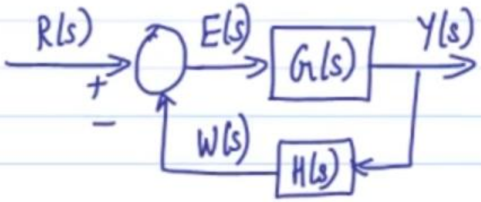
26/2/2018.

Root Locus.

Q: How do the closed loop poles change when a system parameter is varied?



$G(s) = C(s)P(s)$.

Recall that $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$ → CLOSED LOOP TRANSFER FN.

$1 + G(s)H(s) = 0$. → CLOSED LOOP CHARACTERISTIC EQUATION.

$G(s)H(s) = -1$

→ $|G(s)H(s)| = 1$ ⟶ MAGNITUDE CONDITION

→ $\angle G(s)H(s) = \pm 180^\circ(2k+1)$, $k = 0, 1, 2, \ldots$ → ANGLE CONDITION.

Typically, the open loop transfer function can be expressed as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}.$$

$-z_1, -z_2, \ldots, -z_m \Rightarrow$ open loop zeros.

$-p_1, -p_2, \ldots, -p_n \Rightarrow$ open loop poles.

Q: How would the closed loop poles change as K is varied.

Let us first consider the scenario when $K \geq 0$.

$$1 + \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)} = 0.$$

ROOT LOCUS $\rightarrow$ Locus of the closed loop as K is varied.

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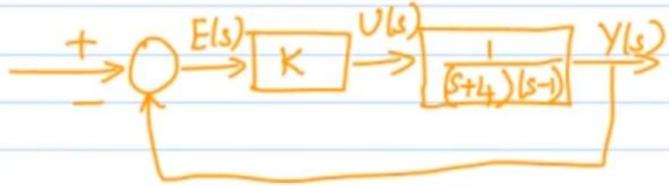
ROOT LOCUS → Locus of the closed loop poles as K is varied.

→ The root locus would have n branches.

Eg.: $G(s)H(s) = \dfrac{K}{(s+4)(s-1)}$. $n = 2$. O.L. poles: $-4, 1$
$m = 0$ O.L. zeros: Nil.



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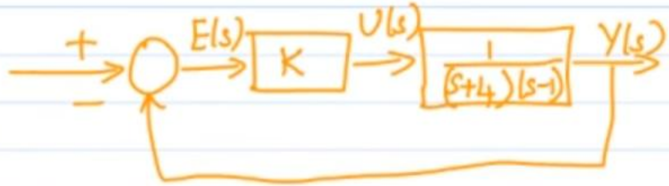
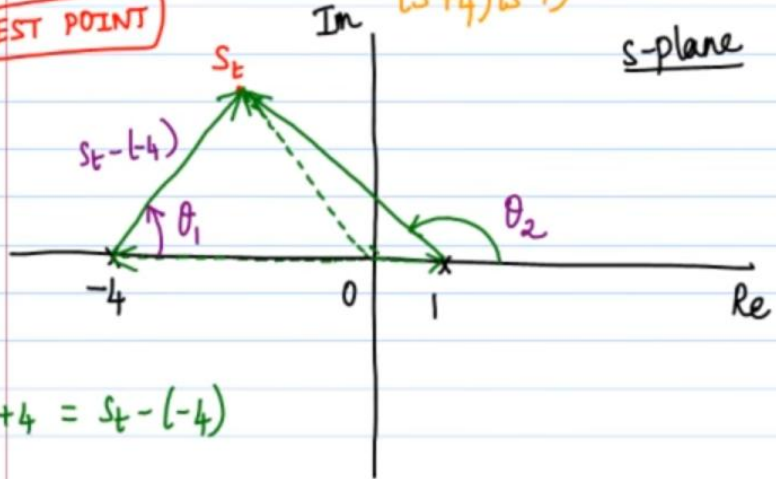


TEST POINT



Q: How do we determine if the test point s_t lies on the root locus?

$$\angle G(s_t)H(s_t) = \angle \frac{K}{(s_t+4)(s_t-1)} = \underbrace{\angle K}_{0} - \underbrace{\angle s_t+4}_{\theta_1} - \underbrace{\angle s_t-1}_{\theta_2}$$

$$= -\theta_1 - \theta_2 \overset{?}{=} \pm 180^\circ(2k+1), \quad k = 0, 1, 2, \ldots$$

