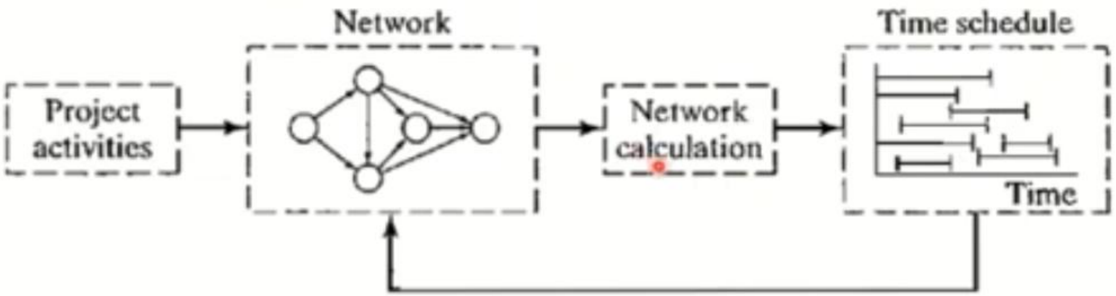
CPM and PERT

- **CPM**: Critical Path Method
- **PERT**: Program Evaluation and Review Technique
  - Both are network based methods to assist in planning, scheduling, and control of projects.
- **Project** can be defined as a collection of interrelated activities and each activity consumes time and resources.
- The objective of CPM and PERT is to provide analytical means for scheduling the activities.



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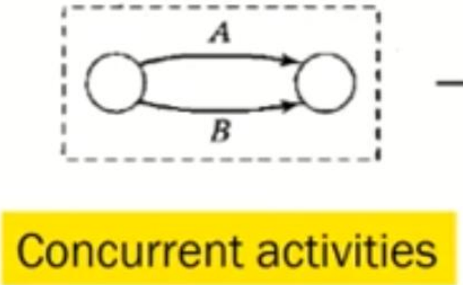
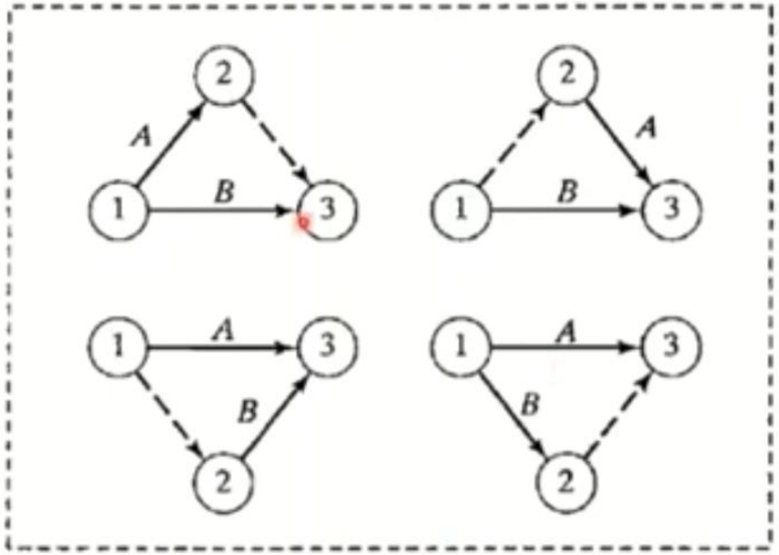
Network Representation

- **Rule 1**: Each activity is represented by, one and only one, arc.
- **Rule 2**: Each activity must be identified by two distinct nodes.

Use of dummy activity to produce unique representation of concurrent activities

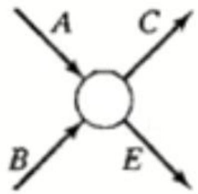




Network Representation

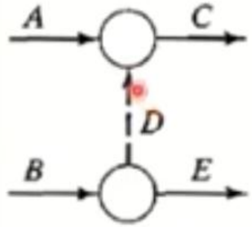
- **Rule 3**: To maintain the correct precedence relationships, the following questions must be answered as each activity is added to the network:
  a) What activities must immediately precede the current activity?
  b) What activities must follow the current activity?
  c) What activities must occur concurrently with the current activity?

