5. Special Cases in Simplex Method

- Degeneracy
- Alternative Optima
- Unbounded Solution
- Nonexistence (or infeasible) solution

Degeneracy

- A tie at minimum ratio (leaving variable)
  - Choose arbitrarily
  - One basic variable become zero in the next iteration (Degeneracy)
- One constraint is redundant
- Example

$$\text{Maximize } z = 3x_1 + 9x_2$$

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Equation form

$z - 3x_1 - 9x_2 = 0$

$x_1 + 4x_2 + x_3 = 8$
$x_1 + 2x_2 + x_4 = 4$
$x_1, x_2 \geq 0$

- Use x_3 and x_4 as slack variables

Degeneracy

Iteration	Basic	x_1	x_2	x_3	x_4	Solution	Ratio
0	z	-3	-9	0	0	0	
	x_3	1	4	1	0	8	8/4=2
	x_4	1	2	0	1	4	4/2=2

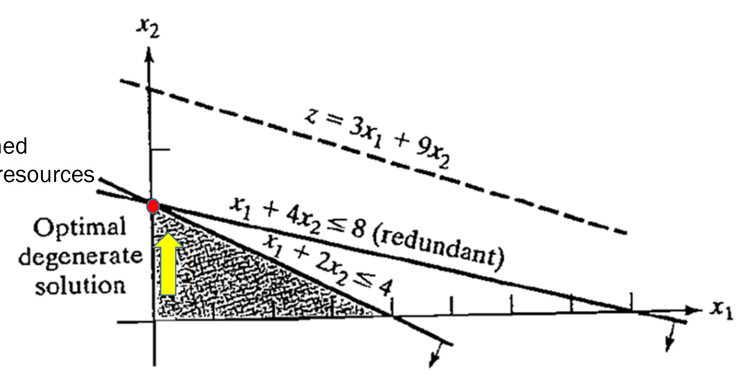
Tie

Cycling
Or
Circling

- Can we stop at iteration 1? No.
  - Temporally degenerate

Degeneracy

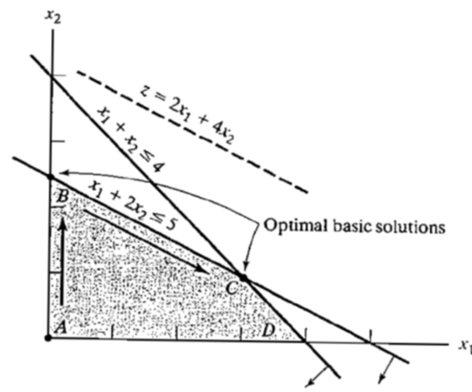
- Overdetermined
- Superfluous resources



Alternate Optima

- Objective function is parallel to non-redundant binding constraint
- Binding constraint: A constraint that is satisfied as an equation at the optimal solution.
- | Maximize $z = 2x_1 + 4x_2$

$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$



Alternate Optima

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	−2	−4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4

- Already get the optima
- Point B in graph

$= 0, x_2 = \frac{5}{2}$, and $z = 10$,

- Optima
- Point C in graph
- Nonzero x_1

$x_1 = 3, x_2 = 1, z = 10$

- All solutions along line BC are optimal.

Unbounded Solution

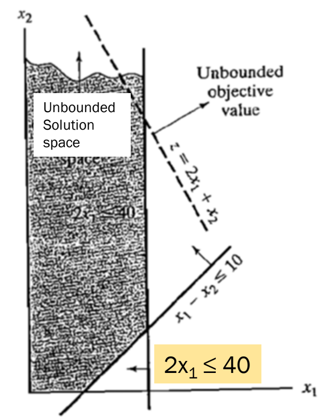
- Objective function value keeps on improving infinitely without violating any constraint
  - At least one variable is unbounded
  - Leads to the conclusion that the model is poorly constructed
- Example

$$\text{Maximize } z = 2x_1 + x_2$$

$$x_1 - x_2 \leq 10$$

$$2x_1 \leq 40$$

$$x_1, x_2 \geq 0$$



Unbounded Solution

Iteration	Basic	X1	X2	X3	X4	Solution	Ratio
0 X1 enters X3 leaves	Z	-2	-1	0	0	0	
	X3	1	-1	1	0	10	10
	X4	2	0	0	1	40	20

- All constraint coefficients under x_3 are either 0 or negative
  - Means no leaving variable and that x_3 can be increased infinitely without violating any constraints.
  - Unbounded problem.

Infeasible Solution

- LP model with inconsistence constraints has no feasible solution.
- This situation will never occur for ≤ type constraints because we can start with slack variables as our basic feasible solutions.
- For other type of constraints, we use artificial variables
  - These artificial variables are forced to become zero at the optima if the model has feasible solution.
  - Otherwise at least one artificial variable will be positive in the optimum iteration

Infeasible Solution

- Example

$$\text{Maximize } z = 3x_1 + 2x_2$$

- Using M-method with M = 100

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	−303	−402	100	0	0	−1200
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	−1	0	1	12

Infeasible Solution

- By allowing R to be positive, the simplex method in essence, has reversed the direction of the inequality from

$$3x_1 + 4x_2 \geq 12 \text{ to } 3x_1 + 4x_2 \leq 12$$

- The result is pseudo-optimal solution.

