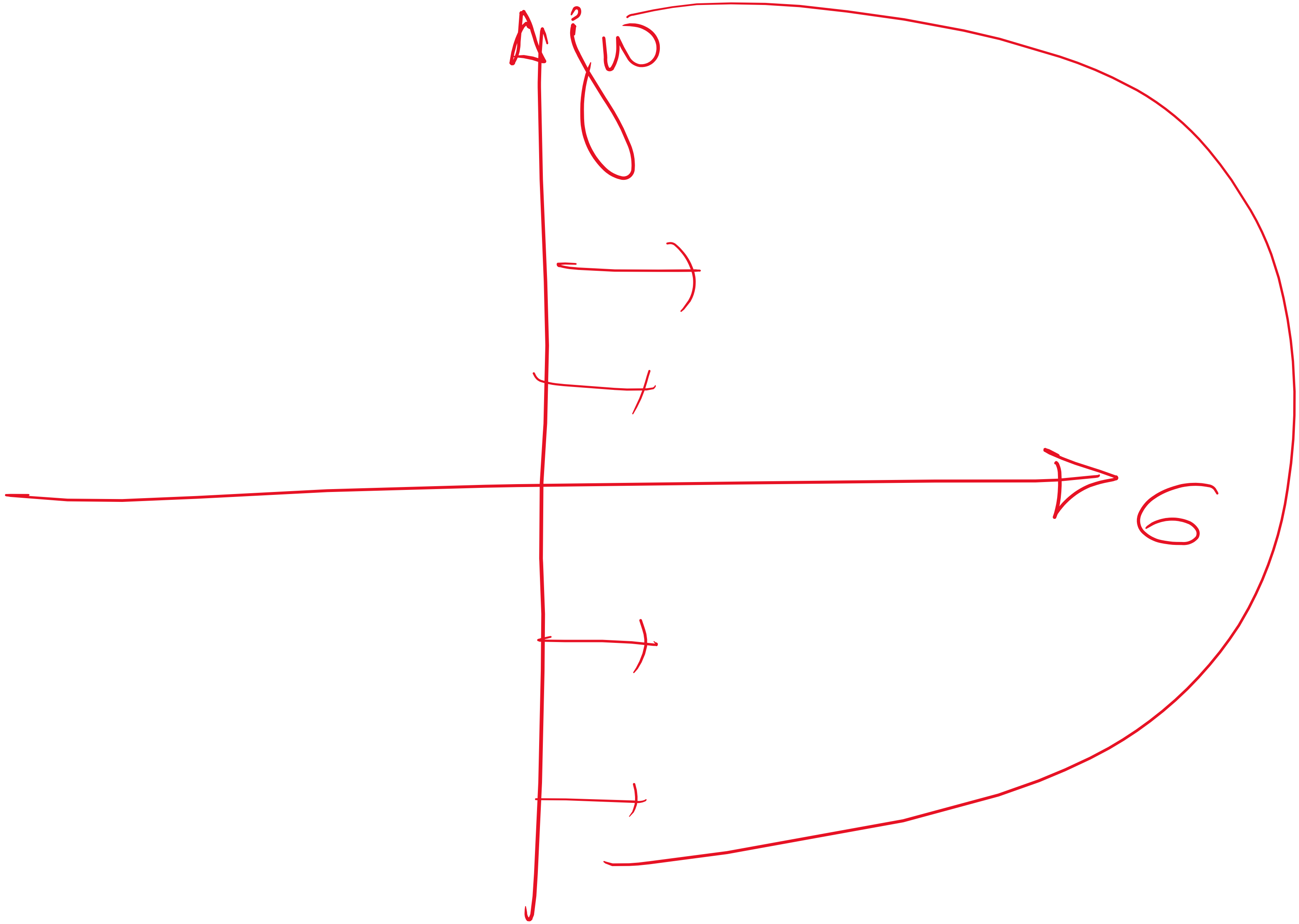


$s^3:\quad 1 \qquad 31 \qquad 0$

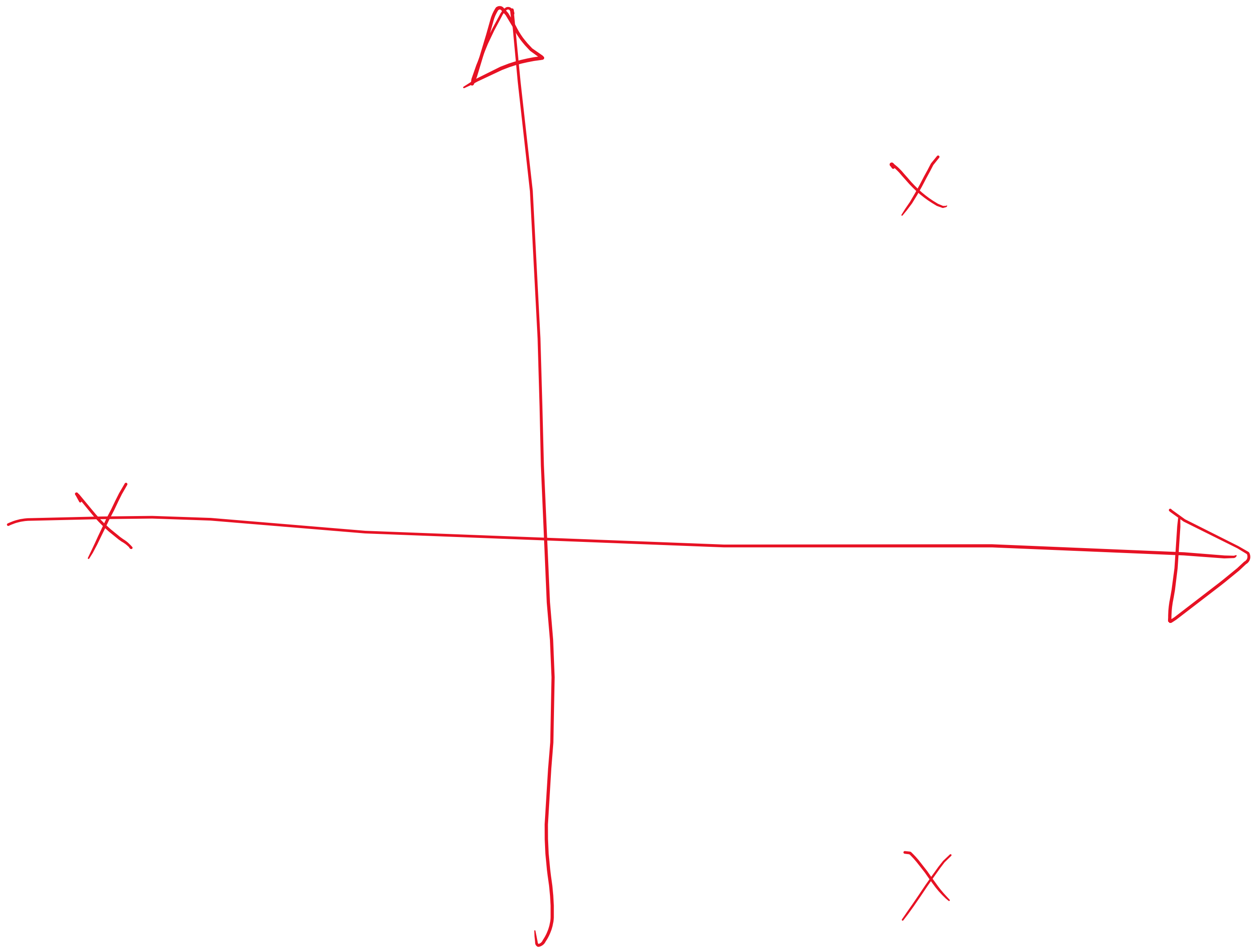
$s^2:\quad \cancel{10}\; 1 \qquad 10\cancel{30}\; 103 \qquad 0$

$$s^1:\quad \frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72 \qquad \frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0 \qquad \frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$$

$$s^0:\quad \frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103 \qquad \frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0 \qquad \frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$$



jω
σ

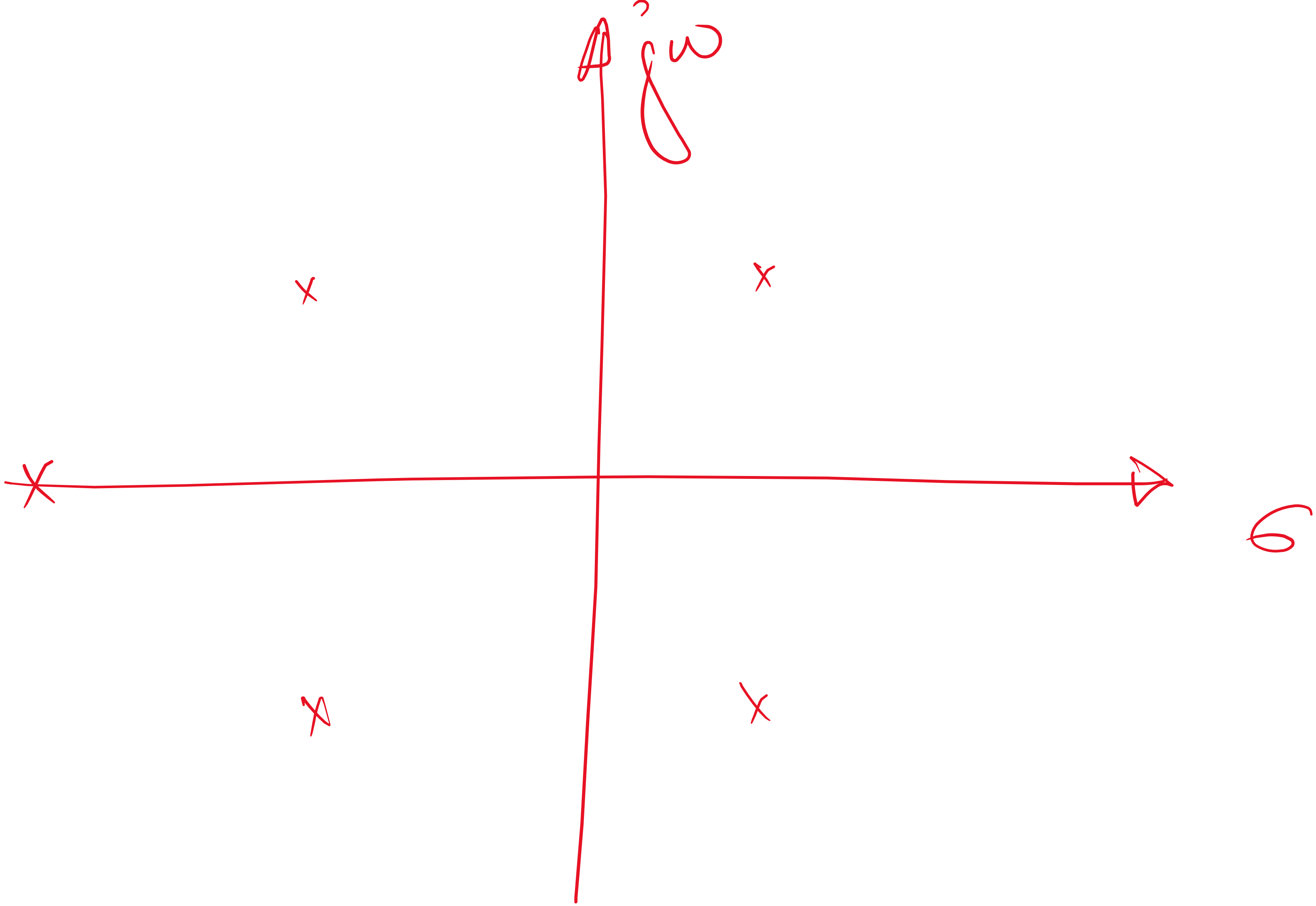


$$T(s) = \frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$$

s^5:	1	3	5
s^4:	2	6	3
s^3:	$\cancel{0}\ \epsilon$	$7/2$	0
s^2:	$\frac{6\epsilon-7}{\epsilon}$	3	0
s^1:	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	0	0
s^0:	3	0	0

Determine Signs

label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
			1
s^3	$\cancel{0}\ \epsilon$	$+$	$-$
		1	2
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
		2	
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$


jω
σ

Lecture #14

18.02.2022

ME 325 Control Systems
(3-0-0-6)

Instructor

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Office: D302

Routh-Hurwitz Criterion: Special Cases

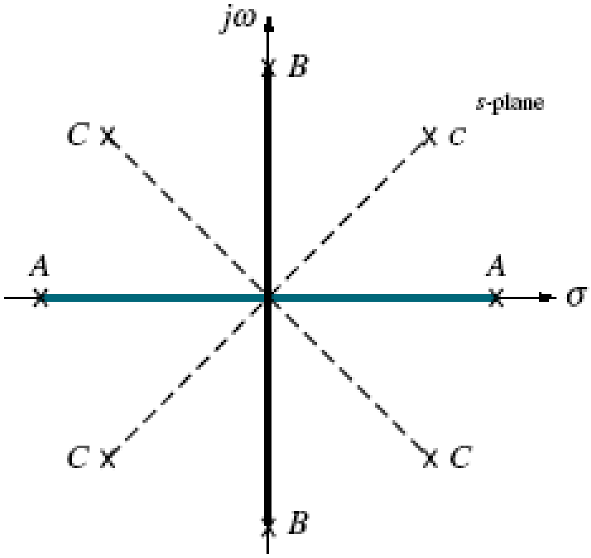
(1) Zero only in the first column and

(2) Entire row is zero

A: Real and symmetrical about the origin

B: Imaginary and symmetrical about the origin

C: Quadrantal and symmetrical about the origin



The final value theorem: steady-state

Let F(s) denote the Laplace transform of the function $f(t)$. The first property is the

Final Value theorem:

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Let us see how this applies to the step response of a general 1st — order system with a pole at $-a$ and without a zero (e.g., the angular velocity response of the DC motor.) We select the system gain such that the steady — state is equal 1.

The step response in the s —domain then is

$$F_1(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a} \qquad \text{also,} \quad sF_1(s) = \frac{a}{(s+a)}$$

Using the partial fraction expansion above, we find the time domain step response as

$$f_1(t) = \left(1 - e^{-at}\right)u(t) \Rightarrow f_1(\infty) = 1$$

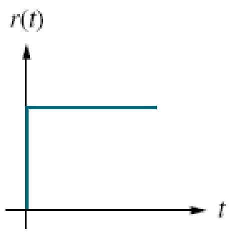
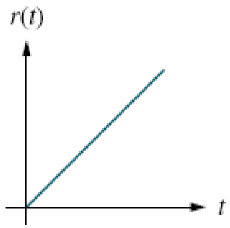
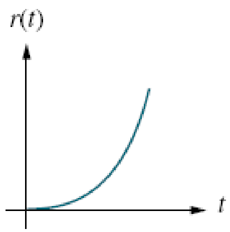
Applying the final value theorem, we find, consistently,

$$f_1(\infty) = \lim_{s\to 0} sF_1(s) = \lim_{s\to 0} \frac{a}{s+a} = 1$$

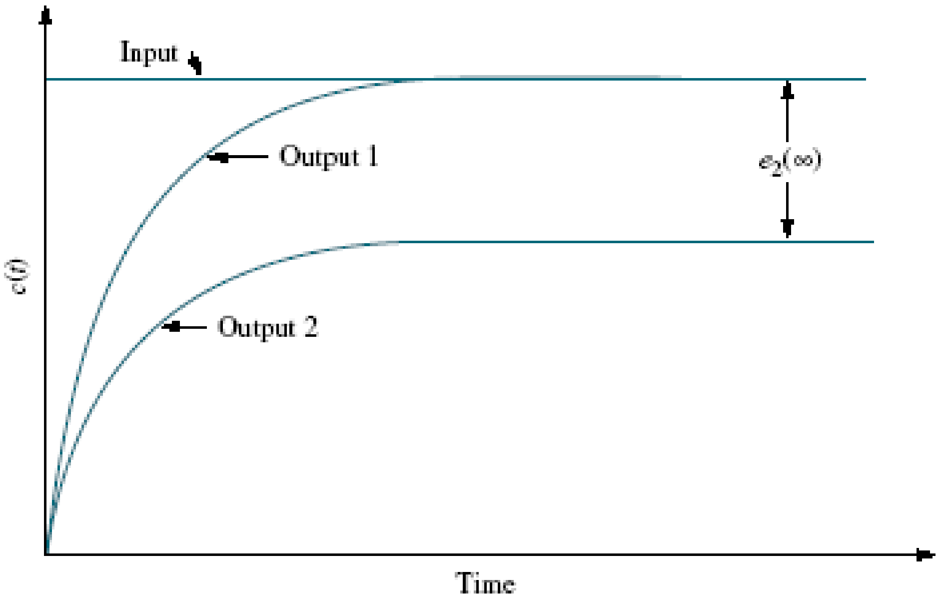
Steady-state Errors

Steady-state error is the difference between the input and the output for a prescribed test input as $t \to \infty$. Test inputs used for steady-state error analysis and design are summarized in the following table.

Test waveforms for evaluating steady-state errors of position control systems

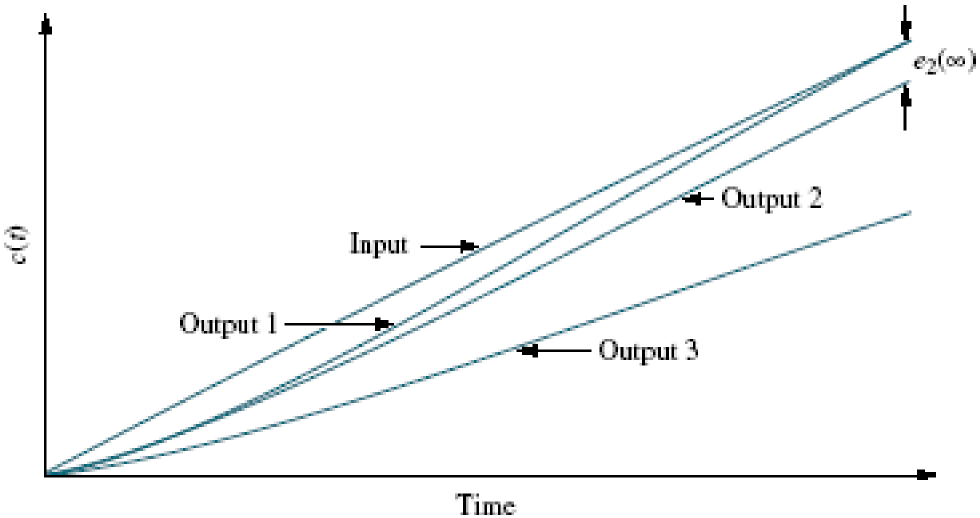
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Steady-state Errors



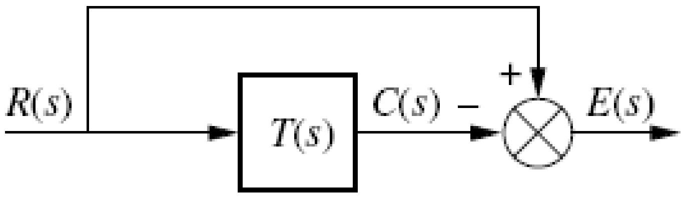
(*a*)

Step input

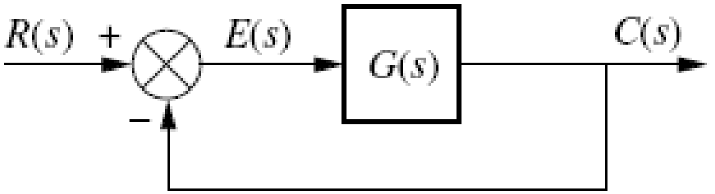


(*b*)

Ramp input

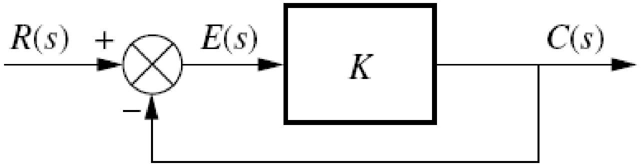


(a)



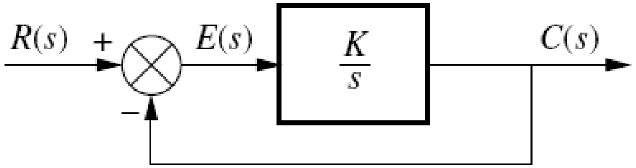
(b)

Closed loop control system errors: (a) general representation; (b) representation for unity feedback system



(a)

Finite steady state error for step input



(b)

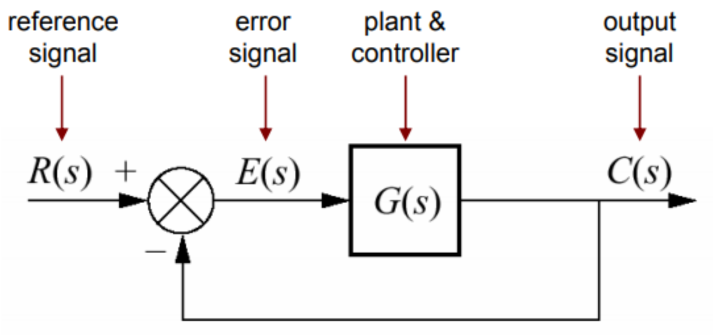
Zero steady state error for step input

Generalising: steady-state error for arbitrary system, unity feedback

Generally, the steady-state error is defined as

Final value theorem

$$e(\infty)=\lim_{t\to\infty}\left[r(t)-c(t)\right]=\lim_{s\to 0}s\left[R(s)-C(s)\right]$$



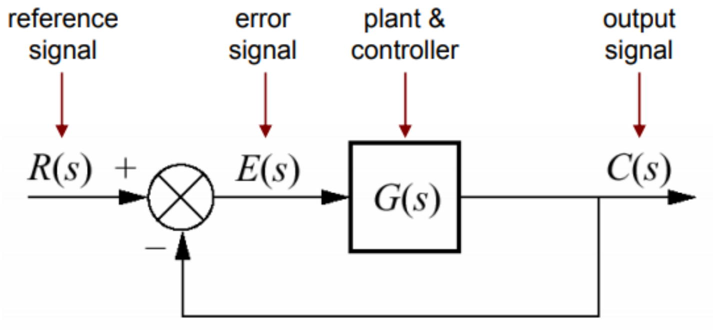
$$e(\infty)=\lim_{t\to\infty}\left[e(t)\right]=\lim_{t\to\infty}\left[r(t)-c(t)\right]=\lim_{s\to 0}s\left[R(s)-C(s)\right]$$

The steady-state error in terms of $T(s)$

$$E(s)=\left[R(s)-C(s)\right]=R(s)\left[1-T(s)\right]$$

$$e(\infty)=\lim_{s\to 0}sR(s)\left[1-T(s)\right]$$

Generalising: steady-state error for arbitrary system, unity feedback



$$e(\infty) = \lim_{t\to\infty}\left[e(t)\right] = \lim_{t\to\infty}\left[r(t) - c(t)\right] = \lim_{s\to 0} s\left[R(s) - C(s)\right]$$

The steady-state error in terms of $G(s)$

$$C(s) = E(s)G(s) \Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

Steady-state error and static error constants

Step Input:

$$R(s)=\frac{1}{s} \qquad e(\infty)=e_{\text{step}}(\infty)=\lim_{s\to 0}\frac{s\left(\frac{1}{s}\right)}{1+G(s)}=\frac{1}{1+\lim_{s\to 0}G(s)}$$

In order to have zero steady-state error $\lim_{s\to 0}G(s)=\infty$

$$G(s)=\frac{(s+z_1)(s+z_1)\cdots}{s^n(s+p_1)(s+p_1)\cdots}$$

for the limit to be infinite, the denominator must be equal to zero as s goes to zero. Thus, $n \geq 1$; that is, at least one pole must be at the origin.

