The Discrete-Time Fourier Transform

While discussing Fourier series, we saw that there are many similarities in analyzing C.T. & D.T. signals.

However, there are impt. differences as well

→ For Example Fourier series of D.T. signals is a finite series as opposed to the infinite series representation required for C.T. periodic signals.

→ Similarly, there are differences between C.T. Fourier Transform and D.T. Fourier Transforms

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Let us try to understand that!

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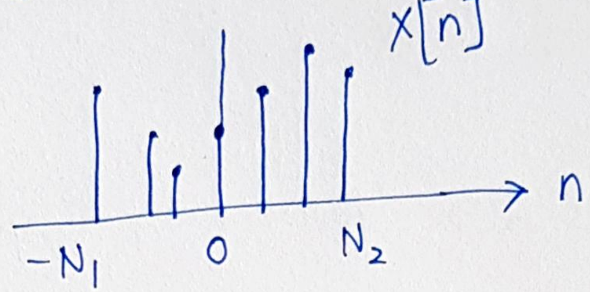
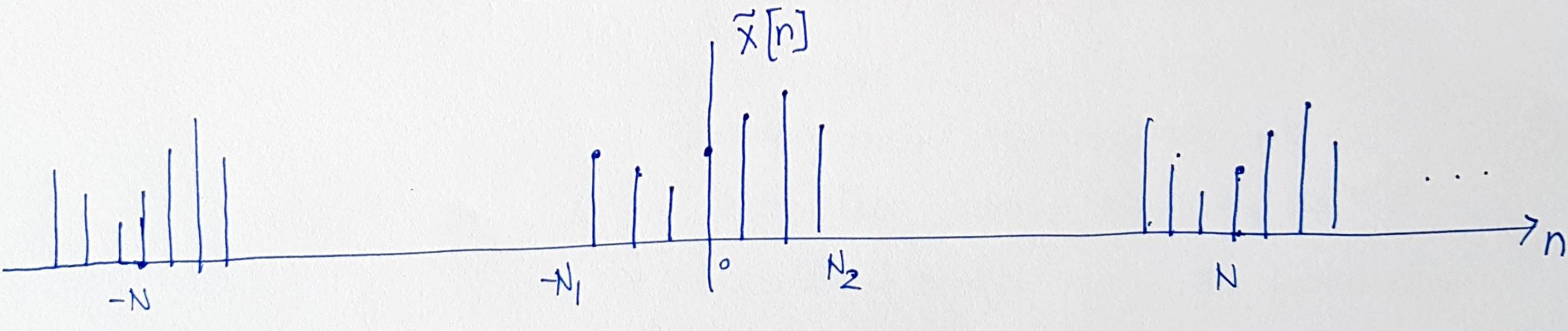
Development of D.T. Fourier Transform

→ Consider a general sequence $x[n]$ that is of finite duration



i.e. for some integers (N_1) and (N_2), $x[n]=0$ outside the range $(-N_1 \leq n \leq N_2)$.

→ From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period as shown :



As we choose the period (N) to be larger,

$\tilde{x}[n]$ is identical to $x[n]$ over a longer interval and as $N \to \infty$

$$\tilde{x}[n] = x[n] \text{ for any finite value of } (n).$$

$\because$ $\tilde{x}[n]$ is periodic with period (N), we can write

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k \, e^{j\frac{2\pi}{N}kn}$$

and

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] \, e^{j\frac{2\pi}{N}kn}$$

<u>Furthermore</u>, $x[n] = \tilde{x}[n]$ over a period that includes the interval $(-N_1 \leq n \leq N_2)$

it is convenient to choose the interval of summation from $(-N_1)$ to (N_2) so that $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation

$$\therefore \quad a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$\because$ $x[n]$ is zero outside the interval $(-N_1 \leq n \leq N_2)$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

Let us define a function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$X(e^{j\omega})$ and $e^{j\omega n}$ are both periodic in (ω) with period (2π).