$s_t + 4 = s_t - (-4)$

Steps Involved in Constructing the Root Locus

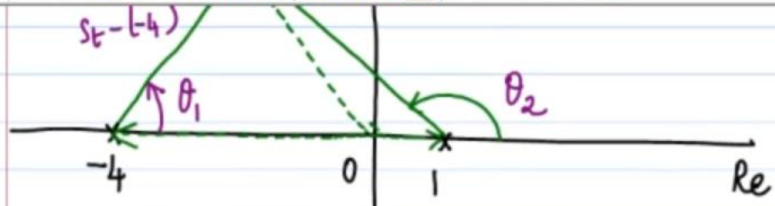
Let us consider a closed loop negative feedback system whose characteristic equation (the roots of this equation are the closed loop poles) is given by

$$1+\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}=0.$$

We shall consider the case when $m \leq n$ with K being a positive parameter. Here $-z_1,\ldots,-z_m$ are the open loop zeros and $-p_1,\ldots,-p_n$ are the open loop poles. The root locus traces the evolution of the roots of this equation with variation in K. The root locus will have n branches/curves. The steps generally followed in constructing the root locus for such a system are:

1. *Locate the open loop zeros and poles in the complex plane:* Note that each of the n curves in the root locus will start from an open loop pole for $K = 0$. Out of these, m curves will terminate at an open loop zero as $K \to \infty$ and the remaining *(n-m)* curves will go to infinity in the complex plane along "asymptotes" as $K \to \infty$.

2. *Locate the root loci that lie on the real axis:* This is determined by the open loop poles and zeros that lie on the real axis. Choose a test point on the real axis. If the total number of real open loop poles and real open loop zeros that lie to the right of this test point is odd, then that point lies on the root locus.



$s_t+4 = s_t-(-4)$

the root locus ?

$$\angle G(s_t)H(s_t) = \angle \frac{K}{(s_t+4)(s_t-1)} = \underbrace{\angle K}_{0} - \underbrace{\angle s_t+4}_{\theta_1} - \underbrace{\angle s_t-1}_{\theta_2}$$

$$= -\theta_1 - \theta_2 \overset{?}{=} \pm 180^\circ(2k+1), \quad k = 0,1,2,\ldots$$

$\boxed{\text{rlocus}}$

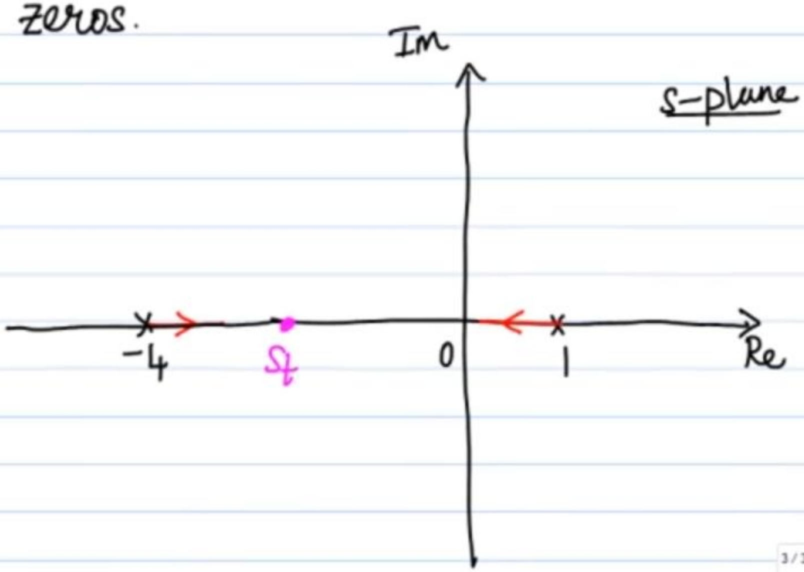
Step 1: Locate the open loop poles and open loop zeros.

open loop poles: $-4, 1$.

open loop zeros: Nil.

The root locus would have 2 branches.

$$1 + \frac{K}{(s+4)(s-1)} = 0. \Rightarrow \boxed{(s+4)(s-1) + K = 0.}$$



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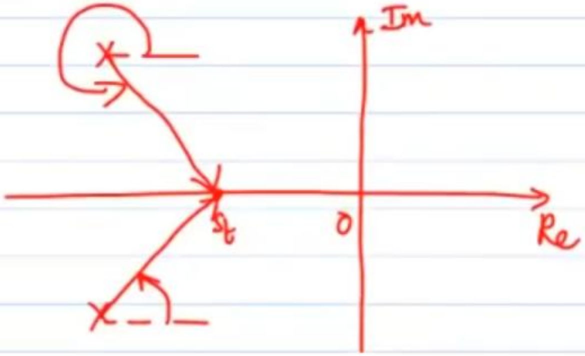
3. *Determine the asymptotes of the root loci:* This step applies only when $m<n$. If the test point is chosen very far away from the origin, then the angle contribution of each open loop pole cancels with that of an open loop zero. Hence, the asymptotes (which exist only when $m<n$) will be straight lines which make an angle with the positive real axis of $\dfrac{\pm 180^{\circ}(2k+1)}{(n-m)}, k = 0,1,2,...$ The point of intersection of the asymptotes on the real axis is given by $-\left[\dfrac{(p_1 + p_2 + ... + p_n) - (z_1 + z_2 + ... + z_m)}{(n-m)}\right]$ (which is the