- **Example**
  - Activity C starts immediately after A and B have been completed.
  - Activity E starts only after B has been completed



(a)

Wrong

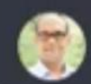


(b)

Example

A publisher has a contract with an author to publish a textbook. The (simplified) activities associated with the production of the textbook are given below. The author is required to submit to the publisher a hard copy and a computer file of the manuscript. Develop the associated network for the project.

Activity	Predecessor(s)	Duration (weeks)
A: Manuscript proofreading by editor	—	3
B: Sample pages preparation	—	2
C: Book cover design	—	4
D: Artwork preparation	—	3
E: Author's approval of edited manuscript and sample pages	*A*, *B*	2
F: Book formatting	*E*	4
G: Author's review of formatted pages	*F*	2
H: Author's review of artwork	*D*	1
I: Production of printing plates	*G*, *H*	2
J: Book production and binding	*C*, *I*	4



Example...

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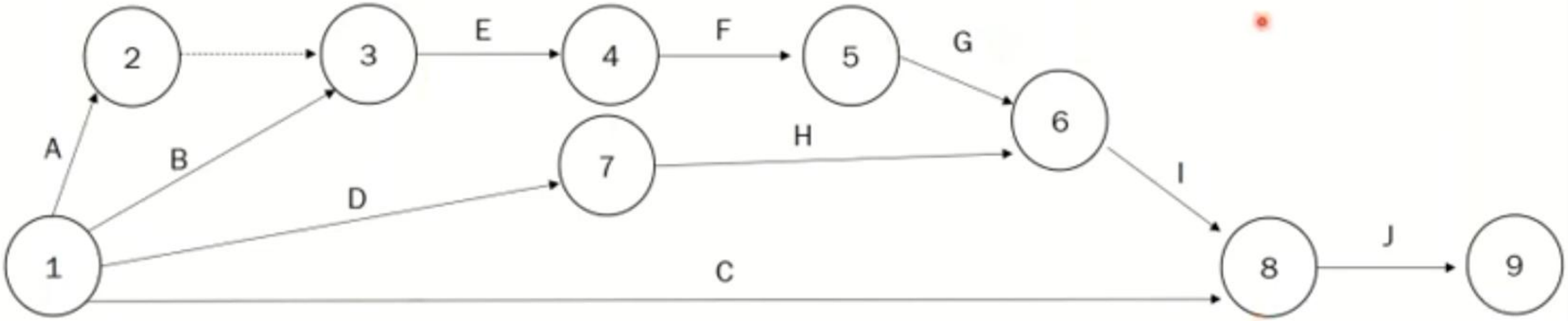

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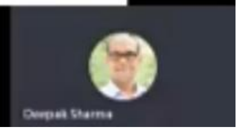
Critical Path Method (CPM)

- Construction of time schedule for the project
  - Total time needed to complete the project.
  - Classification of activities of the project as critical or noncritical.
    - Critical activity: There is no leeway in determining its start and finish time.
    - Noncritical activity: allows some scheduling slack, so that the start time of activity can be advanced or delayed within limits without effecting the completion day of the entire project.
  - Event: A point at which activities are terminated and others are started.
    - In a network, it corresponds to a node.

$$\square_j = \text{Earliest occurrence time of event } j$$
$$\Delta_j = \text{Latest occurrence time of event } j$$
$$D_{ij} = \text{Duration of activity } (i, j)$$

- Two passes in CPM
  - Forward pass for earlier occurrence times
  - Backward pass for latest occurrence times



CPM

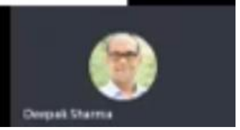
Forward Pass (Earliest Occurrence Times, □). The computations start at node 1 and advance recursively to end node n.

Initial Step. Set $\square_1 = 0$ to indicate that the project starts at time 0.