$\rightarrow e^{j\omega n} = e^{-j(\omega+2\pi)n}$

We see that the coefficients (a_k) are proportional to samples of $X(e^{j\omega})$

i.e. $a_k = \frac{1}{N} X(e^{jk\omega_0})$ where $\boxed{\omega_0 = \frac{2\pi}{N}}$

Thus, $$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0})\, e^{jk\omega_0 n}$$

$\because \boxed{\frac{\omega_0}{2\pi} = \frac{1}{N}}$

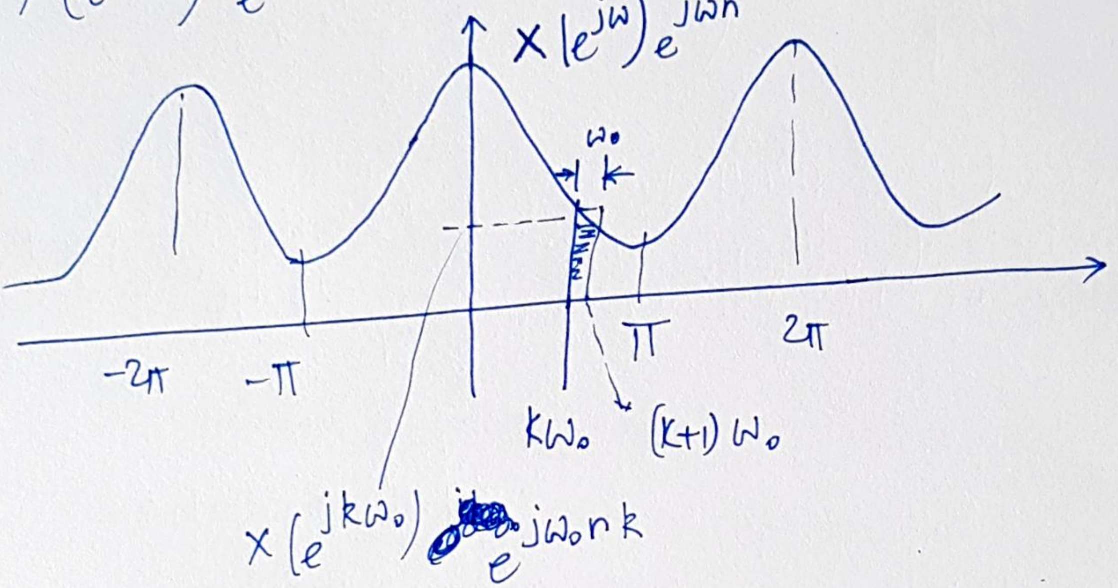
$$\therefore \tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N\rangle} X(e^{jk\omega_0})\, e^{jk\omega_0 n}\, \omega_0$$

Both $X(e^{j\omega})$ and $e^{j\omega n}$ are periodic in (ω) with period (2π).

$\therefore$ The product $(X(e^{j\omega})\, e^{j\omega n})$ is also periodic.

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}\, \omega_0 \quad - (*)$$

Each term in the summation represents the area of the rectangle with height $X\left(e^{jk\omega_0}\right) e^{j\omega_0 n}$ and width ω_0



As $N \to \infty$, $\omega_0 = \frac{2\pi}{N} \to 0$, summation on the LHS of (*) reduces to an integral

$$\tilde{x}[n] \to x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

The interval of integration will always have a width of (2π), $\because$ the summation in (*) is carried out over (N) consecutive intervals each of length $\omega_0 = \frac{2\pi}{N}$

Thus,

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \rightarrow \text{synthesis}$$

(D.T. Aperiodic sequence)

$$\left(\text{DTFT of } x[n]\right) \rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \quad \rightarrow \text{Analysis Equn.}$$

* $X(e^{j\omega})$ is periodic in (ω) with period (2π)

The (DTFT)

[$X(j\omega)$: CTFT is not periodic]

Remarks ① The synthesis Equn. represents $x[n]$ as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $\left(\frac{X(e^{j\omega})\, d\omega}{2\pi}\right)$

