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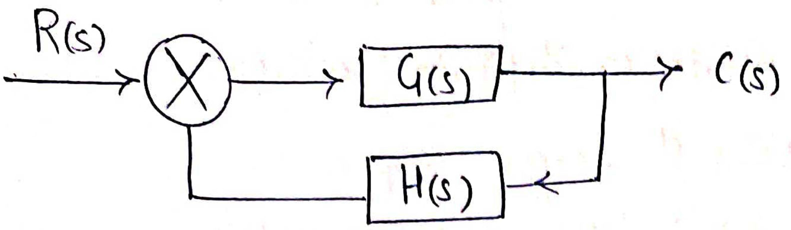
ME 325 → Control Systems



OLTF = $G(s)H(s)$

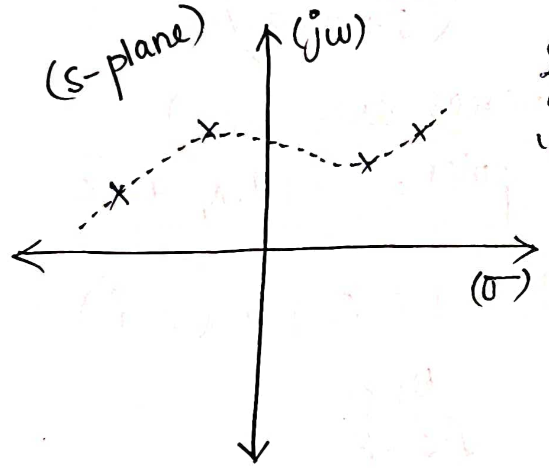
CLTF = $\dfrac{G(s)}{1+G(s)H(s)}$

$$G(s)H(s) = 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

K → (gain); $(s+z_m)$ → roots (zeroes); $(s+p_n)$ → (poles)

(Denominator = 0) } characteristic Eq.

for closed loop char. eq ⇒ $(1+G(s)H(s)=0)$



$\{-\alpha < K < +\alpha\}$

"- - - -" = root locus of the roots.

→ Root locus Method :-

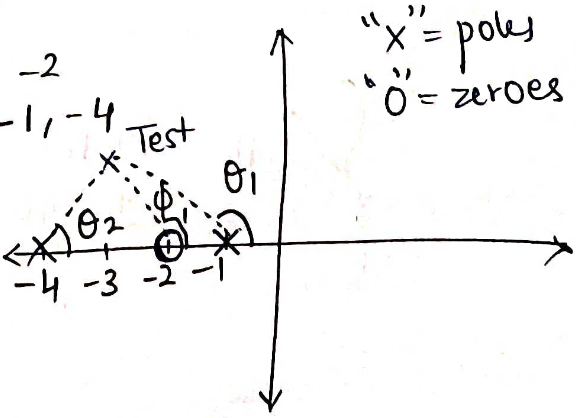
⇒ Conditions

(1) $\angle G(s)H(s) \Rightarrow \pm 180°(2k+1)$ for $k = 0,1,2,3\cdots$ } angle condition

(2) $|GH| = 1$ } magnitude condition.

Eg :- $GH = \dfrac{K(s+2)}{(s+1)(s+4)}$ ⇒ zeroes = −2, poles = −1, −4

"X" = poles
"O" = zeroes



$$\angle G(s)H(s) = \phi_1 - (\theta_1 + \theta_2)$$

General ⇒ $\left(\sum_m \phi_m - \sum_n \theta_n\right) = \psi$

Step (1) Locate the poles and zeroes of OLTF on S-plane.

(2) The root locus branch starts at OL poles & ends at zeroes that maybe at finite or infinite position.

(3) Because of the complex poles & zeroes appear in conjugate pairs, so the root locii are symmetrical about the real axis of the s-plane.

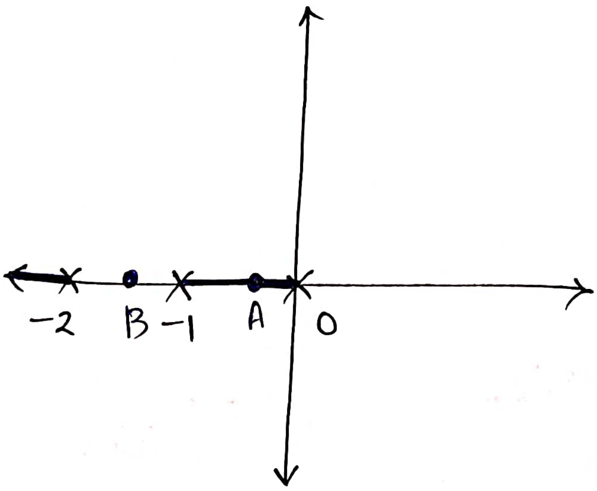
(4) If we include the poles & zeroes at ∞, the no. of open loop poles = no. of open loop zeroes.

Example (1)

$$GH = \left(\frac{k}{s(s+1)(s+2)}\right)$$

∴ zeroes are none

poles = (0, −1, −2)



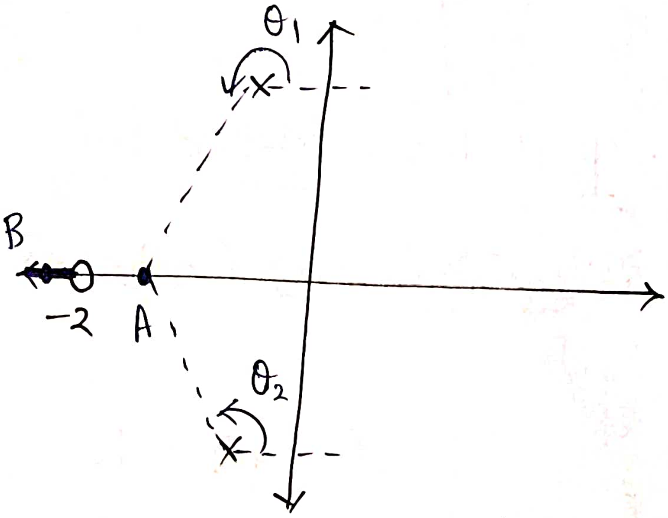
Example (2)

$$GH = \left(\frac{k(s+2)}{s^2+2s+3}\right)$$

∴ zeroes are = (−2)

& poles $= \dfrac{-2 \pm \sqrt{4-12}}{2}$

$= (-1 \pm \sqrt{2}\, i)$



$$\left\{ \angle GH = \begin{matrix} \pm 180 \\ \pm 540^\circ \\ \vdots \end{matrix} \right\}$$

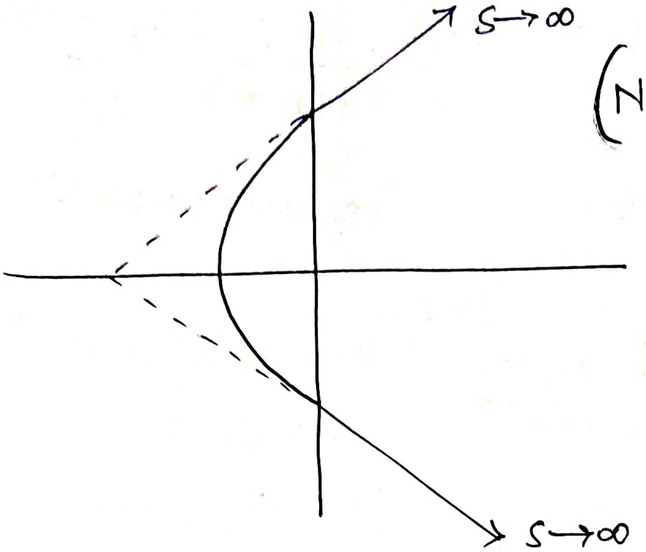
for A: $0^\circ - (\theta_1+\theta_2) = (-360)^\circ$ ✗

but

for B: $180 - 360 = (-180)^\circ$ ✓

(5) determine the asymptotes of the root locus :-

(if ($s \to \infty$) then root locus appeares as a straight line)



(Now we can choose the test points neare the asymptotes!)

