Simple design of fractional delay allpass filters

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ABSTRACT

A novel closed-form method for designing fractional delay allpass filters is proposed. The design uses closed-form formulas and is based on truncating the coefficient vector of a Thiran allpass filter. While the resulting filters are non-optimal, they allow a wider approximation bandwidth than the Thiran allpass filter, which yields a maximally flat delay approximation at the zero frequency. Formulas have been derived to assist in choosing the two parameters, order and prototype order, for the new design. There is practically no upper limit for the filter order, since the method is not prone to numerical problems.

1 INTRODUCTION

Fractional delay filters are useful in numerous digital signal processing applications where accurate time delays are needed or the locations of sampling instants must be changed, such as in telecommunications, music synthesis, and speech coding [1, 2]. Many design methods have been proposed for fractional delay filters of FIR and IIR type [1, 2]. Within the class of IIR filters, digital allpass filters have been a popular choice since their magnitude response is exactly flat and the design can concentrate entirely on the phase response. The transfer function of a digital allpass filter is given by

$$H(z) = \frac{z^{-N} D(z^{-1})}{D(z)} \tag{1}$$

where N is the order of the filter and $D(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}$ is the denominator polynomial with real-valued coefficients a_k, and the numerator polynomial is a reversed version of the denominator.

The design of fractional delay allpass filters is usually based on solving a set of linear equations, such as the least squares method proposed by Lang and Laakso [3, 1], or on an iterative optimization algorithm, such as pseudo-equiripple design techniques [1, 2, 3]. These methods produce optimal or very nearly optimal designs, but their usefulness is limited when high-order filters are needed or when coefficient values should be calculated online in a real-time application. The largest allpass filter order that is possible with current design programs is about 20 or less, depending on specifications [4].

Only one FD allpass filter design method is known that can be implemented using closed-form formulas: the maximally flat group delay method [5] that is based on Thiran's allpole filter design [6]. A drawback of this method is that the fractional delay approximation is excellent only on a narrow band at low frequencies, and a dramatic widening of the bandwidth of good approximation requires the filter order to be increased excessively.

This paper proposes a design method that is based on the Thiran allpass design but overcomes the problem of narrow approximation band while the design is still based on closed-form formulas. Section 2 describes the Thiran allpass filter design method. The new design method is introduced in Section 3. Example designs are presented, and formulas that enable finding a compromise between the approximation bandwidth and peak error of the frequency response magnitude are given. Section 4 concludes the paper.

2 THIRAN ALLPASS FILTER

In 1971, Thiran published a closed-form design method for allpole filters that have a prescribed maximally flat group delay [6]. Fettweis showed that the design formulas can be used for obtaining allpass filters that have the same property [5]. When the desired group delay of an allpass filter is d, it is only necessary to make the substitution $d' = d/2$ in Thiran's formula, since the group delay of an allpass filter is twice that of its denominator (see, e.g., [1]). The Thiran design formula for a fractional delay allpass filter can be written as [5, 1]

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{d+n}{d+k+n} \tag{2}$$

where d is the real-valued delay parameter and $k = 1, 2, 3, ..., N$.

Closed-form formulas that are at most Nth-order rational polynomials of delay d can be obtained from (2). For example, when $N = 2$, the filter coefficients are $a_1 = -2(D-2)/(D+1)$ and $a_2 = (D-1)(D-2)/(D+1)(D+2)$. Here we have introduced a notation $D = N + d$, where D denotes the group delay (in samples) that the allpass filter produces at low frequencies.

In [6] it was shown that the numerator polynomial $D(z)$ of the original Thiran allpole filter has all its zeros inside the unit circle for $d > -0.5$. This implies that the allpass filter designed using Eq. (2) is stable for $d > -1$, because the group delay of the allpass filter is twice that of the numerator (see, e.g., [1]).