② The Fourier Transform $X(e^{j\omega})$

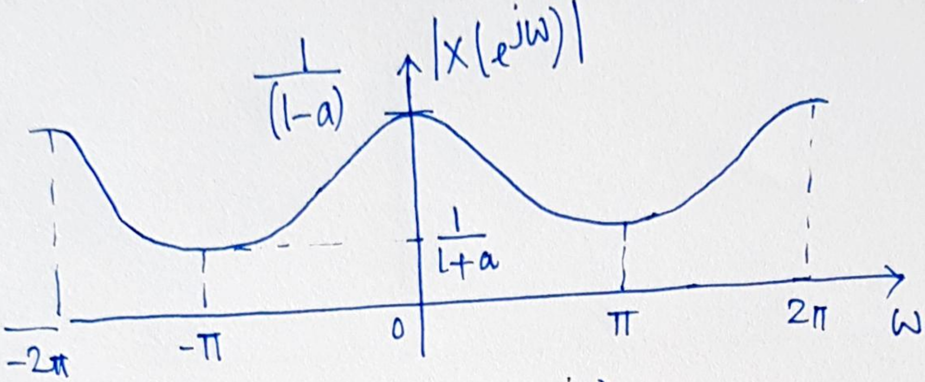
↓ spectrum of $x[n]$

↓ info. about how $x[n]$ is composed of complex exponentials of diff. frequencies !

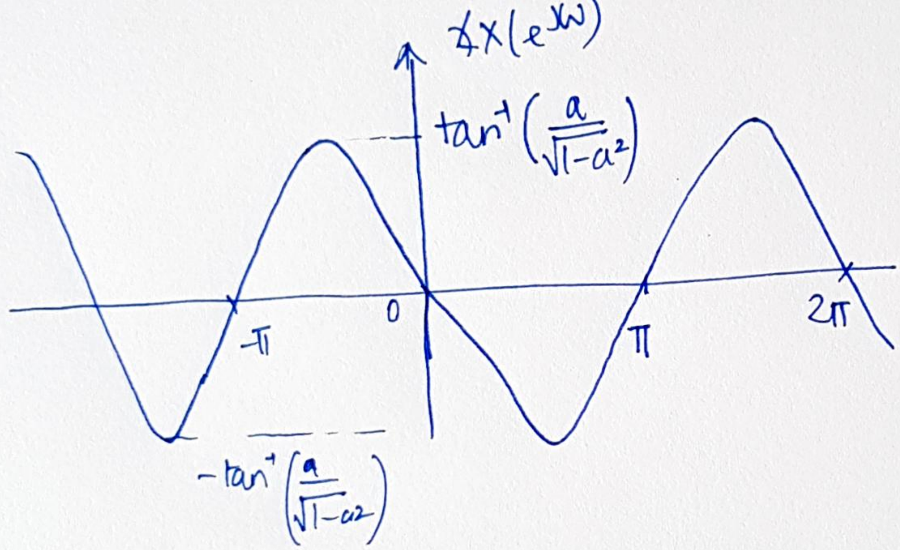
Example ①

Let $x[n] = a^n u[n], \quad |a| < 1$

Then,
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n = \left(\frac{1}{1 - a e^{-j\omega}}\right)$$



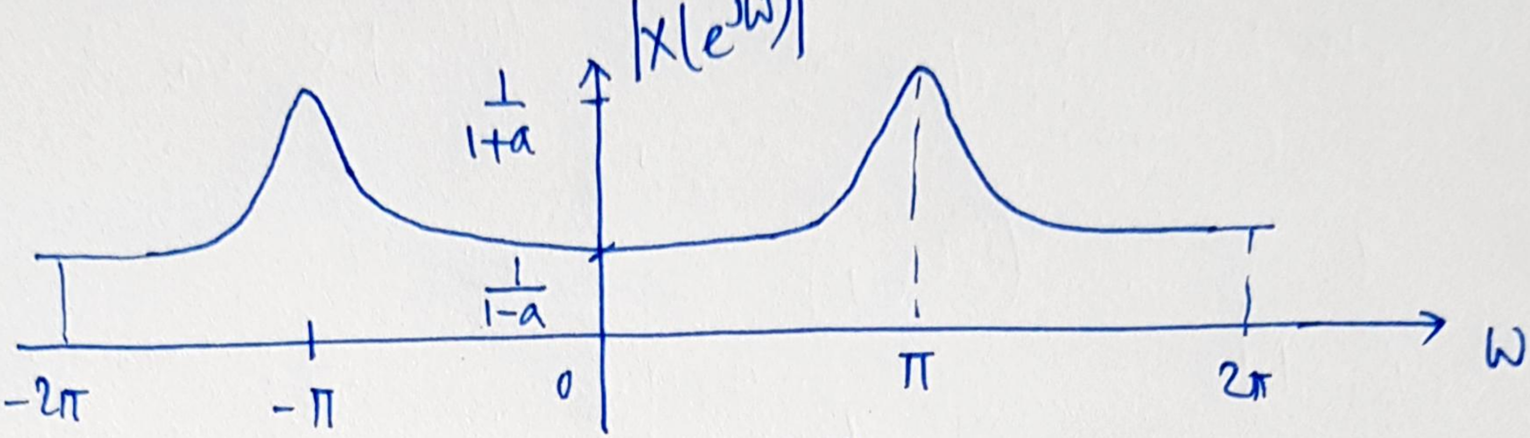
Magnitude of $X(e^{j\omega})$ as a function of (ω) for $\boxed{a > 0}$