$$(s+4)(s-1)$$

$$\frac{1}{K}(s+4)(s-1) + 1 = 0.$$

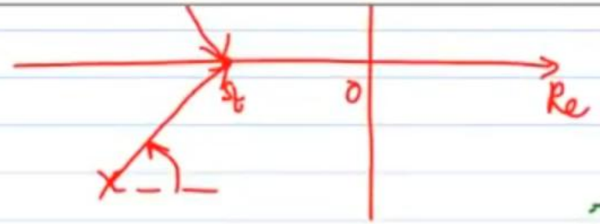
$$(0s+1)(0s+1)$$



$$1 + \frac{K(s+1)}{(s+4)(s-1)} = 0.$$

$$\Rightarrow \frac{1}{K}(s+4)(s-1) + \underset{(0s+1)}{(s+1)} = 0.$$

As $K \to \infty$, $s \to -1$.



$$\Rightarrow \frac{1}{K}(s+4)(s-1)+(s+1)=0.$$
$$(0s+1)$$

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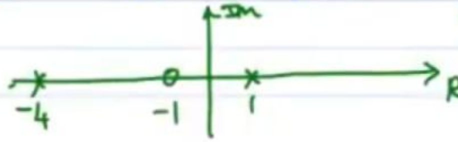


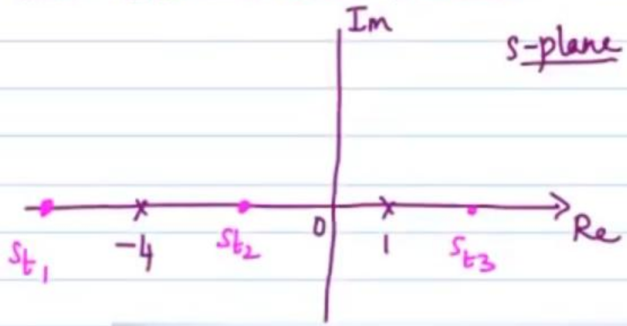
✓ $(-\infty,-4)$
× $(-4,-1)$
$(-1,1)$ ✓
$(1,\infty)$ ×

Step 2: Locate the parts of the real axis that lie on the root locus.

$(-\infty,-4) \rightarrow$ ×

$(-4,1) \rightarrow$ ✓

$(1,\infty) \rightarrow$ ×



$(-4,1)$ lies on the root locus.

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4. *Determine the break-away and break-in points:* A break-away (break-in) point example, will exist on the root locus between two consecutive open loop poles (ze on the real axis with no real open loop zeros (poles) in between them. If characteristic equation is written as $1+K\dfrac{A(s)}{B(s)}=0$, then the solution of the equa $B'(s)A(s)-B(s)A'(s)=0$ will provide the possible values of s at which break-a

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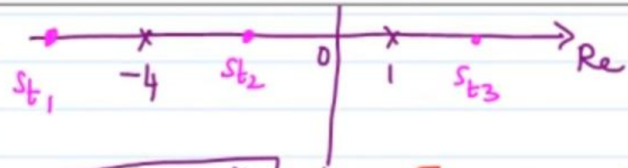
$(-4, 1) \rightarrow$ ✓

$(1, \infty) \rightarrow$ ✗



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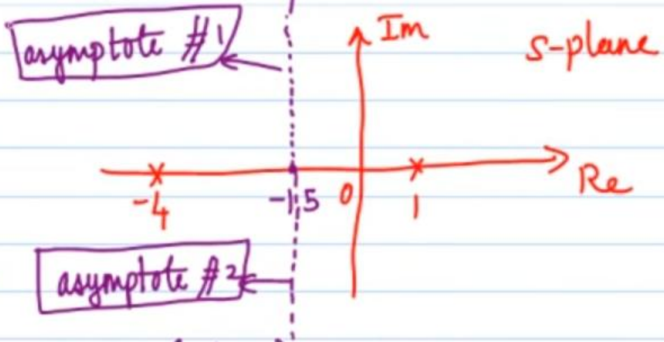

27/2/2018. o.l. poles: $-4, 1$; o.l. zeros: NIL.

Step 3: Asymptotes. $(n-m) = 2-0 = 2 \Rightarrow 2$ asymptotes.

Angle made by the 2 asymptotes $= \pm \dfrac{180^\circ(2k+1)}{(n-m)} = \pm 90^\circ(2k+1)$, $k = 0, 1, 2, \ldots$

$+90^\circ, -90^\circ\ (270^\circ)$.