6. Sensitivity Analysis

- Sensitivity analysis: Change LP parameters (input parameters) within certain limits without altering the optimal solution
- The parameters are usually not exact. With sensitivity analysis, we can ascertain the impact of this uncertainty on the quality of the optimal solution

- Graphical Sensitivity Analysis
  - Two cases
    - Sensitivity of the optimal solution to changes in the availability of resources (RHS of equation)
    - Sensitivity of the optimal solution to changes in unit profit or unit cost (coefficients of the objective function)

Graphical Sensitivity Analysis

JOBCO produces two products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, a unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenues per unit of products 1 and 2 are \$30 and \$20, respectively. The total daily processing time available for each machine is 8 hours.

Letting x_1 and x_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as

$$\text{Maximize } z = 30x_1 + 20x_2$$

subject to

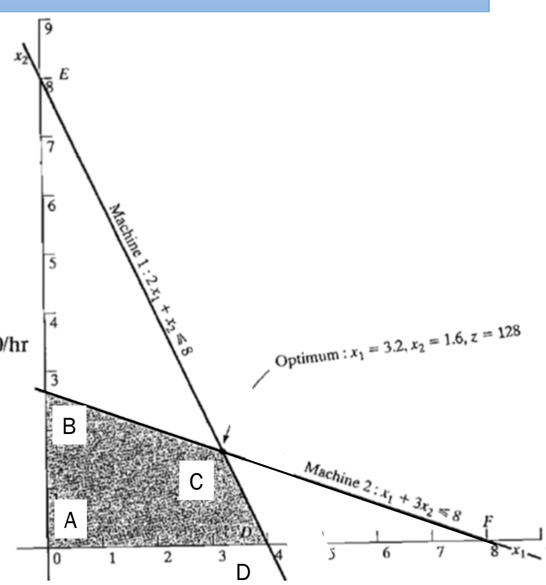
$$\begin{aligned} 2x_1 + x_2 &\le 8 \quad \text{(Machine 1)} \\ x_1 + 3x_2 &\le 8 \quad \text{(Machine 2)} \\ x_1, x_2 &\ge 0 \end{aligned}$$

Graphical Sensitivity Analysis

- Optimal solution using graphical method
- Change RHS of machine 1 equation, that is, from 8 hrs to 9 hrs.
  - Observe change in x_1, x_2 and z
- Rate of change of z

$$\begin{pmatrix} \text{Rate of revenue change} \\ \text{resulting from increasing} \\ \text{machine 1 capacity by 1 hr} \\ \text{(point } C \text{ to point } G\text{)} \end{pmatrix} = \frac{z_G - z_C}{\text{(Capacity change)}} = \frac{142 - 128}{9 - 8} = \$14.00/\text{hr}$$

- Unit worth of resources (\$/hr)
  - Change in the optimal value per unit change in the availability of the resource
- Commonly known as dual or shadow price or Lagrange multiplier



Dual Price and Its Valid Range

- The dual price of \$14/hr remains valid for changes in machine 1 capacity that moves its constraint parallel to itself to any point on the line segment BF.

$$\text{Minimum machine 1 capacity [at } B = (0, 2.67)] = 2 \times 0 + 1 \times 2.67 = 2.67 \text{ hr}$$

$$\text{Maximum machine 1 capacity [at } F = (8, 0)] = 2 \times 8 + 1 \times 0 = 16 \text{ hr}$$

We can thus conclude that the dual price of \$14.00/hr will remain valid for the range

$$2.67 \text{ hrs} \le \text{Machine 1 capacity} \le 16 \text{ hrs}$$

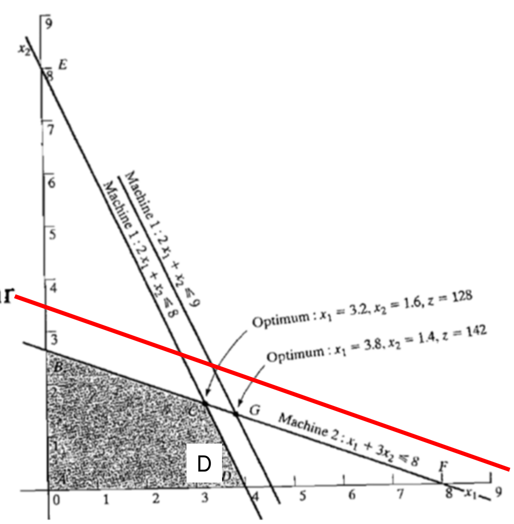
- Outside this range, the dual price will change.