If there are no integrations, then $n = 0$ $\lim\limits_{s\to 0}G(s)=\frac{z_1 z_2\cdots}{p_1 p_2\cdots}$ is finite

For a step input to a unity feedback system, the steady-state error will be zero if there is at least one pure integration in the forward path. If there are no integrations, then there will be a nonzero finite error.

Steady-state error and static error constants

Ramp Input:

$$R(s) = \frac{1}{s^2} \qquad e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s\to 0} \frac{s\left(\frac{1}{s^2}\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s + \lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

In order to have zero steady-state error $\quad \lim_{s\to 0} sG(s) = \infty$

$$G(s) = \frac{(s+z_1)(s+z_1)\cdots}{s^n(s+p_1)(s+p_1)\cdots}$$

for the limit to be infinite, the denominator must be equal to zero as s goes to zero. Thus, $n \geq 2$; that is, at least two poles must be at the origin.

If only one integration exists $\quad \lim_{s\to 0} sG(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \quad$ is finite

If no integration exists $\quad \lim_{s\to 0} sG(s) = 0 \quad$ is infinite

Steady-state error and static error constants

Parabolic Input:

$$R(s) = \frac{1}{s^3} \qquad e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s\to 0} \frac{s\left(\frac{1}{s^3}\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2 + \lim_{s\to 0} s^2 G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

In order to have zero steady-state error $\qquad \lim_{s\to 0} s^2 G(s) = \infty$

$$G(s) = \frac{(s+z_1)(s+z_1)\cdots}{s^n (s+p_1)(s+p_1)\cdots}$$

for the limit to be infinite, the denominator must be equal to zero as s goes to zero. Thus, $n \geq 3$; that is, at least three poles must be at the origin.

If only two integrations exists $\qquad \lim_{s\to 0} s^2 G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \qquad$ is finite

If there is only one or less integration exists $\qquad \lim_{s\to 0} s^2 G(s) = 0 \qquad$ is infinite

Lecture #16

19.02.2020

ME 325 Control Systems
(3-0-0-6)

Instructor

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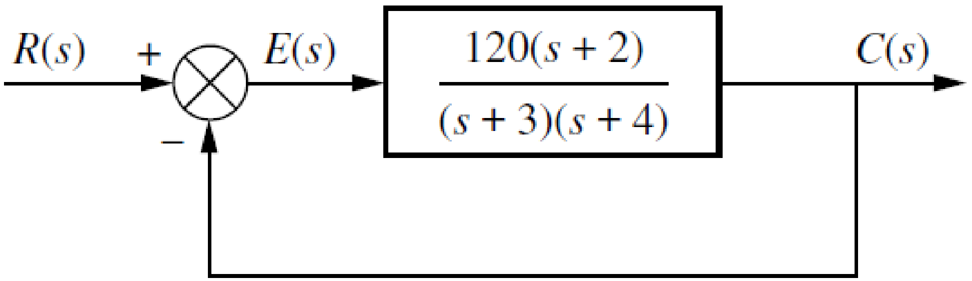
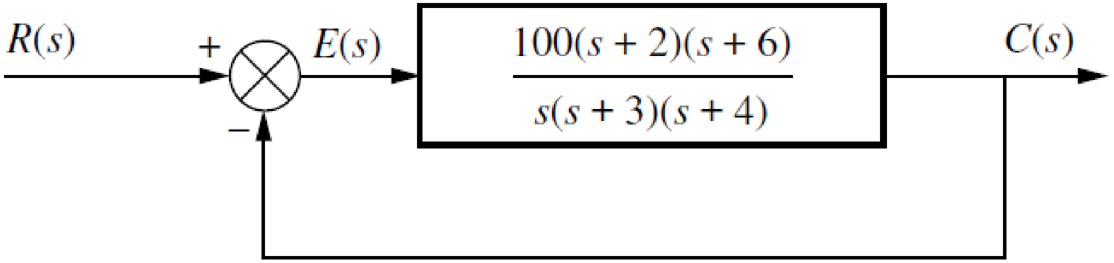
Steady-state Errors for Systems with No integrations

Find the steady - state errors for inputs of $5u(t)$, $5tu(t)$ and $5t^2u(t)$ to the system shown in the following figure. The function $u(t)$ is the unit step.



Find the steady - state errors for inputs of $5u(t)$, $5tu(t)$ and $5t^2u(t)$ to the system shown in the following figure. The function $u(t)$ is the unit step.



Static Error Constants and System Type

For a STEP input, $u(t)$

$$e(\infty)=e_{\text{step}}(\infty)=\frac{1}{1+\lim_{s\to 0}G(s)}$$

For a RAMP input, $tu(t)$

$$e(\infty)=e_{\text{ramp}}(\infty)=\frac{1}{1+\lim_{s\to 0}sG(s)}$$

For a RAMP input, $t^2u(t)$

$$e(\infty)=e_{\text{parabola}}(\infty)=\frac{1}{1+\lim_{s\to 0}s^2G(s)}$$

Static Error Constants

position constant, K_p

$$K_p=\lim_{s\to 0}G(s) \qquad \mathcal{L}\left[u(t)\right]=\frac{1}{s}$$

velocity constant, K_v

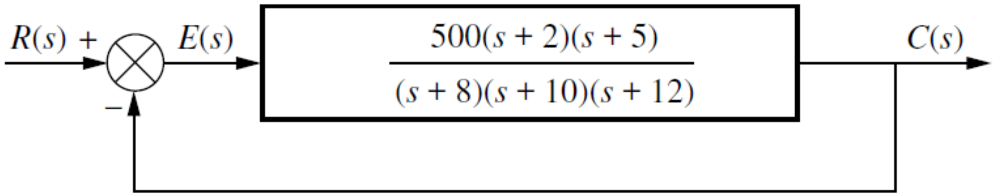
$$K_p=\lim_{s\to 0}sG(s) \qquad \mathcal{L}\left[tu(t)\right]=\frac{1}{s^2}$$

acceleration constant, K_a

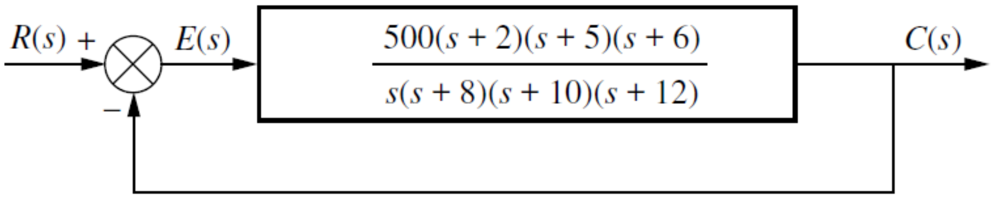
$$K_p=\lim_{s\to 0}s^2G(s) \qquad \mathcal{L}\left[t^2u(t)\right]=\frac{1}{s^3}$$

Steady-State Error via Static Error Constants

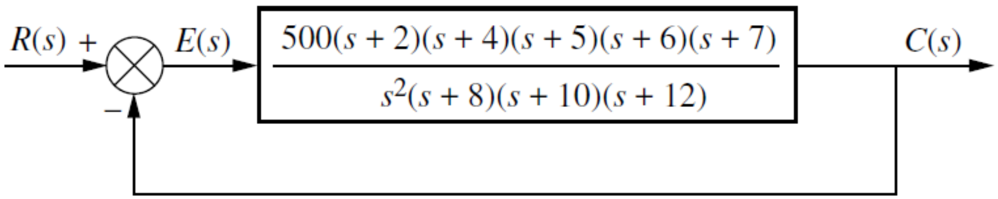
For each system of the following figures evaluate the static error constants and find the expressed error for the standard step, ramp and parabolic inputs.



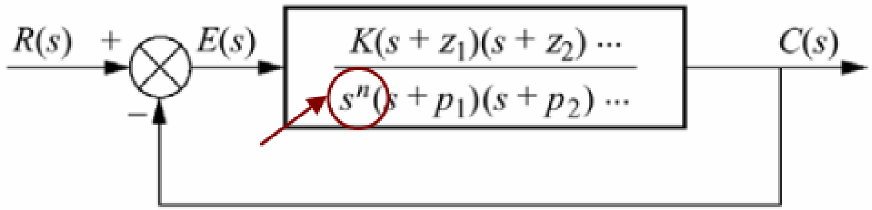
(a)



(b)



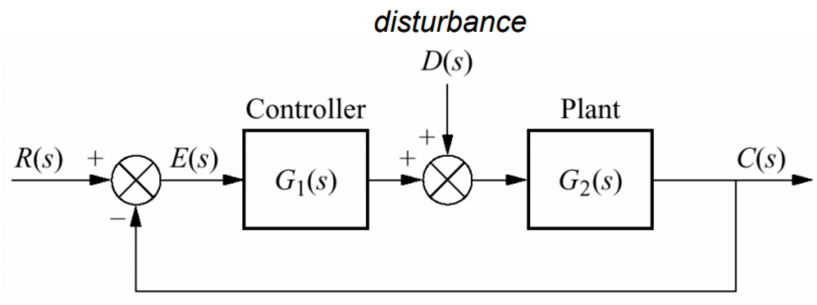
(c)



$n = 0$ Type 0
$n = 1$ Type 1
$n = 2$ Type 2

		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\dfrac{1}{1+K_p}$	$K_p =$ Constant	$\dfrac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\dfrac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\dfrac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\dfrac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\dfrac{1}{K_a}$

Steady-State Error for Disturbances

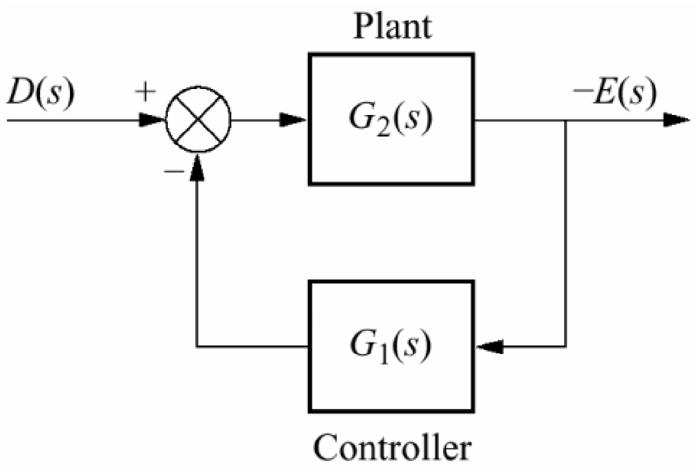


$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

From the summation element

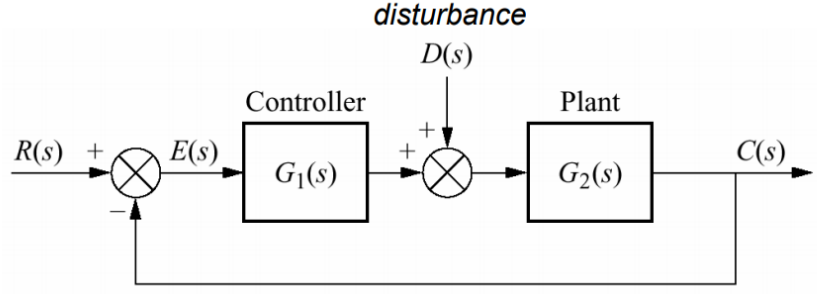
$$E(s) = R(s) - C(s)$$

$$E(s) = \frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s)$$



Equivalent block diagram
with $D(s)$ as input
And – $E(s)$ as output

Steady-State Error for Disturbances



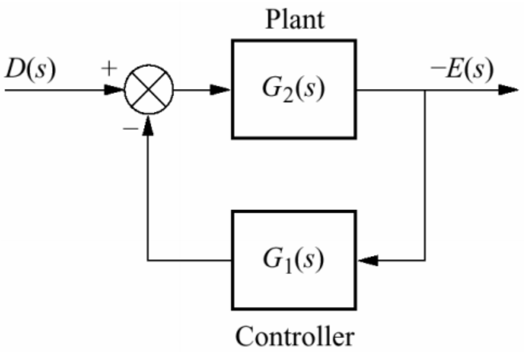
Final value theorem

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s)\right] \equiv e_R(\infty) + e_D(\infty)$$

where

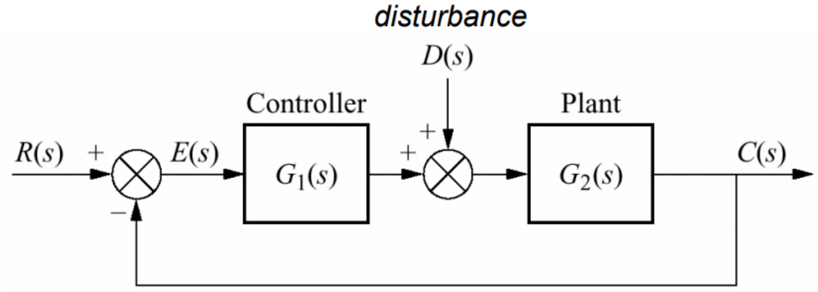
$$e_R(\infty) = \lim_{s\to 0}\left[\frac{sR(s)}{1+G_1(s)G_2(s)}\right]$$

$$e_D(\infty) = \lim_{s\to 0}\left[\frac{sG_2(s)D(s)}{1+G_1(s)G_2(s)}\right]$$



Equivalent block diagram
with $D(s)$ as input
And – $E(s)$ as output

Unit step Disturbance



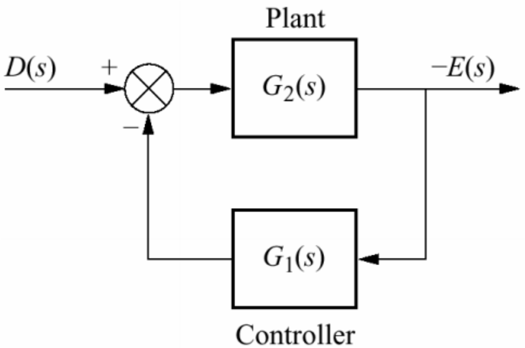
Special case: unit step disturbance $D(s)=\frac{1}{s}$

where

$$e_D(\infty)=-\lim_{s\to 0}\left[\frac{sG_2(s)\times\left(\frac{1}{s}\right)}{1+G_1(s)G_2(s)}\right]$$

$$e_D(\infty)=-\left[\frac{\lim_{s\to 0}G_2(s)}{1+\lim_{s\to 0}G_1(s)G_2(s)}\right]$$

$$e_D(\infty)=-\left[\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)}+\lim_{s\to 0}G_1(s)}\right]$$



Equivalent block diagram
with $D(s)$ as input
And – $E(s)$ as output

If K_1, K_2 are the gains of the controller and plant, respectively, then

$$e(\infty)\downarrow \quad \text{if} \quad K_1\uparrow \ \text{or} \ K_2\downarrow$$

Lecture #17

28.02.2022

ME 325 Control Systems
(3-0-0-6)

Instructor

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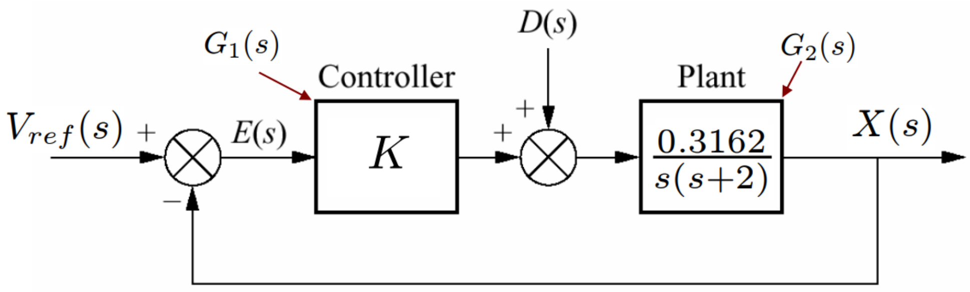
Unit step Disturbance: Example

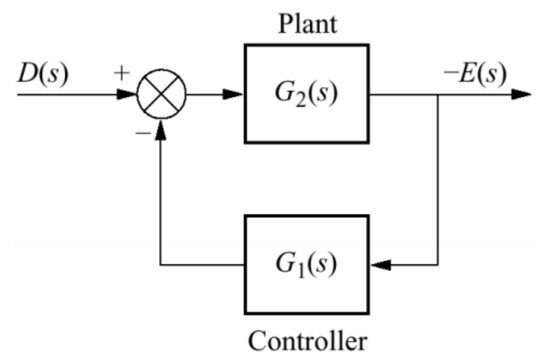


Subject to unit step disturbance $D(s) = \frac{1}{s}$

$$e_D(\infty) = -\left[\frac{1}{\frac{1}{\lim_{s\to 0} G_2(s)} + \lim_{s\to 0} G_1(s)}\right]$$

$$= -\left[\frac{1}{\lim_{s\to 0}\frac{s(s+2)}{0.3162} + \lim_{s\to 0} K}\right]$$

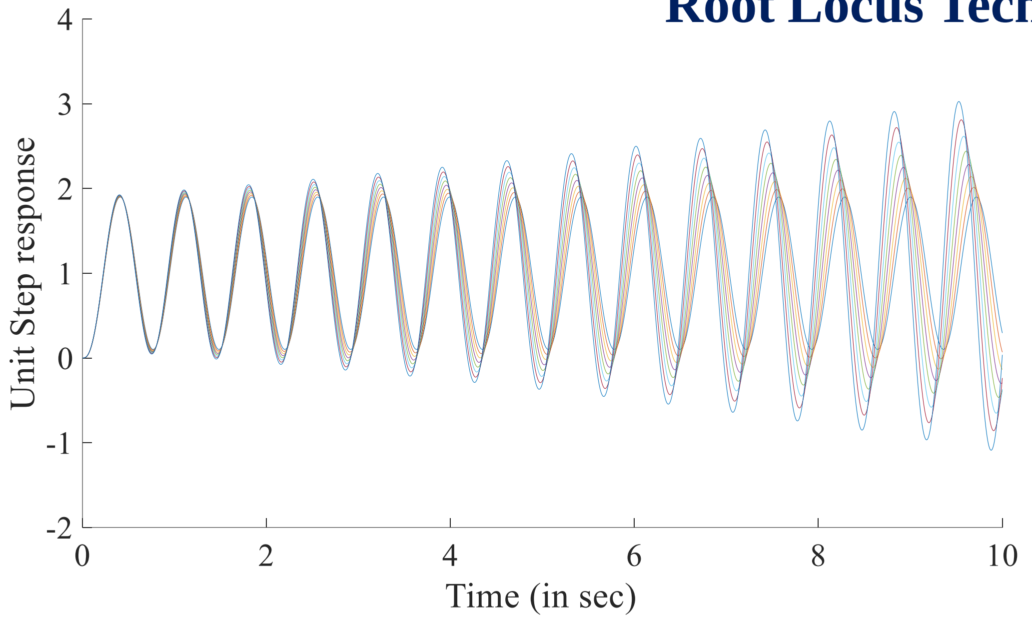
$$= -\frac{1}{K}$$

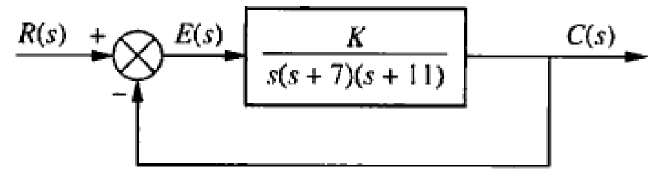


Equivalent block diagram
with $D(s)$ as input
And – $E(s)$ as output

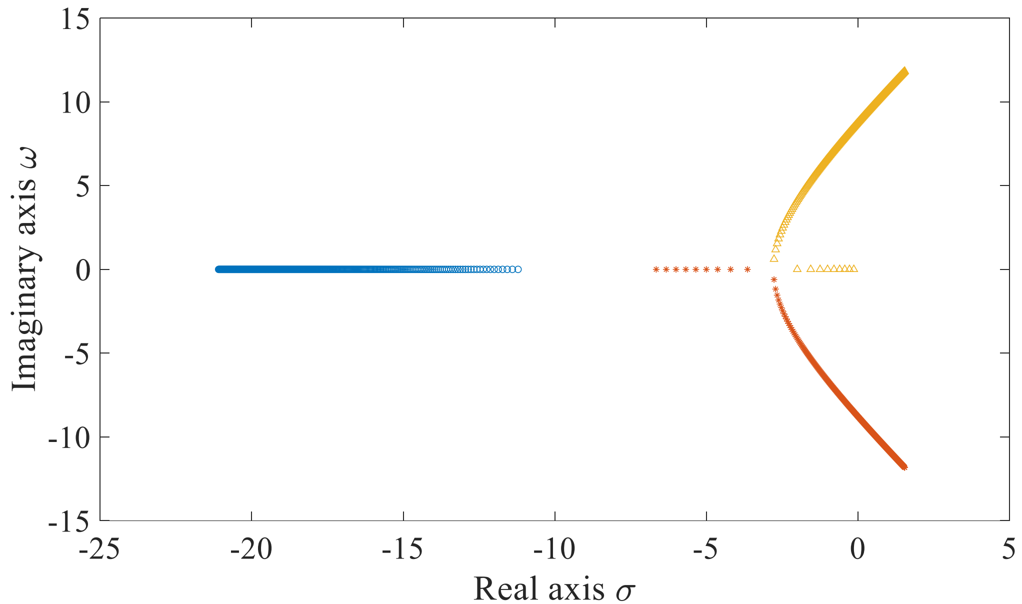
The steady-state error produced by the step disturbance is inversely proportional to the DC gain of $G_1(s)$.