Point of intersection of the 2 asymptotes $= \dfrac{(-4+1)-(0)}{2} = -1.5$

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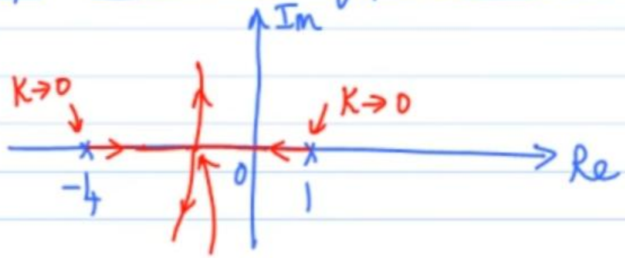
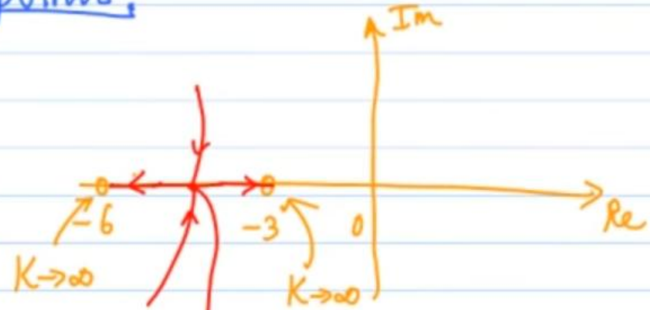
same as calculating $\left[\dfrac{(\text{sum of the open loop poles})-(\text{sum of the open loop zeroes})}{(n-m)}\right]$).

4. *Determine the break-away and break-in points:* A break-away (break-in) point, for example, will exist on the root locus between two consecutive open loop poles (zeros) on the real axis with no real open loop zeros (poles) in between them. If the characteristic equation is written as $1+K\dfrac{A(s)}{B(s)}=0$, then the solution of the equation $B'(s)A(s)-B(s)A'(s)=0$ will provide the possible values of s at which break-away or break-in points may occur. We should select only those roots of this equation that actually lie on the root locus as break-away or break-in points (K will be positive for the roots which will lie on the root locus), i.e., not all the roots of this equation may actually correspond to break-away or break-in points. But a break-away or break-in point, if it exists, will be a root of this equation.

Step 4: Break-away / Break-in points:



Break-away Point



Break-in point

$$d_1(s) = (s+1)(s+2)$$
$$d_1'(s) = (s+1) + (s+2).$$

$$d_2(s) = (s+1)^2 (s+2)$$
$$d_2'(s) = 2(s+1)(s+2) + (s+1)^2$$

Let $G(s)H(s) = K \dfrac{A(s)}{B(s)}$. $\quad A(s) = (s+z_1)\cdots(s+z_m).$

$\qquad B(s) = (s+p_1)\cdots(s+p_n).$

The closed loop characteristic equation is $1 + K \dfrac{A(s)}{B(s)} = 0.$

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Let $G(s)H(s) = K\dfrac{A(s)}{B(s)}$. $A(s) = (s+z_1)\cdots(s+z_m)$.

$B(s) = (s+p_1)\cdots(s+p_n)$.

The closed loop characteristic equation is $\boxed{1 + K\dfrac{A(s)}{B(s)} = 0.}$

$$P_{cl}(s) = 1 + K\frac{A(s)}{B(s)} \Rightarrow P_{cl}'(s) = \frac{dP_{cl}(s)}{ds} = +K\frac{\left(A'(s)B(s) - B'(s)A(s)\right)}{[B(s)]^2} = 0.$$

$$\Rightarrow \boxed{A'(s)B(s) - B'(s)A(s) = 0.}$$

$\rightarrow$ The roots of this eqn. (s_b) are potential break-away / break-in points.

Then, calculate $K = -\dfrac{B(s)}{A(s)}\Big|_{s=s_b}$. Only those values of s_b that result in $K>0$ would lie on the root locus.

5. *Locate the angle of departure (angle of arrival) from an open loop complex pole (at an open loop complex zero):* The angle of departure from an open loop complex pole = 180° - (sum of the angles made by the vectors from other open loop poles to this pole) + (sum of the angles made by the vectors from open loop zeros to this pole). The angle of arrival at an open loop complex zero = 180° - (sum of the angles made by the vectors from other open loop zeros to this zero) + (sum of the angles made by the vectors from open loop poles to this zero).

6. *Determine the points where the root locus may cross the imaginary axis:* These points can be obtained by setting $s = j\omega$ in the characteristic equation and solving for the corresponding values of ω and K.

7. *Sketch the root locus:* Consider a set of test points in the broad neighborhood of the origin and the imaginary axis in the complex plane and sketch the root locus.