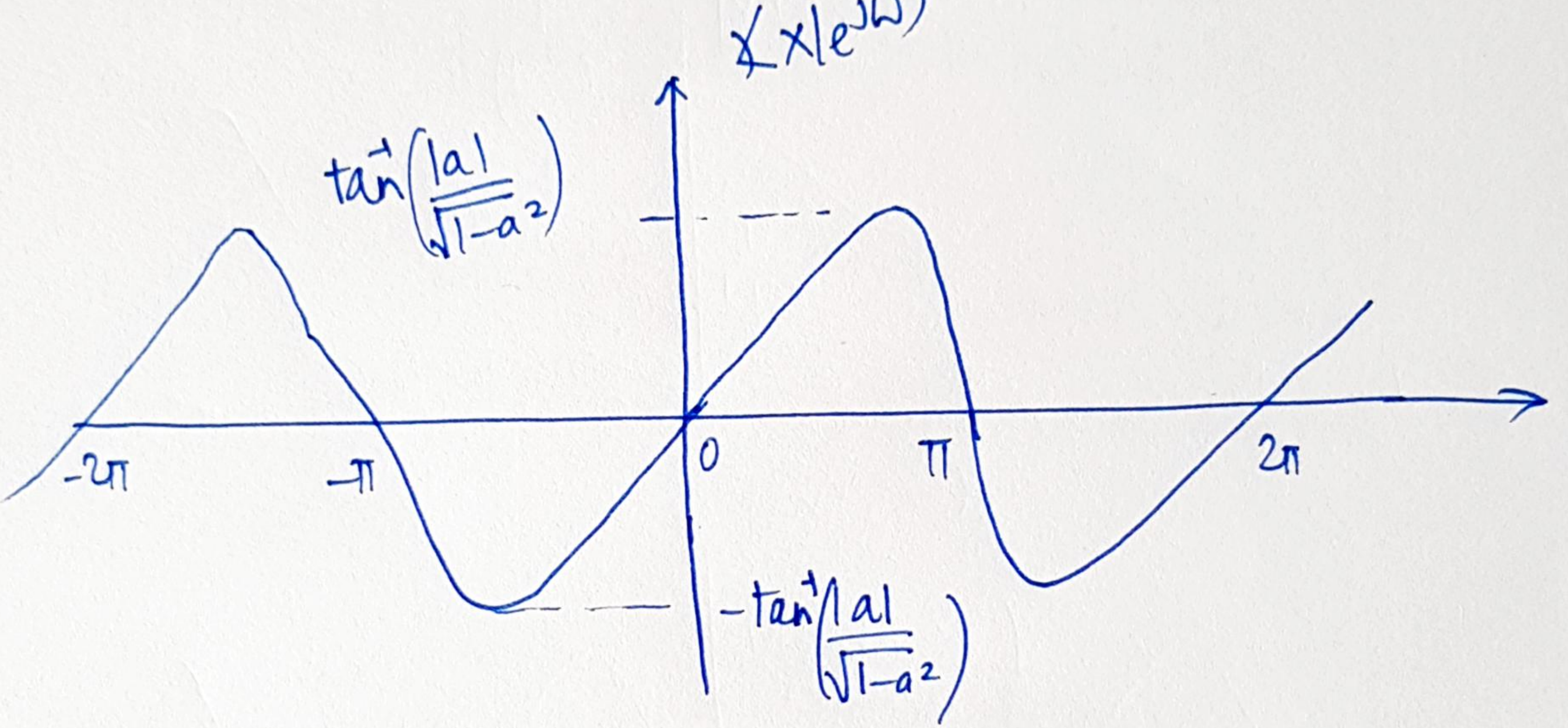


Phase of $X(e^{j\omega})$ for $(a > 0)$

DTFT is periodic in (ω) with period (2π)