Figure 1 shows the group delay error of Thiran allpass filters when $d = -0.5$ and $N = 1, 2, 3, ..., 9$. The frequency variable has

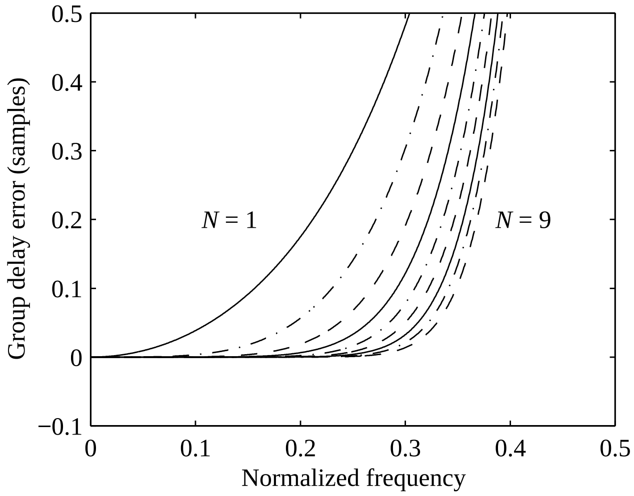

Fig. 1. Group delay error curves of Thiran allpass filters of order N = 1, 2, 3, ..., 9 (left to right) for d = –0.5.

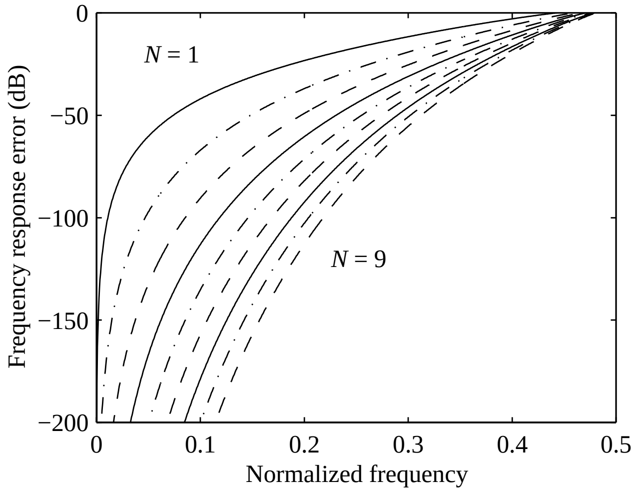

Fig. 2. Frequency response error curves of Thiran allpass filter of order N = 1, 2, 3, ..., 9 (top to bottom) for d = –0.5.

been normalized so that 0.5 corresponds to the Nyquist limit. Note that in Fig. 1 all the group delay error curves are close to zero at low frequencies, as desirable. However, the error increases with frequency, and particularly in the case of low-order filters, the deviation soon becomes large. Also note that when the order of the filter is increased, the bandwidth of good approximation (error smaller than, e.g., 0.1 samples) is not becoming much wider.

It is also of interest to examine the frequency response error (FRE) of the allpass filter as a measure of approximation quality. The following definition is used for the FRE:

$$E(e^{j\omega}) = e^{-j\omega(N+d)} - H(e^{j\omega}) \tag{3}$$

where the first part on the right-hand side represents the frequency response of an ideal fractional delay filter producing a delay of $N + d$ sampling intervals, and the second term is the frequency response of the allpass filter obtained from (1) using coefficients (2), which approximates a constant delay of $N + d$ samples. Figure 2 shows the frequency response error of the Thiran allpass filters of Fig. 1. Notice the sluggish increase of approximation bandwidth as a function of filter order N.

3 NEW DESIGN METHOD

In this section, a new design method is introduced and its properties and usage are discussed.

3.1 Truncating the Coefficient Vector

The coefficient values of the Thiran allpass filter typically decay fast with index k. Sometimes the coefficient values are extremely small. For example, when $d = –0.5$ and $N = 10$, the value of coefficient a_{10} is –0.000001427. In many implementations, coefficients that are this small will be rounded to zero. The contribution of the small coefficients cannot be significant on the properties of the filter. Thus, it seems sensible to neglect the smallest values by truncating the coefficient vector of the allpass filter. The design method proposed in this paper is based on truncating the coefficient vector even further.