Root Locus Technique





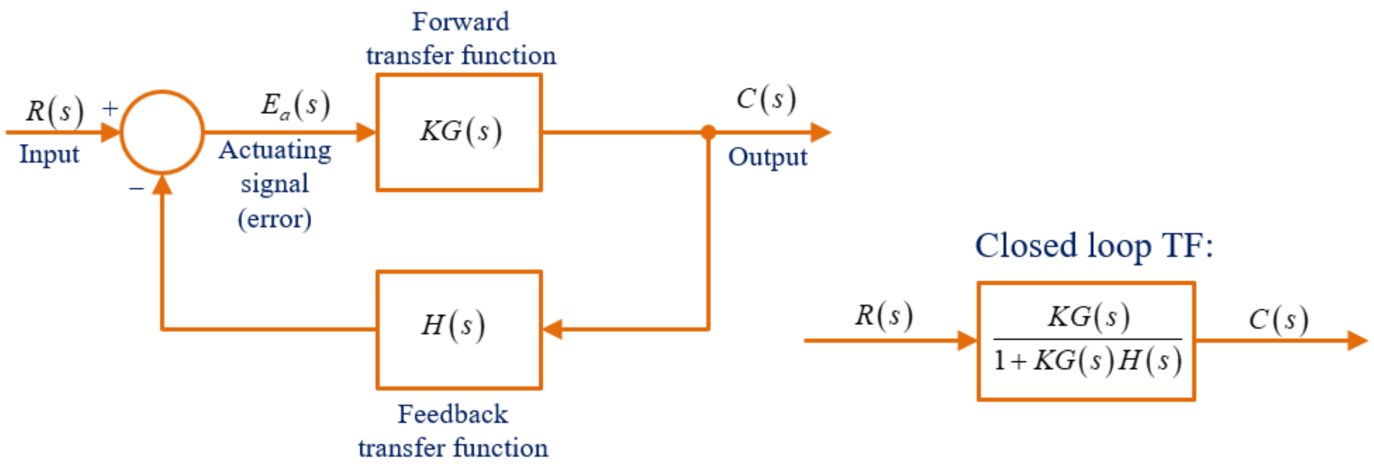
$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$



As K increases from 1386

Root Locus Technique

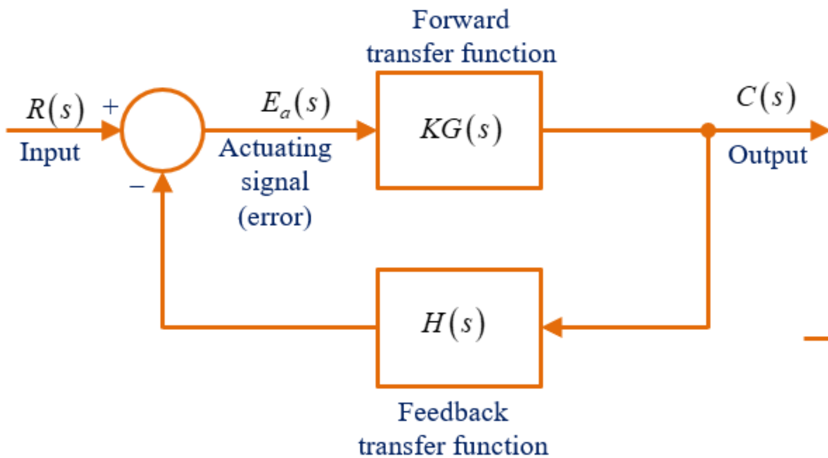
The root locus can be used to describe qualitatively the performance of a system as various parameters are changed. For example, the effect of varying gain upon percent overshoot, settling time, and peak time can be vividly displayed. The qualitative description can then be verified with quantitative analysis.



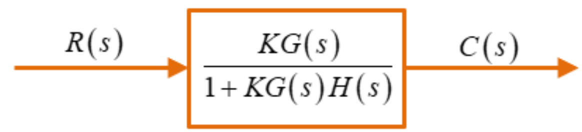
$$G(s)=\frac{N_G(s)}{D_G(s)} \qquad H(s)=\frac{N_H(s)}{D_H(s)} \qquad T(s)=\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}$$

where N and D are factored polynomials and signify numerator and denominator terms, respectively.

Root Locus Technique



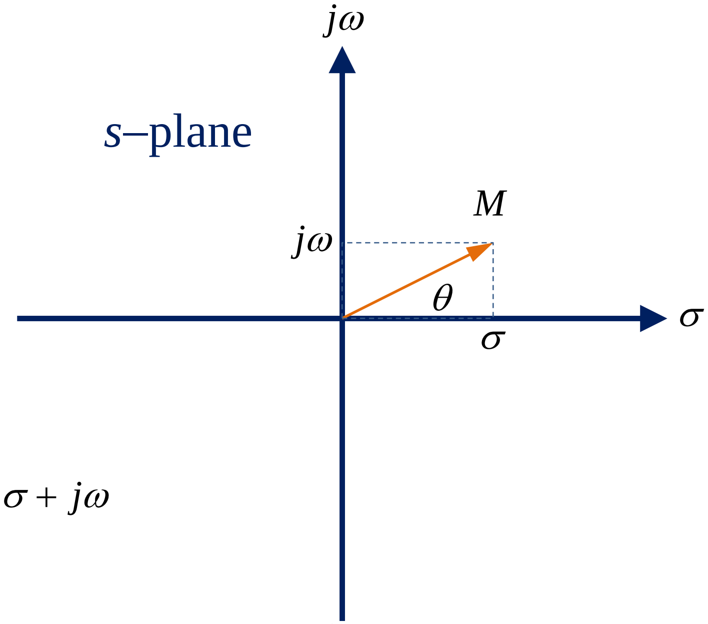
Closed loop TF:



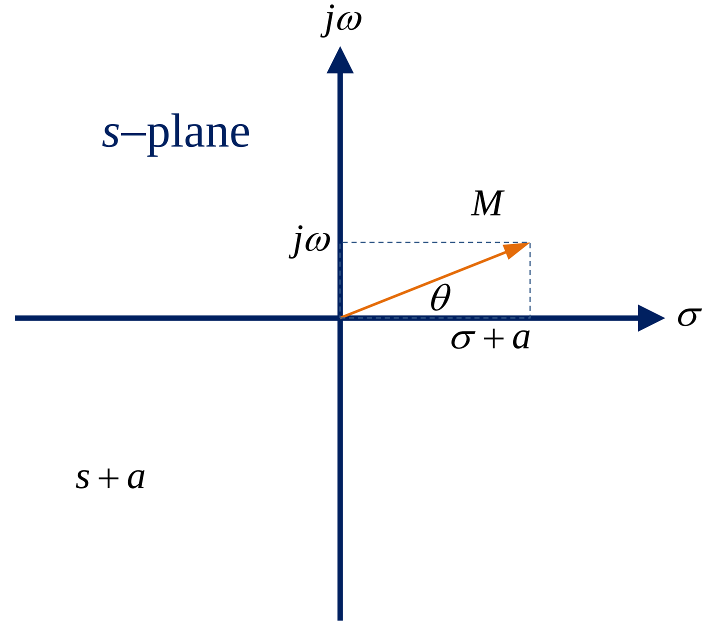
Zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$

$$T(s)=\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}$$

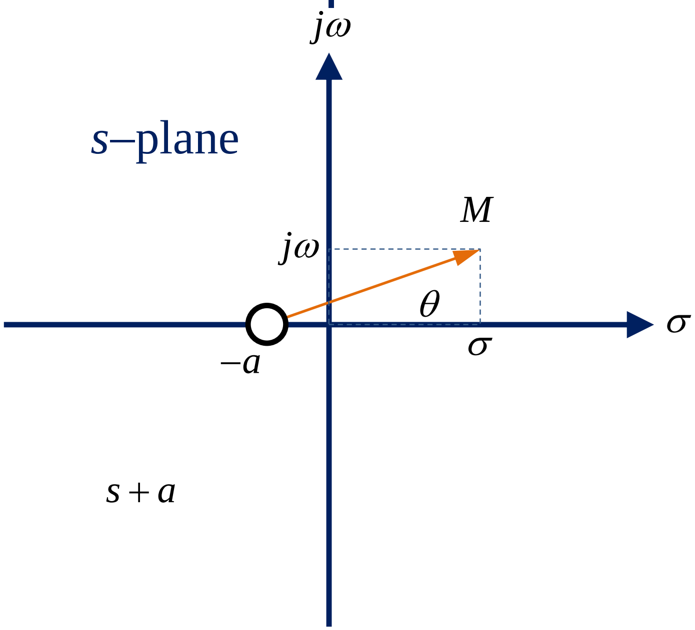
The poles $T(s)$ are not immediately known without factoring the denominator, and they are a function of K.



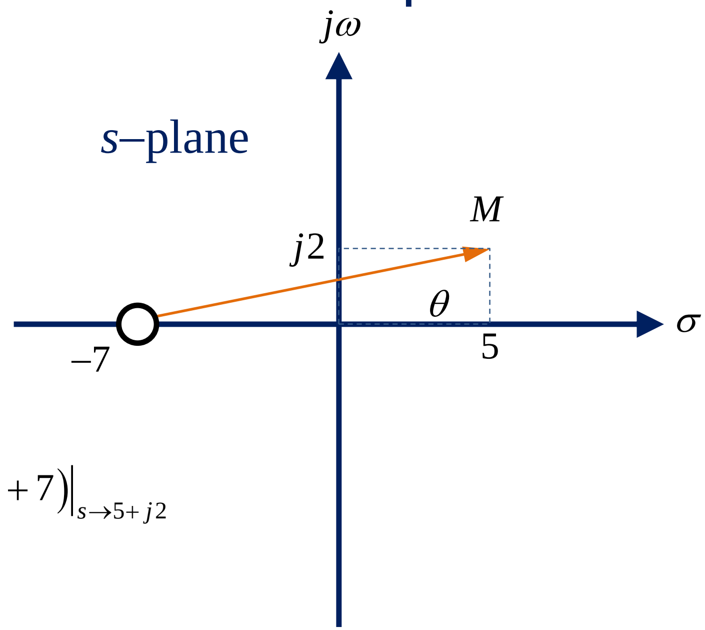
$s = \sigma + j\omega$



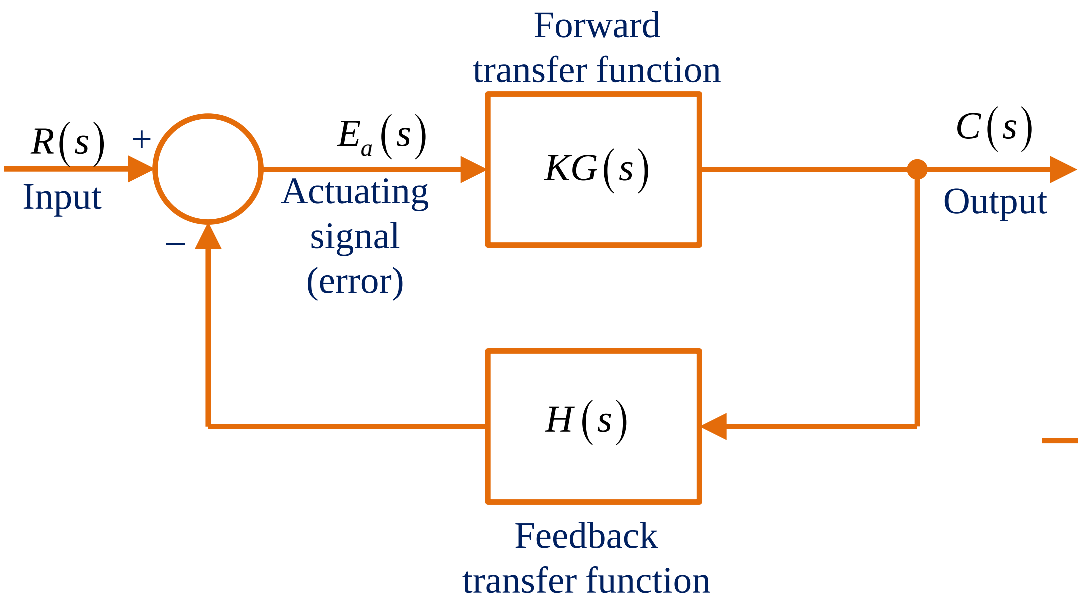
$s + a$



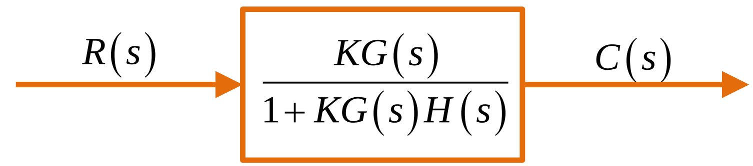
$s + a$



$(s+7)\big|_{s\to 5+j2}$



Closed loop TF:



Open loop TF:

$$KG(s)H(s)$$

Closed-loop pole locations

$$1+KG(s)H(s)=0 \Rightarrow \begin{cases} K = 1/|G(s)H(s)| \\ \angle KG(s)H(s) = (2n+1)180^0 \end{cases}$$

$$F(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{m}(s+p_j)}=\frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}}$$

$$M=\frac{\prod \text{zero length}}{\prod \text{pole length}}=\frac{\prod_{i=1}^{m}\left|(s+z_i)\right|}{\prod_{j=1}^{m}\left|(s+p_j)\right|}$$

where a zero length, $\left|(s+z_i)\right|$, is the magnitude of the vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole length, $\left|(s+p_i)\right|$ is the magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s.

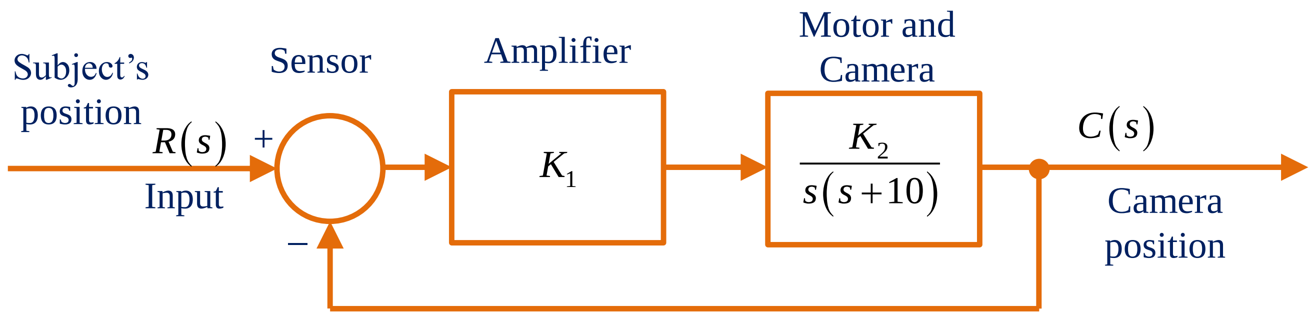
$$\theta=\sum \text{zero angles}-\sum \text{pole angles}$$
$$=\sum_{i=1}^{m}\angle(s+z_i)-\sum_{i=1}^{n}\angle(s+p_i)$$

where a zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of $F(s)$ at $-z_i$, to the point s, and a pole angle is the angle, measured from the positive extension of the real axis, of the vector drawn from the pole of $F(s)$ at $-p_i$ to the point s.

Given

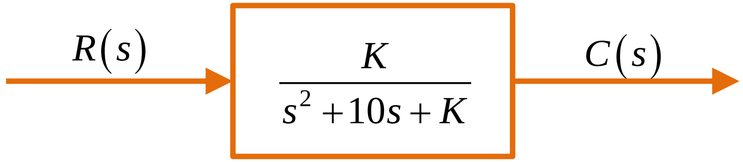
$$F(s)=\frac{(s+1)}{s(s+2)}$$

Find $F(s)$ at the point $s = -3 + j4$

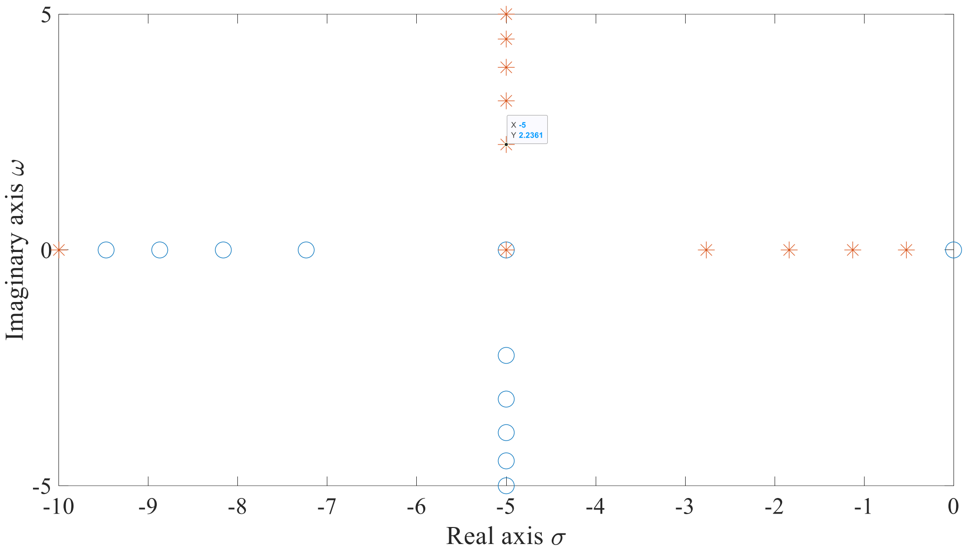


Block diagram of a security camera with auto tracking can used to follow moving objects automatically

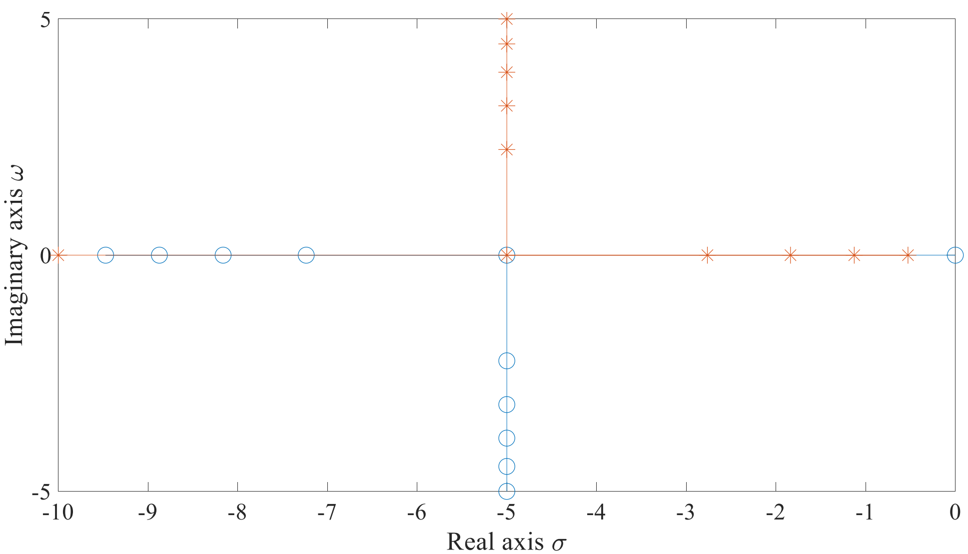
Closed loop TF:



where $K = K_1 K_2$



Pole plot



Root locus

Pole location as function of gain for the system

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

$K \geq 0$

Properties of Root Locus

$$T(s)=\frac{KG(s)}{1+KG(s)H(s)}$$

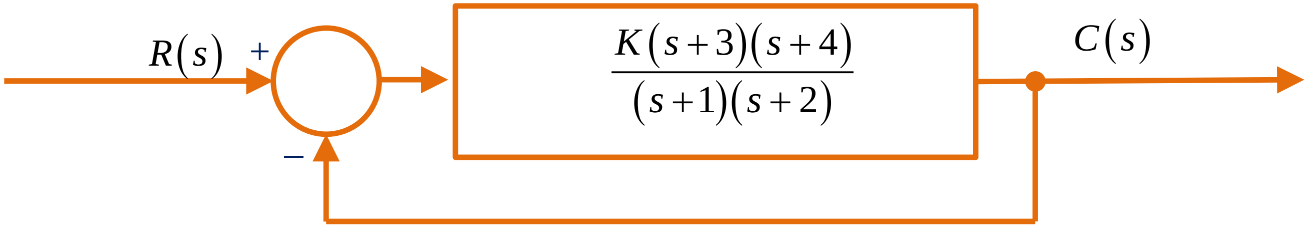
$$KG(s)H(s)=-1=1\angle(2k+1)180^0 \quad k=0,\pm1,\pm2,\pm3,\ldots$$

$$|KG(s)H(s)|=1\Rightarrow K=\frac{1}{|G(s)H(s)|}$$

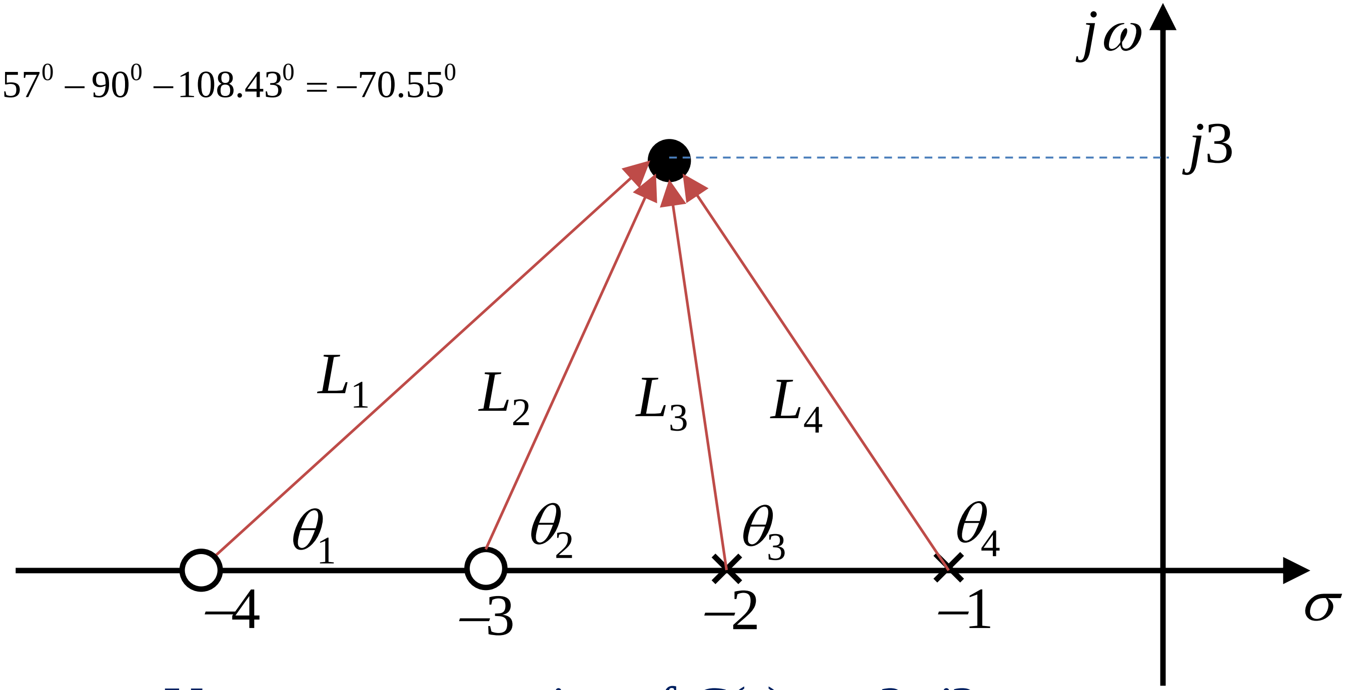
$$\angle KG(s)H(s)=(2k+1)180^0 \quad k=0,\pm1,\pm2,\pm3,\ldots$$

