Sampling

Ques. ① How to represent/reconstruct a continuous time signal by/from its samples?

↳ concept of sampling

② What is sampling Theorem?

③ Identify the condition's under which a continuous time signal can be exactly reconstructed from its samples?

↳ What happens if these condition's are not satisfied?

→ We exploit sampling to convert a C.T. signal to its D.T. version, process the D.T. signal using a D.T. system and then convert back to C.T.

→ <u>sampling Theorem</u> :- A C.T. signal can be completely represented by and recoverable from knowledge of its values or samples at points that are equally spaced in time, provided certain conditions are satisfied.

↳ provides a mechanism to represent a C.T. signal by a D.T. signal

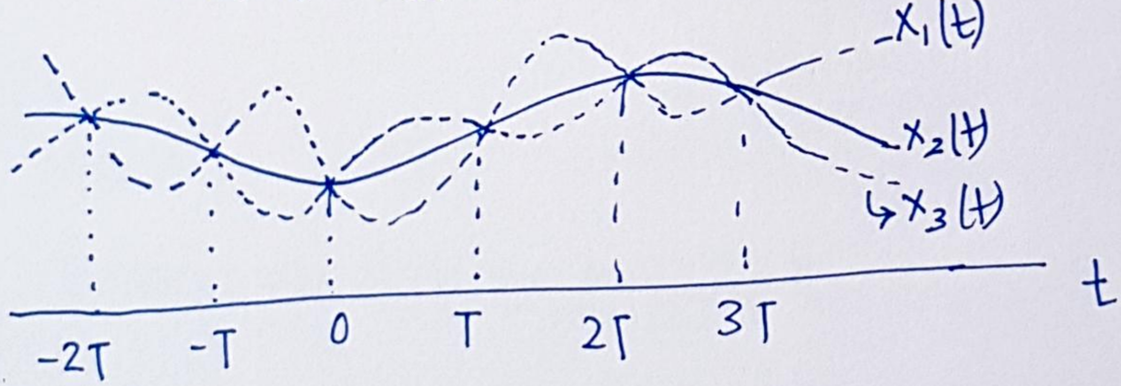
* D.T. versions are easy to work with and prefered over processing of C.T. signals.

Representation of a C.T. signal by its samples :-

- In the absence of any additional conditions or information, we would not expect that a signal could be uniquely specified by a sequence of samples.

- For example, we illustrate three diff. C.T. signals all of which have identical values at integer multiples of T i.e.

$$x_1(kT) = x_2(kT) = x_3(kT)$$



- clearly, an infinite no. of signals can generate a given set of samples.

However,

(i) if a signal is bandlimited, i.e. its Fourier Transform is zero outside a band of frequencies and

(ii) if the samples are taken sufficiently close together in relation to the highest frequency content in the signal

then the samples uniquely specify the signal.

and the C.T. signal can be reconstructed from its samples

$\boxed{\text{SAMPLING THEOREM}}$ Formal Defn.

Little Later!

We next develop the sampling Theorem!

To do so: We need a convenient way to represent the sampling of a C.T. signal at regular intervals!

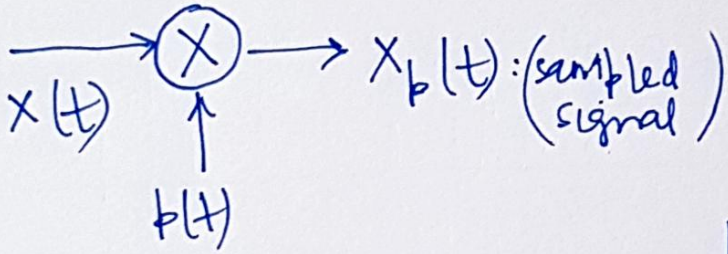
A useful way to do so is through the use of a periodic impulse train multiplied by a C.T. signal $x(t)$ that we wish to sample → mechanism

$\boxed{\text{Impulse Train sampling}}$

Impulse Train Sampling :→

- Let $x(t)$ be a C.T. signal that you wish to sample.
- Let $p(t)$ be a periodic impulse train that you multiply to $x(t)$ in order to sample $x(t)$.