The new design formula for a fractional delay allpass filter is a slightly modified form of Eq. (2):

$$a_k = (-1)^k \binom{M}{k} \prod_{n=0}^{M} \frac{d+n}{d+k+n} \tag{4}$$

where d is the real-valued fractional delay parameter and $k = 1, 2, 3, ..., N$. Usually M is equal to the filter order N but here we propose to choose $M > N$. It is convenient to call M the prototype filter order, since it corresponds to the order of the original allpass filter before truncation. Experiments suggest that the truncated allpass filters ($M > N$) are stable for $d > –1$, i.e., the stability condition is the same as that of the Thiran allpass filter.

3.2 Behavior of Approximation Error

Figure 3 shows the FRE when $d = –0.5$ and $N = 10$. A Thiran allpass filter is compared against fractional delay allpass filters whose coefficients have been truncated from prototype filters of orders $M = 100$, 20, 14, and 11. Interesting properties of the truncated Thiran allpass filters can be observed in Fig. 3. Most importantly, the quality of the truncated allpass filters degrades gracefully since the lobe structure is similar to that of optimal digital filters, such as least squares designs. Closer examination reveals that the results are non-optimal, however.

The level of the highest lobe of the FRE function appears to increase monotonically and smoothly as more and more coefficients are truncated. The approximation bandwidth, which we may define as the largest normalized frequency where the FRE is smaller than or equal to the maximum lobe level, also tends to increase as a function of the number of truncated coefficients. For example in Fig. 3, it is seen that the widest bandwidth (about 0.46) is obtained with a 10th-order filter whose coefficients have been obtained by selecting the 10 first coefficients from a Thiran allpass filter of order $M = 100$. This is also the case for which the maximum FRE is the largest among the example designs given in

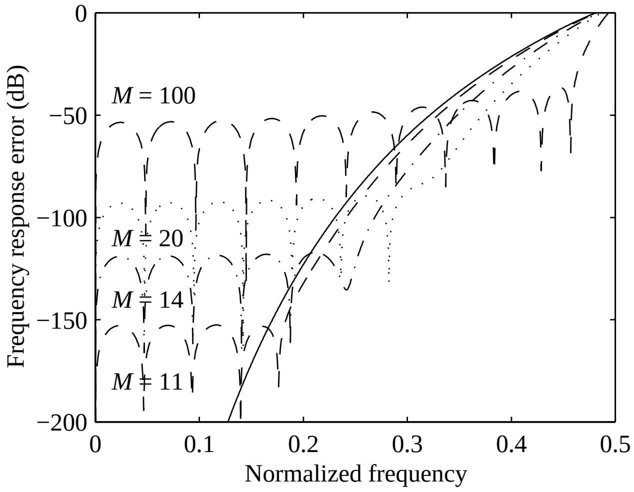

Fig. 3. Frequency response error of several 10th-order fractional delay allpass filters: a Thiran allpass filter of order N = 10 (solid line) and 10th-order truncated allpass filters obtained from prototype filters of order M = 100, 20, 14, and 11 (top to bottom).

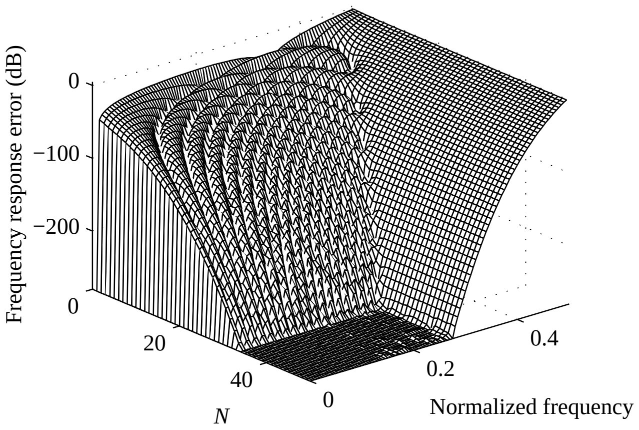

Fig. 4. Frequency response error of a family of truncated Thiran allpass filters obtained from a prototype filter of order M = 50.