Example $$KG(s)H(s)=\frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

Closed loop TF $$T(s)=\frac{K(s+3)(s+4)}{(1+K)s^2+(3+7K)s+(2+12K)}$$



$$\theta_1+\theta_2-\theta_3-\theta_4=56.31^0+71.57^0-90^0-108.43^0=-70.55^0$$

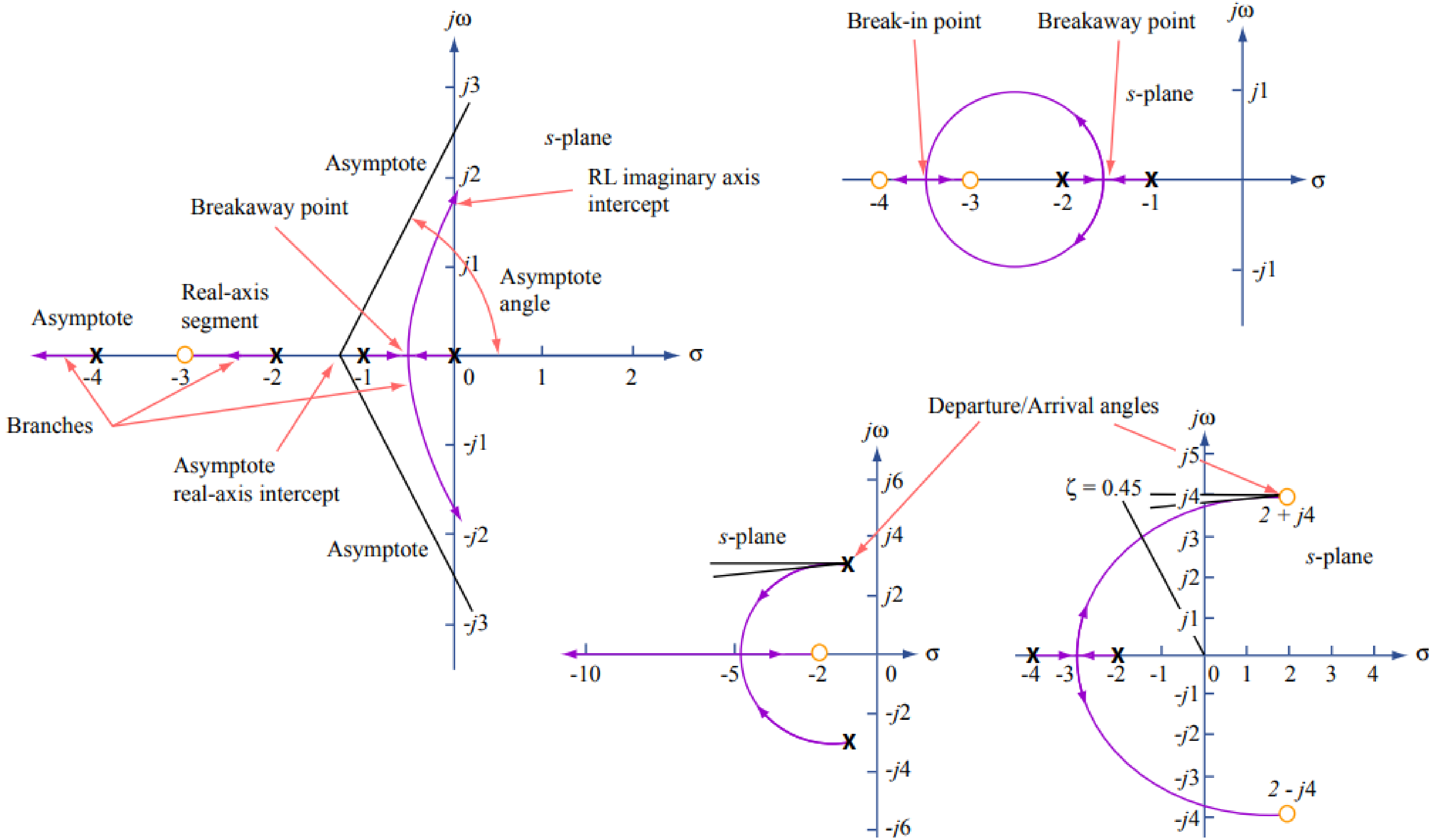


Vector representation of $G(s)$ at $-2+j3$

$$K=\frac{1}{|G(s)H(s)|}=\frac{1}{M}=\frac{\prod \text{pole length}}{\prod \text{zero length}}$$

$$K=\frac{L_3L_4}{L_1L_2}=0.33$$

Root Locus Terminology



Root-locus Sketching Rules

- **Rule 1**: # branches = # poles

  The number of branches of the root locus equals the number of closed-loop pole
- **Rule 2**: symmetrical about the real axis

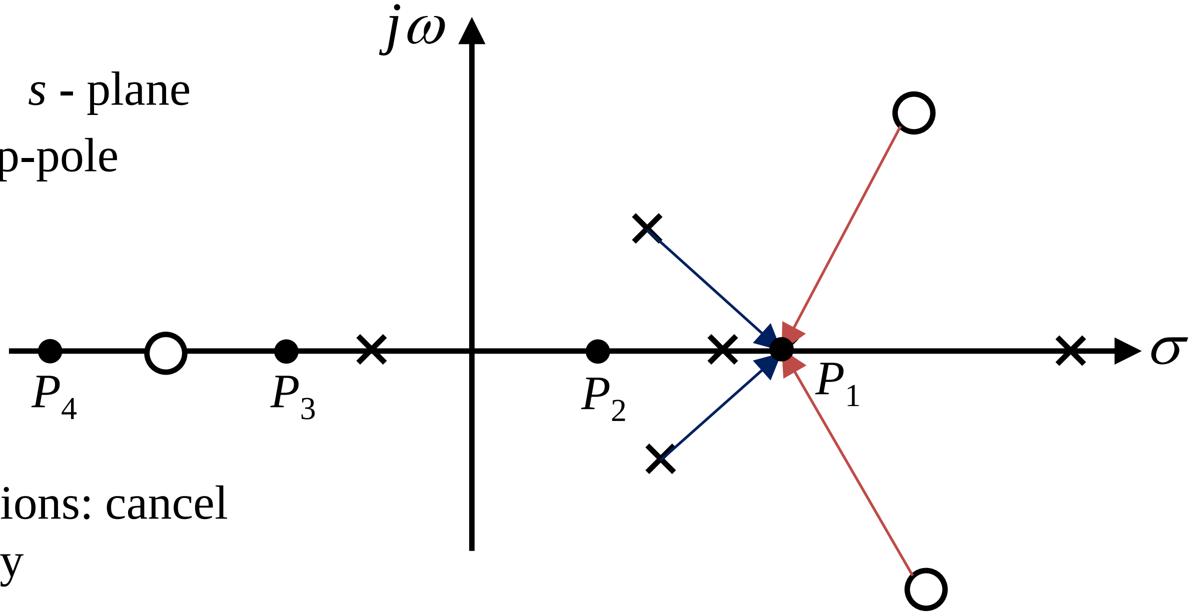
  The root locus is symmetrical about the real axis.
- **Rule 3**: On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real axis, finite open-loop poles and/or finite open-loop zeros.

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

Angle condition for closed loop-pole

$$\angle KG(s)H(s) = (2k+1)180^0$$

$$\angle G(s)H(s) = \sum \text{zeros} - \sum \text{poles}$$



Complex—pole/zero contributions: cancel because of symmetry

Real—pole/zero contributions: each is 0^0 from the left, 180^0 from the right; total contributions from right must be odd number of 180^0's to satisfy angle condition.

Root-locus Sketching Rules

- **Rule 4**: Starting and ending points:

  The root locus begins at the finite and infinite poles of G(s)H(s) and ends at the finite and infinite zeros of G(s)H(s).

$$G(s)=\frac{N_G(s)}{D_G(s)} \qquad H(s)=\frac{N_H(s)}{D_H(s)}$$

Closed loop TF: $$T(s)=\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}$$

If $K \to 0^+$ (small gain limit)

$$T(s)\approx\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+\varepsilon}$$

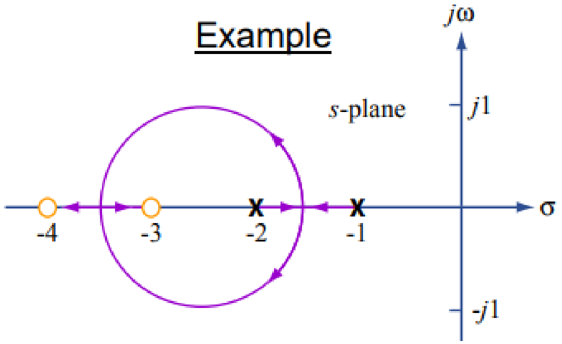
closed—loop denominator is denominator of $G(s)H(s)$ $\Rightarrow$closed—loop poles are the combined poles of $G(s)H(s)$. We conclude that the root locus begins at the poles of $G(s)H(s)$, the open-loop transfer function.

If $K \to +\infty$ (large gain limit)

$$T(s)\approx\frac{KN_G(s)D_H(s)}{\varepsilon+KN_G(s)N_H(s)}$$

closed—loop denominator is numerator of $G(s)H(s)$ $\Rightarrow$closed—loop poles are the combined zeros of $G(s)H(s)$. We conclude that the root locus ends at the zeros of $G(s)H(s)$, the open-loop transfer function.

<u>Example</u>



Root-locus Sketching Rules

- **Rule 5**: Behaviour at infinity: Asymptotes: angles and real-axis intercept

T(s) has a *zero at infinity* if $T(s \to \infty) \to 0$

T(s) has a *pole at infinity* if $T(s \to \infty) \to \infty$

Example:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Clearly, this open—loop transfer function has three poles, 0, −1, −2. It has no *finite* zeros. For large s, we can see that

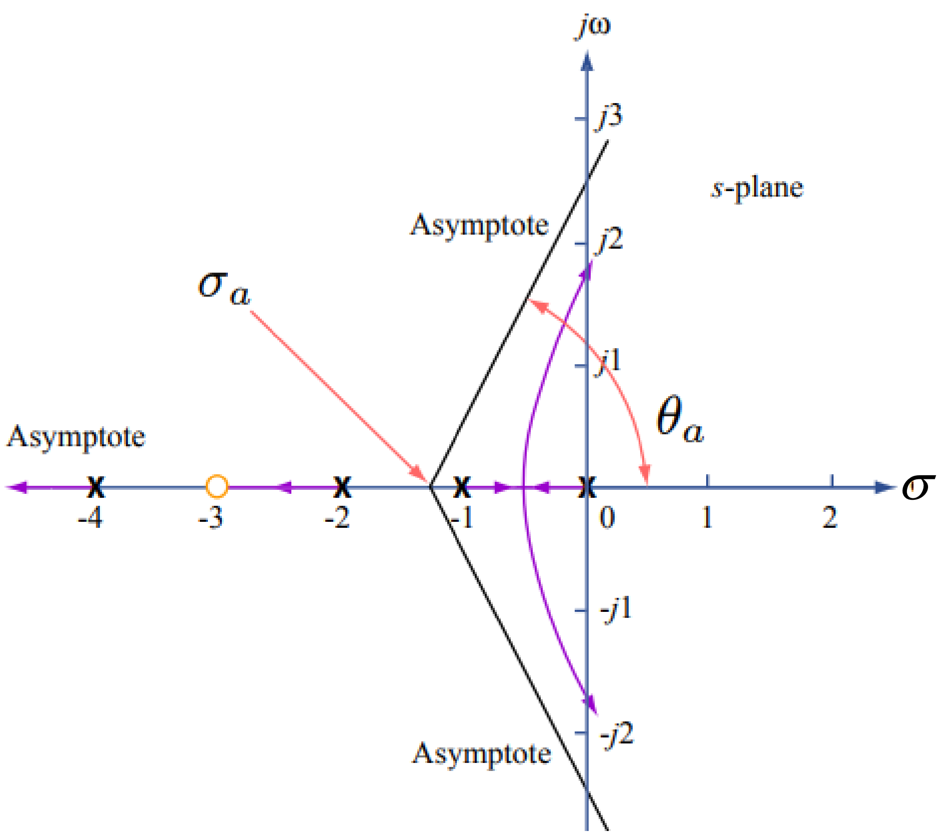
$$KG(s)H(s) \approx \frac{K}{s^3}$$

So this open—loop transfer function has **three** zeros at infinity.

Root-locus Sketching Rules

- **Rule 5**: Asymptotes: angles and real-axis intercept

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a *and angle,* θ_a *as follows:*



$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)180^0}{\#\text{finite poles} - \#\text{finite zeros}} \quad k = 0, \pm 1, \pm 2, \cdots$$

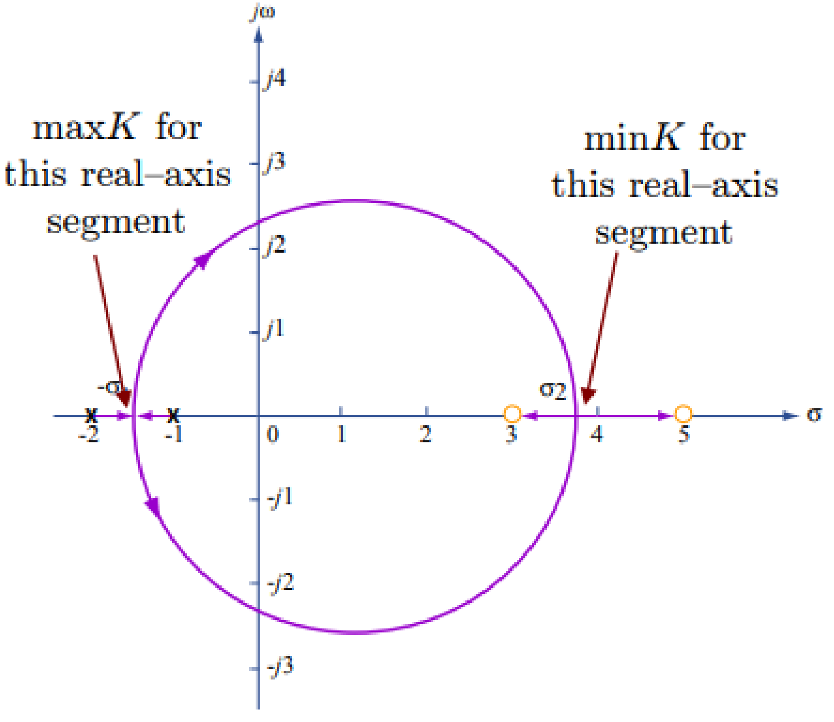
In this example,
poles = {0, –1, – 2, – 4},
zeros = {–1},

$$\sigma_a = \frac{[0+(-1)+(-2)+(-4)]-[(-3)]}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2k+1)180^0}{4-1} = \{60^0, 180^0, 300^0\}$$

Root-locus Sketching Rules

- **Rule 6**: Real axis break-in and breakaway points



In this example,

$$KG(s)H(s)=\frac{K(s-3)(s-5)}{(s+1)(s+2)}=\frac{K\left(s^2-8s+15\right)}{\left(s^2+3s+2\right)}$$

So on the real-axis segments we have

$$K(\sigma)=-\frac{(\sigma+1)(\sigma+2)}{(\sigma-3)(\sigma-5)}=-\frac{\left(\sigma^2+3\sigma+2\right)}{\left(\sigma^2-8\sigma+15\right)}$$

Taking the derivative,

$$\frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma}=-\frac{11\sigma^2-26\sigma-61}{\left(\sigma^2-8\sigma+15\right)^2}$$

and setting $\frac{\mathrm{d}K(\sigma)}{\mathrm{d}\sigma}=0$ we find

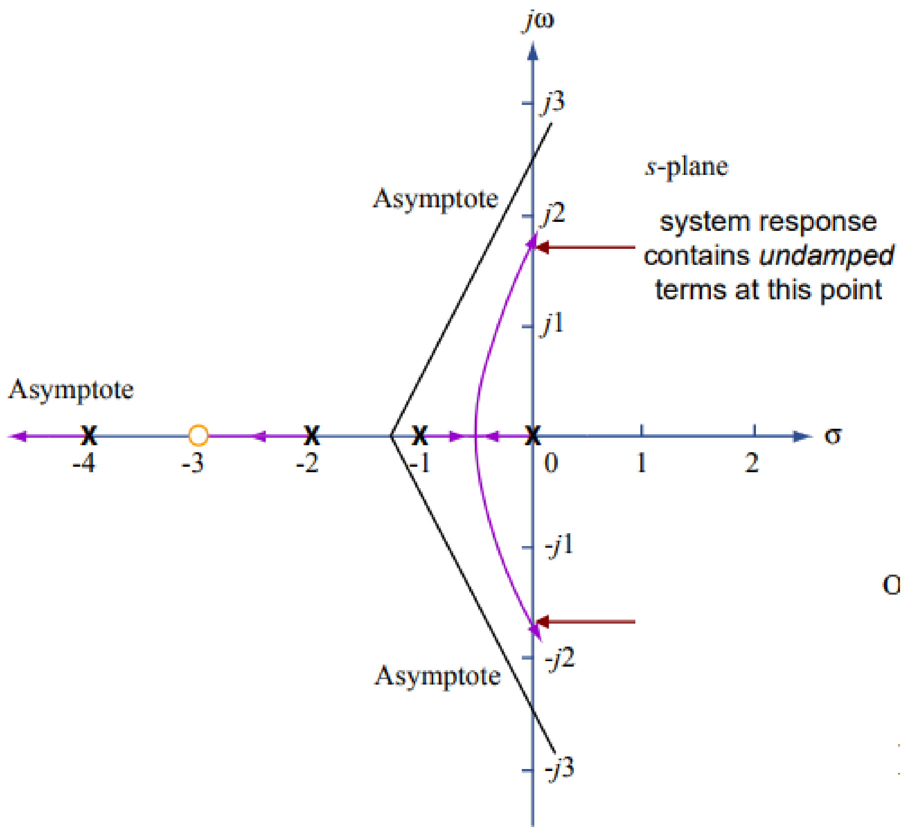
$$\sigma_1=-1.45,\quad \sigma_2=3.82$$

Alternatively, poles = {−1, −2}, zeros = {+3, +5} so we must solve

$$\frac{1}{(\sigma-3)}+\frac{1}{(\sigma-5)}=\frac{1}{(\sigma+1)}+\frac{1}{(\sigma+2)}\Rightarrow 11\sigma^2-26\sigma-61$$

Root-locus Sketching Rules

- **Rule 7**: Imaginary axis crossing



If $s = j\omega$ is a closed—loop pole on the imaginary axis, then

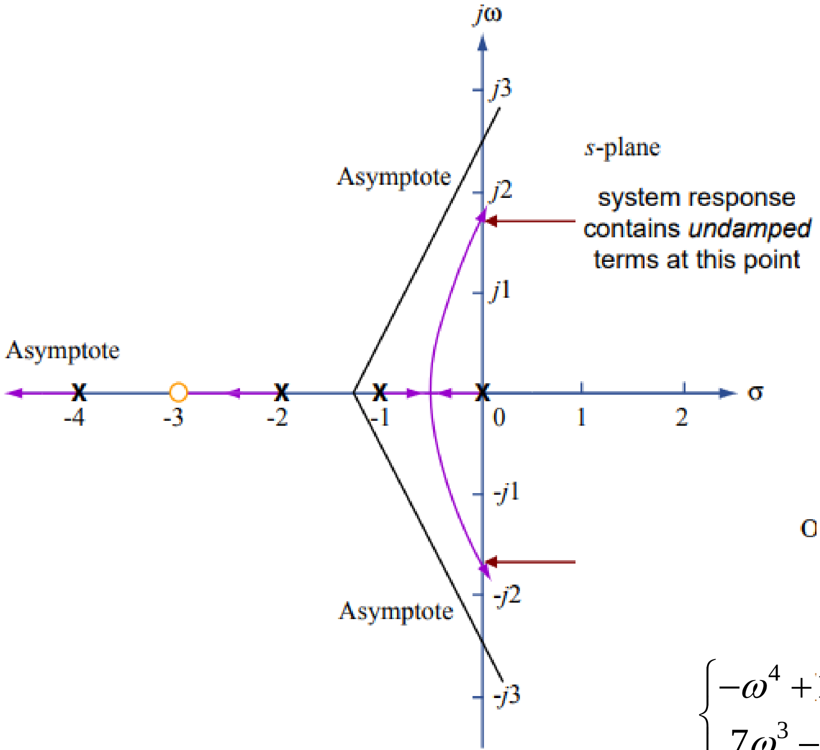
$$KG(j\omega)H(j\omega) = -1$$

The real and imaginary parts of the above equation provide us with a 2 × 2 system of equations, which we can solve for the two unknowns K and ω (i.e., the critical gain beyond which the system goes unstable, and the oscillation frequency at the critical gain.)

The Ruth— Hurwitz criterion can also be used for this purpose.

