Problem 1 (a)

Proof.

$$\begin{aligned} & 2n^2 + 3n + 7 \in \mathcal{O}\left(n^2\right) \\ \implies & 2n^2 + 3n + 7 \leq c \cdot n^2 \\ \implies & \frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{7}{n^2} \leq c \\ \implies & 2 + \frac{3}{n} + \frac{7}{n^2} \leq c \end{aligned}$$

The Big-Oh condition holds for $n \geq 1$ when $c = 12$. Therefore $2n^2 + 3n + 7 \in \mathcal{O}\left(n^2\right)$. □

Problem 1 (b)

Proof.

$$\begin{aligned} & 100n^3 - n^2 + 5n \in \mathcal{O}\left(n^3\right) \\ \implies & 100n^3 - n^2 + 5n \leq c \cdot n^3 \\ \implies & \frac{100n^3}{n^3} - \frac{n^2}{n^3} + \frac{5n}{n^3} \leq c \\ \implies & 100 - \frac{1}{n} + \frac{5}{n^2} \leq c \end{aligned}$$

The Big-Oh condition holds for $n \geq 1$ when $c = 104$. Therefore $100n^3 - n^2 + 5n \in \mathcal{O}\left(n^3\right)$. □

Problem 1 (c)

Proof.

$$\begin{aligned} & 15n^4 + 3n^3 \in \Omega\left(n^4\right) \\ \implies & 15n^4 + 3n^3 \geq c \cdot n^4 \\ \implies & \frac{15n^4}{n^4} + \frac{3n^3}{n^4} \geq c \\ \implies & 15 + \frac{3}{n} \geq c \end{aligned}$$

The Big-Omega condition holds for $n \geq 1$ when $c = 18$. Therefore $15n^4 + 3n^3 \in \Omega\left(n^4\right)$. □

Problem 1 (d)

Proof.

$$
\begin{aligned}
& 2n^2 \log n - 2n^2 \in \Omega\left(n^2\right) \\
\implies & 2n^2 \log n - 2n^2 \geq c \cdot n^2 \\
\implies & \frac{2n^2 \log n}{n^2} - \frac{2n^2}{n^2} \geq c \\
\implies & 2 \log n - 2 \geq c
\end{aligned}
$$

The Big-Omega condition holds for $n \geq 4$ when $c = 2$. Therefore $2n^2 \log n - 2n^2 \in \Omega\left(n^2\right)$. □

Problem 1 (e)

Proof.

$$
\begin{aligned}
& a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \in \Theta\left(n^k\right) \\
\implies & a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \leq c_0 \cdot n^k \\
\implies & \frac{a_k n^k}{n^k} + \frac{a_{k-1} n^{k-1}}{n^k} + \cdots + \frac{a_0}{n^k} \leq c_0 \\
\implies & a_k + \frac{a_{k-1}}{n^1} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_0}{n^k} \leq c_0 \\
& a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \in \Theta\left(n^k\right) \\
\implies & a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \geq c_1 \cdot n^k \\
\implies & \frac{a_k n^k}{n^k} + \frac{a_{k-1} n^{k-1}}{n^k} + \cdots + \frac{a_0}{n^k} \geq c_1 \\
\implies & a_k + \frac{a_{k-1}}{n^1} + \frac{a_{k-2}}{n^2} + \cdots + \frac{a_0}{n^k} \geq c_1
\end{aligned}
$$

The Big-Theta condition holds for $n \geq 1$ when $c_0 = c_1 = \Sigma_{i=0}^{k} a_i$. Therefore $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \in \Theta\left(n^k\right)$. □

Problem 2 (a)

Proof.

$$
f(n) \in \mathcal{O}\left(g(n)\right) \implies f(n) \leq c \cdot g(n) \implies \frac{1}{c} \cdot f(n) \leq g(n)
$$

Let $d = \frac{1}{c}$. We can then rewrite the above expression as:

$$
g(n) \geq d \cdot f(n) \implies g(n) \in \Omega\left(f(n)\right)
$$

□

Problem 2 (b)

Since both functions are positive, $max\{f(n), g(n)\}$ is surely less than $f(n) + g(n)$. It must also be true that $2 \cdot max\{f(n), g(n)\}$ is greater than $f(n) + g(n)$. This gives us $max\{f(n), g(n)\} \leq f(n) + g(n) \leq 2 \cdot max\{f(n), g(n)\}$. Because $f(n) + g(n)$ is bounded on the top and bottom by constants, the Big-Theta condition is satisfied.

Problem 3

Proof.

$$\begin{aligned} f_1(n) + f_2(n) &\leq c_0 \cdot g_1(n) + c_1 \cdot g_2(n) \\ &\leq (c_0 + c_1) \cdot max(g_1(n) + g_2(n)) \end{aligned}$$

$$\begin{aligned} f_1(n) + f_2(n) &\geq c_2 \cdot g_1(n) + c_3 \cdot g_2(n) \\ &\geq (c_2 + c_3) \cdot max(g_1(n) + g_2(n)) \end{aligned}$$

□

Problem 4 (a)

$$\Theta\left(n^{12}\right)$$

Proof.

$$\lim_{n\to\infty} \frac{(2n+15)^{12}}{n^{12}} = 4096$$

□

Problem 4 (b)

$$\Theta\left(n^{\frac{3}{2}}\right)$$

Proof.

$$\lim_{n\to\infty} \frac{\sqrt{4n^3 + 3n^2 + 1}}{n^{\frac{3}{2}}} = 2$$

□

Problem 4 (c)

$$\Theta\left(n^2 \log n\right)$$

Proof.

$$\lim_{n\to\infty} \frac{(n+2)^2 \log n/2}{n^2 \log n} = 1$$

The first term of the expression is lower order, so can be safely ignored as it will approach zero as $n \to \infty$. □

Problem 4 (d)

$$\Theta\left(2^{2n}\right)$$

Proof.

$$\lim_{n\to\infty} \frac{2^{2n+1} + 3^{n-10}}{2^{2n}} = 2$$

□