\# For $a<0$, the corresponding magnitude & phase plots are as shown below:-





Fourier Transform for Periodic Signals

Consider $x[n] = e^{jw_0 n}$ $\longrightarrow$

In C.T. case,

$$e^{jw_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(w - w_0)$$

how?

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw = x(t)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(w - w_0)\, e^{jwt}\, dw = e^{jw_0 t}$$

We may expect something similar in the D.T. case

However, the DTFT of $x[n]$ must be periodic in (w) with period (2π)

$\downarrow$

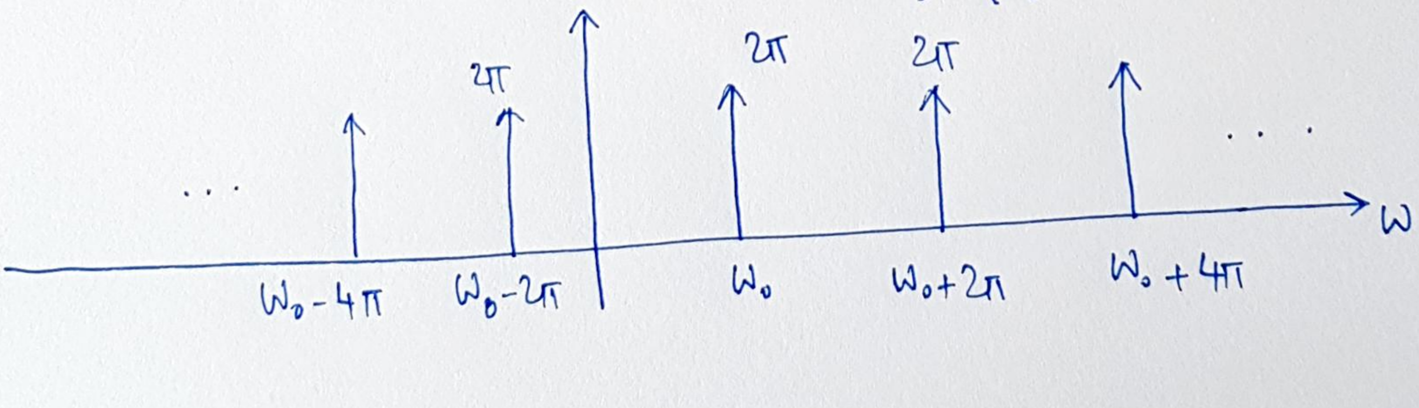
F.T. of $x[n] = e^{jw_0 n}$ must have impulses at w_0, $w_0 \pm 2\pi$, $w_0 \pm 4\pi$ & so on.

While

$$x(t) \rightarrow e^{jw_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{2\pi\,\delta(w - w_0)}_{X(jw)\ \text{(CTFT)}}$$

$$x[n] = e^{jw_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{\sum_{l=-\infty}^{\infty} 2\pi\,\delta(w - w_0 - 2\pi l)}_{X(e^{jw})\ \text{(DTFT)}}$$

FT of $x[n] = e^{jw_0 n}$



Let us check the validity of the DTFT for

$$x[n] = e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)$$

$$\because\ x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)\, e^{j\omega n}\, d\omega \quad -(*)$$

Now, any interval of length (2π) contains exactly one impulse in the summation above

$\therefore$ If the interval of integration chosen includes the impulse located at $(\omega_0 + 2\pi r)$ then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{2\pi} 2\pi\,\delta(\omega-\omega_0-2\pi r)\, e^{j\omega n}\, d\omega$$

$$= e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}.$$

NOW, consider a D.T. periodic sequence $x[n]$ with period (N) and the FS representation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Its F.T.