General Step j. Given that nodes $p, q, \ldots,$ and v are linked *directly* to node j by incoming activities $(p, j), (q, j), \ldots,$ and (v, j) and that the earliest occurrence times of events (nodes) $p, q, \ldots,$ and v have already been computed, then the earliest occurrence time of event j is computed as

$$\square_j = \max\{\square_p + D_{pj}, \square_q + D_{qj}, \ldots, \square_v + D_{vj}\}$$

The forward pass is complete when $\square_n$ at node n has been computed. By definition $\square_j$ represents the longest path (duration) to node j.



CPM

Backward Pass (Latest Occurrence Times, Δ). Following the completion of the forward pass, the backward pass computations start at node n and end at node 1.

Initial Step. Set $\Delta_n = \square_n$ to indicate that the earliest and latest occurrences of the last node of the project are the same.

General Step *j*. Given that nodes $p, q, \ldots,$ and v are linked *directly* to node j by *outgoing* activities $(j, p), (j, q), \ldots,$ and (j, v) and that the latest occurrence times of nodes $p, q, \ldots,$ and v have already been computed, the latest occurrence time of node j is computed as

$$\Delta_j = \min\{\Delta_p - D_{jp}, \Delta_q - D_{jq}, \ldots, \Delta_v - D_{jv}\}$$

The backward pass is complete when Δ_1 at node 1 is computed. At this point, $\Delta_1 = \square_1$ $(= 0)$.

CPM

Based on the preceding calculations, an activity (i, j) will be *critical* if it satisfies three conditions.

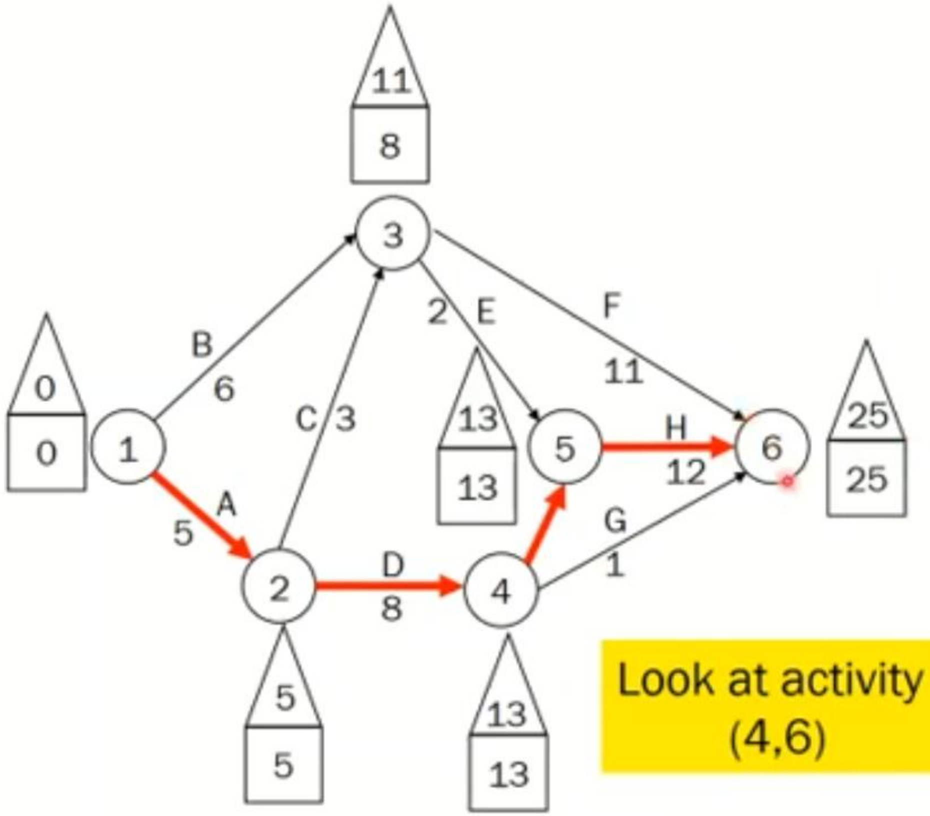
1. $\Delta_i = \square_i$
2. $\Delta_j = \square_j$
3. $\Delta_j - \Delta_i = \square_j - \square_i = D_{ij}$