Root-locus Sketching Rules

- **Rule 7**: Imaginary axis crossing



In this example,

$$KG(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$

$$= \frac{Ks+3K}{s^4+7s^3+14s^2+8s} \Rightarrow$$

$$KG(j\omega)H(j\omega) = \frac{Kj\omega+3K}{\omega^4 - j7\omega^3 - 14\omega^2 + j8\omega}$$

$$KG(j\omega)H(j\omega) = -1$$

$$-\omega^4 + j7\omega^3 + 14\omega^2 - j(8+K)\omega - 3K = 0$$

$$\begin{cases} -\omega^4 + 14\omega^2 - 3K = 0 \\ 7\omega^3 - (8+K)\omega = 0 \end{cases}$$

From the second equation,

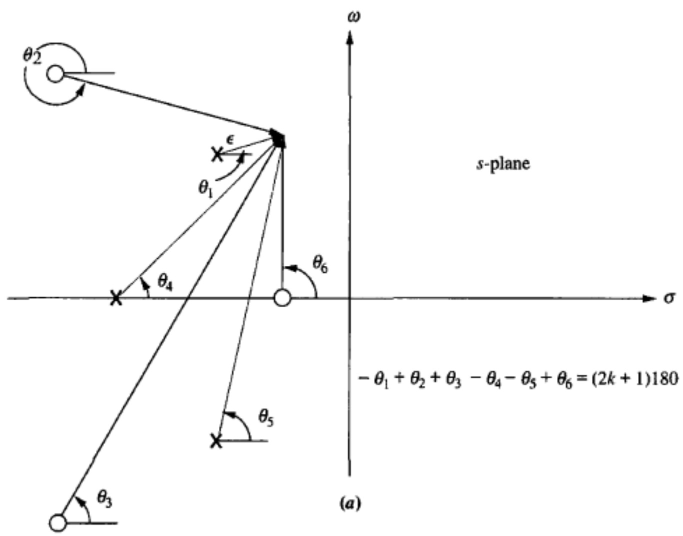
$$\omega^2 = \frac{(8+K)}{7}$$

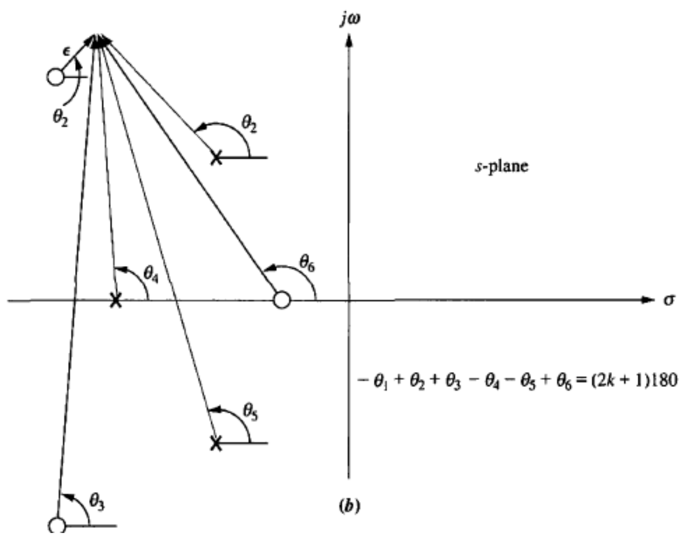
$$-\left(\frac{8+K}{7}\right)^2 + 14\left(\frac{8+K}{7}\right) - 3K = 0 \Rightarrow K^2 + 65K - 720 = 0$$

Of the solutions $K = -74.65$, $K = 9.65$ we can discard the negative one

Thus $K = 9.65$ and $\omega = 1.59$

Angles of Departure and Arrival





Angles of Departure

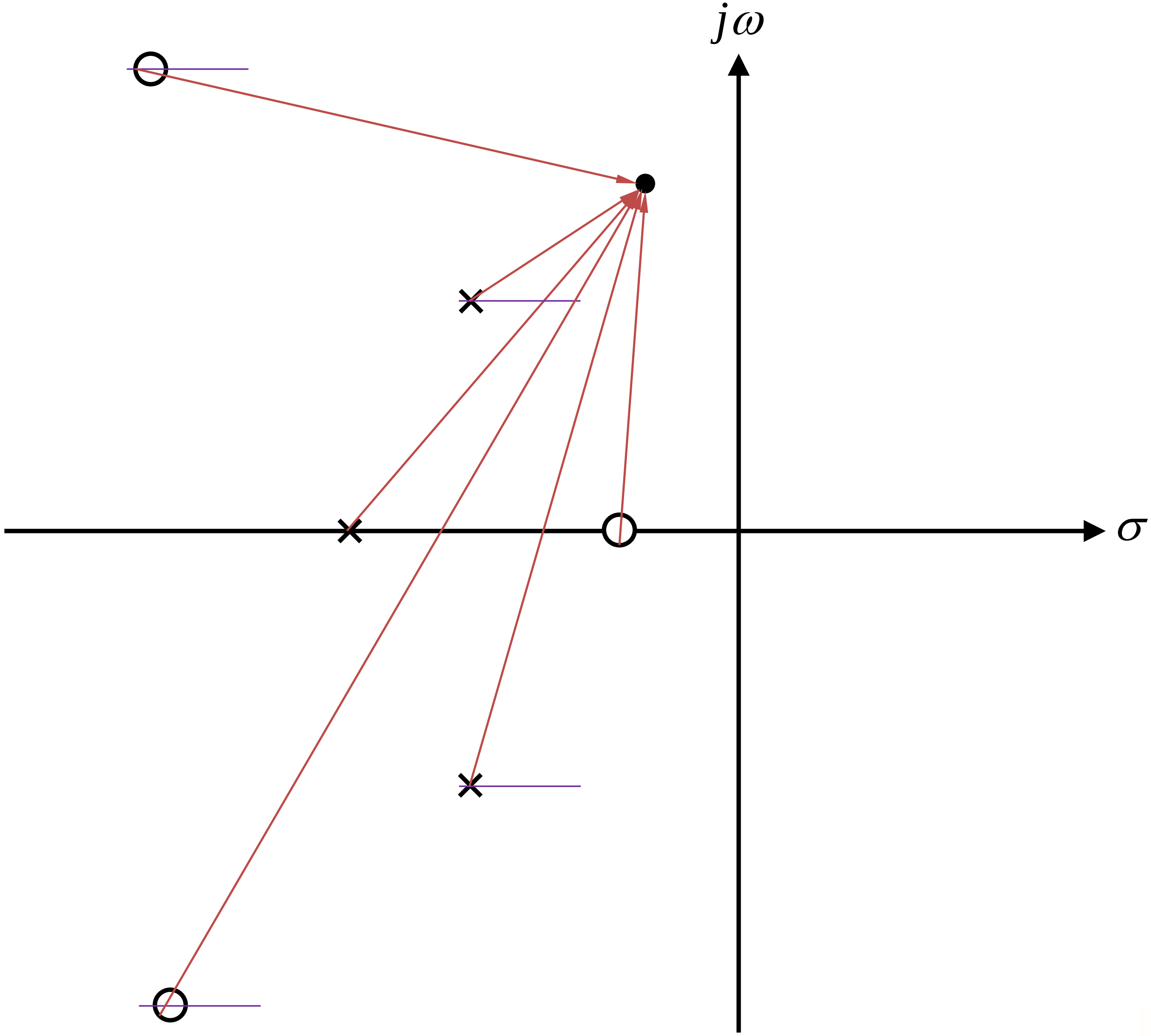
$$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^0$$

$$\theta_1=\theta_2+\theta_3-\theta_4-\theta_5+\theta_6-(2k+1)180^0$$

Angles of Arrival

$$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^0$$

$$\theta_2=\theta_1-\theta_3+\theta_4+\theta_5-\theta_6+(2k+1)180^0$$


$j\omega$
σ

Sketch the root locus plot for the open-loop system $KG(s) = \frac{K}{(s+1)(s+2)(s+4)}$

and find the gain K at which it becomes unstable.

1. Determine and plot the open-loop poles and zeros.
2. Determine and plot the regions of the real axis that lie on the root locus.
3. Determine the number of asymptotes.

   There are no finite zeros, therefore $n - m = 3$.
4. Determine the asymptote angles and centroid, then sketch the asymptotes. For three asymptotes the angles are 60^0, 180^0, 300^0.

   The centroid is

$$\sigma_a = \frac{((\text{sum of the poles}) - (\text{sum of the zeros}))}{(n-m)}$$

$$= \frac{1}{3}((-1-2-4)-(0)) = -\frac{7}{3}$$

These steps were used to produce the following sketch

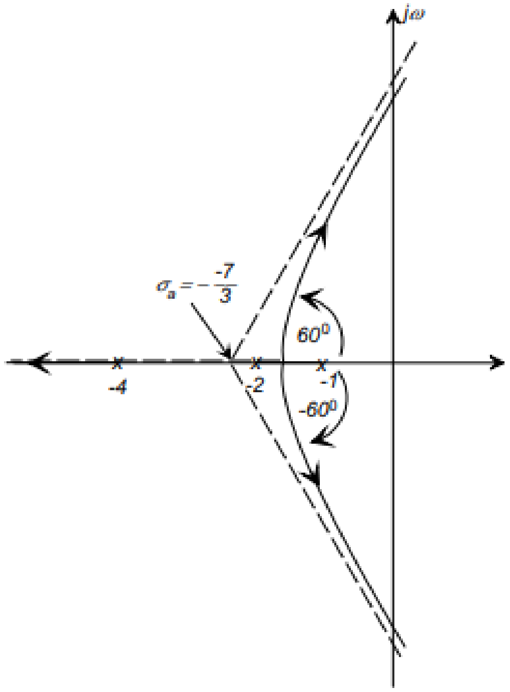
The closed-loop characteristic equation is:

$(s + 1)(s + 2)(s + 4) + K = s^3 + 7s^2 + 14s + 8 + K = 0$

and at the point of marginal stability (when $s = j\omega$)

$-j\omega^3 - 7\omega^2 + j14\omega + 8 + K = 0 + j0$

Equating the real and imaginary parts



$-7\omega^2 + 8 + K = 0$

$-\omega^3 + 14\omega = 0$

giving $\omega = 0, \sqrt{14}$

$K = -8, 90$

Since the root locus is defined only for $K > 0$ we conclude that the system will become unstable for $K > 90$, and the locus will cross the imaginary axis at $s = \pm j\sqrt{14}$ rad/s

Show the effect of PD control on the root-locus of the previous example.

Let $G_c(s) = K_p + K_d s = K(s + b)$, where $b = K_p/K_d$ and $K = K_d$. The PD controller has added a zero at $s = -b$ to the system.

The open-loop transfer function is now

$$KG(s) = \frac{K(s+b)}{(s+1)(s+2)(s+4)}$$

and we have $n - m = 2$. Assume for now that $b = 3$. There will be $n - m = 2$ asymptotes, at angles 90^0 and 270^0.

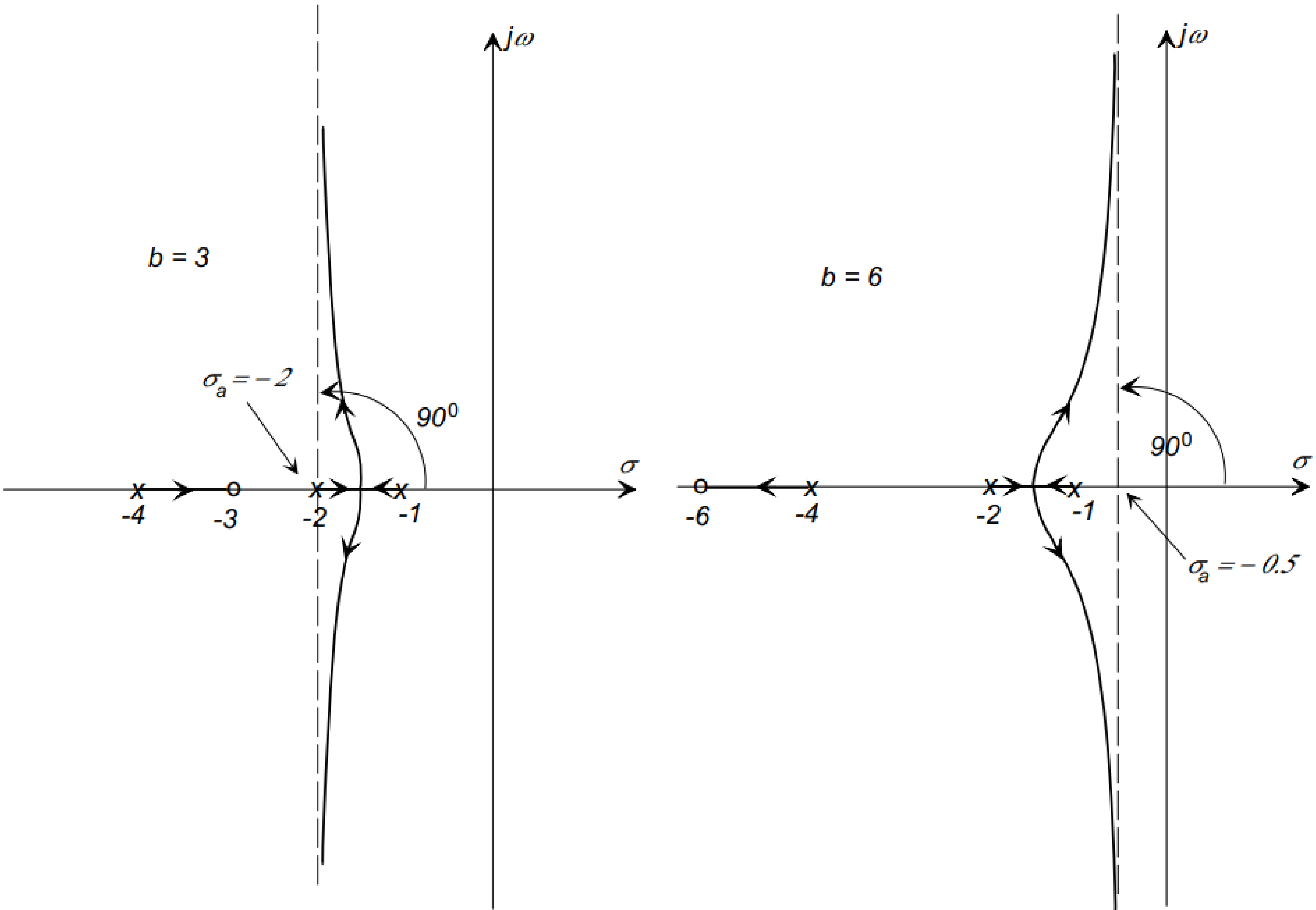
The centroid will be

$$\sigma_a = \frac{1}{2}\left((-1-2-4)-(-3)\right) = -2$$

The centroid will be

$$\sigma_a = \frac{1}{2}\left((-1-2-4)-(-6)\right) = -0.5$$

These two cases are sketched below



(a) P-D zero at *s*= -3 (b) P-D zero at *s*= -6

Notice that as the PD zero moves deeper into the left half plane, it moves the asymptote toward the imaginary axis, meaning that the dominant closed-loop poles become lightly damped.

What will happen if $b > 7$?

where the p_i are the $n-m$ uncancelled poles. With this approximation

$$s^{n-m} = -K, \quad \text{or} \quad s = K^{1/(n-m)}(-1)^{1/(n-m)}$$

The $n-m$ roots of -1 are complex with values

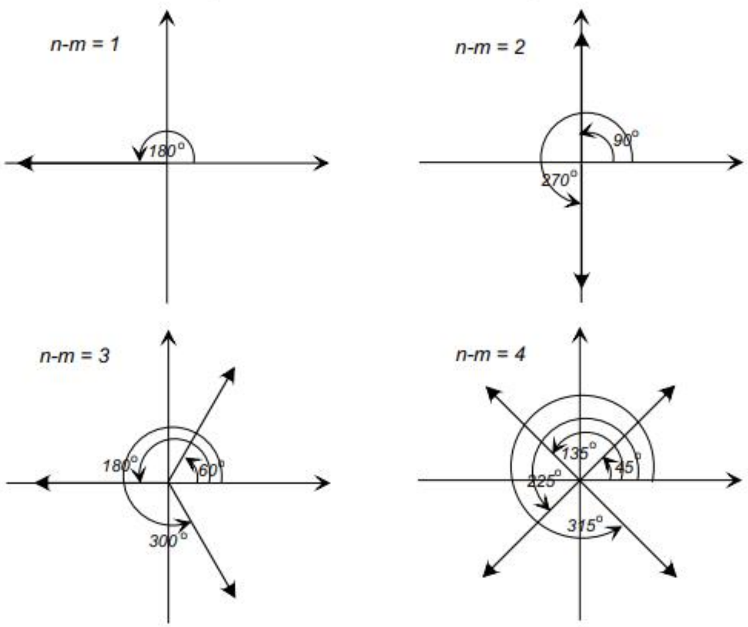
$$s_k = e^{j(2k+1)\pi/(n-m)} \qquad k = 0, 1, \ldots n-m-1$$

that is, they lie equally spaced around the unit circle at angles

$$\theta_k = \frac{(2k+1)\pi}{n-m}, \qquad k = 0, 1, \ldots n-m-1.$$

and as K becomes large, the $n-m$ closed-loop poles approach a set of radial asymptotes at these angles. The asymptotic angles are summarized in the following table:

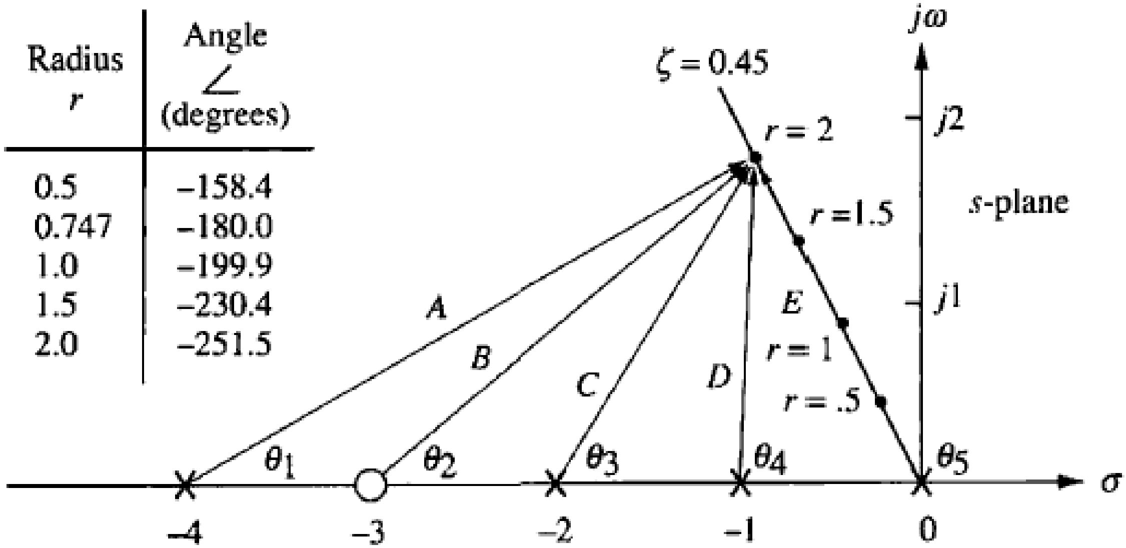
$n-m$	Asymptote Angles
1	180°
2	90°, 270°
3	60°, 180°, 300°
4	45°, 135°, 225°, 315°



As the gain K becomes large, $n-m$ branches of the root locus diverge away from the origin and approach $n-m$ radial asymptotes, at angles $\theta_k = (2k+1)\pi/(n-m)$, for $k = 0 \ldots (n-m-1)$.

Angles of Departure and Arrival

Radius r	Angle $\angle$ (degrees)
0.5	−158.4
0.747	−180.0
1.0	−199.9
1.5	−230.4
2.0	−251.5



$$-\theta_1+\theta_2-\theta_3-\theta_4-\theta_5=-251.5^0$$

$$K=\frac{|A||C||D||E|}{|B|}=1.71$$

In summary, we search a given line for the point yielding a summation of angles (zero angles-pole angles) equal to an odd multiple of 180°. The gain at that point is then found by multiplying the pole lengths drawn to that point and dividing by the product of the zero lengths drawn to that point.

Damping ratio and pole location

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \qquad 0 < \xi < 1$$

$$\sigma_d = \xi\omega_n, \qquad \omega_d = \omega_n\sqrt{1-\xi^2}, \qquad \tan\phi = \frac{\xi}{\sqrt{1-\xi^2}}$$

We can rewrite the step response as

$$1 - \frac{1}{\sqrt{1-\xi^2}} \times e^{-\sigma_d t} \times \cos\left(\omega_d t - \phi\right)$$

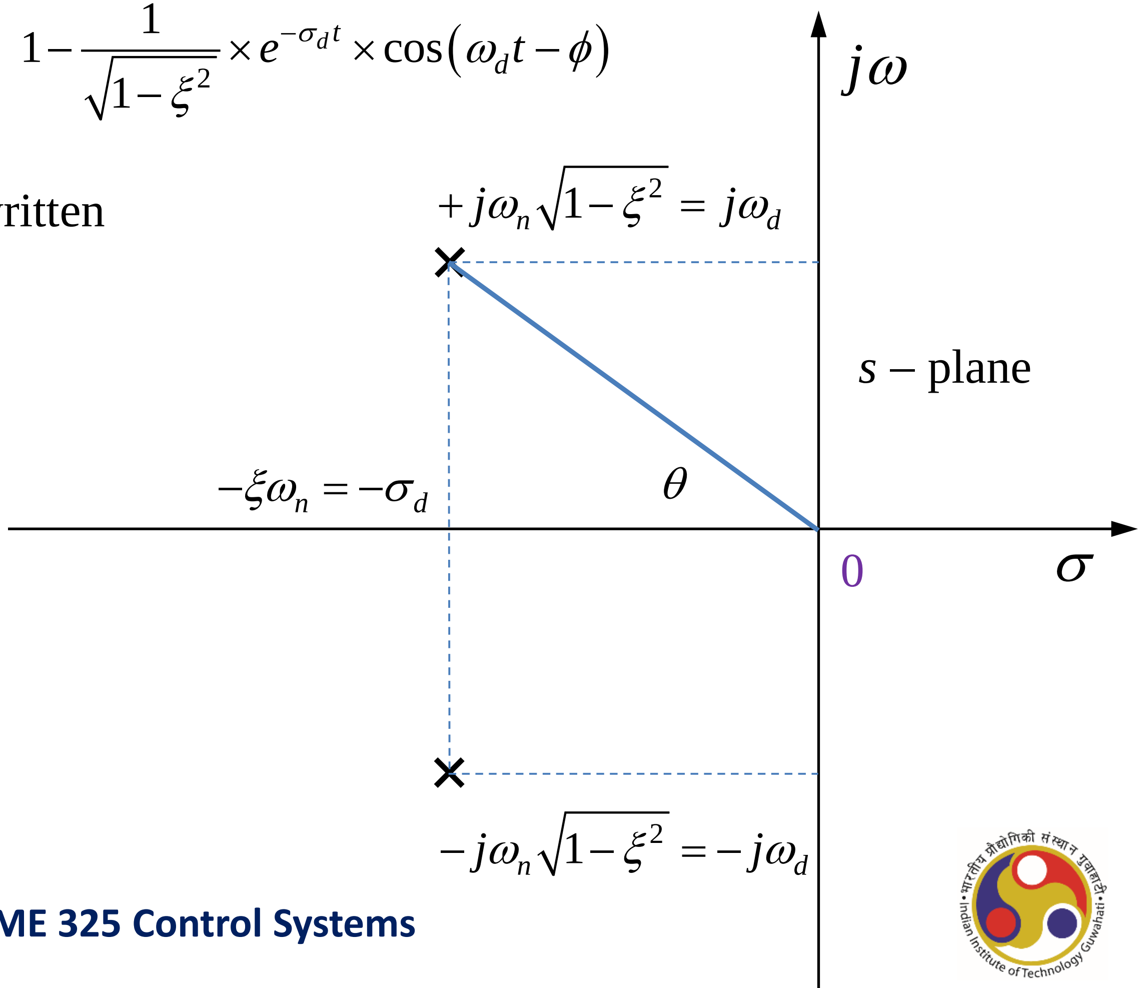
The definition above can be re-written

$$\xi = \frac{\sigma_d}{\omega_n}$$

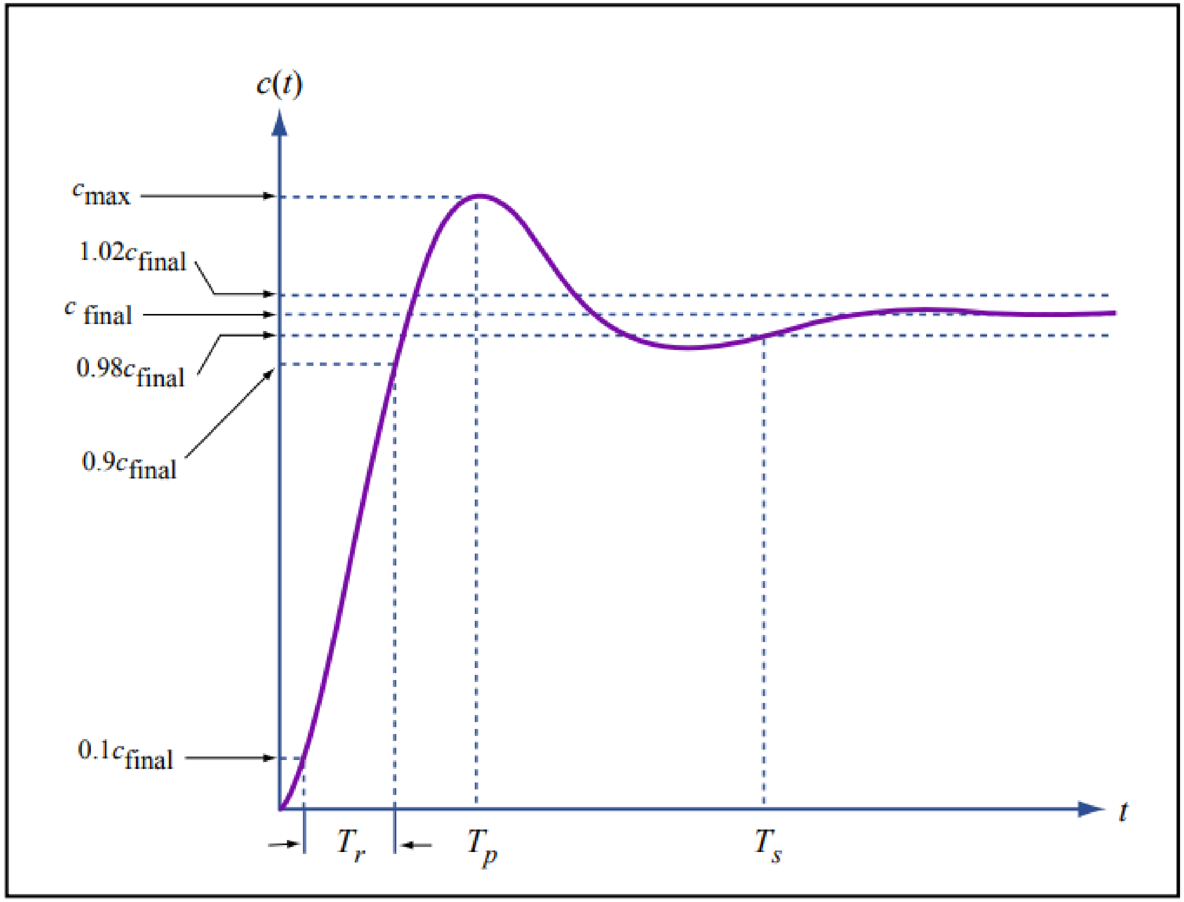
$$\sqrt{1-\xi^2} = \frac{\omega_d}{\omega_n}$$

$$\tan\theta = \frac{\omega_d}{\sigma_d} = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\cos\theta = \xi$$