Mechanism of multiplying $x(t)$ with a periodic impulse train $p(t)$ in order to sample $x(t)$: Impulse Train sampling !



periodic impulse train : sampling function

T: fund. period of $p(t)$ sampling period

fundamental freq. of $p(t)$

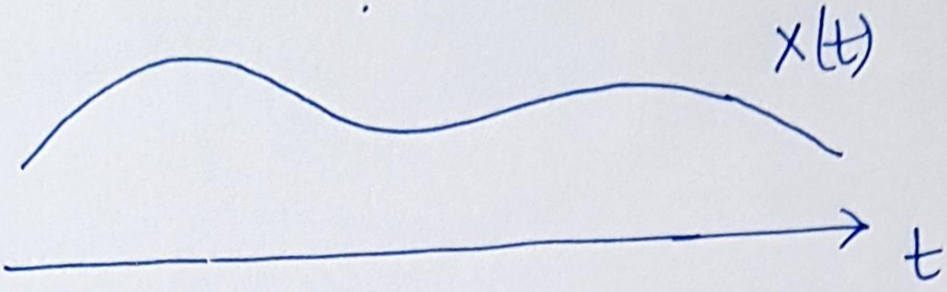
$\omega_s \triangleq \frac{2\pi}{T}$: sampling frequency

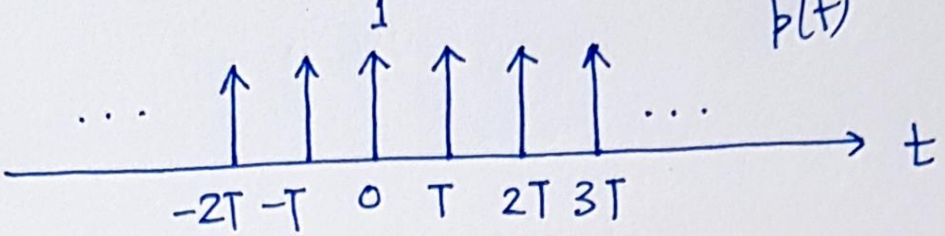
In the time-domain

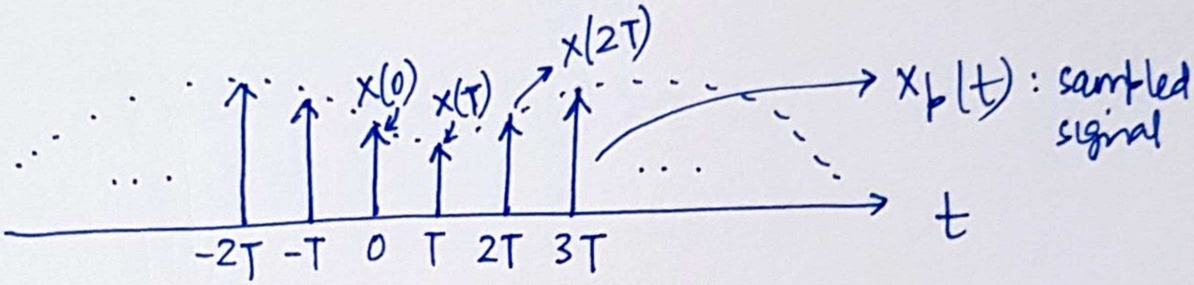
$$x_p(t) = x(t) \cdot p(t)$$

where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

T ↓ sampling period fund. period of $p(t)$.







sampling property of unit impulse: $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

↓

* multiplying the signal $x(t)$ by a unit impulse samples the value of the signal at the point at which the impulse is located.

NOW,

$$x_p(t) = x(t) \cdot p(t)$$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$x_p(t)$: impulse train with the amplitudes of impulses equal to the samples of $x(t)$ at intervals spaced by (T).

Thus.

$$x_p(t) = x(t) \cdot p(t)$$
$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

If we examine this eqn. in the frequency-domain

then $X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right]$

convolution of the F.T. of the original signal and the F.T. of the impulse train

Now, the impulse train $p(t)$ is a periodic signal

its F.T. is itself an impulse train!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \xleftrightarrow{\mathcal{F}} P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

where $\left(\omega_s = \frac{2\pi}{T}\right)$

how

$\hookrightarrow \sum_{n=-\infty}^{\infty} \delta(t-nT) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\cdot\frac{2\pi}{T}\right)$

We know that

$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$

$\because \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega = e^{j\omega_0 t}$

$\therefore$ ~~Any~~ $p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\cdot\omega_0)$

$\sum_{n=-\infty}^{\infty} \delta(t-nT)$

periodic (T)

$\left(=\frac{1}{T}\ \forall k\right)$

(F.S. representation)

$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\cdot\frac{2\pi}{T}\right)$

$\because \left(\omega_0 = \omega_s = \frac{2\pi}{T}\right)$

Furthermore,

$X(j\omega) * \delta(\omega-\omega_0) = X(j(\omega-\omega_0))$

convolution with an impulse simply shifts the signal!

<u>Time-domain</u>

$$x(t) \rightarrow \otimes \rightarrow x_p(t), \quad \uparrow p(t)$$

$$x_p(t) = x(t) \cdot p(t)$$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)$$

$X_p(j\omega)$ is a periodic funcn. of (ω) consisting of a superposition of shifted replicas of $X(j\omega)$ scaled by $(1/T)$.