Dual Price and Its Valid Range

- *The dual price for machine 2 is \$2/hr.*
- Its valid range is

$$\text{Minimum machine 2 capacity [at } D = (4, 0)]$$
$$= 1 \times 4 + 3 \times 0 = 4 \text{ hr}$$

$$\text{Maximum machine 2 capacity [at } E = (8, 0)]$$
$$= 1 \times 0 + 3 \times 8 = 24 \text{ hr}$$



- The range is

$$4 \le Machine\ 2\ capacity \le 24$$

Analysis on the Dual Price

Question 1. If JOBCO can increase the capacity of both machines, which machine should receive higher priority?

The dual prices for machines 1 and 2 are \$14.00/hr and \$2.00/hr. This means that each additional hour of machine 1 will increase revenue by \$14.00, as opposed to only \$2.00 for machine 2. Thus, priority should be given to machine 1.

Question 2. A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of \$10/hr. Is this advisable?

For machine 1, the additional net revenue per hour is $14.00 - 10.00 = \$4.00$ and for machine 2, the net is $\$2.00 - \$10.00 = -\$8.00$. Hence, only the capacity of machine 1 should be increased.

Question 3. If the capacity of machine 1 is increased from the present 8 hours to 13 hours, how will this increase impact the optimum revenue?

The dual price for machine 1 is \$14.00 and is applicable in the range (2.67, 16) hr. The proposed increase to 13 hours falls within the feasibility range. Hence, the increase in revenue is $\$14.00(13 - 8) = \$70.00$, which means that the total revenue will be increased to (current revenue + change in revenue) $= 128 + 70 = \$198.00$.

Analysis on the Dual Price

Question 4. Suppose that the capacity of machine 1 is increased to 20 hours, how will this increase impact the optimum revenue?

The proposed change is outside the range (2.67, 16) hr for which the dual price of \$14.00 remains applicable. Thus, we can only make an immediate conclusion regarding an increase up to 16 hours. Beyond that, further calculations are needed to find the answer . Remember that falling outside the feasibility range does *not* mean that the problem has no solution. It only means that we do not have sufficient information to make an *immediate* decision.

Question 5. We know that the change in the optimum objective value equals (dual price × change in resource) so long as the change in the resource is within the feasibility range. What about the associated optimum values of the variables?

The optimum values of the variables will definitely change. However, the level of information we have from the graphical solution is not sufficient to determine the new values. Section

Changes in the Objective Coefficient

- The optima at C (x_1 =3.2, x_2 = 1.6, z=128)
- Changes in revenue units will change the slope of z
- The optima will remain same if the objective function lies between lines BF and DE.

$$\text{Maximize } z = c_1x_1 + c_2x_2$$

- Find the slopes of

$$x_1 + 3x_2 = 8 \text{ and } 2x_1 + x_2 = 8.$$

$$\frac{1}{3} \leq \frac{c_1}{c_2} \leq \frac{2}{1} \quad \text{or} \quad .333 \leq \frac{c_1}{c_2} \leq 2$$

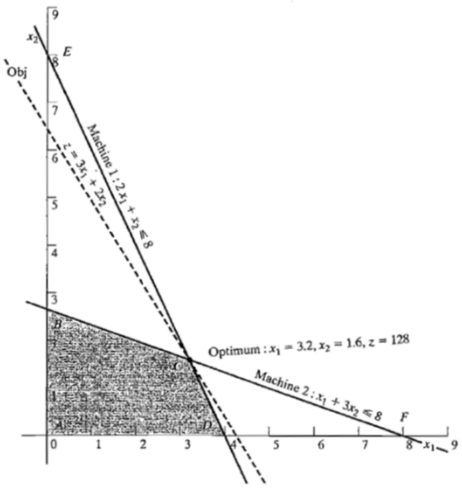


FIGURE 3.13
Graphical sensitivity of optimal solution to changes in the revenue units (coefficients of the objective function)

Changes in the Objective Coefficient

Question 1. Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25, respectively. Will the current optimum remain the same?

The new objective function is

$$\text{Maximize } z = 35x_1 + 25x_2$$

The solution at C will remain optimal because $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ remains within the optimality range (.333, 2). When the ratio falls outside this range, additional calculations are needed to find the new optimum). Notice that although the values of the variables at the optimum point C remain unchanged, the optimum value of z changes to $35 \times (3.2) + 25 \times (1.6) = \$152.00$.

Changes in the Objective Coefficient

Question 2. Suppose that the unit revenue of product 2 is fixed at its current value of $c_2 = \$20.00$. What is the associated range for c_1, the unit revenue for product 1 that will keep the optimum unchanged?

Substituting $c_2 = 20$ in the condition $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$, we get

$$\frac{1}{3} \times 20 \leq c_1 \leq 2 \times 20$$

Or

$$6.67 \leq c_1 \leq 40$$

This range is referred to as the **optimality range** for c_1, and it implicitly assumes that c_2 is fixed at \$20.00.

We can similarly determine the *optimality range* for c_2 by fixing the value of c_1 at \$30.00. Thus,

$$c_2 \leq 30 \times 3 \text{ and } c_2 \geq \frac{30}{2}$$

Or

$$15 \leq c_2 \leq 90$$