ME324 28/09/2022.

Queuing System

The objective of queing analysis is to offer a reasonable satisfactory service to waiting customers.

→ It determines measures of performance of waiting lines.
  * Avg waiting time in que.
  * Productivity of the service facility.
  * Can be used to design service installation.

Elements of Queing system

→ Customer and services
→ Arrival of customer is represented by interarrival time between successive customers.
→ Service is described by the service time per customer
→ Queue size :- Finite or infinite.

→ Queue discipline FCFS, LCFS
  SIRO (Service in Random) etc.

→ Service facility → Single server
  → Parallel server.

→ Server can be arranged
  → series (sequencing?)
  - Network (Router Network)

→ Source → Finite
  → infinite.

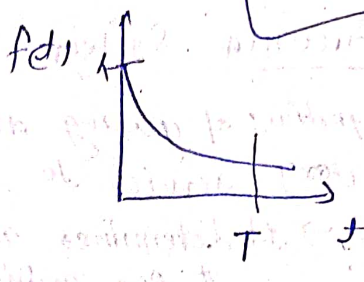
Role of Exponential Distribution

- Arrival of customers is totally random events
- means occurance of an event is not influenced by the length of time, that has elasped, since the occurance of the last event.
- Random interarrival and service time described by exponential distribution.

$$f(t) = \lambda e^{-\lambda t} \quad t>0$$

END GA

Mean $= E\{t\} = \frac{1}{\lambda}$



$$P(t \leq T) = \int_0^T \lambda e^{-\lambda t} dt$$
$$= 1 - e^{-\lambda T}$$

λ: Rate per unit time at which events are generated or occurred

t: time between successive event

S: interval since the occurance of the last event

$$P(t > T+S \mid t > S) = P(t > T)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= P(t > T+S \cap t > S) / P(t > S)$$

$$\Rightarrow \frac{P(t > T+S)}{P(t > S)} = \frac{e^{-\lambda(T+S)}}{e^{-\lambda S}} = e^{-\lambda T} = P(t > T)$$

If A is a subset of B then $P(A \cap B) = P A$.

We know $P(t \leq T) = 1 - e^{-\lambda T}$

Pure Birth and Pure Death model

Pure Birth model

— Only arrivals are allowed.

$P_0(t)$: Probability of no arrival during a time ~~Perid~~ period 't'

Given that → interarrival time is exponential
→ arrival Rate 'λ' customers per unit time.

$$P_0(t) = P(\text{interarrival time} > t)$$
$$= 1 - P(\text{Interarrival time} \leq t)$$
$$= 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$

$P_n(t)$ = Probability of n arrivals during 't'

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 1, 2 \dots$$

Poisson distribution

mean, $E\{n/t\} = \lambda t$

Pure death model

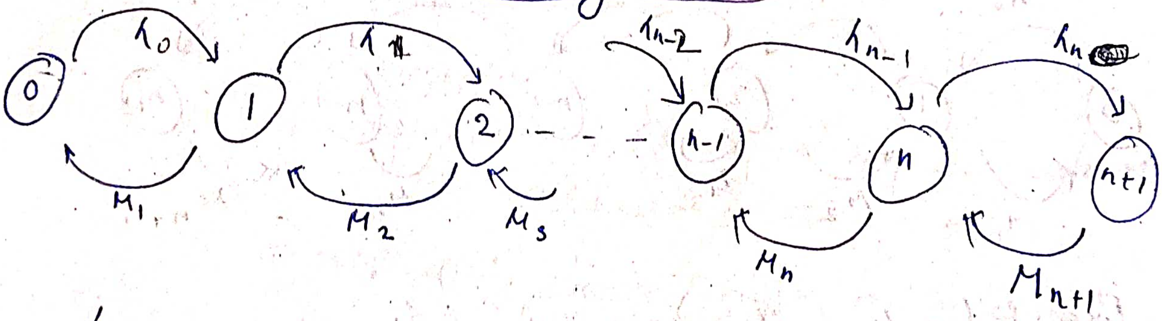
- The system has N customers at time '0' and no new arrival is allowed.
- Only departure can take place

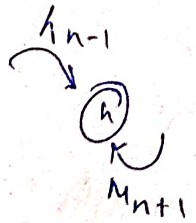
μ : departure rate of customers per unit time.

$$P_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}, \quad n = 1, 2 \dots N$$

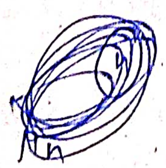
n = remaining no. of customers after time 't'

Generalised Poission Queueing Model:





expected rate of flow into state 'n'



expected rate of flow out of state 'n'