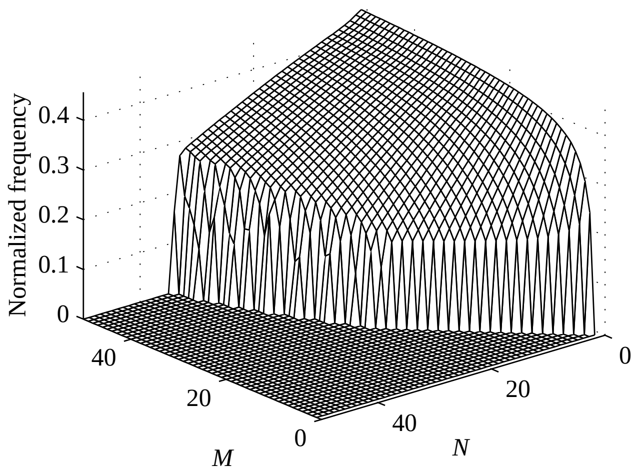

Fig. 5. Normalized bandwidth of the truncated Thiran allpass filters as a function of prototype filter order M and truncated filter order N, when d = –0.5.

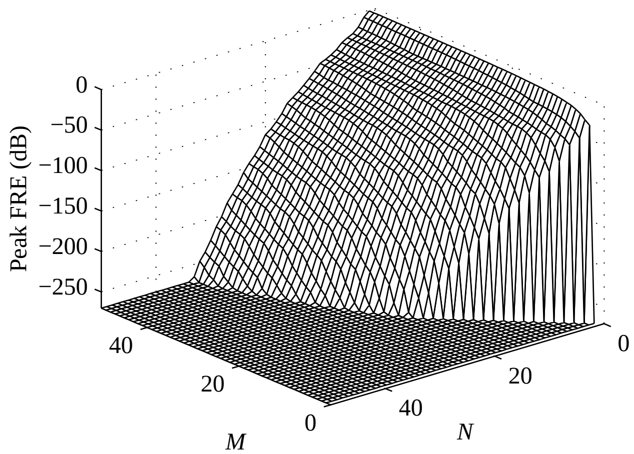

Fig. 6. Peak frequency response error of truncated Thiran allpass filters in the approximation band as a function of prototype filter order M and truncated filter order N, when d = –0.5.

Fig. 3 (about –36 dB). The two properties are connected: when the bandwidth is increased, the error also becomes larger.

Figure 4 shows the frequency response error as a function of truncated filter order when prototype order M is 40 and $d = -0.5$. Here it is clearly seen that the price to be paid for cheaper filter implementation (that is, lower-order allpass filter) is that the error increases. The truncation leads to a larger error at low frequencies, which is visible for cases $N \leq 35$. When $N > 35$, the contribution of the truncation error is smaller than the numerical noise (below –280 dB) in the generation of this figure. The curve at $N = 50$ corresponds to the prototype filter, which is a maximally flat approximation whose frequency response error at $\omega = 0$ is 0, which is equal to $-\infty$ dB.

3.3 How to Choose *N* and *M*

Figures 5 and 6 illustrate the behavior of normalized bandwidth and maximum frequency response error magnitude when $1 \leq M \leq 50$ and $1 \leq N \leq M$. Note that these two-dimensional functions are smooth for small values of N. A discontinuity seen as a cliff in Fig. 5 arises from numerical reasons: when only a few coefficients are truncated, the maximum side lobe vanishes below the numerical noise level (see Figs. 4 and 6) and thus the normalized bandwidth cannot be determined for such cases. The behavior of the smooth part of the functions in Figs. 5 and 6 can be approximated with a two-dimensional polynomial.

We have experimentally devised formulas that describe the behavior of the normalized bandwidth and the maximum FRE of the proposed allpass filter as a function of N and M. These formulas facilitate the design of fractional delay allpass filters when the desired approximation bandwidth and maximum error are given.