Transient response and pole location



Peak time

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Percent overshoot (%OS)

$$\%OS = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)\times 100$$

$$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$$

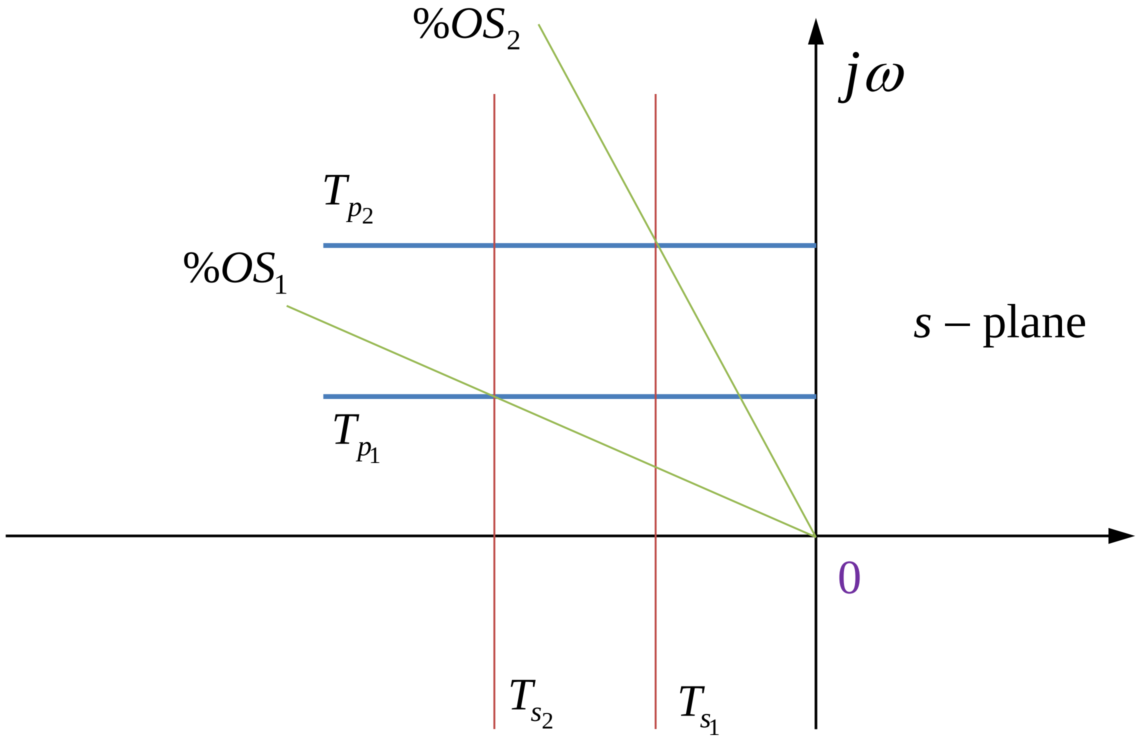
Settling time

(to within ± 2% of steady state)

$$T_s = -\frac{\ln\left(0.02\sqrt{1-\xi^2}\right)}{\xi\omega_n} \approx \frac{4}{\xi\omega_n}$$

(approximation valid for $0 < \xi < 0.9$)

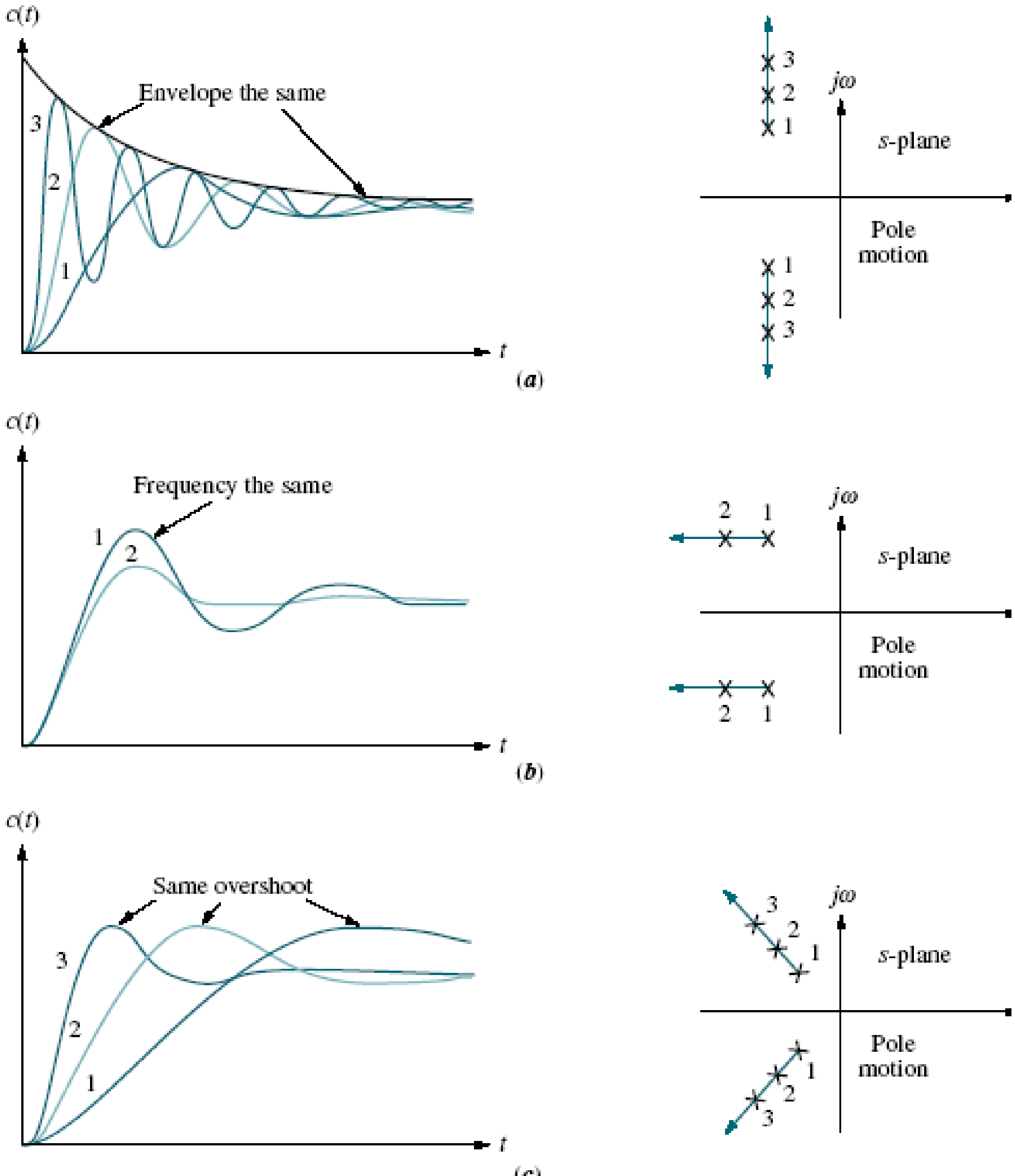
$$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$$



T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the s–plane are lines of constant imaginary value, they are also lines of constant peak time

Settling time is inversely proportional to the real part of the pole. Since vertical lines on the s – plane are lines of constant real value, they are also lines of constant settling time.

Since $\xi = \cos\,\theta$, radial lines are lines of constant ξ. Since percent overshoot is only a function of ξ, radial lines are thus lines of constant percent overshoot, $\%OS$.



Step responses of second-order underdamped systems as poles move: (a) with constant real part; (b) with constant imaginary part; (c) with constant damping ratio

Achieving s desired transient with a given Root Locus

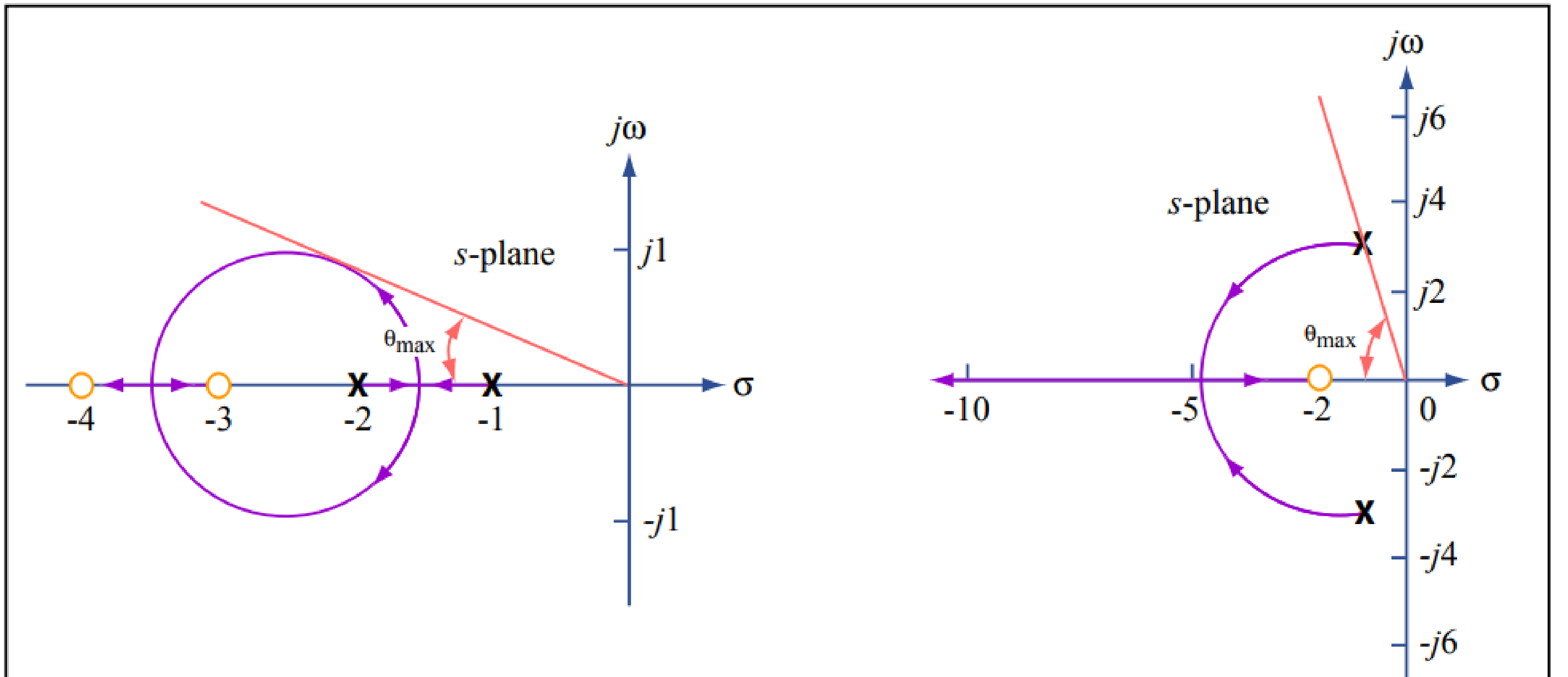


Figure 8.10

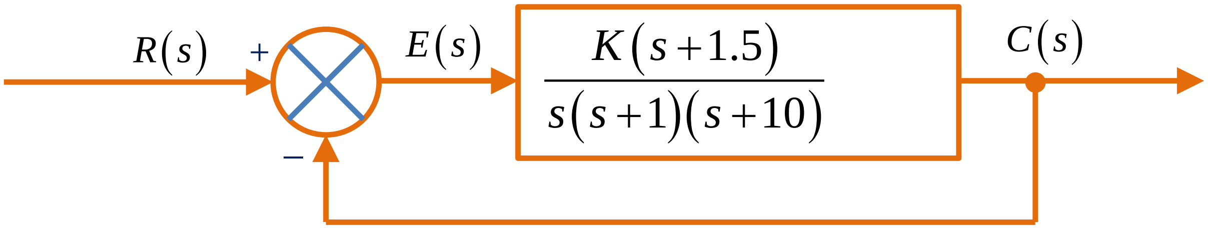
Figure 8.25

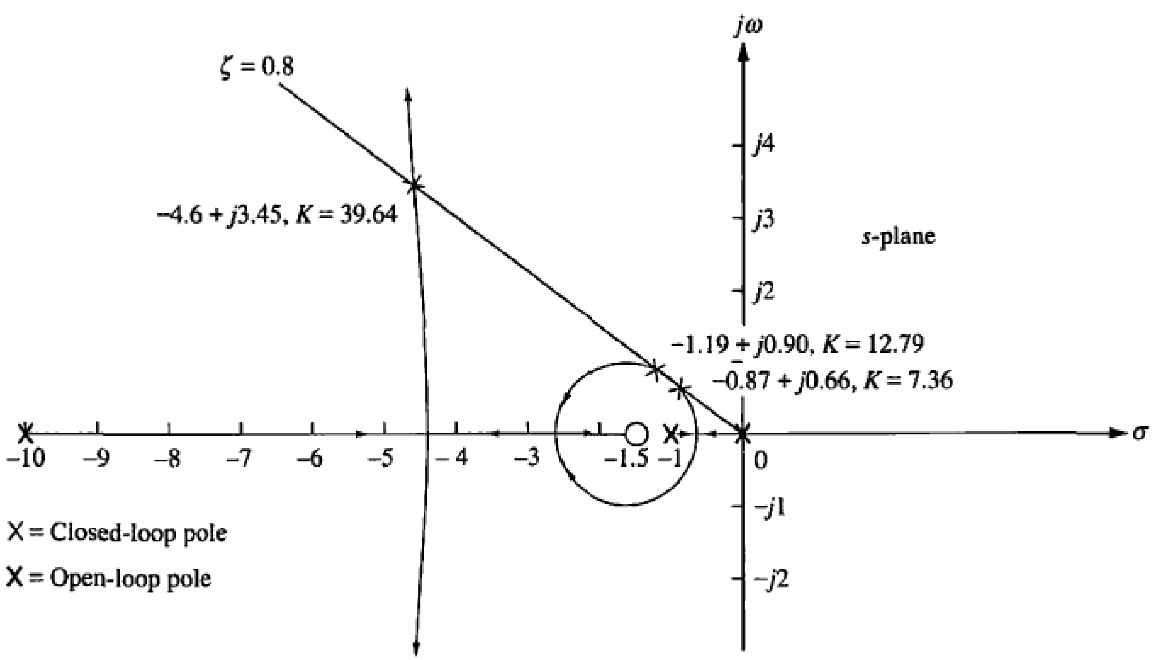
As $\xi\uparrow \Leftrightarrow \theta\downarrow$

- Rise time $T_r \uparrow$ (slower)
- Settling time $T_s \uparrow$ (slower)
- Peak time $T_p \uparrow$ (slower)
- Overshoot %OS $\downarrow$ (smaller)

If the given root locus does not allow the desired transient characteristics to be achieved, then we must modify the root locus by adding poles/zeros (compensator design)

Consider the system shown. Design the value of gain, K, to yield 1.52% overshoot. Also estimate the settling time, peak time, and steady-state error.





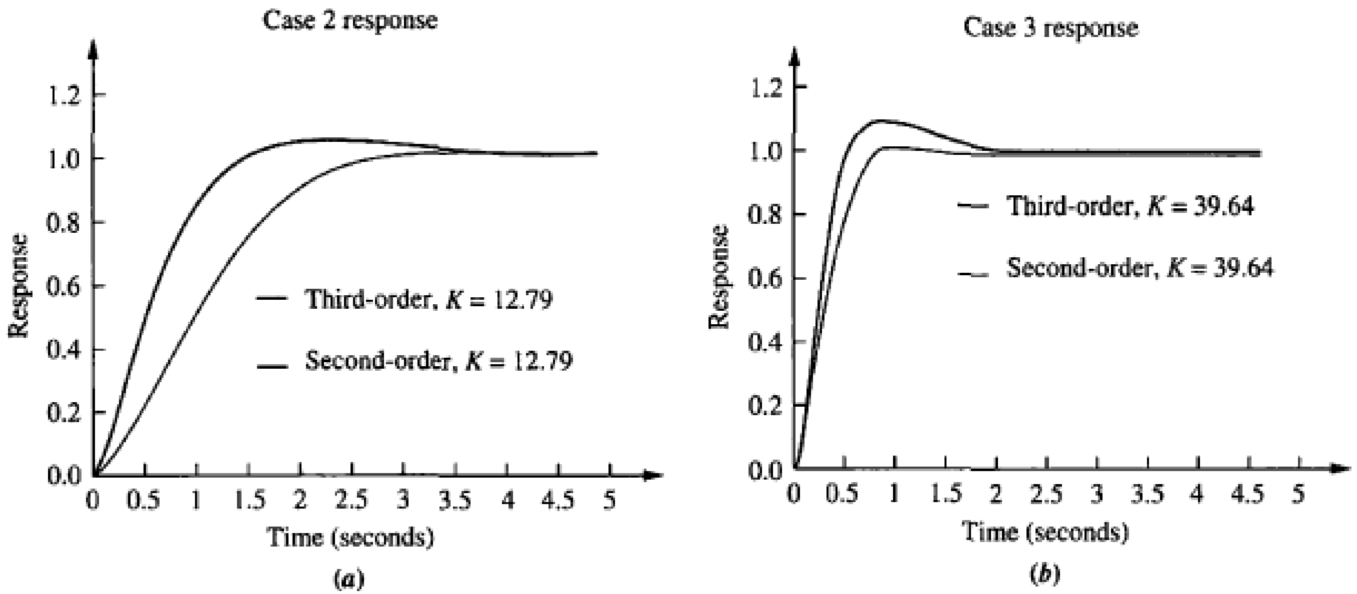
$$T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$T_s = \frac{4}{\xi \omega_n}$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{K(1.5)}{(1)(10)}$$

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.875 \pm j0.657$	$-1.5 + j0$	7.35	-9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.900$	$-1.5 + j0$	12.79	-8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.450$	$-1.5 + j0$	39.64	-1.80	0.87	0.91	5.9



Let us now look at the conditions that would have to exist in order to approximate the behavior of a three-pole system as that of a two-pole system. Consider a three-pole system with complex poles and a third pole on the real axis. Assuming that the complex poles are at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ and the real pole is at $-\alpha_r$, the step response of the system can be determined from a partial-fraction expansion. Thus, the output transform is

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r} \tag{4.57}$$

or, in the time domain,

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t} \tag{4.58}$$

The component parts of $c(t)$ are shown in Figure 4.23 for three cases of α_r. For Case I, $\alpha_r = \alpha_{r_1}$ and is not much larger than $\zeta\omega_n$; for Case II, $\alpha_r = \alpha_{r_2}$ and is much larger than $\zeta\omega_n$; and for Case III, $\alpha_r = \infty$.

Let us direct our attention to Eq. (4.58) and Figure 4.23. If $\alpha_r \gg \zeta\omega_n$ (Case II), the pure exponential will die out much more rapidly than the second-order underdamped step response. If the pure exponential term decays to an insignificant value at the time of the first overshoot, such parameters as percent overshoot, settling time, and peak time will be generated by the second-order underdamped step response component. Thus, the total response will approach that of a pure second-order system (Case III).

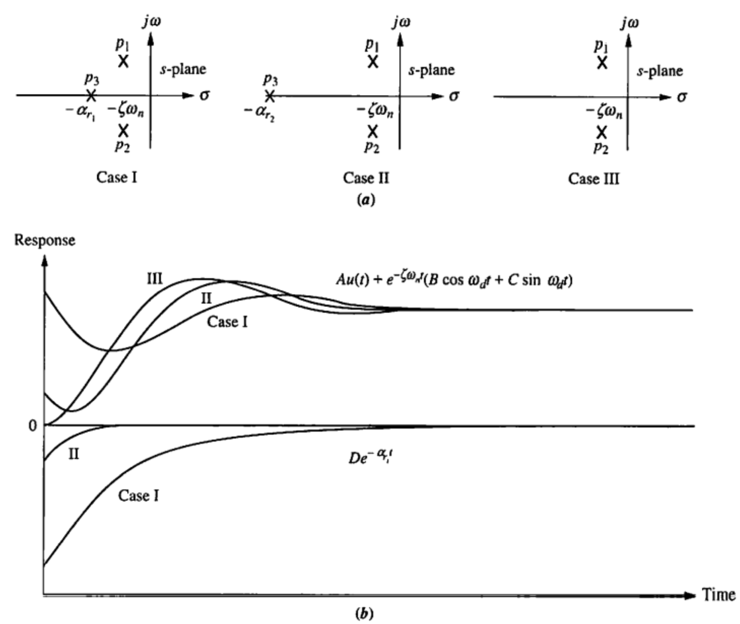


FIGURE 4.23 Component responses of a three-pole system: **a.** pole plot; **b.** component responses: Nondominant pole is near dominant second-order pair (Case I), far from the pair (Case II), and at infinity (Case III)

If α_r is not much greater than $\zeta\omega_n$ (Case I), the real pole's transient response will not decay to insignificance at the peak time or settling time generated by the second-order pair. In this case, the exponential decay is significant, and the system cannot be represented as a second-order system.