Forward Pass

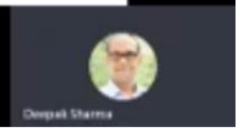
Node 1. Set $\square_1 = 0$
Node 2. $\square_2 = \square_1 + D_{12} = 0 + 5 = 5$
Node 3. $\square_3 = \max\{\square_1 + D_{13}, \square_2 + D_{23}\} = \max\{0 + 6, 5 + 3\} = 8$
Node 4. $\square_4 = \square_2 + D_{24} = 5 + 8 = 13$
Node 5. $\square_5 = \max\{\square_3 + D_{35}, \square_4 + D_{45}\} = \max\{8 + 2, 13 + 0\} = 13$
Node 6. $\square_6 = \max\{\square_3 + D_{36}, \square_4 + D_{46}, \square_5 + D_{56}\}$
$= \max\{8 + 11, 13 + 1, 13 + 12\} = 25$

Backward Pass

Node 6. Set $\Delta_6 = \square_6 = 25$
Node 5. $\Delta_5 = \Delta_6 - D_{56} = 25 - 12 = 13$
Node 4. $\Delta_4 = \min\{\Delta_6 - D_{46}, \Delta_5 - D_{45}\} = \min\{25 - 1, 13 - 0\} = 13$
Node 3. $\Delta_3 = \min\{\Delta_6 - D_{36}, \Delta_5 - D_{35}\} = \min\{25 - 11, 13 - 2\} = 11$
Node 2. $\Delta_2 = \min\{\Delta_4 - D_{24}, \Delta_3 - D_{23}\} = \min\{13 - 8, 11 - 3\} = 5$
Node 1. $\Delta_1 = \min\{\Delta_3 - D_{13}, \Delta_2 - D_{12}\} = \min\{11 - 6, 5 - 5\} = 0$



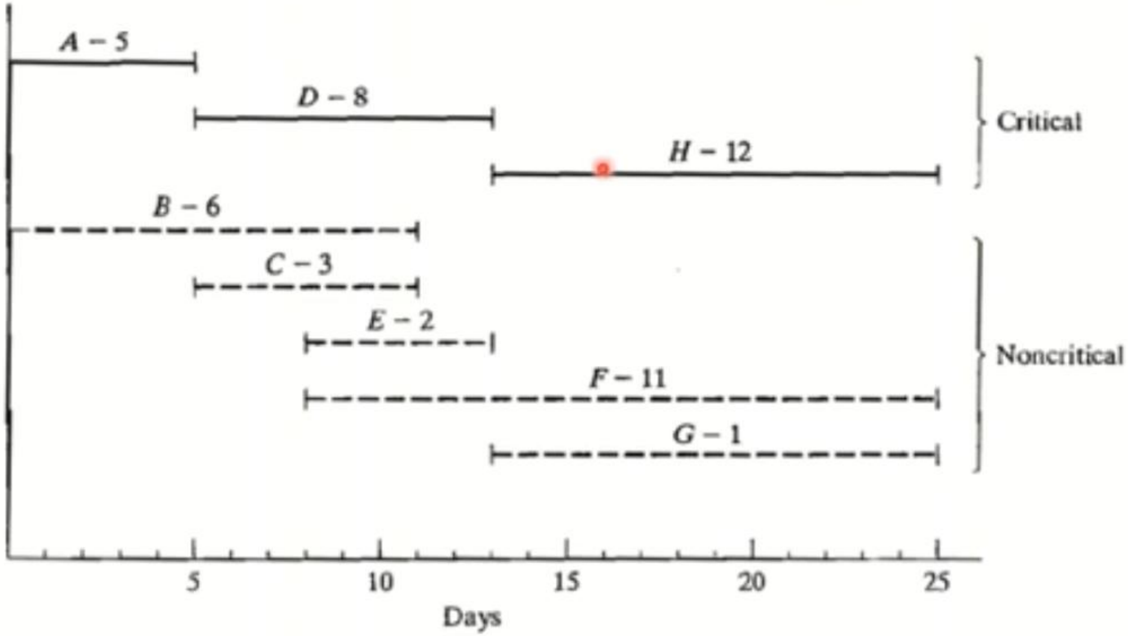
Look at activity (4,6)