& drawing the asymptotes is easy!

$$\psi = \sum_m \phi_m - \sum_n \theta_n$$

$$= (m-n)\,\theta = \pm 180(2k+1)$$

No. of asymptotes $\Rightarrow (n-m)$

& angle of asymptotes $= \left\{ \theta = \dfrac{\pm 180(2k+1)}{(m-n)} \right\}$

$\Rightarrow$ All asymptotes intersect at a point on the real axis.

General Equation :-

$$GH = \frac{k\left\{ s^m + (z_1 + \cdots z_m)\, s^{\overset{m-1}{\cancel{m+1}}} + \cdots (z_1 z_2 \cdots z_m) \right\}}{\left\{ s^n + (p_1 + \cdots p_n)\, s^{n-1} + \cdots (p_1 p_2 \cdots p_n) \right\}}$$

$$= \frac{k}{s^{n-m} + \left\{ (p_1 + \cdots p_n) - (z_1 + \cdots z_m) \right\} s^{n-m-1} + \cdots\cdots}$$

$\boxed{\text{if } s \to (\infty)}$

$$= \frac{k}{\left\{ s + \dfrac{(p_1 + p_2 \cdots p_n) - (z_1 + \cdots z_n)}{(n-m)} \right\}^{n-m}}$$

$$s = \frac{-(p_1 + \cdots p_n) + (z_1 + \cdots z_m)}{(n-m)}$$

} point where all asymptotes intersect.

(real) (real) } sum is always real $\therefore$ asymptotes intersect on the real axis.

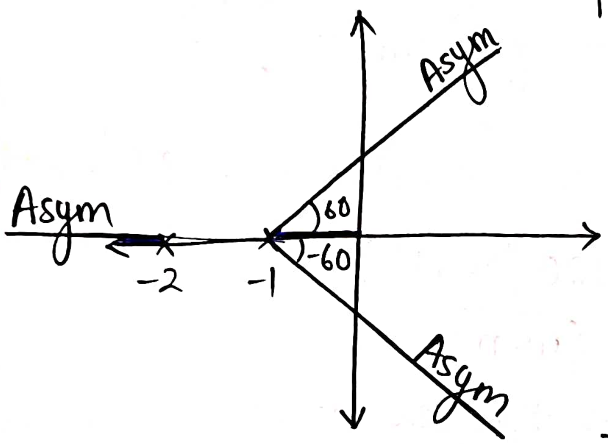
$n =$ no. of poles
$m =$ no. of zeroes } $(n-m) =$ no. of asymptotes

$$\theta = \frac{\pm 180(2K+1)}{(n-m)} \quad ; \quad \underbrace{S = \frac{-(P_1 + \cdots P_n) + (Z_1 + \cdots Z_n)}{(n-m)}}_{\text{intersecting point}}$$

Ex(1)

$\theta = \pm 60(2K+1)$
$= \pm 60, \pm 180°$

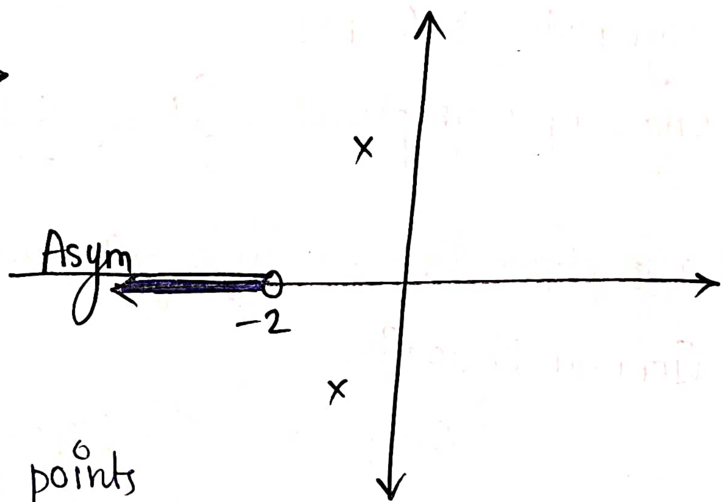
$S = \frac{-3}{3} = (-1)$ } intersecting point



Ex(2)

$\theta = \pm 180(2K+1)$
$= \pm 180° \ldots$

$S = \frac{+2-2}{1} = (0)$



(6) Brake away & brake in points

this is a pt. where multiple roots of the CL char fnc appear.

~~at~~ pt (A) ; if root locus lies betn 2 adjacent O.L. poles on the real axis, then there exists atleast one brake-away point between these two poles.

pt (B) if a root locus lies betn 2 adj O.L. zeroes on the real axis then there always exists atleast one break-in point between the zeroes.

$GH+1=0 \qquad B(s)+KA(s)=0$

$\Rightarrow 1+GH = 1+\dfrac{K(s+z_1)\cdots(s+z_m)}{(s+P_1)\cdots(s+P_n)}$

$\Rightarrow f(s) = B(s)+KA(s)$

$\Rightarrow \dfrac{df(s)}{ds}=0 \Rightarrow B'(s)+KA'(s)=0 \Rightarrow K=\dfrac{-B'(s)}{A'(s)}$

$K=\dfrac{f(s)-B(s)}{A(s)}=\dfrac{-B'(s)}{A'(s)}$

$\therefore \{f(s) = -B'(s)A(s)+B(s)A'(s)=0\}$ ✓

$K=\dfrac{-B(s)}{A(s)} \Rightarrow \left(\dfrac{dK}{ds}=0\right) \rightarrow$ we will get the roots & then we have to check which solns lie on the root locus & those points will only be the

✓ doing this is much more simpler

Example (1) $GH+1=0$

$\Rightarrow K = -(s^3+3s^2+2s) \rightarrow \dfrac{dK}{ds} = (-3s^2-6s+2)$

$\dfrac{dK}{ds}=0 \Rightarrow (s=-0.4226, \cancel{-1.5774})$

(Now check which solns lie on the root locus!)

(-0.4226) lies on root locus

↳ lies betn two poles

$\therefore$ break-away point

(-1.5774) doesn't lie on the root locus $\therefore$ don't consider it!