The next question is, How much farther from the dominant poles does the third pole have to be for its effect on the second-order response to be negligible? The answer of course depends on the accuracy for which you are looking. However, this book assumes that the exponential decay is negligible after five time constants. Thus, if the real pole is five times farther to the left than the dominant poles, we assume that the system is represented by its dominant second-order pair of poles.

What about the magnitude of the exponential decay? Can it be so large that its contribution at the peak time is not negligible? We can show, through a partial-fraction expansion, that the residue of the third pole, in a three-pole system with dominant second-order poles and no zeros, will actually decrease in magnitude as the third pole is moved farther into the left half-plane. Assume a step response, $C(s)$, of a three-pole system:

$$C(s) = \frac{bc}{s(s^2+as+b)(s+c)} = \frac{A}{s} + \frac{Bs+C}{s^2+as+b} + \frac{D}{s+c} \tag{4.59}$$

where we assume that the nondominant pole is located at $-c$ on the real axis and that the steady-state response approaches unity. Evaluating the constants in the numerator of each term,

$$A = 1; \qquad B = \frac{ca-c^2}{c^2+b-ca} \tag{4.60a}$$

$$C = \frac{ca^2-c^2a-bc}{c^2+b-ca}; \quad D = \frac{-b}{c^2+b-ca} \tag{4.60b}$$

As the nondominant pole approaches ∞, or $c \to \infty$,

$$A = 1;\ B = -1;\ C = -a;\ D = 0 \tag{4.61}$$

Thus, for this example, D, the residue of the nondominant pole and its response, becomes zero as the nondominant pole approaches infinity.

The designer can also choose to forgo extensive residue analysis, since all system designs should be simulated to determine final acceptance. In this case, the control systems engineer can use the "five times" rule of thumb as a necessary but not sufficient condition to increase the confidence in the second-order approximation during design, but then simulate the completed design.

Lag compensator

The **lag compensator** is an electrical network which produces sinusoidal output having phase lag when sinusoidal input is given.

$$G_c(s) = \frac{(s + z_c)}{(s + p_c)} \qquad \angle(G_c(s)) = -ve$$

$$z_c > p_c$$

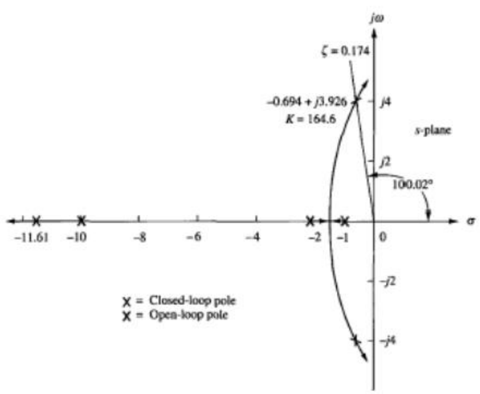
Consider $$z_c = \frac{1}{z_c}, \qquad p_c = \frac{1}{z_c \beta}$$

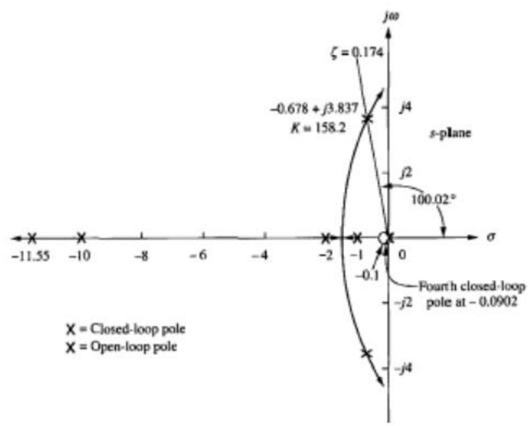
$$G_c(s) = \frac{\left(s + \frac{1}{z_c}\right)}{\left(s + \frac{1}{z_c \beta}\right)} = \beta \frac{(1 + z_c s)}{(1 + z_c s \beta)} \qquad G_c(j\omega) = \beta \frac{(1 + j\omega z_c)}{(1 + j\omega z_c \beta)}$$

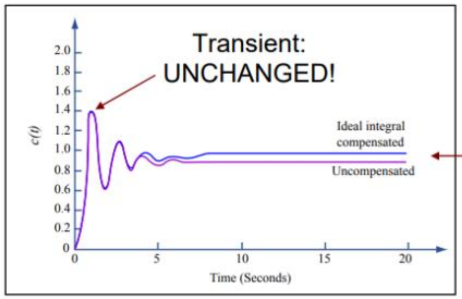
$$k_p = \lim_{s\to 0} G(s) \quad \text{and} \quad k_v = \lim_{s\to 0} sG(s)$$

$$G(s) = \frac{K}{(s+1)(s+2)(s+10)} \qquad k_p = \lim_{s\to 0} G(s) = 8.23 \qquad e(\infty) = \frac{1}{1+K_p} = \frac{1}{1+8.23} = 0.108$$

Compensate the system whose root locus is shown below, to improve the steady-state error by a factor of 10 if the system is operating with a damping ratio of 0.174.







Steady-state error=0
FIXED!

A ten fold improvement means a steady-state error is

$$e(\infty) = \frac{0.108}{10} = 0.0108$$

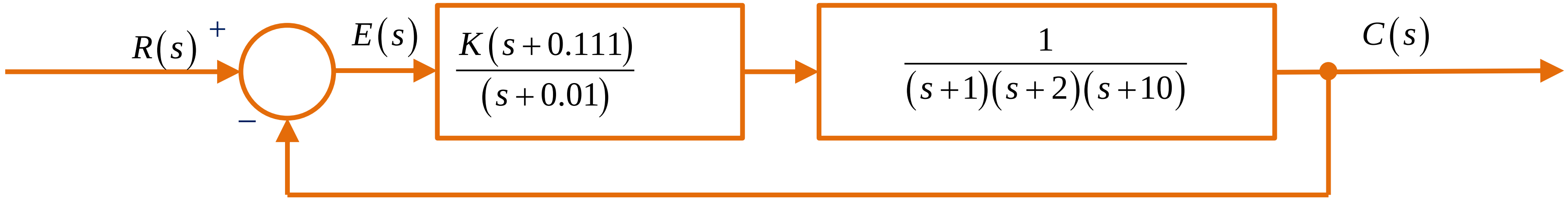
$$e(\infty) = \frac{1}{1+K_p} = 0.0108 \Rightarrow K_p = \frac{1-e(\infty)}{e(\infty)} = 91.59$$

$$K_{pN} = K_{pO}\frac{z_c}{p_c} > K_{pO}$$

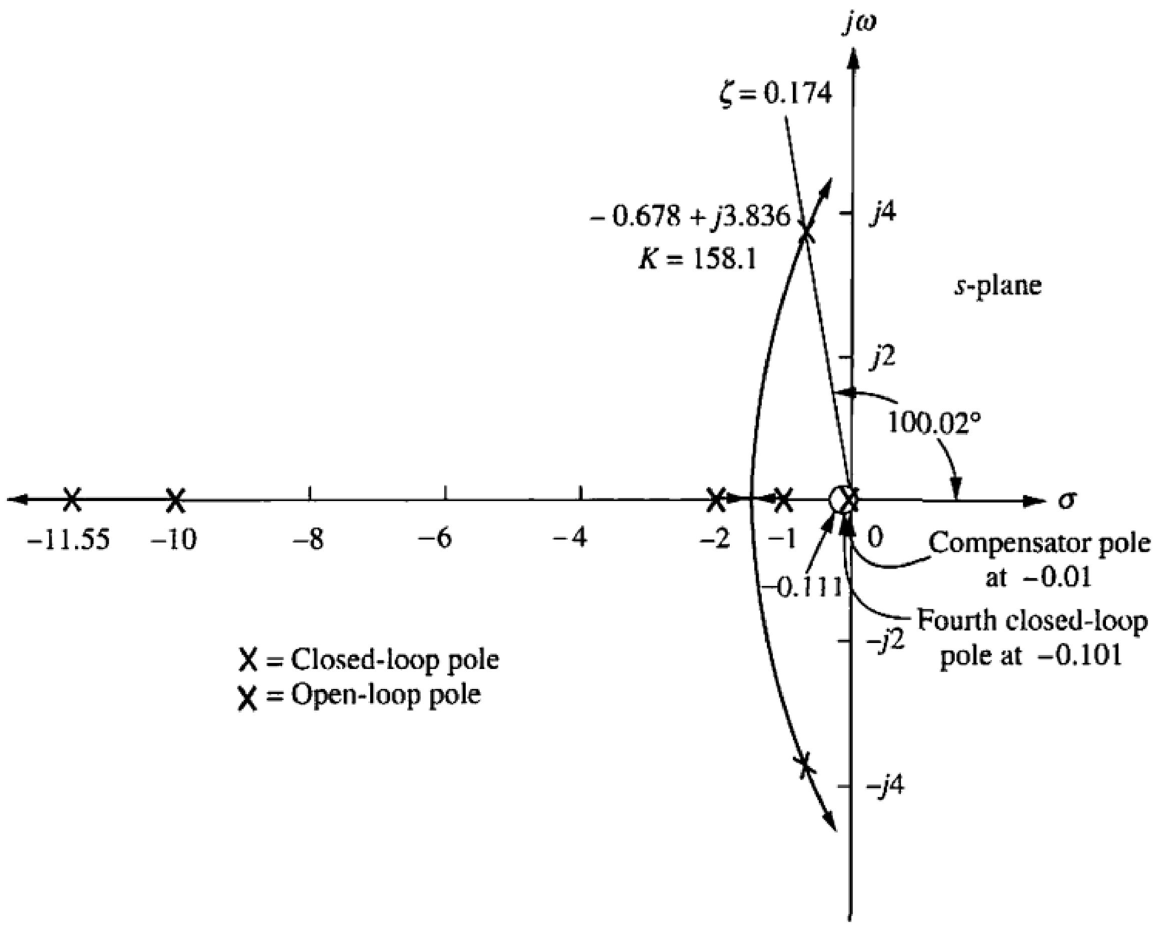
$$\frac{z_c}{p_c} = \frac{K_{pN}}{K_{pO}} = \frac{91.59}{8.23} = 11.13$$

Let us select $p_c = 0.01$

Let us select $z_c = 11.13 \times p_c = 0.111$



R(s)
+
−
E(s)
K(s+0.111) / (s+0.01)
1 / (s+1)(s+2)(s+10)
C(s)



jω
ζ = 0.174
– 0.678 + j3.836
K = 158.1
j4
s-plane
j2
100.02°
σ
−11.55
−10
−8
−6
−4
−2
−1
0
−0.111
Compensator pole at −0.01
Fourth closed-loop pole at −0.101
−j2
−j4
X = Closed-loop pole
X = Open-loop pole

Along the $\xi = 0.174$ line for a multiple of 180^0 and find that the second-order dominant poles are at $-0.678 \pm j3.836$ with a gain, K, of 158.1, The third and fourth closed-loop poles are at -11.55 and -0.101, respectively, and are found by searching the real axis for a gain equal to that of the dominant poles. The fourth pole of the compensated system cancels its zero. This leaves the remaining three closed-loop poles of the compensated system very close in value to the three closed-loop poles of the uncompensated system.

Parameter	Uncompensated	Lag-compensated
Plant and compensator	$\dfrac{K}{(s+1)(s+2)(s+10)}$	$\dfrac{K(s+0.111)}{(s+1)(s+2)(s+10)(s+0.01)}$
K	164.6	158.1
K_p	8.23	87.75
$e(\infty)$	0.108	0.011
Dominant second-order poles	$-0.694 \pm j3.926$	$-0.678 \pm j3.836$
Third pole	-11.61	-11.55
Fourth pole	None	-0.101
Zero	None	-0.111

The transient response of both systems is approximately the same, as is the system gain, but notice that the steady-state error of the compensated system is 1/9.818 that of the uncompensated system and is close to the design specification of a tenfold improvement.

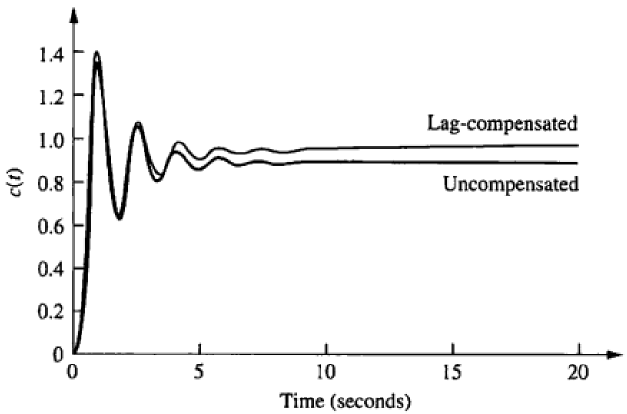


FIGURE 9.13 Step responses of uncompensated and lag-compensated systems for Example 9.2

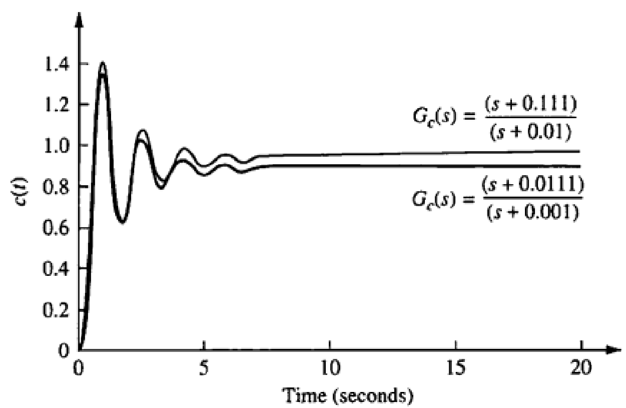


FIGURE 9.14 Step responses of the system for Example 9.2 using different lag compensators

Even though the transient responses of the uncompensated and lag-compensated systems are the same, the lag-compensated system exhibits less steady-state error by approaching unity more closely than the uncompensated system.

We now examine another design possibility for the lag compensator and compare the responses. Let us assume a lag compensator whose pole and zero are 10 times as close to the origin as in the previous design. Even though both responses will eventually reach approximately the same steady-state value, the lag compensator previously designed, $G_c(s) = (s + 0.111)/(s + 0.01)$, approaches the final value faster than the proposed lag compensator, $G_c(s) = (s + 0.0111)/(s + 0.001)$.

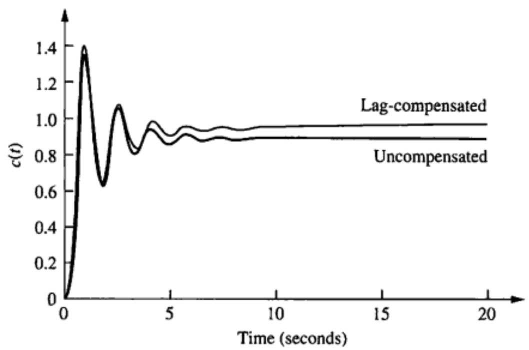


FIGURE 9.13 Step responses of uncompensated and lag-compensated systems for Example 9.2

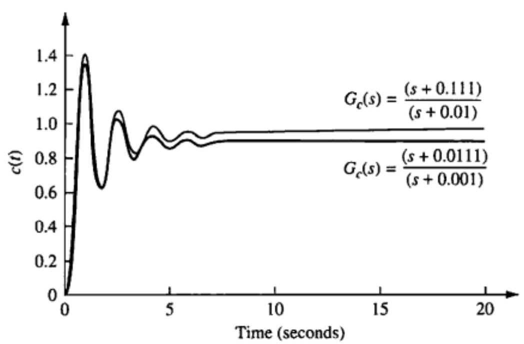
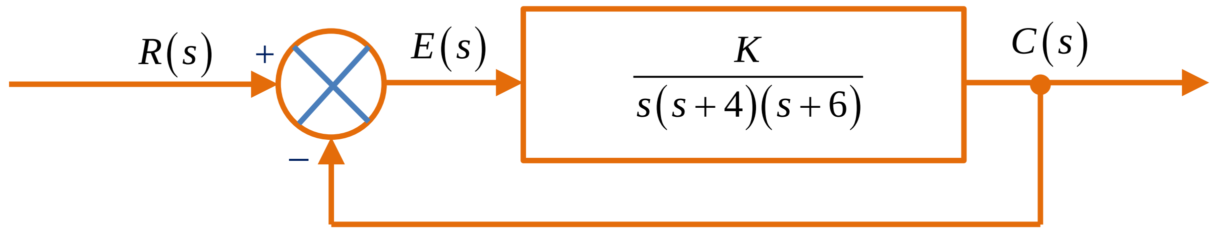


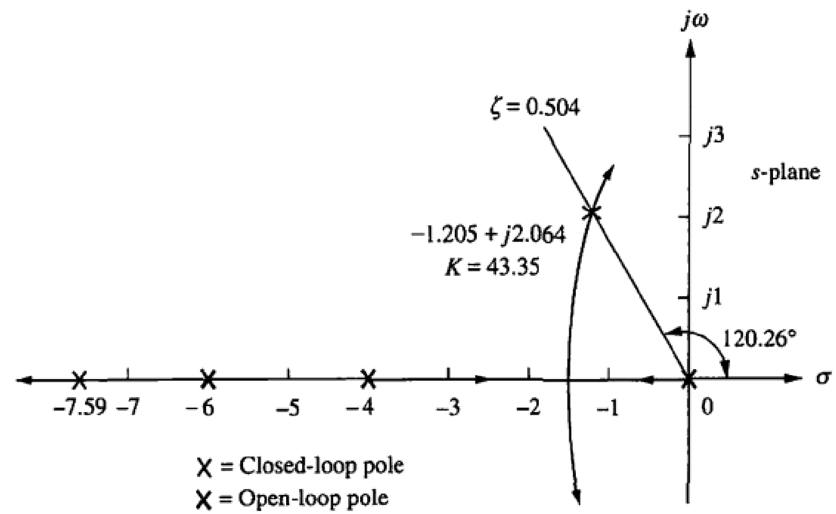
FIGURE 9.14 Step responses of the system for Example 9.2 using different lag compensators

The previously designed lag compensator has a fourth closed-loop pole at -0.101. Using the same analysis for the new lag compensator with its open-loop pole 10 times as close to the imaginary axis, we find its fourth closed-loop pole at -0.01, Thus, the new lag compensator has a closed-loop pole closer to the imaginary axis than the original lag compensator. This pole at —0.01 will produce a longer transient response than the original pole at -0.101, and the steady-state value will not be reached as quickly.

Given the system below, design an ideal derivative compensator to yield a 16% overshoot, with a threefold reduction in settling time.



Let us first evaluate the performance of the uncompensated system operating with 16% overshoot. The root locus for the uncompensated system is shown in the figure. Since 16% overshoot is equivalent to ξ = 0.504, we search along that damping ratio line for an odd multiple of 180^0 and find that the dominant second-order pair of poles is at $-1.205 \pm j2.064$. The settling time of the uncompensated system is



$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{1.205} = 3.320$$

Since our evaluation of percent overshoot and settling time is based upon a second-order approximation, we must check the assumption by finding the third pole and justifying the second-order approximation. Searching beyond – 6 on the real axis for a gain equal to the gain of the dominant, second-order pair, 43.35, we find a third pole at – 7.59, which is over six times as far from the $j\omega$-axis as the dominant, second-order pair. We conclude that our approximation is valid. The transient and steady-state error characteristics of the uncompensated system are summarized in the table below

TABLE 9.3 Uncompensated and compensated system characteristic of Example 9.3

	Uncompensated	Simulation	Compensated	Simulation
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$		$\frac{K(s+3.006)}{s(s+4)(s+6)}$	
Dominant poles	$-1.205 \pm j2.064$		$-3.613 \pm j6.193$	
K	43.35		47.45	
ζ	0.504		0.504	
ω_n	2.39		7.17	
$\%OS$	16	14.8	16	11.8
T_s	3.320	3.6	1.107	1.2
T_p	1.522	1.7	0.507	0.5
K_v	1.806		5.94	
$e(\infty)$	0.554		0.168	
Third pole	−7.591		−2.775	
Zero	None		−3.006	
Comments	Second-order approx. OK		Pole-zero not canceling	

Now we proceed to compensate the system. First we find the location of the compensated system's dominant poles. In order to have a threefold reduction in the settling time, the compensated system's settling time will be one-third of 3.320. The new settling time will be 1.107. Therefore, the real part of the compensated system's dominant, second-order pole is

$$\sigma = \frac{4}{T_s} = \frac{4}{1.107} = 3.613$$

and an imaginary part of

$$\omega_d = 3.613 \tan\left(180^0 - 120.26^0\right) = 6.193$$

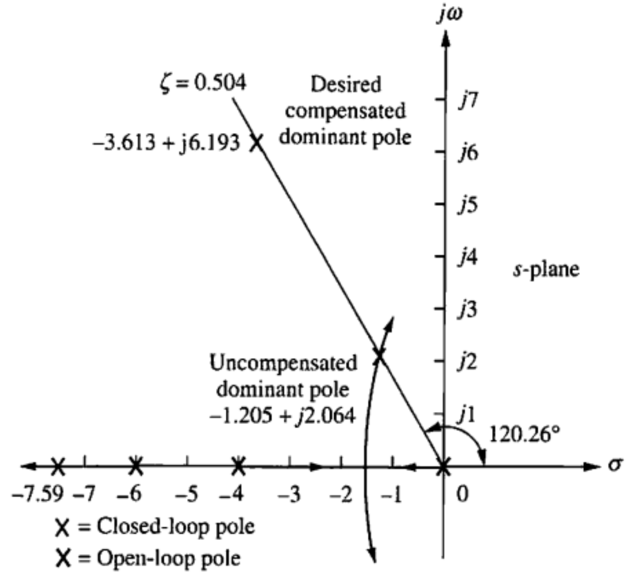


FIGURE 9.19 Compensated dominant pole superimposed over the uncompensated root locus for Example 9.3