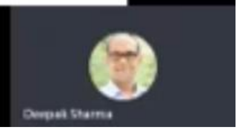


Construction of Time Schedule

- Use information of CPM to make time schedule
- Interval $(\square_i, \Delta_j)$ delineates the (maximum) span during which activity (i,j) may be scheduled without delaying the entire project.

- Critical activities: solid lines
- Noncritical activities: dashed lines
  - Start as early as possible





Determination of Floats

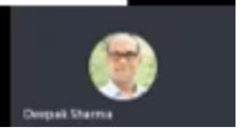
- **Floats** are the slack time available within the allotted span of noncritical activity.
  - Total float $TF_{ij} = \Delta_j - \square_i - D_{ij}$
  - Free float $FF_{ij} = \square_j - \square_i - D_{ij}$

$$\boxed{\text{By definition, } FF_{ij} \leq TF_{ij}.}$$

Red-Flagging Rule. *For a noncritical activity (i, j)*

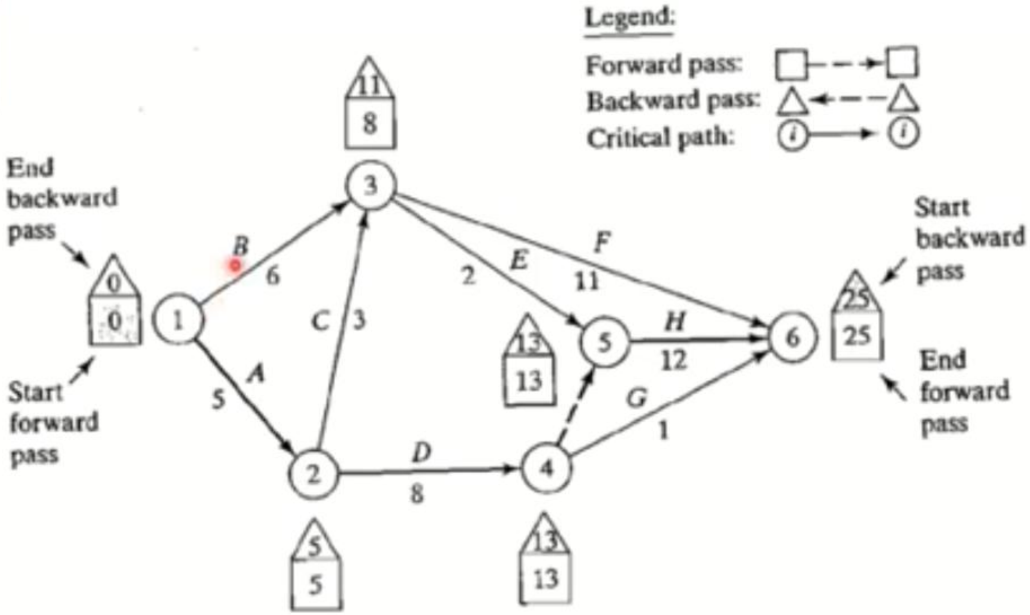
(a) *If $FF_{ij} = TF_{ij}$, then the activity can be scheduled anywhere within its $(\square_j, \Delta_j)$ span without causing schedule conflict.*

(b) *If $FF_{ij} < TF_{ij}$, then the start of the activity can be delayed by at most FF_{ij} relative to its earliest start time $(\square_i)$ without causing schedule conflict. Any delay larger than FF_{ij} (but not more than TF_{ij}) must be coupled with an equal delay relative to $\square_j$ in the start time of all the activities leaving node j.*