Note that, since we are dealing with polynomials with real coefficients, the root l
will be symmetrical with respect to the real axis. Remember that angles are meas
positive in the counterclockwise direction from the positive real axis.

Let us now consider a system with positive feedback whose characteristic equatio

$$1-\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}=0,$$

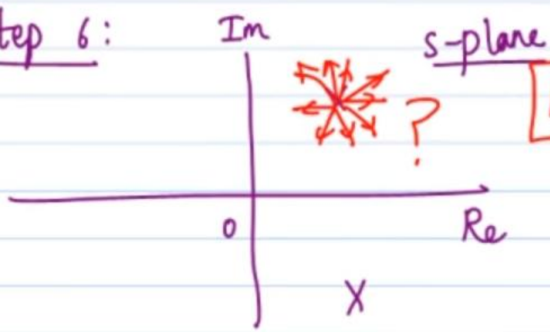
Then, calculate $K = -\left.\dfrac{B(s)}{A(s)}\right|_{s=s_b}$. Only those values of s_b that result in $K>0$ would lie on the root locus.

$$A(s) = 1, \quad B(s) = (s+4)(s-1) \Rightarrow A'(s) = 0, \quad B'(s) = 2s+3.$$

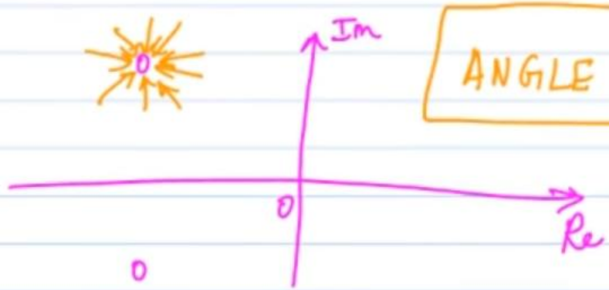
$$\Rightarrow A'(s)B(s) - B'(s)A(s) = 0 \Rightarrow -(2s+3) = 0 \Rightarrow \boxed{s_b = -1.5.}$$

$$K\Big|_{s=s_b} = -\left[\frac{B(s)}{A(s)}\right]_{s=s_b} = -\left[\frac{(s_b+4)(s_b-1)}{1}\right] = -\left[(2.5)(-2.5)\right] = 6.25 > 0.$$

Step 6:



ANGLE OF DEPARTURE



ANGLE OF ARRIVAL

$$K\Big|_{s=s_b} = -\left[\frac{B(s)}{A(s)}\right]_{s=s_b} = -\left[\frac{(s_b+4)(s_b-1)}{1}\right] = -\left[(2.5)(-2.5)\right] = 6.25 > 0.$$

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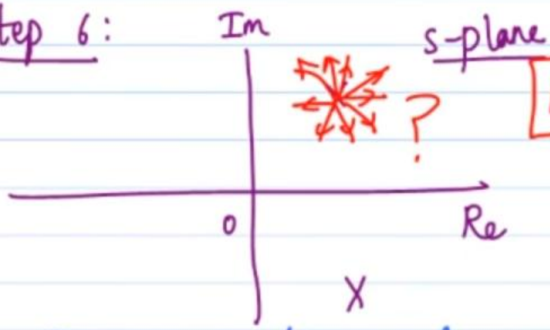
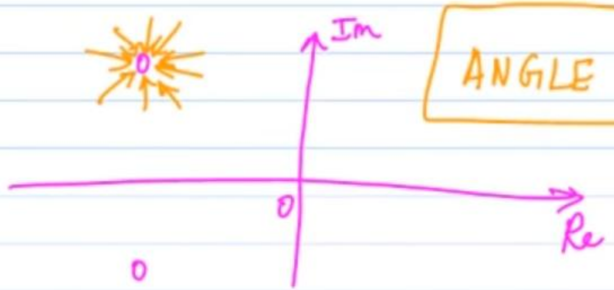


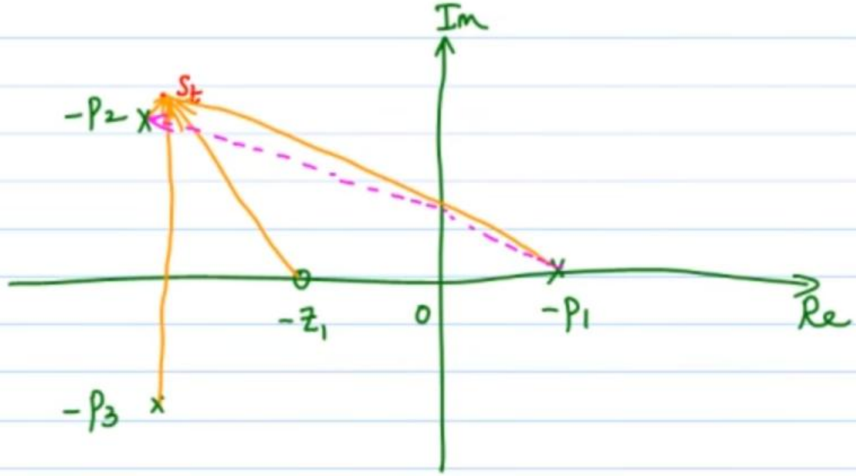
ANGLE OF DEPARTURE



ANGLE OF ARRIVAL

Step 6 applies only when we have complex open loop poles / open loop zeros.