<u>Freq. domain</u>

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$= \frac{1}{2\pi} X(j\omega) * \left[\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k \cdot \frac{2\pi}{T}\right) \right]$$

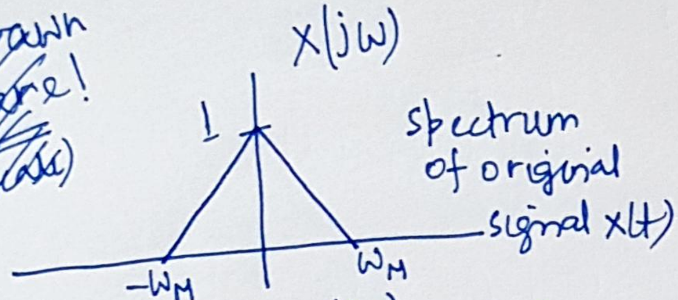
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k \cdot 2\pi/T\right)\right)$$

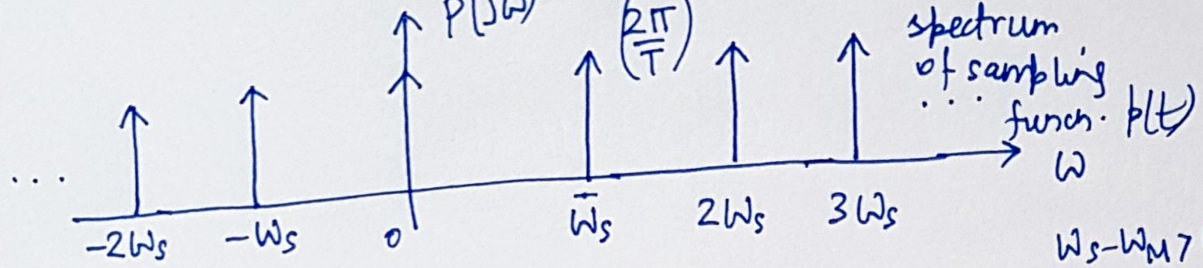
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$

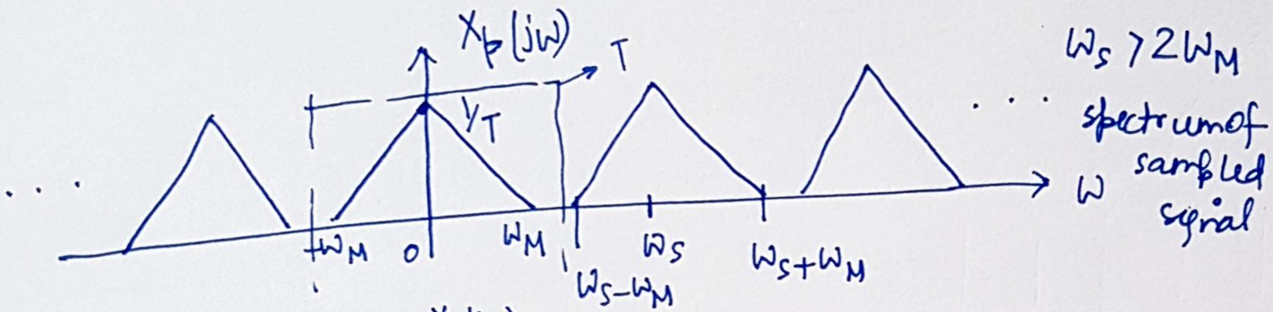
$\downarrow$

F.T. of the sampled signal is the sum of the frequency shifted (replicas) versions of the FT of the original signal $x(t)$.