Example (2) $GH+1=0$

$\dfrac{dK}{ds}=0 \rightarrow s^2+4s+1=0 \rightarrow -3.723$ ✓, $\cancel{-0.2680}$

(-3.723) lies on root locus

↳ another zero considered at "∞" $\therefore$ this pt. lies in betn two zeroes $\therefore$ break-in point.

(7) determine the angle of departure or angle of arrival of the root locus from complex pole or complex "z".

→ for the poles or zeroes on the real axis the departure or arrival is decided by the asymptotes but for complex poles or zeroes it is not yet decided.

→ (a) Angle of departure from a complex pole =

$= (180)^\circ -$ ~~sum~~ $\sum$ (angles of vectors to a complex pole in question from other poles)

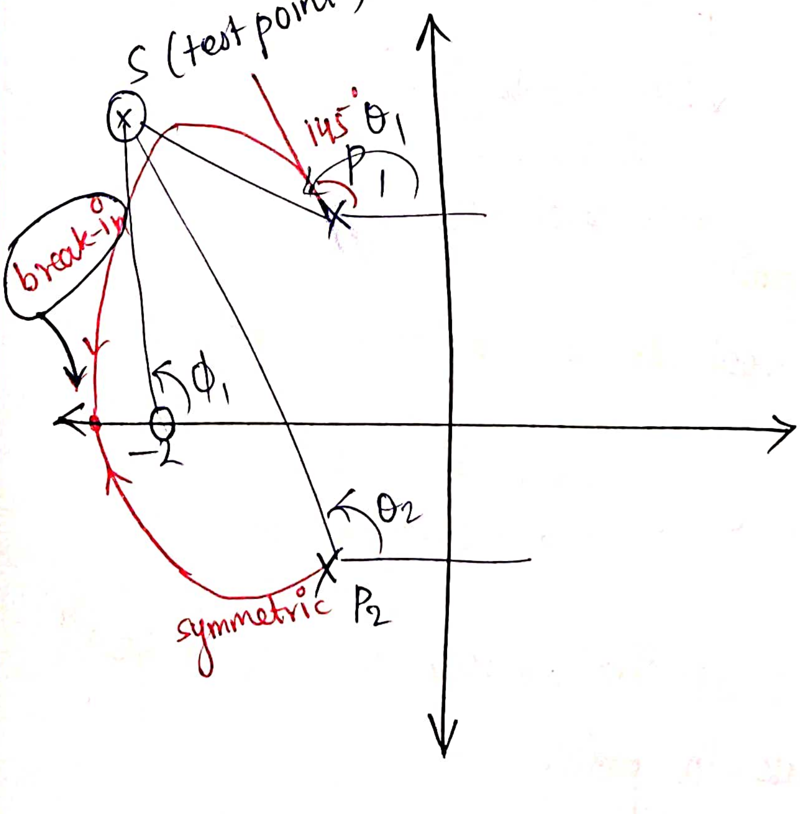
$+ \sum$ (angles of vectors from a complex pole in question from zeroes)

→ (b) Angle of arrival to a complex zero =

$= (180^\circ) - \sum$ (angles of the vectors to a complex zero in question from other zeroes)

$+ \sum$ (angles of vectors to a complex zero in question from poles.)

Example (2)



$(\phi - \theta_1 - \theta_2) = \pm 180(2k+1)$
$= -180$

if test pt taken near to P_1 then $\theta_2 \approx 90^\circ$
& $\phi_1 = 55^\circ$

$\theta_1 = 180 - \theta_2 + \phi$
$= 145^\circ$

(8) find the intersectⁿ pt betⁿ root locus & imaginary axis

$(GH+1=0)$

$\Rightarrow f(s,k)=0$

$(s=\sigma+j\omega) \Rightarrow (f_R + if_I = 0)$

(0)

$f_R = 0$

& $f_I = 0$

Ex (1) $\Rightarrow s^3+3s^2+2s+k=0$

$\hookrightarrow (s=j\omega)$ $\rightarrow$ separate the parts

∴ it will come as

$(k-3\omega^2=0), (2\omega-\omega^3=0)$

no use of $k=0$ ✗

$(k=6)$ ✓

& intersecting pts

$\omega = \pm\sqrt{2}$ $\Big|_{(k=6)}$

∴ $(\omega = \pm\sqrt{2})$ & $(\omega = 0)$

∴ $k=0$ ✗ or $k=\pm 6$

All points have a definite value of "k" !!

GH+1 } definite points on the root locus

GH } arbitrary points

if not intersecting the im axis then $(k<0)$ ✗

& ($\omega \Rightarrow$ img)

not possible

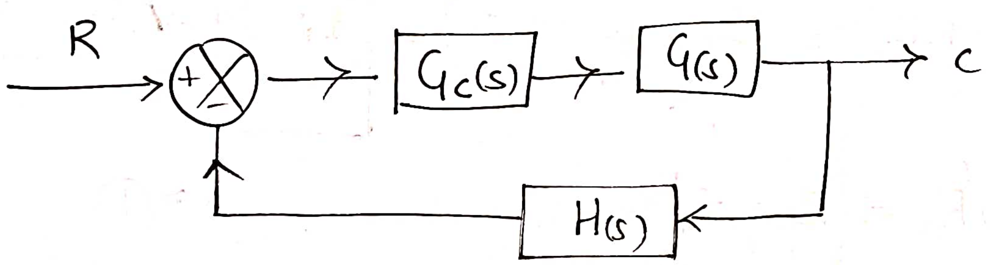
∴ no intersection.

→ Compensator → additional poles & zeroes are added to the open loop transfer function. You must know the effects of doing these.

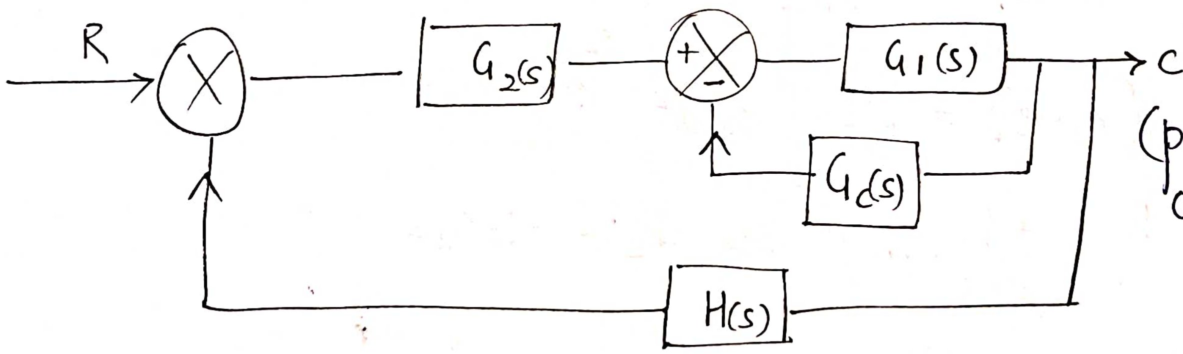
Note :-

(1) The addition of a pole to an OLTF has an effect of pulling the root locus to the right side. This slows down the response, settling time & also the stability of the system may be lost.

(2) The additⁿ of a zero to an OLTF has an effect of pulling the root locus to the left side. The system becomes more stable, along with the speed of response & the settling time.