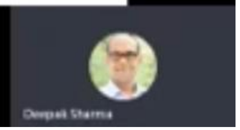


Floats

- Floats for noncritical activities



Noncritical activity	Duration	Total float (*TF*)	Free float (*FF*)
$B(1,3)$	6	$11 - 0 - 6 = 5$	$8 - 0 - 6 = 2$
$C(2,3)$	3	$11 - 5 - 3 = 3$	$8 - 5 - 3 = 0$
$E(3,5)$	2	$13 - 8 - 2 = 3$	$13 - 8 - 2 = 3$
$F(3,6)$	11	$25 - 8 - 11 = 6$	$25 - 8 - 11 = 6$
$G(4,6)$	1	$25 - 13 - 1 = 11$	$25 - 13 - 1 = 11$

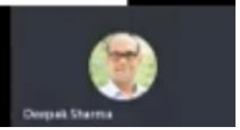


PERT: Program Evaluation and Review Technique

- **Optimistic time estimate**, t_0 or *a*, which occurs when execution goes extremely well.
- **Most likely time estimate**, t_m or *m*, which occurs when execution is done under normal conditions.
- **Pessimistic time**, t_p or *b*, which occurs when execution goes extremely poorly.
- The range *(a, b)* encloses all possible estimates of the duration of an activity.
- Average duration time, $\overline{D}$ or t_e and variance, ϑ

$$\overline{D} = \frac{a + 4m + b}{6}$$

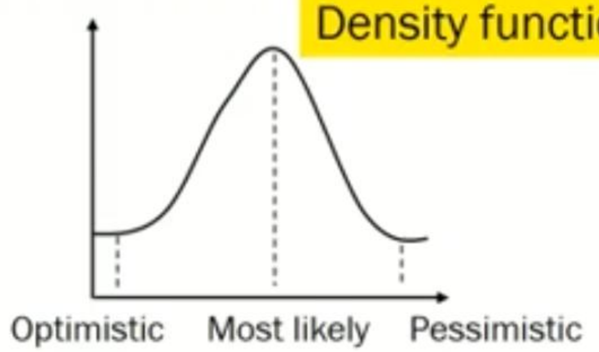
$$v = \left(\frac{b - a}{6}\right)^2$$



PERT

- Probability distribution

Density function: $f(x) = K(x-a)^{\alpha}(b-x)^{\beta}$



- Assumptions
  - The activity durations are independent, that is, the time required to complete an activity will have no bearing on the completion time of any activity of the project.
  - The activity follows β-distribution with t_e and standard deviation $\sigma_1 = \frac{b-a}{6}$

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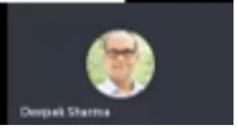
PERT Procedure

1. Draw the project network.
2. Compute the expected duration of each activity t_e.
3. Compute the expected variance ϑ of each activity.
4. Compute the earliest start, earliest finish, latest start, latest finish, and total float of each activity.
5. Determine the critical path and identify critical activities.
6. Computer the expected variance of the project length (also called the variance of the critical path) which is the sum of the variances of all critical activities.
7. Compute the expected standard deviation of the project length σ_c and calculate the normal deviation $\frac{T_S - T_E}{\sigma_c}$
   where, T_S = specified or scheduled time to complete the project
   T_E =normal expected duration (duration of the project)
   σ_c = expected standard deviation of the project length.