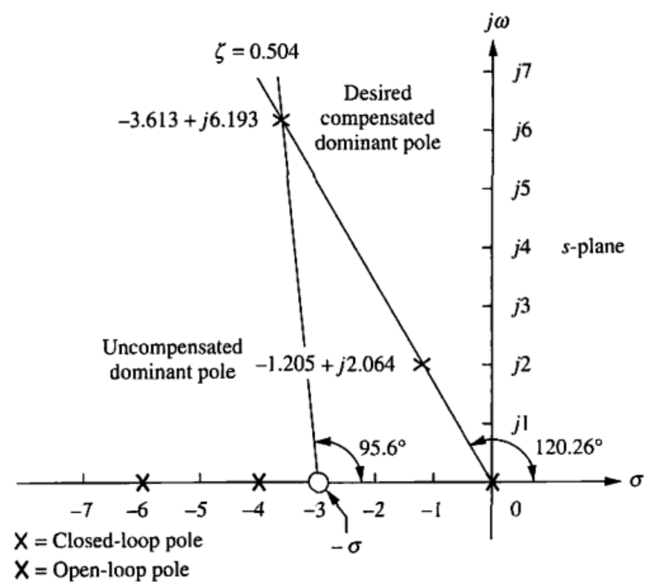


FIGURE 9.20 Evaluating the location of the compensating zero for Example 9.3

The result is the sum of the angles to the design point of all the poles and zeros of the compensated system except for those of the compensator zero itself. The difference between the result obtained and 180° is the angular contribution required of the compensator zero. Using the open-loop poles and the test point, – 3.613 + j6.193, which is the desired dominant second-order pole, we obtain the sum of the angles as –275.6°. Hence, the angular contribution required from the compensator zero for the test point to be on the root locus is $+275.6^0 - 180^0 = 95.6^0$.

$$\frac{6.193}{3.613-\sigma} = \tan\left(180^0 - 95.6^0\right) \Rightarrow \sigma = 3.006$$

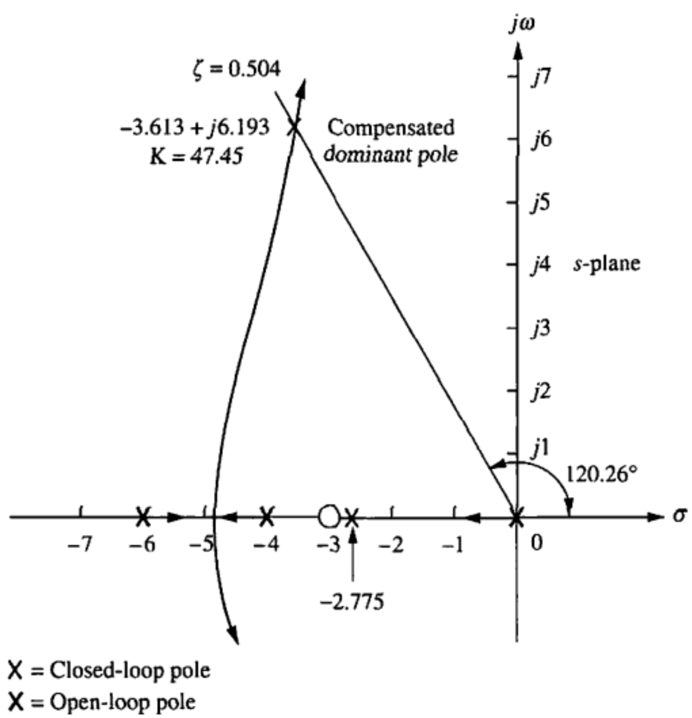


FIGURE 9.21 Root locus for the compensated system of Example 9.3

Lead Compensation

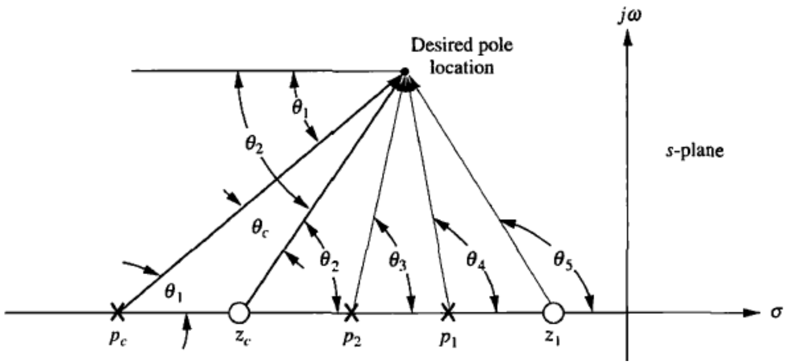
Just as the active ideal integral compensator can be approximated with a passive lag network, an active ideal derivative compensator can be approximated with a passive lead compensator. When passive networks are used, a single zero cannot be produced; rather, a compensator zero and a pole result. However, if the pole is farther from the imaginary axis than the zero, the angular contribution of the compensator is still positive and thus approximates an equivalent single zero. In other words, the angular contribution of the compensator pole subtracts from the angular contribution of the zero but does not preclude the use of the compensator to improve transient response, since the net angular contribution is positive, just as for a single PD controller zero.

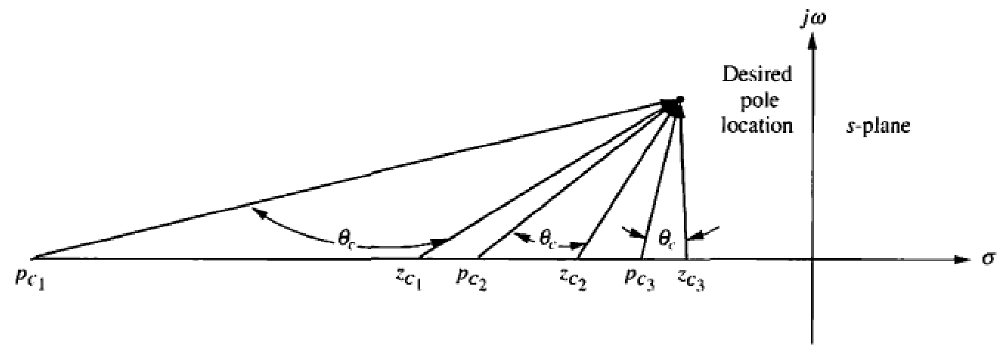
The advantages of a passive lead network over an active PD controller are that (1) no additional power supplies are required and (2) noise due to differentiation is reduced.

If we select a desired dominant, second-order pole on the *s*-plane, the sum of the angles from the uncompensated system's poles and zeros to the design point can be found. The difference between 180^0 and the sum of the angles must be the angular contribution required of the compensator.

$$\theta_2 - \theta_1 - \theta_3 - \theta_4 + \theta_5 = (2k+1)180^0$$

where $\theta_2 - \theta_1 = \theta_c$ is the angular contribution of the lead compensator.





The differences are in the values of static error constants, the gain required to reach the design point on the compensated root locus, the difficulty in justifying a second-order approximation when the design is complete, and the ensuing transient response.

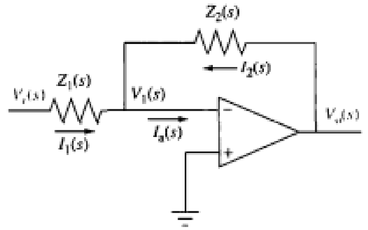


FIGURE 9.60 Operational amplifier configured for transfer function realization

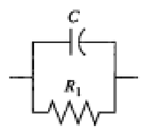
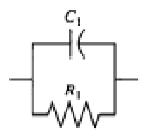
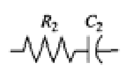
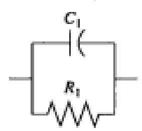
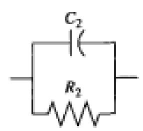
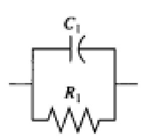
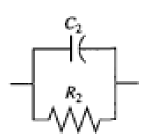
Active-Circuit Realization

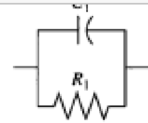
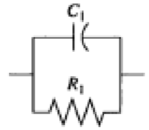
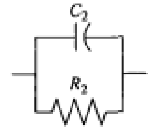
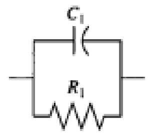
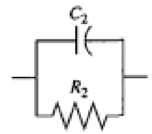
In Chapter 2, we derived

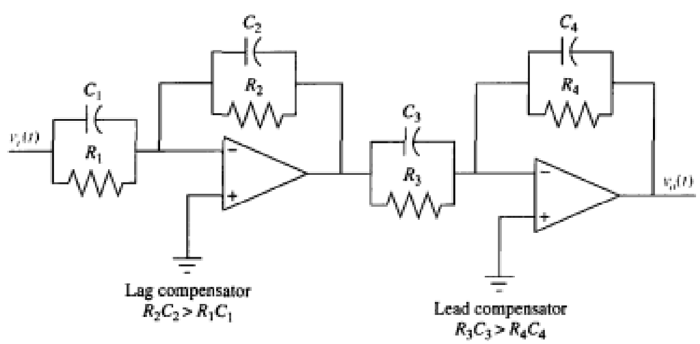
$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}} \tag{9.44}$$

as the transfer function of an inverting operational amplifier whose configuration is repeated here in Figure 9.60. By judicious choice of $Z_1(s)$ and $Z_2(s)$, this circuit can be used as a building block to implement the compensators and controllers, such as PID controllers, discussed in this chapter. Table 9.10 summarizes the realization of PI, PD, and PID controllers as well as lag, lead, and lag-lead compensators using operational amplifiers. You can verify the table by using the methods of Chapter 2 to find the impedances.

TABLE 9.10 Active realization of controllers and compensators, using an operational amplifier

Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Gain			$-\frac{R_2}{R_1}$
Integration			$-\frac{\frac{1}{RC}}{s}$
Differentiation			$-RCs$
PI controller			$-\frac{R_2}{R_1}\frac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD controller			$-R_2C\left(s+\frac{1}{R_1C}\right)$
PID controller			$-\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$
Lag compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$

PID controller			$-\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$
Lag compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation			$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$



GURE 9.61 Lag-lead compensator implemented with operational amplifiers

Passive-Circuit Realization

Lag, lead, and lag-lead compensators can also be implemented with passive networks. Table 9.11 summarizes the networks and their transfer functions. The transfer functions can be derived with the methods of Chapter 2.

The lag-lead transfer function can be put in the following form:

$$G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\alpha T_1}\right)\left(s+\frac{\alpha}{T_2}\right)} \tag{9.50}$$

where $\alpha < 1$. Thus, the terms with T_1 form the lead compensator, and the terms with T_2 form the lag compensator. Equation (9.50) shows a restriction inherent in using this passive realization. We see that the ratio of the lead compensator zero to the lead compensator pole must be the same as the ratio of the lag compensator pole to the lag compensator zero. In Chapter 11 we design a lag-lead compensator with this restriction.

A lag-lead compensator without this restriction can be realized with an active network as previously shown or with passive networks by cascading the lead and lag networks shown in Table 9.11. Remember, though, that the two networks must be isolated to ensure that one network does not load the other. If the networks load each other, the transfer function will not be the product of the individual transfer functions. A possible realization using the passive networks uses an operational amplifier to provide isolation. The circuit is shown in Figure 9.63. Example 9.10 demonstrates the design of a passive compensator.

TABLE 9.11 Passive realization of compensators

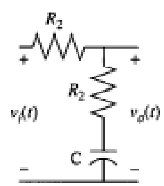
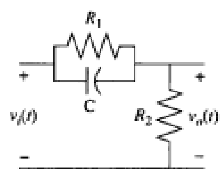
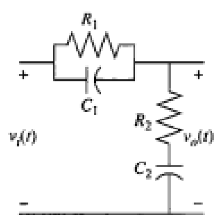
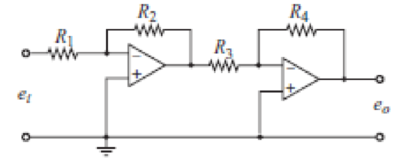
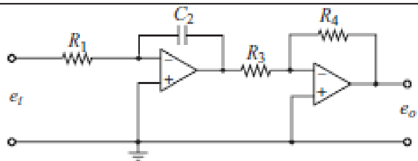
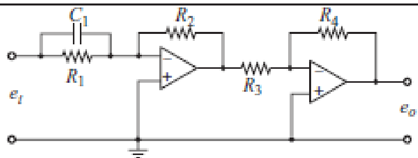
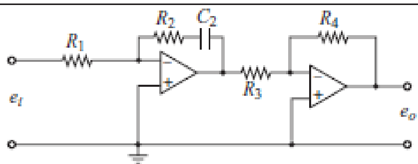
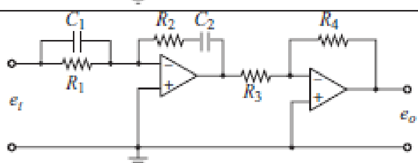
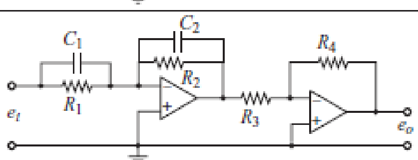
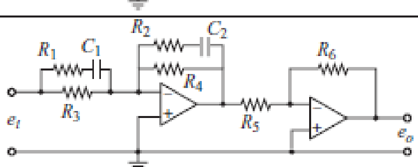
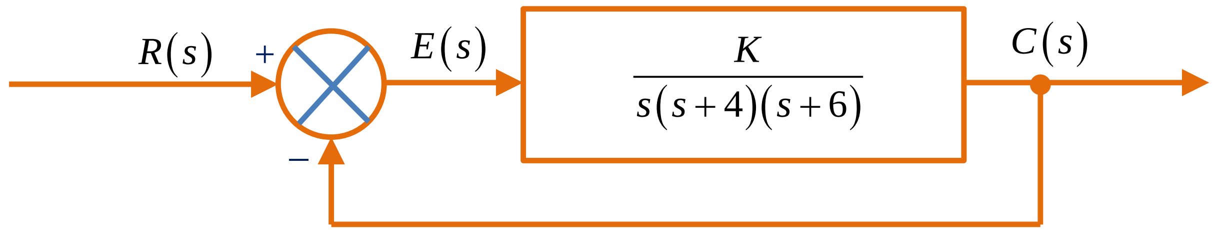
Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation		$\frac{R_2}{R_1+R_2}\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}}$
Lead compensation		$\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}$
Lag-lead compensation		$\frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$

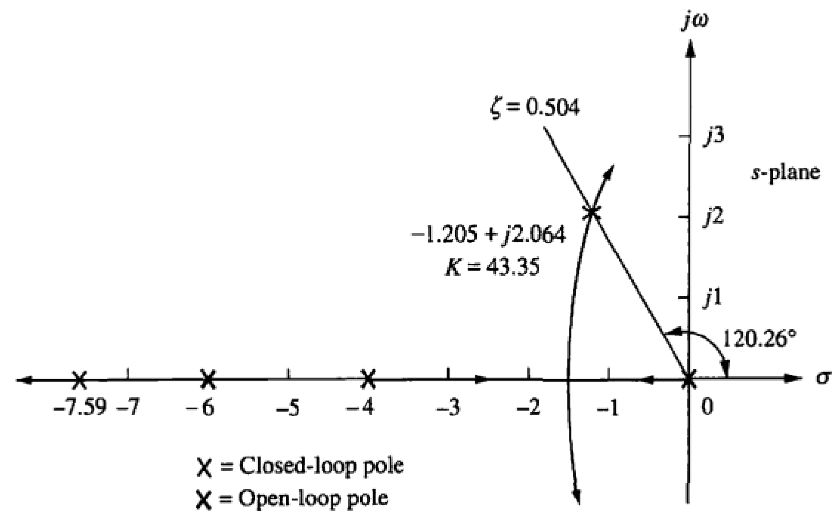
Table 3–1 Operational-Amplifier Circuits That May Be Used as Compensators

	Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational-Amplifier Circuits
1	P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
2	I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
3	PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s+1)$	
4	PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s+1}{R_2C_2s}$	
5	PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$	
6	Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}$	
7	Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1+R_3)C_1s+1](R_2C_2s+1)}{(R_1C_1s+1)[(R_2+R_4)C_2s+1]}$	

Given the system below, design an ideal derivative compensator to yield a 16% overshoot, with a threefold reduction in settling time.

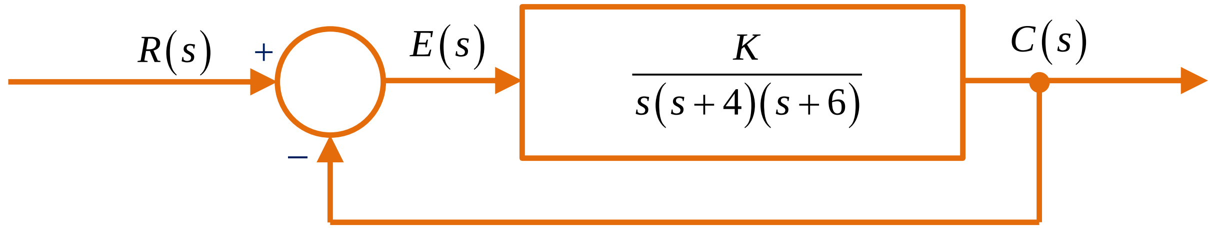


Let us first evaluate the performance of the uncompensated system operating with 16% overshoot. The root locus for the uncompensated system is shown in the figure. Since 16% overshoot is equivalent to ξ = 0.504, we search along that damping ratio line for an odd multiple of 180^0 and find that the dominant second-order pair of poles is at $-1.205 \pm j2.064$. The settling time of the uncompensated system is



$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{1.205} = 3.320$$

Design three lead compensators for the system that will reduce the settling time by a factor of 2 while maintaining 30% overshoot. Compare the system characteristics among the three designs.



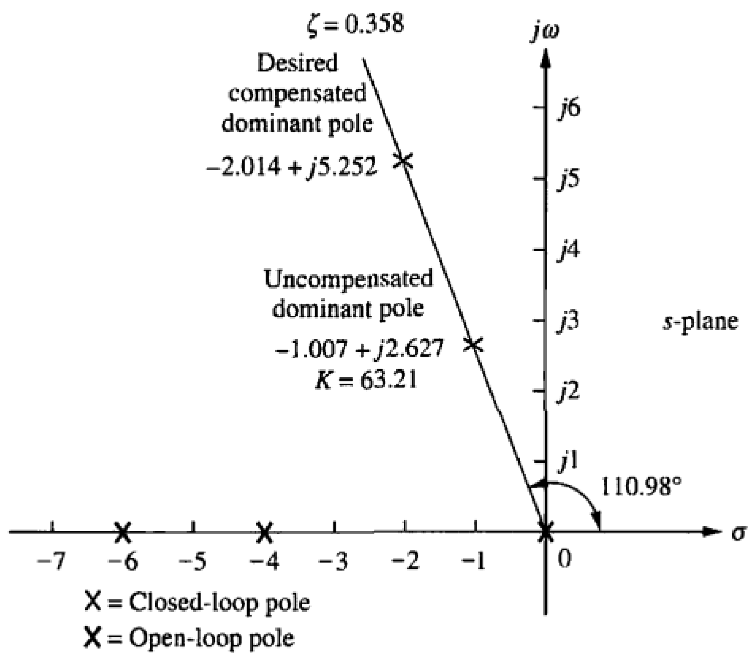


FIGURE 9.26 Lead compensator design, showing evaluation of uncompensated and compensated dominant poles for Example 9.4

30% overshoot is equivalent to a damping ratio of 0.358, we search along the $\xi = 0.358$ line for the uncompensated dominant poles on the root locus. The uncompensated settling time as $T_s = 4/1.007 = 3.972$ s. Next we find the design point. A twofold reduction in settling time yields $T_s = 3.972/2 = 1.986$ s, from which the real part of the desired pole location is $-\ \xi\omega_n = -\ 4/T_s = -\ 2.014$. The imaginary part is ($\omega_d = -\ 2.014 \tan(110.98^0) = 5.252$.

Arbitrarily assume a compensator zero at –5 on the real axis as a possible solution.

Using the root locus program, sum the angles from both this zero and the uncompensated system's poles and zeros, using the design point as a test point. The resulting angle is –172.69^0. The difference between this angle and 180^0 is the angular contribution required from the compensator pole in order to place the design point on the root locus. Hence, an angular contribution of –7.31^0 is required from the compensator pole.

TABLE 9.4 Comparison of lead compensation designs for Example 9.4

	Uncompensated	Compensation a	Compensation b	Compensation c
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$	$\frac{K(s+5)}{s(s+4)(s+6)(s+42.96)}$	$\frac{K(s+4)}{s(s+4)(s+6)(s+20.09)}$	$\frac{K(s+2)}{s(s+4)(s+6)(s+8.971)}$
Dominant poles	$-1.007 \pm j2.627$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$
K	63.21	1423	698.1	345.6
ζ	0.358	0.358	0.358	0.358
ω_n	2.813	5.625	5.625	5.625
$\%OS^*$	30 (28)	30 (30.7)	30 (28.2)	30 (14.5)
T_s^*	3.972 (4)	1.986 (2)	1.986 (2)	1.986 (1.7)
T_p^*	1.196 (1.3)	0.598 (0.6)	0.598 (0.6)	0.598 (0.7)
K_v	2.634	6.9	5.791	3.21
$e(\infty)$	0.380	0.145	0.173	0.312
Other poles	-7.986	-43.8, -5.134	-22.06	-13.3, -1.642
Zero	None	-5	None	-2
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK	No pole-zero cancellation

uncompensated system's poles and zeros, using the design point as a test point. The resulting angle is $-172.69°$. The difference between this angle and $180°$ is the angular contribution required from the compensator pole in order to place the design point on the root locus. Hence, an angular contribution of $-7.31°$ is required from the compensator pole.

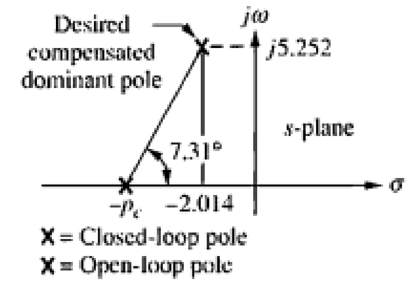
The geometry shown in Figure 9.27 is used to calculate the location of the compensator pole. From the figure,

$$\frac{5.252}{p_c - 2.014} = \tan 7.31° \tag{9.19}$$

from which the compensator pole is found to be

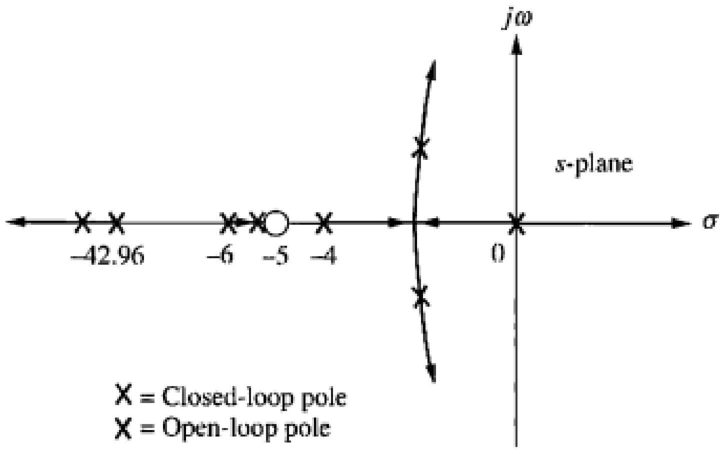
$$p_c = 42.96 \tag{9.20}$$

The compensated system root locus is sketched in Figure 9.28.



Note: This figure is not drawn to scale.

FIGURE 9.27 s-plane picture used to calculate the location of the compensator pole for Example 9.4



Note: This figure is not drawn to scale.

FIGURE 9.28 Compensated system root locus

Thank you