(Series compensator)

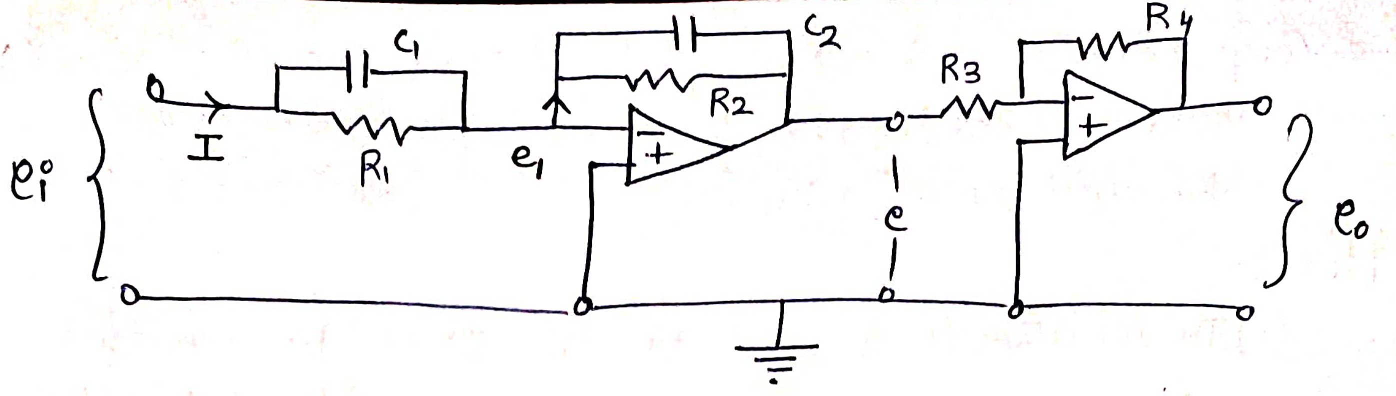


(parallel compensator)

Compensators
- Lead compensator
- Lag compensator
- Lead-Lag compensator
- Velocity feed back
- ....

"most popular"

In our syllabus we would only discuss the series of these compensators



$\cancel{e_i - e_1 = I}$

$$e_i - e_1 = i_1 R_1 \xrightarrow{\text{laplacian}} E_i - E_1 = I_1 R_1 \Rightarrow E_1 \simeq 0$$

$$\therefore (E_i = I_1 R_1)$$

$$(I_1 = E_i / R_1)$$

$$e_i - e_1 = \frac{1}{C_1} \int i_2 \, dt$$

$$E_i = \frac{1}{C_1 s} I_2 \Rightarrow (I_2 = E_i C_1 s)$$

$$I = (I_1 + I_2) = E_i \left(\frac{1}{R_1} + C_1 s \right) = \boxed{E_i \left(\frac{1 + R_1 C_1 s}{R_1} \right)} \rightarrow ①$$

also $e_1 - e = (i_1)' R_2 \Rightarrow E = -I_1' R_2$

$$\Rightarrow I_1' = -E / R_2$$

& $(e_1 - e) = \frac{1}{C_2} \int i_2' \, dt$

$$E = \frac{-1}{C_2 s} (I_2)' \Rightarrow I_2' = -E C_2 s$$

$$I = (I_1' + I_2') = -E \left(\frac{1}{R_2} + C_2 s \right) = \boxed{-E \left(\frac{1 + R_2 C_2 s}{R_2} \right)} \rightarrow ②$$

Equating ① & ②

$$E_i \left(\frac{1 + R_1 C_1 s}{R_1} \right) = -E \left(\frac{1 + R_2 C_2 s}{R_2} \right)$$

$$\therefore \frac{E}{E_i} = -\left(\frac{R_2}{R_1} \right) \left(\frac{1 + R_1 C_1 s}{1 + R_2 C_2 s} \right)$$

$$\therefore \frac{E_o}{E_i} = \frac{R_2 R_4}{R_1 R_3}\left(\frac{1+R_1C_1 s}{1+R_2C_2 s}\right) = \frac{\cancel{R_2} R_4 \cancel{R_1} C_1}{\cancel{R_1} R_3 \cancel{R_2} C_2}\left(\frac{\frac{1}{R_1C_1}+s}{\frac{1}{R_2C_2}+s}\right)$$

$$\therefore \frac{E_o}{E_i} = \left(\frac{R_4 C_1}{R_3 C_2}\right)\left(\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}\right)$$

$$\boxed{\frac{E_o}{E_i} = (K_c)\left(\frac{s+\frac{1}{T}}{s+1/\alpha T}\right)}$$

; where $K_c = \left(\frac{R_4C_1}{R_3C_2}\right)$

$T = (R_1C_1)$

$\alpha T = (R_2C_2)$

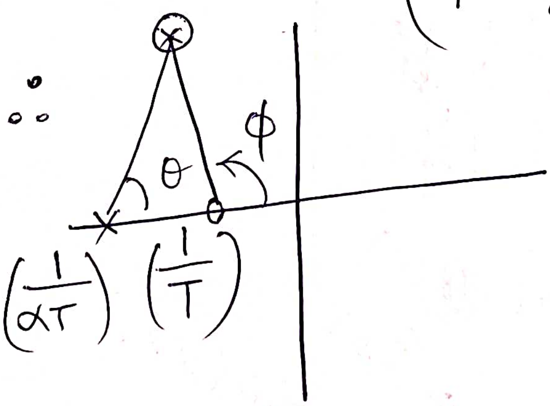
& $\alpha = \left(\frac{R_2C_2}{R_1C_1}\right)$

$$\left(\begin{array}{l} \frac{1}{T} \} \text{ zero} \\ \frac{1}{\alpha T} \} \text{ pole} \end{array}\right)$$

& (K_c = compensator)

if $\alpha < 1 \rightarrow$ then $\left(\frac{1}{T} < \frac{1}{\alpha T}\right)$

$\therefore$



$(\phi - \theta) > 0$

Lead compensator

↳ The steady state output of the compensator has a phase lead to the steady state input of the same compensator.

Lag compensator

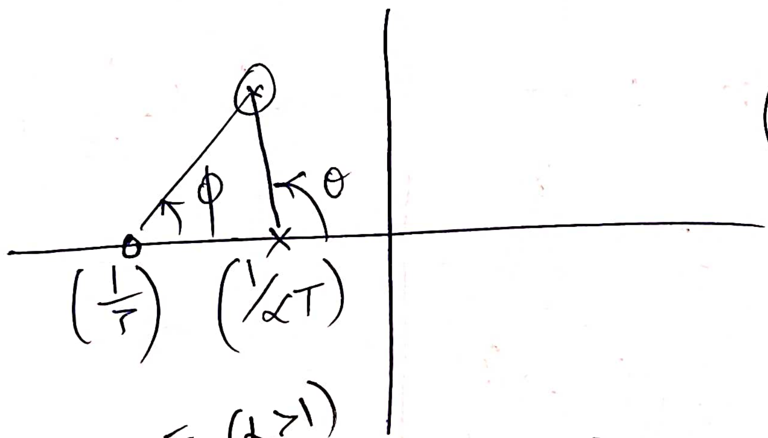
↳ The steady " " " " " " "
phase lag " " " " " " "
same compensator.