PERT Example

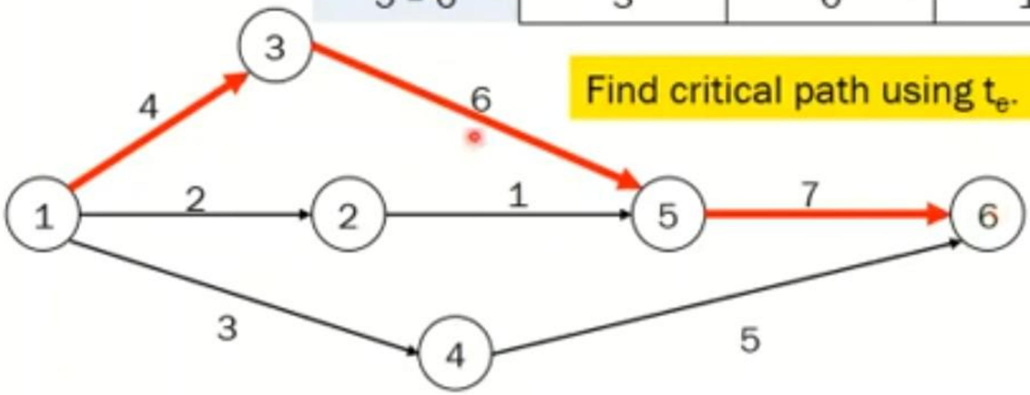
- A small project is composed of activities whose time estimate are listed in the table: Activities are identified by their beginning (*i*) and ending (*j*) node numbers.
  a) Draw the project network.
  b) Find the expected duration and variance for each activity, What is the expected project length?
  c) Calculate the variance and standard deviation of the project length. What is the probability that the project will be completed.
     i. At least 4 weeks earlier than expected?
     ii. No more than 4 weeks later than expected?
  d) If the project due is 19 in weeks, what is the probability of meeting the due date?

PERT Example

$$\overline{D} = \frac{a + 4m + b}{6}$$

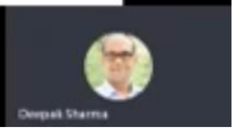
$$v = \left(\frac{b - a}{6}\right)^2$$

Activity	Estimated Duration (weeks)			mean	Variance
i-j	*Optimistic (a)*	*Most likely (m)*	*Pessimistic (b)*	t_e or $\bar{D}$	ϑ
1 - 2	1	1	7	2	1
1 - 3	1	4	7	4	1
1 - 4	2	2	8	3	1
2 - 5	1	1	1	1	0
3 - 5	2	5	14	6	4
4 - 6	2	5	8	5	1
5 - 6	3	6	15	7	4



Find critical path using t_e.

- Duration of project = 17 days
- Variance of the project length = sum of the variances of the critical activities
- $\sigma_c = 3$



PERT Example

Calculate the variance and standard deviation of the project length. What is the probability that the project will be completed.

i. At least 4 weeks earlier than expected?
ii. No more than 4 weeks later than expected?

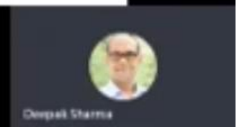
- The standard normal deviation is

$$z = \frac{Due\ date - Expected\ date\ of\ completion}{\sqrt{Variance}}$$

i. $T_S = 13$, $z = (13-17)/3 = -1.33$
$P(T_S \leq 13) = P(z \leq -1.33)$
$= 0.5 - \phi(1.33) = 0.5 - 0.4082 = 0.0918$ or 9.18%

ii. $T_S = 21$, $z = (21-17)/3 = 1.33$
$P(T_S \leq 21) = P(z \leq 1.33)$
$= 0.5 + \phi(1.33) = 0.5 + 0.4082 = 0.9082$ or 90.82%

Z	0.5	0.67	1.00	1.33	2.00
p	0.1915	0.2486	0.3413	0.4082	0.4772



PERT Example

- If the project due is 19 in weeks, what is the probability of meeting the due date?

- When due date is 19 weeks, z = 19-17/3 = 0.67
  $P(T_S \leq 19) = P(z \leq 0.67)$
  $= 0.5 + \phi(0.67) = 0.5 + 0.2486 = 0.7486$ or 74.86%
- The probability of not meeting the due date is $1 - 0.7486 = 0.2514$ or 25.14%.

Z	0.5	0.67	1.00	1.33	2.00
p	0.1915	0.2486	0.3413	0.4082	0.4772