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)}, \quad n = 3, \quad m = 1$$



Angle of departure from $-p_2$:

$$\angle G(s_t)H(s_t) = \pm 180^\circ(2k+1), \quad k = 0, 1, 2, \ldots$$

$$\underbrace{\angle K}_{\substack{0^\circ \\ (K>0)}} + \angle s_t+z_1 - \left[\angle s_t+p_1 + \angle s_t+p_2 + \angle s_t+p_3\right]$$

$$\theta_{dep,(-p_2)} = \pm 180^\circ(2k+1).$$

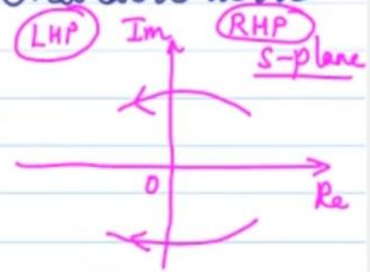
As $s_t \to -p_2$ (s_t is "very" close to $-p_2$). $\quad \angle s_t+p_1 \approx \angle -p_2+p_1$

$$\theta_{dep,(-p_2)} = 180^\circ + \angle -p_2+z_1 - \left[\angle -p_2+p_1 + \angle -p_2+p_3\right]$$

$$\theta_{dep,(-p_3)} = -\theta_{dep,(-p_2)}$$

<u>Step 6</u>: Determine the "cross-over" points (points where the root locus cuts the imaginary axis).

These are found by substituting $\boxed{s = j\omega}$ in the closed loop characteristic equation.



For this example, $G(s)H(s) = \dfrac{K}{(s+4)(s-1)} = \dfrac{K}{s^2+3s-4}$.

The closed loop characteristic equation is $\boxed{1 + G(s)H(s) = 0.}$

$$\Rightarrow 1 + \frac{K}{s^2+3s-4} = 0. \quad \Rightarrow s^2 + 3s + (K-4) = 0.$$

Substitute $s = j\omega \Rightarrow -\omega^2 + 3j\omega + (K-4) = 0 \Rightarrow [-\omega^2 + K - 4] + j[3\omega] = 0$

$\Rightarrow \omega = 0 \Rightarrow K - 4 = 0 \Rightarrow K = 4. \Rightarrow s = 0$ is a
where $K =$