Lead-Lag Compensator :-

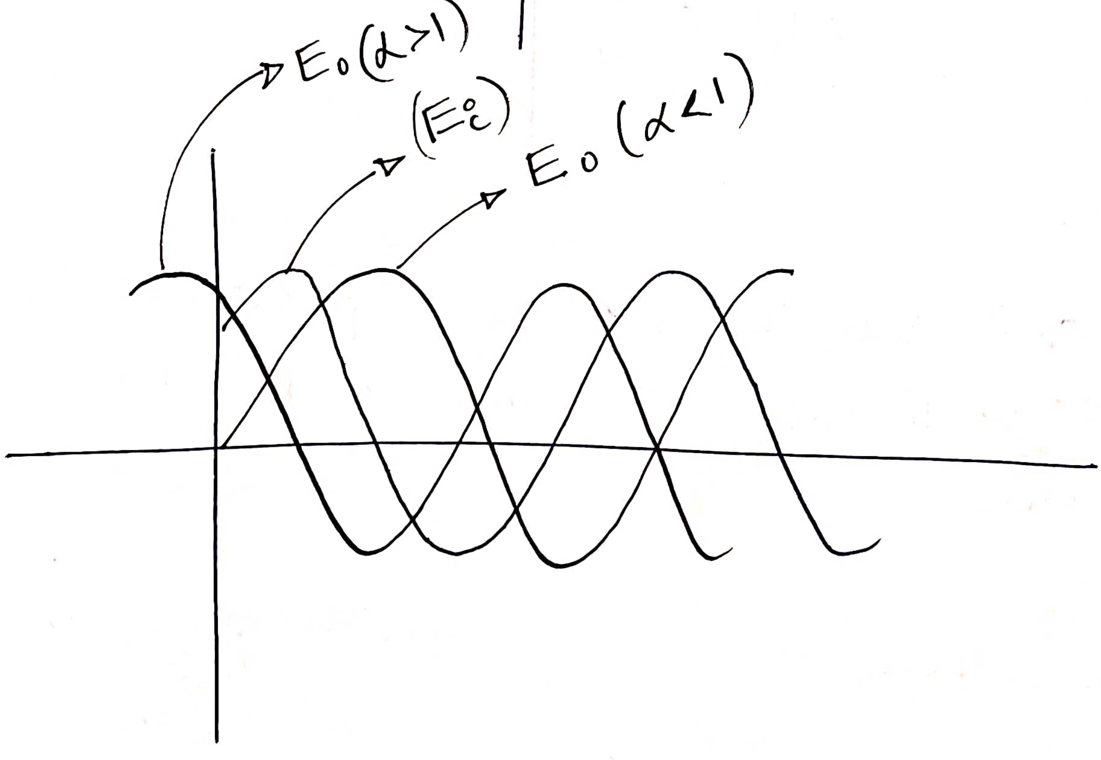
- At the lower freq region, the steady-state output of the comp. has a phase lag to the steady-state input of the same comp.

- At higher freq region, " " " " " " " phase lead " " " " " " " same comp.

→ for $\alpha > 1 \Rightarrow \frac{1}{T} > \frac{1}{\alpha T}$



$(\phi - \theta) < 0$ always.



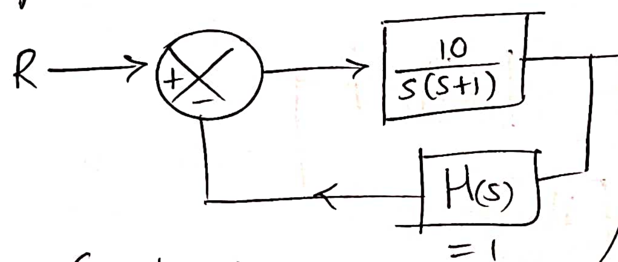
→ Compensator :→

$$G(s) = K_c \left(\frac{s + 1/T}{s + 1/\alpha T} \right) \quad \begin{cases} \alpha < 1 \leftarrow \text{lead} \\ \alpha > 1 \leftarrow \text{lag} \end{cases}$$

Lead compensator

↳ series :- in series to the plant transfer function
↳ with the feed forward transfer fnc is suitable for the design of a system where the system is either unstable for all values of gain or the system is stable but has undesirable transient characteristics.

Example :→

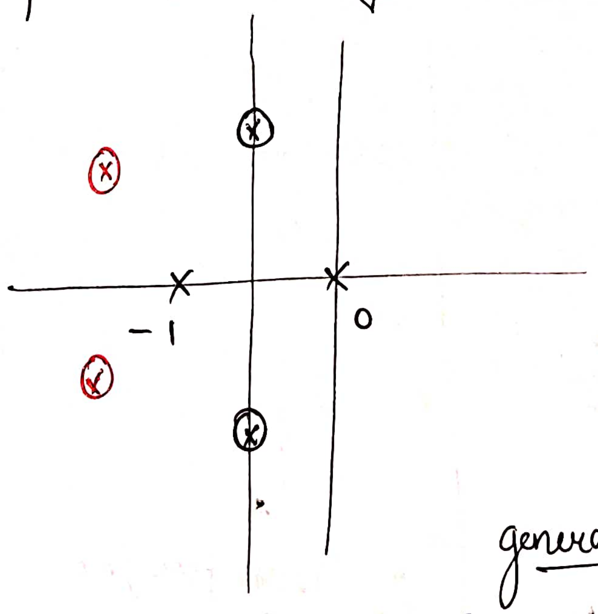


$$GH = \frac{10}{s(s+1)}$$

$$\& \quad \frac{C}{R} = \frac{G}{1+GH} = \frac{\frac{10}{s(s+1)}}{1 + \frac{10}{s(s+1)}} = \left(\frac{10}{s^2+s+10} \right)$$

second order system.

poles = (0 & −1)



$$s^2 + s + 10 = 0$$

$$\hookrightarrow \frac{-1 \pm \sqrt{1-40}}{2} = \left(-0.5 \pm j(3.122) \right)$$

general form

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\therefore \omega_n = \sqrt{10} = (3.1623)$$

$$2\xi\omega_n = 1$$

$$\therefore \xi = (0.1581)$$

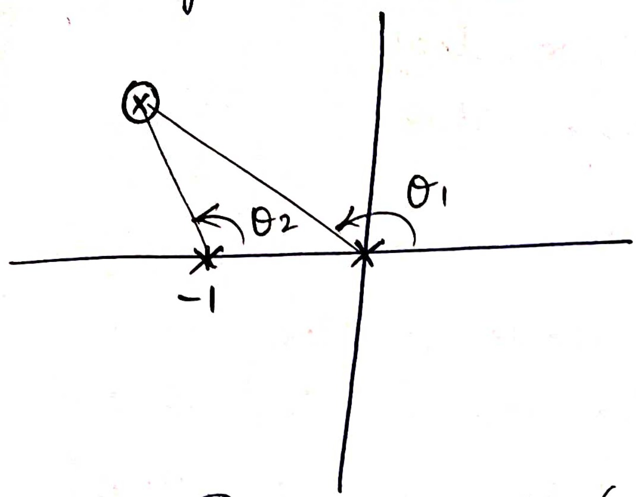
Our desired specification

$\xi = 0.5$, $\omega_n = 3$ rad/s

$$s^2 + (2 \times 0.5 \times 3)s + 9 = 0$$

$$\therefore s = \left(-1.5 \pm j2.598 \right)$$

→ Angle condition : → $\pm 180(2k+1)$



$$\underbrace{\Sigma\phi}_{\text{zeroes}} - \underbrace{\Sigma\theta}_{\text{poles}} = 0 - (\theta_1 + \theta_2)$$