29/07/022

Generalised Poisson Queuing Model

→ Combine both arrival and departure based on the Poisson distribution

→ Model is based on long run or steady state behaviour

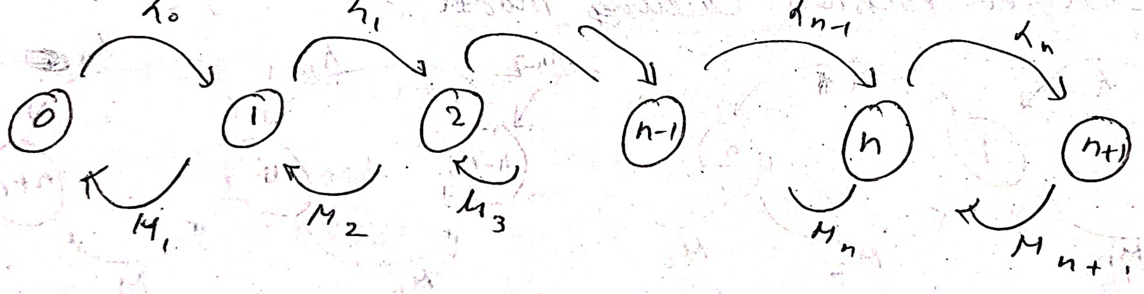
→ Model assumes that both arrival and departure rates are state-dependent meaning they depend on the number of customers in the facility.

n = no. of customers in the system (in queue + in service)

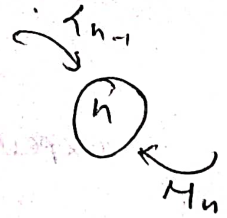
λ_n = Arrival rate given 'n' customers in the system.

μ_n : Departure rate given ————————

P_n = steady-state probability of n customers in the system.



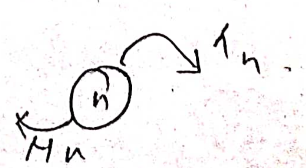
Expected rate of flow into 'n'



$$= \lambda_{n-1} p_{n-1} + \mu_{n+1} P_{n+1}$$

expected rate of flow out of state 'n'

$$= \lambda_n p_n + \mu_n p_n$$



equate, $\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$

$$= (\lambda_n + \mu_n) P_n$$

Let $n=0$,

$$\lambda_0 p_0 = \mu_1 P_1 \quad \text{or} \quad P_1 = \left(\frac{\lambda_0}{\mu_1}\right) P_0$$

$n=1$, $\lambda_0 P_0 + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1$

$$P_2 = \left(\frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}\right) P_0 .$$

$$P_n = \left(\frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1}\right) P_0 .$$

Value of P_0 can be estimated as:

$$\sum_{n=0}^{\infty} P_n = 1$$

Specialised pission Ques

Notation $(a/b/c : d/e/f)$.

a: arrival distribution
b: departure (service time) distribution
c: No. of parallel servers.
d: Que discipline
e: maximum number (finite or infinite) customers allowed in the system.
f: size of calling source (finite or infinite)
λ: arrival rate of customers per unit time.
μ: departure rate ————

Arrival and departure distribution.

M = Markovian (or Poission) distribution

D = constant time.

Que discipline - FCFS, LCFS, SIRO, Generalised distribution

Ex

(M/D/10 : GD/20/∞)

Steady State measures of performance

L_s : expected number of customers in system.
L_q : expected ——————— queue.

W_s : expected waiting time in system.
W_q : expected waiting time in Queue.

$\bar{c}$: expected no. of busy servers.

$$L_s = \sum_{n=1}^{\infty} n P_n \quad , \quad L_q = \sum_{n=c+1}^{\infty} (n-c) P_n$$

Little's Formula

$$L_s = \lambda_{eff} W_s$$
$$L_q = \lambda_{eff} W_q$$

λ_{eff} : effective arrival rate of the system.

$\lambda_{eff} = \lambda$, when all arriving customers can join the system.
otherwise $\lambda_{eff} < \lambda$

$$\begin{Bmatrix} \text{expected waiting} \\ \text{time in system} \end{Bmatrix} = \begin{Bmatrix} \text{Expected waiting} \\ \text{time in queue} \end{Bmatrix} + \begin{Bmatrix} \text{expected service} \\ \text{time} \end{Bmatrix}.$$

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

$$\bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu}$$

Facility utilization $= \frac{\bar{c}}{c}$

No. of spaces = 5

{Poisson's distribution}

arriving rate = 6 car/hour.

Exponential distribution of parking time, mean of 30 min.

$\frac{1}{\mu} = 30$

$\lambda = 6$ car/h. temporary space = 3.

$n = 8 = \{5+3\}$

$\lambda_n = 6$ cars/hour, $n = 0, 1, 2, 3 - - 8$

$$\mu_n = \left\{ \begin{array}{ll} n\left(\frac{60}{30}\right) = 2n, & n = 1, 2, - - 5 \\ 5\left(\frac{60}{30}\right) = 10, & n = 6, 7, 8 \end{array} \right\}$$

$$P_n = \left(\frac{\lambda_{n-1}\lambda_{n-2} - - \lambda_0}{\mu_n \mu_{n-1} - - \mu_1}\right) \cdot P_0$$

$$\frac{\overset{3}{\cancel{6}} \times \overset{3}{\cancel{6}} \times \overset{3}{\cancel{6}}}{\cancel{2} \times 1 \times \cancel{2} \times 2 \times \cancel{2} \times 2 \times 3}$$

$$\sum_{n=0}^{8} P_n = 1$$

Case I $n = 1$, $P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{6}{2n} = \frac{3}{n} P_0$

$n = 2$, $P_2 = \left(\frac{\lambda_1 \lambda_0}{\mu_2 \mu_1}\right) \cdot P_0 = \frac{6^2}{2 \times 2 \times 2 \times 1} P_0 = \frac{3^2}{2!} P_0$

$n = 3$ $P_3 = \frac{3^3}{3!} P_0$

$P_n = \frac{3^n}{n!} P_0$, $n = 1, 2, 3, 4, 5$.