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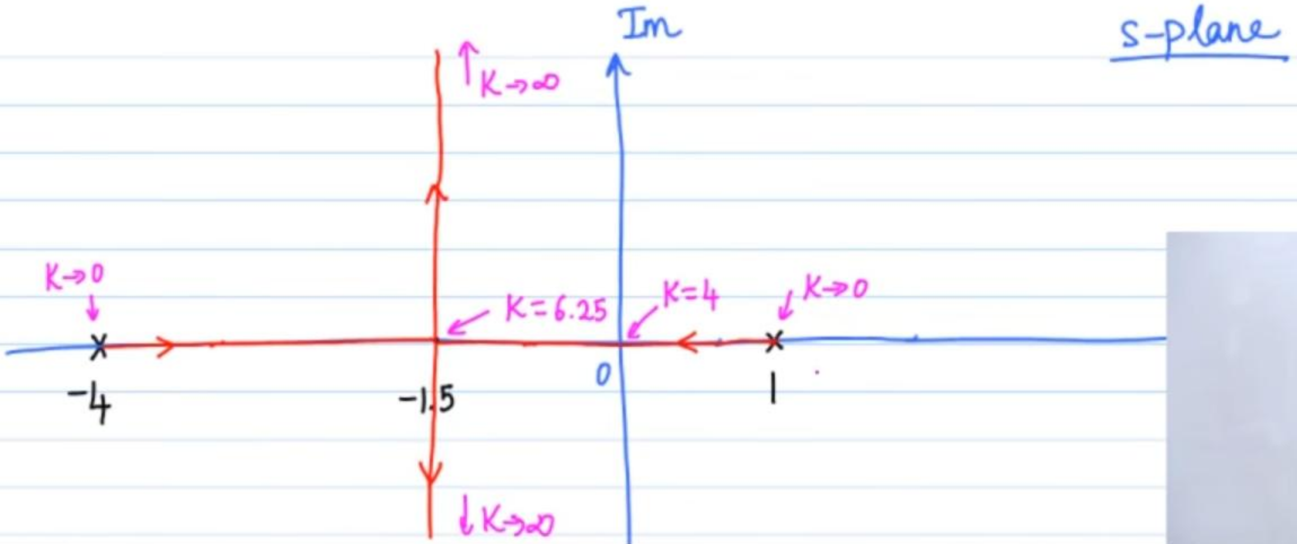
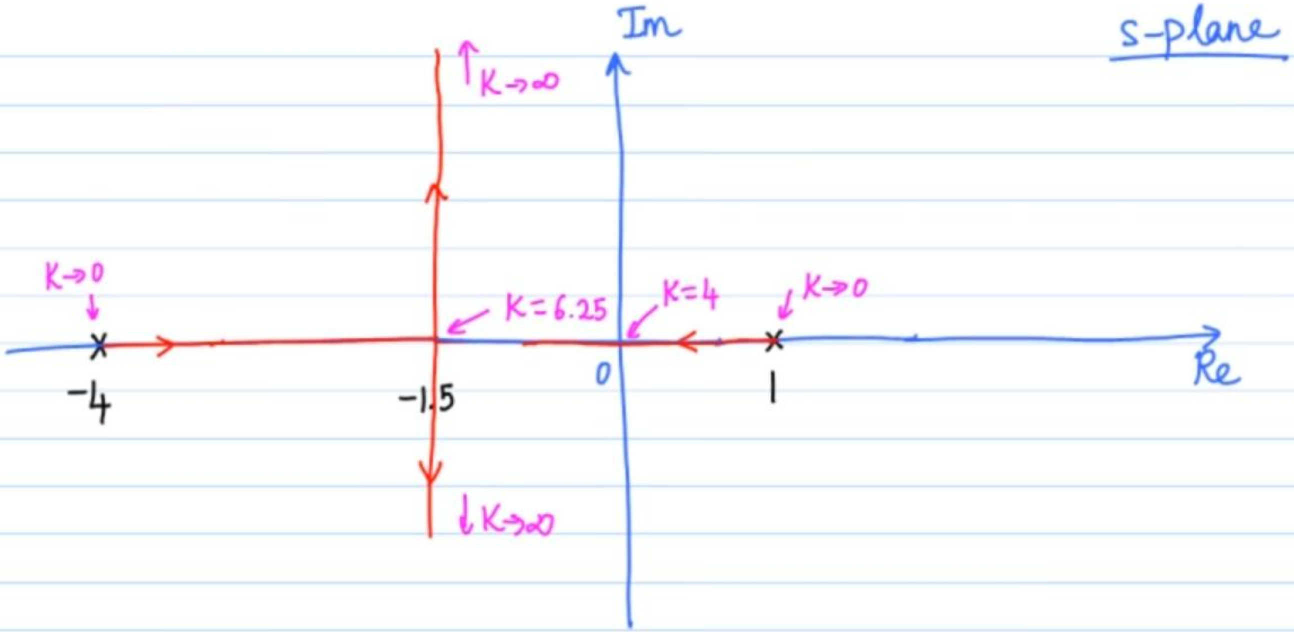
$$\Rightarrow 1 + \frac{K}{s^2+3s-4} = 0. \quad \Rightarrow \quad s^2 + 3s + (K-4) = 0.$$

Substitute $\boxed{s = j\omega} \Rightarrow -\omega^2 + 3j\omega + (K-4) = 0 \Rightarrow \underbrace{[-\omega^2 + K - 4]} + j\,\underbrace{[3\omega]} = 0.$