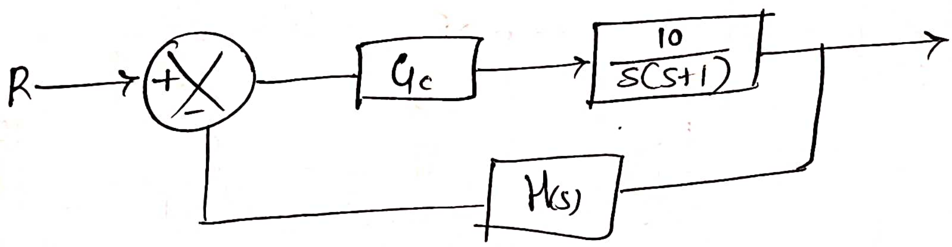
here $\theta_2 = 100.894$
$\theta_1 = 120$

$\therefore \Sigma\phi - \Sigma\theta = \boxed{-220.894}$ close to (-180°)

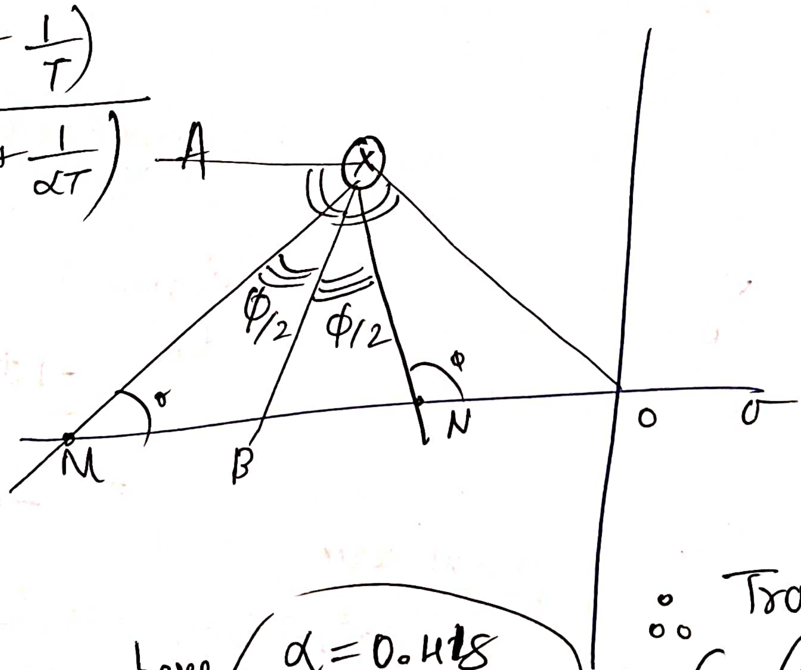
(we need to provide $(+40.894^\circ)$ to the system, then only the root locus would modify!)

as (+ve) ∴ lead compensator to be added.

∴



$$G_c = K_c \frac{\left(s + \frac{1}{T}\right)}{\left(s + \frac{1}{\alpha T}\right)}$$



$M = \left(\frac{1}{\alpha T}\right)$
& $N = (1/T)$

here $\alpha = 0.418$
& $\frac{1}{T} = 1.9432$

∴ Transfer fnc

$$G_c = \left(K_c \frac{s + 1.9432}{s + 4.648}\right)$$

$$G_c(s)G(s) + 1 = 0$$

$$|G_c(s)G(s)| = 1$$

$\Rightarrow K_c$ - value ✓✓

Lag compensator :→

$$G_c(s) = K_c\left(\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}\right); \beta > 1$$

Let a case where the system exhibits satisfactory transient response but unsatisfactory steady state characteristics. This requires an increase of open loop gain but the root locus could not be changed from the dominant closed loop pole. It can be achieved by introducing the lag compensator.

(1) → No angle contribution $\left(\frac{1}{T} \approx \frac{1}{\beta T}\right)$ } this condition can be achived with very high value of (T)!

$\therefore \frac{1}{T}$ & $\frac{1}{\beta T}$ would lie very close to origin

→(2) $(K_c = 1)$

dominant closed loop poles would move over the root locus.

Static velocity error constant :→ $\left[K_v = \lim_{s\to 0} sG(s)\right]$ } compensator added in series

Read previous chapter very clearly!

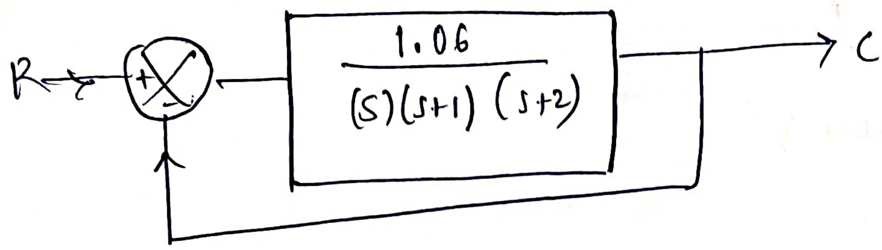
$$= \lim_{s\to 0} sG_c(s)G(s)$$

$$= \lim_{s\to 0} G_c K_v$$

$$= \boxed{K_c \beta K_v}$$

as $K_c = 1 \therefore \quad = (\beta K_v)$

Example :→



$$GH = \frac{1.06}{(s)(s+1)(s+2)} \qquad \frac{C}{R} = \frac{G}{1+GH} \Rightarrow GH+1=0$$

$$\hookrightarrow \frac{C}{R} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

$S_{1,2} \Rightarrow (-0.33 + j(0.5864))$ → dominant pole as second order governs.