$n = 6$, $P_6 = \left(\frac{\lambda_5 - \lambda_0}{\mu_6 - \mu_1}\right) P_0 = \frac{6^6}{10 \times 2^5 \times 5!} P_0$

$= \frac{3^6}{5 \cdot 5!}$

$P_n = \frac{3^n}{5^{n-5} 5!} P_0$, $n = 6, 7, 8$

$$\sum_{n=0}^{\infty} P_n = P_0 + P_0\left[\frac{3}{1!} + \frac{3^2}{2!} + \cdots \frac{3^5}{5!} + \frac{3^6}{5.5!} + \cdots \frac{3^8}{5^3 5!}\right] = 1$$

$\Rightarrow P_0 = 0.04812$

(b) $\lambda_{eff} = ?$

$\lambda = \lambda_{eff} + \lambda_{lost}$

$\lambda_{lost} = P_8 \lambda = 0.1203$

$\lambda_{eff} = 6 - 0.1203 = 5.8737$

30/04/2022

1. $(M/M/1 : GD/\infty/\infty)$

$\lambda_n = \lambda$

$\mu_n = \mu$ $\Big\}$ $n = 1, 2 \cdots$

$\lambda_{eff} = \lambda$; $\lambda_{lost} = 0$.

we, know, $P_n = \left(\frac{\lambda_{n-1} - \lambda_0}{\mu_n - \mu_1}\right) P_0 = \frac{\lambda^n P_0}{\mu^n} = \rho^n p_0$

where, $\rho = \frac{\lambda}{\mu}$

Geometric series.

$\sum_{n=0}^{\infty} P_n = 1$ or $P_0[1 + \rho + \rho^2 + \cdots] = 1$

① If. $\rho < 1$, $p_0\left[\frac{1}{1-\rho}\right] = 1$

$P_0 = 1 - \rho$

$P_n = \rho^n (1-\rho)$ when $\rho < 1$ and $n = 1, 2 \cdots$

② if $\rho \geq 1$, $\lambda > \mu$.

$\rightarrow$ not a steady state.

$$L_s = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n \rho^n (1-\rho) \quad \text{where } \rho < 1$$

$$\frac{d(\rho^n)}{d\rho} = n\rho^{n-1}$$

$$= (1-\rho)\rho \sum_{n=0}^{\infty} n\rho^{n-1}$$

$$= (1-\rho)\rho \sum_{n=0}^{\infty} \frac{d}{d\rho}(\rho^n)$$

$$= (1-\rho)\rho \frac{d}{d\rho}\left(\sum_{n=0}^{\infty} \rho^n\right)$$

$$= (1-\rho)\rho \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$L_s = \frac{\rho}{1-\rho}$$

$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$$

$$W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$$

$$L_q = \lambda_{eff} W_q = \frac{\rho^2}{1-\rho}$$

$$\bar{c} = L_s - L_q = \rho$$

② $(M/M/1 : GD/N/\infty)$

System capacity $= N$.

$$\lambda_n = \begin{cases} \lambda \; ; & n = 0 \cdots N-1 \\ 0 \; ; & n = N, N+1 \cdots \end{cases}$$

$\mu_n = \mu \qquad ; \quad n = 1, 2, \ldots$

using $\rho = \lambda/\mu$

$$p_n = \begin{cases} \rho^n p_0 & ; n \leq N \\ 0 & ; n > N \end{cases}$$

$$\sum_{n=0}^{N} p_n = 1 \qquad \text{or} \qquad p_0 \left[1+\rho+\rho^2 \cdots \rho^N\right] = 1$$

$$p_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{N+1}} & ; \rho \neq 1 \\ \dfrac{1}{N+1} & ; \rho = 1 \end{cases}$$

$$p_n = \begin{cases} \dfrac{(1-\rho)\rho^n}{1-\rho^{N+1}} & ; \rho \neq 1 \\ \dfrac{1}{N+1} & ; \rho = 1 \end{cases}$$

$$\lambda_{eff} = \lambda - \lambda_{lost} = \lambda - \lambda p_N = \lambda(1-p_N)$$

$$L_s = \sum_{n=0}^{N} n p_n = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n \rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \rho \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \rho \frac{d}{d\rho} \left[\frac{1-\rho^{N+1}}{1-\rho}\right]$$

$$= \frac{\rho\left[1-(N+1)\rho^N + N\rho^{N+1}\right]}{(1-\rho)(1-\rho^{N+1})} \quad , \rho \neq 1$$