~~Redrawn here!~~ ~~(To scale)~~

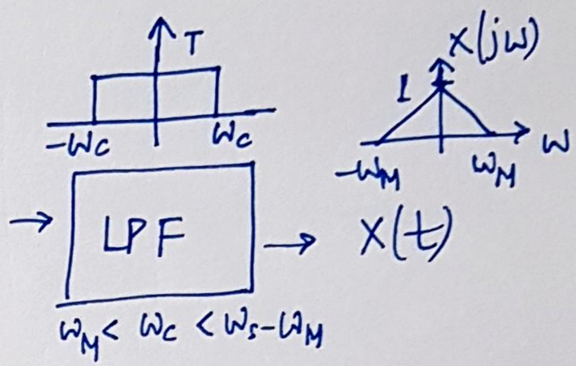




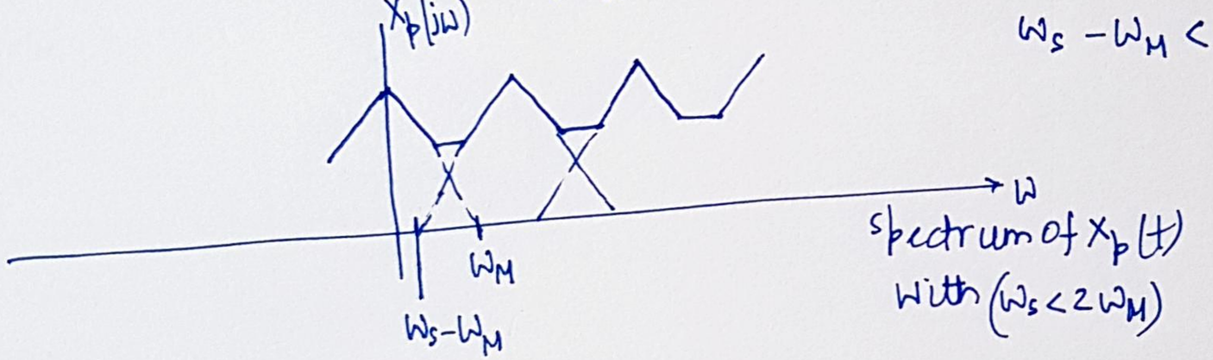


$\omega_s - \omega_M > \omega_M$

$\omega_s > 2\omega_M$



$\omega_s - \omega_M < \omega_M$



Remarks (i) If $\omega_s - \omega_M > \omega_M$ or $\omega_s > 2\omega_M$, there is no overlap between the shifted replicas of $X(j\omega)$.

(ii) If $\omega_s - \omega_M < \omega_M$, there is overlap – distortion – $\boxed{\text{ALIASING}}$

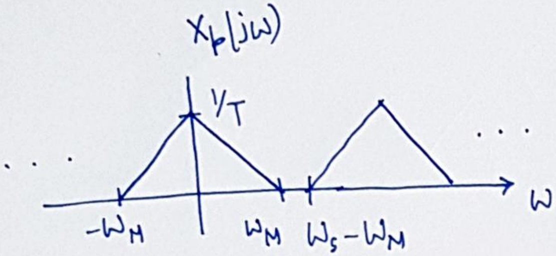
(more in next lecture)

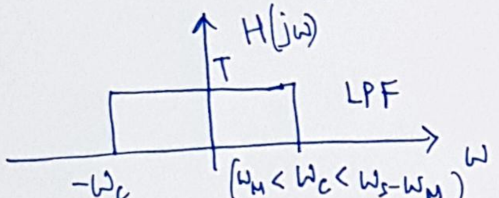
As long as the shifted replicas of the spectrum do not overlap $\omega_s - \omega_M > \omega_M$
or $(\omega_s > 2\omega_M)$

$x(t)$ can be exactly recovered from $x_p(t)$ by means of a LPF with gain (T) and cut-off greater than (ω_M) and less than $(\omega_s - \omega_M)$.

↓

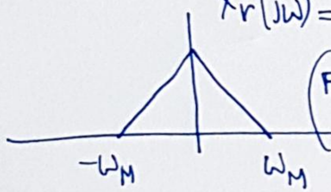
Sampling theorem



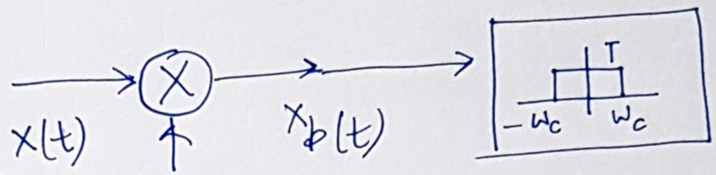


$(\omega_M < \omega_c < \omega_s - \omega_M)$

$X_r(j\omega) = X(j\omega)$



(FAITHFUL RECONSTRUCTION)



$(\omega_s > 2\omega_M)$

highest freq. content (ω_M)

$\omega_M < \omega_c < \omega_s - \omega_M$

$p(t)$: $\sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\omega_s = \frac{2\pi}{T}$

sampling and reconstruction (faithful)

sampling frequency is twice the highest freq. in the original signal!

↓

If $(\omega_s < 2\omega_M)$

↓

ALIASING

↓ distortion

$\omega_s = 2\omega_M$: Nyquist rate

Sampling Theorem

It states that (a) if we have a C.T. signal $x(t)$ and if we have equally spaced samples of that signal $x(nT)$, $n = 0, \pm 1, \pm 2 \ldots$ sampled at the sampling period (T)

(b) and if $x(t)$ is band limited

i.e. $X(j\omega) = 0 \quad , |\omega| > \omega_M$

its Fourier Transform (F.T.) is zero beyond (ω_M)

$\downarrow$ highest frequency contained in $x(t)$

then under the condition that

$$\omega_s \triangleq \frac{2\pi}{T} > 2\omega_M$$

$x(t)$ is uniquely recoverable from the set of samples

i.e. given the set of samples $x(nT)$ one can exactly reconstruct $x(t)$.