& $S_3 \Rightarrow (-2.3386)$

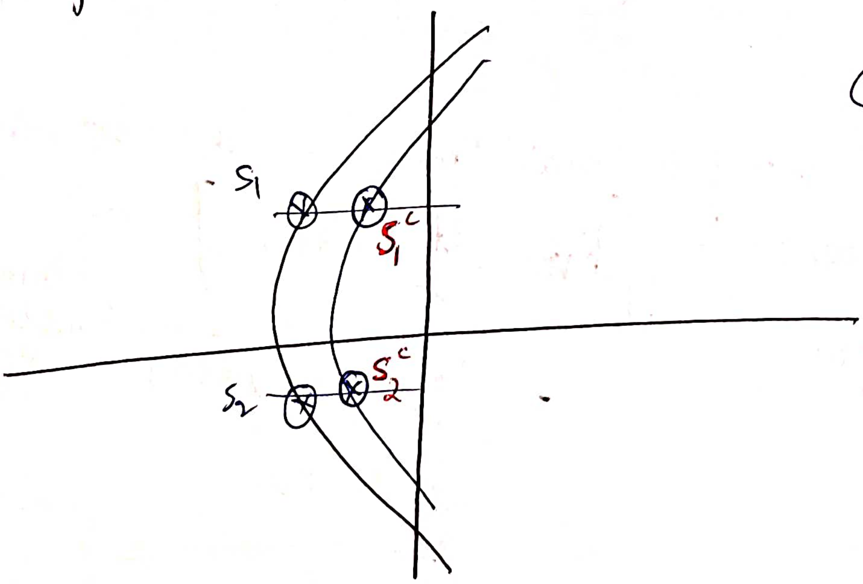
$$K_v = \lim_{s\to 0} sG(s) = (0.53)\ /sec$$

we wan to increase the K_v value to → (5 /sec) ;

for that we add a lag compensator ⇒ $\left(\frac{1}{T}, \frac{1}{\beta T}\right)$

Example let $\frac{1}{T} = -0.05$ assume

$\frac{1}{\beta T} = ?$



$\beta K_v = 5$

$\beta = 10$

$$G_c = K_c \frac{\left(s+\frac{1}{T}\right)}{\left(1+\frac{1}{\beta T}\right)} \Rightarrow G_cG = \frac{(K_c\,1.06)\ s+0.05}{s(s+1)(s+2)\ s+0.005}$$

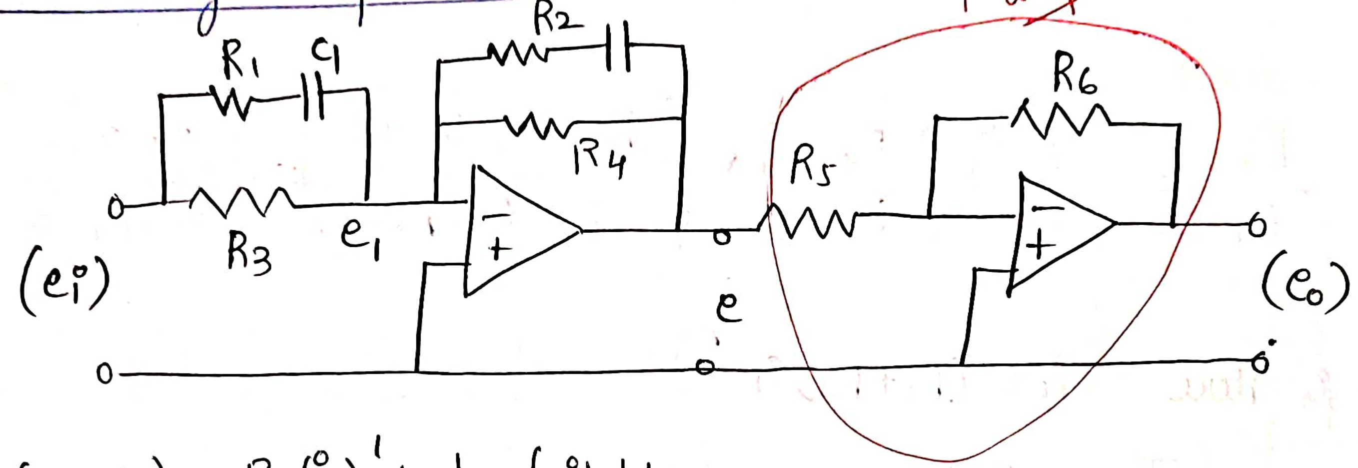
$$|G_cG|_{s=s_1^c} = 1 \Rightarrow K_c = 0.9656$$

Transient system stabilise $\longrightarrow$ Lead
Steady state error reduction $\longrightarrow$ Lag

{ Transient System stabilise → Lead
Steady state error reduction → Lag }

Lead Lag Compensator

inverting amplifier !



$$(e_i - e_1) = R_1 (i_1)' + \frac{1}{C_1}\int i_1' \, dt$$

$$\Rightarrow E_i = \left(R_1 + \frac{1}{C_1 s}\right) I_1' \quad ; (e_1 = 0)$$

$$E_i = \left(\frac{1 + R_1 C_1 s}{C_1 s}\right) I_1' \Rightarrow I_1' = \left(\frac{C_1 s}{1 + R_1 C_1 s}\right) E_i$$

$$I_1 = I_1' + I_2' = \left(\frac{C_1 s}{1 + R_1 C_1 s} + \frac{1}{R_3}\right) E_i$$

$$= \left[\frac{R_3 C_1 s + 1 + R_1 C_1 s}{R_3 (1 + R_1 C_1 s)}\right] E_i$$

$$(e_i - e_1) = R_3 i_2'$$

$$E_i = R_3 I_2'$$

$$I_2' = \left(\frac{E_i}{R_3}\right)$$

by comparision.

$$I_2 = -\left[\frac{(R_2 + R_4) C_2 s + 1}{R_4 (1 + R_2 C_2 s)}\right] E = I_1 = \left[\frac{R_3 C_1 s + 1 + R_1 C_1 s}{R_3 (1 + R_1 C_1 s)}\right] E_i$$

$$\therefore \begin{Bmatrix} \omega_n = 2\ rad/s \\ \xi = 0.125 \end{Bmatrix}$$

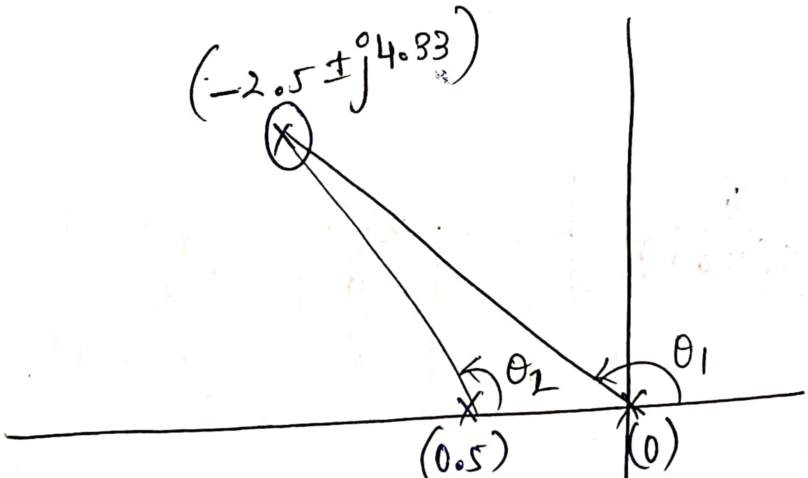
$$\underbrace{K_v}_{\text{static error velocity const}} = \lim_{s \to 0} sG(s) = 8.$$

$$\left.\begin{matrix} \left(\text{if } \omega_n = 5\ rad/s \quad \xi = 0.5 \right) \\ (K_v = 80) \end{matrix}\right\}$$ both lead & lag to be used!

$$\to S_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= (-2.5 \pm j4.33)$$

on putting the values.



$$\underbrace{\Sigma\phi}_{\text{no zero}} - \Sigma\theta$$

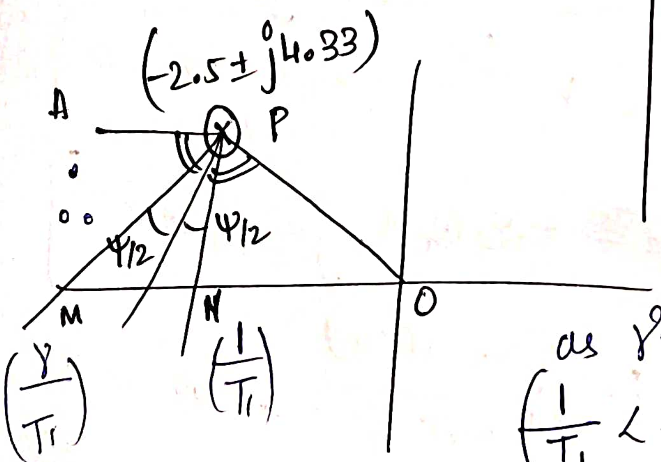
$$= -(\theta_1 + \theta_2)$$

$$= -235°$$

we have nearest angle

$$\pm 180(2K+1) \Rightarrow -180°$$

$$\therefore \psi \Rightarrow 55°$$

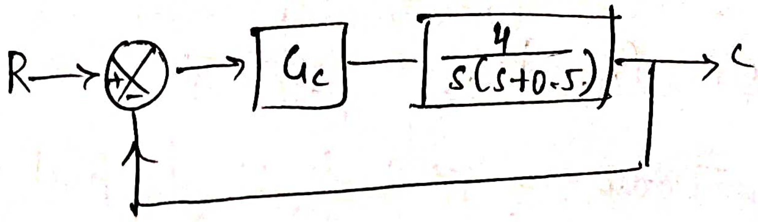


as $\gamma > 1$

$$\left(\frac{1}{T_1} < \frac{\gamma}{T_1}\right)$$

$$\therefore \begin{pmatrix} T_1 = 0.37 \\ \gamma = 3.46 \end{pmatrix}$$

(modified OLTF $\Rightarrow$ $\left(s = -2.5 \pm 4.33\hat{j}\right)$



$$\left|\left(G_c(s)\,G(s)\right)\right| = 1$$

$$K_c\left(\frac{s+1/T_1}{s+\gamma/T_1}\right) \qquad \frac{4}{s(s+0.5)}$$

$K_c = $ ✓

$$K_v = \lim_{s\to 0} s\,G_c(s)\,G(s) = \cancel{K_c\left(\frac{\gamma}{\beta}\right)8} \quad K_c\left(\frac{\beta}{\gamma}\right)8$$

$$\therefore \beta = \left(\frac{80}{8} \times \frac{\gamma}{K_c}\right)$$

$$\therefore \beta = 10 \times (\,,,\,) \quad ✓$$

condition :$\rightarrow$ $$\left|\frac{s+1/T_2}{s+1/\beta T_2}\right| \approx 1 \quad \& \left(0^\circ < \text{angle}\left(\frac{s+1/T_2}{s+\frac{1}{\beta T_2}}\right) \text{angle} < 5^\circ\right)$$

(for getting (T_2) value, trial & error needs to be done properly)

(Because many terms are dependent on each other)

Questions :-

(Q1) $G_{(s)} H_{(s)} = \frac{1}{(s)(s+8)}$

Draw the root locus & find the value of gain "K" for dominant closed loop pole $(-1.6 \pm 1.6j)$

(Q2) $G_{(s)} H_{(s)} = \frac{1}{(s)(s^2+5s+6)}$

Draw the root locus & find the gain value "K" at $(s = -0.7 + j\,0.84)$

(Q3) $G_{(s)} H_{(s)} = \frac{(s+1)}{(s^3+4s^2+6s+1)}$

Only draw the root locus for this!

(Q4) characteristic Equation of a closed loop system is given by

$$= \left(1 + \frac{K}{(s)(s+1)(s+5)} = 0\right) = (1 + GH) = 0 \qquad \left(\frac{C}{R} = \frac{G}{1+GH}\right)$$

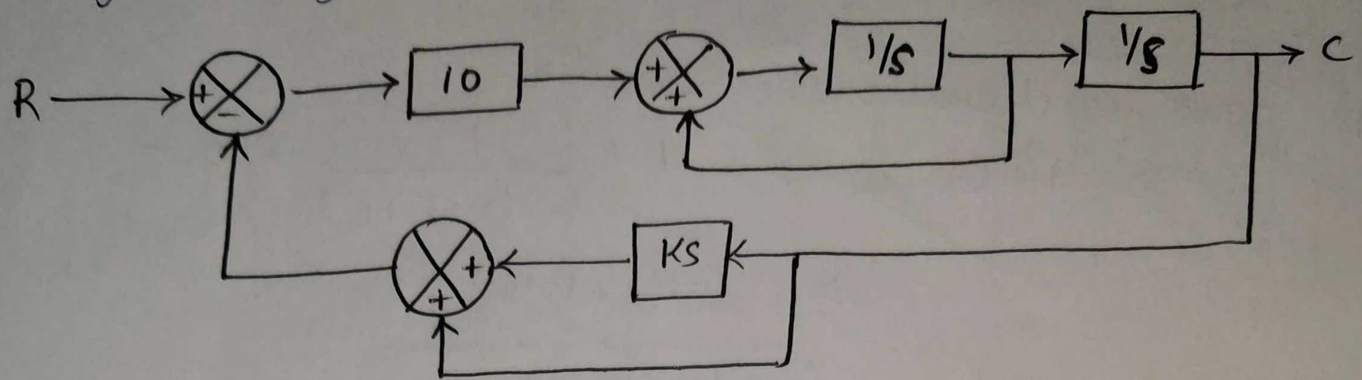
(a) Draw the real axis segment of the root locus

(b) Draw the asymptotes

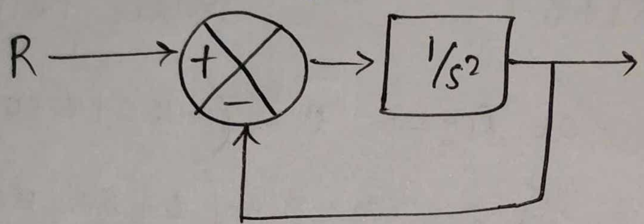
(c) Find the value of "K" where the root locus intersects the imaginary axis.

(Q5) ~~Draw the root locus~~ for a closed loop system, ~~having the root~~ the feed forward transfer fnc is given by $G\cancel{H} = G = \left(\frac{1}{s(s+2)}\right)$
Modify the closed-loop system using a proportional controller & a compensator in series to the plant transfer function for achieving the dominant closed loop pole as $[s = (-1 \pm j)]$ & the steady state error for ramp input as less than (0.2)

(Q5) for a feedback control system as shown in figure. Find the value of gain "K", that results in dominant closed-loop pole, with ~~a~~ a damping ratio ($\xi = 0.5$)



(Q6)
for a given closed loop system, add a lead comp in series with the plan, so that the dominant poles of the closed loop system appear as



$s = (-2 \pm 2j)$