$$X(e^{jw}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta\left(w - \frac{2\pi k}{N}\right)$$

can be constructed from its F.S. coefficients

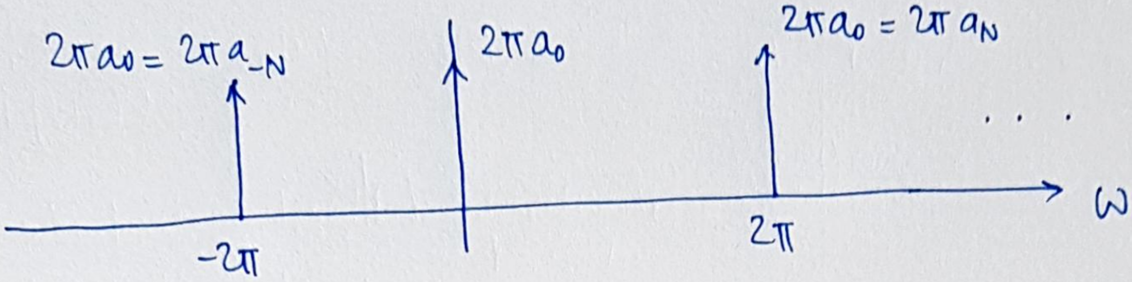
Let us see why this makes sense !

$$x[n] = a_0 + a_1 e^{j\frac{2\pi}{N}n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \cdots$$

$$\cdots + a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n} \quad — \; *** $$

linear combination of signals with $w_0 = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots \frac{(N-1)2\pi}{N}$

F.T. of $a_0 = a_0 e^{j0n}$ [First term in (***)]

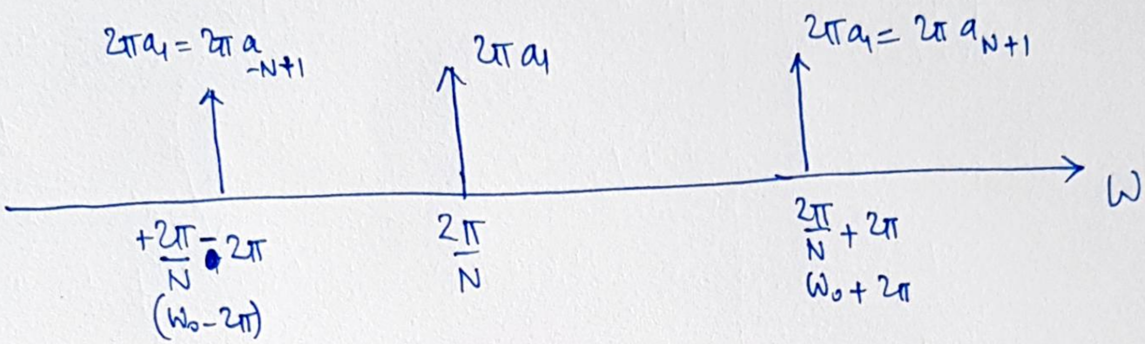


Note that the DTFS coeff. a_k' are periodic in N

$\underline{\underline{a_0 = a_N = a_{-N}}}$

F.T. of $a_1 e^{j\frac{2\pi}{N}n}$

+



+

FT plot of each term in (***)

⋮

$$X(e^{j\omega}) = \ \dots + 2\pi a_0\, \delta\left(\omega - \frac{2\pi \cdot 0}{N}\right) + 2\pi a_1\, \delta\left(\omega - \frac{2\pi}{N} \cdot 1\right) + \dots$$

$$+ 2\pi a_N\, \delta\left(\omega - \frac{2\pi}{N} \cdot N\right) + 2\pi a_{N+1}\, \delta\left(\omega - \frac{2\pi}{N}(N+1)\right) + \dots$$

$$= \ \dots + 2\pi a_0\, \delta(\omega) + 2\pi a_1\, \delta\left(\omega - \frac{2\pi}{N}\right) + \dots +$$

$$2\pi a_N\, \delta(\omega - 2\pi) + 2\pi a_{N+1}\, \delta\left(\omega - \left(2\pi + \frac{2\pi}{N}\right)\right) + \dots$$

Train of impulses occurring at multiples of the fundamental frequency $(2\pi/N)$ with area of impulse located at $\omega = 2\pi k/N$ being $(2\pi a_k)$