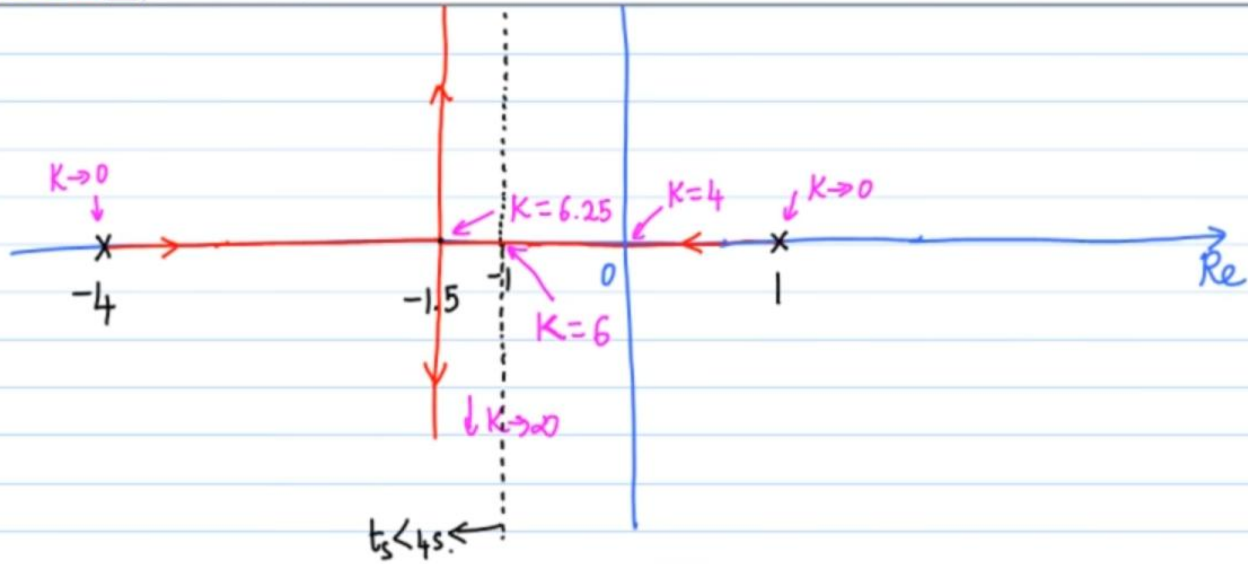
$\Rightarrow \boxed{\omega = 0} \Rightarrow K - 4 = 0 \Rightarrow \boxed{K = 4.} \Rightarrow s = 0$ is a cross-over point where $K = 4$.





The closed loop system stable ∀ $K > 4$.

Q: What range(s) of K would provide a settling time less than 4 s?



$s^2 + 3s + K - 4 = 0.$

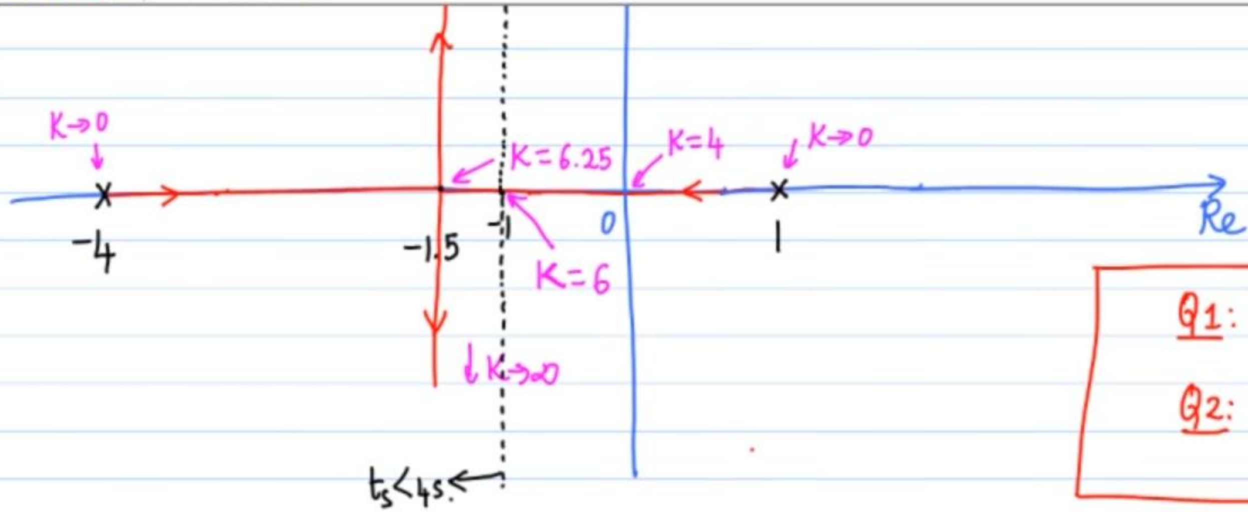
Substitute $s = -1$,

$K = 6.$

The closed loop system stable $\forall \; K > 4$.

Q: What range(s) of K would provide a settling time less than 4 s?

$$t_s = \frac{4}{\xi \omega_n} < 4 \Rightarrow \xi\omega_n > 1 \Rightarrow -\xi\omega_n < -1. \Rightarrow \boxed{K > 6}$$



$s^2 + 3s + K - 4 = 0.$

Substitute $s = -1$,

$K = 6.$

Q1: What happens when $K < 0$?

Q2: What would happen with positive feedback?

The closed loop system stable $\forall\ K > 4$.

Q: What range(s) of K would provide a settling time less than 4 s?

$$t_s = \frac{4}{\xi \omega_n} < 4 \Rightarrow \xi\omega_n > 1 \Rightarrow \boxed{-\xi\omega_n < -1.} \Rightarrow \boxed{K > }$$

Example: