Lectures 8, 9, 10, 11, 12, 13: Duality and Dual Simplex Method

1. Definition of Dual Problem

- Defining Primal (original) LP problem in equation form

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, 2, \ldots, m$$ Nonnegative RHS

$$x_j \geq 0,\ j = 1, 2, \ldots, n$$ Includes the slack, surplus and artificial variables.

Definition of Dual Problem

3. RHS for j-th constraint

Primary Variables							
	x_1	x_2	…	x_j	…	x_n	
Dual Variables	c_1	c_2	…	c_j	…	C_n	Right-hand side
y_1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}	b_1
y_2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}	b_2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
y_m	a_{m1}	a_{m2}	…	a_{mj}	…	a_{mn}	b_m

1. Dual variables for each constraint

2. j-th dual constraint for primal variable

4. Objective coefficients

Primal problem objective[a]	Dual problem: *Objective*	*Constraints type*	*Variables sign*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

[a] All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

1. Definition of Dual Problem

- Example 1

Primal	Primal in equation form
Maximize $z = 5x_1 + 12x_2 + 4x_3$ subject to $x_1 + 2x_2 + x_3 \leq 10$ $2x_1 - x_2 + 3x_3 = 8$ $x_1, x_2, x_3 \geq 0$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$ subject to $x_1 + 2x_2 + x_3 + x_4 = 10$ $2x_1 - x_2 + 3x_3 + 0x_4 = 8$ $x_1, x_2, x_3, x_4 \geq 0$

- Dual problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12$$
$$y_1 + 3y_2 \geq 4$$
$$\left.\begin{array}{r} y_1 + 0y_2 \geq 0 \\ y_1, y_2 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted})$$

Definition of Dual Problem

- Example 2

Primal	Primal in equation form
Minimize $z = 15x_1 + 12x_2$ subject to $x_1 + 2x_2 \geq 3$ $2x_1 - 4x_2 \leq 5$ $x_1, x_2 \geq 0$	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$ subject to $x_1 + 2x_2 - x_3 + 0x_4 = 3$ $2x_1 - 4x_2 + 0x_3 + x_4 = 5$ $x_1, x_2, x_3, x_4 \geq 0$

- Dual problem

$$\text{Maximize } w = 3y_1 + 5y_2$$

$$y_1 + 2y_2 \leq 15$$

$$2y_1 - 4y_2 \leq 12$$

$$\left.\begin{array}{r} -y_1 \quad\quad \leq 0 \\ y_2 \leq 0 \\ y_1, y_2 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \geq 0, y_2 \leq 0)$$

Definition of Dual Problem

- Example 3

Primal	Primal in equation form
Maximize $z = 5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$ $-x_1 + 5x_2 \geq 3$ $4x_1 + 7x_2 \leq 8$ x_1 unrestricted, $x_2 \geq 0$	Substitute $x_1 = x_1^+ - x_1^-$ Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$ subject to $x_1^- - x_1^+ + 2x_2 = 5$ $-x_1^- + x_1^+ + 5x_2 - x_3 = 3$ $4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$ $x_1^-, x_1^+, x_2, x_3, x_4 \geq 0$

- Dual problem

$$\text{Minimize } z = 5y_1 + 3y_2 + 8y_3$$

$$\left.\begin{array}{r} y_1 - y_2 + 4y_3 \geq 5 \\ -y_1 + y_2 - 4y_3 \geq -5 \end{array}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$\left.\begin{array}{r} -y_2 \quad\quad \geq 0 \\ y_3 \geq 0 \\ y_1, y_2, y_3 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \text{ unrestricted}, y_2 \leq 0, y_3 \geq 0)$$

2. Primal-Dual Relationships

- Optimal Dual Solution
  - The optimal solution of either problem yields the optimum solution to the other.
  - No. of variables << no. of constraints, solve dual problem
    - Computational saving
    - Simplex computation largely depends on the no. of constraints.
- Two methods

Gauss-Jordon Elimination Method

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual} \text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Method 2.

$$\begin{pmatrix}\text{Optimal values} \\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of} \\ \text{original objective coefficients} \\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal} \\ \text{inverse}\end{pmatrix}$$

Primal-Dual Relationships

- Example

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

- Primal-Dual problem

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$	Minimize $w = 10y_1 + 8y_2$
subject to	subject to
$x_1 + 2x_2 + x_3 + x_4 = 10$	$y_1 + 2y_2 \ge 5$
$2x_1 - x_2 + 3x_3 + R = 8$	$2y_1 - y_2 \ge 12$
$x_1, x_2, x_3, x_4, R \ge 0$	$y_1 + 3y_2 \ge 4$
	$y_1 \ge 0$
	$y_2 \ge -M$ ($\Rightarrow y_2$ unrestricted)

Primal-Dual Relationships

- The optimal tableau

Basic	x_1	x_2	x_3	x_4	R	Solution
Z	0	0	3/5	29/5	-2/5 + M	274/5
x_2	0	1	-1/5	2/5	-1/5	12/5
x_1	1	0	7/5	1/5	2/5	26/5

- Starting primal variables

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual} \text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Primal-Dual Relationships

- Method 2

Basic	x_1	x_2	x_3	x_4	R	Solution
Z	0	0	3/5	29/5	-2/5 + M	274/5
x_2	0	1	-1/5	2/5	-1/5	12/5
x_1	1	0	7/5	1/5	2/5	26/5

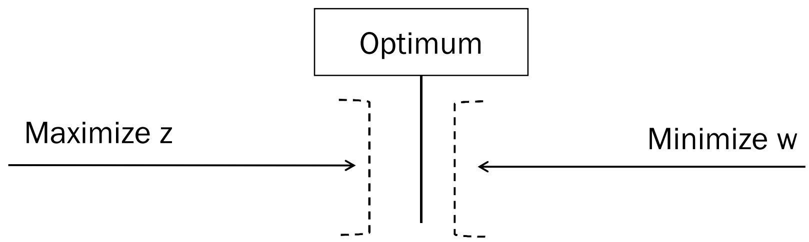
Method 2.

$$\begin{pmatrix}\text{Optimal values} \\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of} \\ \text{original objective coefficients} \\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal} \\ \text{inverse}\end{pmatrix}$$

Primal-Dual Relationships

- Primal-Dual Objective Values

Objective value in the maximization problem	$\leq$	Objective value in the minimization problem



Simplex Tableau Computation

- Generate entire simplex tableau from original data and inverse of matrix

Formula 1: Constraint Column Computations. In any simplex iteration, a left-hand or a right-hand side column is computed as follows:

$$\begin{pmatrix}\text{Constraint column}\\ \text{in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ \text{constraint column}\end{pmatrix}$$

Basic	x_1	x_2	x_3	x_4	R	Solution
Z	0	0	3/5	29/5	-2/5 + M	274/5
x_2	0	1	-1/5	2/5	-1/5	12/5
x_1	1	0	7/5	1/5	2/5	26/5

$$\text{Optimal inverse} = \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

$$\begin{pmatrix}x_1\text{-column in}\\ \text{optimal iteration}\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{original}\\ x_1\text{-column}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

Simplex Tableau Computation

- Formula 1

$$\begin{pmatrix} x_2\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_3\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{7}{5} \end{pmatrix}$$

$$\begin{pmatrix} x_4\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$\begin{pmatrix} R\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} \text{Right-hand side} \\ \text{column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{26}{5} \end{pmatrix}$$

Simplex Tableau Computation

Formula 2: Objective z-row Computations. In any simplex iteration, the objective equation coefficient (reduced cost) of x_j is computed as follows:

$$\begin{pmatrix} \textit{Primal } z\text{-equation} \\ \text{coefficient of variable } x_j \end{pmatrix} = \begin{pmatrix} \text{Left-hand side of} \\ j\text{th } \textit{dual} \text{ constraint} \end{pmatrix} - \begin{pmatrix} \text{Right-hand side of} \\ j\text{th } \textit{dual} \text{ constraint} \end{pmatrix}$$

- The optimal dual variables

$$(y_1, y_2) = \left(\frac{29}{5}, -\frac{2}{5}\right)$$

$$\begin{aligned}
z\text{-cofficient of } x_1 &= y_1 + 2y_2 - 5 &&= \tfrac{29}{5} + 2 \times -\tfrac{2}{5} - 5 &&= 0 \\
z\text{-cofficient of } x_2 &= 2y_1 - y_2 - 12 &&= 2 \times \tfrac{29}{5} - \left(-\tfrac{2}{5}\right) - 12 &&= 0 \\
z\text{-cofficient of } x_3 &= y_1 + 3y_2 - 4 &&= \tfrac{29}{5} + 3 \times -\tfrac{2}{5} - 4 &&= \tfrac{3}{5} \\
z\text{-cofficient of } x_4 &= y_1 - 0 &&= \tfrac{29}{5} - 0 &&= \tfrac{29}{5} \\
z\text{-cofficient of } R &= y_2 - (-M) &&= -\tfrac{2}{5} - (-M) &&= -\tfrac{2}{5} + M
\end{aligned}$$

- Formula 1 and 2 can be applied in any iteration provided inverse is know to us.

3. Addition Simplex Algorithms

- Primal Simplex Method: The simplex method which started with a (basic) feasible solution and remains feasible until the optimal is reached in the last iteration.
- Dual Simplex Method: LP starts at a better than optimal infeasible (basic) solution. Successive iterations remain infeasible and (better than) optimal until feasibility is resorted at the last iteration.
- Generalized Simplex Method: It combines both primal and dual simplex methods. It starts with both nonoptimal and infeasible. Successive iterations are associated with basic feasible or infeasible (basic) solution. At the final iteration, the solution becomes feasible and optimal.

Dual Simplex Method

- Two requirements for starting LP optimal and infeasible
  1. The optimality condition based on reduced cost of the regular (primal) simplex method must satisfy.
  2. All constraints must be of type (≤)
- How to deal with (≥) and (=) types of constraints?

$$x_1 + x_2 = 1$$

is equivalent to

$$x_1 + x_2 \leq 1, x_1 + x_2 \geq 1$$

or

$$x_1 + x_2 \leq 1, -x_1 - x_2 \leq -1$$

Dual Simplex Method

Dual feasibility condition. The leaving variable, x_r, is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.

Dual optimality condition. Given that x_r is the leaving variable, let $\bar{c}_j$ be the reduced cost of nonbasic variable x_j and α_{rj} the constraint coefficient in the x_r-row and x_j-column of the tableau. The entering variable is the nonbasic variable with $\alpha_{rj} < 0$ that corresponds to

$$\min_{\text{Nonbasic } x_j} \left\{ \left| \frac{\bar{c}_j}{\alpha_{rj}} \right|, \alpha_{rj} < 0 \right\}$$

(Ties are broken arbitrarily.) If $\alpha_{rj} \geq 0$ for all nonbasic x_j, the problem has no feasible solution.

Dual Simplex Method

$$\text{Minimize } z = 3x_1 + 2x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 + x_2 + x_3 &\geq 3 \\ -3x_1 + 3x_2 + x_3 &\geq 6 \\ x_1 + x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Nonnegative RHS condition is not Imposed.

In the present example, the first two inequalities are multiplied by -1 to convert them to ($\leq$) constraints. The starting tableau is thus given as:

Reduced cost ≤ 0 for Minimization problem

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3
Ratio	-----	2/3	1	-----	-----	----	

Dual Simplex Method

- Apply Gauss-Jordon Row Operation

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	-1/3	0	-2/3	0	4
x_4	-4	0	-2/3	1	-1/3	0	-1
x_2	-1	1	1/3	0	-1/3	0	2
x_6	2	0	2/3	0	1/3	1	1
Ratio	5/4	-----	1/2	-----	2	-----	

- x_4 will leave and x_3 will enter

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-3	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_6	-2	0	0	1	0	1	0

- All variables are ≥ 0, we get the feasible solution

Is it optimal? (minimization problem)

Generalized Simplex Algorithm

Maximize $z = 2x_3$
subject to

$$\begin{aligned} -x_1 + 2x_2 - 2x_3 &\geq 8 \\ -x_1 + x_2 + x_3 &\leq 4 \\ 2x_1 - x_2 + 4x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

- Starting tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	-2	0	0	0	0
x_4	1	-2	2	1	0	0	-8
x_5	-1	1	1	0	1	0	4
x_6	2	-1	4	0	0	1	10

- Nonoptimal: x_3 is negative reduced cos
- Infeasible: x_4 is negative

Generalized Simplex Algorithm

- Remove infeasibility by applying dual simplex method condition
  - x_4 is leaving variable.
  - x_2 is entering variable (negative coefficient).

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	−2	0	0	0	0
x_2	$-\frac{1}{2}$	1	−1	$-\frac{1}{2}$	0	0	4
x_5	$-\frac{1}{2}$	0	2	$\frac{1}{2}$	1	0	0
x_6	$\frac{3}{2}$	0	3	$-\frac{1}{2}$	0	1	14

Is it feasible?

Is it optimal?
(maximization problem)

- Use primal simplex method steps
- Establish the feasibility first and then satisfy the optimality.

Generalized Simplex Method

- Now apply primal simplex method

	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio
z	0	0	-2	0	0	0	0	
x_2	-1/2	1	-1	-1/2	0	0	4	
x_5	-1/2	0	2	1/2	1	0	0	0
x_6	3/2	0	3	-1/2	0	1	14	14/3
z	-1/2	0	0	1/2	1	0	0	
x_2	-3/4	1	0	-1/4	1/2	0	4	---
x_3	-1/4	0	1	1/4	1/2	0	0	---
X_6	9/4	0	0	-5/4	-3/2	1	14	56/9
z	0	0	0	2/9	2/3	2/9	28/9	
x_2	0	1	0	-2/3	0	1/3	26/3	
x_3	0	0	1	7/18	1/3	1/9	14/9	
x_1	1	0	0	-5/9	-2/3	4/9	56/9	