The normalized approximation bandwidth B (defined in Section 3.2 above) can be approximated in the following way:

$$B = 3.660 - 0.8367N + (-2.055 + 0.5352N)\arctan(M) \quad (5)$$

The coefficients in this equation have been obtained using non-

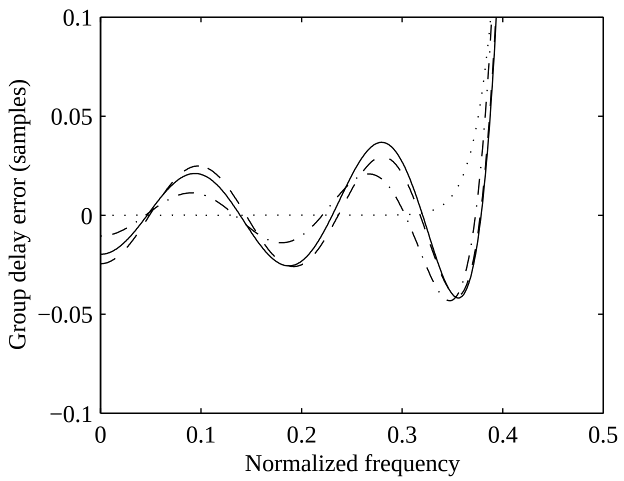

Fig. 7. Group delay error of 5th-order allpass filters for d = –0.5 obtained with different design methods: truncated Thiran (solid line), least squares (dash-dot line), and pseudo-equiripple phase (dashed line). The prototype Thiran allpass filter (dotted line) of order 19 is shown for comparison.

linear least squares optimization in the range $5 \le M \le 100$ with $d = -0.5$, which was assumed to be nearly the worst case. Differences from the actual normalized bandwidth can be as large as about 10 % and even larger near the cliff in Fig. 5. For example, for the cases shown in Fig. 3, Eq. (5) gives the normalized bandwidths 0.44, 0.31, 0.24, and 0.17 for the cases M = 100, 20, 14, and 11, respectively. Finding the proper combination of values of N and M may require several trials—the formula can be used for choosing an initial guess.

The maximum FRE, which we denote by E_{max}, can also be approximated with a simple formula:

$$E_{\max} = p_1(N) + p_2(N)\arctan(M) \tag{6}$$

where $p_1(N)$ and $p_2(N)$ are second-order polynomials of N:

$$p_1 = 86.15 - 80.93N - 14.48N^2 \tag{7}$$

$$p_2 = -60.21 + 50.47N + 9.242N^2 \tag{8}$$

These polynomials were obtained using nonlinear least squares optimization. The differences between the estimates obtained with these polynomials and the actual peak error is typically less than 10 dB and can be larger sometimes, but the results can be beneficial in estimating the behavior of a filter or when searching for suitable values for N and M. For example, in the case N = 10 and M = 100, the peak FRE estimated using Eq. (6) is –35 dB, which is not far from what is seen in Fig. 3 (about –36 dB).

3.4 Design Example and Comparison

As an example, a wideband fractional delay allpass filter is designed. The specifications for the worst case (d = –0.5) are the following: the frequency response error must be smaller than –40 dB at frequencies below $0.4f_s$.

With the design formulas (5) and (6) it is easy to soon find many candidates for the combination of N and M which would fill the specifications. One of the solutions with the smallest possible filter order N should be chosen. The combination M = 19, N = 5 seems to be the optimal choice. For this case, the estimated bandwidth and peak error are 0.42 and –41 dB, respectively. The corresponding actual values are 0.4003 and –42.06 dB.

Figure 7 shows the group delay error for the truncated design. For comparison, results of a least-squares phase design [3] and a pseudo-equiripple phase design [1] are presented in Fig. 7. These filters also fulfill the specifications with filter order N = 5. Note that the difference between these optimal designs and the truncated Thiran filter is not large in this example. Naturally, comparison of allpass filters designed using the proposed method with other techniques reveals that the more elaborate optimal method usually yields a superior approximation.

The main advantages of the new method are thus the ease of the design using closed-form formulas and the possibility to design high-order filters. The maximum filter order obtainable using the other two methods is less than 20, while with the new method orders exceeding 1000 cause no problems.

4 CONCLUSION

The truncation of the coefficient vector of a Thiran allpass filter provides a simple design method which facilitates the design of very high-order fractional delay filters if desired. The values of prototype filter order M and the allpass filter N are practically unlimited, since the design formula is not prone to numerical problems. The method facilitates wideband approximation of fractional delay using lower filter orders than the original Thiran design. The proposed design is based on a closed-form formula.

5 ACKNOWLEDGMENTS

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